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Spectral geometry in dimension 3

by

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In this paper, we investigate the question: to what extent does the spectrum of the Laplacian of a closed manifold M determine its geometry and topology? In dimension 2, one knows classically that the spectrum of M determines the topology of M. In [OPS], Osgood, Phillips, and Sarnak proved the following:

THEOREM [OPS]. *For M a closed surface, the spectrum of M determines the metric on M up to a family of metrics which is compact in the C^∞ topology.*

The situation in dimensions >2 is more complicated. It is now well-known (see [S]) that the spectrum of M may fail to determine even the topology of M. Furthermore, the techniques of [OPS] make heavy use of the assumption of dimension 2 at a number of points, for instance in their use of the structure of conformal classes of metrics on M.

In [BPP1], we studied the questions of finiteness of topological type and compactness of the space of metrics in higher dimensions, under some auxiliary pointwise curvature assumptions. There, the main idea was to employ spectral information together with the curvature assumptions to bring one within the range of the Cheeger Finiteness Theorem and its geometric relatives, in order to recreate the topology and geometry of M. A crucial step was to bound the Sobolev isoperimetric constant

$$C_S(M)=\inf_H \frac{\operatorname{area}(H)}{[\min(\operatorname{vol}(A),\operatorname{vol}(B))]^{1-1/n}},$$

where H runs over hypersurfaces of M which divide M into two pieces A and B.

In [BPP2], we considered the problem of bounding the Cheeger constant $h(M)$ in terms of spectral data alone, without further curvature assumptions. Here, $h(M)$ is given

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by the formula

$$h(M)=\inf_H \frac{\operatorname{area}(H)}{\min(\operatorname{vol}(A),\operatorname{vol}(B))},$$

where H again runs over hypersurfaces which divide M into two parts A and B. There we showed:

THEOREM [BPP2]. *For $n=2$ or 3, there is a positive constant $K(n)$ such that, if*

$$\lambda_1(M)>K(n)\frac{\|\mathrm{Ricc}\|_2}{\sqrt{\operatorname{vol}(M)}},$$

then $h(M)$ is bounded below in terms of spectral data.

Since the Sobolev constant $C_S(M)$ is defined in a manner similar to the Cheeger constant $h(M)$, but exerts far greater control on the local geometry of M, it would appear promising to see if one could improve the techniques of [BPP2] to find conditions under which one could bound C_S spectrally.

In this paper, we carry out a variant of this idea. To state our main result, we first observe that the curvature tensor of any Riemannian manifold has a pointwise orthogonal direct sum decomposition into three parts

$$R(M)=S\oplus\widetilde{\mathrm{Ricc}}\oplus W,$$

where S, $\widetilde{\mathrm{Ricc}}$, and W are the scalar, traceless Ricci, and Weyl tensors respectively. For any dimension n, the first three heat invariants a_0, a_1, a_2 satisfy

$$\begin{aligned} a_0&=\operatorname{vol}(M)\\ a_1&=c_1\int_M S\\ a_2&=c_2\int_M S^2+c_3\int_M\|\widetilde{\mathrm{Ricc}}\|^2+c_4\int_M|W|^2, \end{aligned}$$

where the constants $c_1,\dots,c_4$ depend only on the dimension n, and are positive for $n<6$, see [BGM].

Since the a_i's are spectral invariants for all i, it follows that if we define $\|\mathrm{Ricc}\|_{2,\mathrm{red}}$ by the formula

$$\begin{aligned} \|\mathrm{Ricc}\|_{2,\mathrm{red}}^2&=\int_M\|\widetilde{\mathrm{Ricc}}\|^2+\frac{c_2}{c_3}\left[\left(\int_M S^2\right)-\frac{1}{\operatorname{vol}(M)}\left(\int_M S\right)^2\right]+\frac{c_4}{c_3}\int_M|W|^2\\ &=\frac{a_2}{c_3}-\frac{c_2}{c_3}\left[\frac{a_1^2}{c_1^2a_0}\right], \end{aligned}$$

then $\|\mathrm{Ricc}\|_{2,\mathrm{red}}$ is a spectral invariant, and we have that

$$\|\mathrm{Ricc}\|_{2,\mathrm{red}} \geqslant \|\widetilde{\mathrm{Ricc}}\|_2.$$

More generally, we may define the reduced Riemann tensor $R(M)_{\mathrm{red}}$ by

$$R(M)_{\mathrm{red}} = \left[S - \left(\frac{1}{\mathrm{vol}(M)} \int_M S\right) \cdot \mathbf{1}\right] \oplus \widetilde{\mathrm{Ricc}} \oplus W,$$

where $\mathbf{1}$ is the vector of length 1 in the scalar curvature component. In dimensions two and three, $W=0$ and we may define the reduced Ricci tensor by the same expression with the summand for W absent. We define $\|\mathrm{Ricc}\|_{q,\mathrm{red}}$ by

$$\|\mathrm{Ricc}\|_{q,\mathrm{red}}^q = \int_M |R(M)_{\mathrm{red}}|^q.$$

Note that

$$\frac{\|\mathrm{Ricc}\|_{q,\mathrm{red}}}{\mathrm{vol}(M)^{1/q}} \leqslant \frac{\|\mathrm{Ricc}\|_{2,\mathrm{red}}}{\sqrt{\mathrm{vol}(M)}},$$

for $q<2$, as follows easily from Hölder's inequality.

Note also that $\|\mathrm{Ricc}\|_{2,\mathrm{red}}$ is 0 when M has constant curvature, so that $\|\mathrm{Ricc}\|_{2,\mathrm{red}}$ is a spectral measure of how far away M is from constant curvature.

Our main result is then:

THEOREM 0.1. *For $n=2$ or 3 and any positive integer k, there are constants $Q(n)$ and $K(n,k)$, such that if*

$$\lambda_k > Q(n) \frac{\int_M S}{\mathrm{vol}(M)}$$

and

$$\lambda_k(M) > K(n,k) \frac{\|\mathrm{Ricc}\|_{2,\mathrm{red}}}{\sqrt{\mathrm{vol}(M)}},$$

then the set of manifolds which carry metrics isospectral to M contains only finitely many diffeomorphism types, and the set of such metrics is compact in the C^∞ topology.

It follows in particular that there are spectrally determined open sets about the manifolds of constant curvature for which one has compactness of isospectral sets of metrics:

COROLLARY 0.1. *Let M be a 3-manifold of constant curvature. Then there is a spectrally determined open set U in the space of all metrics on M, such that if $g \in U$, then the set of metrics isospectral to g is compact in the C^∞ topology.*

The corollary follows from the theorem by choosing an eigenvalue $\lambda_k(M)$ sufficiently large so that

$$\lambda_k > Q \frac{\int_M S}{\mathrm{vol}(M)}, \tag{1}$$

the right-hand side being expressible in terms of a_0 and a_1, and setting U to be the neighborhood on which (1) continues to hold, and for which

$$\lambda_k > K(3,k)\frac{\|\mathrm{Ricc}\|_{2,\mathrm{red}}}{\sqrt{\mathrm{vol}(M)}}.$$

M is in this neighborhood, because the constant curvature condition guarantees that $\|\mathrm{Ricc}\|_{2,\mathrm{red}}=0$.

In the case $n=2$, the statement of Theorem 0.1 is contained in the Theorem of [OPS], which is true without any restriction on λ_k, $\|\mathrm{Ricc}\|_{2,\mathrm{red}}$, or $\mathrm{vol}(M)$.

In order to prove Theorem 0.1, we will study the (generalized) Sobolev isoperimetric constants

$$C_S^p(M)=\inf_H \frac{\mathrm{area}(H)}{[\min(\mathrm{vol}(A),\mathrm{vol}(B))]^{1-1/p}},$$

for $n\leqslant p<\infty$, where as before H runs over hypersurfaces which divide M into two parts A and B. Note that the case $p=n$ is the classical Sobolev isoperimetric constant, while $p=\infty$ is the Cheeger constant.

We will show:

THEOREM 0.2. *For all n and k, and for $p>n$, $q>\frac{1}{2}n$, there are constants $Q(n,q)$ and $K(n,k,p,q)$, such that, if*

$$\lambda_k > Q\frac{\int_M S}{\mathrm{vol}(M)}$$

and

$$\lambda_k > K\frac{\|\mathrm{Ricc}\|_{q,\mathrm{red}}}{(\mathrm{vol}(M))^{1/q}},$$

then $C_S^p(M)$ is bounded below in terms of spectral data and $\|\mathrm{Ricc}\|_{q,\mathrm{red}}$.

If, in addition, $q\leqslant 2$ and

$$\lambda_k > K(n,k,p,q)\frac{\|\mathrm{Ricc}\|_{2,\mathrm{red}}}{\sqrt{\mathrm{vol}(M)}},$$

then $C_S^p(M)$ is bounded in terms of spectral data alone.

Theorem 0.2 gives qualitative expression to the idea that as a manifold "stretches out", its low eigenvalues must begin to accumulate below some critical value. In the case of pointwise bounds on the curvature, this idea is well-expressed by Cheng's Inequality [Che], where the "stretching out" is measured by the diameter of M, and the critical value is $\frac{1}{4}(n-1)^2\varkappa+\varepsilon$, where $-(n-1)\varkappa$ is a lower bound for the Ricci curvature of M. The proof of Cheng's Theorem does not carry over to the case of L^p bounds for any finite p, so in our case we must proceed differently.

The idea of the proof of Theorem 0.2 is to construct test functions built up out of the distance function to a surface H which realizes the minimum in the ratio defining the Sobolev constant. Using Gallot's L^p version [Ga] of the Heintze–Karcher Theorem [HK], we will show that if the conditions of Theorem 0.2 are met, then one has sufficient control over the growth of tubes around H to ensure that if C_S^p were sufficiently close to 0, then one could construct $k+1$ functions with disjoint support whose Rayleigh quotient was less than λ_k. This contradiction then establishes a lower bound for C_S^p.

In Section 4 below, we will then show how to pass from Theorem 0.2 to Theorem 0.1. The main point is to give a bootstrap argument which shows that a lower bound C_S^p and an upper bound for $\|\mathrm{Ricc}\|_q$ for $q>\frac{1}{2}p$, together with spectral data, gives an L^∞ bound for Ricc and its covariant derivatives, from which Theorem 0.2 follows readily.

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1. Volumes of tubes

In this section, we begin by studying the volumes of tubes about a hypersurface minimizing the Sobolev constant $C_S^p(M)$.

It is a standard consequence of geometric measure theory that there is a hypersurface H which realizes the minimum of C_S^p. If we denote by Ω the component of $M-H$ of smallest volume, making an arbitrary choice if H divides M into two equal pieces, we then have

$$C_S^p = \frac{\mathrm{area}(H)}{[\mathrm{vol}(\Omega)]^{1-1/p}}.$$

We have the following estimate for the mean curvature $\varkappa(H)$ of H:

LEMMA 1.1. *$\varkappa(H)$ is constant, with*

$$|\varkappa(H)| \leqslant (1-1/p)\frac{C_S^p}{\mathrm{vol}(\Omega)^{1/p}}.$$

Proof. Let f_t be a family of diffeomorphisms of M, with infinitesimal generator ν. Suppose first that the two components of $M-H$ are of unequal volume. From the minimizing property of H, we see that

$$\frac{d}{dt}\left[\frac{\mathrm{area}(f_t(H))}{[\mathrm{vol}(f_t(\Omega))]^{1-1/p}}\right]\Bigg|_{t=0} = 0.$$

But

$$\frac{d}{dt}\left[\frac{\operatorname{area}(f_t(H))}{[\operatorname{vol}(f_t(\Omega))]^{1-1/p}}\right]=[\operatorname{vol}(f_t(\Omega))]^{2(1/p-1)}$$
$$\times(\operatorname{area}'(f_t(H))\operatorname{vol}(f_t(\Omega))^{1-1/p}-\operatorname{area}(f_t(H))(1-1/p)\operatorname{vol}(f_t(\Omega))^{-1/p}\operatorname{vol}'(f_t(\Omega)))$$

so that

$$\frac{d}{dt}(\operatorname{area}(f_t(H)))\Big|_{t=0}-(1-1/p)\frac{\operatorname{area}(H)}{\operatorname{vol}(f_t(\Omega))}\cdot\frac{d}{dt}\operatorname{vol}(f_t(\Omega))\Big|_{t=0}=0.$$

But

$$\frac{d}{dt}\operatorname{area}(f_t(H))\Big|_{t=0}=\int_H(\varkappa(H)N)\cdot\nu$$

and

$$\frac{d}{dt}\operatorname{vol}(f_t(\Omega))\Big|_{t=0}=\int_H N\cdot\nu,$$

where N is the normal vector pointing outward from Ω, so we conclude that

$$\varkappa(H)=(1-1/p)\frac{C_S^p}{\operatorname{vol}(\Omega)^{1/p}},$$

which establishes the lemma in the case where the two components are of unequal volume.

In the case where the two components are of equal volume, we consider separately the cases where ν points inwards on Ω and outwards from Ω. We must then use

$$\frac{d}{dt}\left(\frac{\operatorname{area}(f_t(H))}{\operatorname{vol}(f_t(\Omega))}\right)\geqslant 0$$

in the first case, and

$$\frac{d}{dt}\left(\frac{\operatorname{area}(f_t(H))}{\operatorname{vol}(M-f_t(\Omega))}\right)\geqslant 0$$

in the second case to derive the inequality. □

For each positive number R, let

$$\Omega_R=\{x:\operatorname{dist}(x,\Omega)\leqslant R\}.$$

Ω_R is the tube about Ω of radius R. We will now use the following estimate of Gallot [Ga] to estimate the growth of the volume of Ω_R:

THEOREM [Ga]. *Let Ω be a domain in M, and $H=\partial\Omega$, the boundary of Ω. Then for $q>\frac{1}{2}n$ and for $R,\varepsilon\geqslant 0$, we have*

$$\operatorname{vol}(\Omega_{R+\varepsilon})-\operatorname{vol}(\Omega)\leqslant(e^{B(q)\alpha\varepsilon}-1)\Bigg[\operatorname{vol}(\Omega_R)-\operatorname{vol}(\Omega)+\frac{1}{B(q)\alpha}\operatorname{area}(H)$$
$$+\frac{1}{(B(q)\alpha)^{2q}}\int_H\eta_+(x)^{2q-1}\,dx+\int_{\Omega_{R+\varepsilon}-\Omega}\left(\frac{r_-}{\alpha^2}-1\right)_+^q d\operatorname{vol}\Bigg]. \tag{2}$$

Here,

$$r_-(x)=\sup\left(0,\sup_{X\in T_x(M)-\{0\}}-\frac{\mathrm{Ricc}(X,X)}{\langle X,X\rangle}\right),$$

α is a constant which may be thought of as a "guess" for the curvature of the metric on M, η_+ is $\sup(0,\varkappa)$, where $\varkappa$ is the mean curvature of H (we use slightly different conventions for mean curvature than [Ga]), and

$$B(q)=\left(\frac{q-1}{q}\right)^{1/2}(n-1)^{1-1/2q}\left(\frac{q-1}{q-\frac{1}{2}n}\right)^{1/2-1/2q}.$$

The case where $H=\Omega$ is a point is also allowed, and here we may take $\eta=0$, see [Ga]. We will use this case in Section 4 below.

According to Lemma 1.1, we have

$$\eta_+\leqslant(1-1/p)\frac{C_S^p}{\mathrm{vol}(\Omega)^{1/p}},$$

so that the third term in square brackets in (2) is

$$\begin{aligned}&\leqslant\frac{1}{(B(q)\alpha)^{2q}}\left[\frac{(1-1/p)C_S^p}{\mathrm{vol}(\Omega)^{1/p}}\right]^{2q-1}\mathrm{area}(H)\\&=\frac{1}{(B(q)\alpha)^{2q}}\left[\frac{(1-1/p)C_S^p}{\mathrm{vol}(\Omega)^{1/p}}\right]^{2q-1}C_S^p\,\mathrm{vol}(\Omega)^{1-1/p}\\&=\frac{(1-1/p)^{2q-1}}{(B(q)\alpha)^{2q}}(C_S^p)^{2q}\,\mathrm{vol}(\Omega)^{1-2q/p}.\end{aligned}$$

Thus, we may rewrite Gallot's formula as

$$\begin{aligned}\mathrm{vol}(\Omega_{R+\varepsilon})-\mathrm{vol}(\Omega_R)\leqslant(e^{B(q)\alpha\varepsilon}-1)\biggl[&\mathrm{vol}(\Omega_R)-\mathrm{vol}(\Omega)+\frac{1}{B(q)\alpha}C_S^p\,\mathrm{vol}(\Omega)^{1-1/p}\\&+\frac{(1-1/p)^{2q-1}}{(B(q)\alpha)^{2q}}(C_S^p)^{2q}\,\mathrm{vol}(\Omega)^{1-2q/p}+\int_{\Omega_{R+\varepsilon}-\Omega_R}\left(\frac{r_-}{\alpha^2}-1\right)_+^q d\,\mathrm{vol}\biggr].\end{aligned}\tag{3}$$

Since we will be concerned that the terms involving C_S^p do not blow up when $\mathrm{vol}(\Omega)$ gets small, we will need

$$2q/p\leqslant 1,$$

that is,

$$q\leqslant\tfrac{1}{2}p.$$

As a warm-up to Section 2 below, we will show:

LEMMA 1.2. *Let $p>2q>n$ and suppose that there is a positive constant C such that $\mathrm{vol}(\Omega)>C$. Then C_p^S is bounded below in terms of C, λ_1, $\mathrm{vol}(M)$, and $\|\mathrm{Ricc}\|_q$.*

Proof. We will choose $\varepsilon>0$, and define two test functions $f_{1,\varepsilon}$ and $f_{2,\varepsilon}$ as follows:

$$f_{1,\varepsilon}(x)=\begin{cases}1, & x\in\Omega,\\ 1-(2/\varepsilon)\,\mathrm{dist}(x,\Omega), & \mathrm{dist}(x,\Omega)\leqslant\frac{1}{2}\varepsilon,\\ 0, & \mathrm{dist}(x,\Omega)\geqslant\frac{1}{2}\varepsilon;\end{cases}$$

$$f_{2,\varepsilon}(x)=\begin{cases}0, & \mathrm{dist}(x,\Omega)\leqslant\frac{1}{2}\varepsilon,\\ (2/\varepsilon)\,\mathrm{dist}(x,\Omega)-1, & \frac{1}{2}\varepsilon\leqslant\mathrm{dist}(x,\Omega)\leqslant\varepsilon,\\ 1, & \mathrm{dist}(x,\Omega)\geqslant\varepsilon.\end{cases}$$

Then $f_{1,\varepsilon}$ and $f_{2,\varepsilon}$ have disjoint support, so that

$$\lambda_1\leqslant\max\left(\frac{\int|\mathrm{grad}\,f_{1,\varepsilon}|^2}{\int f_{1,\varepsilon}^2},\frac{\int|\mathrm{grad}\,f_{2,\varepsilon}|^2}{\int f_{2,\varepsilon}^2}\right).$$

But

$$\int f_{1,\varepsilon}^2\geqslant\mathrm{vol}(\Omega),$$
$$\int f_{2,\varepsilon}^2\geqslant\mathrm{vol}(M)-\mathrm{vol}(\Omega_\varepsilon),$$

and

$$\int|\mathrm{grad}\,f_{i,\varepsilon}|^2\leqslant\frac{4}{\varepsilon^2}(\mathrm{vol}(\Omega_\varepsilon)-\mathrm{vol}(\Omega)).$$

We now use the inequality (3) to estimate $\mathrm{vol}(\Omega_\varepsilon)-\mathrm{vol}(\Omega)$:

$$\mathrm{vol}(\Omega_\varepsilon)-\mathrm{vol}(\Omega)\leqslant(e^{B(q)\alpha\varepsilon}-1)$$
$$\times\left[\frac{1}{B(q)\alpha}C_S^p\,\mathrm{vol}(\Omega)^{1-1/p}+\frac{(1-1/p)^{2q-1}}{(B(q)\alpha)^{2q}}(C_S^p)^{2q}\,\mathrm{vol}(\Omega)^{1-2p/q}+\frac{1}{\alpha^{2q}}\cdot\frac{\|\mathrm{Ricc}\|_q^q}{(n-1)^q}\right],$$

where we have used the estimate

$$r_-(x)\leqslant\frac{1}{n-1}|\mathrm{Ricc}(x)|,$$

together with the estimate

$$\left(\frac{r_-}{\alpha^2}-1\right)_+\leqslant\frac{r_-}{\alpha^2}.$$

Let us denote by x_0 the point where the function

$$g(x)=\frac{e^x-1}{x^2}$$

takes its minimum on $(0,\infty)$, and let

$$B(q)\alpha\varepsilon = x_0.$$

This defines ε in terms of α, and we have

$$\begin{aligned}\operatorname{vol}(\Omega_\varepsilon)-\operatorname{vol}(\Omega) &\leqslant (e^{x_0}-1)\\ &\times\left[\frac{1}{B(q)\alpha}C_S^p\operatorname{vol}(\Omega)^{1-1/p}+\frac{(1-1/p)^{2q-1}}{(B(q)\alpha)^{2q}}(C_S^p)^{2q}\operatorname{vol}(\Omega)^{1-2q/p}+\frac{1}{\alpha^{2q}}\cdot\frac{\|\mathrm{Ricc}\|_q^q}{(n-1)^q}\right].\end{aligned}\tag{4}$$

Choose α sufficiently large so that

$$\frac{(e^{x_0}-1)}{\alpha^{2q}}\cdot\frac{\|\mathrm{Ricc}\|_q^q}{(n-1)^q} < \tfrac{1}{10}C \leqslant \tfrac{1}{10}\operatorname{vol}(\Omega).$$

Either (i) the estimate

$$C_1C_S^p\operatorname{vol}(\Omega)^{1-1/p}+C_2(C_S^p)^{2q}\operatorname{vol}(\Omega)^{1-2q/p} \geqslant \tfrac{1}{10}\operatorname{vol}(\Omega)$$

holds for positive constants C_1 and C_2 determined by (4), or else (ii) the opposite inequality holds.

In case (i), we then have the weaker inequality

$$C_1C_S^p\big(\tfrac{1}{2}\operatorname{vol}(M)\big)^{1-1/p}+C_2(C_S^p)^{2q}\big(\tfrac{1}{2}\operatorname{vol}(M)\big)^{1-2q/p} \geqslant \tfrac{1}{10}C,$$

using that $\operatorname{vol}(\Omega)\leqslant\frac{1}{2}\operatorname{vol}(M)$, which gives a lower bound for C_S^p in terms of $\operatorname{vol}(M)$ and C.

In case (ii), we then have that

$$\operatorname{vol}(\Omega_\varepsilon) < \tfrac{6}{5}\operatorname{vol}(\Omega).$$

Thus in case (ii),

$$\operatorname{vol}(M)-\operatorname{vol}(\Omega_\varepsilon) \geqslant 2\operatorname{vol}(\Omega)-\tfrac{6}{5}\operatorname{vol}(\Omega) \geqslant \tfrac{4}{5}\operatorname{vol}(\Omega),$$

using that $\operatorname{vol}(\Omega)\leqslant\frac{1}{2}\operatorname{vol}(M)$, and so

$$\frac{\int|\operatorname{grad} f_{1,\varepsilon}|^2}{\int f_{1,\varepsilon}^2} \leqslant \frac{4}{\varepsilon^2}\cdot\frac{\operatorname{vol}(\Omega_\varepsilon)-\operatorname{vol}(\Omega)}{\operatorname{vol}(\Omega)}$$

and

$$\frac{\int|\operatorname{grad} f_{2,\varepsilon}|^2}{\int f_{2,\varepsilon}^2} \leqslant \frac{5}{4}\cdot\frac{4}{\varepsilon^2}\cdot\frac{\operatorname{vol}(\Omega_\varepsilon)-\operatorname{vol}(\Omega)}{\operatorname{vol}(\Omega)}.$$

On the other hand, by (4),

$$\frac{4}{\varepsilon^2}\cdot\frac{\mathrm{vol}(\Omega_\varepsilon)-\mathrm{vol}(\Omega)}{\mathrm{vol}(\Omega)} \leqslant 4g(x_0)B^2(q)\alpha^2$$
$$\times\left[\frac{C_S^p}{B(q)\alpha}\,\mathrm{vol}(\Omega)^{-1/p}+\frac{(1-1/p)^{2q-1}}{(B(q)\alpha)^{2q}}(C_S^p)^{2q}\,\mathrm{vol}(\Omega)^{-2q/p}+\frac{1}{\alpha^{2q}\,\mathrm{vol}(\Omega)}\cdot\frac{\|\mathrm{Ricc}\|_q^q}{(n-1)^q}\right]$$

so that, using $\mathrm{vol}(\Omega)\geqslant C$,

$$\lambda_1 \leqslant \tfrac{5}{4}\cdot 4g(x_0)B^2(q)\alpha^2$$
$$\times\left[\frac{1}{B(q)\alpha}C_S^pC^{-1/p}+\frac{(1-1/p)^{2q-1}}{B(q)\alpha^{2q}}(C_S^p)^{2q}C^{-2q/p}+\frac{\|\mathrm{Ricc}\|_q^q}{\alpha^{2q}C}\right]. \tag{5}$$

Now choose α so large that

$$4g(x_0)B^2(q)\frac{\|\mathrm{Ricc}\|_q^q}{\alpha^{2q-2}(n-1)^qC} < \tfrac{2}{3}\lambda_1. \tag{6}$$

Either the inequality

$$5g(x_0)B^2(q)\alpha^2\left[\frac{1}{B(q)\alpha}C_S^pC^{-1/p}+\frac{(1-1/p)^{2q-1}}{B(q)\alpha^{2q}}(C_S^p)^{2q}C^{-2q/p}\right] \geqslant \tfrac{1}{6}\lambda_1 \tag{7}$$

holds, implying a lower bound on C_S^p in terms of λ_1, C, $\mathrm{vol}(M)$, and $\|\mathrm{Ricc}\|_q$, or the opposite inequality holds. In the latter case we use this inequality in (5) to obtain a contradiction, since we get an upper estimate for λ_1 which is less than λ_1. Hence (7) holds, so in case (ii), C_S^p is bounded below in terms of C, λ_1, $\mathrm{vol}(M)$, and $\|\mathrm{Ricc}\|_q$.

Hence in either case (i) or case (ii), we have the desired lower bound for C_S^p. □

2. Test functions

From the last section, it is apparent that the heart of the problem in proving Theorem 0.2 is that $\mathrm{vol}(\Omega)$ may go to zero.

Our idea in this section is to construct test functions whose support is on regions built up from level sets of the distance function to Ω, in hopes of showing that if either C_S^p or $\mathrm{vol}(\Omega)$ is too small, we get an estimate for some λ_k which is too small.

To that end, let us fix positive numbers K and D, and define the sequences of numbers

$$A_0, A_1, \dots, A_{2k+3}$$

and

$$B_0, B_1, \dots, B_{2k+3}$$

by:

$$A_0 = B_0 = \mathrm{vol}(\Omega)$$

and

$$A_{i+1} = K[B_i + D],$$
$$B_{i+1} = B_i + A_{i+1} = (1+K)B_i + K \cdot D.$$

Solving the difference equations, it follows that

$$B_i = (1+K)^i B_0 + [(1+K)^i - 1]D$$

and

$$A_i = K(1+K)^{i-1}[B_0 + D].$$

We now set

$$\mathcal{B}_i = \Omega_{R_i}$$

where R_i is such that

$$\mathrm{vol}(\Omega_{R_i}) = B_i,$$

and we set

$$\mathcal{A}_i = \mathcal{B}_i - \mathcal{B}_{i-1} = \Omega_{R_i} - \Omega_{R_{i-1}}.$$

The important point here is that in Gallot's formula, the numbers R_i do not enter in explicitly, but only through the volumes of the Ω_{R_i}'s.

We now view Gallot's inequality as a lower bound on the width of $\mathcal{A}_i$, that is, on $R_i - R_{i-1}$, in the following sense: if we set

$$B_0 = \mathrm{vol}(\Omega)$$

and pick

$$D > \int_M \left(\frac{r_-}{\alpha^2} - 1\right)_+^q - \mathrm{vol}(\Omega),$$

then Gallot's theorem tells us that

$$\varepsilon_i = R_i - R_{i-1}$$

would satisfy

$$e^{B(q)\alpha\varepsilon} \geqslant 1 + K$$

if both C_S^p and $\mathrm{vol}(\Omega)$ were 0, so that an upper estimate for ε_i less than

$$\frac{1}{B(q)\alpha} \log(1+K)$$

would contain an implicit bound for one of C_S^p or $\mathrm{vol}(\Omega)$ away from 0. By Lemma 1.2, then, this would imply a lower bound for C_S^p.

We now construct $k+1$ test functions f_i as follows:

$$f_i(x)=\begin{cases} 1, & x\in \mathcal{A}_{2i}, \\ 1-(2/\varepsilon_i)\,\mathrm{dist}(x,\mathcal{A}_{2i}), & \text{for } 0\leqslant \mathrm{dist}(x,\mathcal{A}_{2i})\leqslant \frac{1}{2}\varepsilon_i, \\ 0, & \text{elsewhere.}\end{cases}$$

The f_i's thus have support contained in $\mathcal{A}_{2i-1}\cup\mathcal{A}_{2i}\cup\mathcal{A}_{2i+1}$, so distinct f_i have disjoint support.

In order for this to make sense, we must make sure that we have not run out of manifold—that is, we must have

$$B_{2k+3}=(1+K)^{2k+3}B_0+[(1+K)^{2k+3}-1]D<\mathrm{vol}(M). \tag{8}$$

Since $B_0=\mathrm{vol}(\Omega)$ and $D>\int_M(r_-/\alpha^2-1)_+^q-\mathrm{vol}(\Omega)$, this will hold when $\mathrm{vol}(\Omega)$ is sufficiently small provided that

$$\int_M\left(\frac{r_-}{\alpha^2}-1\right)_+^q<\frac{\mathrm{vol}(M)}{(1+K)^{2k+3}-1}. \tag{9}$$

Now let us compute the Rayleigh quotients of the f_i's:

$$\int_M|\,\mathrm{grad}(f_i)|^2=\frac{4}{\varepsilon_i{}^2}\cdot A_{2i-1}+\frac{4}{\varepsilon_{i+1}^2}\cdot A_{2i+1}$$

and

$$\int_M f_i^2\geqslant A_{2i}.$$

It follows that if $\varepsilon_i\geqslant(1/B(q)\alpha)\log(1+K)$, then

$$\frac{\int_M|\,\mathrm{grad}\,f_i|^2}{\int_M f_i^2}\leqslant\frac{4B^2(q)\alpha^2}{[\log(1+K)]^2}[(1+K)+(1+K)^{-1}].$$

If we then have the condition

$$\frac{4B^2(q)\alpha^2}{[\log(1+K)]^2}[(1+K)+(1+K)^{-1}]<\lambda_k, \tag{10}$$

together with (9), then we have a contradiction unless either C_S^p or $\mathrm{vol}(\Omega)$ are bounded away from 0, since we have constructed $k+1$ test functions with disjoint support, whose Rayleigh quotients are less than λ_k.

We now give the proof of:

LEMMA 2.1. *Suppose that* $2q>n$ *and that there exist a positive integer* k *and positive numbers* α, *and* K *satisfying*

$$\int_M \left(\frac{r_-}{\alpha^2}-1\right)_+^q < \frac{\mathrm{vol}(M)}{(1+K)^{2k+3}-1} \tag{11}$$

and

$$\frac{4B^2(q)\alpha^2}{[\log(1+K)]^2}[(1+K)+(1+K)^{-1}] < \lambda_k. \tag{12}$$

Then C_S^p *is bounded below in terms of* K, k, $\mathrm{vol}(M)$, λ_1, *and* $\int_M (r_-/\alpha^2-1)_+^q$.

Proof. We assume that there exist $K>0$ and $k>0$ with

$$\int_M \left(\frac{r_-}{\alpha^2}-1\right)_+^q < \frac{\mathrm{vol}(M)}{(1+K)^{2k+3}-1} \tag{13}$$

and

$$\frac{4B^2(q)\alpha^2}{\log(1+K)^2}[(1+K)+(1+K)^{-1}] < \lambda_k. \tag{14}$$

We want to pick a number $D>0$ satisfying

$$\int_M \left(\frac{r_-}{\alpha^2}-1\right)_+^q - \mathrm{vol}(\Omega) < D, \tag{15}$$

$$(1+K)^{2k+3}\,\mathrm{vol}(\Omega)+[(1+K)^{2k+3}-1]D < \mathrm{vol}(M). \tag{16}$$

It will be useful to rewrite (16) as

$$D < \frac{1}{(1+K)^{2k+3}-1}\,\mathrm{vol}(M) - \frac{(1+K)^{2k+3}}{(1+K)^{2k+3}-1}\,\mathrm{vol}(\Omega). \tag{17}$$

If it is not possible to pick such a positive D, then either the left-hand side of (15) is negative, implying the lower bound

$$\mathrm{vol}(\Omega) \geqslant \int_M \left(\frac{r_-}{\alpha^2}-1\right)_+^q,$$

or the right-hand side of (17) is less than or equal to the left-hand side of (15), which gives us the inequality

$$\mathrm{vol}(\Omega) \geqslant \mathrm{vol}(M) - (1+K)^{2k+3}\int \left(\frac{r_-}{\alpha^2}-1\right)_+^q.$$

This last term is positive by assumption. So in either case, if we cannot find such a positive D, we have a lower bound for $\mathrm{vol}(\Omega)$, and hence by Lemma 1.2 a lower bound for C_S^p.

We will now assume that such a D exists and find a lower bound for C_S^p. We construct $k+1$ test functions as described above, with Rayleigh quotients bounded by

$$\frac{\int_M |\operatorname{grad} f_i|^2}{\int_M f_i^2} \leqslant \frac{4}{\varepsilon_i^2}[(1+K)+(1+K)^{-1}].$$

By the max-min principle, at least one of the Rayleigh quotients must be greater than or equal to λ_k. For this choice of i we have by (14) that

$$\varepsilon_i^2 \leqslant 4[(1+K)+(1+K)^{-1}]\lambda_k^{-1} < \frac{\log(1+K)^2}{B^2(q)\alpha^2}$$

with strict inequality in the second line. Thus, for this i, $e^{B(q)\alpha\varepsilon_i}-1<K$.

From our remarks preceding the statement of the lemma, this then gives us a lower bound for C_S^p, and the lemma is proved. □

3. Proof of Theorem 0.2

It remains to play with (11) and (12) in such a way as to obtain usable results. The main point is that one may estimate

$$\int_M \left(\frac{r_-}{\alpha^2}-1\right)_+^q$$

in various ways, in order to apply Lemma 2.1.

First, we will use the estimate

$$r_- \leqslant \frac{|\mathrm{Ricc}_-|}{n-1}$$

to obtain

$$\int_M \left(\frac{r_-}{\alpha^2}-1\right)_+^q \leqslant \int_M \left(\frac{r_-}{\alpha^2}\right)^q \leqslant \int_M \frac{|\mathrm{Ricc}_-|^q}{(n-1)^q\alpha^{2q}}$$

in (11).

For convenience, we will set

$$x = \log(1+K).$$

The inequality (11) then becomes

$$\alpha^{2q} > [e^{(2k+3)x}-1]\frac{\|\mathrm{Ricc}_-\|_q^q}{(n-1)^q \operatorname{vol}(M)},$$

while (12) becomes

$$\alpha^2 < \frac{\lambda_k x^2}{4B^2(q)[e^x + e^{-x}]}.$$

We will be able to choose α satisfying both (11) and (12) provided that

$$\lambda_k > 4B^2(q)\left[\frac{e^x + e^{-x}}{x^2}\right][e^{(2k+3)x} - 1]^{1/q}\frac{\|\mathrm{Ricc}_-\|_q}{(n-1)(\mathrm{vol}(M))^{1/q}}. \tag{18}$$

We thus have:

THEOREM 3.1. *Suppose that there exist x and k such that*

$$\lambda_k > 8B^2(q)\cdot\frac{\cosh(x)}{(n-1)x^2}[e^{(2k+3)x} - 1]^{1/q}\cdot\frac{\|\mathrm{Ricc}_-\|_q}{(\mathrm{vol}(M))^{1/q}}.$$

Then C_S^p is bounded below in terms of x, k, λ_1, λ_k, $\|\mathrm{Ricc}_-\|_q$, and $\mathrm{vol}(M)$.

One application of Theorem 3.1 is when one has a bound for $\|\mathrm{Ricc}_-\|_q$ for large q, for instance when one has pointwise lower bounds for Ricc. In this way, one may retrieve the results of [BPP1] for Ricci curvature bounded from below.

In order to prove Theorem 0.2, we will make use of a somewhat different estimate for

$$\int_M \left(\frac{r_-}{\alpha^2} - 1\right)_+^q.$$

LEMMA 3.1.

$$\int_M \left(\frac{r_-}{\alpha^2} - 1\right)_+^q \leqslant \frac{\|\mathrm{Ricc}\|_{q,\mathrm{red}}^q}{\alpha^{2q}},$$

provided that

$$\alpha^2 \geqslant \frac{1}{n-1}\left|\frac{\int_M S}{\mathrm{vol}(M)}\right|.$$

Proof. Writing

$$\mathrm{Ricc} = S \oplus \widetilde{\mathrm{Ricc}} \oplus W$$

and

$$S = \left(\frac{\int_M S}{\mathrm{vol}(M)}\right) + \left(S - \frac{\int_M S}{\mathrm{vol}(M)}\right),$$

we have that

$$\frac{-\mathrm{Ricc}(X,X)}{\langle X,X\rangle} - (n-1)\alpha^2 \leqslant -\left[S - \frac{\int_M S}{\mathrm{vol}(M)}\cdot\mathbf{1}\right] - \frac{\widetilde{\mathrm{Ricc}}(X,X)}{\langle X,X\rangle}$$

provided that

$$\left|\frac{\int_M S}{\mathrm{vol}(M)}\right| \leqslant (n-1)\alpha^2.$$

It follows that

$$\int_M (r_- - \alpha^2)_+^q \leqslant \frac{1}{(n-1)^q} \|\mathrm{Ricc}\|_{q,\mathrm{red}}^q.$$

We then have

$$\int_M \left(\frac{r_-}{\alpha^2} - 1\right)_+^q \leqslant \frac{\|\mathrm{Ricc}\|_{q,\mathrm{red}}^q}{(n-1)^q \alpha^{2q}},$$

provided that

$$(n-1)\alpha^2 \geqslant \left|\frac{\int_M S}{\mathrm{vol}(M)}\right|.$$

□

Substituting into (11) and (12) gives the conditions

$$\alpha^{2q} > \frac{e^{(2k+3)x} - 1}{(n-1)^q} \cdot \frac{\|\mathrm{Ricc}\|_{q,\mathrm{red}}^q}{\mathrm{vol}(M)}$$

and

$$\alpha^2 < \frac{\lambda_k x^2}{4B^2(q)[e^x + e^{-x}]},$$

provided that

$$(n-1)\alpha^2 > \frac{\int_M S}{\mathrm{vol}(M)}.$$

Hence we have:

THEOREM 3.2. *For any $x>0$, let k be large enough so that*

$$\lambda_k > \frac{1}{n-1} \cdot \frac{\int_M S}{\mathrm{vol}(M)} \cdot 8B^2(q) \frac{\cosh(x)}{x^2}.$$

If, also,

$$\lambda_k > 8B^2(q) \frac{\|\mathrm{Ricc}\|_{q,\mathrm{red}}}{(\mathrm{vol}(M))^{1/q}} \cdot \frac{[e^{(2k+3)x} - 1]^{1/q}}{n-1} \cdot \frac{\cosh(x)}{x^2},$$

then $C_S^p(M)$ is bounded below in terms of k, x, λ_k, $\mathrm{vol}(M)$, and $\|\mathrm{Ricc}\|_{q,\mathrm{red}}$.

Setting

$$Q(n,q) = \frac{1}{n-1} B^2(q) \frac{\cosh(x)}{x^2}$$

and

$$K(n,k,p,q) = 8B^2(q) \frac{[e^{(2k+3)x} - 1]^{1/q}}{n-1} \cdot \frac{\cosh(x)}{x^2},$$

we obtain the statement of Theorem 0.2.

4. The bootstrap

In this section, we complete the proof of Theorem 0.1. We will show:

THEOREM 4.1. *Given $p<4$ and $q>\frac{1}{2}p$, $C>0$, R_0, v, V, and $A_0, A_1, \dots$, the set of n-manifolds M satisfying*

(a) $C_S^p \geqslant C$,

(b) $\|\mathrm{Ricc}\|_q \leqslant R_0$,

(c) $v \leqslant \mathrm{vol}(M) \leqslant V$,

(d) $a_i(M) \leqslant A_i$ *for all* i,

is compact in the C^∞ topology.

Note that for the condition $C_S^p \geqslant C>0$ to be non-vacuous, we must have $p\geqslant n$. Thus this theorem as stated applies only for $n=2$ or 3. We remark that this theorem is valid for larger values of p, but the proof becomes much more delicate, see [Cho]. We will only need the case $2<p<4$.

Before we proceed with the proof, we will show how Theorem 4.1 allows us to complete Theorem 0.1. To begin the discussion, we define the Sobolev constants $C^S(p,q)$ to be the constants occurring in the Sobolev inequalities:

$$\|f\|_{qp/(p-q)} \leqslant C^S(p,q)[\|f\|_q + \|\nabla f\|_q] \quad \text{for } q<p$$

and

$$\|f\|_\infty \leqslant C^S(p,q)[\|f\|_q + \|\nabla f\|_q] \quad \text{for } q>p.$$

In the case where $p=n$, these are the classical Sobolev inequalities, and the existence of such constants is standard, see for instance [Cha].

It will be important to bound the numbers $C^S(p,q)$ from above in terms of the data given in Theorem 4.1. To that end, we observe that a bound from above for $C^S(p,q)$ in terms of a lower bound for C_S^p, an upper bound for $\mathrm{vol}(M)$, and a bound for $\|\mathrm{Ricc}\|_q$ for $q>\frac{1}{2}n$ proceeds in much the same way as the classical case, see [BPP1] for a discussion. The main point to add here is that a bound on $\|\mathrm{Ricc}\|_q$ for $q>\frac{1}{2}n$ allows us to find a number r such that the ball $B(x,r)$ about any point x has volume less than half the volume of M, as one sees readily from Gallot's Theorem applied to $H=\{\mathrm{pt}\}$, while a lower bound for C_S^p for $p<\infty$ gives us a lower bound for $\mathrm{vol}(B(x,r))$, so that there are a bounded number of disjoint balls $B(r,x)$ in M. We may then use the partition argument of [BPP1] to bound $C^S(p,q)$ for $q=1$, and then extend this for all q, by standard arguments (see [BPP1]).

The conditions of Theorem 4.1 now tell us that we have uniform upper bounds for all the numbers $C^S(p,q)$.

Suppose now that we have the bounds

$$\lambda_k > Q(n,q)\frac{\int_M S}{\mathrm{vol}(M)}$$

and

$$\lambda_k > K(n,k,p,q)\frac{\|\mathrm{Ricc}\|_{q,\mathrm{red}}}{\mathrm{vol}(M)^{1/q}}$$

of the assumption of Theorem 0.1, for $q=2$. Then we also have these inequalities for some q satisfying $\frac{3}{2}<q<2$, since the right-hand side is continuous in q.

It follows from Theorem 0.2 that we now have a bound for C_S^p for some p satisfying

$$2q<p<4.$$

We now have bounds for all the terms in Theorem 4.1, because we still have a bound for $\|\mathrm{Ricc}\|_2$, and $2>\frac{1}{2}p$. We may then use Theorem 4.1 to conclude compactness of the space of metrics, completing the proof of Theorem 0.1.

In the proof of Theorem 4.1, we will make frequent use of the following well-known facts:

LEMMA 4.1 [Hölder inequalities]. (a) *For* $1/p+1/q=1$,

$$\int fg \leqslant \|f\|_p\|g\|_q.$$

(b) *If* $a<p<b$, *then*

$$\|f\|_p \leqslant \|f\|_a^{\frac{a}{p}\cdot\frac{b-p}{b-a}}\|f\|_b^{\frac{b}{p}\cdot\frac{p-a}{b-a}}.$$

We will use without further mention simple facts about (b), such as that if a increases while b and p remain fixed, then the exponent of $\|f\|_b$ decreases.

To make the statements of what follows convenient, we will define the symbol "$\preceq$" as follows:

$$A\preceq B \quad \text{if and only if} \quad A\leqslant C_1[C_2+B],$$

where the constants C_1 and C_2 can be computed explicitly.

We then have the following useful but totally elementary inequalities:

(a) If $A\preceq[\sum_{i=1}^k B_i]^\alpha$, then $A\preceq\sup B_i^\alpha$.

(b) If $A\preceq A^\alpha$ for some $\alpha<1$, then $A\preceq 1$.

Finally, we will make use of Gilkey's Theorem on the leading terms in the heat expansion, which we will rewrite in the following way:

THEOREM 4.2 ([Gi], see [A]). *For each $j \geqslant 1$,*

$$\|\nabla^j \mathrm{Rm}\|_2^2 \preceq a_{j+2} + \sum_P \int_M P(|\mathrm{Rm}|, |\nabla \mathrm{Rm}|, \ldots, |\nabla^{j-1}\mathrm{Rm}|),$$

where Rm *denotes the Riemann curvature tensor, P ranges over all monomials in $|\mathrm{Rm}|, \ldots, |\nabla^{j-1}\mathrm{Rm}|$ of weight $2(j+2)$, where the weight of $|\nabla^k \mathrm{Rm}|$ is $k+2$, and the weight of a product of such terms is the sum of the weights of the terms.*

In dimensions 2 and 3, we may use that $W=0$ to everywhere replace the Riemann tensor Rm with the Ricci curvature tensor Ricc.

We may rewrite the Sobolev inequalities in the following way: Suppose that $\|f\|_{p/\alpha} \preceq 1$. Then

$$\|f\|_{p/(\alpha-1)} \preceq \|\nabla f\|_{p/\alpha} \quad \text{if } \alpha > 1$$

and

$$\|f\|_\infty \preceq \|\nabla f\|_{p/\alpha} \quad \text{if } \alpha < 1.$$

Taking $\alpha = \frac{1}{2}p$, and setting $n_p = p/\left(\frac{1}{2}p-1\right) = 2p/(p-2)$ if $p>2$ and $=\infty$ if $p<2$, we have that if $\|f\|_2 \preceq 1$ then

$$\|f\|_{n_p} \preceq \|\nabla f\|_2,$$

and if $\|f\|_{n_p} \preceq 1$, then

$$\|f\|_\infty \preceq \|\nabla f\|_{n_p} \quad \text{for } 2 < p < 4.$$

We will separate the argument into the cases $p<2$ and $2<p<4$.

Case 1: $p<2$. Taking first the case $p<2$, and observing that $\|\mathrm{Ricc}\|_2 \preceq 1$ from the a_2 term in the heat expansion, we have

$$\|\mathrm{Ricc}\|_\infty \preceq \|\nabla \mathrm{Ricc}\|_2$$

and

$$\|\nabla \mathrm{Ricc}\|_2 \preceq \|\mathrm{Ricc}\|_3^{3/2},$$

where the second statement is from Gilkey's Theorem.

But

$$\|\mathrm{Ricc}\|_3^{3/2} \leqslant \|\mathrm{Ricc}\|_2 \|\mathrm{Ricc}\|_\infty^{1/2},$$

so

$$\|\mathrm{Ricc}\|_\infty \preceq 1$$

and

$$\|\nabla \mathrm{Ricc}\|_2 \preceq 1.$$

Now suppose inductively that we have

$$\|\nabla^j \mathrm{Ricc}\|_2 \preceq 1$$

and

$$\|\nabla^k \mathrm{Ricc}\|_\infty \preceq 1 \quad \text{for } k<j.$$

Then

$$\|\nabla^j \mathrm{Ricc}\|_\infty \preceq \|\nabla^{j+1}\mathrm{Ricc}\|_2 \preceq \left[a_{j+3}+\sum_P \int P\right]^{1/2},$$

by Gilkey's Theorem.

In the summation over P, we can replace any term bounded in L^∞ by a constant, and only $|\nabla^k \mathrm{Ricc}|$ for $k<j+1$ appear, so we conclude that only terms which are monomials of the form $|\nabla^j \mathrm{Ricc}|^m$ occur.

From the weight condition, we have that

$$m(j+2) \leqslant 2(j+3),$$

or in other words that $m \leqslant 2$.

But, by Hölder interpolation, we have that

$$\|\nabla^j \mathrm{Ricc}\|_2 \preceq 1,$$

so the last term in the inequality is $\preceq 1$.

Hence, the first and second terms are also $\preceq 1$, and the inductive step is completed.

We conclude that

$$\|\nabla^j \mathrm{Ricc}\|_\infty \preceq 1 \quad \text{for all } j.$$

This concludes the case $p<2$.

Case 2: $2<p<4$. In the case $2<p<4$, we have $\|\mathrm{Ricc}\|_2 \preceq 1$ from the a_2 term, and so

$$\|\mathrm{Ricc}\|_{n_p} \preceq \|\nabla \mathrm{Ricc}\|_2,$$

and

$$\|\nabla \mathrm{Ricc}\|_2 \preceq \|\mathrm{Ricc}\|_3^{3/2}$$

from Gilkey's Theorem, as before.

If $q \geqslant 3$, then we have that

$$\|\mathrm{Ricc}\|_3 \preceq \|\mathrm{Ricc}\|_q \preceq 1.$$

If $q<3$, then we apply Hölder interpolation to get

$$\|\mathrm{Ricc}\|_3 \leqslant \|\mathrm{Ricc}\|_q^a \|\mathrm{Ricc}\|_{n_p}^b$$

where we can calculate b as follows: If $q=\frac{1}{2}p$, then

$$b=\frac{2p}{p-2}\cdot\frac{3-\frac{1}{2}p}{2p/(p-2)-\frac{1}{2}p}=\frac{2}{3}.$$

Hence if $q>\frac{1}{2}p$, $b<\frac{2}{3}$, and we have that

$$\|\mathrm{Ricc}\|_{n_p} \preceq \|\nabla\,\mathrm{Ricc}\|_2 \preceq \|\mathrm{Ricc}\|_{n_p}^{3b/2},$$

so all of these terms are $\preceq 1$.

Now we have

$$\begin{aligned}\|\mathrm{Ricc}\|_\infty &\preceq \|\nabla\,\mathrm{Ricc}\|_{n_p} \preceq \|\nabla^2\,\mathrm{Ricc}\|_2\\ &\preceq \left(a_4+\int |\mathrm{Ricc}|^4+\int |\nabla\,\mathrm{Ricc}|^2|\mathrm{Ricc}|\right)^{1/2}\\ &\preceq \sup(\|\mathrm{Ricc}\|_4^2, \|\nabla\,\mathrm{Ricc}\|_4\|\mathrm{Ricc}\|_2^{1/2}).\end{aligned}$$

But for $p<4$, we have $4<n_p$, and so the first term is $\preceq 1$, and the second term can be written

$$\|\nabla\,\mathrm{Ricc}\|_4\|\mathrm{Ricc}\|_2^{1/2} \preceq \|\nabla\,\mathrm{Ricc}\|_4 \preceq \|\nabla\,\mathrm{Ricc}\|_2^a\|\nabla\,\mathrm{Ricc}\|_{n_p}^b$$

for some $b<1$, and we conclude that

$$\|\nabla\,\mathrm{Ricc}\|_{n_p} \preceq \|\nabla\,\mathrm{Ricc}\|_{n_p}^b,$$

and so all the above terms are $\preceq 1$.

Now assume inductively that, for $j>1$, we have $\|\nabla^k\mathrm{Ricc}\|_\infty \preceq 1$ for all $k<j$, $\|\nabla^j\mathrm{Ricc}\|_{n_p}\preceq 1$, and $\|\nabla^{j+1}\mathrm{Ricc}\|_2\preceq 1$. Then

$$\|\nabla^j\mathrm{Ricc}\|_\infty \preceq \|\nabla^{j+1}\mathrm{Ricc}\|_{n_p} \preceq \|\nabla^{j+2}\mathrm{Ricc}\|_2,$$

and

$$\|\nabla^{j+2}\mathrm{Ricc}\|_2 \preceq \left(a_{j+4}+\sum_P\int P(|\nabla^{j+1}\mathrm{Ricc}|, |\nabla^j\mathrm{Ricc}|)\right)^{1/2}.$$

We now investigate which monomials P may occur. If P involves only one or two terms, then it is bounded, because $\|\nabla^j\mathrm{Ricc}\|_2$ and $\|\nabla^{j+1}\mathrm{Ricc}\|_2$ are both $\preceq 1$.

If there are m terms, then each is of weight at least $j+2$, so the weight condition says that

$$m(j+2) \leqslant 2(j+4),$$

which tells us that $m \leqslant 3$ and $j \leqslant 2$.

For $j=2$, we have the term

$$\left(\int |\nabla^2 \mathrm{Ricc}|^3\right)^{1/2} = \|\nabla^2 \mathrm{Ricc}\|_3^{3/2}.$$

But for $p<6$, $3<n_p$, so this term is $\preceq 1$.

For $j=1$, we get only the term $(\int |\nabla \mathrm{Ricc}|^2 |\nabla^2 \mathrm{Ricc}|)^{1/2}$. But this is

$$\preceq \|\nabla \mathrm{Ricc}\|_4 \|\nabla^2 \mathrm{Ricc}\|_2^{1/2},$$

and $\|\nabla \mathrm{Ricc}\|_4 \preceq 1$ since $4<n_p$ for $p<4$.

We conclude that

$$\|\nabla^2 \mathrm{Ricc}\|_{n_p} \preceq \|\nabla^2 \mathrm{Ricc}\|_2^{1/2} \preceq \|\nabla^2 \mathrm{Ricc}\|_{n_p}^b$$

for some b less than $\frac{1}{2}$, and so all the terms in the inequality are $\preceq 1$.

This completes the inductive step, and we conclude that $\|\nabla^j \mathrm{Ricc}\|_\infty$ is bounded for all j.

We now complete the proof of Theorem 4.1 as in Section 4 of [BPP1].

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