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The sharp Markov property of the Brownian sheet and related processes

by

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0. Introduction

The Brownian sheet $(W_t, t\in\mathbf{R}^2_+)$ has long been known to satisfy Paul Lévy's sharp Markov property with respect to all finite unions F of rectangles (see [W1, Ru]), meaning that

(0.1) $\mathcal{H}(F)$ and $\mathcal{H}(\bar{F}^c)$ are conditionally independent given $\mathcal{H}(\partial F)$,

where $\mathcal{H}(F)=\sigma(W_t,\ t\in F)$ represents the information one can obtain about the sheet by observing it only in the set F. However, (0.1) fails when F is the triangle $\{(t_1,t_2)\in\mathbf{R}^2_+: t_1+t_2<1\}$ [W1], leaving the impression that the sharp Markov property is valid only for a very restricted class of sets. In contrast, the weaker germ-field Markov property, in which one replaces $\mathcal{H}(\partial F)$ by the germ-field $\mathcal{H}^*(\partial F)=\bigcap\mathcal{H}(O)$ (where the intersection is over all open sets containing ∂F), is valid for all open sets in the plane [Ro, Nu]).

One natural explanation for this is the following: in the one-parameter setting, the Markov property of the solution of a stochastic differential equation is closely connected with uniqueness for the initial value problem. Something similar should be true in the plane. Now the Brownian sheet is the solution of a certain hyperbolic partial differential equation [W3], and its Markov property is closely connected to the uniqueness problem for the hyperbolic partial differential equation $\partial^2 u/\partial x\partial y=0$. It is well-known that the boundary data needed to pose the Cauchy problem for this equation are the values of the function on the boundary together with the normal derivative at non-

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characteristic points. For a smooth curve Γ, the normal derivative of the Brownian sheet has been defined by Piterbarg [Pi; Theorem 2], and he has shown that $\mathscr{H}(\partial F)$ and the normal derivative together generate the germ-field. Hence, one can expect the germ-field Markov property.

Of course, for curves which are not smooth, the concept of normal derivative no longer makes sense, but one can still think of the generalized partial derivatives $\partial W/\partial x$ and $\partial W/\partial y$. It can be shown that these generalized partial derivatives correspond to the white noise measures of certain sets, which can be given explicitly. This more down-to-earth description of the minimal σ-field $\mathscr{S}$ (termed minimal splitting field) such that $\mathscr{H}(F)$ and $\mathscr{H}(\bar{F}^c)$ are conditionally independent given $\mathscr{S}$ was given in [W1; W4, Theorem 3.12] for domains with smooth boundaries and in [WZ; Proposition 2] for domains whose boundary consists of piecewise monotone curves.

In this paper, we extend this description of the minimal splitting field to all open sets in the plane, not only for the Brownian sheet but for a wide class of (not necessarily Gaussian) processes with independent planar increments (see Assumption 1.1). The connection with the Cauchy problem and generalized normal derivative is not explored here, though our results suggest a natural definition of characteristic points for non-smooth curves which will be examined in a future paper. Our main objective is to determine which sets F have the sharp Markov property (0.1). For the class of processes satisfying Assumption 1.1 below, sufficient conditions are given for a general open set. For Jordan domains, the sufficient condition turns out to be necessary for the Brownian sheet, yielding a complete answer in this case.

Our approach is as follows: once the minimal splitting field is determined, it is clear that the sharp Markov property will hold if and only if this σ-field is contained in $\mathscr{H}(\partial F)$. One can then determine conditions on the boundary of F for this to be the case. It turns out that there are essentially two ways in which this can happen:

(a) ∂F is essentially horizontal or vertical at most points. This is the case for instance if ∂F is a singular separation line ([DR; Theorem 3.12]; the result of Dalang and Russo was the first instance where the sharp Markov property was shown to hold for a curve containing no vertical or horizontal segment). Here, this result is extended to all singular curves of bounded variation (see Corollary 6.3).

(b) ∂F is rather "thick", e.g. it could have positive two-dimensional Lebesgue measure, or it could be a fractal such as the Sierpinski gasket, or the sample path of a linear Brownian motion.

The necessary and sufficient conditions for a domain bounded by a Jordan curve to satisfy (0.1) are of geometric character, making use of an apparently new condition on planar curves: the Maltese cross condition (see Definition 1.2). In various special cases, this condition reduces to known conditions. For example, if the boundary curve is rectifiable, the Maltese cross condition can be expressed in terms of a parameterization of the curve. This is the natural generalization of the result of Dalang and Russo. If the curve is the graph of a continuous function $y=f(x)$, the Maltese cross condition can be expressed in terms of the Dini-derivatives of f.

From our main result, we can obtain a variety of statements to the effect that the Brownian sheet has the sharp Markov property with respect to almost all Jordan curves, altering the impression mentioned above. The "almost all" can be interpreted both in the sense of Baire category and with respect to various reference measures.

The paper is structured as follows. In Section 1, we present the main assumptions and results. In Section 2, we prove several results concerning sharp field measurability of various random variables. Section 3 gives an explicit description of the minimal splitting field of an arbitrary open set (Theorem 3.3). Section 4 contains sufficient conditions for an open set to have the sharp Markov property (Theorem 4.1), with application to some fractal sets. The proof that the Maltese cross condition implies (0.1) for Jordan domains is given in Section 5 (Theorem 5.6). This condition is proved to be necessary for the Brownian sheet in Section 6 (Theorem 6.1), and the case of rectifiable curves and some extensions are also examined there. Finally, Section 7 contains several theorems to the effect that "the Brownian sheet has the sharp Markov property with respect to almost all Jordan curves".

1. The main results

For the convenience of the reader, the main definitions and results are presented in this section. Throughout this paper, $T=\mathbf{R}_+^2$ will denote the nonnegative quadrant in the plane. The horizontal and vertical axes will be respectively called the x- and y-axes. Two natural orders on T are $\leq$ and $\triangle$, defined by

$$s=(s_1,s_2)\leq t=(t_1,t_2) \quad\Leftrightarrow\quad s_1\leq t_1 \text{ and } s_2\leq t_2,$$

$$s=(s_1,s_2)\,\triangle\, t=(t_1,t_2) \quad\Leftrightarrow\quad s_1\leq t_1 \text{ and } s_2\geq t_2.$$

A continuous curve which is totally ordered for $\leq$ (resp. $\triangle$) is termed *increasing* (resp. *decreasing*). If $t=(t_1,t_2)\in T$, we let $\mathrm{pr}_i(t)=t_i$, $i=1,2$, denote the 1- and 2-projections of t and we put $R_t=\{s\in T: s\leq t\}$.

Lebesgue measure on T will be denoted by m or dt, whereas Lebesgue measure on

R will be denoted by λ. "Measurable sets" will refer to Lebesgue measure, unless indicated otherwise. $\mathscr{B}(T)$ denotes the Borel σ-algebra on T, and $\mathscr{B}_b(T)$ the bounded elements of $\mathscr{B}(T)$.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. If $\mathscr{G} \subset \mathscr{F}$ is a σ-field, we will write $Y \in \mathscr{G}$ to indicate that the random variable Y is $\mathscr{G}$-measurable. A *two-parameter process* is a family $X=(X_t, t \in T)$ of random variables indexed by T. Given $F \subset T$, the *sharp field* $\mathscr{H}(F)$ of F is the σ-field $\mathscr{H}(F)=\sigma\{X_t, t \in F\} \vee \mathscr{N}$, where $\mathscr{N}$ is the σ-field generated by the P-null sets, and the *germ-field* $\mathscr{H}^*(F)$ is defined by

$$\mathscr{H}^*(F) = \bigcap \mathscr{H}(O),$$

where the intersection is over all open sets O containing F. If $R=]s_1, t_1] \times]s_2, t_2]$ is a *rectangle* (by "rectangle" we will always mean "rectangle with sides parallel to the axes"), the *planar increment* $\Delta_R X$ of X over R is

$$\Delta_R X = X_{t_1, t_2} - X_{s_1, t_2} - X_{t_1, s_2} + X_{s_1, s_2}.$$

The process X has *independent planar increments* provided the variables $\Delta_{R_1} X, \dots, \Delta_{R_n} X$ are independent, for all n and for all choices of disjoint rectangles $R_1, \dots, R_n$. The process X is *right-continuous* if for almost all $\omega \in \Omega$ and for all $t \in T$,

$$\lim_{s \to t,\, t \leq s} X_s(\omega) = X_t(\omega).$$

If in addition, the process X is *square-integrable* (i.e. $E(X_t^2) < +\infty$, $\forall t \in T$), then $t \mapsto E(X_t^2)$ is a right-continuous planar distribution function, corresponding to a measure ν_X on $\mathscr{B}(T)$.

If $F = R_1 \cup \dots \cup R_n$, where the R_k are disjoint rectangles, set

$$X(F) = \Delta_{R_1} X + \dots + \Delta_{R_n} X.$$

This defines an additive measure on the set of all finite unions of rectangles, taking values in $L^2(\Omega, \mathscr{F}, P)$. Suppose $E(X_t)=0$, for all $t \in T$. Then $E(X(F)^2) = \nu_X(F)$, so $X(\cdot)$ is ν_X-continuous [DU; Definition I.2.3], and thus has a unique σ-additive extension to $\mathscr{B}_b(T)$ [DU; Theorem I.5.2], which we again denote $X(\cdot)$, so X becomes an L^2-valued measure. In the more modern language of martingale theory, X is a two-parameter martingale and $(\nu_X([0, t_1] \times [0, t_2]), (t_1, t_2) \in T)$ is its expected quadratic variation (see [CW, I]).

In what follows, we will assume X satisfies the following assumption.

ASSUMPTION 1.1. *The process $X=(X_t, t\in T)$ is right-continuous and square-integrable with mean zero. It has independent planar increments, and ν_X is absolutely continuous with respect to Lebesgue measure.*

The Brownian sheet and the Poisson sheet are typical processes which satisfy this assumption. Recall that a *Brownian sheet* is a mean-zero, continuous Gaussian process $(W_t,\ t\in\mathbf{R}^2_+)$, with covariance function

$$E(W_s W_t)=\min(s_1,t_1)\min(s_2,t_2)$$

(see [W4; Chapter 3] for many results about this process). The definition and several properties of the *Poisson sheet* are given in [C; §3, Y]. Assumption 1.1 is also satisfied by many *stable sheets*, that is two-parameter processes with independent planar increments whose increments are stable random variables (see [L; Section 24.4]).

Assumption 1.1 implies in particular that $X(R)=X(\bar{R})$ if R is an open rectangle (as usual, $\bar{R}$ denotes the closure of R). It also allows us to work with Lebesgue measure, rather than with ν_X. Indeed, under this assumption, $X(\cdot)$ can be extended to all bounded Lebesgue measurable sets by setting $X(F\cup N)=X(F)$, when $F\in\mathscr{B}_b(T)$ and $m(N)=0$.

We now turn to the subject of this paper, namely the Markov property of processes satisfying Assumption 1.1. We begin by recalling some classical terminology.

A σ-field $\mathscr{S}$ such that $\mathscr{H}(F)$ and $\mathscr{H}(\bar{F}^c)$ are conditionally independent given $\mathscr{S}$ is termed a *splitting field* for F. When X is a Brownian sheet $W=(W_t,\ t\in T)$, the following properties are well-known.

(1.1) Any splitting field for F contains $\mathscr{H}(F)\cap\mathscr{H}(\bar{F}^c)$ ([Mc; Section 6], [W1]).

(1.2) If F is open, $\mathscr{H}^*(\partial F)$ is a splitting field for F (see [Ro; Chapter 3, § 5] for bounded open sets, [Nu; Theorem 3.1] in the general case).

(1.3) $\mathscr{H}(\partial F)$ is a splitting field for F when F is a finite union of rectangles [Ru; Theorem 7.5].

(1.4) $\mathscr{H}(\partial F)$ is *not* a splitting field when F is the triangular region $\{s\in T\colon s_1+s_2<1\}$ ([W1; W4; p. 399]).

Property (1.2) is known as the *germ-field Markov property* of the Brownian sheet. We say that the process X has the *sharp Markov property* (also known as *Lévy's Markov* property) with respect to $F\subset T$, or that F has the *sharp Markov property* relative to X, provided $\mathscr{H}(\partial F)$ is a splitting field for F (see [W2]). As mentioned in the introduction, because of (1.4), it has widely been assumed in the literature that the Brownian sheet has the sharp Markov property only with respect to a very restricted class of sets (e.g. those in (1.3)). Note that (1.4) is also valid for many other continuous two-parameter processes. The situation of the Poisson sheet is different: the sharp Markov property was shown in [C] to hold for all bounded relatively convex open sets, and it was conjectured there that this was also the case for all bounded open sets.

This conjecture is not addressed here (see however [DW]). Rather, we are interested in showing that the Brownian sheet (and processes which satisfy Assumption 1.1) actually do satisfy the sharp Markov property for a wide class of sets. This is achieved by giving an *explicit* description of the minimal splitting field for an arbitrary open set (see Theorem 3.3). This provides a powerful tool for determining sufficient conditions on an open set for it to have the sharp Markov property (Theorem 4.1). These conditions are easily seen to be satisfied by many sets with a "thick" fractal boundary, and we have in particular

COROLLARY 4.3. *Let D be an open set whose boundary is either the Sierpinski gasket or the Sierpinski carpet. Then D has the sharp Markov property.*

There are also many sets with a "thin" boundary which satisfy the sharp Markov property. We investigate this question in detail for *Jordan domains* D_1, that is domains D_1 for which $\partial D_1 = \Gamma$ is a Jordan curve. Recall that a Jordan curve is a subset of $T \cup \{\infty\}$ which is homeomorphic to the unit circle C. This is equivalent to the existence of a continuous one-to-one parameterization $\varphi: C \to \Gamma$. Indeed, the fact that φ^{-1} is continuous follows from compactness of C and continuity of φ (the image of a closed set under φ is compact).

Let $\mathcal{J}$ be the set of all bounded Jordan curves equipped with the *uniform metric d* defined by

$$d(\Gamma, \tilde{\Gamma}) = \inf \|\varphi - \tilde{\varphi}\|_\infty = \inf \sup_{x \in C} \|\varphi(x) - \tilde{\varphi}(x)\|, \tag{1.5}$$

where the infimum is over all parameterizations φ and $\tilde{\varphi}$ of Γ and $\tilde{\Gamma}$, respectively. This is indeed a metric. To get the triangle inequality, suppose $\Gamma^1, \Gamma^2, \Gamma^3 \in \mathcal{J}$, $\varepsilon > 0$ and φ^1, φ^2, ψ^2 and ψ^3 are respectively parameterizations of Γ^1, Γ^2, Γ^2 and Γ^3 such that

$$\|\varphi^1 - \varphi^2\|_\infty \le d(\Gamma^1, \Gamma^2) + \varepsilon \quad \text{and} \quad \|\psi^2 - \psi^3\| \le d(\Gamma^2, \Gamma^3) + \varepsilon.$$

Then $\varphi^3 = \psi^3 \circ (\psi^2)^{-1} \circ \varphi^2$ is another parameterization of Γ^3 such that

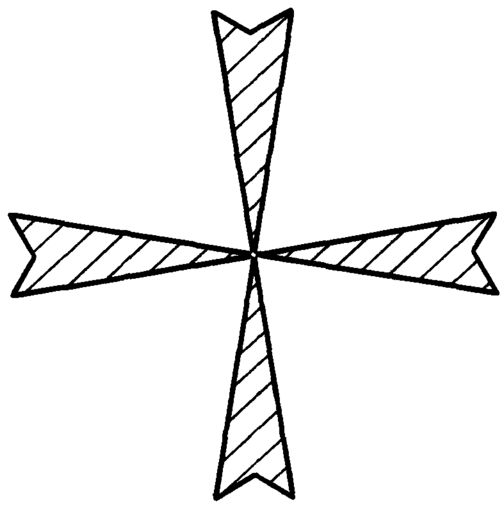


Fig. 1.1. A Maltese cross (a small indentation at the end of each branch of the cross is indicated for historical accuracy!).

$$\|\varphi^2-\varphi^3\|_\infty = \|\psi^2\circ(\psi^2)^{-1}\circ\varphi^2-\psi^3\circ(\psi^2)^{-1}\circ\varphi^2\|_\infty = \|\psi^2-\psi^3\|_\infty,$$

and thus

$$d(\Gamma^1,\Gamma^3)\le\|\varphi^1-\varphi^3\|_\infty\le d(\Gamma^1,\Gamma^2)+d(\Gamma^2,\Gamma^3)+2\varepsilon.$$

Recall that a Jordan curve Γ splits $\mathbf{R}^2$ into two open connected domains $D_1(\Gamma)$ and $D_2(\Gamma)$, and it is the boundary of both [N; Theorem 10.2].

THEOREM 7.3. *"Almost every" Jordan domain has the sharp Markov property, where "almost every" can be interpreted in the following sense. Let $\mathcal{G}$ be the set of all $\Gamma\in\mathcal{J}$ such that $\mathcal{H}(D_1(\Gamma))$ and $\mathcal{H}(D_2(\Gamma))$ are not conditionally independent given $\mathcal{H}(\Gamma)$. Then $\mathcal{G}$ has first Baire category.*

It is also possible to obtain a similar statement where the "almost every" refers to a probability measure on the set of Jordan curves. A natural choice of this measure is defined by Burdzy and Lawler [BL] as follows. Let $(B_u, u\in[0,1])$ be a planar Brownian motion, starting at the origin, defined on an auxiliary probability space $(\Omega', \mathcal{F}', P')$, and let $Z_u=B_u-uB_1$ be the associated Brownian bridge with endpoints at the origin. Let $D(\omega')$ be the unbounded connected component of the complement of the curve $u\to Z_u(\omega')$, $0\le u\le 1$. According to [BL; Theorem 1.5 (ii)], the boundary $\Gamma(\omega')$ of $D(\omega')$ is a Jordan curve P'-a.s. This induces a probability measure Q' on $\mathcal{J}$, for which we have the following result.

THEOREM 7.6. *For Q'-almost all $\Gamma \in \mathcal{J}$, $\mathcal{H}(D_1(\Gamma))$ and $\mathcal{H}(D_2(\Gamma))$ are conditionally independent given $\mathcal{H}(\Gamma)$.*

It turns out that we can give sufficient conditions on a Jordan curve for its two complementary domains to have the sharp Markov property; these are necessary when X is the Brownian sheet. To state them we need a few definitions and properties of Maltese crosses.

DEFINITION 1.2. (a) *Let $t \in \mathbf{R}^2$. The* Maltese cross of slope $\alpha>0$, radius $h>0$ *and* centered at t *is the set $M_\alpha(t,h)$ defined by*

$$M_\alpha(t,h) = \{s \in \mathbf{R}^2: |s_2 - t_2| < \alpha |s_1 - t_1| < \alpha h \ \text{ or } \ |s_1 - t_1| < \alpha |s_2 - t_2| < \alpha h\},$$

and for $\alpha=0$, we set

$$M_0(t,h) = \bigcap_{\alpha>0} M_\alpha(t,h).$$

(b) *The* Maltese cross condition *is satisfied at $t \in \Gamma$ if*

$$\Gamma \cap M_\alpha(t,h) \neq \varnothing, \quad \forall h>0, \ \forall \alpha>0.$$

The cross condition *is satisfied if*

$$\Gamma \cap M_0(t,h) \neq \varnothing, \quad \forall h>0.$$

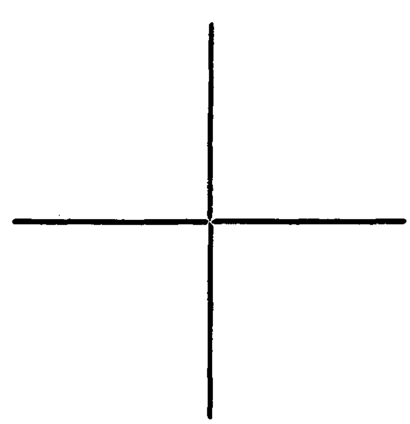


Fig. 1.2. A cross.

(c) *$M(\Gamma)$ is the set of $t\in\Gamma$ for which the Maltese cross condition is* not *satisfied, and $M_0(\Gamma)$ is the set of $t\in\Gamma$ for which the cross condition is* not *satisfied. For $\alpha\geq 0$ and $h>0$, put*

$$M_\alpha(\Gamma, h) = \{t\in\Gamma : \Gamma\cap M_\alpha(t,h) = \varnothing\}.$$

For $\alpha>0$, $M_\alpha(t, h)$ is open and does not contain t. The set has, roughly, the shape of a Maltese cross (see Figure 1.1).

For $\alpha=0$, $M_0(t, h)$ is shaped like a conventional cross: two crossed lines centered at t with t itself removed (see Figure 1.2). Clearly $M(\Gamma)\subset M_0(\Gamma)$ and

$$M_0(\Gamma) = \bigcup_{h>0} M_0(\Gamma, h).$$

The interest of this definition lies in Theorems 5.6 and 6.1 below. Some explanation concerning this condition is in order. Notice that the Maltese cross condition holds at $t_0=\varphi(u_0)$ if and only if

$$\liminf_{u\to u_0} \min\left(\frac{|\varphi_1(u)-\varphi_1(u_0)|}{|\varphi_2(u)-\varphi_2(u_0)|}, \frac{|\varphi_2(u)-\varphi_2(u_0)|}{|\varphi_1(u)-\varphi_1(u_0)|}\right) = 0. \tag{1.6}$$

Now when φ is differentiable at t, this means that the tangent to Γ at t is either horizontal or vertical. The Maltese cross condition is thus analogous to a condition on the tangent to Γ, but it does not require that the tangent exist.

If Γ is the graph of a continuous function φ_2 and if $\varphi(t)=(t_1, \varphi_2(t_1))$, then the Maltese cross condition is essentially a condition on the Dini derivatives of φ_2 (see [S; Chapter IV, § 2]):

$$\liminf_{h\to 0} \frac{|\varphi_2(t_1+h)-\varphi_2(t_1)|}{h} = 0 \quad\text{or}\quad \limsup_{h\to 0} \frac{|\varphi_2(t_1+h)-\varphi_2(t_1)|}{h} = +\infty.$$

THEOREM 5.6. *Let $(X_t, t\in T)$ satisfy Assumption 1.1, and let D be a Jordan domain with boundary Γ. Assume*

$$\lambda\{\mathrm{pr}_i(M(\Gamma))\} = 0, \quad i=1 \text{ or } 2. \tag{1.7}$$

Then D has the sharp Markov property.

The fact that one can choose either $i=1$ or $i=2$ in (1.7) is due to the property that

$$\lambda\{\mathrm{pr}_1(M(\Gamma))\} > 0 \iff \lambda\{\mathrm{pr}_2(M(\Gamma))\} > 0,$$

which is a straightforward consequence of Lemma 5.5(b).

In many cases, condition (1.7) is easy to check. For instance, if Γ is rectifiable, with a one-to-one parameterization $\varphi=(\varphi_1, \varphi_2)$: $[0, 1]\to T$, then φ_1 and φ_2 have bounded variation [S; Chapter 4, (8.2)], and so φ_i is canonically associated with a signed measure $d\varphi_i$ on $[0, 1]$, $i=1, 2$. We will show in Corollary 6.3 that

$$\lambda\{\mathrm{pr}_i(M(\Gamma))\} = 0 \quad \Leftrightarrow \quad d\varphi_1 \text{ and } d\varphi_2 \text{ are mutually singular.}$$

The above theorem shows for instance that there are many *unbounded* domains for which the Poisson sheet has the sharp Markov property: it suffices that (1.7) hold and that Γ pass through the point at infinity.

We know by [C; Theorem 4.1] that the Poisson sheet has the sharp Markov property with respect to many Jordan domains which do not satisfy (1.7). For the Brownian sheet, the situation is very different.

THEOREM 6.1. *Let $D\subset T\cup\{\infty\}$ be a Jordan domain with boundary Γ, and let $(X_t, t\in T)$ be a Brownian sheet. Then D has the sharp Markov property if and only if $\lambda\{\mathrm{pr}_i(M(\Gamma))\}=0$, $i=1$ or 2.*

Even for the Brownian sheet, the condition $\lambda\{\mathrm{pr}_i(M(\partial D))\}=0$ is not necessary for general domains D, though we conjecture that a slight modification of it is (see Remark 6.2).

2. Sharp field measurability and Vitali covering

In this section, we prove several statements concerning the sharp field of certain sets. Most of these are proved using the Vitali Covering Theorem (see 2.2). They will be useful in the following sections, but their proofs can be skipped until the reader is convinced they are really useful.

LEMMA 2.1. (a) *Let F and F_n, $n\in\mathbf{N}$, be measurable subsets of T, all contained in some fixed compact set. If the F_n are disjoint and $m(F\triangle\bigcup_{n\in\mathbf{N}}F_n)=0$ (where $\triangle$ denotes the symmetric difference), then $\sum_{n\in\mathbf{N}}X(F_n)$ converges in $L^2(\Omega, \mathcal{F}, P)$, and is equal to $X(F)$ a.s.*

(b) *For any set F, $\mathcal{H}(F)=\mathcal{H}(\bar{F})$.*

Proof. Set $G_m=\bigcup_{1\leq n\leq m}F_n$. Then $m(F\triangle G_m)\to 0$ as $m\to\infty$. By Assumption 1.1 and the dominated convergence theorem, $\nu(F\triangle G_m)\to 0$ as $m\to\infty$, so

$$\lim_{n\to\infty} E\left(\left(X(F)-\sum_{n=1}^{m} X(F_n)\right)^2\right)=0,$$

proving (a).

As for (b), since $F\subset\bar{F}$, we only need to show that X_t is $\mathscr{H}(F)$-measurable for each $t\in\bar{F}\setminus F$. Now for each such t, there is a sequence $(t^n, n\in\mathbf{N})$ of elements of F converging to t. But then $m(R_t\triangle R_{t^n})\to 0$ as $n\to\infty$, so by (a)

$$X_t = X(R_t) = \lim_{n\to\infty} X(R_{t^n}) = \lim_{n\to\infty} X_{t^n}$$

in $L^2(\Omega, \mathscr{F}, P)$. This completes the proof. □

Note that the conclusions of this lemma are not valid in general without Assumption 1.1. Indeed, if X is a Poisson point process on the line $s_2=1$ (i.e. X_t is the number of random points in the set $R_t\cap\{s\in T: s_2=1\}$) and if $F=[0,1[^2$, then $\mathscr{H}(F)$ is trivial but $\mathscr{H}(\bar{F})$ is not.

The following theorem is drawn from [S; Chapter IV, § 3]. The special case that we will be using is stated here for the convenience of the reader. Let $B(t,r)$ denote the open ball centered at t of radius r. A family $\mathscr{E}$ of sets *covers a set F in the sense of Vitali* provided for each $t\in F$ and $r>0$, there is $E\in\mathscr{E}$ with $t\in E\subset B(t,r)$.

VITALI COVERING THEOREM 2.2. *Let F be a Lebesgue measurable set in $\mathbf{R}$ (resp. $\mathbf{R}^2$), and let $\mathscr{E}$ be a family of closed non-degenerate intervals of $\mathbf{R}$ (resp. squares of $\mathbf{R}^2$) that covers F in the sense of Vitali. Fix $\varepsilon>0$. Then there is a finite or countable sequence (E_n) of disjoint elements of $\mathscr{E}$ such that $\lambda(F\setminus\bigcup_{n\in\mathbf{N}}E_n)=0$ (resp. $m(F\setminus\bigcup_{n\in\mathbf{N}}E_n)=0$) and $\lambda(F\triangle\bigcup_{n\in\mathbf{N}}E_n)<\varepsilon$ (resp. $m(F\triangle\bigcup_{n\in\mathbf{N}}E_n)<\varepsilon$).*

Most texts only give the first statement in 2.2. However the second statement follows from the first: it suffices to consider only sets in $\mathscr{E}$ which are contained in a fixed open set $O\supset F$ with $\lambda(O\setminus F)<\varepsilon$ (resp. $m(O\setminus F)<\varepsilon$).

For $F\subset T$, and $i=1, 2$ set

$$S^i(F)=\{(t_1,t_2)\in T: \exists(s_1,s_2)\in F \text{ with } s_i=t_i,\ s_{3-i}\geq t_{3-i}\}.$$

For $i=1$, this set is the "vertical shadow" of F, and for $i=2$ it is the "horizontal shadow". An example is shown in Figure 2.1. Observe that if F is open (resp. compact), then $S^i(F)$ is open (resp. compact).

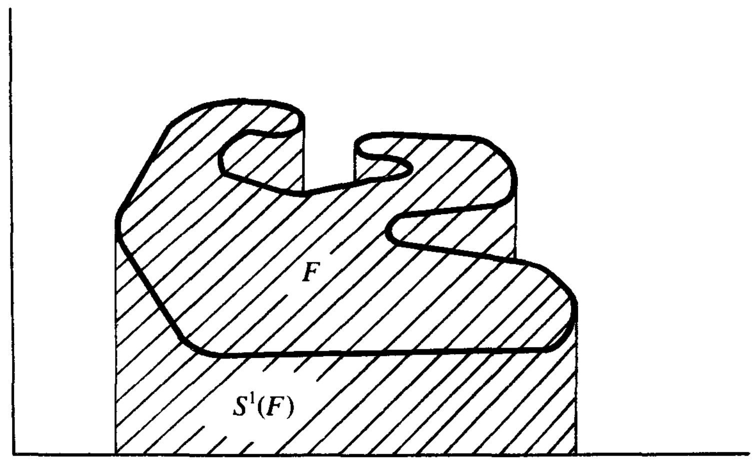


Fig. 2.1. The vertical shadow of F.

PROPOSITION 2.3. *Let F be a bounded Borel subset of T which is totally ordered for $\leq$ (resp. $\triangle$). Then $X(S^1(F))+X(S^2(F))$ (respectively $X(S^1(F))-X(S^2(F))$) is $\mathscr{H}(F)$-measurable.*

Proof. We assume that F is totally ordered for $\leq$ (modifications for the other case will be indicated below). Then the intersection of F with any line of the form $t_1+t_2=c$ is either empty or contains exactly one point. Let $L(F)$ be the union of the x-axis and the set

$$\{s \in T\colon \exists t \in F \text{ such that } t \triangle s\}.$$

According to [W3; Theorem 2.7], the boundary of $L(F)$ is a continuous curve C with a parameterization $Z=(Z^1,Z^2)\colon \mathbf{R}_+\to\mathbf{R}_+^2$ such that $Z(0)=(0,0)$, $u\mapsto Z(u)$ is increasing for $\leq$, and $Z^1(u)+Z^2(u)=u$. It is easily seen that any open interval $I\subset\mathbf{R}_+$ has the property

$$\lambda\{\mathrm{pr}_1(Z(I))\}+\lambda\{\mathrm{pr}_2(Z(I))\}=\lambda(I). \tag{2.1}$$

By a standard monotone class argument, we see that (2.1) holds for all Borel subsets of C.

Fix $\varepsilon>0$, and using Assumption 1.1, let $\delta>0$ be such that $m(G)<\delta$ implies $\nu_X(G)<\varepsilon$. Now set $B=\{(u,0)\colon u=s_1+s_2,\ (s_1,s_2)\in F\}$ (this is the 45 degree projection of F onto the x-axis: see Figure 2.2). By (2.1), $\lambda(B)=\lambda(\mathrm{pr}_1(F))+\lambda(\mathrm{pr}_2(F))$. Define

$$h_i(x)=\inf\{Z^{3-i}(u)\colon Z^i(u)\geq x\},\quad x\in\mathbf{R}_+,\ i=1,2.$$

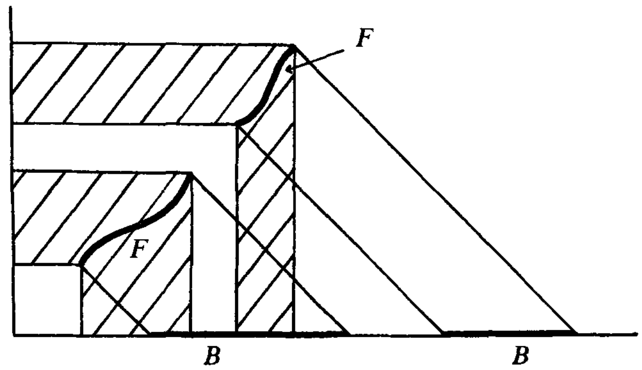


Fig. 2.2. Note that Proposition 2.3 is obvious for smooth sets F.

Since F is bounded, there is M such that $h^i(x) \leq M$ when $x \leq \sup \mathrm{pr}_i(Z(B))$, $i=1,2$.

Observe that the set $\mathscr{I}$ of all intervals $[a, b]$, $a<b$, such that a and b are both in B is a Vitali covering of the set of points of density of B (see [S; Chapter IV, (10.2)]). Since $\delta/M>0$, let $(I_n, n \in \mathbf{N})$ be a sequence of disjoint intervals of $\mathscr{I}$ with the properties guaranteed by Theorem 2.2. Then

$$\lambda\Big(B \setminus \bigcup_{n\in\mathbf{N}} I_n\Big) = 0 \quad \text{and} \quad \lambda\Big(\bigcup_{n\in\mathbf{N}} I_n\Big) \leq \lambda(B) + \delta/M.$$

Assume $I_n = [a_n, b_n]$ and $a_n = s_1^n + s_2^n$, $b_n = t_1^n + t_2^n$, where $(s_1^n, s_2^n) \in F$, $(t_1^n, t_2^n) \in F$. We have

$$\begin{aligned} m\Big\{\bigcup_{n\in\mathbf{N}} \big((R_{t^n} \setminus R_{s^n}) \setminus (S^1(F) \cup S^2(F))\big)\Big\} &= \int_{\mathrm{pr}_1\left(Z\left(\bigcup_{n\in\mathbf{N}} I_n\right)\setminus F\right)} h_1(x)\,dx + \int_{\mathrm{pr}_2\left(Z\left(\bigcup_{n\in\mathbf{N}} I_n\right)\setminus F\right)} h_2(x)\,dx \\ &\leq M\lambda\Big(\bigcup_{n\in\mathbf{N}} I_n \setminus B\Big) \\ &\leq \delta. \end{aligned}$$

It follows that

$$\begin{aligned} &E\Big\{\Big[\sum_{n\in\mathbf{N}} (X_{t^n} - X_{s^n}) - (X(S^1(F)) + X(S^2(F)))\Big]^2\Big\} \\ &\quad = \nu_X\Big\{\bigcup_{n\in\mathbf{N}} \big((R_{t^n} \setminus R_{s^n}) \setminus (S^1(F) \cup S^2(F))\big)\Big\} \\ &\quad < \varepsilon. \end{aligned}$$

Since ε is arbitrary, we can conclude that $X(S^1(F)) + X(S^2(F)) \in \mathscr{H}(F)$.

If F had been totally ordered for $\triangle$, we would have worked with lines $t_1-t_2=c$, and replaced s_1+s_2 by s_1-s_2 in the definition of B. The remainder of the argument is similar. The proof is complete. □

PROPOSITION 2.4. *Let F be a bounded Borel subset of T. Then $X(F)$ is $\mathscr{H}(F)$-measurable.*

In principle, the X-measure of F is obtained by covering F with small squares, so it is clear that $X(F)$ is $\mathscr{H}^*(F)$-measurable. The trick to showing $X(F)\in\mathscr{H}(F)$ is to arrange the cover so that the corners of the squares belong to F. For this we need two lemmas. The first is a straightforward extension of Lusin's Theorem to functions with values in a separable Hilbert-space.

LEMMA 2.5. *Let $g(x)$, $0\leq x\leq N$, be a measurable function with values in a separable Hilbert space $\mathscr{L}$, and fix $\varepsilon>0$. Then there is a compact subset $K\subset[0,N]$ such that $\lambda(K)>N-\varepsilon$ and $g|_K$ is continuous.*

Proof. Let $(\mathscr{L}_n, n\in\mathbf{N})$ be an increasing sequence of finite-dimensional subspaces which span $\mathscr{L}$, and let $g_n(x)$ be the projection of $g(x)$ on $\mathscr{L}_n$. With the obvious identification, we may consider g_n as a function with values in some $\mathbf{R}^k$ (with $k=\dim E_n$). By Lusin's Theorem [S; Chapter III, (7.1)], there is a compact set $K_n\subset[0,N]$ such that $\lambda(K_n)>N-\varepsilon 2^{-n-1}$ and $g_n|_{K_n}$ is continuous. Let $K'=\bigcap_{n\in\mathbf{N}}K_n$ and note that $\lambda(K')>N-\varepsilon/2$. Next let $f_n(x)=\|g(x)-g_n(x)\|$. The sequence $(f_n, n\in\mathbf{N})$ is real-valued and converges pointwise to zero. By Egoroff's Theorem [S; I, (9.6)], there is a compact set $K''\subset[0,N]$ such that $\lambda(K'')>N-\varepsilon/2$ and $(f_n|_{K''}, n\in\mathbf{N})$ converges uniformly to zero. Put $K=K'\cap K''$. Then $\lambda(K)>N-\varepsilon$ and $g|_K$ is the uniform limit of continuous functions. The lemma is proved. □

LEMMA 2.6. *Let $F\subset[0,N]^2$ be Borel. Then there is a measurable subset $F'\subset F$, with $m(F\setminus F')=0$, for which there is a Vitali covering that consists of squares with sides parallel to the axes and having all four corners in F.*

Proof. It is sufficient to show that for each $\varepsilon>0$, there is a measurable subset K of $[0,N]$ with $K\subset\mathrm{pr}_1(F)$, $\lambda(\mathrm{pr}_1(F)\setminus K)<\varepsilon$, such that the statement of the lemma is valid with F replaced by $F_\varepsilon=F\cap(K\times[0,N])$. So fix $\varepsilon>0$ and apply Lemma 2.5 to $g(x)$ given by $g(x,y)=I_F(x,y)$ considered as a function of x with values in $L^2([0,N],d\lambda)$. Let K be the resulting compact set consisting of points of continuity of g, with $\lambda(K)>N-\varepsilon$. Let $F(s_1)$ denote the vertical section of F at s_1, i.e. $F(s_1)=\{s_2\colon (s_1,s_2)\in F\}$, and for $B\subset[0,N]$, set $B+h=\{x\in[h,N]\colon x-h\in B\}$. Define

$$F(s_1,h)=F(s_1)\cap F(s_1+h)\cap(F(s_1)+h)\cap(F(s_1+h)+h).$$

The lemma will be proved if we show that

$$\lim_{h\searrow 0,\, h\in K-s_1}\lambda(F(s_1)\setminus F(s_1,h))=0,\quad \forall s_1\in K. \tag{2.2}$$

Indeed, we can then set $F'_\varepsilon=\bigcap_{h>0}G^h_\varepsilon$, where

$$G^h_\varepsilon=\{s\in F_\varepsilon\colon \exists h'<h \text{ such that } (s_1+h',s_2)\in F,\ (s_1,s_2-h')\in F,\ (s_1+h',s_2-h')\in F\}$$

(observe that G^h_ε is the projection of a Borel subset of $\mathbf{R}^3$, and thus is analytic [DM;II.13], hence measurable [DM;III.33]), and by (2.2) and Fubini's Theorem, $m(F_\varepsilon\setminus F'_\varepsilon)=0$.

So we now prove (2.2). Note that for any s_1,

$$\lambda(F(s_1)\triangle(F(s_1)+h))=\int_0^N|I_F(s_1,y)-I_F(s_1,y-h)|\,dy\to 0 \tag{2.3}$$

as $h\downarrow 0$, since $I_F(s_1,\cdot)\in L^1([0,N])$; this is a standard property of translates of L^1-functions. In addition,

$$\begin{aligned}\lambda(F(s_1)\triangle F(s_1+h))&=\int_0^N(I_F(s_1,y)-I_F(s_1+h,y))\,dy\\&=\|I_F(s_1,\cdot)-I_F(s_1+h,\cdot)\|^2_{L^2([0,N])}\\&\to 0\end{aligned} \tag{2.4}$$

for $s_1\in K$ when $h\downarrow 0$ in such a way that $s_1+h\in K$ (by choice of K). Finally, for $s_1\in K$, we have

$$\begin{aligned}\lambda(F(s_1)\triangle(F(s_1+h)+h))&=\int_0^N|I_F(s_1,y)-I_F(s_1+h,y-h)|\,dy\\&\le\int_0^N|I_F(s_1,y)-I_F(s_1,y-h)|\,dy+\int_0^N|I_F(s_1,y-h)-I_F(s_1+h,y-h)|\,dy\\&\le\lambda(F(s_1)\triangle(F(s_1)+h))+\int_0^{N-h}|I_F(s_1,y)-I_F(s_1+h,y)|\,dy\\&\le\lambda(F(s_1)\triangle(F(s_1)+h))+\lambda(F(s_1)\triangle F(s_1+h))\\&\to 0\end{aligned} \tag{2.5}$$

as $h \downarrow 0$ in such a way that $s_1+h \in K$ (by (2.3) and (2.4)). But (2.3), (2.4) and (2.5) clearly imply (2.2), completing the proof. □

Proof of Proposition 2.4. Let F' be the subset of F given by Lemma 2.6, and $\mathscr{C}$ the Vitali covering of F' by squares with corners in F. Fix $\eta>0$, and let $\varepsilon>0$ be such that $\nu_X(A)<\eta$ whenever $m(A)<\varepsilon$ (ε exists by Assumption 1.1). Applying Theorem 2.2, we get a sequence $(F_n, n \in \mathbf{N})$ of disjoint elements of $\mathscr{C}$ such that

$$m\Big(F \triangle \bigcup_{n\in\mathbf{N}} F_n\Big) < \varepsilon.$$

Thus

$$E\bigg(\Big(X(F)-X\Big(\bigcup_{n\in\mathbf{N}} F_n\Big)\Big)^2\bigg) = \nu_X\Big(F\triangle\Big(\bigcup_{n\in\mathbf{N}} F_n\Big)\Big) < \eta.$$

Now since $X(\cdot)$ is σ-additive, we get

$$X\Big(\bigcup_{n\in\mathbf{N}} F_n\Big) = \sum_{n\in\mathbf{N}} X(F_n) = \sum_{n\in\mathbf{N}} \Delta_{F_n} X \in \mathscr{H}(F),$$

since all four corners of F_n belong to F. Since η is arbitrary, $X(F) \in \mathscr{H}(F)$. The proposition is proved. □

APPROXIMATION LEMMA 2.7. *Let $f: [a,b] \to \mathbf{R}_+$ be measurable and bounded by $M>0$, and let A be a measurable subset of $[a,b]$. Set*

$$\hat{A} = \{t \in T\colon t_1 \in A,\ 0 \le t_2 \le f(t_1)\}.$$

Fix $\varepsilon>0$. Suppose that $\mathscr{I}$ is a Vitali covering of A by non-degenerate closed intervals I with at least one extremity $a_I \in A$, and that for each $x \in A$ and $\eta>0$, there is $I \in \mathscr{I}$ with length $<\eta$ and $a_I = x$. Then there is a sequence of disjoint intervals I^1, I^2, ... in $\mathscr{I}$ such that

$$E\bigg\{\Big(X(\hat{A}) - \sum_{n\in\mathbf{N}} X(I^n \times [0, f(a_{I^n})])\Big)^2\bigg\} < \varepsilon. \tag{2.6}$$

Proof. By Assumption 1.1, there is $\delta>0$ such that $m(G)<\delta$ implies $\nu_X(G)<\varepsilon/2$. Now by Lusin's Theorem [S; Chapter III, (7.1)], there is a compact set $K \subset [a,b]$ such that $\lambda(K) > b-a-\delta/M$ and $f|_K$ is continuous. Set $B = A \cap K$. Then by Fubini's Theorem, $m(\hat{B}\triangle\hat{A})<\delta$, so $\nu_X(\hat{B}\triangle\hat{A})<\varepsilon/2$. It is thus sufficient to show that there is a sequence I^1, I^2, ... of disjoint intervals in $\mathscr{I}$ such that

$$m\{\hat{B}\triangle \bigcup_{n\in\mathbf{N}} (I^n\times[0,f(a_{I^n})])\}<\delta,$$

where a_{I^n} is the extremity of I^n which lies in A.

Since f is uniformly continuous on K, let $\eta>0$ be such that

$$|s_1-t_1|<\eta,\ s_1,t_1\in K \quad\Rightarrow\quad |f(s_1)-f(t_1)|<\delta/(2(b-a)).$$

Set

$$\mathcal{I}_\delta=\{I=[\alpha,\beta]\in\mathcal{I}: a_I\in B, |\beta-\alpha|<\eta\}.$$

Then $\mathcal{I}_\delta$ is a Vitali covering of B, so by Theorem 2.2, there is a sequence $(I^n,\ n\in\mathbf{N})$ of disjoint intervals in $\mathcal{I}_\delta$ such that $\lambda(B\setminus\bigcup_{n\in\mathbf{N}}I^n)=0$ and $\lambda(B\triangle\bigcup_{n\in\mathbf{N}}I^n)<\delta/(2M)$. Thus

$$\begin{aligned} m\left\{\hat{B}\triangle \bigcup_{n\in\mathbf{N}}(I^n\times[0,f(a_{I^n})])\right\} &= \int_a^b \left| f(s)I_B(s)-\sum_{n\in\mathbf{N}} f(a_{I^n})I_{I^n}(s)\right| ds\\ &\le \sum_{n\in\mathbf{N}}\left(\int_{I^n\cap B}|f(s)-f(a_{I^n})|\,ds+M\lambda(I^n\setminus B)\right)\\ &\le \delta/2+\delta/2\\ &=\delta. \end{aligned}$$

This completes the proof. □

The following is an easy consequence of Lemma 2.7.

Proposition 2.8. *Using the notation of Lemma* 2.7, *let* Γ *be a set containing the graph of* $f|_A$, *i.e.* $\{(t_1,t_2)\in T: t_1\in A, t_2=f(t_1)\}\subset\Gamma$. *Assume that*

(2.7) *for λ-almost all $x\in A$, $(x,f(x))$ is an accumulation point of* $(\mathbf{R}_+\times\{f(x)\})\cap\Gamma$. *Then $X(\hat{A})$ is $\mathcal{H}(\Gamma)$-measurable.*

Proof. Set

$$\mathcal{I}=\{[s_1,t_1]: (s_1\in A \text{ and } (t_1,f(s_1))\in\Gamma) \text{ or } (t_1\in A \text{ and } (s_1,f(t_1))\in\Gamma)\},$$

and for $I=[s_1,t_1]\in\mathcal{I}$, set $\psi(I)=f(s_1)$ if $s_1\in A$, $\psi(I)=f(t_1)$ otherwise. Note that $I\times[0,\psi(I)]$ is a rectangle whose two upper corners belong to Γ. Thus $X(I\times[0,\psi(I)])$ is $\mathcal{H}(\Gamma)$-measurable. Now by (2.7), $\mathcal{I}$ is a Vitali covering of a subset A' of A with $\lambda(A\setminus A')=0$, which also satisfies the assumption of Lemma 2.7. So by this lemma, $X(\hat{A})=X(\hat{A}')$ is arbitrarily close in L^2-norm to random variables which are $\mathcal{H}(\Gamma)$-measurable. This completes the proof. □

3. Characterization of the minimal splitting field for an open set

In this section, we shall describe the generators of the minimal splitting field for an arbitrary open set.

In the study of sets with complicated boundaries, we will need the following "hitting times". For $G \subset T$, define maps T_G and L_G by

$$T_G(t) = \begin{cases} \inf\{v \geq t_2 : (t_1, v) \in G\} & \text{if } \{\ \} \neq \emptyset, \\ +\infty & \text{otherwise,} \end{cases}$$

$$L_G(t) = \begin{cases} \sup\{v \leq t_2 : (t_1, v) \in G\} & \text{if } \{\ \} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Note that T_G corresponds to the first entrance time of G along the half-line $\{t_1\} \times [t_2, +\infty[$, whereas L_G corresponds to the last exit of G along the segment $\{t_1\} \times [0, t_2]$. We have the following lemma.

LEMMA 3.1. (a) *Assume G is open. Then T_G is upper-semicontinuous (u.s.c.) and L_G is lower-semicontinuous (l.s.c.).*

(b) *Assume G is closed. Then T_G is l.s.c. and L_G is u.s.c.*

Proof. Observe that

$$\{t \in T : T_G(t) < y\} = S^1(G \cap (\mathbf{R}_+ \times [0, y[))$$

which is an open subset of T. This proves the first statement in (a). The other three statements of the lemma can be proved similarly. Details are left to the reader. □

Throughout this section, we work with a fixed non-empty open set D_1 (not necessarily bounded). We are going to determine the generators of the minimal splitting field for D_1. Set $D_2 = (\bar{D}_1)^c$, $\Gamma = \partial D_1 \cap \partial D_2$. In order to avoid trivialities, we assume that the open set D_2 is not empty. Note that $\Gamma = \partial \bar{D}_1 = \partial D_2 = \partial \bar{D}_2$, and that by Lemma 2.1, $\mathscr{H}(D_i) = \mathscr{H}(\bar{D}_i)$, $i = 1, 2$. We let $\dot{D}_i$ denote the interior of $\bar{D}_i$ (in general, $\dot{D}_1$ may be distinct from D_1 but it always turns out that $\dot{D}_2 = D_2$). Then $\partial \dot{D}_i = \partial \bar{D}_i = \Gamma$.

Define two open sets S_1 and S_2 by

$$S_1 = \dot{D}_1 \cap S^1(\dot{D}_2), \quad S_2 = \dot{D}_2 \cap S^1(\dot{D}_1),$$

and define maps p and τ with domain $S_1 \cup S_2$ by

$$p(t) = \begin{cases} T_{\dot{D}_2}(t) & \text{if } t \in S_1, \\ T_{\dot{D}_1}(t) & \text{if } t \in S_2, \end{cases}$$

and $\tau(t)=(t_1, p(t))$. Note that p never takes the value $+\infty$ and that τ projects $S_1 \cup S_2$ onto Γ. Taking $\dot{D}_1$ instead of D_1 in the definition of p makes a significant difference (consider, for instance, the case $D_1=([0,1[\times[0,2[)\setminus(A\times[1,2[)$, where A is a Cantor set such that $\lambda(A)>0$). The following technical properties of the map τ will be important.

LEMMA 3.2. (a) *τ is Borel.*

(b) *For any open set $F\subset S_1\cup S_2$, $\tau(F)$ is Borel.*

(c) $\tau(S_1)\cap\tau(S_2)=\emptyset$.

Proof. (a) This is clear since $\tau(t)=(t_1,p(t))$ and p is u.s.c. by Lemma 3.1.

(b) Since any open set is a countable union of closed rectangles, it is sufficient to prove (b) in the case $F=[a,b]\times[c,d]\subset S_1$. Then

$$\tau(F)=\{t\in T\colon a\le t_1\le b,\, t_2=p(t_1,d)\},$$

which is the graph of the u.s.c. map $p(\cdot\,,d)$, and (b) is proved.

(c) Assume $s\in S_1$, $t\in S_2$, and $\tau(s)=\tau(t)$. Then $s_1=t_1$, so we can assume for instance that $s_2<t_2$. But then the definition of τ implies $p(s)<t_2<p(t)$, contradicting equality of $p(s)$ and $p(t)$. □

For any subset B of $\mathbf{R}_+$ and $d\ge 0$, we set $B^{(d)}=B\times\{d\}$. If $B^{(d)}\subset S_i$, $i=1,2$, we set

$$V(B^{(d)})=\{(t_1,t_2)\in T\colon t_1\in B, 0\le t_2\le p(t_1,d)\}$$

(see Figure 3.1). With these notations, we can describe the minimal splitting field for D_1

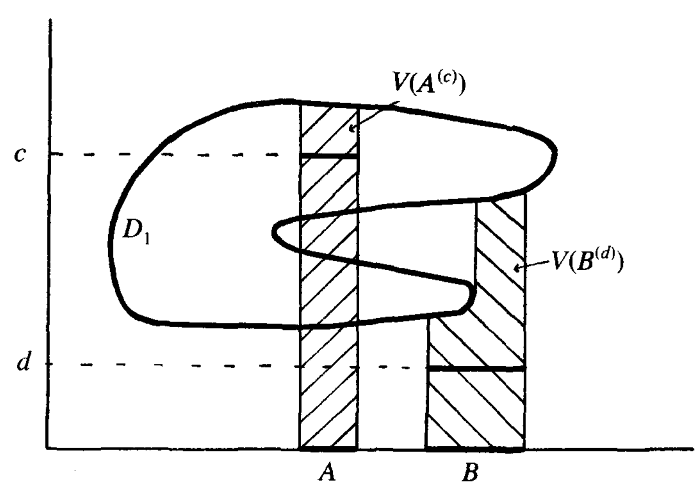


Fig. 3.1.

and D_2. Set

$$\mathcal{M}(D_1) = \mathcal{H}(\Gamma) \vee \sigma\{X(V(B^{(d)})): B^{(d)} \subset S_i, B = [a,b], a < b, d > 0, i = 1,2\}.$$

THEOREM 3.3. *Let $(X_t, t \in T)$ satisfy Assumption 1.1, and let D_1 be any open subset of T, $D_2 = (\bar{D}_1)^c$. Then*

$$\mathcal{M}(D_1) = \mathcal{H}(D_1) \cap \mathcal{H}(D_2),$$

and this is the minimal splitting field for $\mathcal{H}(D_1)$ and $\mathcal{H}(D_2)$.

Remark 3.4. (a) In the case of the Brownian sheet and for domains with smooth boundary, this result is contained in [W4; Theorem 3.12], and for domains whose boundary is a piecewise monotone curve, in [WZ; Proposition 2]. It may be advantageous to the reader to compare our statement with these references, in which the variables $X(V(B^{(d)}))$ are replaced by the X-measure of vertical and horizontal shadows of portions of Γ. This description is not valid in general: Example 3.5 below illustrates exactly what difference there is between the shadow description and ours.

(b) The proofs in [W3; WZ] are rather short. Here, we use similar ideas, but much technical effort is needed to handle, for instance, the case where $m(\Gamma)>0$. The results of the previous section will be handy here.

(c) We now have a powerful tool for proving that $\mathcal{H}(\Gamma)$ is a splitting field: it suffices to show that $X(V(B^{(d)})) \in \mathcal{H}(\Gamma)$, when $B^{(d)} \subset S_i$, $i=1,2$. Since $V(B^{(d)})$ is the region below the graph of an u.s.c. function, it is possible to do this in many cases, as the following sections illustrate.

(d) One must take care when comparing Theorem 3.3 to other results in the literature. For the Brownian sheet, Rozanov [Ro; Chapter 3, § 5.3] gives a chacterization of the minimal splitting field of a bounded open set. However, his definition of a splitting field $\mathcal{S}$ is

(3.1) $\mathcal{H}^*(\bar{D}_1)$ is conditionally independent of $\mathcal{H}^*(D_1^c)$ given $\mathcal{S}$,

and the minimal splitting fields is then $\mathcal{H}^*(\Gamma)$ (note that if $H^*(\Gamma)$ is a splitting field in this sense, it is necessarily minimal by (1.1), since $\mathcal{H}^*(\Gamma) \subset \mathcal{H}^*(\bar{D}_1) \cap \mathcal{H}^*(D_1^c)$). Now $H^*(\Gamma) = \mathcal{H}^*(\bar{D}_1) \cap \mathcal{H}^*(\bar{D}_2)$ is in general distinct from $\mathcal{M}(\Gamma) = \mathcal{H}(\bar{D}_1) \cap \mathcal{H}(\bar{D}_2) = \mathcal{H}(D_1) \cap \mathcal{H}(D_2)$. This is the case for instance in Example 3.5.

Example 3.5. Let A be a Cantor set in $[0,1]$ such that $\lambda(A)>0$, and let $I_1, I_2, \ldots$ be the disjoint open intervals whose union is A^c. Set

$$D_1 = \bigcup_{n\in\mathbf{N}} (I_n\times I_n)\cup([0,1]\times[0,1])^c,$$

$$D_2 = \bar{D}_1^c.$$

Note that $D_2\subset[0,1]\times[0,1]$, and in particular, that this set is bounded. The common boundary Γ of D_1 and D_2 is the union of the boundary of $[0,1]^2$, the boundaries of the $I_n\times I_n$, and a subset of the diagonal whose projection on the x-axis is A.

Let E be the portion of D_2 below the diagonal. Then E is exactly the vertical shadow of a portion of Γ, but it will be a consequence of Proposition 6.7 below that $X(E)\notin\mathscr{H}(D_1)$, and thus is not an element of $\mathscr{M}(D_1)=\mathscr{M}(D_2)$. On the other hand, $X(E)$ is easily seen to belong to $\mathscr{H}^*(\Gamma)$. It will be clear from Theorem 4.1 that $\mathscr{H}(\Gamma)$ is the minimal splitting field in this case, since if $B\subset\mathbf{R}_+$ and $B^{(d)}\subset E$, then $V(B^{(d)})$ is the domain below the graph of an u.s.c. function which takes the value 1 on $B\cap A$ and is constant on each $B\cap I_n$, for each $n\in\mathbf{N}$. □

The proof of the Theorem 3.3 relies on several preliminary statements.

PROPOSITION 3.6. (a) *For $i=1,2$, for all measurable subsets B of $[a,b]$, and for all $d>0$ such that $[a,b]\times\{d\}\subset S_i$, $V(B^{(d)})$ is bounded and $X(V(B^{(d)}))$ is $\mathscr{M}(D_1)$-measurable.*

(b) $\mathscr{M}(D_1)\subset\mathscr{H}(D_1)\cap\mathscr{H}(D_2)$.

Proof. The map $x\mapsto p(x,d)$ is u.s.c., so it is bounded on the closed interval $[a,b]$, hence $V(B^{(d)})$ is bounded. Note that $B\mapsto V(B^{(d)})$ preserves unions and intersections. Since $X(\cdot)$ is countably additive, a standard monotone class argument [DM; I.19] yields (a) for Borel sets B. But then Assumption 1.1 yields (a) for any measurable B.

To show (b), it is sufficient by Proposition 2.4 and Lemma 2.1 (b) to show that $X(V(A^{(d)}))\in\mathscr{H}(D_1)\cap\mathscr{H}(D_2)$, for each $d>0$ and each closed interval A for which $A^{(d)}$ is in either S_1 or S_2.

If $A^{(d)}$ is in S_i, let us show that $X(V(A^{(d)}))\in\mathscr{H}(D_{3-i})$. Let $(D^n,\ n\in\mathbf{N})$ be an increasing sequence of finite unions of open rectangles such that $\bigcup_{n\in\mathbf{N}}D^n=\dot{D}_{3-i}$. Set $f_n(x)=T_{D^n}(x,d)$, and

$$A_n=\{x\in\mathbf{R}_+: f_n(x)<+\infty\}.$$

Since f_n is u.s.c., A_n is open. Now $A\subset\bigcup_{n\in\mathbf{N}}A_n$ and A is compact, so there is $n_0\in\mathbf{N}$ such that $A_{n_0}\supset A$. Since $f_{n+1}\leq f_n$ and $\sup\{f_{n_0}(x)\colon x\in A\}$ is finite, the f_n are uniformly bounded on A for $n\geq n_0$. Define

$$V_n=\{t\in T: t_1\in A_n, 0\leq t_2\leq f_n(t_1)\}.$$

It is easy to see that $f_n \downarrow p(\cdot, d)$, so that by the above $m(V_n \triangle V(A^{(d)})) \to 0$. Thus $E([X(V_n) - X(V(A^{(d)}))]^2) \to 0$, and it suffices to check that $X(V_n) \in \mathscr{H}(D_{3-i})$. Since D^n is a finite union of rectangles, f_n is a step function, so V_n is a finite union of rectangles R_n of the form $R_n = I_n \times [0, b_n]$, where the I_n are disjoint intervals and b_n is the constant value of f_n on I_n. Since both upper corners of R_n belong to D_{3-i}, $X(R_n) \in \mathscr{H}(D_{3-i})$, and so we have shown that $X(V(A^{(d)})) \in \mathscr{H}(D_{3-i})$.

The proof that $X(V(A^{(d)})) \in \mathscr{H}(D_i)$ uses similar ideas but is simpler because we do not need the compactness argument. Set

$$q(t) = T_\Gamma(t) = \inf\{v \geq t_2 : (t_1, v) \in \Gamma\}.$$

By Lemma 3.1, $q(\cdot, d)$ is l.s.c. on A, so we can find an increasing sequence of step functions f_n which increase to $q(\cdot, d)$; we can even require that the graph of each f_n is in $\dot{D}_i$. Set $F = \bigcup_{n \in \mathbf{N}} F_n$, where

$$F_n = \{t \in T : t_1 \in A, 0 \leq t_2 \leq f_n(t_1)\}.$$

Since f_n is a step function, F_n is a finite union of rectangles with upper corners in $\dot{D}_i$, so $X(F_n) \in \mathscr{H}(\dot{D}_i) = \mathscr{H}(D_i)$. Since $X(F_n) \to X(F)$ in L^2, $X(F) \in \mathscr{H}(D_i)$. Now

$$X(V(A^{(d)})) = X(V(A^{(d)}) \setminus F) + X(F) \quad \text{and} \quad V(A^{(d)}) \setminus F \subset D_i \cup \Gamma = \bar{D}_i,$$

so by Proposition 2.4 and Lemma 2.1 (b), $X(V(A^{(d)})) \in \mathscr{H}(D_i)$. This completes the proof. □

Let us define a map U by $U(t) = \tau^{-1}(\tau(t))$. U maps a point in S_i onto a countable union of open segments, all contained in the vertical line through t (when D_i has smooth boundary, $U(t)$ is usually a single segment). Note that $U(s)$ and $U(t)$ are either identical or disjoint, and in particular, if $s_1 \neq t_1$, then $U(s)$ and $U(t)$ are disjoint. One consequence of this is that if L is any horizontal line segment contained in D_i and if we restrict ourselves to subsets $F \subset L$, then $F \mapsto U(F)$ preserves set operations.

LEMMA 3.7. *Fix $i \in \{1, 2\}$ and let L be a horizontal line segment of the form $L = [a, b] \times \{d\}$. Suppose $L \subset S_i$. If F is a measurable subset of L, then $U(F^{(d)})$ is measurable and bounded and $X(U(F^{(d)}))$ is $\mathscr{M}(D_1)$-measurable.*

Proof. Since $F \mapsto U(F)$ preserves set operations, it is sufficient to prove the lemma when F is a subinterval of L. So in fact, we only need to show that $X(U(L)) \in \mathscr{M}(D_1)$

(note that $U(L)$ is Borel by Lemma 3.2). Define

$$\beta(u) = L_{\dot{D}_{3-i}}(u, d).$$

Then $U(L)=(F_1 \setminus F_2)\setminus F_3$, where

$$F_1 = V(L),$$

$$F_2 = \{t \in T: a < t_1 < b,\ 0 \leq t_2 < \beta(t_1)\},$$

$$F_3 = (F_1 \setminus F_2) \cap \Gamma.$$

Note that F_1 is bounded, and all of these sets are Borel. Now $X(F_1) \in \mathcal{M}(D_1)$ by definition, and $X(F_3) \in \mathcal{H}(\Gamma) \subset \mathcal{M}(D_1)$, by Proposition 2.4, so it only remains to show that $X(F_2) \in \mathcal{M}(D)$.

The proof of this is somewhat similar to part (b) of Proposition 3.6. Let $(D^n,\ n \in \mathbf{N})$ be an increasing sequence of finite unions of rectangles such that $\bigcup_{n \in \mathbf{N}} D^n = \dot{D}_{3-i}$. Set $\beta_n(u) = L_{D^n}(u, d)$ and

$$F^n = \{t \in T: a \leq t_1 \leq b, 0 \leq t_2 \leq \beta_n(u)\}.$$

Then $F^n \uparrow F_2$, and each F^n is a finite union of rectangles of the form $I_k^n \times [0, b_k^n]$, where $I_1^n, I_2^n, \ldots$ are disjoint intervals. Set

$$G^n = \bigcup_{k \in \mathbf{N}} V(I_k^n \times \{b_k^n\}),$$

$$\tilde{G}^n = \bigcup_{k \in \mathbf{N}} \{t \in T: t_1 \in I_k^n, \beta(u) \leq t_2 \leq p(u, b_k^n)\}.$$

Then G^n and $\tilde{G}^n$ are increasing sequences which increase to G and $\tilde{G}$, respectively, and we have $G = F_2 \cup \tilde{G}$, $\tilde{G} \cap F_2 = \emptyset$. Thus $X(F_2) = X(G) - X(\tilde{G})$. Since $\tilde{G} \subset \Gamma$, $X(\tilde{G}) \in \mathcal{H}(\Gamma)$ by Proposition 2.4, so the proof will be complete provided we show that $X(G) \in \mathcal{M}(D_1)$. But $X(G)$ is the L^2-limit of

$$X(G^n) = \sum_{k \in \mathbf{N}} X(V(I_k^n \times \{b_k^n\})),$$

which is $\mathcal{M}(D_1)$-measurable by definition. The lemma is proved. □

PROPOSITION 3.8. *Let $\mathcal{U}$ be the family of Borel subsets F of T with the property $F = U(F)$. If $F \in \mathcal{U}$ is bounded, then $X(F) \in \mathcal{M}(D_1)$.*

Note that for domains D_1 with smooth boundaries, the statement that $F = U(F)$ is

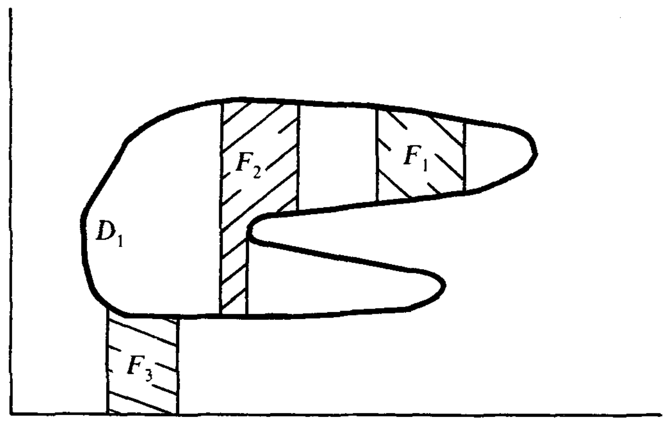


Fig. 3.2. Three sets F_i with $F_i = U(F_i)$, $i = 1, 2, 3$.

essentially "F is a domain bounded on each side by vertical lines and above and below by portions of the boundary": see Figure 3.2.

Proof of Proposition 3.8. Let $M > 0$ be such that $F \subset [0, M]^2$. Fix $\varepsilon > 0$, and let $\tilde{D}$ be a finite union of rectangles contained in $S_i \cap [0, M]^2$ such that $m((S_i \cap [0, M]^2) \setminus \tilde{D}) < \varepsilon$. Set $F' = F \cap \tilde{D}$. Then $U(F') \subset F$ and $m(F \setminus U(F')) < \varepsilon$. Since ε is arbitrary, it is sufficient to prove that $X(U(F')) \in \mathcal{M}(D_1)$. It is clear that we may assume that

$$\tilde{D} = [u, v] \times \bigcup_{l=1}^{n} I_l,$$

where $u < v$ and $I_l = [a_l, b_l]$, with $a_1 < b_1 < a_2 < \ldots < a_n < b_n$. Let G_l be the intersection of F with $[u, v] \times \{b_l\}$. Then G_l is measurable. Now define

$$A_l = \{x \in G_l : p(x, b_l) < a_{l+1}\}, \quad 1 \leq l < n.$$

By Lemma 3.7, $X(U(A_l \times \{b_l\})) \in \mathcal{M}(D_1)$. Now since $F = U(F)$, we have

$$(G_l \setminus A_l) \times I_l \subset U(G_{l+1} \times I_{l+1}),$$

from which it follows that

$$U(F') = \bigcup_{l=1}^{n} U(A_l \times \{b_l\}).$$

Since the union is disjoint, the conclusion follows. □

PROPOSITION 3.9. *Fix* $t\in\Gamma$. *Then* $X(R_t\cap\dot{D}_i)\in\mathcal{M}(D_1)$, $i=1,2$.

Proof. Set $B=\mathrm{pr}_1(\Gamma\cap([0,t_1]\times\{t_2\}))$. Then B is closed, so $[0,t_1]\setminus B=\bigcup_{n\in\mathbf{N}}I_n$, where the $I_n=]a_n,b_n[$ are disjoint open intervals. Fix $n\in\mathbf{N}$. We begin by showing that $X(F_n\cap\dot{D}_i)\in\mathcal{M}(D_1)$, where $F_n=I_n\times[0,t_2]$. There are two cases to distinguish.

Case 1. $I_n\times\{t_2\}\subset\dot{D}_i$. In this case,

$$F_n\cap\dot{D}_{3-i}\subset S_{3-i}\quad\text{and}\quad F_n\cap\dot{D}_{3-i}\in\mathcal{U}.$$

Now

$$\begin{aligned}X(F_n\cap\dot{D}_i)&=X_{b_n,t_2}-X_{a_n,t_2}-X(F_n\cap\dot{D}_{3-i})-X(F_n\cap\Gamma)\\&\in\mathcal{M}(D_1)\end{aligned}$$

by Propositions 2.4 and 3.8 (since (b_n,t_2) and (a_n,t_2) belong to Γ; this is where we use the fact that $t\in\Gamma$).

Case 2. $I_n\times\{t_2\}\subset\dot{D}_{3-i}$. Then

$$F_n\cap\dot{D}_i\subset S_i\quad\text{and}\quad F_n\cap\dot{D}_i\in\mathcal{U},$$

so $X(F_n\cap\dot{D}_i)\in\mathcal{M}(D_1)$ by Proposition 3.8.

Now set $F=B\times[0,t_2]$. The proposition will be proved provided we show that $X(F\cap\dot{D}_i)\in\mathcal{M}(D_1)$. Set

$$\begin{aligned}\beta_i(u)&=L_{\dot{D}_i}(u,t_2),\quad i=1,2,\\B_1&=\{v\in B:\beta_2(v)<\beta_1(v)\},\\B_2&=\{v\in B:\beta_1(v)\le\beta_2(v)\},\\R_i&=B_i\times[0,t_2],\quad i=1,2.\end{aligned}$$

We are going to write $F\cap D_i$, as a disjoint union and difference of sets, each of which will have the property that its X-measure belongs to $\mathcal{M}(D_1)$. Observe that

$$\begin{aligned}F\cap\dot{D}_i&=(R_1\cap\dot{D}_i)\cup(R_2\cap\dot{D}_i)\\&=(R_{3-i}\cap\dot{D}_i)\cup[R_i\setminus((R_i\cap\dot{D}_{3-i})\cup(R_i\cap\Gamma))].\end{aligned}\tag{3.2}$$

Now by definition, $R_{3-i}\cap\dot{D}_i\in\mathcal{U}$, $i=1,2$, and this is a bounded Borel set, so by Proposition 3.8, $X(R_{3-i}\cap\dot{D}_i)\in\mathcal{M}(D_1)$, and, equivalently, $X(R_i\cap\dot{D}_{3-i})\in\mathcal{M}(D_1)$, $i=1,2$. Furthermore, by Proposition 2.4, $X(R_i\cap\Gamma)\in\mathcal{M}(D_1)$. Finally, to see that $X(R_i)\in\mathcal{H}(\Gamma)$, we apply

Proposition 2.8 to the function $f(u)=t_2 I_{B_i}(u)$. If u is a point of density of B_i, $(u,f(u))$ is an accumulation point of $\{(s_1,s_2)\in\Gamma: s_2=f(u)\}$, so this proposition implies in particular that $X(R_i)\in\mathscr{H}(\Gamma)$. Now since

$$X(F\cap\dot{D}_i)=X(R_{3-i}\cup\dot{D}_i)+(X(R_i)-X(R_i\cap\dot{D}_{3-i})-X(R_i\cap\Gamma))$$

by (3.2), the proof is complete. □

LEMMA 3.10. *Set*

$$\mathscr{G}_i=\sigma\{X(O), O\subset\dot{D}_i, O \text{ open}\}$$

$$\mathscr{G}_i^*=\sigma(\{X(R_t\cap\dot{D}_i), t\in\Gamma\}\cup\{X(F), F\in\mathscr{U}, F\subset S_i\}\cup\{X(R_t\cap\Gamma), t\in T\}).$$

Then $\mathscr{G}_i$ and $\mathscr{G}_{3-i}^*$ are independent and $\mathscr{H}(D_i)=\mathscr{G}_i\vee\mathscr{G}_{3-i}^*$, $i=1,2$.

Proof. We only carry out the proof for $i=1$, since the case $i=2$ is similar. It follows from Propositions 2.4, 3.9, 3.8 and 3.6(b) that $\mathscr{G}_1\vee\mathscr{G}_2^*\subset\mathscr{H}(D_1)$, and from Assumption 1.1 that $\mathscr{G}_1$ and $\mathscr{G}_2^*$ are independent.

To see that $\mathscr{H}(D_1)\subset\mathscr{G}_1\vee\mathscr{G}_2^*$, we show that $X_t\in\mathscr{G}_1\vee\mathscr{G}_2^*$ for each fixed $t\in D_1$. Set

$$s_1=\inf\{u>0: [u, t_1]\times\{t_2\}\subset\dot{D}_1\}.$$

Then $(s_1,t_2)\in\Gamma$, and

$$X_t=Z_1+Z_2+Y_1+Y_2+Y,$$

where

$$Z_i=X(R_{s_1,t_2}\cap\dot{D}_i),\quad Y_i=X((R_t\setminus R_{s_1,t_2})\cap\dot{D}_i,\quad Y=X(R_t\cap\Gamma).$$

By Assumption 1.1, we can replace the rectangles above by their interiors without modifying the values of Z_i and Y_i, $i=1,2$, so it follows from the definition that Z_1, $Y_1\in\mathscr{G}_1$ and Z_2, $Y\in\mathscr{G}_2^*$. Since $t\in D_1$, we have

$$(R_t\setminus R_{s_1,t_2})\cap\dot{D}_2\in\mathscr{U},\quad (R_t\setminus R_{s_1,t_2})\cap\dot{D}_2\subset S_2,$$

so $Y_2\in\mathscr{G}_2^*$. Thus $X_t\in\mathscr{G}_1\vee\mathscr{G}_2^*$, and the proof is complete. □

Before proving Theorem 3.3, we recall an elementary fact about conditional expectations.

Fact 3.11. Let Z be an integrable random variable with values in $\mathbf{R}^n$, defined on some probability space $(\Omega, \mathscr{F}, P)$, and let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two σ-algebras such that $\sigma(Z)\vee\mathscr{H}_2$ is independent of $\mathscr{H}_1$. Then $E(Z|\mathscr{H}_1\vee\mathscr{H}_2)=E(Z|\mathscr{H}_2)$.

Proof of Theorem 3.3. Fix $t^1, \dots, t^n \in D_2$, and let $h\colon \mathbf{R}^n \to \mathbf{R}$ be bounded and Borel. Since $\mathscr{M}(D_1)\subset\mathscr{H}(D_1)$ by Proposition 3.6(b), $\mathscr{M}(D_1)$ will be a splitting field for $\mathscr{H}(D_1)$ and $\mathscr{H}(D_2)$ provided we show that

$$E(h(X_{t^1}, \dots, X_{t^n})|\mathscr{H}(D_1)) = E(h(X_{t^1}, \dots, X_{t^n})|\mathscr{M}(D_1))$$

(see [DM; II.45]). It is even enough to show that

$$E(h(X_{t^1}, \dots, X_{t^n})|\mathscr{H}(D_1)) \in \mathscr{M}(D_1).$$

We are only going to write out the proof for $n=1$, but it will be obvious that the same proof is valid for all $n \in \mathbf{N}$. Set $t^1=t$, and let $s=(s_1, s_2)$ be defined by

$$s_1 = \inf\{u < t_1 \colon [u, t_1]\times\{t_2\} \in \dot{D}_2\}, \quad s_2 = t_2.$$

Then $s \in \Gamma$, and we have

$$X_t = Z + Y_1 + Y_2 + Y_3,$$

where

$$Z = X(R_s), \quad Y_i = X((R_t \setminus R_s)\cap \dot{D}_i), \ i=1,2, \quad Y_3 = X((R_t\setminus R_s)\cap\Gamma).$$

So

$$E(h(X_t)|\mathscr{H}(D_1)) = E(g(Z, Y_1, Y_2, Y_3)|\mathscr{H}(D_1)), \tag{3.3}$$

where $g\colon \mathbf{R}^4 \to \mathbf{R}$ is defined by

$$g(z, y_1, y_2, y_3) = h(z+y_1+y_2+y_3).$$

Thus we only need to show that the right-hand side of (3.3) is $\mathscr{M}(D_1)$-measurable whenever g is a bounded Borel function on $\mathbf{R}^4$. By a standard monotone class argument (see [DM; I.21]) it is sufficient to do this when g has the special form

$$g_0(z)\, g_1(y_1)\, g_2(y_2)\, g_3(y_3),$$

where $g_i\colon \mathbf{R}\to\mathbf{R}$ is bounded Borel, $i=0, \dots, 3$. By Lemma 2.1(b), $Z \in \mathscr{H}(D_1)$, while by Proposition 2.4, $Y_1 \in \mathscr{H}(D_1)$ and $Y_3 \in \mathscr{H}(\Gamma)\subset\mathscr{H}(\bar{D}_1)=\mathscr{H}(D_1)$, so using Lemma 3.10 we have

$$E(g(Z, Y_1, Y_2, Y_3)|\mathscr{H}(D_1)) = g_0(Z)\, g_1(Y_1)\, g_3(Y_3)\, E(g_2(Y_2)|\mathscr{G}_1 \vee \mathscr{G}_2^*).$$

Applying 3.11, we see this is

$$= g_0(Z) g_1(Y_1)\, g_3(Y_3)\, E(g_2(Y_2)|\mathscr{G}_2^*).$$

Now $Z \in \mathscr{M}(D_1)$ by definition, $Y_1 \in \mathscr{M}(D_1)$ since $(R_t \setminus R_s) \cap \dot{D}_1 \in \mathscr{U}$, and by Proposition 2.4, $Y_3 \in \mathscr{H}(\Gamma) \subset \mathscr{M}(D_1)$ by definition. Since $\mathscr{G}_2^* \subset \mathscr{M}(D_1)$ by Propositions 3.9, 3.8 and 2.4, we have shown that $\mathscr{M}(D_1)$ is a splitting field for $\mathscr{H}(D_1)$ and $\mathscr{H}(D_2)$. By (1.1) and Proposition 3.6(b), $\mathscr{M}(D_1)$ is the minimal splitting field and $\mathscr{M}(D_1) = \mathscr{H}(D_1) \cap \mathscr{H}(D_2)$. This completes the proof. □

4. The sharp Markov property for domains with thick boundary

In this section, we show that there are many interesting examples of open sets which satisfy the sharp Markov property.

THEOREM 4.1. *Let D be an open set and suppose that for $i=1,2$ and $[a, b] \times \{d\} \subset S_i$,*

(4.1) $$\lambda(\{u \in [a, b]: p(u, d) \text{ is an isolated point of } (\mathbf{R}_+ \times \{p(u, d)\}) \cap \partial D\}) = 0.$$

Then D has the sharp Markov property.

Proof. We need to show that $\mathscr{H}(\partial D)$ is a splitting field for $\mathscr{H}(D)$ and $\mathscr{H}(\bar{D}^c)$. Since $\mathscr{H}(\partial D) \subset \mathscr{H}(D)$ by Lemma 2.1(b), $\mathscr{H}(\partial D)$ will be a splitting field provided $\mathscr{H}(\partial D) \supset \mathscr{M}(D)$. Indeed, in this case, for $G \in \mathscr{H}(\bar{D}^c)$, we have $P(G|\mathscr{H}(D)) = P(G|\mathscr{M}(D))$, so $P(G|\mathscr{M}(D)) = P(G|\mathscr{H}(\partial D))$, since $P(G|\mathscr{M}(D)) \in \mathscr{H}(\partial D)$ and this random variable satisfies the appropriate integral conditions.

Since $\Gamma = \partial \bar{D} \subset \partial D$, it is sufficient, according to Theorem 3.3, to show that $X(V(B^{(d)})) \in \mathscr{H}(\partial D)$, for each $B = [a, b]$ and $d \in \mathbf{R}_+$ such that $B^{(d)} \subset S_i$, $i=1$ or 2. Define $f: B \to \mathbf{R}$ by $f(u) = p(u, d)$. By (4.1) and Proposition 2.8, $X(V(B^{(d)})) = X(\hat{B}) \in \mathscr{H}(\partial D)$, and the theorem is proved. □

A very simple application of Theorem 4.1 yields the sharp Markov property for finite unions of rectangles. Of course, since the boundary of a finite union of rectangles consists of finitely many vertical and horizontal segments, most of the results of

Section 2 are not needed, and only part of Assumption 1.1 comes into play. This gives us a new proof of the following corollary, due to Russo [Ru; Theorem 7.5] in the bounded case.

COROLLARY 4.2. *Assume* $(X_t, t \in T)$ *is a process with independent planar increments, and* D *is a finite union of (not necessarily bounded) rectangles with sides parallel to the coordinate axes. Then* D *has the sharp Markov property.*

Proof. If $[a,b]\times\{d\}\subset S^i$, $i=1$ or 2, then $p(u,d)$ will always lie on one of the horizontal segments of ∂D, and thus will not be an isolated point of $(\mathbf{R}_+\times\{p(u,d)\})\cap\partial D$. So the statement follows from Theorem 4.1. □

There are many other interesting cases where condition (4.1) is satisfied. In particular, many open sets whose boundary is a fractal satisfy (4.1). We only consider two:

the Sierpinski gasket Γ_1 (see [M; p. 142]). The only horizontal section of Γ_1 which contains isolated points is the section through the apex, which is a singleton;

the Sierpinski carpet Γ_2 (see [M; p. 144]). In this case, no horizontal section of Γ_2 contains isolated points.

Condition (4.1) is thus clearly satisfied by both Γ_1 and Γ_2, so that by Theorem 4.1, we have

COROLLARY 4.3. *Let* D *be an open set whose boundary is either the Sierpinski carpet or the Sierpinski gasket. Then* D *has the sharp Markov property.*

Note that there are many open sets D such that $\partial\bar{D}=\Gamma_i$, $i=1$ or 2, and so Γ_i is the common boundary of D and $\bar{D}^c$. Indeed, let $S^1, S^2 \ldots$ be the open triangles (respectively squares) which one removes to get the Sierpinski gasket (resp. carpet). Let $Y_1, Y_2, \ldots$ be i.i.d. Bernouilli random variables with $P\{Y_k=0\}=P\{Y_k=1\}=1/2$. Set $D=\bigcup S^k$, where the union is over those k for which $Y_k=1$. Clearly, $\Gamma_1=\partial\bar{D}$ (resp. $\Gamma_2=\partial\bar{D}$), for almost all realizations of $Y_1, Y_2, \ldots$.

Corollary 4.3 also illustrates the importance of distinguishing $\dot{D}$ and D. For instance, to get the Sierpinski gasket, one starts with an initial (closed) triangle T^0, and removes a sequence of open triangles $S^1, S^2, \ldots$. Let $D=\bigcup_{n\in\mathbf{N}} S^n$. Then ∂D is the whole Sierpinski gasket, but $\partial\dot{D}=\partial T^0$. Observe that D has the sharp Markov property by Corollary 4.2, but $\dot{D}$ does not (by Theorem 6.1 below).

It is tempting to conjecture that if the boundary of D has Hausdorff dimension strictly greater than 1, then D satisfies the sharp Markov property. However, this is false since some portion of the boundary might be, say, a diagonal line segment. We might suppose that D satisfies the following stronger condition.

(4.2) Every open set that contains one point in $\partial\bar{D}$ also contains a subset of $\partial\bar{D}$ with Hausdorff dimension >1.

Does (4.2) imply that D has the sharp Markov property? The answer is no, as the example below shows.

Example 4.4. Let A be an unbounded Cantor set in $\mathbf{R}_+$ with positive measure, and let $I_1, I_2, \ldots$ be the disjoint open intervals whose union is A^c (since A is unbounded, each I_n is bounded). In each square $I_n \times I_n$, build a Sierpinski carpet whose "outer rim" is $I_n \times I_n$ (its Hausdorff dimension is ~ 1.89 [M; Plate 145, p. 144]). Now let D_1 be an open set which consists of the union of

$$\{s \in T: s_1 > s_2\} \setminus \bigcup_{n \in \mathbf{N}} (I_n \times I_n)$$

and "half" the squares which one removes to build each of the Sierpinski carpets (choose them at random, as above). Set $D_2 = \bar{D}_1^c$. Then $\Gamma = \partial\bar{D}_1 = \partial D_1 = \partial D_2$ is the union of the carpets and the subset of the diagonal whose projection on the x-axis is A. This set clearly satisfies (4.2), and yet the sharp Markov property can be shown to fail (use Proposition 6.7).

If D_1 is an open set whose boundary is a separation line (see [DR; §2]), the horizontal sections of $\partial\bar{D}_1$ may each contain exactly one point, and yet the sharp Markov property may hold [DR; Theorem 3.12]. This corresponds to the case of "thin" boundaries. In the next section, we investigate the case where Γ is a Jordan curve.

5. Sufficient conditions for Jordan domains: the Maltese cross condition

Throughout the rest of this paper, we will assume that D_1 is a Jordan domain, that is $\partial D_1 = \Gamma$ is a Jordan curve in $\mathbf{R}^2 \cup \{\infty\}$. It will be convenient to assume that Γ is parameterized by a function φ defined on $[0,1]$ instead of on the unit circle, that is

$$\Gamma = \{\varphi(u): u \in [0,1]\},$$

where $\varphi = (\varphi_1, \varphi_2): [0,1] \to \mathbf{R}^2 \cup \{\infty\}$ is continuous, one-to-one on $[0,1[$ and $\varphi(0) = \varphi(1)$. In the terminology of [N], which we shall use below, Γ is a *directed loop*.

The two complementary open domains D_1 and D_2 of Γ may both be unbounded if

Γ passes through ∞, and Γ may have positive measure (see [D; XIII.21, Problem 2, p. 221] or [Ha; §36, p. 233]).

A standard property of Jordan domains is that $D_i = \dot{D}_i$ and $\partial D_i = \Gamma$ [N, Theorem 10.2]. If D_1 is bounded, then $S_1 = D_1$. In Section 3, the maps defined on $S_1 \cup S_2$ by

$$p(t) = T_{\bar{D}_i}(t) \text{ if } t \in S_{3-i}, i = 1, 2, \quad \tau(t) = (t_1, p(t))$$

were of primary importance. For Jordan domains, it turns out that it is more convenient to work with the closely related maps

$$q(t) = T_\Gamma(t) \quad \text{and} \quad \varrho(t) = (t_1, q(t)) \quad \text{on} \quad S_1 \cup S_2.$$

The relationship between p and q is made precise below. In fact, it will turn out that for Jordan domains, $p(t)$ and $q(t)$ are equal for most t (see Lemma 5.3; however, this is not necessarily true for general domains. See Example 3.4, for instance.)

LEMMA 5.1. *The lower semicontinuous regularization of p is q and*

$$q(t) = \liminf_{s \to t, s \neq t} q(s).$$

Proof. Lower semicontinuity of q follows from Lemma 3.1 and $q \leq p$ by definition. Fix $t = (t_1, t_2) \in S_1 \cup S_2$. Then $(t_1, q(t)) \in \partial D_2$, so there is a sequence $(s^n, n \in \mathbf{N})$ of points in D_2 which converge to t. We have

$$\begin{aligned} q(t) &\leq \liminf_{s \to t, s \neq t} q(s) \\ &\leq \liminf_{n \to \infty} q(s_1^n, t_2) \\ &\leq \liminf_{n \to \infty} p(s_1^n, t_2) \\ &\leq \liminf_{n \to \infty} s_2^n \\ &= q(t). \end{aligned}$$

This completes the proof. □

The above lemma implies in particular that p and q coincide at points of continuity of p, which are also points of continuity of q. In fact, much more is true. To prove this, we need a property of Jordan curves.

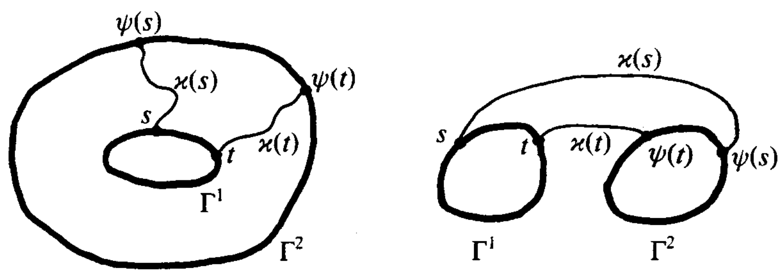


Fig. 5.1. Two possible relative positions of Γ^1 and Γ^2.

MONOTONICITY LEMMA 5.2. *Let Γ^1 and Γ^2 be two disjoint Jordan curves in $\mathbf{R}^2$, with continuous parameterizations φ^i: $[0,1]\to\Gamma^i$, which are one-to-one on $[0,1[$ and such that $\varphi^i(0)=\varphi^i(1), i=1,2$. Fix $F\subset\Gamma^1$, and assume that for each $t\in F$, there is $\psi(t)\in\Gamma^2$ and a simple arc $\varkappa(t)$ with extremities t and $\psi(t)$ such that*

$$\Gamma^1\cap\varkappa(t)=\{t\} \quad \textit{and} \quad \Gamma^2\cap\varkappa(t)=\{\psi(t)\},$$

and

$$s,t\in F,\ s\neq t \ \Rightarrow\ \varkappa(s)\cap\varkappa(t)=\emptyset.$$

If $\varphi^1(0)\in F$ and $\psi(\varphi^1(0))=\varphi^2(0)$, *then $g=(\varphi^2)^{-1}\circ\psi\circ\varphi^1$ is monotone on $(\varphi^1)^{-1}(F)$.*

Proof. $\Gamma^1\cup\Gamma^2$ has three complementary domains D^1, D^2 and D^3, two of them, say D^1 and D^2, are Jordan domains with boundary Γ^1 and Γ^2 respectively, and $\partial D^3=\Gamma^1\cup\Gamma^2$ (see [N; Theorem V.11.3] and Figure 5.1).

Observe that for $t\in F, \varkappa(t)$ lies entirely in $\bar{D}^3$. For otherwise, $\varkappa(t)$ would connect a point in D^i to a point in $D^j, i\neq j$, without meeting $\Gamma^1\cup\Gamma^2$, and this is impossible.

Now fix $0<a<b<1$, and set $\bar{s}=\varphi^1(a), \bar{t}=\varphi^1(b)$. We may assume without loss of generality that $g(a)<g(b)$. We will show that

$$a<u<b \ \Leftrightarrow\ g(a)<g(u)<g(b). \tag{5.1}$$

This will complete the proof, for if $u<v$, we use (5.1) to compare $g(u)$ with $g(a)$ and $g(b)$, and then (5.1) with u, v and either a or b to get $g(u)<g(v)$.

Given the symmetry of the problem, we only need to prove the "$\Rightarrow$" part of (5.1).

By [N; Chapter V.11, Example 3], $D^3\setminus(\varkappa(\bar{s})\cup\varkappa(\bar{t}))$ consists of two complementary Jordan domains E^1 and E^2, one of which, say E^1, satisfies $\partial E^1=\varphi^1([a,b])\cup\varkappa(\bar{s})\cup\varkappa(\bar{t})\cup\Gamma_1^2$ where Γ_1^2 is a subarc of Γ^2 with extremities $\psi(\bar{s})$ and $\psi(\bar{t})$.

Observe that

$$t\in\varphi^1([a,b]) \;\Rightarrow\; \varkappa(t)\subset\bar{E}^1 \;\Rightarrow\; \psi(t)\in\Gamma_1^2.$$

For otherwise, $\varkappa(t)$ would connect points in $\bar{E}^1$ to points in E^2 without meeting $\varkappa(\bar{s})\cup\varkappa(\bar{t})$, and this is impossible since $\varkappa(t)\subset\bar{D}_3$. The second implication is clear.

Similarly, we have

$$t\notin\varphi^1([a,b]) \;\Rightarrow\; \varkappa(t)\subset\bar{E}^2.$$

But then, since $\varphi^1(0)\notin\varphi^1([a,b])$, $\psi(\varphi^1(0))=\varphi^2(0)\notin\Gamma_1^2$. Thus $\Gamma_1^2=\varphi^2([g(a),g(b)])$, proving

$$a<u<v \;\Rightarrow\; g(a)<g(u)<g(b)$$

and completing the proof. □

Lemma 5.2 is useful in the proof of the following.

LEMMA 5.3. *Fix $i=1$ or 2 and suppose $[a,b]\times\{d\}\subset S_i$. Define $g(u)=\varphi^{-1}(\varrho(u,d))$, $u\in[a,b]$. Let Γ' be the sub-arc of Γ with extremities $\varrho(a,d)$ and $\varrho(b,d)$ which contains $\varrho((a+b)/2,d)$. If $\varphi(0)\notin\Gamma'$, then*

(a) *g is monotone;*

(b) *$g(\cdot),p(\cdot,d)$ and $q(\cdot,d)$ have the same points of continuity;*

(c) *p and q coincide at these points of continuity;*

(d) *$p(\cdot,d)$ and $q(\cdot,d)$ have both left and right limits at each $x\in\,]a,b[$.*

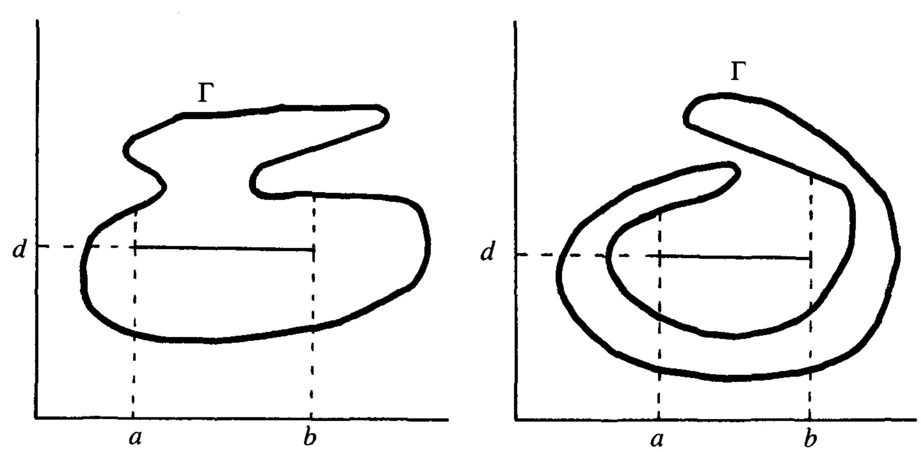


Fig. 5.2. Two possible relative positions of Γ and $[a,b]\times\{d\}$.

Proof. Fix $c<d$ such that $R=[a,b]\times[c,d]\subset S_i$. Then ∂R is a Jordan curve such that $\partial F\cap\Gamma=\emptyset$. Set $F=[a,b]\times\{d\}$, and for $t\in F$, let $\varkappa(t)$ be the vertical segment with extremities t and $\varrho(t)$. Since $\varphi(0)\notin\Gamma'$, we can apply Lemma 5.2 to the Jordan curves $\Gamma^1=\partial R, \Gamma^2=\Gamma$, to get (a). Let D^3 be as in the proof of Lemma 5.2.

For the rest of the proof we will assume without loss of generality that g is increasing. Since φ is one-to-one and continuous, and $\varphi(0)\notin\Gamma'$, it is clear that g and $q(\cdot,d)$ have the same points of continuity. We must show the same is true of g and $p(\cdot,d)$.

Define $h(\cdot)$ by

$$h(x)=\varphi^{-1}(\tau(x,d)),\quad a\le x\le b.$$

We claim for $x\in]a,b[$ that

$$g(x-)\le h(x)\le g(x+). \tag{5.2}$$

This will complete the proof. Indeed, (5.2) implies that h is monotone, and that $p(x,d)=q(x,d)$ and $p(\cdot,d)$ is continuous at x whenever $q(\cdot,d)$ is. From Lemma 5.1, we get (b) and (c), and (d) follows from monotonicity of g and h. It remains to prove (5.2).

Let Γ_n be the subarc of Γ' with extremities $\varrho(x-1/n,d)$ and $\varrho(x+1/n,d)$. Let us write $a=x-1/n, b=x+1/n$. Fix $n\in\mathbf{N}$. Without loss of generality, we can assume $\Gamma_n=\Gamma'$. Let $\tilde{L}$ be the vertical segment from (x,d) to $\tau(x,d)$. As in the proof of Lemma 5.2, let E_1 and E_2 be the two complementary open domains of $D^3\setminus(\varkappa(a,d)\cup\varkappa(b,d))$, and assume that E_1 is the one that satisfies $\partial E_1=F\cup\varkappa(a,d)\cup\varkappa(b,d)\cup\Gamma'$. By the definition of τ, $\tilde{L}\subset\bar{E}^1\cup\bar{E}^2\subset D_i\cup\Gamma$. The initial part of $\tilde{L}$, namely the open line from (x,d) to $\varrho(x,d)$, is in E^1, as we have seen in the proof of 5.2. Suppose $\tau(x,d)\in\Gamma\setminus\Gamma'\subset\bar{E}^2$. Then let

$$z=\inf\{y>d\colon (x,y)\in\bar{E}^2\}.$$

Evidently $z>d$ and $(x,z)\in\bar{E}^1\cap\bar{E}^2$. But this is a contradiction since $\bar{E}^1\cap\bar{E}^2\subset\varkappa(a,d)\cup\varkappa(b,d)$ and $\tilde{L}$ does not intersect this set. It follows that $\tau(x,d)\in\Gamma'$, and hence that $g(x-1/n)<h(x)<g(x+1/n)$. Let $n\to\infty$ to get (5.2). The proof is complete. □

Recall Definition 1.2, which defines the Maltese cross condition and the related sets $M(\Gamma)$ and $M_0(\Gamma)$.

LEMMA 5.4. *$M(\Gamma)$ and $M_0(\Gamma)$ are Borel.*

Proof. $M(\Gamma)$ is Borel since

$$M(\Gamma) = \bigcup_{h, \alpha \in \mathbf{Q}_+^*} M_\alpha(\Gamma, h)$$

and $M_\alpha(\Gamma, h)$ is easily seen to be closed ($\mathbf{Q}_+^*$ denotes the set of positive rational numbers). To see that $M_0(\Gamma)$ is Borel, we only need to show that $M_0(\Gamma, h)$ is Borel, since

$$M_0(\Gamma) = \bigcup_{h \in \mathbf{Q}_+^*} M_0(\Gamma, h).$$

Now

$$M_0(\Gamma, h) = \bigcup_{\substack{O \text{ open} \\ O \supset M_0((0,0), h)}} \{t \in \Gamma : \Gamma \cap (t+O) = \varnothing\}.$$

Since $\mathbf{R}^2$ is a separable metric space, the above union can be made countable. Since each set appearing in the right-hand side of the union is closed, this completes the proof. □

We shall say that a curve Γ^1 *dominates* a curve Γ^2 provided $\Gamma^2 \subset S^1(\Gamma^1)$.

FUNDAMENTAL LEMMA 5.5. (a) *Suppose* $\lambda\{\mathrm{pr}_1(M_0(\Gamma))\}>0$. *Then there is a simple subarc* Γ' *of* Γ *with extremities* $\bar{s}$ *and* $\bar{t}$, *say, and two continuous monotone curves* Γ_L *and* Γ_U, *both with extremities* $\bar{s}$ *and* $\bar{t}$, *such that*

(5.3) Γ_U *dominates* Γ' *and* Γ' *dominates* Γ_L;

(5.4) $S^1(\Gamma_U) \setminus S^1(\Gamma_L)$ *is a disjoint union of rectangles whose boundaries are contained in* $\Gamma_U \cup \Gamma_L$;

(5.5) $\lambda(\mathrm{pr}_1(\Gamma' \cap \Gamma_U \cap \Gamma_L))>0$;

(*see Figure* 5.3).

(b) *Suppose that* $\lambda\{\mathrm{pr}_1(M(\Gamma))\}>0$. *Then in addition to* (5.3)–(5.5), *there is an* $\alpha>0$ *and a closed set* $F \subset \Gamma' \cap \Gamma_U \cap \Gamma_L$ *such that* $\lambda(\mathrm{pr}_1(F))>0$ *and*

(5.6) *if* $s \in F, t \in \Gamma'$, *then* $\alpha<|t_2-s_2|/|t_1-s_1|<1/\alpha$;

(5.7) Γ' *has a tangent at each* $s \in F$.

Proof. We first localize. Since $M_0(\Gamma)=\bigcup_{h\in \mathbf{Q}_+} M_0(\Gamma, h)$, there is $h>0$ such that $\lambda\{\mathrm{pr}_1(M_0(\Gamma, h))\}>0$. Moreover, it is clear that there exists an open square R of side less than h whose intersection with $M_0(\Gamma, h)$ has a 1-projection with positive measure. Fix

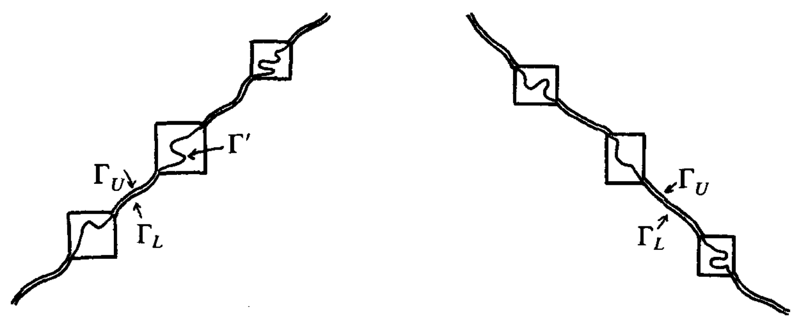


Fig. 5.3. Two possibilities for Γ'.

such an R and let $I \subset [0,1[$ be a closed interval such that $\varphi(I) \subset R$ and $\lambda(\mathrm{pr}_1(F_0)) > 0$, where $F_0 = \varphi(I) \cap M_0(\Gamma, h)$.

The proof is based on one simple remark.

(5.8) Let $s, t \in F_0$ and suppose that $s = \varphi(u)$, $t = \varphi(v)$ with $u < v$ in I. Let J be the open interval $]u, v[$. Then if R_m is the open rectangle having two opposite corners at s and t, we have $\varphi(J) \subset R_m$.

Indeed, $M_0(s, h)$ consists of four branches, each having length greater than the side of R, so that it divides R into four disjoint rectangles. By the same token, $M_0(s, h) \cup M_0(t, h)$ divides R into nine disjoint rectangles, and the middle one, R_m, has s and t at opposing corners (see Figure 5.4). Now $\varphi(J)$ is a continuous curve with

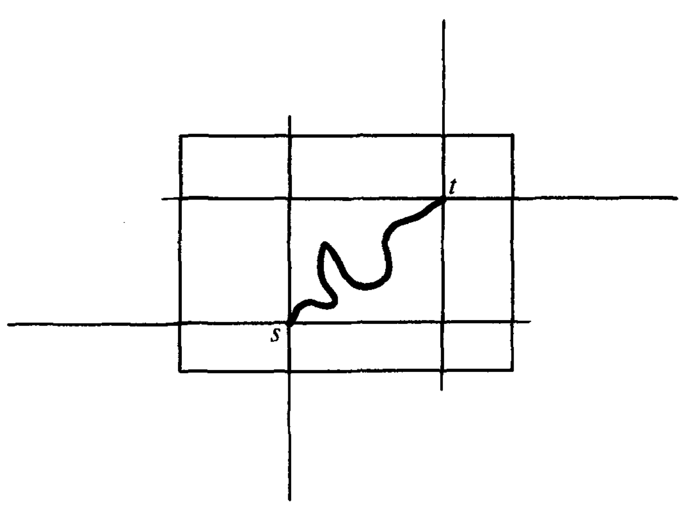


Fig. 5.4.

extremities s and t. It does not intersect ∂R since $J \subset I$, and it does not intersect $M_0(s,h) \cup M_0(t,h)$ since s and t are in $M_0(\Gamma,h)$. A moments thought now shows that it must be contained in R_m, as we claimed.

If we now take three points in F_0, say $r=\varphi(u), s=\varphi(v), t=\varphi(w)$ with $u<v<w$ in I, then (5.8) implies that s is contained in the rectangle having two opposite corners at r and t. Thus the three points can be totally ordered by one of the orders $\leq$ or $\triangle$. It follows that the whole set F_0 can be totally ordered by the same order.

We will assume for the rest of this proof that the order is $\leq$ (the argument above shows that the restriction of φ to $B=\varphi^{-1}(F_0)$ is monotone: for we either have $r \leq t$ or $t \leq r$. In the first case we have $\varphi(u) \leq \varphi(v) \leq \varphi(w)$, which implies that $\varphi|_B$ is increasing, and in the second, $\varphi(w) \leq \varphi(v) \leq \varphi(u)$, which implies $\varphi|_B$ is decreasing). By reparametrizing Γ if necessary, we may then suppose that $\varphi|_B$ is increasing with respect to $\leq$.

Let us shrink things slightly. There is a closed subset F_1 of F_0 whose 1-projection still has positive measure. Let $B_1=\varphi^{-1}(F_1)$, let $\underline{u}=\inf\{u: u \in B_1\}$, and let $\bar{u}=\sup\{u: u \in B_1\}$. Set $K=[\underline{u}, \bar{u}]$, and let $\Gamma'=\varphi(K)$. Then Γ' has extremities $\underline{s}=\varphi(\underline{u})$ and $\bar{t}=\varphi(\bar{u})$.

To construct Γ_L and Γ_U, first let

$$\Lambda_L = \{t \in T: \exists s \in F_1 \text{ such that } s \triangle t\},$$

$$\Lambda_U = \{t \in T: \exists s \in F_1 \text{ such that } t \triangle s\},$$

and then let Γ_L and Γ_U be the upper left boundary of Λ_L and the lower right boundary of Λ_U respectively. According to [W3; Theorem 2.7] these are monotone non-decreasing curves. An alternate description is the following: $\dot{K} \setminus B_1$ is open, and hence is a disjoint union of open intervals: $\dot{K}_1 \setminus B_1 = \bigcup_n]u_n, v_n[$. Let R_n be the closed rectangle whose lower left corner is $\varphi(u_n)$ and whose upper right corner is $\varphi(v_n)$. Then

$$\Gamma' \subset F_1 \cup \bigcup_n R_n$$

and Γ_U consists of F_1 together with the left and top boundaries of each of the R_n, and Γ_L consists of F_1 together with the bottom and right boundaries of each of the R_n. Now (5.3) and (5.4) are clear, and (5.5) follows since $\Gamma_L \cap \Gamma_U \cap \Gamma' = F_1$.

Next, if $\lambda\{\mathrm{pr}_1(M(\Gamma))\}>0$, we use the same reduction as before to find $\alpha>0$ and $h>0$ for which $\lambda\{\mathrm{pr}_1(M_\alpha(\Gamma,h))\}>0$. Since $M_\alpha(\Gamma,h) \subset M_0(\Gamma,h)$, we get (5.3)–(5.5). But now, by the definition of the Maltese cross $M_\alpha(t,h)$, (5.6) clearly holds for each $t \in F_1 \cap M_\alpha(\Gamma,h)$ since $\Gamma \cap M_\alpha(t,h)=\emptyset$.

Finally, note that after removing from Γ_L (resp. Γ_U) at most countably many vertical segments, one is left with the graph of a monotone function ψ_L (resp. ψ_U), with the property that $\psi_L \leq \psi_U$. So at points x where $\psi_L(x)=\psi_U(x)$ and where both these functions are differentiable, their derivatives must coincide. Since $\psi_L(x)=\psi_U(x)$, $\forall x \in \mathrm{pr}_1(F_1)$, we conclude that Γ' has a tangent at λ-almost all $\psi_L(x)$, $x \in \mathrm{pr}_1(F_1)$. If we take a slightly smaller set $F \subset F_1$ with the same measure, we can satisfy (5.7). The proof is complete. □

We now state the main result of this section.

THEOREM 5.6. *Let $(X_t, t \in T)$ satisfy Assumption* 1.1, *and let Γ be a Jordan curve with complementary open domains D_1 and D_2. Assume*

$$\lambda\{\mathrm{pr}_i(M(\Gamma))\} = 0, \quad i = 1 \text{ or } 2. \tag{5.9}$$

Then $\mathscr{H}(D_1)$ and $\mathscr{H}(D_2)$ are conditionally independent given $\mathscr{H}(\Gamma)$.

Remark 5.7. (a) We are not assuming that Γ is bounded, nor that $\Gamma \subset \mathbf{R}^2_+$. Of course, $(X_t, t \in \mathbf{R}^2_+)$ can be extended to all of $\mathbf{R}^2$ by setting $X_t = 0$, if $t \in \mathbf{R}^2 \setminus \mathbf{R}^2_+$, and thus the behavior of Γ in $\mathbf{R}^2 \setminus \mathbf{R}^2_+$ is irrelevant.

(b) A straightforward extension of Theorem 5.6 can be made by considering a domain D_1 whose boundary consists of countably many disjoint Jordan curves $(\Gamma_n, n \in \mathbf{N})$. In this case, (5.9) becomes

$$\lambda\left\{\mathrm{pr}_i\left(\bigcup_{n \in \mathbf{N}} M(\Gamma_n)\right)\right\} = 0, \quad i = 1 \text{ or } 2.$$

Proof of Theorem 5.6. By Theorem 3.3, it suffices to show that if B is an interval, $d > 0$, and if $B^{(d)}$ is in either $\dot{D}_1 \cap S^1(\dot{D}_2)$ or $\dot{D}_2 \cap S^1(\dot{D}_1)$, then $X(V(B^{(d)})) \in \mathscr{H}(\Gamma)$.

By Lemma 5.3 (a), (b) and (c), the maps $p(\cdot, d)$ and $q(\cdot, d)$ coincide except on a countable set, so we may replace p by q in the definition of $V(B^{(d)})$. Let $f(x) = q(x, d)$ and, to simplify notation, if $A \subset B$ let

$$\hat{A} = V(A^{(d)}) = \{t \in T : t_1 \in A, 0 \leq t_2 \leq f(t_1)\}.$$

We will decompose B into a number of disjoint sets B_n and show that $X(\hat{B}_n) \in \mathscr{H}(\Gamma)$ for each n. This will imply the theorem since $X(\hat{B}) = \Sigma_n X(\hat{B}_n)$.

Let B_1 be the set of $u \in B$ such that f is either discontinuous or has a strict local extremum at u. Let B_2 be the set of $u \in B \setminus B_1$ such that $(u, f(u))$ is an accumulation point of $\Gamma \cap (\mathbf{R} \times \{f(u)\})$, and let B_3 be the set of u in $B \setminus (B_1 \cup B_2)$ at which f is strictly monotone.

Recall that f is *strictly monotone* at u if there exists $h > 0$ such that either

(5.10) $$u-h<v<u<w<u+h \quad \Rightarrow \quad f(v)<f(u)<f(w)$$

or

(5.11) $$u-h<v<u<w<u+h \quad \Rightarrow \quad f(v)>f(u)>f(w).$$

Note that the B_n are measurable, being the projections of Borel sets. Since f can have at most countably many local extrema or discontinuities by Lemma 5.3, B_1 is countable. Thus $X(\hat{B}_1)$ vanishes a.s., and is trivially in $\mathscr{H}(\Gamma)$. Furthermore, $X(\hat{B}_2) \in \mathscr{H}(\Gamma)$ by Proposition 2.8.

Leaving aside for the moment the question of whether $X(\hat{B}_3) \in \mathscr{H}(\Gamma)$, let us show $B=B_1 \cup B_2 \cup B_3$. Suppose $t_1 \in B \setminus B_3$, and show $t_1 \in B_1 \cup B_2$. Now f is not strictly monotone at t_1, and we must have one of the following:

(5.12 a) f has a strict local extremum at t_1;

(5.12 b) $(t_1, f(t_1))$ is an accumulation point of the intersection of $\mathbf{R} \times \{f(t_1)\}$ with the graph of f.

(5.12 c) There exists a monotone sequence $(u^n, n \in \mathbf{N})$ converging to t_1 such that for all n, $f(u^{2n})>f(t_1)>f(u^{2n+1})$.

If (5.12 a) holds, then $t \in B_1$, and if (5.12 b) holds, $t \in B_2$. Thus suppose (5.12 c) holds.

Now if f is not continuous at t_1, we have $t_1 \in B_1$. If f is continuous at t_1, then $(u^n, f(u^n))$ converges to $t=(t_1, f(t_1))$. Let $v_0, v_1, \dots$ be such that $t=\varphi(v_0)$, and $(u^n, f(u^n))=\varphi(v_n)$. As φ is continuous and one-to-one, v_n must converge to v_0. Now (u^n) is monotone, and we may assume without loss of generality that it is decreasing.

By Lemma 5.3, (v_n) is also monotone, and we may suppose it is decreasing as well. Let Γ_n be the arc $\{\varphi(v): v_0 \leq v \leq v_{2n}\}$. The segment $B^{(d)}$ lies entirely inside D_1 or D_2, and hence does not intersect Γ. By the definition of $q(\cdot, d)$, Γ cannot intersect the open vertical segment from (u, d) to $(u, f(u))$ for any $u \in B$. Thus Γ_n can intersect the polygonal path from t to (t_1, d) to (u^{2n}, d) to $(u^{2n}, f(u^{2n}))$ only at the two endpoints. (This path is the solid curve in Figure 5.5.)

On the other hand, Γ_n is a continuous curve starting at t, and passing through $(u^{2n+1}, f(u^{2n+1}))$ and $(u^{2n}, f(u^{2n}))$. Thus it must pass at least once through the open horizontal segment $]t_1, u^{2n}[\times\{t_2\}$ (the dotted line in Figure 5.5). This is true for all n, hence (5.12 b) must hold, and $t_1 \in B_2$. In all cases, we have shown $t_1 \in B_1 \cup B_2$, and thus $B=B_1 \cup B_2 \cup B_3$.

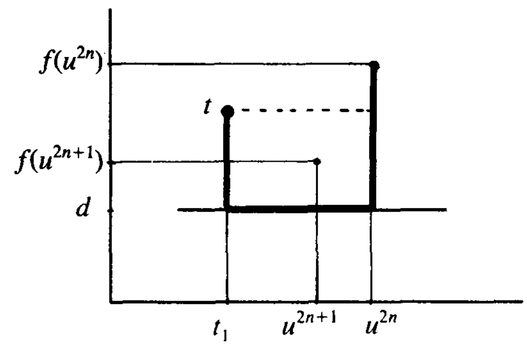


Fig. 5.5.

It only remains to show that $X(\hat{B}_3) \in \mathscr{H}(\Gamma)$. By hypothesis, the Maltese cross condition is satisfied at $(u, f(u))$ for almost every $u \in B_3$, so by Assumption 1.1, we can replace B_3 by a smaller set with the same measure, which we again denote B_3, so that for each $u \in B_3$ there is $h>0$ such that

(5.13 a) Γ satisfies the Maltese cross condition at $(u, f(u))$;

(5.13 b) either (5.10) or (5.11) holds at u;

(5.13 c) $\Gamma \cap (]u-h, u+h[\times \{f(u)\}) = \{(u, f(u))\}$.

Fix $h>0$ and let B_h^+ (resp. B_h^-) be the set of $u \in B_3$ for which (5.10), (5.13 a) and (5.13 c) (resp. (5.11), (5.13 a) and (5.13 c)) hold. It is then enough to show that $X(\hat{B}_h^+)$ and $X(\hat{B}_h^-)$ are in $\mathscr{H}(\Gamma)$. Let us omit the subscript h and consider

$$A^{\pm} = I \cap B_h^{\pm},$$

where I is a fixed interval of length less than h. (Indeed, B_h is a finite union of such sets and $B_h^+ \cup B_h^- \uparrow B_3$.)

The two sets $A^{\pm}$ are handled the same way, so we will only deal with A^+ here. The restriction to an interval of length less than h means that (5.10) applies to any pair of points in A^+. Thus

(5.14) $f|_{A^+}$ is strictly increasing.

Let $A_0^+ = A^+ \cap \{u: (u, f(u)) \notin M_0(\Gamma)\}$, i.e., the subset of A^+ such that Γ satisfies the cross condition at $(u, f(u))$. Since $u \notin B_2$, $(u, f(u))$ must be an accumulation point of

$\Gamma\cap\{t: t_1=u, t_2>f(u)\}$. The graph $\tilde{G}$ of $f|_{A_0^+}$ is totally ordered for $\leq$, and it is also the graph of $f^{-1}|_{f(A_0^+)}$, so we can apply Proposition 2.8 to the horizontal shadow $S^2(\tilde{G})$ of $\tilde{G}$ to see that $X(S^2(\tilde{G}))\in\mathscr{H}(\Gamma)$. Now $S^1(\tilde{G})=\hat{A}_0^+$, so by Proposition 2.3, $X(\hat{A}_0^+)\in\mathscr{H}(\Gamma)$.

Now let $A_1^+=A^+\setminus A_0^+$. We have reduced the proof to the problem of showing that $X(\hat{A}_1^+)\in\mathscr{H}(\Gamma)$. If $u\in A_1^+$, then (5.13) and (5.14) hold and in addition, for some $\delta>0$,

$$\Gamma\cap(\{u\}\times]f(u)-\delta, f(u)+\delta[)=\{(u,f(u))\}. \tag{5.15}$$

Let $A_{1,\delta}^+$ be the set of $u\in A_1^+$ which satisfy (5.15) for some fixed δ. By taking h and/or δ smaller if necessary, we may assume that $h=\delta$. Let

$$C=A_{1,h}^+\cap\{u: f(u)\in J\}$$

where J is a given interval of length less than h. It is enough to show that $X(\hat{C})\in\mathscr{H}(\Gamma)$.

Let G be the graph of $f|_C$. By construction, $G\subset M_0(\Gamma,h)$, which puts us in the situation of Lemma 5.5. Let $\Gamma_0=\{\varphi(u): u\in L\}$, where $L\subset]0,1[$ is the smallest interval such that $G\subset\Gamma_0$. Then Γ_0 must look like the first picture in Figure 5.3. In particular, from (5.8), if $r,t\in G, s\in\Gamma_0$ are such that $r_1<s_1<t_1$, then $r\leq s\leq t$.

The salient points we have established can be expressed succintly in terms of G.

(5.16a) G is the graph of a function and $G\subset M(\Gamma)$;

(5.16b) if $r,t\in G$ and $s\in\Gamma_0$ are such that $r_1<s_1<t_1$, then $r_2<s_2<t_2$.

These are the only facts we will use about Γ in what follows.

If $t\in G$, then for any α and $h>0$, $M_\alpha(t,h)$ intersects Γ. That means at least one of its four branches does. We will handle them separately, starting with the two horizontal branches. Let

$$C_1=\left\{u\in C: \liminf_{t\in\Gamma, t_1\downarrow u}\left|\frac{t_2-f(u)}{t_1-u}\right|=0\right\}.$$

Fix $\varepsilon>0$ and let $\mathscr{I}_\varepsilon$ be the class of intervals $[a,b]$ with $a\in C_1$, $0<b-a<h$, for which there exists v such that

$$(b,v)\in\Gamma \quad\text{and}\quad |v-f(a)|<\varepsilon|b-a|. \tag{5.17}$$

Then $\mathscr{I}_\varepsilon$ is a Vitali cover of C_1 and Lemma 2.7 applies: there is a sequence of intervals $[a_n,b_n]$ in $\mathscr{I}_\varepsilon$ such that $E((X(\hat{C}_1)-Y)^2)<\varepsilon$, where $Y=\Sigma_{n\in\mathbf{N}}X([a_n,b_n]\times[0,f(a_n)])$. For each

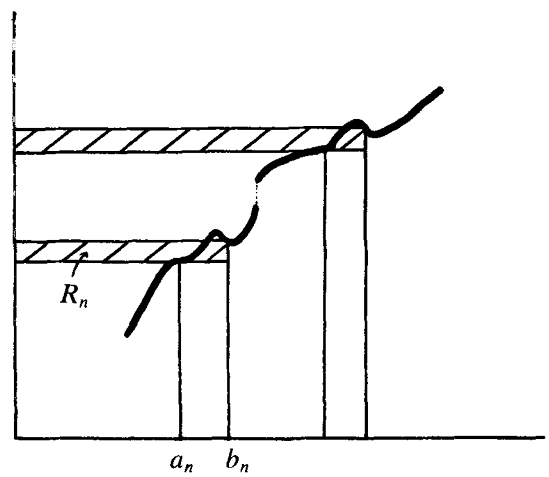


Fig. 5.6.

n, choose $v=v_n$ to satisfy (5.17) with $b=b_n$. Set

$$Z=\sum_{n\in\mathbf{N}}(X_{b_n,v_n}-X_{a_n,f(a_n)}).$$

Z is clearly $\mathscr{H}(\Gamma)$-measurable. By (5.16b), if $a_m<a_n$, then $f(a_m)<v_m<f(a_n)<v_n$, so that the intervals $[f(a_n),v_n]$ are disjoint. Let $R_n=[0,b_n]\times[f(a_n),v_n]$ and notice that

$$Z-Y=\sum_{n\in\mathbf{N}}X(R_n),$$

and that the rectangles R_n are disjoint. (See Figure 5.6. This is the key observation; most of the work in this proof was to set it up.) Thus

$$E((Z-Y)^2)=\sum_{n\in\mathbf{N}}E(X(R_n)^2). \tag{5.18}$$

Now the area of $\bigcup_{n\in\mathbf{N}}R_n$ is bounded by

$$\sum_{n\in\mathbf{N}}b_n(v_n-f(a_n))\le\varepsilon\sum_{n\in\mathbf{N}}b_n(b_n-a_n)$$

by (5.17). The diameter of C_1 is less than h, so if $b=\sup C_1$, this is

$$\le\varepsilon bh.$$

Now let $\varepsilon \to 0$. This goes to zero, hence by Assumption 1.1, $E((Z-Y)^2) \to 0$, so $Z-Y \to 0$ in L^2, hence $Z \to X(\hat{C}_1) \in \mathscr{H}(\Gamma)$.

Next let

$$C_2 = \left\{ u \in C \setminus C_1 : \liminf_{t \in \Gamma, t_1 \uparrow u} \left| \frac{f(u)-t_2}{u-t_1} \right| = 0 \right\}.$$

We proceed exactly as above except that we derive $\mathscr{I}_\varepsilon$ using intervals whose right, rather than left, endpoint is in C_2. Once again the rectangles R_n are disjoint and we conclude that $X(\hat{C}_2) \in \mathscr{H}(\Gamma)$. (The reason for handling C_1 and C_2 separately is simply that the rectangles defined in case 2 may not be disjoint from those in case 1.)

This takes care of the horizontal branches of the Maltese cross. The other two cases correspond to the vertical branches, and they follow by symmetry. If we interchange the horizontal and vertical coodinates, this interchanges horizontal and vertical branches of the crosses, while (5.16) remains true. If C_3 and C_4 are the corresponding sets for the vertical branches, and if G_3 and G_4 are the subsets of G over C_3 and C_4 respectively, the arguments above establish that $X(S^2(G_3))$ and $X(S^2(G_4))$ are in $\mathscr{H}(\Gamma)$. Then Proposition 2.3 implies that $X(\hat{C}_3)$ and $X(\hat{C}_4)$ are also $\mathscr{H}(\Gamma)$-measurable, since $\hat{C}_i = S^1(G_i)$. This finishes the proof. □

6. Necessary conditions for the Brownian sheet

In the previous section, we showed that, for any process satisfying Assumption 1.1, the Maltese cross condition is sufficient to ensure that a Jordan domain has the sharp Markov property. However, this condition is not always necessary. For instance, if X is the Poisson sheet, then the sharp Markov property is known to hold for a large class of domains whose boundaries do not satisfy the Maltese cross condition (see [C; Theorem 4.1]). The same is true of many jump processes, since in this case the Markov property is related to global properties concerning the way discontinuities of the process propagate: see [DW]. However, if we restrict ourself to the Brownian sheet, it turns out that for Jordan domains, the Maltese cross condition is indeed necessary as well as sufficient. The main result of this section is the following theorem.

THEOREM 6.1. *Let $\Gamma \subset T \cup \{\infty\}$ be a Jordan curve with complementary open domains D_1 and D_2, and let $(X_t, t \in T)$ be a Brownian sheet. Then $\mathscr{H}(D_1)$ and $\mathscr{H}(D_2)$ are conditionally independent given $\mathscr{H}(\Gamma)$ if and only if $\lambda\{\mathrm{pr}_1(M(\Gamma))\} = 0$.*

Remark 6.2. (a) If $D_1 \subset T$ is an open set, and $\Gamma = \partial \bar{D}_1$, then D_1 can satisfy the sharp

Markov property even though $\lambda\{\mathrm{pr}_1(M(\Gamma))\}>0$. This is the case in Example 3.5, where $M(\Gamma)$ is the subset of the diagonal whose projection on the x-axis is the Cantor set A.

(b) We conjecture that in general, the necessary and sufficient condition for the sharp Markov property to hold in the case of the Brownian sheet is

$$\lambda\{\mathrm{pr}_1(M(\tau(S_1)\cup\tau(S_2)))\}=0.$$

Before beginning the proof of Theorem 6.1, we give a few corollaries which provide easily verifiable criteria in various special cases. For instance, in the case where Γ is rectifiable [S; Chapter IV, § 8], the Maltese cross condition can be expressed in terms of the (one-to-one) parameterization $\varphi=(\varphi_1,\varphi_2)$ of Γ. Recall that Γ is rectifiable if and only if both φ_1 and φ_2 have bounded variation [S; Chapter 4, (8.2)]. So in this case, φ_i is canonically associated with a signed measure on $[0,1]$, denoted $d\varphi_i$, $i=1,2$. We let $|d\varphi_i|$ denote the total variation measure associated with $d\varphi_i$. Recall that two signed measures μ_1 and μ_2 are *mutually singular* if and only if $|\mu_1|$ and $|\mu_2|$ are mutually singular [H; Chapter 6, § 30]: we denote this $\mu_1\perp\mu_2$.

COROLLARY 6.3. *Let $(X_t, t\in T)$ be a Brownian sheet, and let Γ be a rectifiable Jordan curve with continuous one-to-one parameterization $\varphi=(\varphi_1,\varphi_2)$: $[0,1]\to T\cup\{\infty\}$. Let D_1 and D_2 be the two complementary open domains bounded by Γ. Then $\mathscr{H}(D_1)$ and $\mathscr{H}(D_2)$ are conditionally independent given $\mathscr{H}(\Gamma)$ if and only if the signed measures $d\varphi_1$ and $d\varphi_2$ are mutually singular.*

Proof. By Theorem 6.1, all we have to show is that

$$\lambda\{\mathrm{pr}_1(M(\Gamma))\}=0 \quad\Leftrightarrow\quad d\varphi_1\perp d\varphi_2. \tag{6.1}$$

For this, we recall that if φ_1 and φ_2 have bounded variation, then

$$\psi_i(x)=\lim_{h\to 0}\frac{\varphi_{3-i}(x+h)-\varphi_{3-i}(x)}{\varphi_i(x+h)-\varphi_i(x)} \tag{6.2}$$

exists and is finite for $|d\varphi_i|$-almost all x, $i=1,2$, and

$$d\varphi_1\perp d\varphi_2 \quad\Leftrightarrow\quad \psi_1(x)=0\ \ |d\varphi_1|\text{-a.e.} \quad\Leftrightarrow\quad \psi_2(x)=0\ \ |d\varphi_2|\text{-a.e.} \tag{6.3}$$

Both (a) and (b) are well-known if $d\varphi_i$ is Lebesgue-measure [S; Chapter IV, (7.1)]. Since we have found no reference to the general case, we give a sketch of the proof.

Let A_1^i (resp. A_2^i) be the set of points of increase (resp. decrease) of φ_i, and set

$A_3^i=[0,1]\setminus(A_1^i\cup A_2^i)$. Since A_3^i consists of local extrema of φ_i and points x such that the set $\{y:\varphi_i(y)=\varphi_i(x)\}$ is infinite, A_3^i is a $|d\varphi_i|$-null set [S; Chapter IX, (6.4)]. Now on A_1^i and A_2^i one can first prove a result similar to that of [S; Chapter IV, (5.1)], and then repeat the proof of [S; Chapter IV, (5.4)], in each case using the Vitali Covering Theorem 2.4 for the non-negative measure $|d\varphi_1|+|d\varphi_2|$, instead of for Lebesgue measure (this more general form of the Vitali Covering Theorem can be found in [DS; III, 12.3]). This proves (6.2); details are left to the reader.

The proof of (6.3) involves the same decomposition of $[0,1]$. Each A_j^i, $j=1,2,3$, is handled as in [S; Chapter IV, (7.1)]. Again details are left to the reader.

In order to prove (6.1), first assume that $\lambda\{\mathrm{pr}_1(M(\Gamma))\}=0$. Define $A=\{x\in[0,1]: 0<|\psi_1(x)|<+\infty\}$. Looking back to (1.6), we see that $A\subset M(\Gamma)$, and so $\lambda(\{\varphi_1(x): x\in A\})=0$. By [S; Chapter IX, (6.4)], this implies that A has $|d\varphi_1|$-measure zero. Thus $|\psi_1(x)|\in\{0,+\infty\}$, for $|d\varphi_1|$-almost all x. By (6.2), we get $\psi_1(x)=0$ $|d\varphi_1|$-a.e. so $d\varphi_1\perp d\varphi_2$ by (6.3).

Now assume that $\lambda\{\mathrm{pr}_1(M(\Gamma))\}>0$, and let us show that $d\varphi_1$ and $d\varphi_2$ are not mutually singular. Indeed, let $\Gamma'=\varphi([x_0,x_1])$, F and $\alpha>0$ be given by Lemma 5.5. By (5.6) and (5.7), we have $0<\psi_1(x)<+\infty$, for $x\in\varphi^{-1}(F)$. So by (6.3), we only need to show that $\varphi^{-1}(F)$ has positive $|d\varphi_1|$-measure. Define

$$L(x)=\max_{x_0\leq u\leq x}\varphi_1(u).$$

By Lemma 5.5, $L(x)=\varphi_1(x)$ when $x\in\varphi^{-1}(F)$, so

$$\lambda(\{L(x):x\in\varphi^{-1}(F)\})>0.$$

By [S; Chapter IX, (6.4)], this is equivalent to saying that $\varphi^{-1}(F)$ has positive dL-measure. But since L is absolutely continuous with respect to $|d\varphi_1|$, $\varphi^{-1}(F)$ also has positive $|d\varphi_1|$-measure. This proves the corollary. □

If φ_1 or φ_2 does not have bounded variation, the measures $d\varphi_i$ may not be defined. However, for curves which are not too irregular, there are two measures naturally associated with Γ which allow one to extend Corollary 6.3. Indeed, consider the following regularity assumption on Γ.

Assumption 6.4. For λ-almost all $r\in\mathbf{R}_+$, the intersections of Γ with the horizontal line $\mathbf{R}_+\times\{r\}$ and with the vertical line $\{r\}\times\mathbf{R}_+$ are finite.

Since Γ is a Jordan curve, this is equivalent to requiring that φ_1 and φ_2 satisfy

Banach's condition (T1) [S; Chapter IX, §6]. It is satisfied by functions of bounded variation [S; Chapter IX, (6.2)], but the converse is false.

Let q_1, ϱ_1 be the maps denoted q and ϱ at the beginning of Section 5, and let q_2, ϱ_2 be the horizontal counterparts, that is

$$q_2(t) = \inf\{u \geq t_1 : (u, t_2) \in \Gamma\}, \quad \varrho_2(t) = (q_2(t), t_2).$$

Now let μ_i be the image on Γ of two-dimensional Lebesgue measure m under the map ϱ_i, i.e.

$$\mu_i(F) = m(\varrho_i^{-1}(F)), \quad F \in \mathscr{B}(\Gamma).$$

Null sets of μ_i are identified by the following lemma (whose conclusion is false without Assumption 6.4).

LEMMA 6.5. *Suppose Γ satisfies Assumption 6.4. Then for $i=1,2$ and $F \in \mathscr{B}(\Gamma)$,*

$$\mu_i(F) > 0 \quad \Leftrightarrow \quad \lambda(\mathrm{pr}_i(F)) > 0.$$

Proof. For $s=(s_1, s_2) \in T$, define

$$l_i(s) = \begin{cases} \inf\{s_{3-i} - t_{3-i} : (t_1, t_2) \in \Gamma, s_i = t_i, t_{3-i} < s_{3-1}\} & \text{if } \{\ \} \neq \emptyset, \\ s_{3-i} & \text{otherwise.} \end{cases}$$

Now the set $A_i = \{x \in \mathbf{R}_+ : \Gamma \cap (\{x\} \times \mathbf{R}_+) \text{ is infinite}\}$ has Lebesgue measure zero by Assumption 6.4, and so

$$\begin{aligned} \mu_i(F) &= m(\varrho_i^{-1}(F)) \\ &= \int_{\mathrm{pr}_i(F)} dt_i \int_{\mathbf{R}_+} dt_{3-i} I_{\varrho_i^{-1}(F)} \\ &= \int_{\mathrm{pr}_i(F) \setminus A_i} dt_i\, k(t_i), \end{aligned}$$

where $k(t_i) = \Sigma\, l_i(s)$ and the summation is over all $s \in F$ with $s_i = t_i$. Since $k(t_i) > 0$ for $t_i \in \mathrm{pr}_i(F) \setminus A_i$, the conclusion of the lemma follows. □

COROLLARY 6.6. *Let $(X_t, t \in T)$ be a Brownian sheet, and let Γ be a Jordan curve satisfying Assumption 6.4 with complementary open domains D_1 and D_2. Then $\mathscr{H}(D_1)$ and $\mathscr{H}(D_2)$ are conditionally independent given $\mathscr{H}(\Gamma)$ if and only if μ_1 and μ_2 are mutually singular.*

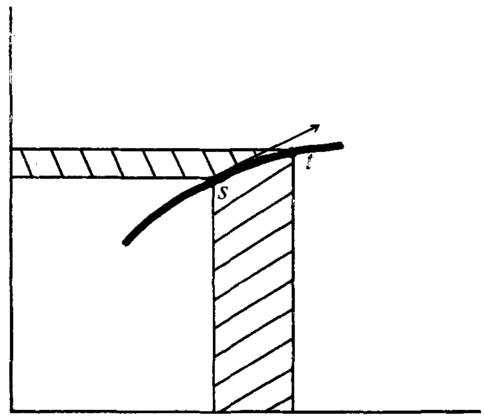


Fig. 6.1.

Proof. Set $t_0=\varphi(0)$. For $s, t \in \Gamma \setminus \{t_0\}$, let $\Gamma[s, t]$ denote the simple arc in Γ with extremities s and t which does not contain t_0. Define α_i: $[0, 1] \to \mathbf{R}$ by $\alpha_i(x)=\mu_i(\varphi([0, x]))$, i.e., α_i is the inverse image of μ_i under φ. By (6.2), applied to α_1 and α_2,

$$\lim_{h\to 0} \frac{\alpha_2(x+h)-\alpha_2(x)}{\alpha_1(x+h)-\alpha_1(x)}$$

exists $d\alpha_1$-a.s., and thus

$$\psi(0) = \lim_{t\to s} \mu_2(\Gamma[s, t])/\mu_1(\Gamma[s, t])$$

exists and is finite for μ_1-almost all $s \in \Gamma$.

Now suppose that at $s=(s_1, s_2) \in \Gamma$, Γ admits a tangent vector (d_1, d_2) which is not vertical, that is $d_1 \neq 0$. It is easily seen (see Figure 6.1) that $\psi(s)=s_1 d_2/(s_2 d_1)$. The proof of this fact, which uses only elementary calculus, is omitted.

The corollary is now easily proved. Indeed, if $\lambda\{\mathrm{pr}_1(M(\Gamma))\}>0$, then by Lemma 5.5(b), Γ has a tangent, which is not vertical or horizontal, on a subset F for which $\mu_1(F)>0$ (by Lemma 6.5). By the above, μ_1 and μ_2 are not mutually singular (note that this implication does not use Assumption 6.4).

Now assume that μ_1 and μ_2 are not mutually singular. Observe that

$$\mu_i\{\Gamma \setminus \varrho_i(S^i(\Gamma))\}=0,$$

and according to Assumption 6.4 and Lemma 6.5,

$$\mu_i\{t\in\Gamma: \{(s_1, s_2)\in\Gamma: s_i = t_i\} \text{ is infinite}\} = 0.$$

But since μ_1 and μ_2 are not mutually singular, we have $\psi(s)>0$ on a set $G\subset\Gamma$ with $\mu_1(G)>0$, or, equivalently by Lemma 6.5, with $\lambda(\mathrm{pr}_1(G))>0$. By the above, we may assume that

$$G\subset \varrho_1(S^1(\Gamma))\cap\varrho_2(S^2(\Gamma))\cap\{t\in\Gamma: \{(s_1, s_2): s_i = t_i\} \text{ is finite, } i=1,2\}.$$

Thus $G\subset M_0(\Gamma)$, so by Lemma 5.5 (a), there is a subarc Γ^1 of Γ and monotone curves Γ_L and Γ_U satisfying (5.3)–(5.5). Since $\psi>0$ on G, the slope of Γ at each point of $\Gamma^1\cap G$ is finite and non-zero. So $G\subset M(\Gamma)$, and thus $\lambda\{\mathrm{pr}_1(M(\Gamma))\}>0$. This completes the proof. □

Note that it is not difficult to provide counterexamples which show that Corollary 6.6 is false without Assumption 6.4.

We now turn to the proof of Theorem 6.1. To begin with, by Theorem 5.6, we only need to prove necessity of the condition $\lambda\{\mathrm{pr}_1(M(\Gamma))\}=0$. We will show that this reduces to the following statement concerning monotone curves.

PROPOSITION 6.7. *Let $(X_t, t\in T)$ be a Brownian sheet. Fix $a, b>0$ and set $R=[0,a]\times[0,b]$. Let Γ_L and Γ_U be two continuous increasing (resp. decreasing) curves in R such that*

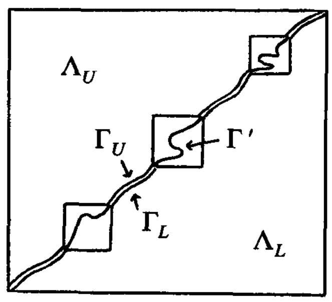
(6.4) *Γ_U dominates Γ_L and $(\Gamma_L\cup\Gamma_U)\cap\partial R=\{(0,0), (a,b)\}$ (resp. $=\{(0,b), (a,0)\}$);*

(6.5) *$S^1(\Gamma_U)\setminus S^1(\Gamma_L)$ is a (countable) disjoint union of rectangles whose boundaries are contained in $\Gamma_L\cup\Gamma_U$;*

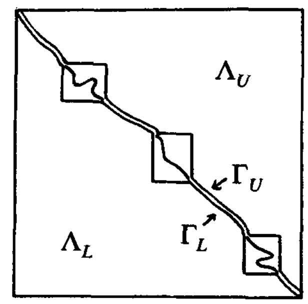
(6.6) *$\lambda(\mathrm{pr}_1(\Gamma_L\cap\Gamma_U))>0$ and at λ-almost all $x\in\mathrm{pr}_1(\Gamma_L\cap\Gamma_U)$, the (common) tangent of Γ_L and Γ_U has slope $\psi(x)\neq 0$.*

Set $\Lambda_L=S^1(\Gamma_L)$. Then $X(\Lambda_L)\notin\mathscr{H}(\Gamma_L\cup\Gamma_U\cup\partial R)$.

Proof of Theorem 6.1. As indicated above, we need only prove the "only if" part. Assume $\lambda\{\mathrm{pr}_1(M(\Gamma))\}>0$. By Lemma 5.5, we obtain the existence of a subarc Γ' of Γ and two monotone curves Γ_L and Γ_U satisfying (5.3)–(5.5), a set F and $\alpha>0$ such that (5.6) and (5.7) hold, and a sequence $(R_n, n\in\mathbf{N})$ of rectangles with boundary contained in $\Gamma_U\cup\Gamma_L$ whose union is $S^1(\Gamma_U)\setminus S^1(\Gamma_L)$.



Case 1



Case 2

Fig. 6.2.

Let $A=\mathrm{pr}_1(F)$. Since F is totally ordered (for $\leq$ or $\triangle$), we can write $F=\{\beta(x): x\in A\}$, for some monotone function β. Let A' be the set of points of density of A, and fix $x_0\in A'$. Since Γ is non-self-intersecting, there is $\delta>0$ such that the distance between $\beta(x_0)$ and $\Gamma\setminus\Gamma'$ is at least 2δ. Choose $x_1\in A'$ such that $|\beta(x_1)-\beta(x_0)|<\delta$, and let R_0 be the rectangle with sides parallel to the axes and with two opposing corners at $\beta(x_0)$ and $\beta(x_1)$. By our choice of R_0, $\lambda(\mathrm{pr}_1(F_0))>0$, where $F_0=F\cap R_0$. Moreover, $\Gamma\cap R_0=\Gamma'\cap R_0$, and $\Gamma_L\cap\partial R_0=\Gamma_U\cap\partial R_0=\{\beta(x_0),\beta(x_1)\}$. In particular, $\partial R_0\cap\partial R_n=\emptyset$, $\forall n\in\mathbf{N}$ (this follows from the fact that a point of density of A is necessarily a limit from both sides of points of A). There are two possible cases, as in Figure 6.2.

Let $\Gamma_0=\Gamma_L\cap R_0$. By Lemma 3.7 and Theorem 3.3, $X(S^1(\Gamma_0))\in\mathscr{H}(D_1)\cap\mathscr{H}(D_2)$, which is the minimal splitting field for D_1 and D_2. But we are going to deduce from Proposition 6.7 that $X(S^1(\Gamma_0))\notin\mathscr{H}(\Gamma)$, which will complete the proof.

Let us play the Devil's advocate and suppose that $X(S_1(\Gamma_0))\in\mathscr{H}(\Gamma)$. Then

$$\begin{aligned}E(X(S^1(\Gamma_0))|\mathscr{H}(\Gamma)\vee\mathscr{H}(\partial R_0)) &= X(S^1(\Gamma_0))\\ &= X(S^1(\Gamma_0)\cap R_0)+X(S^1(\Gamma_0)\cap R_0^c)\\ &\equiv Y_1+Y_2.\end{aligned}$$

Now $Y_2\in\mathscr{H}(\partial R_0)$, so

$$Y_2=E(Y_2|\mathscr{H}(\Gamma)\vee\mathscr{H}(\partial R_0)),$$

which implies that $Y_1=E(Y_1|\mathscr{H}(\Gamma)\vee\mathscr{H}(\partial R_0))$ as well.

Define a new Brownian sheet $W=(W_t, t\in R_0)$ by

$$W_t = \int_{R_t \cap R_0} dX_s, \quad t \in R_0,$$

and set

$$\tilde{W}_t = \int_{R_t \setminus F_C} dX_s, \quad t \in T.$$

Let

$$\mathscr{G} = \sigma\{W_t, t \in (\Gamma \cap R_0) \cup \partial R_0\},$$

$$\tilde{\mathscr{G}} = \sigma\{\tilde{W}_t, t \in (\Gamma \setminus R_0) \cup \partial R_0\}.$$

It follows from the properties of white noise that W and $\tilde{W}$ are independent. Then $\mathscr{G}$ and $\tilde{\mathscr{G}}$ are independent as well, and in fact, $\sigma(Y_1) \vee \mathscr{G}$ is independent of $\tilde{\mathscr{G}}$, so by 3.11,

$$\begin{aligned} Y_1 &= E(Y_1 | \mathscr{H}(\Gamma) \vee \mathscr{H}(\partial R_0)) \\ &= E(Y_1 | \mathscr{G} \vee \tilde{\mathscr{G}}) \qquad (6.7) \\ &= E(Y_1 | \mathscr{G}). \end{aligned}$$

Set $\mathscr{G}(E) = \sigma\{W_t, t \in E\}$. Since W is a Brownian sheet, it satisfies the sharp Markov property with respect to finite unions of rectangles (see Corollary 4.2). Note that $\partial(\bigcup_{k=1}^n R_k) = \bigcup_{k=1}^n \partial R_k$, so by the Markov property,

$$\mathscr{G}\left(\bigcup_{k=1}^n R_k\right) \perp \mathscr{G}\left(\left(\bigcup_{k=1}^n R_k\right)^c\right) \Big| \mathscr{G}\left(\bigcup_{k=1}^n \partial R_k\right)$$

where we write $\mathscr{A} \perp \mathscr{B} | \mathscr{C}$ as shorthand for "$\mathscr{A}$ and $\mathscr{B}$ are conditionally independent given $\mathscr{C}$". Passing to the limit, we see that

$$\mathscr{G}\left(\bigcup_{k=1}^\infty R_k\right) \perp \mathscr{G}\left(\left(\bigcup_{k=1}^\infty R_k\right)^c\right) \Big| \mathscr{G}\left(\bigcup_{k=1}^\infty \partial R_k\right).$$

Since $\mathscr{G}(\partial R_0) \vee \mathscr{G}(F_0) \subset \mathscr{G}((\bigcup_{k=1}^\infty R_k)^c)$, we can enlarge the conditioning field (see [C; Lemma 2.2]) to see that

$$\mathscr{G}\left(\bigcup_{k=1}^\infty R_k\right) \perp \mathscr{G}\left(\left(\bigcup_{k=1}^\infty R_k\right)^c\right) \Big| \mathscr{G}\left(\bigcup_{k=1}^\infty \partial R_k\right) \vee \mathscr{G}(F_0) \vee \mathscr{G}(\partial R_0). \qquad (6.8)$$

Observe that $Y_1 \in \mathscr{G}((\bigcup_{k=1}^\infty R_k)^c)$, and that by (6.7), $Y_1 = E(Y_1 | \mathscr{A})$, for any σ-field $\mathscr{A} \supset \mathscr{G}$.

Thus

$$Y_1 = E\left(Y_1 \middle| \mathscr{G}\left(\bigcup_{k=1}^{\infty} R_k \right) \vee \mathscr{G}\left(\bigcup_{k=1}^{\infty} \partial R_k \right) \vee \mathscr{G}(F_0) \vee \mathscr{G}(\partial R_0) \right)$$
$$= E\left(Y_1 \middle| \mathscr{G}\left(\bigcup_{k=1}^{\infty} \partial R_k \right) \vee \mathscr{G}(F_0) \vee \mathscr{G}(\partial R_0) \right)$$

by (6.8). It follows that

(6.9) $$Y_1 = E(Y_1 | \mathscr{G}(\Gamma_L) \vee \mathscr{G}(\Gamma_U) \vee \mathscr{G}(\partial R_0)).$$

Notice that Y_1 is $\mathscr{G}(R_0)$-measurable, so (6.9) only involves W, which is a Brownian sheet, hence we can reduce to the case $R_0=[0,a]\times[0,b]$. But now Proposition 6.7 implies that (6.9) is *not* true. This is the desired contradiction. □

We will now head toward a proof of Proposition 6.7. The proof relies specifically on the fact that the Brownian sheet is a Gaussian process, and uses ideas similar to those developed by Dalang and Russo [DR] in a simpler setting. Though we could refer to [DR] from time to time, we prefer for the convenience of the reader to give full details here.

In order to stress that we are working with a Brownian sheet, we write $(W_t, t\in T)$ instead of $(X_t, t\in T)$, throughout the remainder of this section.

Fix $a>0$, $b>0$ and let $R=[0,a]\times[0,b]$ be a rectangle. Γ_L and Γ_U are the monotone curves of Proposition 6.7, R_n denotes the rectangles bounded by the two curves, and $\Delta=\Gamma_L\cup\Gamma_U$. These are all subsets of R. Let $\Lambda_L=S^1(\Gamma_L)$ be the part of R below Γ_L and let Λ_U be the part of R above Γ_U. We let ϱ_1 and ϱ_2 denote the vertical and horizontal projections, respectively, on Δ (rather than on Γ, as before). There are two different cases, that in which the curves are increasing, and that in which they are decreasing (see Figure 6.2). We will treat them together as much as possible, but we will have to consider them separately from time to time.

Our first step is to derive representations of the sharp fields of certain sets. For each $h\in L^2(R,\lambda)$ let us denote

$$W(h) = \int h(t)\, dW_t.$$

Let $\mathscr{L}(F)$ be the closed linear subspace of $L^2(\Omega,\mathscr{F},P)$ which is generated by $\{W_t, t\in F\}$. Since we are dealing with the Brownian sheet, which is a Gaussian process, we can deal with the linear spaces $\mathscr{L}(F)$ instead of the $\mathscr{H}(F)$. Now it is well known that

$\mathscr{L}(R)$ is isomorphic to $L^2(R, dt)$ under the isometry

$$h \mapsto W(h).$$

Let us look at some different linear spaces. Consider $\mathscr{L}(\partial R)$. This is isomorphic to a closed subspace of $L^2(R, dt)$. To see which one, let $\partial^+ R = ([0, a] \times \{b\}) \cup (\{a\} \times [0, b])$ be the upper-right boundary of R and let λ^+ be Lebesgue measure on $\partial^+ R$ normalized on each of the two segments so that $\lambda^+([0, a] \times \{b\}) = ab = \lambda^+(\{a\} \times [0, b])$. Let g be defined on $\partial^+ R$. Then we define $\tilde{g}$ by

$$\tilde{g}(u, v) = g(u, b) - g(a, v). \tag{6.10}$$

PROPOSITION 6.8. *Let $L_0^2(\partial R)$ be the class of functions $g \in L^2(\partial^+ R, \lambda^+)$ which satisfy*

$$\int_0^a g(u, b)\, du = 0. \tag{6.11}$$

Then the map $g \mapsto W(\tilde{g})$ is an isometry between $L_0^2(\partial R)$ and $\mathscr{L}(\partial R)$.

Proof. If $t \in \partial^+ R$, set $g_t(u, v) = 1$ if $(u, v) \in \partial^+ R$, $u \le t_1$ and $v \ge t_2$, and set it equal to zero otherwise. Then $\tilde{g}_t = I_{R_t}$, so $W(\tilde{g}_t) = W_t$. Note that the class of functions $\{g_t, t \in \partial^+ R\}$ generates the Borel functions on $\partial^+ R$.

Let $g \in L^2(\partial^+ R, \lambda^+)$. Note that we can add or subtract a constant from g without changing $\tilde{g}$, so that by replacing g by $g - (1/a) \int_0^a g(u, b)\, du$ if necessary, we may assume that g satisfies (6.11). In that case,

$$\int\int_R g(u, b)\, g(a, v)\, du\, dv = \left(\int_0^a g(u, b)\, du\right)\left(\int_0^b g(a, v)\, dv\right) = 0,$$

which implies that

$$\begin{aligned} E(W(\tilde{g})^2) &= \|\tilde{g}\|^2_{L^2(R)} \\ &= \int\int_R (g(u, b) - g(a, v))^2\, du\, dv \\ &= \int\int_R (g^2(u, b) + g^2(a, v))^2\, du\, dv \\ &= \|g\|^2_{L_0^2}. \end{aligned}$$

The map $g \mapsto W(\tilde{g})$ is linear and preserves norms, so it extends to an isometry of the Hilbert spaces generated by the W_t on one side and the g_t on the other, completing the proof. □

Consider the representation of $\mathscr{L}(\Delta)$. There are two cases: Case 1, in which Γ_L is increasing, and Case 2, in which Γ_L is decreasing. In each case we can represent an element of $\mathscr{L}(\Delta)$ by a function defined on Δ, but the form is different in the two cases. Given h on Δ, let us define a function $\hat{h}$ on R. In Case 1, we define

$$\hat{h}(t) = \begin{cases} h(\varrho_1(t)) & \text{if } t \in \Lambda_L, \\ h(\varrho_2(t)) & \text{if } t \in \Lambda_U, \\ h(\varrho_1(t)) + h(\varrho_2(t)) & \text{if } t \in R_n, n = 1, 2, \dots . \end{cases} \tag{6.12}$$

In Case 2, we define

$$\hat{h}(t) = \begin{cases} h(\varrho_1(t)) - h(\varrho_2(t)) & \text{if } t \in S_1(\Delta), \\ 0 & \text{otherwise.} \end{cases}$$

Let μ_i, $i=1,2$, be the image of Lebesgue measure under ϱ_i, i.e. $\mu_i(F) = m(\varrho_i^{-1}(F))$, $F \in \mathscr{B}(\Delta)$.

PROPOSITION 6.9. (a) *Assume that Case 1 obtains. Let $L_1^2(\Delta)$ be the class of $h \in L^2(\Delta, \mu_1 + \mu_2)$ such that*

$$\int_{R_n} h(\varrho_1(t))\, dt = 0, \quad \forall n \in \mathbf{N}. \tag{6.13}$$

Then $\mathscr{L}(\Delta)$ is isomorphic to $L_1^2(\Delta)$ and the map $h \mapsto W(\hat{h})$ is an isometry.

(b) *Assume Case 2 obtains. Let $Q = R_{t^0}$ for some $t^0 \in \Gamma_L$. Let $L_2^2(\Delta)$ be the class of $h \in L^2(\Delta, \mu_1 + \mu_2)$ which satisfy* (6.13) *and which also satisfy*

$$\int_{Q} h(\varrho_1(t))\, dt = 0. \tag{6.14}$$

Then $\mathscr{L}(\Delta)$ is isomorphic to $L_2^2(\Delta)$ and there exists a constant $K>0$ such that the map $h \mapsto W(\hat{h})$ satisfies

$$K\|h\|^2 \leq \|W(\hat{h})\|^2 \leq 2\|h\|^2. \tag{6.15}$$

Before proving this, we need the following real variable lemma (which could be deduced from [DR; Theorem 3.3]).

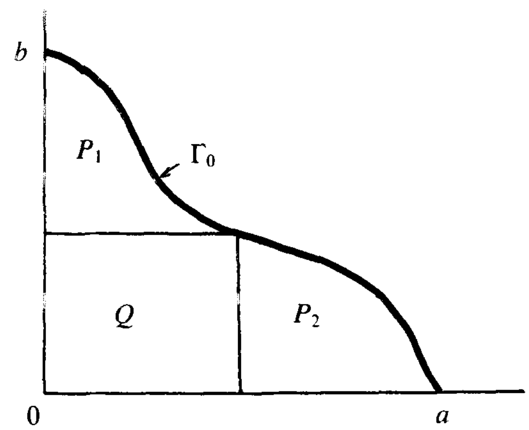


Fig. 6.3. P_1, P_2 and Q.

LEMMA 6.10. *Let $a>0$, $b>0$, and let Λ be a domain in $[0,a]\times[0,b]$ which is bounded by the coordinate axes and by a continuous decreasing curve Γ_0. Let $0<c<a$. Then there exists a constant K, depending only on a, b, c and Γ_0, such that for any pair h and g of square-integrable functions defined on $[0,\infty)$ satisfying*

$$\int_0^c h(u)\,du=0$$

(*no such condition is required of g*), *we have*

$$K\int\int_\Lambda (h(u)^2+g(v)^2)\,du\,dv \le \int\int_\Lambda (h(u)+g(v))^2\,du\,dv \le 2\int\int_\Lambda (h(u)^2+g(v)^2)\,du\,dv. \tag{6.16}$$

In the special case $\Gamma_0=\partial^+R$, this statement remains valid with $c=a$.

Proof. Fix d such that $(c,d)\in\Gamma_0$ and let $Q=[0,c]\times[0,d]$. Because Γ_0 is decreasing, $Q\subset\bar{\Lambda}$. Let $P_1=\Lambda\cap([0,c]\times[d,b])$, and $P_2=\Lambda\cap(]c,a]\times[0,d])$, so $\Lambda=Q\cup P_1\cup P_2$ (see Figure 6.3).

Note that

$$\int\int_Q h(u)\,g(v)\,du\,dv=\left(\int_0^c h(u)\,du\right)\left(\int_0^d g(v)\,dv\right)=0,$$

so that

$$\int\int_Q (h(u)+g(v))^2\,du\,dv=\int\int_Q (h(u)^2+g(v)^2)\,du\,dv. \tag{6.17}$$

Moreover,

$$\iint_{P_1} h(u)^2\,du\,dv \le (b-d)\int_0^c h(u)^2\,du,$$

and a similar equation holds for the integral of g^2 over P_2, so

$$\iint_{P_1} h(u)^2\,du\,dv \le (b/d-1)\iint_Q h(u)^2\,du\,dv, \tag{6.18}$$

$$\iint_{P_2} g(v)^2\,du\,dv \le (a/c-1)\iint_Q g(v)^2\,du\,dv. \tag{6.19}$$

To handle the integral of g^2 over P_1, write $g=(h+g)-h$:

$$\iint_{P_1} g(v)^2\,du\,dv \le 2\iint_{P_1}(h+g)^2\,du\,dv + 2\iint_{P_1} h(u)^2\,du\,dv.$$

Apply (6.18) to the last term, and treat the integral of h^2 over P_2 analogously to see that

$$\iint_{P_1} g(v)^2\,du\,dv \le 2\iint_{P_1}(h+g)^2\,du\,dv + 2(b/d-1)\iint_Q h(u)^2\,du\,dv; \tag{6.20}$$

$$\iint_{P_2} h(u)^2\,du\,dv \le 2\iint_{P_2}(h+g)^2\,du\,dv + 2(a/c-1)\iint_Q g(v)^2\,du\,dv. \tag{6.21}$$

By (6.17),

$$\iint_\Lambda (h(u)^2+g(v)^2)\,du\,dv = \iint_Q (h+g)^2\,du\,dv + \iint_{P_1\cup P_2}(h(u)^2+g(v)^2)\,du\,dv.$$

The last integral on the right is dominated by the sum of the right-hand sides of (6.18)–(6.21), so this is

$$\le 2\iint_\Lambda (h+g)^2\,du\,dv + 3\iint_Q ((b/d-1)\,h(u)^2 + (a/c-1)\,g(v)^2)\,du\,dv.$$

It is clear from (6.17) that $\iint_Q h^2 \le \iint_\Lambda (h+g)^2$, so this is

$$\le (3b/d+3a/c-4)\iint_\Lambda (h+g)^2\,du\,dv.$$

This proves (6.16). In the special case $\Gamma_0=\partial^+R$ and $c=a$, (6.16) follows directly from (6.17). □

Proof of Proposition 6.9. In both Case 1 and Case 2, if $t\in\Delta$, there exists a function h_t on Δ such that $\hat{h}=I_{R_t}$. In Case 2, for example, h_t is given by $h_t(s)=-1$ if $s\in\Delta$ and $t\underline{\Delta}s$ and $h_t(s)=0$ otherwise. We leave Case 1 to the reader. It is not difficult to see that the smallest class of functions which is closed under addition, scalar multiplication, and a.e. convergence and which contains the h_t is the class of Borel functions on Δ.

In Case 1, if g_n equals 1 on the upper boundary segment of R_n, -1 on the right boundary segment and is 0 elsewhere, then $\hat{g}_n\equiv 0$, so that, as the map $h\mapsto\hat{h}$ is linear, one can subtract a multiple of g_n from h in order to satisfy (6.13). Suppose then that $h\in L_1^2(\Delta)$. Notice that

$$m(R_n)\int_{R_n} h(\varrho_1(t))\,h(\varrho_2(t))\,dt=\left(\int_{R_n} h(\varrho_1(t))\,dt\right)\left(\int_{R_n} h(\varrho_2(t))\,dt\right)=0.$$

Thus

$$\begin{aligned}\|W(\hat{h})\|^2&=\int_R \hat{h}(t)^2\,dt\\&=\int_{\Lambda_L}\hat{h}(t)^2\,dt+\int_{\Lambda_U}\hat{h}(t)^2\,dt+\sum_n\int_{R_n}\hat{h}(t)^2\,dt\\&=\int_{\Lambda_L}h(\varrho_1(t))^2\,dt+\int_{\Lambda_U}h(\varrho_2(t))^2\,dt+\sum_n\int_{R_n}(h(\varrho_1(t))^2+h(\varrho_2(t))^2)\,dt\\&=\int_\Delta h^2\,d(\mu_1+\mu_2).\end{aligned}$$

The map $h\mapsto W(\hat{h})$ is linear and preserves norms, so it extends to an isometry of the Hilbert spaces generated by the W_t on one side and the h_t on the other. This proves (a).

Now suppose Case 2 obtains. Let g_0 equal one on Γ_L, zero elsewhere, and let g_n equal 1 on the upper-right boundary of R_n and 0 elsewhere. Then $\hat{g}_0\equiv\hat{g}_n\equiv 0$, so that we can subtract multiples of g_0 and g_n from h without changing $\hat{h}$. Thus we may assume without loss of generality that h satisfies (6.13) and (6.14).

Suppose $h\in L_2^2(\Delta)$. Then

$$\begin{aligned}E(W(\hat{h})^2)&=\|\hat{h}\|^2\\&=\int_{\Lambda_L}(h(\varrho_1(t))-h(\varrho_2(t)))^2\,dt+\sum_n\int_{R_n}(h(\varrho_1(t))-h(\varrho_2(t)))^2\,dt.\end{aligned}\tag{6.22}$$

Apply Lemma 6.10 to the first term to see that this is

$$\geq K\int_{\Lambda_L}(h(\varrho_1(t))^2+h(\varrho_2(t))^2)\,dt+\sum_n\int_{R_n}(h(\varrho_1(t))^2+h(\varrho_2(t))^2)\,dt.$$

We can rewrite this in terms of the measures μ_1 and μ_2:

$$\geq \min(1,K)\int_\Delta h^2 d(\mu_1+\mu_2)=\min(1,K)\,\|h\|^2.$$

It also follows from (6.22) that $\|\hat{h}\|^2\leq 2\|h\|^2$, which proves (6.15).

But now, we have seen that $h\mapsto W(\hat{h})$ is a linear map between $\{W_t, t\in\Delta\}$ and a subset of $L_2^2(\Delta)$. By (6.15), this map is bi-continuous, so it extends to the closed Hilbert spaces generated by the two sets. Since the subset $\{h_t, t\in\Delta\}$ generates the Borel functions on Δ, we conclude that the closure of their span is $L_2^2(\Delta)$ itself. The proposition is proved. □

COROLLARY 6.11. *If $X\in\mathscr{L}(\Delta\cup\partial R)$, there exist Borel functions h_0 on Δ and g_0 on ∂^+R such that*

$$X=W(\hat{h}_0+\tilde{g}_0).$$

Proof. We know $X=W(\xi)$ for some $\xi\in L^2(R,dt)$. Random variables of the form $Y+Z$, where $Y\in\mathscr{L}(\Delta)$ and $Z\in\mathscr{L}(\partial R)$, are dense in $\mathscr{L}(\Delta\cup\partial R)$ so there exist sequences $(Y_m)\subset\mathscr{L}(\Delta)$ and $(Z_m)\subset\mathscr{L}(\partial R)$ such that $X=\lim_{m\to\infty}(Y_m+Z_m)$. Thus there are $h_m\in L_1^2(\Delta)$ (respectively $L_2^2(\Delta)$) and $g_m\in L_0^2(\partial R)$ such that $Y_m+Z_m=W(\hat{h}_m+\tilde{g}_m)$. Consequently,

$$\xi=\lim_{m\to\infty}(\hat{h}_m+\tilde{g}_m),$$

where the limit is in $L^2(R,dt)$.

We claim that ξ is of the form $\xi=\hat{h}_0+\tilde{g}_0$ for some h_0 on Δ and g_0 on ∂^+R. Note that

$$\lim_{m,l\to\infty}\int(\hat{h}_l-\hat{h}_m+\tilde{g}_l-\tilde{g}_m)^2\,dt=0. \tag{6.23}$$

To simplify notation, let $h=h_l-h_m$ and $g=g_l-g_m$. In Case 1, refer to (6.10) and (6.12) to see that we can rewrite the integral in (6.23) as

$$
\begin{aligned}
\int\!\!\int_R (\hat{h}+\tilde{g})^2\,dt = &\int\!\!\int_{\Lambda_U} (h(\varrho_2(u,v))+g(u,b)-g(a,v))^2\,du\,dv \\
&+\int\!\!\int_{\Lambda_L} (h(\varrho_1(u,v))+g(u,b)-g(a,v))^2\,du\,dv \\
&+\sum_n \int\!\!\int_{R_n} (h(\varrho_1(u,v))+h(\varrho_2(u,v))+g(u,b)-g(a,v))^2\,du\,dv.
\end{aligned}
\tag{6.24}
$$

Observe that $h(\varrho_2(u,v))$ is a function of v alone on Λ_U (equal to $h(\varrho_2(0,v))$), and $h(\varrho_1(u,v))$ is a function of u alone on Λ_L. We are going to apply Lemma 6.10 to the first two integrals in (6.24). This is possible since rotating Λ_U and Λ_L by ± 90 degrees transforms them into regions to which the lemma applies. In order to satisfy the hypothesis of Lemma 6.10, fix $0<c<\min(a,b)$ and set $\alpha=\alpha_l-\alpha_m$, $\beta=\beta_l-\beta_m$, where

$$\alpha_m=\int_0^c g_m(u,b)\,du,\quad \beta_m=\int_0^c g_m(a,v)\,dv,\quad m\in\mathbf{N}.$$

Then by Lemma 6.10, there are constants $K_U>0$ and $K_L>0$ such that (6.24) is

$$
\begin{aligned}
&\geq K_U\int\!\!\int_{\Lambda_U} [(g(u,b)-\alpha)^2+(h(\varrho_2(0,v))-g(a,v)+\alpha)^2]\,du\,dv \\
&\quad +K_L\int\!\!\int_{\Lambda_L} [(g(a,v)-\beta)^2+(h(\varrho_1(u,0))+g(u,b)-\beta)^2]\,du\,dv \\
&\quad +\sum_n \int\!\!\int_{R_n} [h(\varrho_1(u,v))+h(\varrho_2(u,v))+\alpha-\beta+(g(u,b)-\alpha)-(g(a,v)-\beta)]^2\,du\,dv.
\end{aligned}
$$

In particular,

$$\int\!\!\int_{\Lambda_U} (g_m(u,b)-\alpha_m)^2\,du\,dv \quad\text{and}\quad \int\!\!\int_{\Lambda_L} (g_m(a,v)-\beta_m)^2\,du\,dv$$

converge to zero. This implies that the one-variable functions $g_m(\cdot\,,b)-\alpha_m$ and $g_m(a,\cdot\,)-\beta_m$ converge to zero in measure and in L^2 for the measures ν_1 and ν_2 respectively, where

$$d\nu_1(u)=(b-r_1(u))\,du,\quad d\nu_2(v)=(a-r_2(v))\,dv,$$

and

$$r_1(u)=\sup\{t_2:(u,t_2)\in\Gamma_U\},\quad r_2(u)=\sup\{t_1:(t_1,v)\in\Gamma_L\}.$$

Making use of the second part of (6.4), we see that $r_1(u)<b$ and $r_2(u)<a$, so ν_1 (resp. ν_2) is equivalent to Lebesgue measure on $[0,a]$ (resp. $[0,b]$). Thus $(g_m(\cdot,b))$ (resp. $(g_m(a,\cdot)))$ is a Cauchy sequence in the topology of convergence in Lebesgue measure on $[0,a]$ (resp. $[0,b]$), and it is also a Cauchy sequence in $L^2([0,a-\varepsilon],d\lambda)$ (resp. $L^2([0,b-\varepsilon],d\lambda)$), for each $\varepsilon>0$.

It follows that there is a Borel function g_0 on ∂^+R such that $g_m(\cdot,b)-\alpha_m\to g_0(\cdot,b)$ in measure on $[0,a]\times\{b\}$, and $g_m(a,\cdot)-\beta_m\to g_0(a,\cdot)$ in measure on $\{a\}\times[0,b]$.

Now look at the other terms. We see from these that $h_m(\varrho_2(0,v))+\alpha_m-\beta_m$ converges in measure on Λ_U, and $h_m(\varrho_1(u,0))+\alpha_m-\beta_m$ converges in measure on Λ_L, which implies the existence of a Borel function $\bar h_0$ on $\varrho_2(\Lambda_U)\cup\varrho_1(\Lambda_L)$ such that $h_m(\cdot)+\alpha_m-\beta_m$ converges to $\bar h_0$ in $(\mu_1+\mu_2)$-measure on $\varrho_2(\Lambda_U)\cup\varrho_1(\Lambda_L)$.

Finally, looking at the integrals over R_n, we see that $h_m(\varrho_1(u,v))+h_m(\varrho_2(u,v))+\alpha_m-\beta_m$ converges in $L^2(R_n,dt)$. By (6.13) and Lemma 6.10, it follows that $h_m(\varrho_1(u,v))$ and $h_m(\varrho_2(u,v))+\alpha_m-\beta_m$ converge in $L^2(R_n,dt)$, so there is a Borel function h_0^i on $\varrho_i(R_n)$, $i=1,2$, such that $h_m(\cdot)\to h_0^1(\cdot)$ in μ_1-measure on $\varrho_1(R_n)$, and $h_m(\cdot)+\alpha_m-\beta_m\to h_0^2(\cdot)$ in μ_2-measure on $\varrho_2(R_n)$. Now define h_0 on Δ by

$$h_0(t)=\bar h_0(t)I_{\varrho_1(\Lambda_L)\cup\varrho_2(\Lambda_U)}(t)+h_0^1I_{\varrho_1(R_n)}(t)+h_0^2I_{\varrho_2(R_n)}(t),\quad t\in\Delta.$$

It now only remains to check that $\xi=\hat h_0+\tilde g_0$. Note that on Λ_U, $\hat h_0+\tilde g_0$ is the limit in measure of

$$\begin{aligned}h_m(\varrho_2(u,0))+\alpha_m-\beta_m+g_m(u,b)-\alpha_m-(g(a,v)-\beta_m)&=\hat h_m(u,v)+\tilde g_m(u,v)\\&\to\xi(u,v),\end{aligned}$$

so $\hat h_0+\tilde g_0$ and ξ coincide on Λ_U. In the same way, these two functions also coincide on Λ_L and on R_n, $n\in\mathbf{N}$.

The proof in Case 2 is similar, except that (6.24) becomes

$$\begin{aligned}\int_R(\hat h+\tilde g)\,dt=&\int\int_{\Lambda_U}(g(u,b)-g(a,v))^2\,du\,dv\\&+\int\int_{\Lambda_L\cup\bigcup_n R_n}(h(\varrho_1(u,v))+g(u,b)-h(\varrho_2(u,v))-g(a,v))^2\,du\,dv\end{aligned}$$

since the support of $\hat{h}$ is in Λ_L. Apply Lemma 6.10 as before to construct the functions h_0 and g_0. Details are left to the reader. □

This brings us to the proof of Proposition 6.7.

Proof of Proposition 6.7. Suppose $W(\Lambda_L) \in \mathscr{L}(\Delta \cup \partial R)$. By Corollary 6.11 there exist functions h on Δ and g on $\partial^+ R$ such that $I_{\Lambda_L} = \hat{h} + \hat{g}$ a.e. on R. Consider Case 1. $I_{\Lambda_L} = 0$ in Λ_U, so

$$h(\varrho_2(u, v)) + g(u, b) - g(a, v) = 0$$

or

$$g(u, b) = g(a, v) - h(\varrho_2(u, v))$$

for a.e. $(u, v) \in \Lambda_U$. The left-hand side depends on u, the right-hand side on v (for $u \mapsto \varrho_2(u, v)$ is constant). Therefore both sides are equal to a constant, say α:

$$g(\cdot\,, b) = \alpha \quad \text{a.e.,} \tag{6.25}$$

$$h(\varrho_2(u, v)) = g(a, v) - \alpha \quad \text{a.e.} \tag{6.26}$$

On the other hand, $I_{\Lambda_L} = 1$ on Λ_L, so for a.e. $(u, v) \in \Lambda_L$,

$$h(\varrho_1(u, v)) + g(u, b) - g(a, v) = 1,$$

or, using (6.25),

$$g(a, v) = h(\varrho_1(u, v)) + \alpha - 1.$$

As before, both sides are equal to a constant, say β, hence

$$h(\varrho_1(u, v)) = 1 + \beta - \alpha \tag{6.27}$$

for a.e. u. From (6.26), then, if $(u, v) \in \Lambda_U$, we have $h(\varrho_2(u, v)) = \beta - \alpha$ for a.e. v. Note that if $(u, v) \in \Lambda_L$, then $\varrho_1(u, v) \in \Gamma_L$, and if $(u, v) \in \Lambda_U$, then $\varrho_2(u, v) \in \Gamma_U$. Thus, in terms of the measures μ_1 and μ_2, we have

$$\begin{aligned} h &= 1 + \beta - \alpha \quad \mu_1\text{-a.e. on } \Gamma_L; \\ h &= \beta - \alpha \quad \mu_2\text{-a.e. on } \Gamma_U. \end{aligned} \tag{6.28}$$

But by (6.6) and (6.3), μ_1 and μ_2 are *not* orthogonal on $\Gamma_U \cap \Gamma_L$, so this is a contradiction. This finishes the proof in Case 1.

In Case 2, note that $\hat{h}$ and I_{Λ_L} vanish in Λ_U, so

$$g(u, b)-g(a, v)=0,$$

hence $g(u, b)$ and $g(a, v)$ are equal to the same constant, say α. In $\Lambda_L, I_{\Lambda_L}=1$, so

$$h(\varrho_1(u, v))-h(\varrho_2(u, v))+\alpha-\alpha=1.$$

As before, both functions must be constant a.e., which means that there are *distinct* constants c_1 and c_2 such that $h=c_1$ μ_1-a.e. and $h=c_2$ μ_2-a.e. Since μ_1 and μ_2 are not orthogonal, we must have $c_1=c_2$, which is a contradiction. This completes the proof. □

7. The sharp Markov property of most Jordan curves

We are now in a position to show that curves which satisfy the sharp Markov property are the rule rather than the exception. We will prove several precise statements to the effect that "almost every" curve has the sharp Markov property. The "almost every" can be interpreted both in the sense of Baire category and with respect to certain reference measures. We shall consider two cases: the case of curves of the form $y=f(x)$, where $f\colon \mathbf{R}_+\to\mathbf{R}_+$ is continuous, and bounded Jordan curves.

Equip $C(\mathbf{R}_+, \mathbf{R}_+)$ with the metric of uniform convergence on compact sets. For $f\in C(\mathbf{R}_+, \mathbf{R}_+)$ set

$$D_1(f)=\{t\in\mathbf{R}^2\colon t_1<0 \text{ or } (t_1\geq 0 \text{ and } t_2<f(t_1))\},$$

$$D_2(f)=\{t\in\mathbf{R}^2\colon t_1>0 \text{ and } t_2>f(t_1)\},$$

$$\Gamma(f)=\partial D_1(f)=\partial D_2(f).$$

THEOREM 7.1. *Let F be the set of all $f\in C(\mathbf{R}_+, \mathbf{R}_+)$ such that $\mathcal{H}(D_1(f))$ and $\mathcal{H}(D_2(f))$ are not conditionally independent given $\mathcal{H}(\Gamma(f))$. Then F is a meager set (or set of first Baire category), i.e. "almost all" $f\in C(\mathbf{R}_+, \mathbf{R}_+)$ determine domains with the sharp Markov property.*

Proof. To begin with, $C(\mathbf{R}_+, \mathbf{R}_+)$ with the above metric is a complete space, and is thus of second Baire category by the Baire Category Theorem [R; Chapter 7, Section 7.16]. Now the domain $D_1(f)$ is a Jordan domain in the sense of Section 5, the Jordan curve being the union of the graph of f and $\{(0, y)\colon y\geq f(0)\}$, and passing through the point at infinity (see Figure 7.1). It is thus sufficient by Theorem 5.6 to show that the set of all $f\in C(\mathbf{R}_+, \mathbf{R}_+)$ for which (5.9) fails is meager. Now each f for which (5.9) fails has a

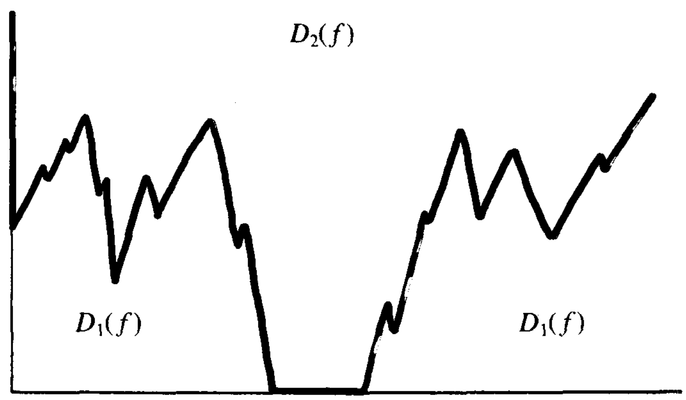


Fig. 7.1.

finite upper-right Dini derivative at at least one point $x \in \mathbf{R}_+$ (in fact, on a set of positive measure). But the set of all such f is meager (see e.g. [Royden, Chapter 7, Section 7, Problem 30.c]). □

Natural measures on $C(\mathbf{R}_+, \mathbf{R}_+)$ can be obtained from reflecting linear Brownian motion, or from the measure induced on $C(\mathbf{R}_+, \mathbf{R}_+)$ by the positive part $(B_t^+, t \geq 0)$ of a linear Brownian motion. Let Q denote either of these two measures.

THEOREM 7.2. *Let F be as in Theorem* 7.1. *Then F is a Q-null set.*

Proof. By [DEK; Theorem 1], no points of a Brownian sample path are points of increase or decrease. Since f has only countably many local extrema, $\Gamma(f)$ satisfies (4.1) for Q-almost all f. By Theorem 4.1, this gives the conclusion. □

Similar theorems can be given for Jordan curves. Recall that we equip the set $\mathcal{J}$ of bounded Jordan curves with the uniform metric d defined in (1.5). For $\Gamma \in \mathcal{J}$, let $D_1(\Gamma)$ and $D_2(\Gamma)$ be the two complementary open domains of Γ.

THEOREM 7.3. *Let $\mathcal{G}$ be the set of all $\Gamma \in \mathcal{J}$ such that $\mathcal{H}(D_1(\Gamma))$ and $\mathcal{H}(D_2(\Gamma))$ are not conditionally independent given $\mathcal{H}(\Gamma)$. Then $\mathcal{G}$ is a meager set.*

Let $\mathcal{C}$ be the set of all $\Gamma \in \mathcal{J}$ which define a Jordan curve consisting of finitely many vertical and horizontal segments. The proof of Theorem 7.3 uses the following property of $\mathcal{C}$.

LEMMA 7.4. *$\mathcal{C}$ is dense in $\mathcal{J}$.*

Proof. We thank the referee for suggesting the following proof, which is simpler than the authors' original one. Let $D_1(\Gamma)$ be the bounded component of $\mathbf{R}^2\setminus\Gamma$ (recall that Γ is bounded). Since $D_1(\Gamma)$ is simply connected, the Riemann Mapping Theorem (see e.g. [A; Chapter 6, Theorem 1]) implies the existence of an analytic one-to-one mapping of the unit open disc D_0 onto $D_1(\Gamma)$. This mapping extends to a homeomorphism Θ of $\bar{D}_0$ onto $D_1(\Gamma)\cup\Gamma$ [Po; Theorem 9.10]. Let $C=\partial D_0$. Since Θ is a homeomorphism, $\Theta|_C$ is continuous and one-to-one from C onto Γ. Let $C_n=(1-1/n)\,C$ and define a Jordan curve Γ_n by $\Gamma_n=\Theta(C_n)$. Parameterize Γ and Γ_n by

$$\varphi\colon C\to\Gamma,\quad \varphi(x)=\Theta(x),$$

$$\varphi_n\colon C\to\Gamma_n,\quad \varphi_n(x)=\Theta((1-1/n)\,x).$$

Then $d(\Gamma_n,\Gamma)\to 0$ since Θ is uniformly continuous on $\bar{D}_0$. Thus Γ is the uniform limit of a sequence of analytic Jordan curves. It remains to show that every analytic Jordan curve belongs to the closure of $\mathscr{E}$. Since an analytic Jordan curve is a finite union of monotone curves, the straightforward proof of this fact can be obtained using the argument following Corollary 4.4 of [CW]. The lemma is proved. □

Proof of Theorem 7.3. It is sufficient to show that $\mathscr{G}$ is contained in a countable union of closed sets whose complements are dense in $\mathscr{J}$. Set

$$\mathscr{G}_{\alpha,h}=\{\Gamma\in\mathscr{J}\colon \exists t\in\Gamma \text{ such that } \Gamma\cap M_\alpha(t,h)=\varnothing\}.$$

By Theorem 5.6, $\mathscr{G}$ is contained in the union of the $\mathscr{G}_{\alpha,h}$, $\alpha, h\in\mathbf{Q}_+^*$. To see that the $\mathscr{G}_{\alpha,h}$ are closed, let $(\Gamma_k, k\in\mathbf{N})$ be a sequence of elements of $\mathscr{G}_{\alpha,h}$ converging to $\Gamma\in\mathscr{J}$, and let us show that $\Gamma\in\mathscr{G}_{\alpha,h}$. Indeed, if $t^k\in\Gamma_k$ satisfies $\Gamma_k\cap M_\alpha(t^k,h)=\varnothing$, then the sequence $(t^k, k\in\mathbf{N})$ is bounded, so there is a subsequence converging to $t\in\Gamma$. We again denote this subsequence $(t^k, k\in\mathbf{N})$, and show that $\Gamma\cap M_\alpha(t,h)=\varnothing$.

Suppose not. Then there is $s\in\Gamma$ such that

$$|s_2-t_2|<\alpha|s_1-t_1|<\alpha h \quad\text{or}\quad |s_1-t_1|<\alpha|s_2-t_2|<\alpha h. \tag{7.1}$$

Let $s^k\in\Gamma_k$ satisfy $s^k\to s$ as $k\to\infty$. Then for large enough k, (7.1) is satisfied with s^k and t^k instead of s and t, respectively, implying $s^k\in\Gamma_k\cap M_\alpha(t^k,h)$, a contradiction.

Finally, $\mathscr{G}^c_{\alpha,h}$ is dense in $\mathscr{J}$ by Lemma 7.4, since $\mathscr{E}\subset\mathscr{G}^c_{\alpha,h}$, for each $\alpha, h\in\mathbf{Q}_+^*$. This completes the proof. □

Theorem 7.3 would not be very meaningful if the set $\mathscr{J}$ itself were meager! Since $\mathscr{J}$ is not complete (for instance, a sequence of ellipses could converge to a segment) the Baire Category Theorem cannot be applied. However, we have the following theorem.

THEOREM 7.5. *The set $\mathcal{J}$ with the uniform metric (see* (1.5)) *is not meager.*

Proof. By the definition of "meager" [R; Chapter VII, Section 7], it is sufficient to show that if $(O_n, n\in\mathbf{N})$ is a sequence of dense open subsets of $\mathcal{J}$, then $\bigcap_{n\in\mathbf{N}} O_n \neq \emptyset$.

Observe that if φ is the parameterization of some Jordan curve, then φ^{-1} is uniformly continuous, so we have

$$\forall \varepsilon>0,\ \exists\delta(\varphi,\varepsilon)>0\colon\ |\varphi(x)-\varphi(y)|<\delta(\varphi,\varepsilon)\ \Rightarrow\ |x-y|<\varepsilon. \tag{7.2}$$

Now fix $\Gamma_1\in O_1$. Since O_1 is open, there is $r_1>0$ such that $B(\Gamma_1, r_1)\subset O_1$. Let φ_1 be a parameterization of Γ_1, and set $\delta_1=\delta(\varphi_1, 1)$. Since O_2 is dense in $\mathcal{J}$, there is $\Gamma_2\in O_2\cap B(\Gamma_1, s_1)$, where $s_1=\min(r_1,\delta_1)/8$, and since this set is open, there is $r_2>0$, $r_2<s_1$, such that $B(\Gamma_2, r_2)\subset O_2\cap B(\Gamma_1, s_1)$. We now proceed by induction.

Let φ_n be a parameterization of Γ_n, and set $\delta_n=\delta(\varphi_n, 1/n)$. At step $n+1$, there is $\Gamma_{n+1}\in O_{n+1}\cap B(\Gamma_n, s_n)$, where

$$s_n=\min(r_n, \min_{m\leq n}\delta_m)/8,$$

and since this set is open, there is $r_{n+1}>0, r_{n+1}<s_n$, such that

$$B(\Gamma_{n+1}, r_{n+1})\subset O_{n+1}\cap B(\Gamma_n, s_n). \tag{7.3}$$

Now observe that if $m, n\geq N$, then

$$\|\varphi_n-\varphi_m\|_\infty \leq \min(r_N/8;\delta_N/4)\to 0$$

as $N\to\infty$, so $(\varphi_n, n\in\mathbf{N})$ is a Cauchy sequence for the uniform norm. Thus, there is a continuous function $\varphi\colon C\to\mathbf{R}^2_+$ such that $\lim_{n\to\infty}\|\varphi_n-\varphi\|=0$. We are going to show that φ is one-to-one and thus $\Gamma=\varphi(C)\in\mathcal{J}$.

Indeed, assume that there are $x, y\in C$, $x\neq y$, such that $\varphi(x)=\varphi(y)$. Fix $n\in\mathbf{N}$ such that $|x-y|>1/n$, and fix $m\in\mathbf{N}$, $m\geq n$, such that $|\varphi_m(x)-\varphi_m(y)|<\delta_n/4$. Then

$$\begin{aligned}
|\varphi_n(x)-\varphi_n(y)| &\leq |\varphi_n(x)-\varphi_m(x)|+|\varphi_m(x)-\varphi_m(y)|+|\varphi_m(y)-\varphi_n(y)|\\
&\leq \delta_n/4+\delta_n/4+\delta_n/4\\
&<\delta_n\\
&=\delta\left(\varphi_n, \frac{1}{n}\right),
\end{aligned}$$

so $|x-y|<1/n$ by (7.2), a contradiction. Thus $\Gamma\in\mathcal{J}$.

It now only remains to be shown that Γ is in the intersection of all the O_n. For each $n \in \mathbf{N}$, note that $|\varphi-\varphi_n| \le r_n/4$, so $\Gamma \in B(\Gamma_n, r_n/4)$, and this ball is contained in O_n by (7.3). This completes the proof. □

Remark 7.6. At first glance, it might seem more natural to equip $\mathcal{J}$ with the Hausdorff metric rather than the uniform metric. However, if we used the Hausdorff metric, the space $\mathcal{J}$ itself would be a meager set.

Recall from Section 1 the definition of the probability measure Q' on $\mathcal{J}$. We have the following Jordan curve analogue of Theorem 7.2. (We would like to thank T. Mountford, who brought reference [M] to our attention, and K. Burdzy, who showed us a different proof [B1].

THEOREM 7.7. *For Q'-almost all $\Gamma \in \mathcal{J}$, $\mathcal{H}(D_1(\Gamma))$ and $\mathcal{H}(D_2(\Gamma))$ are conditionally independent given $\mathcal{H}(\Gamma)$.*

Proof. Suppose the contrary. By Theorem 5.6, there would be a set with positive Q'-probability on which $\lambda\{\mathrm{pr}_1(M(\Gamma))\}>0$. By Lemma 5.5, for each such Γ, there is a subset F of Γ, totally ordered by $\le$ or $\triangle$, such that $\lambda(\mathrm{pr}_1(F))>0$ and Γ has a tangent at each point of F. By [M; Theorem 2(iii)], it follows that F does not have null harmonic measure in $D_i(\Gamma)$, $i=1$ and 2. But this contradicts Theorem 2.6(i) of [B], where it is shown that Q'-a.s., the set of points of Γ which are not "twist points" [B; Section 2] has null harmonic measure. This proves the theorem. □

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