Acta Math., 175 (1995), 75–121

Harmonic analysis for certain representations of graded Hecke algebras

by

ERIC M. OPDAM

University of Leiden
Leiden, The Netherlands

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0. Introduction

Consider the following data: a Euclidean space $\mathfrak{a}$, a root system $R\subset\mathfrak{a}^*$, a choice of positive roots $R_+\subset R$, and a multiplicity function k on the roots. Let $\mathfrak{h}$ denote the complexification of $\mathfrak{a}$. In his paper [2] Cherednik attaches commuting operators D_ξ ($\xi\in\mathfrak{h}$) to these data. The D_ξ act on functions defined on $\mathfrak{h}$ and are invariant for translations in the lattice $2\pi i\mathbf{Z}R^\vee$ ($R^\vee\subset\mathfrak{a}$ is the coroot system). He shows that, together with the Weyl group W, these operators generate an operator algebra that is isomorphic to the graded Hecke algebra $\mathbf{H}$ associated to R_+ and the root labels k_α (the graded Hecke algebra was defined by Lusztig in [15]). As a result, many natural function spaces on $\mathfrak{h}$ have the structure of an $\mathbf{H}$-module. For example, the spaces $C_c^\infty(\mathfrak{a})$ and $C^\infty(T)$ (where $T=i\mathfrak{a}/2\pi iQ^\vee$ with $Q^\vee=\mathbf{Z}R^\vee$) are $\mathbf{H}$-modules in this way. Moreover these spaces are equipped with natural inner products which are invariant for two different $*$-structures ("$*$" and "$+$", respectively) on $\mathbf{H}$. The purpose of this paper is to study these two $\mathbf{H}$-modules from the viewpoint of harmonic analysis. We will study decompositions of these modules into irreducible $\mathbf{H}$-modules which are unitary with respect to $*$ and $+$, respectively.

It turns out that the irreducible modules which occur in these decompositions all have the property that the subspace of W-invariant vectors is 1-dimensional. Such modules will be called W-spherical modules over $\mathbf{H}$. This terminology is motivated by the fact that the pair $(\mathbf{H}, W)$ behaves like a Gelfand pair: the space of W-invariant vectors of an irreducible $\mathbf{H}$-module is at most 1-dimensional (cf. Proposition 1.2).

Let us give a brief outline of this paper. The first three sections are introductory, but contain some new proofs of existing results. Section 2 is almost entirely due to Heckman [8] and contains an elegant proof of the commutativity of Cherednik's operators "without calculations". Section 3 contains short proofs of results of Cherednik [3] and Matsuo [19]. As a result we obtain a detailed description of the holomorphic eigenfunctions of the Cherednik operators. In Section 4 we discuss the $+$- and $*$-structure on $\mathbf{H}$, and a family of unitary irreducible W-spherical modules for each of these. In Section 5 we show how the results of Sections 1, 2 and 4 can be used to solve the spectral problem for the D_ξ acting on $C^\infty(T)$. This results in a complete set of orthogonal polynomials $E(\lambda)$ ($\lambda \in P$, the weight lattice) in $C^\infty(T)$. The L_2 norms and values at $e \in T$ can easily be calculated using induction on k. For each $\lambda \in P$, the span of the polynomials $E(w\lambda)$ ($w \in W$) is an irreducible W-spherical $\mathbf{H}$-module. The associated spherical functions are the so called Jacobi polynomials. In this way, the results of this section generalize the results of [20] in the sense that we no longer restrict ourselves to W-invariant polynomials. It is noticable that this simplifies the proofs somewhat. It seems likely that this extension to noninvariant polynomials is also applicable in the case of Macdonald polynomials [18]. This would generalize the results of [4], and explain the Macdonald constant term conjectures in terms of unitary structures of modules over the affine Hecke algebra.

The rest of the paper is devoted to the decomposition of $C_c^\infty(\mathfrak{a})$. Section 6 is a technical and preparatory section containing results on the growth behaviour and the asymptotic behaviour of the eigenfunctions $G(\lambda)$ for the D_ξ. The uniform growth estimates for $G(\lambda)$ can be obtained from the results of Section 3 and a study of the KZ equation. Our analysis of the KZ equation at this point is analogous to the analysis of de Jeu [13] of the Dunkl operators. The asymptotic behaviour of $G(\lambda)$ can be obtained from known results for the hypergeometric function and results in Section 3. In Section 7 we define a transform $\mathcal{F}$ for functions on $\mathfrak{a}$ that corresponds to the decomposition of $C_c^\infty(\mathfrak{a})$ in a family of induced modules for $\mathbf{H}$. We also introduce a wave packet operator $\mathcal{J}$ and we study the basic properties of $\mathcal{F}$ and $\mathcal{J}$. The Paley–Wiener theorem for the transform $\mathcal{F}$ is discussed in Section 8. If $x \in \mathfrak{a}$ we denote by C_x the convex hull of the orbit Wx. We define a Paley–Wiener space $\pi(M_x)$ (for a precise definition we refer the reader to Definitions 8.1, 8.2 and 8.3) and we show that $\mathcal{F}(C_c^\infty(C_x))\subset\pi(M_x)$ and $\mathcal{J}(\pi(M_x))\subset C_c^\infty(C_x)$. The proofs of these statements are analogous to Helgason's proof of the Paley–Wiener theorem for Riemannian symmetric spaces ([11, Chapter IV, §7]). Finally, in Section 9 we show that $\mathcal{J}\mathcal{F}=\mathrm{id}$ and $\mathcal{F}\mathcal{J}=\mathrm{id}$, and we give explicit inversion formulas and Plancherel theorems. The key step in the proof of $\mathcal{J}\mathcal{F}=\mathrm{id}$ is the beautiful idea due to van den Ban and Schlichtkrull [1] to use Peetre's characterization of differential operators. The results of Section 8 are of crucial importance here.

In order to put our results in perspective it is enlightening to compare these with the theory of the spherical transform on a Riemannian symmetric space. It should be made clear that the harmonic analysis on Riemannian symmetric spaces is the main source of inspiration for the results presented here. The theory of the spherical transform is generalized in two ways in this paper and it is worthwhile to discuss both these steps.

Firstly, we replace the spherical function on a Riemannian symmetric space $X=G/K$ by the more general notion of hypergeometric function associated to the root system R (the restricted root system of X) and a multiplicity function k. If $2k=m$, the root multiplicity function of X, then this hypergeometric function reduces to (the restriction to a Cartan subspace of) the spherical function, but in general it is no longer associated to a geometric object such as X. (This procedure was studied in the papers [10] and [6] and simplified considerably since then by the work of Dunkl [5] and Heckman [7]; we refer the reader to [9] for an up to date account of these matters.) Our results imply that the inversion formula and the Plancherel formula for the spherical transform on X still hold when the spherical functions are replaced by hypergeometric functions, provided that the labels k_α are nonnegative real numbers.

The second generalization consists of the passage from the W-invariant functions (on $\mathfrak{a}$ or T) to arbitrary functions. As we explained above, we work with W-spherical modules over $\mathbf{H}$, embedded in the function spaces $C_c^\infty(\mathfrak{a})$ and $C^\infty(T)$. The hypergeometric function is just a W-spherical vector of such a module. It turns out that there is no need to restrict oneself to the W-spherical part only. All the formulas that are relevant to harmonic analysis (special values, asymptotic behaviour, inversion and Plancherel formula) are equally simple and elegant with respect to properly chosen bases of the W-spherical modules.

Let us conclude this introduction with two problems that seem to be interesting for further research. First of all, the results mentioned above indicate that there is a relation between the K-spherical representations of G and the irreducible W-spherical modules of $\mathbf{H}(k)$ (where k corresponds to the root multiplicities m_α of X). Is there a more direct way to exhibit this relation? Secondly, we have avoided the situation where $k_\alpha<0$ in this paper. If $k_\alpha<0$ (but small) the spectral problem is well-posed, and many interesting new phenomena arise in the noncompact case. The decomposition of $C_c^\infty(\mathfrak{a})$ now involves lower-dimensional spectral series which are related to irreducible unitary spherical **H**-modules that arise from unitary induction of "discrete series" representations of parabolic subsystems of positive rank. We hope to analyse this further in a forthcoming paper (joint work with G. J. Heckman).

Acknowledgements. The author would like to thank E. van den Ban, I. Dolgachev, G. J. Heckman, M. de Jeu and E. Looijenga for valuable discussions.

1. Notations and preliminaries

The first part of this section serves to fix notations. The setup is similar to the setup in [22]. In the second part of this section we will review some elementary facts of the representation theory of the graded Hecke algebra. Let $\mathfrak{a}$ be a Euclidean space of dimension n and $R\subset\mathfrak{a}^*$ (the dual of $\mathfrak{a}$) be an integral root system. We do not assume that R is reduced, and we will write R^0 for the inmultiplicable roots in R and R_0 for the indivisible roots in R. Denote by W the associated Weyl group. If $\alpha\in R$ then we use the notation $\alpha^\vee\in\mathfrak{a}$ for the element in $\mathfrak{a}$ that satisfies $\lambda(\alpha^\vee)=2(\alpha,\lambda)/(\alpha,\alpha)$. The set $R^\vee=\{\alpha^\vee\}\subset\mathfrak{a}$ is called the coroot system (and its elements are called coroots). We define $Q=\mathbf{Z}.R$, the root lattice of R, and $Q^\vee=Q(R^\vee)$. We will also need the so called weight lattice $P=P(R)=\mathrm{Hom}_{\mathbf{Z}}(Q^\vee,\mathbf{Z})\subset\mathfrak{a}^*$. Let us denote by $\mathfrak{h}$ the complexification $\mathbf{C}\otimes_{\mathbf{R}}\mathfrak{a}$ of $\mathfrak{a}$. The complex torus H is given by $H=Q^\vee\otimes_{\mathbf{Z}}\mathbf{C}^\times$. We write A for the real split part of H, and T for the compact part of H, so that we have the decomposition $H=AT$. The Weyl group acts on H in a natural way (via the W-action on $Q^\vee$). If we put

$$\begin{aligned} h^\lambda &: H\to\mathbf{C}^\times,\\ h &= \varkappa\otimes z\to z^{\lambda(\varkappa)} \end{aligned}\tag{1.1}$$

(where $\lambda\in\mathfrak{h}^*$, the dual of $\mathfrak{h}$), then this defines a single valued function if and only if $\lambda\in P$. The set $\{h^\lambda\}_{\lambda\in P}$ exhausts the algebraic characters of H, and the $\mathbf{C}$-linear span of these characters is the ring $\mathbf{C}[H]$ of regular functions on H. The regular points of H for the action of W are $H^{\mathrm{reg}}=\{h\in H\mid\Delta(h)=\prod_{\alpha\in R_+^0}(h^{\alpha/2}-h^{-\alpha/2})\neq0\}$, where R_+ is a choice of positive roots (the function Δ is called the Weyl-denominator). The denominator formula of Weyl asserts that

$$\Delta(h)=\sum_{w\in W}\det(w)h^{w\delta}\tag{1.2}$$

where

$$\delta=\frac{1}{2}\sum_{\alpha\in R_+^0}\alpha.\tag{1.3}$$

We also choose a basis $(\alpha_1, ..., \alpha_n)$ for R_+, and let $(\lambda_1, ..., \lambda_n) \subset P$ be the corresponding basis of fundamental weights. The subset $Q_+ \subset Q$ $(P_+ \subset P)$ is by definition the $\mathbf{Z}_+$-span of $(\alpha_i)_{i=1}^n$ $((\lambda_i)_{i=1}^n)$ (where $\mathbf{Z}_+ = 0, 1, 2, ...$) and is referred to as "the positive roots (weights)". Corresponding to these notions of positivity we also have $\mathfrak{a}_+$ $(=\{x \in \mathfrak{a} \mid \alpha(x) > 0\ \forall \alpha \in R_+\})$, $\mathfrak{a}_+^*$ $(=\{\lambda \in \mathfrak{a}^* \mid \lambda(\alpha^\vee) > 0\ \forall \alpha \in R_+\})$, A_+ etc. Given a choice of positive roots we have a partial order on $\mathfrak{h}^*$ (defined by $\lambda \leqslant \mu \Leftrightarrow \mu - \lambda \in Q_+$) and on W (the Bruhat order). Finally, the choice of the positive roots also determines a length function l on W.

It is well-known that

$$\mathbf{C}[H]^W = \mathbf{C}[z_1, ..., z_n] \tag{1.4}$$

where

$$z_i = \sum_{w \in W/W_{\lambda_i}} h^{w\lambda_i} \tag{1.5}$$

(where W_{λ_i} is the subgroup of W that stabilizes λ_i). The map

$$\begin{aligned} \mathrm{pr}: H &\to \mathbf{C}^n \\ h &\to (z_1(h), ..., z_n(h)) \end{aligned} \tag{1.6}$$

parametrizes the W-orbits in H, and is ramified along the discriminant $\{z \in \mathbf{C}^n \mid d(z) = \Delta^2(h) = 0\}$ of R.

Let $\mathcal{K}$ denote the linear space of multiplicity functions, i.e. the space of W-invariant complex functions on R. If $k \in \mathcal{K}$ we define

$$\varrho(k) = \varrho(R_+, k) = \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha. \tag{1.7}$$

We associate a multiplicity k^0 (k_0) of R^0 (R_0) to a given $k \in \mathcal{K}$ in such a way that

$$\varrho(R_+, k) = \varrho(R_{0+}, k_0) = \varrho(R_+^0, k^0).$$

The graded Hecke algebra was introduced by Lusztig in [15]. The facts discussed below are completely elementary. For the most part they can be found in [15] or [2].

One can associate a graded Hecke algebra $\mathbf{H}$ to the following data: a Euclidean space $\mathfrak{a}$, a reduced integral root system R in $\mathfrak{a}$, a positive subset R_+ in R, and a multiplicity function k on R. Let $S(\mathfrak{h})$ denote the symmetric algebra of $\mathfrak{h}$ and let r_i denote the simple reflection in the simple root α_i of R. Then $\mathbf{H}(R_+, k)$ is the unique associative algebra over $\mathbf{C}$ with the following properties:

(1) As a $\mathbf{C}$-vector space, $\mathbf{H} = S(\mathfrak{h}) \otimes \mathbf{C}[W]$.

(2) The maps $S(\mathfrak{h}) \to \mathbf{H}$, $p \to p \otimes e$ and $\mathbf{C}[W] \to \mathbf{H}$, $w \to 1 \otimes w$ are algebra homomorphisms.

(3) $(p \otimes e) \cdot (1 \otimes w) = p \otimes w\ \forall p \in S(\mathfrak{h})\ \forall w \in W$.

(4) $(1 \otimes r_i) \cdot (\xi \otimes e) - (r_i(\xi) \otimes e) \cdot (1 \otimes r_i) = -k_i \alpha_i(\xi)\ \forall \xi \in \mathfrak{h}\ \forall i$.

We often identify $S(\mathfrak{h})$ and $\mathbf{C}[W]$ with their images in $\mathbf{H}$ via the maps indicated in the above description. (Consequently we may use either 1 or e in order to denote the unit element $1\otimes e$ of $\mathbf{H}$!)

PROPOSITION 1.1. (1) $\forall \xi\in\mathfrak{h}\ \forall w\in W$:

$$w\cdot\xi\cdot w^{-1}=w(\xi)+\sum_{\alpha\in R_+\cap wR_-} k_\alpha\alpha(w\xi)r_\alpha.$$

(2) $\forall p\in S(\mathfrak{h})\ \forall i$:

$$r_i\cdot p-p^{r_i}\cdot r_i=-k_i\Delta_i(p),$$

where $\Delta_i(p)=(p-p^{r_i})/\alpha_i^\vee$.

(3) $\mathbf{H}=S(\mathfrak{h})\cdot\mathbf{C}[W]=\mathbf{C}[W]\cdot S(\mathfrak{h})$.

(4) *The center* $Z(\mathbf{H})$ *of* $\mathbf{H}$ *equals* $S(\mathfrak{h})^W$.

Proof. (1) We use induction on the length of w. Write $w=r_iw_1$ with $l(w_1)<l(w)$. In this situation we have $R_+\cap wR_-=r_i(R_+\cap w_1R_-)\cup\{\alpha_i\}$. Using the induction hypothesis and the above description of the product in $\mathbf{H}$ this readily leads to (1).

(2) Use induction on the degree of p.

(3) Immediate from the above description of $\mathbf{H}$.

(4) Using (1) it is easy to see that $Z(\mathbf{H})\subset S(\mathfrak{h})$. Now apply (2) in order to conclude (4). □

In the next proposition we collect some useful elementary facts about finite-dimensional representations of $\mathbf{H}$.

PROPOSITION 1.2. *Let* V *be a finite-dimensional* $\mathbf{H}$*-module. If* $\lambda\in\mathfrak{h}^*$ *we define* $V^\lambda=\{v\in V\mid \xi v=\lambda(\xi)v\ \forall\xi\in\mathfrak{h}\}$.

(1) $\exists\lambda\in\mathfrak{h}^*$ *such that* $V^\lambda\neq 0$.

(2) $\forall\lambda\in\mathfrak{h}^*$: $\lambda(\alpha_i^\vee)r_i+k_i$ *maps* V^λ *to* $V^{r_i\lambda}$.

(3) *If* $\lambda(\alpha^\vee)\neq\pm k_\alpha\ \forall\alpha\in R$ *then* $\dim(V^\lambda)=\dim(V^{w\lambda})\ \forall w\in W$.

(4) *Let* $\lambda(\alpha^\vee)\neq 0,\pm k_\alpha\ \forall\alpha\in R$. *If* V *has dimension* $|W|$ *and has central character* $\chi_\lambda: p\in S(\mathfrak{h})^W\to p(\lambda)$ *then* V *is irreducible.*

(5) *Put* $I_\lambda=\mathrm{Ind}_{S(\mathfrak{h})}^{\mathbf{H}}(\mathbf{C}_\lambda)=\mathbf{H}\otimes_{S(\mathfrak{h})}\mathbf{C}_\lambda$. *Then* I_λ *has central character* χ_λ *and is isomorphic to the regular representation of* $\mathbf{C}[W]$ *when restricted to* $\mathbf{C}[W]\subset\mathbf{H}$. *It has the following universal property: for any* $\mathbf{H}$*-module* V *and* $v\in V^\lambda$ *there exists a unique* $\mathbf{H}$*-module morphism* $I_\lambda\to V$ *such that* $1\otimes 1\to v$.

(6) *Let* λ *be regular. The nonzero* $\mathbf{H}$*-module morphism* $I_{r_i\lambda}\to I_\lambda$ *determined by* $1\otimes 1\to(\lambda(\alpha_i^\vee)r_i+k_i)\otimes 1$ *(cf. (2)) is an isomorphism if and only if* $\lambda(\alpha_i^\vee)\neq\pm k_i$.

(7) *Let* V *be an irreducible* $\mathbf{H}$*-module. The dimension of the space of* W*-invariant vectors* V^W *is at most one.*

Proof. Straightforward and left to the reader (use (5) to prove (7)). □

The next theorem is less elementary. Although we will not really need this result in this paper it clarifies the definitions in Section 7 somewhat. For its proof we refer the reader to [2] (also see [24]). We note that the case where λ is regular simply follows from the above proposition (use (6) for the "only if" part).

THEOREM 1.3. *I_λ is irreducible if and only if $\lambda(\alpha^\vee)\neq\pm k_\alpha$ $\forall\alpha\in R$.*

The next proposition will be useful for many computations.

PROPOSITION 1.4. *Let λ be regular and such that $\lambda(\alpha^\vee)\neq\pm k_\alpha$ $\forall\alpha\in R$. We define $v_w\in I_\lambda$ inductively on the length of $w\in W$ as follows: $v_e=1\otimes 1\in I_\lambda^\lambda$, and if $w<r_iw$ for some $w\in W$ and simple reflection r_i then*

$$v_{r_iw}=\frac{w\lambda(\alpha_i^\vee)}{w\lambda(\alpha_i^\vee)+k_i}r_iv_w+\frac{k_i}{w\lambda(\alpha_i^\vee)+k_i}v_w. \tag{1.8}$$

(1) *$0\neq v_w\in I_\lambda^{w\lambda}$, hence $\{v_w\}_{w\in W}$ is a basis for I_λ.*

(2) *Formula (1.8) holds for all $w\in W$ and simple reflections r_i.*

(3) *Put $\phi_{w'}=|W|^{-1}\sum_{w\in W}wv_{w'}$. Then $\phi_{w'}=\phi$ is independent of $w'\in W$, and*

$$\phi=|W|^{-1}\sum_{w\in W}\prod_{\alpha\in R_+}\left(1-\frac{k_\alpha}{w\lambda(\alpha^\vee)}\right)v_w.$$

Moreover, ϕ spans I_λ^W.

(4) *Put $\phi_e^-=|W|^{-1}\sum_{w\in W}(-1)^{l(w)}wv_e$ and $\phi_{w_0}^-=|W|^{-1}\sum_{w\in W}(-1)^{l(w)}ww_0v_{w_0}$ (where w_0 denotes the longest element of W). Then*

$$\phi_e^-=|W|^{-1}\prod_{\alpha\in R_+}\left(1+\frac{k_\alpha}{\lambda(\alpha^\vee)}\right)\sum_{w\in W}(-1)^{l(w)}v_w$$

and

$$\phi_{w_0}^-=|W|^{-1}\prod_{\alpha\in R_+}\left(1-\frac{k_\alpha}{\lambda(\alpha^\vee)}\right)\sum_{w\in W}(-1)^{l(w)}v_w.$$

Proof. (1) and (2) follow from straightforward calculations and are left to the reader. As for (3), we first show the independence of w' using (2):

$$\begin{aligned}\phi_{r_iw'}&=|W|^{-1}\sum_{w\in W}wv_{r_iw'}\\&=|W|^{-1}\sum_{w\in W}\left(\frac{w'\lambda(\alpha_i^\vee)}{w'\lambda(\alpha_i^\vee)+k_i}\right)wr_iv_{w'}+\left(\frac{k_i}{w'\lambda(\alpha_i^\vee)+k_i}\right)wv_{w'}=\phi_{w'}.\end{aligned}$$

Now we calculate the coefficient b_w of v_w in ϕ. If we use (2) we see that the only term in $\phi=\phi_{w_0w}=|W|^{-1}\sum_{w'\in W} w'v_{w_0w}$ that contributes to b_w is $|W|^{-1}w_0v_{w_0w}$. Repeated application of (2) now gives the asserted formula for b_w. From Proposition 1.2 (5) we see that I_λ^W has dimension 1, hence this subspace is spanned by ϕ.

Let us finally consider (4). It is easy to check that $\psi=\sum_{w\in W}(-1)^{l(w)}v_w$ is a skew element of I_λ. Hence by Proposition 1.2 (5) both ϕ_e^- and $\phi_{w_0}^-$ are multiples of ψ. The determination of the multiplicative constants is similar to the argument we used in (3) and is left to the reader. □

2. Cherednik's operators

In this section we will discuss certain operators introduced by Cherednik in [2]. Cherednik analysed these operators in more detail in his paper [3] and the results of this section can all be found there. Instead of simply referring to these papers we choose to give an account here of a different approach due to Heckman [8]. Heckman's method is very direct and fits nicely into the framework of this paper. It is a pleasure to thank him for his kind permission to use this material here.

Definition 2.1. Let $R\in\mathfrak{a}^*$ be a root system and $k\in\mathcal{K}$ a multiplicity such that $k_\alpha^0\geqslant 0$ $\forall\alpha\in R^0$. Let dt be the Haar measure on T that is normalized by $\int_T dt=1$ and let $\delta_k(t)=\prod_{\alpha\in R}(1-t^\alpha)^{k_\alpha}$. We define a Hermitean inner product $(\cdot,\cdot)_k$ on $\mathbf{C}[P]$ by

$$(f,g)_k=\int_T f(t)\overline{g(t)}\delta_k(t)\,dt.$$

Definition 2.2 (see [2]). Let $R_+\in R$ be a choice of positive roots, $k\in\mathcal{K}$ an arbitrary multiplicity function and let $\xi\in\mathfrak{h}$. The Cherednik operator $D_\xi=D_\xi(R_+,k)$ is the differential difference operator on $\mathfrak{h}$ defined by

$$D_\xi=\partial_\xi+\sum_{\alpha\in R_+}k_\alpha\alpha(\xi)\frac{1}{1-e^{-\alpha}}(1-r_\alpha)-\varrho(k)(\xi).$$

($\forall\lambda\in\mathfrak{h}^*$ we define the function e^λ on $\mathfrak{h}$ by $e^\lambda(\xi)=e^{\lambda(\xi)}$ $\forall\xi\in\mathfrak{h}$.)

PROPOSITION 2.3 ([2, Proposition 3.8]). *If $k_\alpha^0\geqslant 0$ and $\xi\in\mathfrak{a}$ then $D_\xi(k)$ is symmetric with respect to $(\cdot,\cdot)_k$.*

Proof. This is a straightforward calculation, left to the reader. □

If $\lambda\in\overline{\mathfrak{a}_+^*}$ we use the notation $W_\lambda=\{w\mid w\lambda=\lambda\}$ and $W^\lambda=\{w\mid l(ww')\geqslant l(w)\ \forall w'\in W_\lambda\}$. As is well-known, W^λ is a complete set of representatives for the right cosets of W_λ in W. Let $R_\lambda=\{\alpha\in\mathbf{R}\mid(\alpha,\lambda)=0\}$ be the parabolic subsystem of R associated with W_λ.

Definition 2.4 ([8]). For $\lambda \in P$ we denote by λ^* the unique dominant weight in the W orbit of λ. Let $w^\lambda \in W^{\lambda^*}$ be the unique element such that $\lambda = w^\lambda \lambda^*$. Define a partial ordering $\leqslant_W$ on P as follows:

$$\lambda \leqslant_W \mu \iff \begin{cases} (1)\ \lambda^* \leqslant \mu^*, \\ (2)\ \text{if } \lambda^* = \mu^* \text{ then } w^\lambda \leqslant w^\mu. \end{cases}$$

PROPOSITION 2.5 ([8]). *The operators $D_\xi(R_+, k)$ act on $\mathbf{C}[P]$ and are upper triangular with respect to $\leqslant_W$, i.e. $D_\xi(e^\lambda) = \sum_{\mu \leqslant_W \lambda} a_{\lambda,\mu} e^\mu$.*

Proof. Easy and left to the reader. □

Definition 2.6 ([8]). If $k_\alpha \geqslant 0$ and $\lambda \in P$ we define $E(\lambda, k) \in \mathbf{C}[P]$ by the conditions:
(1) $E(\lambda, k) = e^\lambda + \sum_{\mu <_W \lambda} c_{\lambda,\mu} e^\mu$.
(2) $\forall \mu <_W \lambda$: $(E(\lambda, k), e^\mu)_k = 0$.

COROLLARY 2.7 ([8]). *The $E(\lambda, k)$ are simultaneous eigenfunctions for the operators $D_\xi(R_+, k)$ and form a basis of $\mathbf{C}[P]$.*

Proof. Use Proposition 2.5. □

LEMMA 2.8. *Let D be a linear operator acting on meromorphic functions on $\mathfrak{h}$ and of the form $D = \sum_{w \in W} D_w w$ where D_w is a linear differential operator with meromorphic coefficients on $\mathfrak{h}$. If D vanishes on $\mathbf{C}[P]$ then $D_w = 0$ $\forall w \in W$.*

Proof. By induction on the highest order d of the D_w. Let $d = 0$. If $x \in \mathfrak{h}$ and $\Delta(x) \neq 0$ then x and wx are different on H if $w \neq e$. Hence there exists a $p \in \mathbf{C}[P]$ such that $p(wx) = \delta_{w,e}$. Now from $D(p^w) = 0$ $\forall w \in W$ it follows that $D_w(x) = 0$ $\forall w \in W$. In the general case we notice that (cf. (1.5)) $[D_w, z_i] = 0$ $\forall i = 1, \ldots, n$ $\forall w \in W$ by the induction hypothesis. But $\{z_i - z_i(x)\}_{i=1}^n$ is a set of coordinate functions at x if $\Delta(x) \neq 0$. This implies that $d = 0$, and we have returned to the first case. □

COROLLARY 2.9 ([3, Theorem 2.4]). *Let $\mathbf{A}$ denote the associative complex algebra with 1 of linear operators acting on holomorphic functions on $\mathfrak{h}$ generated by the $D_\xi(R_+, k)$ and by $w \in W$. The linear map $\sum_{w \in W} \xi \otimes w \to \sum_{w \in W} D_\xi(R_+, k) w$ from $\mathfrak{h} \otimes \mathbf{C}[W] \to \mathbf{A}$ can be extended in a unique way to an isomorphism $\sum_{w \in W} p_w \otimes w \to \sum_{w \in W} p_w(D(k)) w$ of the graded Hecke algebra $\mathbf{H}(R_+^0, k^0)$ to $\mathbf{A}$.*

Proof. By Corollary 2.7 and Lemma 2.8 the D_ξ commute with each other (because they can be diagonalized simultaneously on $\mathbf{C}[P]$). It is straightforward to check that $r_i D_\xi(R_+, k) - D_{r_i(\xi)}(R_+, k) r_i = -k^0_{\alpha_i} \alpha_i(\xi)$ $\forall \alpha_i \in R$ simple. Hence the linear map $\sum_{w \in W} \xi \otimes w \to \sum_{w \in W} D_\xi(R_+, k) w$ can be extended uniquely to an epimorphism of algebras as indicated. To show the injectivity of this map we use Lemma 2.8 again.

Suppose that $\sum_{w\in W} p_w(D(k))w=0$ in $\mathbf{A}$. If we write $\sum_{w\in W} p_w(D(k))w=\sum_{w\in W} D_w w$, then $D_w=0$ $\forall w\in W$ by Lemma 2.8. On the other hand, let w' be such that the degree of $p_{w'}$ is maximal and let q denote its highest degree part. Then the highest order part of $D_{w'}$ equals $q(\partial)$, hence $q=0$. Consequently, $p_w=0$ $\forall w\in W$. □

It is easy to calculate the eigenvalue of the $E(\lambda,k)$:

PROPOSITION 2.10 ([8]). *Define* $\varepsilon:\mathbf{R}\to\{\pm1\}$ *by* $\varepsilon(x)=x/|x|$ *if* $x\neq 0$ *and* $\varepsilon(0)=-1$. *Given* $\lambda\in P$ *we put* $\tilde\lambda=\lambda+\frac12\sum_{\alpha\in R_+} k_\alpha\varepsilon(\lambda(\alpha^\vee))\alpha$. *Then*

$$D_\xi(k)E(\lambda,k)=\tilde\lambda(\xi)E(\lambda,k). \tag{2.1}$$

If $\lambda\in P_+$ *then* $\tilde\lambda=w_\lambda(\lambda+\varrho)$ *where* w_λ *denotes the longest element* W_λ. *Moreover, if* $\lambda\in P_+$ *and* $w\in W^\lambda$ *then* $(w\lambda)\tilde{}=w\tilde\lambda$.

Proof. The statement about the eigenvalue follows immediately from the formula

$$D_\xi(k)(e^\lambda)=\tilde\lambda(\xi)e^\lambda+\sum_{\mu<_W\lambda} a_{\lambda,\mu}e^\mu$$

and this can be checked directly from Definition 2.2. We have $w_\lambda(R_{\lambda,+})=-R_{\lambda,+}$ and $w_\lambda(R_+\backslash R_{\lambda,+})=R_+\backslash R_{\lambda+}$. Therefore:

$$w_\lambda(\varrho(k))=\frac12\sum_{\alpha\in R_+}k_\alpha w_\lambda(\alpha)=\frac12\sum_{\alpha\in R_+}\varepsilon(\lambda(\alpha^\vee))k_\alpha\alpha$$

and we see that $\tilde\lambda=w_\lambda(\lambda+\varrho)$. The last statement of the proposition is a consequence of the following well-known description of W^λ: $w\in W^\lambda$ if and only if $w(\alpha)\in R_+$ $\forall\alpha\in R_{\lambda,+}$. □

COROLLARY 2.11 ([8]). $\{E(\lambda,k)\}_{\lambda\in P}$ *is an orthogonal basis of* $\mathbf{C}[P]$ *with respect to* $(\cdot,\cdot)_k$.

Proof. The eigenvalues are distinct since $\lambda\neq\mu$ implies (if $k_\alpha\geqslant0$ $\forall\alpha$) that $\tilde\lambda\neq\tilde\mu$. □

We close this section by relating the above operators and their eigenfunctions to the hypergeometric differential operators and the Jacobi polynomials $P(\lambda,k)$. We refer the reader to [10, Definition 2.5 (for the Harish-Chandra homomorphism) and Definition 2.13], and to [6] (for the Jacobi polynomials).

Due to the commutativity of the operators $D_\xi(k)$ we can extend the map $\mathfrak{h}\to\mathrm{End}(\mathbf{C}[P])$ in a unique way to an algebra homomorphism $S(\mathfrak{h})\to\mathrm{End}(\mathbf{C}[P])$ (cf. the proof of Corollary 2.9). The image of $p\in S(\mathfrak{h})$ will be denoted by $p(D(k))$. If we do not want to specialize at any value of $\mathcal{K}$ in particular we will write $p(D)$.

THEOREM 2.12. (1) *Let $\lambda \in P_+$. Then $E(\lambda, k)$ is W_λ-invariant, and*

$$P(\lambda, k) = \sum_{w \in W^\lambda} E^w(\lambda, k).$$

Here E^w denotes the function on T defined by $E^w(t) = E(w^{-1}t)$.

(2) *Let $p \in S(\mathfrak{h})^W$ and denote by D_p the (W-invariant) differential operator on $\mathbf{C}[P]$ that coincides with $p(D)$ when restricted to $\mathbf{C}[P]^W$. Then D_p is the hypergeometric differential operator such that $\gamma(D_p) = p$ (where γ is the Harish-Chandra homomorphism).*

Proof. (1) The W_λ-invariance is a consequence of Definition 2.6 and the fact that if $\lambda \in P_+$ and $\mu <_W \lambda$ then $w\mu <_W \lambda$ $\forall w \in W$. Hence $\sum_{w \in W^\lambda} E^w(\lambda, k)$ is W-invariant and has leading term e^λ. By Corollary 2.11 it fits the orthogonality description of the $P(\lambda, k)$ as in [6].

(2) Because of Proposition 1.1 (4) and Corollary 2.9 we see that D_p is a W-invariant differential operator on $\mathbf{C}[P]$. By (1) and Proposition 2.10 we have

$$\begin{aligned} D_p(k)P(\lambda, k) &= p(\tilde{\lambda})P(\lambda, k) \\ &= p(w_\lambda(\lambda + \varrho))P(\lambda, k) \\ &= p(\lambda + \varrho)P(\lambda, k) \\ &= \gamma(k)^{-1}(p)P(\lambda, k). \end{aligned}$$

Since differential operators on $\mathbf{C}[P]$ are determined by their action on $\mathbf{C}[P]^W$ this completes the proof. □

3. The Knizhnik–Zamolodchikov connection

In this section we want to study the eigenfunction problem for the Cherednik operators. This is of course essential to the study of the spectral problem for these operators. However, the spectral problem on the compact torus T can be solved using the polynomials that were introduced in the previous sections. The reader might want to skip this section temporarily and read about the compact problem first (this problem is addressed in the next two sections).

The material that is discussed here can for the most part be found in the papers [19] and [3] but we will follow a different and more direct route. The goal is to establish a precise relation between hypergeometric functions and eigenfunctions of the D_ξ. We do this using the Knizhnik–Zamolochikov (KZ in the sequel)-connection as an intermediate step. An interesting feature of this method is that we do not need the integrability of the KZ-connection. In fact this turns out to be a simple corollary.

Let us begin by fixing some notations. Let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathfrak{h}^{\mathrm{reg}}$.

Definition 3.1. Let $V=\mathcal{O}\otimes\mathbf{C}[W]$, $\lambda\in\mathfrak{h}^*$ and $k\in\mathcal{K}$. The KZ-connection $\nabla(\lambda,k)$ on V is defined by the following covariant differentiation ($\xi\in\mathfrak{h}$):

$$\begin{aligned}&\nabla_\xi(\lambda,k)(\phi\otimes w)\\&\quad=\partial_\xi\phi\otimes w+\frac{1}{2}\sum_{\alpha\in R_+}k_\alpha\alpha(\xi)\left(\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\phi\otimes(1-r_\alpha)w+\phi\otimes r_\alpha\varepsilon_\alpha(w)\right)-w\lambda(\xi)\phi\otimes w.\end{aligned}$$

(Here $\varepsilon_\alpha(w)=-\operatorname{sign}(w^{-1}\alpha)w$.)

If $X\in\mathfrak{h}^{\mathrm{reg}}$ is a W-invariant set then we have a natural action of W on $\mathcal{O}(X)$. When $x\in\mathfrak{h}^{\mathrm{reg}}$ then we will use the notation $\mathcal{O}_{Wx}$ for the "multi-germs" at the orbit Wx, i.e. $\mathcal{O}_{Wx}=\bigoplus_{w\in W}\mathcal{O}_{wx}$. When $\phi\in\mathcal{O}_{Wx}$ we write $p_x(\phi)=\phi_x$ for the projection of ϕ to the direct summand $\mathcal{O}_x$ of $\mathcal{O}_{Wx}$. We define an action π_1 of W on $\mathcal{O}_{Wx}$ by means of the formula $\pi_1(w)\phi=\phi^w=\phi\circ w^{-1}$. Let π_2 denote the action of W on $\mathbf{C}[W]$ by multiplication on the left. We have an action of $W\times W$ on $V_{Wx}=\mathcal{O}_{Wx}\otimes\mathbf{C}[W]$ via $\pi_1\otimes\pi_2$. The restriction of $\pi_1\otimes\pi_2$ to the diagonal subgroup $\Delta\subset W\times W$ is denoted by π, and the subspace of Δ-invariant elements of V_{Wx} is denoted by V_{Wx}^Δ. Let p_x be the projection onto the summand V_x of V_{Wx} with respect to the decomposition $V_{Wx}=\bigoplus_{w\in W}V_{wx}$, and let p_e denote the projection onto the summand $\mathcal{O}_{Wx}\otimes e\simeq\mathcal{O}_{Wx}$ of V_{Wx} with respect to the decomposition $V_{Wx}=\bigoplus_{w\in W}\mathcal{O}_{Wx}\otimes w$. Let $f\in\mathbf{C}[W]$ be the element $f=\sum_{w\in W}w$. Clearly,

$$\pi(f)\colon V_x\xrightarrow{\sim}V_{Wx}^\Delta\quad\text{with inverse }p_x\tag{3.1}$$

and

$$\pi(f)\colon\mathcal{O}_{Wx}\xrightarrow{\sim}V_{Wx}^\Delta\quad\text{with inverse }p_e.\tag{3.2}$$

The next lemma is the key lemma of this section.

LEMMA 3.2. *Let $\Psi\in V_{Wx}^\Delta$ and put $\Phi=p_x\Psi$ and $\psi=p_e\Psi$. Then*

$$\nabla_\xi(\lambda,k)\Psi=\sum_{w\in W}\pi(w)(\nabla_{w^{-1}\xi}\Phi)=\sum_{w\in W}((D_{w^{-1}\xi}-w\lambda(\xi))\psi)^w\otimes w.$$

Proof. The first equality is equivalent to $\pi(w)\circ\nabla_\xi\circ\pi(w^{-1})=\nabla_{w\xi}$ and this is easy using the definition of ∇. For the second equality we first check that

$$\begin{aligned}\pi_1(w)\circ D_{w^{-1}\xi}\circ\pi_1(w^{-1})&=D_\xi+\frac{1}{2}\sum_{\alpha\in R_+}k_\alpha\alpha(\xi)\pi_1(r_\alpha)(1-\operatorname{sign}(w^{-1}\alpha))\\&=\partial_\xi+\frac{1}{2}\sum_{\alpha\in R_+}\left(\left(\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\right)(1-\pi_1(r_\alpha))-\operatorname{sign}(w^{-1}\alpha)\pi_1(r_\alpha)\right)\end{aligned}$$

using Proposition 1.1 (1). Then the assertion follows from Definition 3.1 and the fact that $\pi_1(r_\alpha)=\pi_2(r_\alpha)$ on V_{Wx}^Δ. $\square$

Definition 3.3. Via the algebra $\mathbf{A}$ (see Proposition 2.9) we consider $\mathcal{O}_{Wx}$ as an $\mathbf{H}$-module. Splitting this module according to the action of the center we obtain the following $\mathbf{H}(k)$-modules: $S(\lambda,k)=\{\psi\in\mathcal{O}_{Wx} \mid p(D(k))\psi=p(\lambda)\psi\ \forall p\in S(\mathfrak{h})^W\}$ (here $\lambda\in\mathfrak{h}^*$).

COROLLARY 3.4. *$\forall(\lambda,k)\in\mathfrak{h}\times\mathcal{K}$:*

$$\pi(f)\colon V_x^{\nabla(\lambda,k)} \xrightarrow{\sim} V_{Wx}^{\Delta,\nabla(\lambda,k)} \quad \textit{with inverse } p_x$$

and

$$\pi(f)\colon S(\lambda,k)^\lambda \xrightarrow{\sim} V_{Wx}^{\Delta,\nabla(\lambda,k)} \quad \textit{with inverse } p_e.$$

Proof. Immediate using (3.1), (3.2) and Lemma 3.2. □

At this point it is clear that it is useful to investigate the $\mathbf{H}$-module S.

LEMMA 3.5. *Put $S(\lambda,k)_x^W=\{\phi\in\mathcal{O}_x \mid D_p(k)\phi=p(\lambda)\phi\ \forall p\in S(\mathfrak{h})^W\}$, the local solution space of the system of hypergeometric differential equations. Then*

$$\pi_1(f)\colon S_x^W \xrightarrow{\sim} S^W \quad \textit{with inverse } p_x.$$

Hence generically in the parameters (λ,k) we have that $\dim S(\lambda,k)^W \geqslant |W|$.

Proof. Clear using Theorem 2.12 (2). The statement about the dimension of the local solution space of the hypergeometric system is a basic feature of this system and is proved by substitution of formal power series as in [11, Chapter IV, §5]. (Of course we even know that this dimension equals $|W|$ for all parameter values but we do not need this here (see [10, Corollary 3.9]).) □

COROLLARY 3.6. *Suppose that $\lambda(\alpha^\vee)\neq 0, \pm\left(k_\alpha+\frac{1}{2}k_{\alpha/2}\right)\ \forall\alpha\in R^0$. Then*

$$S(\lambda,k)\simeq I_\lambda^{|W|}.$$

Proof. Put $d=\dim S^\lambda$. By Corollary 3.4 $d\leqslant|W|=\operatorname{rank}(\nabla)$. Note that the integrability of ∇ is equivalent to $d=|W|$. From the conditions on (λ,k) we deduce using Proposition 1.2 (4), (5) that $\forall\psi\in S(\lambda,k)^W\setminus\{0\}$: $\mathbf{H}\cdot\psi\simeq I_\lambda$. Hence $S(\lambda,k)\simeq I_\lambda^{d'}$ where $d'=\dim S(\lambda,k)^W$ (using Proposition 1.2). Hence by Lemma 3.5, $d'=d=|W|$ generically. Thus ∇ is integrable and $d=|W|$ for all (λ,k). □

COROLLARY 3.7. *The KZ-connection is integrable.*

Finally we study the projection $\pi_1(f)\colon S^\lambda\to S^W$ in detail.

Definition 3.8. Let $\mathcal{H}$ denote the harmonic polynomials on $\mathfrak{h}^*$, and define $q\in \mathbf{C}(\mathfrak{h}^*\times\mathcal{K})\otimes\mathcal{H}$ by the condition that $\forall\lambda\in\mathfrak{h}^*$ and $k\in\mathcal{K}$ such that $\lambda(\alpha^\vee)\neq 0, \pm(k_\alpha+\frac{1}{2}k_{\alpha/2})$, $q(\lambda,k)\in\mathcal{H}$ is the unique harmonic element such that

$$q(\lambda,k,w\lambda)=\delta_{w,e}\prod_{\alpha\in R^0_+}\frac{\lambda(\alpha^\vee)}{\lambda(\alpha^\vee)-k_\alpha-\frac{1}{2}k_{\alpha/2}}\quad \forall w\in W.$$

LEMMA 3.9. (1) *The function* $(\lambda,k,\mu)\to\prod_{\alpha\in R^0_+}(\lambda(\alpha^\vee)-(k_\alpha+\frac{1}{2}k_{\alpha/2}))q(\lambda,k,\mu)$ *is a polynomial.*

(2) *Suppose that* $\lambda(\alpha^\vee)\neq 0,\pm(k_\alpha+\frac{1}{2}k_{\alpha/2})$. *Then* $q(w\lambda,k)\colon I^W_\lambda\to I^{w\lambda}_\lambda$ *is the inverse of* $f\colon I^{w\lambda}_\lambda\to I^W_\lambda$ $\forall w\in W$.

Proof. (1) follows from [26, Chapter 4, Exercise 71 (f)] and (2) is immediate using Proposition 1.4. □

Definition 3.10. We define $D(\lambda,k)=D_{q(\lambda,k)}(k)$, i.e. $D(\lambda,k)$ is the differential operator that coincides with $q(\lambda,k,D(k))$ on W-invariant functions.

COROLLARY 3.11. *If* $\lambda(\alpha^\vee)\neq 0,\pm(k_\alpha+\frac{1}{2}k_{\alpha/2})$ $\forall\alpha\in R^0$ *then* $D(\lambda,k)\colon S(\lambda,k)^W\xrightarrow{\sim} S(\lambda,k)^\lambda$ *is an isomorphism, with inverse* $\pi_1(f)$.

Proof. Clear by Corollary 3.6 and Lemma 3.9. □

COROLLARY 3.12 ([19, Theorem 5.4.1] and [3, Theorem 4.7]). *If* $\lambda(\alpha^\vee)\neq k_\alpha+\frac{1}{2}k_{\alpha/2}$ $\forall\alpha\in R^0_+$ *then* $p_e\pi_2(f)\colon V_x^{\nabla(\lambda,k)}\xrightarrow{\sim} S(\lambda,k)^W_x$ *is an isomorphism. Its inverse is given by* $\phi\to\bar{D}(\lambda,k)\phi=\sum_{w\in W}D^w(\lambda,k)\phi\otimes w$.

Proof. Since $\dim S(\lambda,k)^W_x=\dim V_x^{\nabla(\lambda,k)}=|W|$ $\forall(\lambda,k)$ (so now we use [10, Corollary 3.9]) it is sufficient to show that $p_e\pi_2(f)$ maps $V_x^{\nabla(\lambda,k)}$ to $S(\lambda,k)^W_x$ and that $\bar{D}(\lambda,k)p_e\pi_2(f)=\mathrm{id}$ on $V_x^{\nabla(\lambda,k)}$. Since $\bar{D}(\lambda,k)$ is holomorphic outside the hyperplanes $\lambda(\alpha^\vee)=k_\alpha+\frac{1}{2}k_{\alpha/2}$ $(\alpha\in R^0_+)$ (Lemma 3.9 (1)) and $V_x^{\nabla(\lambda,k)}$ depends holomorphically on the parameters (λ,k) it suffices to show this for generic parameters. But then it is clear from the previous results and the observation that $p_e\pi_2(f)=p_x\pi_1(f)p_e\pi(f)$ and $\bar{D}(\lambda,k)=p_x\pi(f)D(\lambda,k)\pi_1(f)$. □

Now we are in the position to give a precise statement about the relation between eigenfunctions of the D_ξ which are analytic on $\mathfrak{a}$ and hypergeometric functions.

Definition 3.13. For each irreducible representation δ of W, let d_δ be the lowest embedding degree of δ in $\mathbf{C}[\mathfrak{h}]$, and let $\varepsilon_\delta(k)=\sum_{\alpha\in R_+}k_\alpha(1-\chi_\delta(r_\alpha)/\chi_\delta(e))$. (Here χ denotes the character.) Let $\mathcal{K}_+=\{k\in\mathcal{K}\mid \mathrm{Re}(\varepsilon_\delta(k))+d_\delta>0\ \forall\delta\in\widehat{W},\delta\neq\mathrm{triv}\}$. Note that $\mathcal{K}_+$ is an open neighbourhood of $\{k\in\mathcal{K}\mid \mathrm{Re}(k_\alpha)\geqslant 0\ \forall\alpha\in R\}$.

LEMMA 3.14. *If $k\in\mathcal{K}_+$ and ϕ is holomorphic in a neighbourhood of $\mathfrak{a}$ and a nonzero solution of the eigenfunction equations $D_\xi(k)\phi=\lambda(\xi)\phi$ $\forall\xi$ then $\phi(0)\neq 0$.*

Proof. Let $\{\xi_i\}$ denote an orthonormal basis of $\mathfrak{a}$ and let $\{\xi_i^*\}$ be the dual basis. The lowest homogeneous part of the operator $\sum_{i=1}^n \xi_i^* D_{\xi_i}(k)$ at the origin is equal to $E(k)=\sum_{i=1}^n \xi_i^*\partial_{\xi_i}+\sum_{\alpha\in R_+} k_\alpha(1-r_\alpha)$. The element $\sum_{\alpha\in R_+} k_\alpha(1-r_\alpha)$ of the group algebra of W is central. Hence it acts on an irreducible representation δ of W by scalar multiplication, and it is easy to see that this scalar is in fact equal to $\varepsilon_\delta(k)$. By the definition of $\mathcal{K}_+$ the operator $E(k)$ has no polynomials in its kernel other than the constants when $k\in\mathcal{K}_+$. □

THEOREM 3.15. *There exists an open neighbourhood U of $0\in\mathfrak{a}$ such that there exists a holomorphic function G on $\mathfrak{h}^*\times\mathcal{K}_+\times(\mathfrak{a}+iU)$ with the following properties:*

(1) $G(\lambda,k,0)=1$,

(2) $\forall\xi\in\mathfrak{h}$: $D_\xi(k)G(\lambda,k)=\lambda(\xi)G(\lambda,k)$.

These properties determine G completely. G can be continued meromorphically to $\mathfrak{h}^*\times\mathcal{K}\times(\mathfrak{a}+iU)$. It can be expressed in terms of the hypergeometric function F as follows: $G(\lambda,k)=|W|D(\lambda,k)F(\lambda,k)$.

Proof. We will use the following two properties of the hypergeometric function F. First of all, $F(\lambda,k,0)=1$ (cf. [22, Theorem 6.1]). Secondly, there exists an entire function f on $\mathcal{K}$ such that the function $(\lambda,k,x)\to f(k)F(\lambda,k,x)$ is holomorphic in $(\lambda,k,x)\in\mathfrak{h}^*\times\mathcal{K}\times(\mathfrak{a}+iU)$ (cf. [22, Proposition 3.8]). Define G by $G(\lambda,k)=|W|D(\lambda,k)F(\lambda,k)$. Let us show that G has the asserted properties. By Corollary 3.11 we know that (2) is satisfied and that $|W|^{-1}\sum_{w\in W}G^w=F$. Hence $G(\lambda,k,0)=F(\lambda,k,0)=1$, which proves that property (1) holds. Let us now prove that G is holomorphic in $\mathfrak{h}^*\times\mathcal{K}_+\times(\mathfrak{a}+iU)$. From Lemma 3.9 and the second property of F mentioned above we see that the function G is meromorphic and that its singular set is the zero set of a function that depends on (λ,k) only. Let $S\times(\mathfrak{a}+iU)$ denote this singular set. Suppose that $S\cap(\mathfrak{h}^*\times\mathcal{K}_+)\neq\varnothing$. Choose a regular element (λ_0,k_0) of S, and let ϕ be an irreducible holomorphic function in a neighbourhood V of (λ_0,k_0) such that $V\cap S=\{\phi=0\}$. Let $l\in\mathbf{N}$ be the smallest integer such that $\widetilde{G}=\phi^l G$ extends holomorphically to $V\times(\mathfrak{a}+iU)$. Then $\widetilde{G}(\lambda,k,0)=0$ $\forall(\lambda,k)\in V\cap S$, and hence also $\widetilde{G}(\lambda,k,x)=0$ $\forall(\lambda,k,x)\in(V\cap S)\times(\mathfrak{a}+iU)$ by Lemma 3.14. This contradicts the minimality of l. The uniqueness assertion is a consequence of the fact that F is the unique holomorphic solution of the hypergeometric equations in a neighbourhood of 0 with $F(0)=1$, combined with Corollary 3.11. □

Let us consider the general rank one case now, i.e. the case where R is of type BC_1. If α denotes the linear functional on $\mathfrak{a}\simeq\mathbf{R}$ that plays the role of the simple root of R then $R_+=\{\alpha,2\alpha\}$ and $R=\{\pm\alpha,\pm2\alpha\}$. The lattice $Q^\vee$ is the $\mathbf{Z}$-span of the vector $(2\alpha)^\vee$. The equation on $\mathfrak{h}\simeq\mathbf{C}$ for the hypergeometric function associated with BC_1 is symmetric with respect to translations in the lattice $2\pi i Q^\vee$ and with respect to multiplication by -1 (cf. [10, §4]). The quotient of $\mathfrak{h}$ with respect to these symmetries is isomorphic to $\mathbf{C}$, and we can take $z=\frac{1}{2}-\frac{1}{4}(e^\alpha+e^{-\alpha})$ as a coordinate on this quotient space. In this way we find that the hypergeometric function F_{BC_1} that is associated with BC_1 compares to the classical hypergeometric function in the following way:

$$F_{\mathrm{BC}_1}(\lambda,k,x)=F(a,b,c;z(x)),$$

where the relations between the parameters (a,b,c) and (λ,k) are given by

$$\begin{aligned} a&=\tfrac{1}{2}(\lambda+\varrho(k))(\alpha^\vee),\\ b&=\tfrac{1}{2}(-\lambda+\varrho(k))(\alpha^\vee),\\ c&=\tfrac{1}{2}+k_\alpha+k_{2\alpha}. \end{aligned}$$

It is not difficult to see that

$$q(\lambda,k,\mu)=\frac{\lambda(\alpha^\vee)}{2\lambda(\alpha^\vee)-2k_\alpha-4k_{2\alpha}}+\frac{\mu(\alpha^\vee)}{2\lambda(\alpha^\vee)-2k_\alpha-4k_{2\alpha}}.$$

Hence the operator $D(\lambda,k)$ is given by the formula

$$D(\lambda,k)=\frac{1}{2}+\frac{1}{2(\lambda(\alpha^\vee)-k_\alpha-2k_{2\alpha})}\,\partial_{\alpha^\vee}=\frac{1}{2}-\frac{1}{2b}\cdot\frac{d}{d\alpha}.$$

Now we can find the function $G(\lambda,k)$ by application of $2D(\lambda,k)$ to $F_{\mathrm{BC}_1}(\lambda,k)$. If we use the above relation as well we obtain the following formula:

$$G(\lambda,k,x)=F(a,b,c;z(x))+\frac{1}{2b}\sinh(\alpha(x))F'(a,b,c;z(x)).$$

4. Invariant Hermitean structures

In this section we study two different $*$-structures for the graded Hecke algebra and a family of irreducible unitary modules for each of these.

Let R be a reduced integral root system and let $k\in\mathcal{K}$ be a real multiplicity function. Fix a positive system $R_+\subset R$. The first $*$-structure "$+$" that we consider on $\mathbf{H}(R_+,k)$ is simply defined by $\xi^+=\bar{\xi}$ and $w^+=w^{-1}$. One easily checks that this extends uniquely to an anti-linear anti-involution of $\mathbf{H}$. Let $\lambda\in\overline{\mathfrak{a}^*_+}$. Recall the notations of Definition 2.4. Let $\mathbf{P}_\lambda\subset\mathbf{H}$ denote the "parabolic subalgebra" generated by $\xi\in\mathfrak{h}$ and $w\in W_\lambda$. Hence $\mathbf{P}_\lambda=S(\mathfrak{h})\otimes\mathbf{C}[W_\lambda]$ (as a vector space). Denote by $\mathbf{C}_{\tilde{\lambda}}$ the 1-dimensional $\mathbf{P}_\lambda$-module defined by (see Proposition 2.10 for the definition of $\tilde{\lambda}$)

$$\begin{aligned} \xi\cdot 1&=\tilde{\lambda}(\xi) \quad &\forall\lambda\in\mathfrak{h},\\ r_i\cdot 1&=1 &\forall r_i\in W_\lambda. \end{aligned}$$

(One easily checks that this defines a $\mathbf{P}_\lambda$-module.) Finally we define an $\mathbf{H}$-module $V_{\tilde\lambda}$ by

$$V_{\tilde\lambda} = \mathrm{Ind}_{\mathbf{P}_\lambda}^{\mathbf{H}}(\mathbf{C}_{\tilde\lambda}) = \mathbf{H} \otimes_{\mathbf{P}_\lambda} \mathbf{C}_{\tilde\lambda}.$$

Clearly $V_{\tilde\lambda} = \mathbf{C}[W/W_\lambda]$ as a W-module.

THEOREM 4.1. *Let $k_\alpha \geqslant 0$ $\forall \alpha \in R$, and $\lambda \in \overline{\mathfrak{a}_+^*}$.*

(1) $V_{\tilde\lambda}^\mu \neq \{0\} \Leftrightarrow \exists w \in W^\lambda$ *such that* $\mu = w\tilde\lambda\,(=(w\lambda)^{\sim})$. *If* $w \in W^\lambda$ *then* $\dim_{\mathbf{C}}(V_{\tilde\lambda}^{w\tilde\lambda}) = 1$.

(2) $V_{\tilde\lambda}$ *is irreducible.*

(3) *There exists a unique basis* $\{v_w\}_{w\in W^\lambda}$ *of* $V_{\tilde\lambda}$ *such that*

(i) $v_e = 1 \otimes 1$,

(ii) $r_i v_w = \dfrac{w\tilde\lambda(\alpha_i^\vee)+k_i}{w\tilde\lambda(\alpha_i^\vee)} v_{r_i w} - \dfrac{k_i}{w\tilde\lambda(\alpha_i^\vee)} v_w \quad \forall i = 1, 2, \dots, n,$

(iii) $\xi v_w = w\tilde\lambda(\xi) v_w \quad \forall \xi \in \mathfrak{h}$.

(*Here we tacitly used the notation $v_w = 0$ if $w \notin W^\lambda$.*)

(4) *There exists a positive definite Hermitean form $(\cdot,\cdot)$ on $V_{\tilde\lambda}$ such that $\xi^* = \bar\xi$ and $w^* = w^{-1}$ in $V_{\tilde\lambda}$. This form is unique up to scaling. In terms of the basis $\{v_w\}_{w\in W^\lambda}$ one has ($a = a(\lambda, k) \in \mathbf{R}_+$ is a scaling factor):*

$$(v_w, v_{w'}) = \frac{a(\lambda,k)\delta_{w,w'}}{\prod_{\alpha\in R_+}(1-k_\alpha/w\tilde\lambda(\alpha^\vee))}.$$

(5) *Define $\forall w' \in W^\lambda$: $\phi_{w'} = |W|^{-1}\sum_{w\in W} w v_{w'}$. Then $\phi_{w'} = \phi$ is independent of $w' \in W^\lambda$ and spans $V_{\tilde\lambda}^W$. Moreover,*

$$\phi = \sum_{w\in W^\lambda} b_w v_w$$

where

$$b_w = |W|^{-1} \prod_{\alpha\in R_+} \left(1 - \frac{k_\alpha}{w\tilde\lambda(\alpha^\vee)}\right). \tag{4.1}$$

(6) $(wv_e, w'w_0 v_{w_0 w_\lambda}) = |W_\lambda|^{-1} a(\lambda,k)\delta_{wW_\lambda, w'W_\lambda}$ $\forall w, w' \in W$.

(7) $\|\phi\|^2 = |W|^{-1} a(\lambda, k)$.

(8) *If λ is regular then*

$$\|\phi_e^-\|^2 = |W|^{-1} a(\lambda,k) \prod_{\alpha\in R_+} \frac{\tilde\lambda(\alpha^\vee)+k_\alpha}{\tilde\lambda(\alpha^\vee)-k_\alpha}.$$

(ϕ_e^- *was defined in Proposition* 1.4 (4) *as an element of* $I_{\tilde\lambda}$, *and* $I_{\tilde\lambda} = V_{\tilde\lambda}$ *if λ is regular.*)

Proof. (1) Let w_λ denote the longest element in W_λ. Recall that $\tilde{\lambda}=w_\lambda(\lambda+\varrho)$ (cf. Proposition 2.10). Hence

$$\begin{aligned} &\tilde{\lambda}(\alpha^\vee)>k_\alpha \quad \text{if } \alpha\in R_+\backslash R_{\lambda,+}, \\ &\tilde{\lambda}(\alpha^\vee)\leqslant -k_\alpha \quad \text{if } \alpha\in R_{\lambda,+}. \end{aligned} \tag{4.2}$$

Note that in particular $\tilde{\lambda}(\alpha^\vee)=-k_\alpha \Leftrightarrow \alpha\in R_{\lambda,+}$ and simple. Let $w\in W^\lambda$, and choose r_{i_1} such that $l(r_{i_1}w)<l(w)$. Thus $w^{-1}(\alpha_{i_1})<0$, implying that $w(\alpha)\neq\alpha_{i_1}$ $\forall\alpha\in R_{\lambda,+}$. Hence $r_{i_1}w\in W^\lambda$. Now suppose that $V_{\tilde{\lambda}}^{r_{i_1}w\tilde{\lambda}}\neq\{0\}$, and choose $0\neq v_{r_{i_1}w}\in V_{\tilde{\lambda}}^{r_{i_1}w\tilde{\lambda}}$. Observe that $r_{i_1}w\tilde{\lambda}(\alpha_{i_1}^\vee)=\tilde{\lambda}(-w^{-1}\alpha_{i_1}^\vee)>k_{i_1}$ since $-w^{-1}\alpha_{i_1}\in R_+\cap w^{-1}R_-\subset R_+\backslash R_{\lambda,+}$ (use (4.2)). This implies that the element v_w defined by

$$v_w=\frac{r_{i_1}w\tilde{\lambda}(\alpha_{i_1}^\vee)}{r_{i_1}w\tilde{\lambda}(\alpha_{i_1}^\vee)+k_{i_1}}\left(r_{i_1}+\frac{k_{i_1}}{r_{i_1}w\tilde{\lambda}(\alpha_{i_1}^\vee)}\right)v_{r_{i_1}w} \tag{4.3}$$

is nonzero and it follows from Proposition 1.2 (2) that $v_w\in V_{\tilde{\lambda}}^{w\tilde{\lambda}}$. By induction on the length of w we conclude that $V_{\tilde{\lambda}}^{w\tilde{\lambda}}\neq 0$ $\forall w\in W^\lambda$. On the other hand we know that the dimension of $V_{\tilde{\lambda}}$ equals $|W^\lambda|$. This proves (1). In order to prove (3) we first of all note that the uniqueness of such a basis is immediately clear from (3)(i) and (3)(ii). Next we define the basis elements v_w $(w\in W^\lambda)$ by induction on the length of w using (4.3) and the initial value $v_e=1\otimes 1$. We have to show that this basis satisfies (3)(ii). If $r_iw>w$ and $r_iw, w\in W^\lambda$ then (3)(ii) is a restatement of (4.3). If $r_iw, w\in W^\lambda$ but $r_iw<w$, (3)(ii) follows by applying r_i to both sides of the first case. Finally, if $w\in W^\lambda$ but $r_iw\notin W^\lambda$ then $\alpha_i=w(\alpha_j)$ for a certain $\alpha_j\in R_\lambda$ simple, so that $(r_i+k_i/w\tilde{\lambda}(\alpha_i^\vee))v_w=(r_i-1)v_w=0$ or equivalently, $r_iv_w=v_w$, in accordance with (3)(ii). This proves (3). As to (2), observe that any nontrivial submodule of $V_{\tilde{\lambda}}$ must contain at least one of the elements v_w $(w\in W_\lambda)$. Using (3)(ii) we see that this immediately implies that the submodule coincides with $V_{\tilde{\lambda}}$, proving (2).

(4) Take $a(\lambda,k)=1$. Define a Hermitean form by

$$(v_w,v'_w)=\delta_{w,w'}\prod_{\alpha\in R_+}\left(1-\frac{k_\alpha}{w\tilde{\lambda}(\alpha^\vee)}\right)^{-1}.$$

Clearly $\xi^*=\bar{\xi}$ with respect to this form. In order to prove $r_j^*=r_j$ $\forall j=1,2,\dots,n$ we restrict our attention to the 2-dimensional subspace $V_{\tilde{\lambda}}^{w\tilde{\lambda}}\oplus V_{\tilde{\lambda}}^{r_jw\tilde{\lambda}}$ if $w, r_jw\in W^\lambda$, and to $V_{\tilde{\lambda}}^{w\tilde{\lambda}}$ if $w\in W^\lambda, r_jw\notin W^\lambda$. The second case is trivial since $r_j|V_{\tilde{\lambda}}^{w\tilde{\lambda}}=+1$ in this situation. Let us consider the first case. It is sufficient to show that the $+1$ and -1 eigenspaces of r_j in

$V_{\tilde{\lambda}}^{w\tilde{\lambda}} \oplus V_{\tilde{\lambda}}^{r_j w\tilde{\lambda}}$ are perpendicular:

$$
\begin{aligned}
&(v_w + r_j v_w, v_w - r_j v_w)\\
&= \left(\left(1+\frac{k_j}{w\tilde{\lambda}(\alpha_j^\vee)}\right)v_{r_jw} + \left(1-\frac{k_j}{w\tilde{\lambda}(\alpha_j^\vee)}\right)v_w, -\left(1+\frac{k_j}{w\tilde{\lambda}(\alpha_j^\vee)}\right)v_{r_jw} + \left(1+\frac{k_j}{w\tilde{\lambda}(\alpha_j^\vee)}\right)v_w\right)\\
&= -\left(1-\frac{k_j}{r_jw\tilde{\lambda}(\alpha_j^\vee)}\right)^2 \|v_{r_jw}\|^2 + \left(1-\frac{k_j}{r_jw\tilde{\lambda}(\alpha_j^\vee)}\right)\left(1-\frac{k_j}{w\tilde{\lambda}(\alpha_j^\vee)}\right)\|v_w\|^2 = 0.
\end{aligned}
$$

(5) Similar to Proposition 1.4 (3), except for the determination of the coefficients b_w. We argue as follows. Using (3) one easily checks that $\sum_{w\in W^\lambda} b_w v_w \in V_{\tilde{\lambda}}^W$. Observe that (3)(ii) implies that application of $w\in W$ does not alter the sum of the coefficients with respect to the basis $\{v_w\}_{w\in W^\lambda}$. Thus because $V_{\tilde{\lambda}}^W$ has dimension 1 all we need to show is that $\sum_{w\in W^\lambda} b_w = 1$. Indeed,

$$\sum_{w\in W^\lambda} b_w = |W|^{-1} \sum_{w\in W^\lambda} \prod_{\alpha\in R_+} \left(1-\frac{k_\alpha}{w\tilde{\lambda}(\alpha^\vee)}\right) = |W|^{-1} \sum_{w\in W} \prod_{\alpha\in R_+} \left(1-\frac{k_\alpha}{w\tilde{\lambda}(\alpha^\vee)}\right)$$

(since we have $w\notin W^\lambda \Rightarrow \exists \alpha_i \in R_\lambda$ such that $-w\alpha_i \in R_+ \Rightarrow \exists \alpha\in R_+$ such that $w\tilde{\lambda}(\alpha^\vee) = -\tilde{\lambda}(\alpha_i^\vee) = k_\alpha$). Hence

$$\sum_{w\in W^\lambda} b_w = |W|^{-1} \prod_{\alpha\in R_+} \tilde{\lambda}(\alpha^\vee)^{-1} \sum_{w\in W} (-1)^{l(w)} \prod_{\alpha\in R_+} (w\tilde{\lambda}(\alpha^\vee) - k_\alpha) = 1.$$

(6) We show that $(wv_e, v_{w_0w_\lambda}) = |W_\lambda|^{-1} a\delta_{wW_\lambda, w_0w_\lambda W_\lambda}$ (note that $w_0w_\lambda \in W^\lambda$ is the longest element). We may assume that $w\in W^\lambda$, and repeated application of (3) shows that if $w\in W^\lambda$ then

$$wv_e = \prod_{\alpha\in R_+\cap w^{-1}R_-} \left(1+\frac{k_\alpha}{\tilde{\lambda}(\alpha^\vee)}\right) v_w + \sum_{w'<w} c_{w'} v_{w'}.$$

Hence

$$
\begin{aligned}
(wv_e, v_{w_0w_\lambda}) &= a\delta_{w,w_0w_\lambda} \frac{\prod_{\alpha\in R_+\setminus R_{\lambda,+}} (1+k_\alpha/\tilde{\lambda}(\alpha^\vee))}{\prod_{\alpha\in R_+}(1-k_\alpha/w_0w_\lambda\tilde{\lambda}(\alpha^\vee))}\\
&= \frac{a\delta_{w,w_0w_\lambda}}{\prod_{\alpha\in R_{\lambda,+}}(1+k_\alpha/\varrho(k)(\alpha^\vee))}\\
&= \frac{a\delta_{w,w_0w_\lambda}}{|W_\lambda|}.
\end{aligned}
$$

In the last equality we used the well-known identity

$$|W_\lambda| = \prod_{\alpha\in R_{\lambda,+}} \left(1+\frac{k_\alpha}{\varrho(k)(\alpha^\vee)}\right). \tag{4.4}$$

(This formula simply follows from identity 2.8 of Macdonald's paper [16] when we take $u_\alpha = e^{k_\alpha}$ and evaluate at $\varrho(k)$.)

(7) $(\phi,\phi) = |W|^{-2}\sum_{w,w'\in W}(wv_e, w'w_0 v_{w_0 w_\lambda}) = |W|^{-1}a$.

(8) Similar to the proof of (7) we have $(\phi_e^-, \phi_{w_0}^-) = |W|^{-1}a$. Now use Proposition 1.4(4). □

The second $*$-structure on $\mathbf{H}$ we will investigate is given by $w^* = w^{-1}$ $(\forall w\in W)$ and $\xi^* = -w_0\cdot\overline{w_0(\xi)}\cdot w_0$ $(\forall \xi\in\mathfrak{h})$. Again it is easy to check that this can be extended to $\mathbf{H}$ as an anti-linear anti-involution (provided $k\in\mathbf{R}$).

THEOREM 4.2. *Let $k_\alpha\in\mathbf{R}$ and $\lambda\in i\mathfrak{a}^*$.*

(1) *Define a positive definite Hermitean form $(\cdot,\cdot)$ on I_λ by means of*

$$(w_1 v_e, w_2 v_e) = \delta_{w_1,w_2}.$$

Then $(\cdot,\cdot)$ is invariant with respect to $*$.

(2) *If $\phi\in I_\lambda^W$ is the spherical vector $\phi = |W|^{-1}\sum_{w\in W} wv_e$ then $\|\phi\|^2 = |W|^{-1}$.*

(3) *If $\lambda\in i\mathfrak{a}^{*,\mathrm{reg}}$ then $(w_1 v_{w_3}, w_2 v_{w_3}) = \delta_{w_1,w_2}$ $\forall w_1,w_2,w_3\in W$.*

(4) *If $\lambda\in i\mathfrak{a}^{*,\mathrm{reg}}$ then*

$$(v_{w_1}, w_0 v_{w_0 w_2}) = \frac{\delta_{w_1,w_2}}{\prod_{\alpha\in R_+}(1-k_\alpha/w_1\lambda(\alpha^\vee))}.$$

Proof. (1) The unitarity of $w\in W$ is immediate from the definition. Thus we need to show that

$$(\xi w_1 v_e, w_2 v_e) = (w_1 v_e, \xi^* w_2 v_e) \quad \forall w_1,w_2\in W,\ \xi\in\mathfrak{a}. \tag{4.5}$$

By the conjugation formula Proposition 1.1(1) we have

$$\xi w_1 = w_1\cdot w_1^{-1}(\xi) + \sum_{\{\alpha\in R_+\,|\,w_1 r_\alpha < w_1\}} k_\alpha w_1\alpha(\xi) w_1 r_\alpha \tag{4.6}$$

and

$$\xi^* w_2 = -w_0\cdot w_0(\xi)\cdot w_0 w_2 = -w_2\cdot w_2^{-1}(\xi) - \sum_{\{\alpha\in R_+\,|\,w_2 r_\alpha > w_2\}} k_\alpha w_2\alpha(\xi) w_2 r_\alpha. \tag{4.7}$$

By these formulas it follows immediately that (4.5) holds if $w_1 = w_2$ or if $w_1\neq w_2$ and $w_1 r_\beta \neq w_2$ $\forall\beta\in R$. So let us assume that $\exists\beta\in R_+$: $w_1 r_\beta = w_2$. In this case (4.5) reduces to (using (4.6) and (4.7))

$$\sum_{\{\alpha\in R_+\,|\,w_1 r_\alpha < w_1\}} k_\alpha w_1\alpha(\xi)\delta_{w_1 r_\alpha, w_2} = -\sum_{\{\alpha\in R_+\,|\,w_2 r_\alpha > w_2\}} k_\alpha w_2\alpha(\xi)\delta_{w_1, w_2 r_\alpha}. \tag{4.8}$$

But $w_1 r_\alpha = w_2 \Leftrightarrow \alpha=\beta$ and also $w_1 = w_2 r_\alpha \Leftrightarrow \alpha=\beta$. Therefore, if $w_1 r_\beta > w_1$, then $w_2 r_\beta < w_2$ and we see that both sides of (4.8) are equal to 0. If on the other hand $w_1 r_\beta < w_1$ then $w_2 r_\beta > w_2$. Since $w_2\beta = w_1 r_\beta \beta = -w_1\beta$ both sides of (4.8) reduce to $k_\beta w_1 \beta(\xi)$ in this case. This proves (1).

(2) Immediate from the definitions.

(3) By the universal property of I_λ we may define a morphism $m: I_\lambda \to I_{w_3\lambda}$ by $m(v_e) = v_{w_3^{-1}}$. By Proposition 1.4 we see that then $m(v_w) = v_{ww_3^{-1}}$ $\forall w \in W$, so in particular $m(v_{w_3}) = v_e$. Hence $(w_1 v_{w_3}, w_2 v_{w_3}) = \delta_{w_1,w_2}$ also defines a $*$-invariant form on I_λ, which must be equal to the form defined in (1) up to multiplication by a constant c since I_λ is irreducible (Proposition 1.2). But by Proposition 1.4 we have $\phi = |W|^{-1}\sum_{w\in W} wv_e = |W|^{-1}\sum_{w\in W} wv_{w_3}$, so that the norm of ϕ is the same with respect to both forms. Hence $c=1$ and (3) is proved.

(4) If $w_1 \neq w_2$ then we see that $(v_{w_1}, w_0 v_{w_0 w_2}) = 0$ using the fact that

$$(\xi v_{w_1}, w_0 v_{w_0 w_2}) = (v_{w_1}, \xi^* w_0 v_{w_0 w_2}).$$

By repeated application of (1.8) we see that

$$v_{w_0 w_1} = \prod_{\alpha\in R_+}\left(1+\frac{k_\alpha}{w_1\lambda(\alpha^\vee)}\right)^{-1} w_0 v_{w_1} + \sum_{w\neq w_0} b_w w v_{w_1} \tag{4.9}$$

for certain constants b_w. Hence we can evaluate $(v_{w_1}, w_0 v_{w_0 w_1})$ using (3) and (4.9), proving (4). $\square$

5. Harmonic analysis on T

Let T be the torus $T = i\mathfrak{a}/2\pi i Q^\vee$. Let $k \in \mathcal{K}$ be such that $k^0_\alpha = k_\alpha + \frac{1}{2}k_{\alpha/2} \geqslant 0$ $\forall \alpha \in R^0$. Clearly, we obtain an action of $\mathbf{H} = \mathbf{H}(R^0_+, k^0)$ (the graded Hecke algebra associated with $R^0_+ \subset R$ and multiplicity k^0) on $\mathbf{C}[P]$ via the operators $D_\xi(k)$ and the action of W. By Proposition 2.3 we know that the inner product

$$(f,g)_k = \int_T f(t)\overline{g(t)}|\delta_k(t)|\,dt = \int_T f(t)\overline{g(t)}\prod_{\alpha\in R}(1-t^{-\alpha})^{k_\alpha}\,dt$$

is invariant with respect to the $+$-structure on $\mathbf{H}$.

LEMMA 5.1. (1) *Let $\lambda \in P_+$. The subspace $\langle E(w\lambda, k)\rangle_{w\in W} \subset \mathbf{C}[P]$ is an $\mathbf{H}$-submodule.*

(2) *In fact, $\langle E(w\lambda,k)\rangle_{w\in W} \simeq V_{\tilde\lambda}$ as an $\mathbf{H}$-module. Fix an $\mathbf{H}$-module morphism*

$$j: V_{\tilde\lambda} \to \langle E(w\lambda,k)\rangle_{w\in W}$$

by $j(v_e)=E(\lambda,k)$. *Then* j *is an isomorphism.*

(3) $E(w\lambda,k)=\prod_{\alpha\in R_+\cap w^{-1}R_-}\left(\dfrac{\tilde\lambda(\alpha^\vee)+\frac12 k_{\alpha/2}+k_\alpha}{\tilde\lambda(\alpha^\vee)+\frac12 k_{\alpha/2}}\right)j(v_w)$ $(\forall w\in W^\lambda)$.

(4) $P(\lambda,k)=|W^\lambda|j(\phi)$.

Proof. (1) and (2). It is clear that $\langle E(w\lambda,k)\rangle_{w\in W}$ is an $\mathbf{H}$-submodule, since the center of $\mathbf{H}$ acts on this space by means of the homomorphism $\chi_\lambda: p(D_\xi(k))\to p(\lambda)$ and $\chi_\lambda=\chi_\mu \Leftrightarrow \lambda\in W\cdot\mu$. We showed that $E(\lambda,k)$ is W_λ invariant in Theorem 2.12 (1). Hence $j: V_{\tilde\lambda}\to\mathbf{H}\cdot E(\lambda,k)$ defined by $h\cdot v_e\to h\cdot E(\lambda,k)$ is a well-defined homomorphism. But $V_{\tilde\lambda}$ is irreducible and has dimension $|W^\lambda|$ so (2) follows.

(3) It is easy to verify that

$$E(r_iw\lambda)=\left(r_i+\frac{k_{\alpha_i}+\frac12 k_{\alpha_i/2}}{(w\lambda)^{\tilde{}}(\alpha_i^\vee)}\right)E(w\lambda)\quad\text{if } r_iw\lambda>w\lambda \text{ and } w\in W^\lambda.$$

Comparing this to Theorem 4.1 (3) and using the trivial formula

$$\prod_{\alpha\in R_+^0\cap w^{-1}R_-^0}\left(\frac{\tilde\lambda(\alpha^\vee)+k_\alpha+\frac12 k_{\alpha/2}}{\tilde\lambda(\alpha^\vee)}\right)=\prod_{\alpha\in R_+\cap w^{-1}R_-}\left(\frac{\tilde\lambda(\alpha^\vee)+\frac12 k_{\alpha/2}+k_\alpha}{\tilde\lambda(\alpha^\vee)+\frac12 k_{\alpha/2}}\right)$$

gives the result.

(4) Note that $P(\lambda,k)=|W_\lambda|^{-1}\sum_{w\in W}w\cdot E(\lambda,k)$ (see Theorem 2.12 (1)). □

Lemma 5.1 makes it possible to use the results of the previous section for the purpose of solving the spectral problem for the operators $D_\xi(k)$ on T. The formulation of the results is short and elegant when one uses the following generalizations of Harish-Chandra's c-function. Let

$$\delta_w(\alpha)=\begin{cases}0 & \text{if } w(\alpha)>0,\\ 1 & \text{if } w(\alpha)<0\ (\alpha\in R_+, w\in W).\end{cases}$$

Define

$$c_w^*(\lambda,k)=\prod_{\alpha\in R_+}\frac{\Gamma(-\lambda(\alpha^\vee)-\frac12 k_{\alpha/2}-k_\alpha+\delta_w(\alpha))}{\Gamma(-\lambda(\alpha^\vee)-\frac12 k_{\alpha/2}+\delta_w(\alpha))}$$

and

$$\tilde c_w(\lambda,k)=\prod_{\alpha\in R_+}\frac{\Gamma(\lambda(\alpha^\vee)+\frac12 k_{\alpha/2}+\delta_w(\alpha))}{\Gamma(\lambda(\alpha^\vee)+\frac12 k_{\alpha/2}+k_\alpha+\delta_w(\alpha))}.$$

So $\tilde c(\lambda,k)=\tilde c_e(\lambda,k)$ and $c^*(\lambda,k)=c_{w_0}^*(\lambda,k)$.

Let us make some general remarks about the solution of the spectral problem to be presented here, before we go into details. By Lemma 5.1 we already see that $\mathbf{C}[P]$

decomposes into an orthogonal direct sum of the mutually inequivalent irreducible W-spherical $\mathbf{H}$-submodules $j(V_{\tilde{\lambda}})$ $(\lambda\in P_+)$. This reduces our task to the determination of the normalization constant $a(\lambda,k)$ that was introduced in Theorem 4.1 (4), in such a way that j becomes an isometry. This problem will be solved by a simple inductive procedure. Although we could have determined this constant also by referring to the results of [20] for the W-invariant polynomials, we preferred to include this inductive procedure because it results in nicer proofs and a better understanding of the nature of Macdonald's conjectures [17]. More precisely, the inductive procedure describes how the closed formulas for the L_2-norms and the values at the identity of the orthogonal polynomials $E(\lambda,k)$ arise from a repeated use of the structure of $V_{\tilde{\lambda}}$ as an irreducible W-spherical unitary $(\mathbf{H},+)$-module. The formulas describing these stuctures for $V_{\tilde{\lambda}}$ were given in Theorem 4.1.

The next lemma plays a pivotal role in the induction step.

LEMMA 5.2. *Recall the definition of* $\phi_e^-\in I_\lambda$ *from Proposition* 1.4 (4). *We have*

$$j(\phi_e^-(\lambda+\delta,k))=|W_\lambda|^{-1}\Delta j(\phi(\lambda,k+1)) \tag{5.1}$$

(*where* Δ *denotes the Weyl denominator* (1.3) *and* 1 *is the multiplicity defined by* $1_\alpha=1$ *if* $\alpha\in R^0$ *and* $1_\alpha=0$ *else*). *Consequently,*

$$a(\lambda,k+1)=|W_\lambda|^2a(\lambda+\delta,k)\prod_{\alpha\in R_+^0}\frac{(\lambda+\varrho+\delta)(\alpha^\vee)+k_\alpha+\frac12k_{\alpha/2}}{(\lambda+\varrho+\delta)(\alpha^\vee)-k_\alpha-\frac12k_{\alpha/2}}.$$

Proof. The first assertion follows directly from the divisibility of skew polynomials by Δ and the definition of the $E(\lambda,k)$ using orthogonality. The second assertion follows from the first by Theorem 4.1 (7) and (8). □

THEOREM 5.3. *Let* $w\in W^\lambda$ *and let* w_λ *denote the longest element of* W_λ. *Let* $\lambda\in P_+$.

(1) $\|E(w\lambda,k)\|_k^2=\dfrac{c^*_{ww_\lambda}(-(\lambda+\varrho),k)}{\tilde{c}_{ww_\lambda}(\lambda+\varrho,k)}$.

(2) $E(w\lambda,k,e)=\dfrac{\tilde{c}_{w_0}(\varrho,k)}{\tilde{c}_{ww_\lambda}(\lambda+\varrho,k)}$.

Proof. (1). We may assume that R is connected. It is sufficient to prove the statement when $k_\alpha\in\mathbf{Z}_+$ $\forall\alpha\in R$. First of all we claim that (1) is equivalent to the statement that the embedding of $V_{\tilde{\lambda}}$ in $L_2(T,|\delta_k|\,dt)$ via j is an isometry if we take

$$a(\lambda,k)=|W_\lambda|^2\frac{c^*(-(\lambda+\varrho),k)}{\tilde{c}(\lambda+\varrho,k)}$$

in Theorem 4.1 (4). Namely, given this value of a we calculate the value of $\|E(w\lambda,k)\|_k^2$ by means of Theorem 4.1 (4) and Lemma 5.1 (3) as follows:

$$\|E(w\lambda,k)\|_k^2=|W_\lambda|^2\frac{c^*(-(\lambda+\varrho),k)}{\tilde{c}(\lambda+\varrho,k)}\prod_{\alpha\in R_+\cap w^{-1}R_-}\left(\frac{\tilde{\lambda}(\alpha^\vee)+\frac{1}{2}k_{\alpha/2}+k_\alpha}{\tilde{\lambda}(\alpha^\vee)+\frac{1}{2}k_{\alpha/2}}\right)$$
$$\times\prod_{\alpha\in R_+\cap w^{-1}R_+}\left(\frac{\tilde{\lambda}(\alpha^\vee)-\frac{1}{2}k_{\alpha/2}}{\tilde{\lambda}(\alpha^\vee)-\frac{1}{2}k_{\alpha/2}-k_\alpha}\right).$$

Now use the well-known formula (cf. (4.4))

$$|W_\lambda|=\prod_{\alpha\in R_{\lambda,+}}\left(\frac{\varrho(\alpha^\vee)+\frac{1}{2}k_{\alpha/2}+k_\alpha}{\varrho(\alpha^\vee)+\frac{1}{2}k_{\alpha/2}}\right)\tag{5.2}$$

and $\tilde{\lambda}=w_\lambda(\lambda+\varrho)$ (implying that $\tilde{\lambda}(\alpha^\vee)=-\varrho(\alpha^\vee)$ if $\alpha\in R_\lambda$) in combination with

$$w_\lambda(R_+\cap w^{-1}R_-)\amalg R_{\lambda,+}=R_+\cap(ww_\lambda)^{-1}R_-$$

and

$$w_\lambda(R_+\cap w^{-1}R_+)=(R_+\cap(ww_\lambda)^{-1}R_+)\amalg R_{\lambda,-}.$$

This leads to (1), proving the claim. It is easy to check that this value of $a(\lambda,k)$ satisfies the relation asserted in Lemma 5.2. Applying this relation sufficiently many times we may assume that one of the root multiplicities is 0. Let R_1 be a root subsystem of R such that $k=0$ on $R\backslash R_1$ and such that the rank of R_1 equals the rank of R. Then $D_\xi(R_+,k)=D_\xi(R_{1,+},k)$ and it follows that $E(R_+,\lambda,k)=E(R_{1,+},\lambda,k)$. But this means that we may now omit the roots in $R\backslash R_1$ altogether and proceed with R_1. Repeating this we end up with the situation where $k=0$, and here (1) is obviously true.

(2) This is proved by a similar induction process. Put $j(\phi(\lambda,k))(e)=b(\lambda,k)$ $\forall\lambda\in P_+$. First of all we note that the assertion is equivalent to

$$b(\lambda,k)=\frac{\tilde{c}_{w_0}(\varrho,k)}{\tilde{c}_{w_\lambda}(\lambda+\varrho,k)}$$

since $|W|^{-1}\sum_{w\in W}E(\lambda,k)^w(e)=E(\lambda,k,e)$. Next we observe that this is true if $k=0$ and that omission of conjugacy classes of roots having multiplicity 0 does not change either side of this formula. To do the induction step proceed as follows. From Theorem 1.4 (4) we obtain

$$j(\phi_e^-(\lambda+\delta,k))=|W|^{-1}\prod_{\alpha\in R_+^0}\left(1+\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{(\lambda+\delta)\check{}(\alpha^\vee)}\right)\sum_{w\in W}(-1)^{l(w)}j(v_w(\lambda+\delta,k)).\tag{5.3}$$

If we apply the operator $D=\prod_{\alpha\in R_+^0} D_{\alpha^\vee}(k)$ to (5.3) this becomes

$$Dj(\phi_e^-(\lambda+\delta,k)) = |W|^{-1} \prod_{\alpha\in R_+^0} \big((\lambda+\delta+\varrho(k))(\alpha^\vee)+k_\alpha+\tfrac{1}{2}k_{\alpha/2}\big) \sum_{w\in W} j(v_w(\lambda+\delta,k)). \tag{5.4}$$

It is clear from the definition of v_w and of ϕ (see Theorem 4.1) that $j(v_w(\lambda+\delta,k))(e)= j(\phi(\lambda+\delta,k))(e)=b(\lambda+\delta,k)$. This results in the following formula when we evaluate (5.4) at e:

$$Dj(\phi_e^-(\lambda+\delta,k))(e) = b(\lambda+\delta,k) \prod_{\alpha\in R_+^0} \big((\lambda+\delta+\varrho(k))(\alpha^\vee)+k_\alpha+\tfrac{1}{2}k_{\alpha/2}\big). \tag{5.5}$$

A moment's thought shows that this formula can be generalized as follows. Let f be an arbitrary W-invariant holomorphic germ at e. Then

$$D(fj(\phi_e^-(\lambda+\delta,k)))(e) = f(e)b(\lambda+\delta,k) \prod_{\alpha\in R_+^0} \big((\lambda+\delta+\varrho(k))(\alpha^\vee)+k_\alpha+\tfrac{1}{2}k_{\alpha/2}\big). \tag{5.6}$$

Now take $\lambda=0$ in (5.6) and put $f=j(\phi(\lambda,k+1))$. Observe that $j(\phi_e^-(\delta,k))=\Delta$. We thus obtain

$$D(\Delta j(\phi(\lambda,k+1)))(e) = b(\lambda,k+1)b(\delta,k) \prod_{\alpha\in R_+^0} \big((\delta+\varrho(k))(\alpha^\vee)+k_\alpha+\tfrac{1}{2}k_{\alpha/2}\big). \tag{5.7}$$

Finally use Lemma 5.2 to compare the right hand sides of (5.5) and (5.7). This leads to the recurrence formula

$$b(\lambda,k+1) = |W_\lambda| \frac{b(\lambda+\delta,k)}{b(\delta,k)} \prod_{\alpha\in R_+^0} \frac{(\lambda+\delta+\varrho(k))(\alpha^\vee)+k_\alpha+\frac{1}{2}k_{\alpha/2}}{(\delta+\varrho(k))(\alpha^\vee)+k_\alpha+\frac{1}{2}k_{\alpha/2}}$$

for $b(\lambda,k)$. Using (5.2) one easily verifies that the asserted value for $b(\lambda,k)$ satisfies this recurrence relation. By the above remarks and the induction procedure as in the proof of (1) this proves (2). □

6. Asymptotic expansions and growth estimates

In this section we develop two types of growth estimates for the eigenfunctions $G(\lambda,k;x)$. First of all we give a majorizing function for $|G(\lambda,k)|$, and closely related locally uniform bounds for $|\partial_{\xi^\alpha} G(\lambda,k)|$ on A. This part was inspired by the analogous results of de Jeu [13] in the case of Dunkl operators. The methods we use are also completely similar, although there are some complications that cause the results here to be a bit weaker than those in the Dunkl case. The second part of this section deals with asymptotic expansions of $G(\lambda,k;x)$ in Weyl chambers, in the spirit of Harish–Chandra's treatment of asymptotic behaviour of spherical functions [11].

PROPOSITION 6.1. *Let $k_\alpha \geqslant 0\ \forall\alpha$. Then*

(1) $|G(\lambda,k;x)|\leqslant |W|^{1/2}e^{\max_w \mathrm{Re}(w\lambda(x))}$ *if* $x\in\mathfrak{a}$.

In fact, we have more generally:

(2) $|G(\lambda,k;z)|\leqslant |W|^{1/2}e^{-\min_w \mathrm{Im}(w\lambda(y))+\max_w w\varrho(y)+\max_w \mathrm{Re}(w\lambda(x))}$ *if* $z=x+iy$ *with* $x,y\in\mathfrak{a}$ *and* $|\alpha(y)|\leqslant\pi\ \forall\alpha\in R$.

Proof. Put $\phi_w(z)=G(\lambda,k;w^{-1}z)$; from Lemma 3.2 we see that $\sum_w \phi_w\otimes w$ is $\nabla(\lambda,k)$-flat. By Definition 3.1 this means that

$$\partial_\xi\phi_w=-\frac{1}{2}\sum_{\alpha>0}k_\alpha\alpha(\xi)\left(\frac{1+e^{-\alpha(z)}}{1-e^{-\alpha(z)}}(\phi_w-\phi_{r_\alpha w})-\mathrm{sgn}(w^{-1}\alpha)\phi_{r_\alpha w}\right)+(w\lambda,\xi)\phi_w.$$

Assume $k_\alpha\geqslant 0\ \forall\alpha$, and take complex conjugates:

$$\partial_\xi\bar\phi_w=-\frac{1}{2}\sum_{\alpha>0}k_\alpha\alpha(\bar\xi)\left(\frac{1+e^{-\alpha(\bar z)}}{1-e^{-\alpha(\bar z)}}(\bar\phi_w-\bar\phi_{r_\alpha w})-\mathrm{sgn}(w^{-1}\alpha)\bar\phi_{r_\alpha w}\right)+(w\bar\lambda,\bar\xi)\bar\phi_w.$$

Hence

$$\begin{aligned}\partial_\xi\sum_w|\phi_w|^2&=\sum_w((\partial_\xi\phi_w)\bar\phi_w+\phi_w(\partial_\xi\bar\phi_w))\\&=-\frac{1}{2}\sum_{\substack{\alpha>0\\ w}}\left(k_\alpha\alpha(\xi)\left(\frac{1+e^{-\alpha(z)}}{1-e^{-\alpha(z)}}(\phi_w-\phi_{r_\alpha w})\bar\phi_w-\mathrm{sgn}(w^{-1}\alpha)\phi_{r_\alpha w}\bar\phi_w\right)\right.\\&\left.+k_\alpha\alpha(\bar\xi)\left(\frac{1+e^{-\alpha(\bar z)}}{1-e^{-\alpha(\bar z)}}(\bar\phi_w-\bar\phi_{r_\alpha w})\phi_w-\mathrm{sgn}(w^{-1}\alpha)\bar\phi_{r_\alpha w}\phi_w\right)\right)+2\sum_w\mathrm{Re}(w\lambda(\xi))|\phi_w|^2.\end{aligned}$$

For each fixed α we first add the terms with index w and $r_\alpha w$. We obtain

$$\begin{aligned}\partial_\xi\sum_w|\phi_w|^2=&-\frac{1}{4}\sum_{\substack{\alpha>0\\ w}}k_\alpha\left(\alpha(\xi)\frac{1+e^{-\alpha(z)}}{1-e^{-\alpha(z)}}+\alpha(\bar\xi)\frac{1+e^{-\alpha(\bar z)}}{1-e^{-\alpha(\bar z)}}\right)|\phi_w-\phi_{r_\alpha w}|^2\\&+\sum_{\substack{\alpha>0\\ w}}k_\alpha\,\mathrm{sgn}(w^{-1}\alpha)\,\mathrm{Im}(\alpha(\xi))\,\mathrm{Im}(\bar\phi_w\phi_{r_\alpha w})+2\sum_w\mathrm{Re}(w\lambda(\xi))|\phi_w|^2.\end{aligned}$$

Using $z=x+iy$ we rewrite this as follows:

$$\begin{aligned}\partial_\xi\sum_w|\phi_w|^2=&-\frac{1}{2}\sum_{\substack{\alpha>0\\ w}}k_\alpha\left(\frac{\mathrm{Re}(\alpha(\xi))(1-e^{-2\alpha(x)})+2\,\mathrm{Im}(\alpha(\xi))e^{-\alpha(x)}\sin\alpha(y)}{|1-e^{-\alpha(z)}|^2}\right)|\phi_w-\phi_{r_\alpha w}|^2\\&+\sum_{\substack{\alpha>0\\ w}}k_\alpha\,\mathrm{sgn}(w^{-1}\alpha)\,\mathrm{Im}(\alpha(\xi))\,\mathrm{Im}(\bar\phi_w\phi_{r_\alpha w})+2\sum_w\mathrm{Re}(w\lambda(\xi))|\phi_w|^2.\quad(6.1)\end{aligned}$$

First we take $x\in\mathfrak{a}^{\mathrm{reg}}$ and $\xi\in\mathfrak{a}^{\mathrm{reg}}$ such that x and ξ belong to the same Weyl chamber. Let $\mu\in\{w\operatorname{Re}\lambda\}_{w\in W}$ such that $\mu(\xi)=\max_w \operatorname{Re}(w\lambda(\xi))$. Formula (6.1) implies

$$\begin{aligned}\partial_\xi\Big(e^{-2\mu(x)}\sum_w|\phi_w(z)|^2\Big)&=-\frac{1}{2}\sum_{\substack{\alpha>0\\ w}}k_\alpha\frac{\alpha(\xi)(1-e^{-2\alpha(x)})}{|1-e^{-\alpha(z)}|^2}|\phi_w-\phi_{r_\alpha w}|^2e^{-2\mu(x)}\\&\quad+2\sum_w(w\operatorname{Re}\lambda-\mu)(\xi)|\phi_w|^2e^{-2\mu(x)}\leqslant 0.\end{aligned}$$

Hence $e^{-2\max_w\operatorname{Re}(w\lambda(x))}\sum_w|\phi_w(z)|^2\leqslant\sum_w|\phi_w(iy)|^2$ if $x\in\mathfrak{a}^{\mathrm{reg}}$, and by continuity this estimate holds $\forall x\in\mathfrak{a}$. Now $|\phi_e(z)|\leqslant\big(\sum_w|\phi_w(z)|^2\big)^{1/2}$, hence the above formula shows that

$$|G(\lambda,k;x+iy)|\leqslant e^{\max_w\operatorname{Re}(w\lambda(x))}\Big(\sum_w|\phi_w(iy)|^2\Big)^{1/2}\tag{6.2}$$

if $|\alpha(y)|\leqslant\pi$ $\forall\alpha\in R$ (thus avoiding problems of multivaluedness of $G(\lambda,k;x+iy)$). Note that this estimate already proves (1) when we substitute $y=0$ and use Theorem 3.15 (1).

In order to prove (2) we take $y\in\mathfrak{a}^{\mathrm{reg}}$ such that $|\alpha(y)|\leqslant\pi$ $\forall\alpha\in R$, and $\eta\in\mathfrak{a}^{\mathrm{reg}}$ belonging to the same chamber, and let $\xi=i\eta$. Note that $\operatorname{Re}(w\lambda(\xi))=-\operatorname{Im}(w\lambda(\eta))$, and take $\mu\in\{w\operatorname{Im}\lambda\}_{w\in W}$ such that $-\operatorname{Im}(w\lambda(\eta))\leqslant-\mu(\eta)$ $\forall w\in W$. Observe that

$$\begin{aligned}\Big|\sum_{\substack{\alpha>0\\ w}}k_\alpha\operatorname{sgn}(w^{-1}\alpha)\operatorname{Im}(\alpha(\xi))\operatorname{Im}(\bar\phi_w\phi_{r_\alpha w})\Big|&\leqslant\sum_{\substack{\alpha>0\\ w}}k_\alpha|\alpha(\eta)|\cdot|\phi_w|\cdot|\phi_{r_\alpha w}|\\&\leqslant 2\max_w(w\varrho,\eta)\sum_w|\phi_w|^2.\end{aligned}$$

Choose $\nu\in\{w\varrho\}_{w\in W}$ such that $(\nu,\eta)=\max_w(w\varrho,\eta)$. Using (6.1) we obtain (with $F(iy)=e^{2(\mu-\nu)(y)}\sum_w|\phi_w(iy)|^2$)

$$\begin{aligned}(\partial_\xi F)(iy)&=-\sum_{\substack{\alpha>0\\ w}}k_\alpha\Big(\frac{\alpha(\eta)\sin\alpha(y)}{|1-e^{-\alpha,iy}|^2}\Big)|\phi_w-\phi_{r_\alpha w}|^2\cdot e^{2(\mu-\nu)(y)}\\&\quad+\Big(\sum_{\substack{\alpha>0\\ w}}k_\alpha\operatorname{sgn}(w^{-1}\alpha)\operatorname{Im}(\alpha(\xi))\operatorname{Im}(\bar\phi_w\phi_{r_\alpha w})-2(\nu,\eta)\sum_w|\phi_w|^2\Big)e^{2(\mu-\nu)(y)}\\&\quad+\Big(2\sum_w(\mu-w\operatorname{Im}\lambda)(\eta)|\phi_w|^2\Big)e^{2(\mu-\nu)(y)}\leqslant 0\end{aligned}$$

(since $\alpha(\eta)\sin\alpha(y)>0$ (if η,y belong to the same chamber and moreover $|\alpha(y)|\leqslant\pi$ $\forall\alpha$)). Hence we see that $F(iy)\leqslant F(0)$. Together with (6.2) and Theorem 3.15 (1) this proves (2).

COROLLARY 6.2 (cf. [13]). *Let* $k_\alpha \geqslant 0$ $\forall\alpha$, *and let* τ *denote a multi-index. Then* $\forall\varepsilon>0$ $\exists C_\varepsilon>0$ *such that* $\forall x\in\mathfrak{a}, \lambda\in i\mathfrak{a}^*, |\lambda|>\varepsilon$:

$$|\partial_{\xi^\tau} G(\lambda,k;x)| \leqslant C_\varepsilon |\lambda|^{|\tau|}.$$

In particular, if $K\subset\mathfrak{a}$ *is compact* $\exists C_K>0$ *such that* $\forall x\in K, \lambda\in i\mathfrak{a}^*$:

$$|\partial_{\xi^\tau} G(\lambda,k;x)| \leqslant C_K(1+|\lambda|^{|\tau|}).$$

Proof. Choose $\delta>0$ such that $T_\varepsilon=\{z\,|\,|z_i|=\delta/\varepsilon\}\subset\{z=x+iy\,|\,|\alpha(y)|\leqslant\pi\ \forall\alpha\in R\}$. By Cauchy's formula,

$$\partial_{\xi^\tau} G(\lambda,k;x) = C_\tau \int_{x+T_{|\lambda|}} \frac{G(\lambda,k,z)\,dz}{(z-x)^\tau}.$$

If $z\in x+T_{|\lambda|}$ then $|z-x|^\tau=\delta^{|\tau|}|\lambda|^{-|\tau|}$ and by Proposition 6.1 (2) we have $|G(\lambda,k;z)|\leqslant C$ (independent of $\lambda\in i\mathfrak{a}^*$ and $x\in\mathfrak{a}$). This proves the result. □

Next we consider the asymptotic behaviour of $G(\lambda,k;x)$ when $x\in\mathfrak{a}_-$, $x\to\infty$. We study this behaviour simply by means of the relation between $G(\lambda,k;x)$ and the hypergeometric function $F(\lambda,k;x)$ and the existing knowledge on the asymptotic behaviour of F. Let us first recall what is known about the behaviour of F:

THEOREM 6.3 ([10]; the different sign in the argument of the c-function is due to a change of sign in the definition of $\tilde{c}$). *Assume that* $\mathrm{Re}(k_\alpha)\geqslant 0$ $(\forall\alpha)$, *that* λ *is regular and that* $\lambda(\varkappa^\vee)+1\neq 0$ $(\forall\varkappa\in Q\backslash\{0\})$. *If* $x\in\mathfrak{a}_-$ *then*

$$F(\lambda,k,x)=\sum_{w\in W} c(-w\lambda,k)\phi(w\lambda+\varrho(k),k;x)$$

where ϕ *is of the following form*:

$$\phi(\lambda+\varrho(k),k;x)=e^{(\lambda+\varrho(k))(x)}\sum_{\varkappa\in Q_+}\Delta_\varkappa(\lambda,k)e^{\varkappa(x)}.$$

The coefficients Δ *satisfy the following properties*:

(1) $\Delta_0(\lambda,k)=1$.

(2) $\Delta_\varkappa(\lambda,k)$ *is a rational function, with poles only at hyperplanes of the form*

$$\lambda(\varkappa^\vee)+1=0 \quad (\varkappa\in Q_+\backslash\{0\}).$$

(3) *Let* $x_0\in\mathfrak{a}_+$. *Then* $\exists K_{x_0}\in\mathbf{R}_+$ *such that* $\forall\varkappa\in Q_+$ $\forall\lambda\in\overline{\mathfrak{a}^*_+}+i\mathfrak{a}^*$:

$$|\Delta_\varkappa(\lambda,k)|\leqslant K_{x_0}e^{\varkappa(x_0)}.$$

Proof. (1) By definition.

(2) See [10, formula (3.15)].

(3) See [11, Lemma 5.6]. □

We need a preparatory lemma:

LEMMA 6.4. *Choose an orthonormal basis* $(\xi_1, \ldots, \xi_n)$ *of coordinates on* $\mathfrak{a}$. *We use the multi-index notation* $\partial_{\xi^\beta} = (\partial/\partial\xi_1)^{\beta_1} \ldots (\partial/\partial\xi_n)^{\beta_n}$. *Let* $p \in S(\mathfrak{h})$, *then* D_p *has an asymptotic expansion of the following form on* $\mathfrak{a}$:

$$D_p = \partial(p(\cdot - \varrho(k))) + \sum_{\substack{\beta \in \mathbf{Z}_+^n \\ \varkappa \in Q_+ \setminus \{0\}}} C_{\alpha,\varkappa} e^{\varkappa} \partial_{\xi^\beta}.$$

Here $C_{\alpha,\varkappa} \in \mathbf{C}$, *and there are only finitely many* $\alpha \in \mathbf{Z}_+^n$ *such that* $\exists \varkappa \in Q_+$ *such that* $C_{\alpha,\varkappa} \neq 0$. *Moreover, there exists a* $N = N_p \in \mathbf{Z}_+$ *and a* $C \in \mathbf{R}_+$ *such that* $|C_{\alpha,\varkappa}| \leqslant C(1+|\varkappa|)^N$.

Proof. D_p is a differential operator with coefficients in the ring of functions generated by $1/(1-e^\alpha)$ $(\alpha \in R)$, and thus we have an asymptotic expansion with the properties stated above. The only thing that remains to be shown is the precise form of the leading term. It is sufficient to prove this statement when p is a monomial. We use induction on the degree of p. Assume that p is of the form $\xi p'$. Then

$$D_{\xi p'} = (\partial_\xi - (\varrho(k), \xi)) D_{p'} + \sum_{\alpha \in R_+} k_\alpha \alpha(\xi) \frac{1}{1-e^{-\alpha}} (D_{p'} - r_\alpha \circ D_{p'} \circ r_\alpha)$$

(since the right hand side is a differential operator which has the same restriction to W-invariant functions as the operator $p(D)$). Now take the leading term and use the induction hypothesis. □

COROLLARY 6.5. *Assume that* $\mathrm{Re}(k_\alpha) \geqslant 0$ $\forall \alpha \in R_+$, *that* λ *is regular and that*

$$\lambda(\varkappa^\vee) + 1 \neq 0 \quad \forall \varkappa \in Q \setminus \{0\}.$$

Define

$$\widetilde{G}(\lambda, k; x) = \prod_{\alpha \in R_+^0} (\lambda(\alpha^\vee) - k_\alpha - \tfrac{1}{2} k_{\alpha/2}) G(\lambda, k; x).$$

If $x \in \mathfrak{a}_-$, $\widetilde{G}$ *has an asymptotic expansion as follows*:

$$\widetilde{G}(\lambda, k; x) = \sum_{w \in W} c(-w\lambda, k) \Psi(w; \lambda + \varrho(k), k; x)$$

with

$$\Psi(w, \lambda + \varrho(k), k; x) = \sum_{\varkappa \in Q_+} \Lambda_\varkappa(w; \lambda, k) e^{(w\lambda + \varrho(k) + \varkappa)(x)}.$$

The coefficients Λ *satisfy the following properties*:

(1) $\Lambda_0(w; \lambda, k) = \delta_{w,1} |W| \prod_{\alpha \in R_{0+}} \lambda(\alpha^\vee)$.

(2) *$\Lambda_\varkappa(w;\lambda,k)$ is a rational function, with poles only at hyperplanes of the form*

$$(w\lambda,\varkappa^\vee)+1=0 \quad (\varkappa\in Q_+\backslash\{0\}).$$

(3) *Let $x_0\in\mathfrak{a}_+$. Then $\exists K_{x_0}\in\mathbf{R}_+$ and an $M\in\mathbf{Z}_+$ such that $\forall\varkappa\in Q_+$:*

$$|\Lambda_\varkappa(w;\lambda,k)|\leqslant K_{x_0}(1+|\lambda|)^M e^{\varkappa(x_0)} \quad \text{if } w\lambda\in\overline{\mathfrak{a}_+^*}+i\mathfrak{a}^*.$$

Proof. By Theorem 3.15 we see that

$$\widetilde{G}(\lambda,k;x)=\sum_{w\in W} c(-w\lambda,k)\Psi(w;\lambda+\varrho(k),k;x)$$

where

$$\Psi(w;\lambda+\varrho(k),k;x)=\widetilde{D}(\lambda,k)\phi(w\lambda+\varrho(k),k;x)$$

with

$$\widetilde{D}(\lambda,k)=|W|\prod_{\alpha\in R_+^0}\left((\lambda,\alpha^\vee)-k_\alpha-\tfrac{1}{2}k_{\alpha/2}\right)D(\lambda,k).$$

By Lemma 3.9 and Definition 3.10 we conclude that $\widetilde{D}(\lambda,k)$ is polynomial in λ, and from Lemma 6.4 we may conclude that $\widetilde{D}(\lambda,k)$ has the following asymptotic expansion on $\mathfrak{a}_-$:

$$\begin{aligned}\widetilde{D}(\lambda,k)&=|W|\prod_{\alpha\in R_+^0}\left(\lambda(\alpha^\vee)-k_\alpha-\tfrac{1}{2}k_{\alpha/2}\right)\partial(q(\lambda,\cdot-\varrho(k)))+\sum_{\substack{\beta,\gamma,\varkappa\\(\varkappa\in Q_+\backslash\{0\})}} C_{\beta,\gamma,\varkappa}e^\varkappa\lambda^\beta\partial_{\xi^\gamma}\\&=\sum_{\substack{\beta,\gamma,\varkappa\\(\varkappa\in Q_+)}} C_{\beta,\gamma,\varkappa}e^\varkappa\lambda^\beta\partial_{\xi^\gamma}.\end{aligned}$$

Moreover, $\exists C\in\mathbf{R}_+$, $N\in\mathbf{Z}_+$ such that $|C_{\beta,\gamma,\varkappa}|\leqslant C(1+|\varkappa|)^N$. From these observations (1) and (2) follow easily. As for (3), let $x_0\in\mathfrak{a}_+$. The above expression for Ψ leads to

$$|\Lambda_\varkappa(w;\lambda,k)|\leqslant\sum_{\substack{\beta,\gamma\\ \mu,\nu\in Q_+\\ \mu+\nu=\varkappa}}|C_{\alpha,\beta,\mu}\lambda^\beta(w\lambda+\varrho(k)+\nu)^\gamma\Delta_\nu(w\lambda,k)|.$$

Use that $|\mu|,|\nu|\leqslant C_1|\varkappa|$ if $\mu,\nu\in Q_+$ and $\mu+\nu=\varkappa$. Also use that $|(w\lambda+\varrho(k)+\nu)^\gamma|\leqslant C_2(1+|\lambda|)^{|\gamma|}(1+|\nu|)^{|\gamma|}$. Finally we may choose $C_3\in\mathbf{R}_+$ such that (since $w\lambda\in\overline{\mathfrak{a}_+^*}+i\mathfrak{a}^*$)

$$|\Delta_\nu(w\lambda,k)|\leqslant C_3e^{\nu(x_0)/4}\leqslant C_3e^{-\nu(x_0)/4}e^{\varkappa(x_0)/2}.$$

Inserting all these upper bounds, we obtain

$$\begin{aligned}|\Lambda_\varkappa(w;\lambda,k)|&\leqslant C_4\sum_{\substack{\beta,\gamma\\ \{\nu\in Q_+:\,\nu\leqslant\varkappa\}}}(1+|\lambda|)^{|\beta|+|\gamma|}(1+|\varkappa|)^N(1+|\nu|)^{|\gamma|}e^{-\nu(x_0)/4}\cdot e^{\varkappa(x_0)/2}\\&\leqslant K_{x_0}(1+|\lambda|)^Me^{\varkappa(x_0)}.\end{aligned}$$

This proves (3). ☐

7. The Cherednik transform

In this and subsequent sections we will examine the decomposition of $C_c^\infty(\mathfrak{a})$ as an $\mathbf{H}$-module, and some related results (Paley–Wiener theorem, inversion formulas, Plancherel formula). The methods that we use are based on the work of van den Ban and Schlichtkrull on the most continuous part of the Plancherel formula for semisimple symmetric spaces [1]. The relevance of these methods to the case of hypergeometric functions was observed by Heckman in [9].

Let $\lambda\in\mathfrak{h}^*$. By the universal property of I_λ, there exists a unique $\mathbf{H}$-module morphism $\mathcal{J}_\lambda: I_\lambda\to C^\infty(\mathfrak{a})$ such that $\mathcal{J}_\lambda(e\otimes 1)(x)=G(\lambda,k;x)$. In the next proposition we have included (2) because it elucidates the situation, but formally this property will not be used in what follows.

PROPOSITION 7.1. (1) $\mathcal{J}_\lambda(w\otimes 1)=G^w(\lambda,k;\cdot)$.
(2) *If* $\lambda(\alpha^\vee)\neq k_\alpha^0$ $(\forall\alpha\in R^0)$ *then* $\mathcal{J}_\lambda$ *is a monomorphism.*
(3) *If* $\lambda(\alpha^\vee)\neq k_\alpha^0$ $(\forall\alpha\in R^0)$ *and* $\lambda\in\mathfrak{h}^{*,\mathrm{reg}}$ *then* $\mathcal{J}_\lambda(v_w)=G(w\lambda,k,\cdot)$.

Proof. (2) is a consequence of the irreducibility criterion Theorem 1.3, and the rest is trivial. □

Definition 7.2. Fix a nondegenerate sesquilinear pairing $(\cdot,\cdot): I_\lambda\times I_{-\bar\lambda}\to\mathbf{C}$ by means of $(w\otimes 1, w'\otimes 1)=\delta_{w,w'}$.

PROPOSITION 7.3. (1) $(\cdot,\cdot)$ *is* $*$*-invariant.*
(2) *If* $\lambda(\alpha^\vee)\neq 0,\pm k_\alpha^0$ $\forall\alpha\in R_+^0$ *then*

$$(v_w(\lambda), w_0 v_{w_0w'}(-\bar\lambda))=\frac{\delta_{w,w'}}{\prod_{\alpha\in R_+^0}(1-(k_\alpha+\frac{1}{2}k_{\alpha/2})/w\lambda(\alpha^\vee))}.$$

Proof. Analogous to Theorem 4.2. □

Definition 7.4. Let X be the vector bundle $\coprod_{\lambda\in i\mathfrak{a}_+^*} I_\lambda$ and $X_{\mathbf{C}}$ be the vector bundle $\coprod_{\lambda\in i\mathfrak{a}_+^*+\mathfrak{a}^*} I_\lambda$. Denote by Σ $(\Sigma_{\mathbf{C}})$ the space of sections of X $(X_{\mathbf{C}})$.

The fibres of $X_{\mathbf{C}}$ are $\mathbf{H}$-modules. Consequently, $\Sigma_{\mathbf{C}}$ and Σ are also $\mathbf{H}$-modules in a natural way. Moreover, the fibres of X carry $*$-invariant positive definite Hermitean forms. If $v\in I_\lambda$ we denote $|v|=(v,v)^{1/2}$. In the next definition we will identify some $\mathbf{H}$-submodules of Σ and $\Sigma_{\mathbf{C}}$ which will be useful later on.

Definition 7.5. We use the notation Σ^k to indicate the space of k times continuously differentiable sections $(k=0,1,\dots,\infty)$. By Σ_b we mean the bounded sections, and by Σ_c we indicate the compactly supported sections. The notation $\Sigma_{\mathbf{C}}^\varepsilon$ is used for the space of holomorphic sections. Let ω denote the completion of the Borel measure on $i\mathfrak{a}^*$ that corresponds to the volume form $(-i)^n d\lambda$. We define a measure ν on $i\mathfrak{a}^*$ by

$$d\nu(\lambda)=\frac{(2\pi)^{-n}\tilde{c}_{w_0}^2(\varrho(k),k)}{\tilde{c}(\lambda,k)\tilde{c}(w_0\lambda,k)}\,d\omega(\lambda).$$

(This is the well-known spherical Plancherel measure from the theory of Riemannian symmetric spaces when k is associated to the multiplicity function of a Riemannian symmetric pair.) If $p\in[1,\infty)$ we define the p-norm $\|\sigma\|_p$ of a measurable section $\sigma\in\Sigma$ as follows:

$$\|\sigma\|_p=\left(\int_{i\mathfrak{a}_+^*}|\sigma(\lambda)|^p\,d\nu(\lambda)\right)^{1/p}.$$

We also define $\|\sigma\|_\infty$ as usual. This gives rise to the Banach spaces Σ_p and a pairing of Σ_p and Σ_q if $p^{-1}+q^{-1}=1$.

Remark 7.6. Observe that $\Sigma_c^\infty\subset\Sigma_p$ is dense $\forall p\in(1,\infty)$.

Definition 7.7. Let $\mu=\mu_k$ denote the measure on $\mathfrak{a}$ ($k\in\mathcal{K}$ fixed) given by

$$d\mu(x)=|\delta_k(x)|\,dx=\prod_{\alpha\in R_+}\left|2\sinh\left(\tfrac{1}{2}\alpha(x)\right)\right|^{2k_\alpha}dx.$$

LEMMA 7.8. *Let $f,g\in C^\infty(\mathfrak{a})$, and let one of these functions be compactly supported. Then $\forall\xi\in\mathfrak{h}$ and $k\in\mathcal{K}$ such that $k_\alpha^0\geqslant 0$ $\forall\alpha\in R^0$:*

$$\int_{\mathfrak{a}}(D_\xi f)g\,d\mu=\int_{\mathfrak{a}}f(D_{-w_0(\xi)}g^{w_0})^{w_0}\,d\mu.$$

Consequently, the Hermitean inner product $(f,g)_k=\int_{\mathfrak{a}}f\bar{g}\,d\mu_k$ on $C_c^\infty(\mathfrak{a})$ is $*$-invariant when $C_c^\infty(\mathfrak{a})$ is considered as $\mathbf{H}$-module via the action of W and the operators D_ξ.

Proof. We calculate the formal transpose of D_ξ with respect to $d\mu$:

$$\begin{aligned}D_\xi^t&=\delta_k^{-1}\circ\partial_{-\xi}\circ\delta_k+\sum_{\alpha\in R_+}k_\alpha\alpha(\xi)(1-r_\alpha)\frac{1}{1-e^{-\alpha}}-\varrho(\xi)\\&=\partial_{-\xi}-\sum_{\alpha\in R_+}k_\alpha\alpha(\xi)\frac{e^{\alpha/2}+e^{-\alpha/2}}{e^{\alpha/2}-e^{-\alpha/2}}+\sum_{\alpha\in R_+}k_\alpha\alpha(\xi)\frac{1}{1-e^{\alpha}}(-1-r_\alpha)+\varrho(\xi)\\&=\partial_{-\xi}+\sum_{\alpha\in R_-}k_\alpha\alpha(\xi)\frac{1}{1-e^{-\alpha}}(1-r_\alpha)-(-\varrho,-\xi)=w_0\circ D_{-w_0\xi}\circ w_0.\end{aligned}$$

This proves the result. □

Now we come to the definition of the Cherednik transform and the associated wave packet operator. Given $\lambda\in i\mathfrak{a}^*$ and $v\in I_\lambda$, we know from Proposition 6.1 (1) that $\mathcal{J}_\lambda(v)\in C_b^\infty(\mathfrak{a})$ (provided that $k_\alpha\geqslant 0$ $\forall\alpha\in R$). Hence the following definition makes sense:

Definition 7.9. Let $f\in L_1(\mathfrak{a},\mu)$ and let $k_\alpha\geqslant 0$ $\forall\alpha\in R$. We define the Cherednik transform $\mathcal{F}(f)\in\Sigma$ as follows: $\mathcal{F}(f)(\lambda)$ is the unique element of I_λ such that $\forall v\in I_\lambda$: $(\mathcal{F}(f)(\lambda),v)=\int_{\mathfrak{a}} f(x)\overline{\mathcal{J}_\lambda(v)(x)}\,d\mu(x)$.

In the rank one case this transform is closely related to the Jacobi transform. Let us use the notations that were introduced at the end of Section 3, where we expressed the function G in terms of the classical hypergeometric function. We use α as coordinate on $\mathfrak{a}$. The orthogonality measure μ is given by the formula

$$\begin{aligned} d\mu(\alpha) &= (2\cosh\alpha-2)^{k_\alpha+k_{2\alpha}}(2\cosh\alpha+2)^{k_{2\alpha}}\,d\alpha \\ &= (2\cosh\alpha-2)^{c-1/2}(2\cosh\alpha+2)^{a+b-c+1/2}\,d\alpha. \end{aligned}$$

In I_λ $(\lambda\in i\mathfrak{a}^*)$ we have the basis $(e\otimes 1, r\otimes 1)$ and the spherical vector $\phi=\frac{1}{2}(e\otimes 1+r\otimes 1)$. If we write $(f\in L_1(\mathbf{R},\mu))$

$$\mathcal{F}f(\lambda)=\mathcal{F}_e f(\lambda)e\otimes 1+\mathcal{F}_r f(\lambda)r\otimes 1$$

then

$$\mathcal{F}_e f(\lambda)=\int_{\mathbf{R}} f(\alpha)G(-\lambda,k,\alpha)\,d\mu(\alpha)$$

and

$$\mathcal{F}_r f(\lambda)=\int_{\mathbf{R}} f(\alpha)G(-\lambda,k,-\alpha)\,d\mu(\alpha).$$

If f happens to be symmetric, i.e. $f(\alpha)=f(-\alpha)$, we may use the expression of G in terms of the classical hypergeometric function (see the end of Section 3) in order to obtain

$$\begin{aligned} \mathcal{F}f(\lambda) &= \left\{\int_{\mathbf{R}} f(\alpha)F\big(a,b,c;\tfrac{1}{2}(1-\cosh\alpha)\big)\,d\mu(\alpha)\right\}2\phi \\ &= \{2^{3/2}\Gamma(c)\hat{f}_{c-1,a+b-c}(\lambda(\alpha^\vee))\}\phi, \end{aligned}$$

where $\hat{f}_{\alpha,\beta}$ denotes the Jacobi transform (in the notation of [14]). The results that we will derive for the Cherednik transform in general in the next sections were long known in the case of the Jacobi transform (cf. [14] and the references given therein).

Let us proceed now with the general theory.

PROPOSITION 7.10. (1) *If* $f\in C_c^\infty(\mathfrak{a})$ *then* $\mathcal{F}(f)$ *extends to* $\Sigma_{\mathbf{C}}^{\varepsilon}$.

(2) $\forall f\in L_1(\mathfrak{a},\mu)$: $|\mathcal{F}(f)(\lambda)|\leqslant|W|\cdot\|f\|_1$ $(\forall\lambda\in i\mathfrak{a}_+^*)$.

(3) $\mathcal{F}$ *maps* $L_1(\mathfrak{a},\mu)$ *to the Banach space* Σ_b^0 *and is continuous with operator norm* $\|\mathcal{F}\|\leqslant|W|$.

(4) $\mathcal{F}\colon C_c^\infty\to\Sigma_b^0$ *is a morphism of* $\mathbf{H}$*-modules.*

Proof. (1) We extend the definition of $\mathcal{F}(f)$ as follows: $\mathcal{F}(f)(\lambda)$ is the unique element of I_λ such that $\forall v \in I_{-\bar{\lambda}}$: $(\mathcal{F}(f)(\lambda), v) = \int_{\mathfrak{a}} f(x)\overline{\mathcal{J}_{-\bar{\lambda}}(v)(x)}\, d\mu(x)$. Clearly, $\mathcal{F}(f)(\lambda) = \sum_w \mathcal{F}_w(f)(\lambda)\cdot(w\otimes 1)$ with

$$\mathcal{F}_w(f)(\lambda) = \int_{\mathfrak{a}} f(x) G(-\lambda, k, w^{-1}x)\, d\mu(x)$$

and this is obviously holomorphic in $\lambda \in \mathfrak{h}^*$ if $f \in C_c^\infty(\mathfrak{a})$.

(2) Note that by Proposition 6.1 (1) one has $(\forall \lambda \in i\mathfrak{a}^*, v \in I_\lambda, x \in \mathfrak{a})$

$$|\mathcal{J}_\lambda(v)(x)| \leqslant |W|\cdot|v|.$$

Hence $|(\mathcal{F}(f)(\lambda), v)| \leqslant |W|\cdot|v|\cdot\|f\|_1$, implying that

$$|\mathcal{F}(f)(\lambda)| \leqslant |W|\cdot\|f\|_1.$$

(3) Immediate from (1) and (2).

(4) Use Lemma 7.8 and the fact that $\mathcal{J}_\lambda$ is a morphism of $\mathbf{H}$-modules. □

In the proof we used that $|\mathcal{J}_\lambda(\sigma(\lambda))(x)| \leqslant |W|\cdot|\sigma(\lambda)|$ if $\lambda \in i\mathfrak{a}^*$, $x \in \mathfrak{a}$ and $\sigma(\lambda) \in I_\lambda$. From this we see that the following makes sense:

Definition 7.11. Let $\sigma \in \Sigma_1$. We define

$$\mathcal{J}(\sigma)(x) = \int_{i\mathfrak{a}_+^*} \mathcal{J}_\lambda(\sigma(\lambda))(x)\, d\nu(\lambda).$$

PROPOSITION 7.12. (1) *If $\sigma \in \Sigma_c^0$ then $\mathcal{J}(\sigma)$ is real analytic on $\mathfrak{a}$.*

(2) $|\mathcal{J}(\sigma)(x)| \leqslant |W|\cdot\|\sigma\|_1\ \forall \sigma \in \Sigma_1$.

(3) *$\mathcal{J}$ maps Σ_1 to the Banach space $C_b^0(\mathfrak{a})$ and is continuous with norm $\|\mathcal{J}\| \leqslant |W|$.*

(4) *$\mathcal{J}: \Sigma_c^0 \to C^\epsilon(\mathfrak{a})$ is a morphism of $\mathbf{H}$-modules.*

(5) $\mathcal{J} = \mathcal{F}^*|\Sigma_1$.

Proof. (1), (2), (3): similar to Proposition 7.10.

(4) If $\sigma \in \Sigma_c^0$ we may interchange D_ξ and the integral over $i\mathfrak{a}_+^*$.

(5) Let $f \in L_1(\mathfrak{a}, \mu)$ and let $\phi \in \Sigma_1$. Then

$$(\mathcal{F}(f), \phi) = \int_{i\mathfrak{a}_+^*} (\mathcal{F}(f)(\lambda), \phi(\lambda))\, d\nu(\lambda) = \int_{i\mathfrak{a}_+^*} \int_{\mathfrak{a}} f(x)\overline{\mathcal{J}_\lambda(\phi(\lambda))(x)}\, d\nu(\lambda).$$

Since $|\mathcal{J}_\lambda(\phi(\lambda))(x)| \leqslant |W|\cdot|\phi(\lambda)|$ $(\forall x \in \mathfrak{a})$ we may apply Fubini's theorem:

$$(\mathcal{F}(f), \phi) = \int_{\mathfrak{a}} \int_{i\mathfrak{a}_+^*} f(x)\overline{\mathcal{J}_\lambda(\phi(\lambda))(x)}\, d\nu(\lambda)\, d\mu(x) = (f, \mathcal{J}(\phi)). \qquad □$$

8. The Paley–Wiener theorem

Definition 8.1. Given $x\in\mathfrak{a}$, let C_x denote the convex hull of Wx. Define the support function H_x on $\mathfrak{a}^*$ as follows: $H_x(\lambda)=\sup_{y\in C_x}\lambda(y)$. An entire function ϕ on $\mathfrak{h}^*$ is said to have Paley–Wiener type x if $\forall N\in\mathbf{Z}_+$ $\exists C\in\mathbf{R}_+$ such that

$$|\phi(\lambda)|\leqslant C(1+|\lambda|)^{-N}e^{H_x(-\operatorname{Re}\lambda)}.$$

The space of functions of Paley–Wiener type x is denoted by $\mathrm{PW}(x)$. We also use the notation $\mathrm{PW}=\bigcup_{x\in\mathfrak{a}}\mathrm{PW}(x)$.

We will now give $\mathrm{PW}(x)$ the structure of an $\mathbf{H}$-module. Let $Z=\mathbf{C}[\mathfrak{h}^*]^W\subset\mathbf{H}$ denote the center of $\mathbf{H}$, and let $M=(\xi_1,\dots,\xi_n)\subset\mathbf{C}[\mathfrak{a}^*]$ be the maximal ideal corresponding to $0\in\mathfrak{h}^*$, and put $m=M\cap Z$. Denote by $\widehat{Z}$ the m-adic completion of Z and by $\widehat{\mathbf{C}[\mathfrak{h}^*]}$ the M-adic completion of $\mathbf{C}[\mathfrak{h}^*]$. Since $\mathbf{C}[\mathfrak{h}^*]$ is a free Z-module of finite rank, we have that $\widehat{\mathbf{C}[\mathfrak{h}^*]}=\mathbf{C}[\mathfrak{h}^*]\otimes_Z\widehat{Z}$. Following the construction in [15] we define the completion of $\mathbf{H}$ by $\widehat{H}=\widehat{Z}\otimes_Z\mathbf{H}$. As a $\widehat{\mathbf{C}[\mathfrak{h}^*]}$-module, $\widehat{H}\simeq\widehat{\mathbf{C}[\mathfrak{h}^*]}\otimes_{\mathbf{C}}\mathbf{C}[W]$. Its algebra structure is determined by the following rules:

(1) $\widehat{\mathbf{C}[\mathfrak{h}^*]}\otimes 1\simeq\widehat{\mathbf{C}[\mathfrak{h}^*]}$,

(2) $1\otimes\mathbf{C}[W]\simeq\mathbf{C}[W]$,

(3) $(f\otimes 1)\cdot(1\otimes w)=f\otimes w$,

(4) $(1\otimes r_i)\cdot(f\otimes 1)-(f^{r_i}\otimes r_i)=-(k_{\alpha_i}+2k_{2\alpha_i})\Delta_i(f)$,

where $\Delta_i(f)=1/\alpha_i^\vee(f-f^{r_i})$ $(f\in\widehat{\mathbf{C}[\mathfrak{h}^*]})$. It is an elementary exercise (using for example the classical Paley–Wiener theorem) to show that $\Delta_i(f)\in\mathrm{PW}(x)$ if $f\in\mathrm{PW}(x)$. Therefore we see that

$$H_{\mathrm{PW}(x)}=\mathrm{PW}(x)\otimes_{\mathbf{C}}\mathbf{C}[W]\subset\widehat{H}$$

is a subalgebra, and also an $\mathbf{H}$-module.

Definition 8.2. For all $x\in\mathfrak{a}$, we define the $\mathbf{H}$-module M_x by $M_x=\mathrm{Ind}_{C[W]}^{H_{\mathrm{PW}(x)}}(\mathrm{triv})$ (and of course we also define $M=\mathrm{Ind}_{C[W]}^{H_{\mathrm{PW}}}(\mathrm{triv})$).

Via the identification $\mathrm{PW}(x)\simeq\mathrm{PW}(x)\otimes 1=M_x$ we have equipped $\mathrm{PW}(x)$ with the structure of an $\mathbf{H}$-module.

Definition 8.3. $\pi\colon M\to\Sigma^\epsilon$ is the map such that $\forall\lambda\in i\mathfrak{a}_+^*$:

$$\pi(f\otimes 1)(\lambda)=\sum_w f(w\lambda)\prod_{\alpha\in R_+^0}\left(1-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{w\lambda(\alpha^\vee)}\right)v_w.$$

PROPOSITION 8.4. (1) *π is a monomorphism of* $\mathbf{H}$*-modules.*

(2) *Define functions $g_w^{w'}$ on $i\mathfrak{a}_+^*$ by means of the equation* ($\lambda \in i\mathfrak{a}_+^*$)

$$\pi(f\otimes 1)(\lambda)=\sum_w g_w^{w'}(\lambda) w w_0 v_{w_0 w'}.$$

Then $g_w^{w'}(\lambda)=(w^{-1}\cdot f)(w'\lambda)$. *In particular, $g_w^{w'}$ extends to an element of* $\mathrm{PW}(x)$ *if* $f\in\mathrm{PW}(x)$.

Proof. (1) Let $\xi\in\mathfrak{h}$ and $f\otimes 1\in M$. Then

$$\pi(\xi f\otimes 1)(\lambda)=\sum_w w\lambda(\xi) f(w\lambda)\prod_{\alpha\in R_+^0}\left(1-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{w\lambda(\alpha^\vee)}\right)v_w=\xi\cdot\pi(f\otimes 1)(\lambda).$$

Now let $r=r_\alpha$ be a simple reflection. Then

$$\begin{aligned}\pi(r\cdot(f\otimes 1))(\lambda)&=\pi\big((f^r r-(k_\alpha+\tfrac{1}{2}k_{\alpha/2})\Delta_\alpha(f))\otimes 1\big)(\lambda)\\ &=\pi\big((f^r-(k_\alpha+\tfrac{1}{2}k_{\alpha/2})\Delta_\alpha(f))\otimes 1\big)(\lambda)\\ &=\sum_w\left(\left(1+\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{w\lambda(\alpha^\vee)}\right)f(rw\lambda)-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{w\lambda(\alpha^\vee)}f(w\lambda)\right)\\ &\qquad\times\prod_{\alpha\in R_+^0}\left(1-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{w\lambda(\alpha^\vee)}\right)v_w\\ &=\sum_w f(w\lambda)\prod_{\alpha\in R_+^0}\left(1-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{w\lambda(\alpha^\vee)}\right)\\ &\qquad\times\left(\left(1+\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{w\lambda(\alpha^\vee)}\right)v_{rw}-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{w\lambda(\alpha^\vee)}v_w\right)\\ &=r\cdot\pi(f\otimes 1)(\lambda).\end{aligned}$$

(The last equality holds because of the definition of v_w, see Proposition 1.4.)

(2) By the W-equivariance of π if suffices to show this for $g_e^{w'}$; as in the proof of Proposition 1.4 one easily checks that indeed $g_e^{w'}(\lambda)=f(w'\lambda)$. □

COROLLARY 8.5. (1) *We can define a $*$-invariant Hermitean inner product on M by means of* $(f\otimes 1, g\otimes 1)=(\pi(f\otimes 1),\pi(g\otimes 1))$. *Then*

$$\begin{aligned}(f\otimes 1, g\otimes 1)&=\int_{i\mathfrak{a}^*} f(\lambda)\overline{w_0\cdot g(w_0\lambda)}\prod_{\alpha\in\mathbf{R}_+^0}\left(1-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{\lambda(\alpha^\vee)}\right)d\nu(\lambda)\\ &=\sum_{w\in W}\int_{iw'\mathfrak{a}_+^*} w\cdot f(\lambda)\overline{w\cdot g(\lambda)}\,d\nu(\lambda).\end{aligned}$$

(*The last equality holds for any choice of* $w'\in W$.)

(2) $\mathcal{J}$ *is well-defined on* $\pi(M)$.

Proof. Notice that

$$\prod_{\alpha\in R_+^0}\left(1-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{\lambda(\alpha^\vee)}\right)d\nu(\lambda)=\prod_{\alpha\in R_+}\left(\frac{-\lambda(\alpha^\vee)+k_\alpha+\frac{1}{2}k_{\alpha/2}}{\lambda(\alpha^\vee)+\frac{1}{2}k_{\alpha/2}}\right)d\nu(\lambda)$$
$$=\frac{(2\pi)^{-n}\tilde{c}^2_{w_0}(\varrho(k),k)}{\tilde{c}(\lambda,k)\tilde{c}_{w_0}(w_0\lambda,k)}\,d\lambda.$$

The function $(2\pi)^{-n}\tilde{c}^2_{w_0}(\varrho(k),k)/\tilde{c}(\lambda,k)\tilde{c}_{w_0}(w_0\lambda,k)$ is regular on $i\mathfrak{a}^*$ (if $k_\alpha\geqslant 0\ \forall\alpha$). Moreover, we have the following well-known elementary estimates of the c-functions (see [11, Proposition 7.2]):

(a) $|\tilde{c}(\lambda,k)|^{-1}\leqslant c_1+c_2|\lambda|^{\sum_{\alpha\in R_+}k_\alpha}$ $(\lambda\in\overline{\mathfrak{a}^*_+}+i\mathfrak{a}^*)$,

(b) $|\tilde{c}_{w_0}(\lambda,k)|^{-1}\leqslant c_1+c_2|\lambda|^{\sum_{\alpha\in R_+}k_\alpha}$ $(\lambda\in\overline{\mathfrak{a}^*_+}+i\mathfrak{a}^*)$.

From these estimates (1) and (2) easily follow (using Theorem 4.2, Definition 8.3 and Proposition 8.4 (2)). □

THEOREM 8.6 (Paley–Wiener theorem). *Let $k_\alpha>0\ \forall\alpha\in R$.*

(1) *$\mathcal{F}$ maps $C_c^\infty(C_x)$ to $\pi(M_x)$ $\forall x\in\mathfrak{a}$,*

(2) *$\mathcal{J}$ maps $\pi(M_x)$ to $C_c^\infty(C_x)$ $\forall x\in\mathfrak{a}$.*

(We use the notation $C_c^\infty(C_x)$ for the space of C^∞ functions on $\mathfrak{a}$ having support inside C_x.)

Proof. (1) Let $f\in C_c^\infty(C_x)$, and let $(\forall\lambda\in\mathfrak{h}^{*,\mathrm{reg}})$

$$\mathcal{F}(f)(\lambda)=\sum_w d_w(f)(\lambda)\prod_{\alpha\in R_+^0}\left(1-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{w\lambda(\alpha^\vee)}\right)v_w.$$

Using Proposition 7.3 and the definition of $\mathcal{F}$ we get

$$d_w(f)(\lambda)=(\mathcal{F}(f)(\lambda),w_0v_{w_0w})=\int_{\mathfrak{a}}f(y)G(-w_0w\lambda,k,w_0y)\,d\mu(y)=d_e(f)(w\lambda).$$

Hence we need to show that $d_e(f)\in\mathrm{PW}(x)$. It is clear that $d_e(f)$ is entire and from Proposition 6.1 (1) we conclude that

$$|d_e(f)(\lambda)|\leqslant C_0e^{H_x(-\operatorname{Re}\lambda)}$$

(where $C_0\geqslant|W|^{1/2}\|f\|_1$). If $p\in S(\mathfrak{h})$ arbitrary then, using Lemma 7.8:

$$|p(\lambda)|\cdot|d_e(f)(\lambda)|=|d_e(p(D)(f))(\lambda)|\leqslant C_1e^{H_x(-\operatorname{Re}\lambda)}.$$

This implies the result.

(2) If $\phi\in M_x$ then

$$\begin{aligned}\mathcal{J}(\pi(\phi))(y) &= \int_{i\mathfrak{a}^*_+} \mathcal{J}_\lambda(\pi(\phi(\lambda))(y)\,d\nu(\lambda)\\ &= \int_{i\mathfrak{a}^*} \phi(\lambda)G(\lambda,k,y) \prod_{\alpha\in R^0_+}\left(1-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{\lambda(\alpha^\vee)}\right)d\nu(\lambda).\end{aligned}$$

From the estimates in the proof of Corollary 8.5 and Corollary 6.2 we see that $\mathcal{J}(\pi(\phi))\in C^\infty(\mathfrak{a})$. Next we need to prove that $\mathcal{J}(\pi(\phi))(y)=0$ if $y\notin C_x$. By the W-equivariance of $\mathcal{J}\pi$ it suffices to show this when $y\in\mathfrak{a}_-$. We can now almost copy Helgason's proof of the Paley–Wiener theorem for Riemannian symmetric spaces [11]. Assume $y\in\mathfrak{a}_-$ and $y\notin C_x$. Using now that $k_\alpha>0$ $\forall\alpha\in R$ we have the following estimate (similar to the ones in the proof of Corollary 8.5):

$$\frac{1}{|\prod_{\alpha\in R^0_+}\lambda(\alpha^\vee)c(\lambda,k)|} \leqslant C_1+C_2|\lambda|^{\sum_{\alpha\in R_+}k_\alpha} \quad \text{if } \lambda\in\overline{\mathfrak{a}^*_+}+i\mathfrak{a}^*. \tag{8.1}$$

Using Corollary 6.5 (3) we obtain

$$\begin{aligned}\mathcal{J}(\pi(\phi))(y) &= \int_{i\mathfrak{a}^*}\phi(\lambda)\widetilde{G}(\lambda,k,y)\frac{1}{\prod_{\alpha\in R^0_+}\lambda(\alpha^\vee)}\,d\nu(\lambda)\\ &= \sum_{\varkappa\in Q_+}\sum_{w\in W}\int_{i\mathfrak{a}^*}\phi(\lambda)\Lambda_\varkappa(w;\lambda,k)e^{(w\lambda+\varrho(k)+\varkappa)(y)}\frac{(2\pi)^{-n}|W|^{-1}(-i)^n\,d\lambda}{\prod_{\alpha\in R^0_+}\lambda(\alpha^\vee)c(w\lambda,k)}\end{aligned}$$

(where we used that $c(\lambda,k)c(w_0\lambda,k)=c(w\lambda,k)c(-w\lambda,k)$ $\forall w\in W$ and that $c_{w_0}(\varrho(k),k)=|W|^{-1}c(\varrho(k),k)$). Hence it suffices to show that $\forall w,\varkappa$:

$$u_{w,\varkappa}=\int_{i\mathfrak{a}^*}\phi(\lambda)\Lambda_\varkappa(w;\lambda,k)e^{w\lambda(y)}\frac{d\lambda}{\prod_{\alpha\in R^0_+}\lambda(\alpha^\vee)c(w\lambda,k)}=0.$$

Choose $\eta\in\mathfrak{a}^*_+$. We know that $\Lambda_\varkappa(w;\lambda,k)$ and $c(w\lambda,k)^{-1}\prod_{\alpha\in R^0_+}\lambda(\alpha^\vee)^{-1}$ are holomorphic for $w\lambda\in\overline{\mathfrak{a}^*_+}+i\mathfrak{a}^*$. Using the Paley–Wiener estimates for ϕ, Corollary 6.5 (3) and (8.1) we see that

$$\frac{|\phi(\lambda)\Lambda_\varkappa(w;\lambda,k)e^{w\lambda(y)}|}{|\prod_{\alpha\in R^0_+}\lambda(\alpha^\vee)c(w\lambda,k)|}\leqslant C(1+|\lambda|)^{-n-1}e^{\mathrm{Re}(w\lambda(y))+H_x(-\mathrm{Re}\,w\lambda)}$$

if $w\lambda\in\mathfrak{a}^*_++i\mathfrak{a}^*$. Thus we may apply Cauchy's theorem to $u_{w,k}$, changing the contour of integration from $i\mathfrak{a}^*$ to $i\mathfrak{a}^*+tw^{-1}\eta$ $(t\in\mathbf{R}_+)$. By the above estimate for the integrand of $u_{w,\varkappa}$ we get

$$|u_{w,\varkappa}|\leqslant C'e^{t(\eta(y)+H_x(-\eta))}.$$

We may assume that $x\in\mathfrak{a}_-$. Then $H_x(-\eta)=-\eta(x)$. Hence $t(\eta(y)+H_x(-\eta))=t\eta(y-x)$. If $y\notin C_x$ and $x,y\in\mathfrak{a}_-$ then $y\notin x+\mathbf{R}_+R^\vee_+$. Thus we can choose $\eta\in\mathfrak{a}^*_+$ in such a way that $\eta(y-x)<0$. Now let $t\to\infty$ and we conclude that $u_{w,\varkappa}=0$ $\forall w,\varkappa$. $\square$

9. Inversion formulas and the Plancherel formula

In this section we will assume that $k_\alpha > 0$ $\forall \alpha \in R$ unless stated otherwise. We will treat the inversion formulas and the Plancherel formula for the Cherednik transform. Our proof is similar to Rosenberg's proof [25] of the spherical Plancherel formula for a noncompact symmetric space. However, in our more general situation there is no underlying group structure and therefore it is not enough to prove the inversion formula at the origin only (as is the case in Rosenberg's proof). This is the reason why we have to use the argument of van den Ban and Schlichtkrull [1] at this point.

By Theorem 8.6 we may define:

Definition 9.1. Let $K=\mathcal{J}\mathcal{F}: C_c^\infty(\mathfrak{a}) \to C_c^\infty(\mathfrak{a})$.

PROPOSITION 9.2. (1) *K is symmetric with respect to the inner product* $(f,g)=\int f\bar{g}\,d\mu$.

(2) *K is an* $\mathbf{H}$*-module morphism.*

(3) $K(C_c^\infty(C_x)) \subset C_c^\infty(C_x)$ $\forall x \in \mathfrak{a}$.

Proof. (1) Immediate from Proposition 7.12 (5). (2) Immediate from Proposition 7.10 (4) and Proposition 7.12 (4). (3) Clear by Theorem 8.6. □

LEMMA 9.3. *If* $f \in C_c^\infty(\mathfrak{a})$ *then* $\mathrm{supp}(Kf) \subset \bigcup_{w\in W} w(\mathrm{supp}(f))$.

Proof. First we construct a basis $\mathcal{B}$ for the Euclidean topology on $\mathfrak{a}$ with the property that if $B\in\mathcal{B}$ and $f\in C_c^\infty(B)$ then $\mathrm{supp}(Kf)\subset\bigcup_{w\in W}\overline{wB}$. We introduce the following notations. If $x\in\mathfrak{a}$ we define $\mathfrak{a}_x^+=\sum_{\{\alpha\in R\mid\alpha(x)>0\}}\mathbf{R}_+\alpha^\vee$ and ${}^+\mathfrak{a}_x=\sum_{\{\alpha\in R\mid\alpha(x)\geqslant 0\}}\mathbf{R}_+\alpha^\vee$. The interior of the convex hull of Wx is denoted by F_x, and one easily checks that $F_x=\bigcap_{w\in W}w(x-\mathfrak{a}_x^+)$. Now let ${}_xF=\bigcup_{w\in W}w(x+{}^+\mathfrak{a}_x)$. We claim that these sets satisfy the following property: $y\notin{}_xF \Rightarrow \overline{F_y}\cap{}_xF=\varnothing$. Let us prove this claim. Assume that $y\notin{}_xF$ and $\exists z\in\overline{F_y}\cap{}_xF$. Since $F_y=wF_y=F_{wy}$ and ${}_xF=w_xF={}_{wx}F$ $\forall w\in W$ we may also assume that $z\in x+{}^+\mathfrak{a}_x$, and that $\mathfrak{a}_y^+\subset{}^+\mathfrak{a}_x$. But then $z\in(x+{}^+\mathfrak{a}_x)\cap\overline{F_y}\subset(x+{}^+\mathfrak{a}_x)\cap(y-\overline{\mathfrak{a}_y^+})\Rightarrow y\in x+{}^+\mathfrak{a}_x+\overline{\mathfrak{a}_y^+}=x+{}^+\mathfrak{a}_x\subset{}_xF$, in contradiction to the assumptions. This proves the claim. Next we prove

$$f\in C_c^\infty({}_xF) \implies \mathrm{supp}(Kf)\subset\overline{{}_xF}. \tag{9.1}$$

Namely, assume that $\exists y\notin{}_xF$ such that $Kf(y)\neq 0$. Then $\exists g\in C_c^\infty(F_y)$ with $(g,Kf)\neq 0$. But by the symmetry of K (Proposition 9.2 (1)) this means that $(Kg,f)\neq 0$. Since $\mathrm{supp}(Kg)\subset\overline{F_y}$ (by the Paley–Wiener theorem 8.6) and $\mathrm{supp}(f)\subset{}_xF$ and $\overline{F_y}\cap{}_xF=\varnothing$ (by the above claim) this is a contradiction, proving (9.1). We now come to the definition of $\mathcal{B}$. Define $\mathcal{B}=\{B_{x,y}\}_{x,y\in\mathfrak{a}}$ with $B_{x,y}=F_x\cap\{y+{}^+\mathfrak{a}_y\}$. Since $\bigcup_{w\in W}wB_{x,y}=\bigcup_w B_{x,wy}=F_x\cap{}_yF$ we see that if $f\in C_c^\infty(B_{x,y})$ then $\mathrm{supp}(Kf)\subset\overline{F_x}\cap\overline{{}_yF}=\bigcup_w w\overline{B_{x,y}}$. $\mathcal{B}$ contains arbitrarily small neighbourhoods of x ($\forall x\in\mathfrak{a}\setminus\{0\}$) of the form $B_{(1+\varepsilon)x,(1-\varepsilon)x}$ ($0<\varepsilon<1$), and the sets $B_{x,0}=F_x$ form a basis of neighbourhoods for $0\in\mathfrak{a}$. All in all the collection $\mathcal{B}$ fulfils the description from the beginning of this proof.

The rest of the proof is a standard application of partitions of unity. Given $f\in C_c^\infty(\mathfrak{a})$ and $y\notin\bigcup_w w(\mathrm{supp}(f))$, we can find a finite covering of $\mathrm{supp}(f)$ with elements from $\mathcal{B}$, $\mathrm{supp}(f)\subset\bigcup_i B_i$ say, such that $\forall w\in W$: $wy\notin\bigcup_i \overline{B_i}$. By partition of unity there exists a decomposition $f=\sum_i\phi_i$ with $\phi_i\in C_c^\infty(B_i)$, so that

$$\mathrm{supp}(Kf)=\mathrm{supp}\left(\sum_i K\phi_i\right)\subset\bigcup_i\mathrm{supp}(K\phi_i)\subset\bigcup_i\bigcup_w w\overline{B_i}\not\ni y. \qquad \square$$

COROLLARY 9.4. *There exist differential operators D_w ($w\in W$) on $\mathfrak{a}^{\mathrm{reg}}$, locally of finite order and with coefficients in $C^\infty(\mathfrak{a}^{\mathrm{reg}})$ such that $\forall f\in C_c^\infty(\mathfrak{a})$:*

$$(Kf)|\mathfrak{a}^{\mathrm{reg}}=\sum_{w\in W}D_w w(f|\mathfrak{a}^{\mathrm{reg}}).$$

Proof. If $f\in C_c^\infty(\mathfrak{a})$, $x\in\mathfrak{a}^{\mathrm{reg}}$ and $U\ni x$ is a ball in $\mathfrak{a}^{\mathrm{reg}}$ such that $\overline{U}\subset\mathfrak{a}^{\mathrm{reg}}$ we conclude from Lemma 9.3 that $Kf|U=K\tilde{f}|U$ if $\tilde{f}\in C_c^\infty(\mathfrak{a}^{\mathrm{reg}})$ is such that $\tilde{f}=f$ on WU. Hence it suffices to show that $K=\sum_w D_w w$ on $C_c^\infty(\mathfrak{a}^{\mathrm{reg}})$. From Lemma 9.3 it is clear that on $C_c^\infty(\mathfrak{a}^{\mathrm{reg}})$ we have a decomposition $K=\sum_w K_w w$ such that $K_w\colon C_c^\infty(\mathfrak{a}^{\mathrm{reg}})\to C_c^\infty(\mathfrak{a}^{\mathrm{reg}})$ is support preserving. By Peetre's theorem [23], K_w is a differential operator with the asserted properties. $\square$

LEMMA 9.5. *With the notation of the previous corollary, D_w has finite order ($\forall w\in W$), the order of D_e is strictly larger than the order of D_w ($w\neq e$) and the highest order part of D_e has locally constant coefficients on $\mathfrak{a}^{\mathrm{reg}}$.*

Proof. We first study the restriction of K to the $\mathbf{H}$-submodule $C_c^\infty(\bigcup_w wU)$ of $C_c^\infty(\mathfrak{a})$ where $U\subset\mathfrak{a}^{\mathrm{reg}}$ is some open ball such that $\overline{U}\subset\mathfrak{a}^{\mathrm{reg}}$. The differential operators D_w have finite order on U, and $K=\sum_w D_w w\colon C_c^\infty(\bigcup_w wU)\to C_c^\infty(\bigcup_w wU)$ is an $\mathbf{H}$-module morphism (Proposition 9.2 (2)). Let

$$\sigma(K)=\sum_w\sigma_d(D_w)w=\sum_{\{\beta:|\beta|=d\}}\sum_{w\in W}a_{w,\beta}\partial(\xi^\beta)w$$

denote the highest order part of K on U. Note that K, and thus $\sigma(K)$, is W-invariant. Hence *modulo terms of order* $\leqslant d$ we have ($\forall\xi\in\mathfrak{a}$)

$$\begin{aligned}0=[D_\xi,K]&=[\partial_\xi,\sigma(K)]+\sum_{\alpha\in R_+}k_\alpha\alpha(\xi)\left[\frac{1}{1-e^{-\alpha}},\sigma(K)\right](1-r_\alpha)\\&=[\partial_\xi,\sigma(K)]+\sum_{\beta,w}a_{w,\beta}\partial(\xi^\beta)(\partial_\xi-\partial_{w\xi})w.\end{aligned}$$

Hence $\sigma(D_e)$ is W-invariant and $a_{w,\beta}=0$ $\forall w\neq e$. In particular, the symbol $\sigma_{d-1}(D_w)$ is well-defined if $w\neq e$. Now, *modulo terms of order* $\leqslant d-1$ we have $(\forall\xi\in\mathfrak{a})$

$$\begin{aligned}
0&=[D_\xi,K]\\
&=[\partial_\xi,\sigma_d(D_e)]+\Big[\partial_\xi,\sum_{w\neq e}\sigma_{d-1}(D_w)w\Big]+\sum_{\alpha\in R_+}k_\alpha\alpha(\xi)\Big[\frac{1}{1-e^{-\alpha}}(1-r_\alpha),\sigma_d(D_e)\Big]\\
&=[\partial_\xi,\sigma_d(D_e)]+\sum_{w\neq e}\sigma_{d-1}(D_w)(\partial_\xi-\partial_{w\xi})w+\sum_{\alpha\in R_+}k_\alpha\alpha(\xi)\Big[\frac{1}{1-e^{-\alpha}},\sigma_d(D_e)\Big](1-r_\alpha)\\
&=\sum_{\{\beta:|\beta|=d\}}\partial_\xi(a_{e,\beta})\partial(\xi^\beta)+\sum_{w\neq e}\sigma_{d-1}(D_w)(\partial_\xi-\partial_{w\xi})w.
\end{aligned}$$

Hence $a_{e,\beta}$ is locally constant on U, and apparently we also have $\sigma_{d-1}(D_w)=0$ $\forall w\neq e$ (but we will not need this fact in the sequel). □

LEMMA 9.6. *Let $\pi^{-1}C^\infty(\mathfrak{a})$ denote the ring of functions generated by $\prod_{\alpha\in R_+^0}\alpha^{-1}$ and $C^\infty(\mathfrak{a})$. Let D_w be a differential operator of finite order on $\mathfrak{a}^{\mathrm{reg}}$ with coefficients in $C^\infty(\mathfrak{a}^{\mathrm{reg}})$ $(\forall w\in W)$. If $K=\sum_w D_w w$ has the property that $K(f|\mathfrak{a}^{\mathrm{reg}})\in\pi^{-1}C^\infty(\mathfrak{a})$ $\forall f\in C^\infty(\mathfrak{a})$ then the coefficients of D_w are in $\pi^{-1}C^\infty(\mathfrak{a})$ $(\forall w\in W)$.*

Proof. By induction on the maximal order d of $D_w(w\in W)$. Let $d=0$, and write $K=\sum a_w w$ with $a_w\in C^\infty(\mathfrak{a}^{\mathrm{reg}})$. Let $(h_w)_{w\in W}$ be a basis of harmonic polynomials on $\mathfrak{a}$. Clearly, if $x\in\mathfrak{h}^{\mathrm{reg}}$ then $\det(x)=\det((h_w^{w'})_{w,w'})(x)\neq0$, so that $\det^{-1}\in\pi^{-1}C^\infty(\mathfrak{a})$. Since $K(h_w)=\sum_{w'}a_{w'}h_w^{w'}\in\pi^{-1}C^\infty(\mathfrak{a})$ we see that $a_w\in\pi^{-1}C^\infty(\mathfrak{a})$ by Cramer's rule. Let us now consider the induction step. Let $(p_i)_{i=1,\dots,n}$ be a set of generators for $\mathbf{C}[\mathfrak{h}]^W$. Then

$$[K,p_i]=\sum_w[D_w,p_i]w$$

and if $D_w=\sum_{\{\beta:|\beta|\leqslant d\}}a_{w,\beta}\partial(\xi^\beta)$ then

$$\sigma_{d-1}[D_w,p_i]=\{\sigma_d(D_w),p_i\}=\sum_{\{\beta:|\beta|=d\}}\sum_{j=1}^n a_{w,\beta}\frac{\partial p_i}{\partial x_j}\,\partial\Big(\frac{\partial\xi^\beta}{\partial\xi_j}\Big)$$

(by using the standard formula for the Poisson bracket in coordinates). Thus by the induction hypothesis we obtain $\forall\beta\in\mathbf{Z}_+^n$ with $|\beta|=d-1$ and $\forall i\in\{1,\dots,n\}$:

$$b_{\beta,i}=\sum_{j=1}^n(\beta_j+1)a_{w,\beta+e_j}\frac{\partial p_i}{\partial x_j}\in\pi^{-1}C^\infty(\mathfrak{a}).$$

Using Cramer's rule and the fact that $\det(\partial p_i/\partial x_j)=c\prod_{\alpha\in R_+^0}\alpha$ $(c\neq0)$ we see that $a_{w,\beta+e_j}\in\pi^{-1}C^\infty(\mathfrak{a})$ $(\forall w,j$ and β with $|\beta|=d-1)$. We may now subtract $\sigma_d(K)$ and apply the induction hypothesis once more to finish the proof. □

COROLLARY 9.7. *There exists a* $p \in S(\mathfrak{h})^W$ *such that* $\forall f \in C_c^\infty(\mathfrak{a})$, $K(f)=p(D)f$.

Proof. It is sufficient to prove that $\exists p \in S(\mathfrak{h})^W$ such that $K(f)|\mathfrak{a}^{\mathrm{reg}}=p(D)(f)|\mathfrak{a}^{\mathrm{reg}}$ $\forall f \in C_c^\infty(\mathfrak{a})$. By Corollary 9.4 and Lemma 9.5, $K(f)|\mathfrak{a}^{\mathrm{reg}}=\sum_w D_w w(f|\mathfrak{a}^{\mathrm{reg}})$ with D_w differential operators in $\mathfrak{a}^{\mathrm{reg}}$ with C^∞-coefficients, $\mathrm{ord}(D_w)<\mathrm{ord}(D_e)$ $\forall w \neq e$ and such that the coefficients of $\sigma(D_e)$ are locally constant. Clearly the condition of Lemma 9.6 is satisfied. (Using Lemma 9.3 we extend the action of K to $C^\infty(\mathfrak{a}^{\mathrm{reg}})$ by defining $Kf(x)=Kg(x)$ where $g \in C_c^\infty(\mathfrak{a})$ and $g=f$ in a neighbourhood of Wx.) Hence we know that the coefficients of D_w are in $\pi^{-1}C^\infty(\mathfrak{a})$. In particular we see that $\sigma(D_e)$ has constant coefficients. On the other hand we know that $\sigma(D_e)$ is W-invariant, hence we can find a homogeneous $p' \in S(\mathfrak{h})^W$ such that $\mathrm{ord}(K-p'(D))<\mathrm{ord}(K)$ on $C^\infty(\mathfrak{a}^{\mathrm{reg}})$. Define K' by $K'=K-p'(D)$. This is an operator on $C_c^\infty(\mathfrak{a})$ of the form $K'=\sum_{w\in W} D'_w w$ with $D'_w \in \pi^{-1}C_c^\infty(\mathfrak{a})$, which commutes with the action of $\mathbf{H}$. Hence we may again apply Lemma 9.5 to conclude that $\mathrm{ord}(D'_w)<\mathrm{ord}(D'_e)$ $\forall w \neq e$ and that the coefficients of $\sigma(D'_e)$ are constant. Replacing K by K' and induction on the order completes the proof. □

Our next task is to prove that actually $p=1$. Obviously it is sufficient to only consider the restriction of K to $C_c^\infty(\mathfrak{a})^W$. In this situation we can simply refer to [1], since the only information that is used in their proof is information on the asymptotic behaviour of Eisenstein integrals. The analogous asymptotic results for $F(\lambda, k, x)$ are well-known (and in fact much simpler than in their situation) (cf. Theorem 6.3). Needless to say, the fact that $p=1$ is a reflection of the relation between the asymptotic behaviour of F (cf. Theorem 6.3) and the measure ν (cf. Definition 7.5).

In order to give precise references, let us introduce a notation. Given $\phi \in C_c^\infty(\mathfrak{a}^{\mathrm{reg}})^W$, we define ($t \in \mathbf{R}_+, \lambda \in i\mathfrak{a}^*$): $\phi_{t,\lambda}(x)=t^{-n/2}|\delta_k(x)|^{-1/2}e^{\lambda(x)}\phi(x/t)$. Then

LEMMA 9.8 ([1, Lemma 12.15]). *If* $\phi, \psi \in C_c^\infty(\mathfrak{a}^{\mathrm{reg}})^W$ *then* (*recall the inner product* $(\cdot,\cdot)_k$ *that was defined in Lemma* 7.8)

$$\lim_{t\to\infty}(p(D)\phi_{t,\lambda}, \psi_{t,\lambda})_k = p(\lambda)(\phi, \psi)_0.$$

LEMMA 9.9 ([1, Corollary 13.3]). *If* $\phi, \psi \in C_c^\infty(\mathfrak{a}^{\mathrm{reg}})^W$ *then*

$$\lim_{t\to\infty}(\mathcal{F}(\phi_{t,\lambda}), \mathcal{F}(\psi_{t,\lambda}))_{\Sigma_2} = (\phi, \psi)_0.$$

THEOREM 9.10. *On* $C_c^\infty(\mathfrak{a})$ *we have* $K=\mathcal{J}\mathcal{F}=\mathrm{id}$.

Proof. By Corollary 9.7 we have $K=p(D)$. Choose $\phi, \psi \in C_c^\infty(\mathfrak{a}_{\mathrm{reg}})^W$ such that

$$(\phi, \psi)_0 = \int_{\mathfrak{a}} \phi\psi\, dx \neq 0.$$

Then by Lemma 9.8 and Lemma 9.9 we have

$$\begin{aligned}p(\lambda)(\phi,\psi)_0&=\lim_{t\to\infty}(p(D)\phi_{t,\lambda},\psi_{t,\lambda})_k\\&=\lim_{t\to\infty}(\mathcal{J}\mathcal{F}\phi_{t,\lambda},\psi_{t,\lambda})_k\\&=\lim_{t\to\infty}(\mathcal{F}\phi_{t,\lambda},\mathcal{F}\psi_{t,\lambda})_{\Sigma_2}=(\phi,\psi)_0.\end{aligned}$$

Hence $p=1$. □

Let us now consider the operator $K'=\pi^{-1}\mathcal{F}\mathcal{J}\pi\colon M\to M$.

LEMMA 9.11. *There exists an entire function f on $\mathfrak{h}^*$ such that $K'=m_f$ (multiplication by f) on M. Moreover, f is W-invariant.*

Proof. This is analogous to the well-known treatment of the Fourier inversion formula on the Schwartz space due to Gårding (cf. [12, Vol. I, Lemma 7.1.4]). We note that K' is an $\mathbf{H}$-module morphism, so in particular that $m_\xi K'=K'm_\xi$ $\forall\xi\in\mathfrak{h}$. It follows that $\forall\lambda\in\mathfrak{h}^*$, K' maps the maximal ideal i_λ associated to λ into itself. Hence we define $f(\lambda)\in\mathbf{C}\simeq\mathrm{End}(\mathbf{C})$ as the value of the quotient map of K' on $M/i_\lambda\simeq\mathbf{C}$. Since $f\phi\in M$ $\forall\phi\in M$ it is clear that f is entire. It is also clear that f is invariant for the action of W on $M=\mathrm{Ind}_{\mathbf{C}[W]}^{H_{\mathrm{PW}}}(\mathrm{triv})$, and it is an easy exercise to see that this coincides with invariance with respect to the ordinary W-action on PW. □

LEMMA 9.12. *$K'=\mathrm{id}$ on M.*

Proof. Since $G(\lambda,k,0)=1$ $\forall\lambda$ we have $\forall\lambda\in i\mathfrak{a}^*_+$ and $\phi\in M^W$:

$$\mathcal{J}_\lambda(\pi\phi(\lambda))(0)=\phi(\lambda)\sum_{w\in W}\prod_{\alpha\in R_+^0}\left(1-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{w\lambda(\alpha^\vee)}\right)G(w\lambda,k,0)=|W|\phi(\lambda)$$

(see Definition 8.3; in the second equality we used identity (4.4)). Hence $\forall\phi\in M^W$ we have

$$\int_{i\mathfrak{a}^*}\phi(\lambda)\,d\nu(\lambda)=\mathcal{J}(\pi(\phi))(0)=\mathcal{J}\mathcal{F}\mathcal{J}(\pi(\phi))(0)=\mathcal{J}(\pi(f\phi))(0)=\int_{i\mathfrak{a}^*}f(\lambda)\phi(\lambda)\,d\nu(\lambda).$$

Since f is W-invariant and entire this implies that $f\equiv1$. □

In the next theorem we summarize our results:

THEOREM 9.13. *Let $k_\alpha>0$ $\forall\alpha\in R$. Recall the definitions of Sections 7 and 8.*

(1) *$\forall x_0\in\mathfrak{a}$, $\pi^{-1}\mathcal{F}$ is an isomorphism of the $\mathbf{H}$-module $C_c^\infty(C_{x_0})$ onto M_{x_0}. Its inverse is $\mathcal{J}\pi$. We have the following explicit formulas: $\forall f\in C_c^\infty(\mathfrak{a})$ we have*

$$\pi^{-1}\mathcal{F}f(\lambda)=\int_{\mathfrak{a}}f(x)G(-w_0\lambda,k,w_0x)\,d\mu(x)\quad(\forall\lambda\in\mathfrak{h}^*)$$

and $\forall\phi\in M$ *we have*

$$\mathcal{J}\pi(\phi)(x)=\int_{i\mathfrak{a}^*}\phi(\lambda)G(\lambda,k,x)\prod_{\alpha\in R_+^0}\left(1-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{\lambda(\alpha^\vee)}\right)d\nu(\lambda).$$

(2) *Define* $\mathcal{F}_w^{w'}f$ $(w,w'\in W,\ f\in C_c^\infty(\mathfrak{a}))$ *as the function on* $iw'\mathfrak{a}_+^*$ *such that* $\forall\lambda\in i\mathfrak{a}_+^*$:

$$\mathcal{F}f(\lambda)=\sum_w\mathcal{F}_w^{w'}f(w'\lambda)wv_{w'}.$$

$\mathcal{F}_w^{w'}f$ *is the restriction to* $iw'\mathfrak{a}_+^*$ *of the analytic continuation of* $\mathcal{F}_w^e f$. *Moreover,* $\forall\lambda\in iw'\mathfrak{a}_+^*$:

$$\mathcal{F}_w^{w'}f(\lambda)=((w_0w^{-1})\cdot\pi^{-1}\mathcal{F}f)(w_0\lambda)=\int_{\mathfrak{a}}f(x)G(-\lambda,k,w^{-1}x)\,d\mu(x).$$

The inversion formula now takes the following form ($w'\in W$ *arbitrary*):

$$f(x)=\sum_w\int_{\lambda\in i\mathfrak{a}_+^*}\mathcal{F}_w^{w'}f(w'\lambda)G(w'\lambda,k,w^{-1}x)\,d\nu(\lambda).$$

(3) *Equip* $C_c^\infty(\mathfrak{a})$ *with the inner product* $(f,g)=\int_{\mathfrak{a}}f\bar{g}\,d\mu$ *and* $\pi(M)$ *with the inner product* $(\sigma,\tau)=\int_{i\mathfrak{a}_+^*}(\sigma(\lambda),\tau(\lambda))_{I_\lambda}d\nu(\lambda)$. *Then* $\mathcal{F}$ *is an isometry. Explicitly we have the following Parseval-type formulas* ($f,g\in C_c^\infty(\mathfrak{a})$):

$$\begin{aligned}\int_{\mathfrak{a}}f\bar{g}\,d\mu&=\sum_w\int_{i\mathfrak{a}_+^*}\mathcal{F}_w^{w'}f(w'\lambda)\overline{\mathcal{F}_w^{w'}g(w'\lambda)}\,d\nu(\lambda)\\&=\int_{i\mathfrak{a}^*}\pi^{-1}\mathcal{F}f(\lambda)\overline{(w_0\cdot\pi^{-1}\mathcal{F}g)(w_0\lambda)}\prod_{\alpha\in R_+^0}\left(1-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{\lambda(\alpha^\vee)}\right)d\nu(\lambda).\end{aligned}$$

(4) *If* $f\in C_c^\infty(\mathfrak{a})^W$ *then*

$$\pi^{-1}\mathcal{F}f(\lambda)=\int_{\mathfrak{a}}f(x)F(-\lambda,k,x)\,d\mu(x).$$

Clearly $\pi^{-1}\mathcal{F}f\in M^W=\mathrm{PW}^W$. *Conversely, if* $\phi\in\mathrm{PW}^W$, *then*

$$\mathcal{J}\pi(\phi)(x)=|W|\int_{i\mathfrak{a}_+^*}\phi(\lambda)F(\lambda,k,x)\,d\nu(\lambda).$$

We can rewrite the Parseval identity for W*-invariant functions as follows. Let* $f,g\in C_c^\infty(\mathfrak{a})^W$. *Then*

$$\int_{\mathfrak{a}}f\bar{g}\,d\mu=|W|\int_{i\mathfrak{a}_+^*}\pi^{-1}\mathcal{F}f(\lambda)\overline{\pi^{-1}\mathcal{F}g(\lambda)}\,d\nu(\lambda).$$

(5) $\mathcal{F}$ *extends in a unique way to an isometric isomorphism of* $L_2(\mathfrak{a},\mu)$ *onto* $\Sigma_2\simeq\int_{i\mathfrak{a}_+^*}^{\oplus}I_\lambda\,d\nu(\lambda)$.

Proof. (1) The explicit formulas follow directly from the definitions of $\mathcal{F}$ and $\mathcal{J}$ and π. The stated results follow from Theorem 8.6, Theorem 9.10 and Lemma 9.12.

(2) According to (1) we know that $\mathcal{J}\mathcal{F}=\mathrm{id}$. The explicit formulas follow directly from the definitions of $\mathcal{F}$ and $\mathcal{J}$.

(3) Use Proposition 7.12 (5) and the inversion formula $\mathcal{J}\mathcal{F}=\mathrm{id}$. The explicit formula in terms of $\pi\mathcal{F}$ is a consequence of Corollary 8.5.

(4) Use the following formulas (see Proposition 1.4):

$$\begin{aligned} F(\lambda,k,x)=\mathcal{J}_\lambda(\phi)(x) &= \frac{1}{|W|}\sum_w G(\lambda,k,wx) \\ &= \frac{1}{|W|}\sum_w \prod_{\alpha\in R_+^0}\left(1-\frac{k_\alpha+\frac{1}{2}k_{\alpha/2}}{w\lambda(\alpha^\vee)}\right)G(w\lambda,k,x) \end{aligned}$$

(and $\sum_w \prod_{\alpha\in R_+^0}\left(1-(k_\alpha+\frac{1}{2}k_{\alpha/2})/w\lambda(\alpha^\vee)\right)=|W|$).

(5) We need to show that $\pi(M)$ is dense in Σ_2. Since $\Sigma_c^\infty\subset\Sigma_2$ is dense, it suffices to show that $\forall\sigma\in\Sigma_c^\infty$, there exists a sequence $\{\sigma_k\}$ in $\pi(M)$ such that $\sigma_k\to\sigma$ in Σ_2. Extend π in the obvious way to $C^\infty(i\mathfrak{a}^*)$, and give $C_c^\infty(i\mathfrak{a}^{*,\mathrm{reg}})$ the structure of an $\mathbf{H}$-module with $*$-invariant inner product via $\pi\colon C_c^\infty(i\mathfrak{a}^{*,\mathrm{reg}})\xrightarrow{\sim}\Sigma_c^\infty\subset\Sigma_2$. Hence $(\phi,\psi)=\sum_w\int_{i\mathfrak{a}_+^*} w\cdot\phi\,\overline{w\cdot\psi}\,d\nu$ $\forall\phi,\psi\in C_c^\infty(i\mathfrak{a}^{*,\mathrm{reg}})$. We know that $M\supset M^W=\mathrm{PW}^W$, and it is easy to see (using the ordinary Fourier transform) that PW^W is dense in the Schwartz space $S(i\mathfrak{a}^*)^W$. Thus, if $\phi\in C_c^\infty(i\mathfrak{a}^{*,\mathrm{reg}})^W$ then we can find a sequence $\phi_k\to\phi$ (in the Schwartz topology) with $\phi_k\in\mathrm{PW}^W$. This implies that $p\phi_k\to p\phi$ in $L_2(i\mathfrak{a}^*,\nu)$ for all polynomials p. Using the relations in $\mathbf{H}$, we see that for any W-invariant function ψ and $w\in W$ there exists a polynomial p_w such that

$$w\cdot p\psi=p_w\psi.$$

Hence $w\cdot p\phi_k\to w\cdot p\phi$ in $L_2(i\mathfrak{a}^*,\nu)$ ($\forall p$, polynomial and $w\in W$). It follows that $p\phi_k\to p\phi$ with respect to the inner product (ϕ,ψ). Next we observe that if $\{h_w\}_{w\in W}$ is a basis of harmonic polynomials on $\mathfrak{h}^*$ and $\phi\in C_c^\infty(i\mathfrak{a}^{*,\mathrm{reg}})$, there exist $\phi_w\in C_c^\infty(i\mathfrak{a}^{*,\mathrm{reg}})^W$ such that $\phi=\sum_w h_w\phi_w$ (use that $\det(h_w(w'x))\neq0$ if $x\in\mathfrak{h}^{*,\mathrm{reg}}$). Combined with the above remarks this implies that $\pi(C_c^\infty(i\mathfrak{a}^{*,\mathrm{reg}}))$ is in the closure of $\pi(M)$ in Σ_2. But clearly, $\pi(C_c^\infty(i\mathfrak{a}^{*,\mathrm{reg}}))=\Sigma_c^\infty$ and we are done. □

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ERIC M. OPDAM
Department of Mathematics
University of Leiden
P.O. Box 9512
2300 RA Leiden
The Netherlands
opdam@wi.leidenuniv.nl

Received November 10, 1993