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Asymptotic expansions of matrix coefficients of Whittaker vectors at irregular singularities

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0. Introduction

Singularities of systems of linear differential equations are usually classified into two classes: the regular type and the irregular type. When only one variable is involved, both types of singularities have been studied extensively in the literature. Some general tools have been developed, e.g., asymptotic expansions [Wa], and there are abundant families of examples, e.g., the confluent hypergeometric functions which include the classical Whittaker functions and Bessel functions [WW]. But no powerful general tools are available to handle irregular singularities in several variables.

An example is the system of differential equations satisfied by Whittaker functions on a semi-simple Lie group split over $\mathbf{R}$, which has irregular singularities at ∞ in every direction in the positive Weyl chamber. Since the Fourier coefficients of an automorphic form along the nilpotent radical of a parabolic subgroup are expressed in terms of Whittaker functions, a better understanding of their growth in every direction would be useful in the study of automorphic forms. In [MW], it was conjectured that the growth condition in the definition of automorphic form is superfluous for real semi-simple Lie groups with reduced real rank at least 2. In the same paper Miatello and Wallach [MW] have given a family of examples and one of the key steps in the estimates follows from the compactness of a certain set. This fails to be true in general, for example, $\mathrm{SL}(3,\mathbf{R})$. It seems that this failure may be compensated for by a better understanding of Whittaker functions. The present work is an initial probe to examine the phenomenon of irregular singularities through specific examples and a preparation for an understanding of the growth condition satisfied by automorphic forms.

The classical Whittaker functions have been studied in great detail in [WW]. In that reference, a convergent series expansion near 0 (on the negative chamber) and an asymptotic series expansion at ∞ (on the positive chamber) are given. Motivated by the theory of automorphic forms a general theory of Whittaker functions (vectors) was developed from the view point of representation theory. The C^∞-continuous Whittaker vectors (Jacquet's Whittaker vectors) first introduced by Jacquet in [J] are defined by analytic continuation of certain integrals. The algebraic notion of Whittaker vector was introduced by Kostant in [K1]. They are functionals on the algebraic dual of K-finite vectors of a representation of a Lie group G. In the case of principal series representation, he has proved that the dimension of the space of Whittaker vectors is the order of the little Weyl group and the dimension of the space of C^∞ Whittaker vectors is at most one (hence the C^∞-continuity characterizes Jacquet's Whittaker vectors). Though Kostant's Whittaker vectors are defined on the K-finite vectors, in [GW1], Goodman and Wallach have shown that they extend to continuous functionals on a space of Gevrey vectors.

The work of Kostant [K1] and Goodman and Wallach [GW1] mentioned above is intimately connected to the theory of the quantized system of generalized non-periodic Toda lattice type. In [K2], Kostant integrated the quantized system of non-periodic Toda lattices by representation theory. In [GW2], [GW3], [GW4], Goodman and Wallach studied both the periodic and non-periodic types under the same frame-work. In [GW2], the structure of the commutant of the Hamiltonian and in [GW4], the joint spectral decomposition of those commutants were examined. The present dissertation is influenced by their work.

A fully developed and powerful tool in dealing with irregular singularities in the theory of ordinary differential equations are asymptotic series expansions (see [Wa]). This becomes one of our basic tools because following the procedure described in [GW2] one may study the restriction of Whittaker functions on rays. Another inspiration is Zuckerman's conjecture that we will explain later. This led us to use a method similar to the characteristic method in the theory of differential equations. The problem is thereby reduced to the analysis of a problem in algebraic geometry which is related to a deep theorem of Kostant on principal nilpotents [K2]. What follows are more details to illustrate our approach and motivation.

Let G be a split semi-simple Lie group over $\mathbf{R}$ and let $G=NAK$ be an Iwasawa decomposition G. Let $\mathfrak{g}$, $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{k}$ be respectively the Lie algebras of G, N, A and K. Let $M=\{k\in K \mid kak^{-1}=a, a\in A\}$. Then one has $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{a}\oplus\mathfrak{k}$. Let $\Delta=\Delta(\mathfrak{g},\mathfrak{a})$ be the root system of $(\mathfrak{g},\mathfrak{a})$ and Δ^+ be the positive root system associated to $\mathfrak{n}$ and set $\varrho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. If $l=\operatorname{rank}\mathfrak{g}$, then let $\{\alpha_1,\dots,\alpha_l\}=\Pi$ be the set of simple positive roots and for each i, choose a root vector $X_i\in\mathfrak{g}_{\alpha_i}\backslash\{0\}$. Let $\eta\colon\mathfrak{n}\to\mathbf{C}$ be a generic character, i.e. $\eta(X_i)\neq 0$, $i=1,\dots,l$. For $\nu\in\mathfrak{a}_{\mathbf{C}}^*$, (π_ν, H) will denote the corresponding spherical principal series. π_ν is a representation of G on $H=L^2(M/K)$ and the action is defined by

$$\pi_\nu(x)(f)(u)=a(ux)^{\nu+\varrho}f(k(ux))$$

for $f\in H$, $x\in G$, $u\in K$. Here $g=n(g)a(g)k(g)$, $n(g)\in N$, $a(g)\in A$, $k(g)\in K$. Let X denote the space of all K-finite vectors in H and X^* its algebraic dual. We have an action π_ν^* of $\mathfrak{g}$ on X^* defined by $\pi_\nu^*(z)\phi=-\phi\circ\pi_\nu(z)$ for $z\in\mathfrak{g}$, $\phi\in X^*$. Then the space of Whittaker vectors is $\mathrm{Wh}(v)=\{v^*\in X^* \mid \pi_v^*(Z)v^*=\eta(Z)v^* \text{ for } Z\in\mathfrak{n}\}$. It is a theorem of Kostant that $\dim \mathrm{Wh}(v)=|W(A)|$, where $W(A)$ is the Weyl group of (G,A) ([K1]). Though Whittaker vectors are functionals on K-finite vectors, Goodman and Wallach [GW1] had shown that they can be extended to continuous functionals on a space of Gevrey vectors. Therefore $\phi_{v^*}(g)=v^*(\pi_\nu(g)1_\nu)$, for $v^*\in\mathrm{Wh}(\nu)$ ($1_\nu\in H$ is the constant function 1 on K), is an analytic function on G. This function is called a Whittaker function and we use $W(\nu)$ to denote the space of all such functions. Observe that a Whittaker function is determined by its restriction on A. When

$$G=\mathrm{SL}(2,\mathbf{R}),\quad \eta\begin{bmatrix}0&1\\0&0\end{bmatrix}=i,\quad A=\left\{a_t=\begin{bmatrix}e^{t/2}&0\\0&e^{-t/2}\end{bmatrix}\,\middle|\, t\in\mathbf{R}\right\},$$

if ϕ is a Whittaker function, then $F=e^{-\varrho}\phi|_A$ as a function of $z=2e^t$ satisfies Whittaker's differential equation ([WW])

$$F''(z)+\left[-\tfrac{1}{4}+\left(\tfrac{1}{4}-\nu^2\right)z^{-2}\right]F(z)=0.$$

The singularities of this equation are at 0 and ∞ which are respectively regular and irregular. For generic ν, $\{M_{0,\nu}(z), M_{0,-\nu}(z)\}$ is a basis for the solution such that

$$M_{0,\nu}(z)=z^{\nu+1/2}\sum_{i=0}^{\infty}c_i(\nu)z^i,$$

and converges uniformly on $t\leqslant t_0$.

On the other hand, there is a basis for the solutions $\{I_+, I_-\}$ such that

$$I_\pm\sim e^{\pm z/2}\sum_{i=0}^{\infty}d_i^\pm(\nu)z^{-i}$$

as $t\to\infty$. The difference between these two types of results is due to the type of singularities. Notice, also, that the growth of leading terms in $I_\pm$ does not depend on ν.

The first expansion, that is, on the negative chamber has been generalized by Goodman and Wallach [GW1] to the case that G is a split semi-simple Lie group.

THEOREM (cf. [GW1]). *For generic* $\nu\in\mathfrak{a}_{\mathbf{C}}^*$, $\mathrm{Wh}(\nu)$ *has a basis* $\{\widetilde{\omega}_s(\nu)\mid s\in W\}$ *such that for* $v\in X$,

$$\widetilde{\omega}_s(\nu)(\pi_\nu(a)v)=a^{s\nu+\varrho}\sum_{\mu\in L^+}a^\mu q_{\mu,s}(\nu)(v)$$

with $q_{\mu,s}\in H'_\infty$. *Here* L^+ *is the positive weight lattice. The series converges uniformly on the sets*

$$A^-(t)=\{\exp H \mid H\in\mathfrak{a},\, \alpha(H)\leqslant t \text{ for } \alpha\in\Delta^+\}.$$

Furthermore, for all $a\in A$, a^{α_i} *large for each* i,

$$|a^{-s\nu-\varrho}\widetilde{\omega}_s(\nu)(\pi_\nu(a)v)|\leqslant C_1\exp\left(C_2\sum_{i=1}^{l}a^{\alpha_i}\right)$$

with $C_1, C_2>0$.

But except for some special directions, the behavior of a Whittaker function on the other chambers is more mysterious. The difficulty arises from the presence of irregular singularities. We choose the positive Weyl chamber A^+ as our object of investigation since on A^+ all singularities are irregular. Nevertheless, the last part of the above theorem gives us a bound on how fast the Whittaker functions grow on A^+.

When $G=\mathrm{SL}(n,\mathbf{R})$, Zuckerman has given the following conjecture: *Consider the tridiagonal matrix*

$$Z(p,x)=\begin{bmatrix} p_1 & -x_1^2 & & & & \\ 1 & p_2 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -x_{n-1}^2 \\ & & & & 1 & p_n \end{bmatrix}, \quad x_i=e^{\alpha_i}.$$

Set $f_k=\mathrm{tr}\, Z(p,x)^k$, $k=1,\dots,n$. *Set* $S=-\sum jp_j$. *Then there is a branch of solutions* $p(x)$ *of the system of algebraic equations* $f_k(p(x),x)=0$, $k=1,\dots,n$, *such that* $e^{-(\varrho+S)}\phi|_A$ *is of moderate growth on* A^+.

Let L_k be the quantization of f_k. When $G=\mathrm{SL}(3,\mathbf{R})$, L_2 is the Hamiltonian H and $\{L_k\}$ generate the commutant of H. Suppose $e^{tS}\sum_{k=0}^\infty u_kt^{-k}$ is an asymptotic expansion (if it exists) of a joint eigenfunction of the operators L_k, in the direction of $H_{\tilde{\varrho}}$, $\alpha_i(H_{\tilde{\varrho}})=1$, then one can verify that

$$\lim_{t\to\infty}\sum_k(-t)^{-\deg L_k}e^{-tS}T_{\tau H_{\tilde{\varrho}}}L_kT_{-\tau H_{\tilde{\varrho}}}e^{tS}\left(\sum_{j=0}^\infty u_jt^{-j}\right)$$
$$=\det\begin{bmatrix} \partial S/\partial h_1 & -x_1^2 & 0 \\ 1 & \partial S/\partial h_2 & -x_2^2 \\ 0 & 1 & \partial S/\partial h_3 \end{bmatrix}, \quad t=e^\tau.$$

Here T_v, $v\in\mathfrak{a}$, is the translation operator. This observation motivates us to use a method similar to the characteristic method in the theory of differential equations.

The precise statement of our main result concerning the growth of Whittaker functions on A^+ when G is split, semi-simple is given in §7. Roughly speaking, we have shown that there exist functions (leading exponents) $S^{(1)}, ..., S^{(w)}$ defined on a Zariski open dense subset U of A and that there is a basis $\{\phi^{(1)}, ..., \phi^{(w)}\}$ of $W(\nu)$ such that $e^{-(\varrho+S^{(m)})}\phi^{(m)}|_A$ is of moderate growth on each ray $\{x_0+\tau H_{\tilde{\varrho}}\,|\,\tau\geqslant 0\}$, $x_0\in\log U$. Here $\tilde{\varrho}\in\mathfrak{a}^*$ is given by $\langle\tilde{\varrho},\alpha_i\rangle=1$, $i=1, ..., l$. The leading exponents $S^{(1)}, ..., S^{(w)}$ can be determined by using an analogous construction as in Zuckerman's conjecture. Furthermore, the growth rate of $e^{-(\varrho+S^{(m)})}\phi^{(m)}|_A$ on each ray $\{x_0+\tau H_{\tilde{\varrho}}\,|\,\tau\geqslant 0\}$, as a function in x_0, is a rational function of $S^{(1)}, ..., S^{(w)}$.

One might also consider Toda lattices of periodic type and find asymptotic expansions along the same direction. In other words, one can define a similar system of differential equations associated with an affine Lie algebra $\mathfrak{g}$ which arises from a simple Lie algebra $\mathfrak{g}_0$. For $\mathfrak{g}$ of a certain type, the associated system has a Hamiltonian which is the same as the Hamiltonian for the system associated to $\mathfrak{g}_0$ except that it has one more term which decays exponentially in the direction $\tilde{\varrho}$. To see that one can "ignore" this term, we regard the system $\mathfrak{g}_0$ as the system for $\mathfrak{g}$ associated to a non-generic character η of $\mathfrak{u}$ which one may think of as the limit of a family of generic characters.

The organization of this paper is as follows. In §1 we describe the system of differential equations satisfied by a Whittaker function and set up an integrable connection associated to this system. We then study the solutions of this system when restricted to an irregular direction in the positive Weyl chamber. In §2, we follow the modified procedures in the general theory of asymptotic expansions of solutions of an ordinary differential equation at an irregular singularity to compute the leading exponents of asymptotic expansions of a basis of Whittaker functions at a fixed direction when G is $\mathrm{SL}(3,\mathbf{R})$. In §3, motivated by the calculation in §2, some specific shearing transforms are used in the general case to reduce the problem of finding those leading exponents in the asymptotic expansions to the problem of diagonalizing a certain matrix. §4 is then devoted to diagonalizing this matrix by a method similar to the method of characteristics which leads to a problem in algebraic geometry which we deal with in §5 and §6. Our main theorem and its proof are given in §7. In the last section, we show how one can apply the results in previous sections to affine Lie algebras. A very short tour of the general theory of asymptotic expansions of ordinary differential equations at irregular singularities is included as an appendix.

Finally, the author is indebted to Nolan Wallach for very useful conversations on the subject of this paper.

1. The system of differential equations satisfied by Whittaker functions

Let G be a split real reductive Lie group and $G=NAK$ be an Iwasawa decomposition. Let $\mathfrak{g}$, $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{k}$ denote the Lie algebras of the G, N, A and K, respectively. Then one has $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{a}\oplus\mathfrak{k}$. Set $\mathfrak{s}=\mathfrak{a}+\mathfrak{n}$.

If X is a Lie algebra, then the universal enveloping algebra of X is denoted by $U(X)$. By the Poincaré–Birkhoff–Witt theorem (PBW), one has a direct sum decomposition

$$U(\mathfrak{g})=U(\mathfrak{s})\oplus U(\mathfrak{g})\mathfrak{k}. \tag{1.1}$$

Let $p: U(\mathfrak{g})\to U(\mathfrak{s})$ be the canonical projection defined by (1.1). It is well-known that $p|_{U(\mathfrak{g})^{\mathfrak{k}}}$ is an algebra homomorphism.

The derived algebra $[\mathfrak{n},\mathfrak{n}]$ of $\mathfrak{n}$ is an ideal of $\mathfrak{s}$. Set $\mathfrak{b}=\mathfrak{s}/[\mathfrak{n},\mathfrak{n}]$ and $\mathfrak{u}=\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]$. Let $\pi: U(\mathfrak{s})\to U(\mathfrak{b})$ be the canonical quotient homomorphism. There is an algebra homomorphism $\tau: U(\mathfrak{s})\to U(\mathfrak{s})$ that extends $H\mapsto H+\varrho(H)\cdot 1$ on $\mathfrak{a}$ and is the identity map on $\mathfrak{n}$. Here $\varrho(H)=\frac{1}{2}\operatorname{tr}\operatorname{ad}H|_{\mathfrak{n}}$ for $H\in\mathfrak{a}$. Define $\tilde{\gamma}: U(\mathfrak{g})^{\mathfrak{k}}\to U(\mathfrak{b})$ by setting $\tilde{\gamma}=\pi\circ\tau\circ p$. The restriction of the canonical projection of $\mathfrak{b}$ to $\mathfrak{b}/\mathfrak{u}$ induces an isomorphism $\mathfrak{a}\xrightarrow{\sim}\mathfrak{b}/\mathfrak{u}$ and the inverse map induces a homomorphism $\mu: U(\mathfrak{b})\to U(\mathfrak{a})$. Then $\gamma=\mu\circ\tilde{\gamma}: U(\mathfrak{g})^{\mathfrak{k}}\to U(\mathfrak{a})$ is the usual Harish-Chandra homomorphism. It is well known that $\gamma: U(\mathfrak{g})^{\mathfrak{k}}\to U(\mathfrak{a})^W$ is a surjective homomorphism. Here W is the Weyl group of $(\mathfrak{g},\mathfrak{a})$.

Let θ be the Cartan involution on $\mathfrak{g}$ associated with $\mathfrak{k}$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition. Let σ be the corresponding projection onto $\mathfrak{p}$, then $\sigma(X)=\frac{1}{2}(X-\theta X)$. Let B be a G-invariant symmetric bilinear form on $\mathfrak{g}$ such that $-B(\cdot,\theta(\cdot))$ is positive-definite on $\mathfrak{p}$. We obtain a positive-definite inner product on $\mathfrak{s}$ by setting

$$\langle X,Y\rangle=-B(\sigma(X),\theta(\sigma Y)), \tag{1.2}$$

for $X,Y\in\mathfrak{s}$.

Let $\Pi=\{\alpha_1,\dots,\alpha_l\}$ be the simple root system of $(\mathfrak{g},\mathfrak{a})$ defined by $\mathfrak{n}$, $l=\operatorname{rank}\mathfrak{g}$. We choose $\{X_\alpha\}_{\alpha\in\Pi}$ so that it forms an orthonormal basis for $\mathfrak{u}$ (here we regard X_α as an element in $\mathfrak{u}$ through the canonical quotient map). If $\{h_i\}_{i=1,\dots,l}$ is an orthonormal basis for $\mathfrak{a}$ and C is the Casimir operator in $U(\mathfrak{g})$, then one has

$$\tilde{\gamma}(C)=\sum_{i=1}^{l}h_i^2+\sum_{\alpha\in\pi}X_\alpha^2-\langle\varrho,\varrho\rangle. \tag{1.3}$$

Set $\Omega=\sum_{i=1}^{l}h_i^2+\sum_{\alpha\in\pi}X_\alpha^2$ and let $U(\mathfrak{b})^\Omega=\{x\in U(\mathfrak{b})\,|\,[x,\Omega]=0\}$. Then $\tilde{\gamma}(U(\mathfrak{b}))^{\mathfrak{k}}=U(\mathfrak{b})^\Omega$ since one has

THEOREM 1.1 ([GW2]). *$\mu: U(\mathfrak{b})^{\Omega} \to U(\mathfrak{a})^W$ is an algebra isomorphism. Moreover, if $\{u_i\}_{i=1,\dots,l}$ is a set of homogeneous algebraic independent generators for $U(\mathfrak{a})^W$, then there exist unique elements $\omega_1, \dots, \omega_l \in U(\mathfrak{b})^{\Omega}$ such that $\mu(\omega_i)=u_i$ and $U(\mathfrak{b})^{\Omega}=$ $\mathbf{R}[\omega_1, \dots, \omega_l]$.*

Let $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and (π_ν, H) be the spherical principal series representation of G associated with ν. Let X^ν be the space of K-finite vectors and $(X^\nu)^*$ its algebraic dual. Given a unitary generic character $\eta: \mathfrak{n} \to \mathbf{C}$, i.e., $\eta(X_\alpha) \neq 0$ for all $\alpha \in \Pi$. Then the space of all Whittaker vectors associated with π_ν and η is

$$\mathrm{Wh}_\eta(X^\nu) = \{v^* \in (X^\nu)^* \mid \pi_\nu^*(x)v^* = \eta(x)v^* \text{ for all } x \in \mathfrak{n}\}.$$

Here $(\pi_\nu^*(x)v^*)(w) = v^*(\pi_\nu(-x)w)$ for $w \in X^\nu$. Let $1_\nu \in H$ be the constant function on K. Then the space of all Whittaker functions associated with π_ν and η is

$$W_\eta(\nu) = \{\phi \in C^\infty(G) \mid \phi(g) = v^*(\pi_\nu(g)1_\nu) \text{ for some } v^* \in \mathrm{Wh}_\eta(X^\nu)\}.$$

Though Whittaker vectors are functionals on K-finite vectors in [GW1], Goodman and Wallach have shown that they extend to continuous functionals on a space of Grevey vectors and as a consequence, $v^*(\pi_\nu(g)1_\nu)$ is a smooth function on G. Observe that a Whittaker function ϕ is completely determined by its restriction on A. Set $\phi^\varrho = e^{-\varrho}\phi|_A$. We define a representation π_η of $\mathfrak{b}$ on $C^\infty(\mathfrak{a})$ by

$$(\pi_\eta(H)f)(x) = \frac{d}{dt}\Big|_{t=0} f(x+tH) \tag{1.5a}$$

and

$$(\pi_\eta(X_\alpha)f)(x) = -\eta(X_\alpha)e^{\alpha(x)}f(x) \tag{1.5b}$$

for $x \in \mathfrak{a}$, $H \in \mathfrak{a}$. Then ϕ^ϱ is characterized by

$$\pi_\eta(\tilde{\gamma}(u))\phi^\varrho = \chi_\nu(u)\phi^\varrho \tag{1.6}$$

for all $u \in U(\mathfrak{g})^{\mathfrak{k}}$, where $\chi_\nu = \nu \circ \gamma$ (cf. [GW2]). If $\omega_1, \dots, \omega_l$ are chosen as in Theorem 1.1, then

$$\pi_\eta(\omega_i)\phi^\varrho = \chi_{\nu,i}\phi^\varrho, \quad i = 1, \dots, l, \tag{1.7}$$

with $\chi_{\nu,i} = \chi_\nu(u_i)$ and $\tilde{\gamma}(u_i) = \omega_i$, an equivalent system with finitely many equations.

By using the representation π_η, we may regard elements in $U(\mathfrak{b})^{\Omega}$ as differential operators with coefficients in the ring of functions $\mathcal{R} = \mathbf{R}[e^{\alpha_1}, \dots, e^{\alpha_l}]$. For $H \in \mathfrak{a}$, let $\partial(H)$ be the differential operator defined by

$$\partial(H)f(x) = \frac{d}{dt}\Big|_{t=0} f(x+tH).$$

We extend this map ∂ to an isomorphism of $S(\mathfrak{a})$, the symmetric algebra of $\mathfrak{a}$, with $D(\mathfrak{a})$, the differential operators with constant coefficients. We will therefore identify $S(\mathfrak{a})$ with $D(\mathfrak{a})$.

Now it is well-known that the space of W-harmonics $\mathcal{H}$ in $S(\mathfrak{a})$ is of dimension $w=|W|$. We choose a basis $\{e_i\}_{i=1,\dots,w}$ of $\mathcal{H}$ such that $e_1=1$ and each e_j is homogeneous. Set $E_i=\partial e_i$, $i=1,\dots,w$, and $\mathcal{B}=\pi_\eta(U(\mathfrak{b}))^\Omega$. Let $\mathcal{Q}$ be the algebra of differential operators generated by $\mathcal{R}$ and $S(\mathfrak{a})$. Then we have the following algebraic analogue of "separation of variables" for operators in $\mathcal{Q}$.

PROPOSITION 1.2 ([GW4]). *If $D\in\mathcal{Q}$, then there exist $\omega_{ij}\in\mathcal{B}$ and $f_i\in\mathcal{R}$ such that*

$$D=\sum f_i E_j \omega_{ij}. \tag{1.8}$$

Every element x of $U(\mathfrak{b})^\Omega$ can be written in the form $\sum_{c_{m,n}} h_1^{m_1}\dots h_l^{m_l} X_{\alpha_1}^{n_1}\dots X_{\alpha_l}^{n_l}$, $m=(m_1,\dots,m_l)$, $n=(n_1,\dots,n_l)$ and it is said to be homogeneous of degree d if $\sum m_i+\sum n_i=d$ whenever $c_{m,n}\neq 0$. Let $\{\omega_i\}$ be a vector space basis of $U(\mathfrak{b})^\Omega$ which consists of homogeneous elements. For $H\in\mathfrak{a}$, one has

$$(\partial H)E_i=\sum u_{ij}^k(H)E_j\pi_\eta(\omega_k) \tag{1.9}$$

for some $u_{ij}^k(H)\in\mathcal{R}$ by Proposition 1.2. Therefore

$$(\partial H)E_i\phi^\varrho=\sum u_{ij}^k(H)\chi_\nu(\omega_k)E_j\phi^\varrho. \tag{1.10}$$

Set $F=[E_1\phi^\varrho,\dots,E_w\phi^\varrho]^t$ and $\Gamma_H=(\Gamma_{H,ij})_{i,j=1,\dots,w}$ with $\Gamma_{H,ij}=\sum_k u_{ij}^k(H)\chi_\nu(\omega_k)$. Then (1.10) can be rewritten as

$$(\partial H)F=\Gamma_H F. \tag{1.11}$$

If we define a connection ∇ on the trivial vector bundle $\mathbf{C}^w$ over $\mathfrak{a}_\mathbf{C}$ by $\nabla_H=\partial H-\Gamma_H$, then it can be shown that it is integrable ([GW2]). The integrability of ∇ is equivalent to the following assertion: given any $v_0\in\mathbf{C}^w$ and $z_0\in\mathfrak{a}_\mathbf{C}$, there exists a solution F of the system (1.11) such that $F(z_0)=v_0$. (The uniqueness of a solution with given initial condition is a standard result.)

It is clear that any solution of the system (1.6) will be converted to a solution of the system (1.11). Conversely, if $F=[f_1,\dots,f_w]^t$ satisfies (1.11), then it can be shown ([GW2]) that f_1 is a solution of the system (1.6) and $f_i=E_i f_1$, $i=1,\dots,w$. In other words, (1.6) and (1.11) are equivalent systems.

Our concern is the behavior of Whittaker functions on the positive chamber and the equivalent system (1.11) enables us to restrict our attention to a fixed direction. Let

$x_0 \in \mathfrak{a}_{\mathbf{C}}$ be a fixed point, $v \in \mathfrak{a}_{\mathbf{C}}$ a fixed direction. Set $\Phi_v(x_0;\tau)=F(x_0+\tau v)$. Then one has

$$\frac{d\Phi_v}{d\tau}(x_0;\tau)=(\partial v)F(x_0+\tau v)=\Gamma_v(x_0+\tau v)\Phi_v(x_0;\tau), \tag{1.12}$$

which is a system of ordinary differential equations in τ.

Let $\tilde{\varrho}$ be such that $\langle\tilde{\varrho},\alpha_i\rangle=1$, $i=1,\dots,l$, and $H_0=H_{\tilde{\varrho}}$ is defined by $\langle\alpha,\tilde{\varrho}\rangle=\alpha(H_{\tilde{\varrho}})$. Put $v=H_0$ in (1.12), then one has

$$\frac{d\Phi}{d\tau}(x_0;\tau)=\Gamma_{H_0}(x_0+\tau H_0)\Phi(x_0;\tau). \tag{1.13}$$

For simplicity, we drop H_0 in the notation $\Phi_{H_0}(x_0,\tau)$ and $\Gamma_{H_0}(x_0+\tau H_0)$.

Now (1.8) can be obtained from the linear isomorphism

$$U(\mathfrak{u})\otimes\mathcal{H}\otimes U(\mathfrak{b})^{\Omega}\to U(\mathfrak{b})$$

given by $z\otimes e\otimes w\mapsto zew$ (more precisely, for every $j\geqslant 0$,

$$U_j(\mathfrak{b})=\sum_{r+s+t=j} U_r(\mathfrak{u})\cdot\mathcal{H}_s\cdot U_t(\mathfrak{b})^{\Omega}\Big)$$

by applying the representation π_η. In particular, if, as elements in $U(\mathfrak{b})$,

$$H_0e_i=\sum v_{ij}^k e_j\omega_k \tag{1.14}$$

with $v_{ij}^k\in U_{s_{ij}^k}(\mathfrak{u})$, $s_{ij}^k=\deg e_i+1-\deg e_j-\deg\omega_k$, then

$$\partial H_0\cdot E_i=\sum \pi_\eta(v_{ij}^k)E_j\pi_\eta(\omega_k) \tag{1.15}$$

and $u_{ij}^k=\pi_\eta(v_{ij}^k)$ is homogeneous of degree s_{ij}^k. Hence,

$$\Gamma_{ij}(x_0+\tau H_0)=\sum u_{ij}^k(x_0+\tau H_0)\chi_\nu(\omega_k)=\sum e^{s_{ij}^k\tau}u_{ij}^k(x_0)\chi_\nu(\omega_k).$$

We make the change of variable $t=e^\tau$ in (1.13), then

$$\frac{d\Phi}{dt}(x_0;t)=A(x_0;t)\Phi(x_0;t) \tag{1.16}$$

with $A(x_0;t)_{ij}=\sum t^{s_{ij}^k-1}u_{ij}^k(x_0)\chi_\nu(\omega_k)$. The ordinary differential equation (1.16) has an irregular singularity at $t=+\infty$. Such a system has a fundamental matrix of solutions with an asymptotic expansion as $t\to\infty$ (cf. the appendix).

2. Example: $\mathrm{SL}(3,\mathbf{R})$

In this section, we will follow the procedures given in the last section to get the linear system of differential equations (1.14) and calculate the leading exponents in the asymptotic expansions of its solutions. For the rest of this section G will denote $\mathrm{SL}(3,\mathbf{R})$. Nevertheless, most of the following calculations will be made in $\mathrm{GL}(3,\mathbf{R})$ or $\mathfrak{gl}(3,\mathbf{R})$ for the sake of simplicity and in order to match the notation used in Zuckerman's conjecture for $\mathrm{GL}(n,\mathbf{R})$ described in the introduction.

Let E_{ij} be the elementary matrix with the (i,j)th entry 1 and all other entries zero. Let $h_i=E_{ii}$, $i=1,2,3$. Let $\mathfrak{a}$ be the $\mathbf{R}$-span of h_1-h_2 and h_2-h_3, and then $U(\mathfrak{a})^W$ is generated by 1, $\sum h_ih_j$ and $h_1h_2h_3$. For $i=1,2$, $X_i=E_{i,i+1}$ is a root vector for the root α_i, here $\alpha_i\left(\sum c_jh_j\right)=c_i-c_{i+1}$. Then $\Omega=\sum_{i=1}^3 h_i^2+\sum_{j=1}^2 X_j^2$.

Following the recipe given in [GW2], we can obtain a set of generators for $U(\mathfrak{b})^\Omega$ as an algebra, $\left\{L^2-\sum h_ih_j-\frac{1}{2}\sum X_i^2,\ L_3=h_1h_2h_3-\frac{1}{2}X_2^2h_1-\frac{1}{2}X_1^2h_3\right\}$. Then the partial differential equations satisfied by a Whittaker function are

$$D_i=\phi^\varrho=\chi_i\phi^\varrho,\quad i=2,3,$$

with

$$D_2=\pi_\eta(L_2)=\partial\left(\sum h_ih_j\right)+\sum e^{2\alpha_i},$$
$$D_3=\pi_\eta(L_3)=\partial(h_1h_2h_3)+e^{2\alpha_2}\partial h_1+e^{2\alpha_1}\partial h_3$$

for some χ_i, $i=2,3$. Notice that here we assume without loss of generality that $\eta(X_j)=\pm\sqrt{-1}$, $j=1,2$, because we can conjugate η by an element in A. Also, we can drop the factor $\frac{1}{2}$ by a translation on $\mathfrak{a}$.

LEMMA. *Let $y_i=h_{i+1}$, $i=1,2$, then*

$$e_0=1,\quad e_1=y_1,\quad e_2=y_2,$$
$$e_3=y_1(y_1+2y_2),\quad e_4=y_2(y_2+2y_1),$$
$$e_5=y_1y_2(y_1+y_2)$$

form a basis of the space of harmonics in $S(\mathfrak{a})$.

Since $S(\mathfrak{a})\simeq S(\mathfrak{a})^W\otimes\mathcal{H}$, $\mathcal{H}$ the space of all harmonics, we have $H_0e_i=\sum v_{ij}e_j$ for some $v_{ij}\in S(\mathfrak{a})^W$. In fact,

$$[v_{ij}]_{0\leqslant i,j\leqslant 5}=\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0\\ \frac{1}{3}\omega_2 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & 0\\ \frac{1}{3}\omega_2 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & 0\\ \frac{1}{6}\omega_3 & \frac{2}{3}\omega_2 & \frac{1}{3}\omega_2 & 0 & 0 & \frac{3}{2}\\ -\frac{1}{6}\omega_3 & \frac{1}{3}\omega_2 & \frac{2}{3}\omega_2 & 0 & 0 & \frac{3}{2}\\ 0 & -\frac{1}{9}\omega_3 & \frac{1}{9}\omega_3 & \frac{2}{9} & 0 & 0 \end{bmatrix},$$

where

$$\omega_2 = \sigma_1^2 - 3\sigma_2,$$
$$\omega_3 = 27\sigma_3 - 9\sigma_1\sigma_2 + 2\sigma_1^3,$$

and σ_i is given by $\prod_{i=1}^{3}(t-h_i) = \sum_{i=0}^{3}(-1)^i \sigma_i t^{3-i}$. Let $D_1 = \sum \partial h_i$. We use the identities (as differential operators)

$$\partial\omega_2 = D_1^2 - 3D_2 + 3\sigma, \quad \sigma = \sum e^{2\alpha_i},$$
$$\partial\omega_3 = (27D_3 - 9D_2D_1 + 2D_1^3) + 9(e^{2\alpha_1} - 2e^{2\alpha_2})E_1 + 9(2e^{2\alpha_1} - e^{2\alpha_2})E_2$$

to replace ω_2, ω_3 in v_{ij} by expressions with lower degree in the $S(\mathfrak{a})$ component. We continue this procedure and eventually get expressions as in (1.8) or (1.9).

Following the procedure described in §1, we obtain the equation (cf. (1.16))

$$\frac{d\Phi(x_0;t)}{dt} = A(x_0;t)\Phi(x_0;t)$$

with

$$A(x_0;t) = t^3(a_0 + A_2 t^{-2} + A_4 t^{-4}),$$
$$A_0 = 3\sigma^2 E_{61},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{2\alpha_2} + \frac{1}{2}e^{2\alpha_1} & -\frac{1}{2}e^{2\alpha_2} + 4e^{2\alpha_1} & 0 & 0 & 0 \\ 0 & 4e^{2\alpha_2} - \frac{1}{2}e^{2\alpha_1} & \frac{7}{2}e^{2\alpha_2} - e^{2\alpha_1} & 0 & 0 & 0 \\ \chi_2\sigma & 0 & 0 & \frac{5}{3}e^{2\alpha_2} - \frac{4}{3}e^{2\alpha_1} & -\frac{4}{3}e^{2\alpha_2} + \frac{5}{3}e^{2\alpha_1} & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ \frac{1}{3}\chi_2 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{1}{3}\chi_2 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{6}\chi_3 & \frac{2}{3}\chi_3 & \frac{1}{3}\chi_2 & 0 & 0 & \frac{3}{2} \\ \frac{1}{6}\chi_3 & \frac{1}{3}\chi_2 & \frac{2}{3}\chi_2 & 0 & 0 & \frac{3}{2} \\ 0 & -\frac{1}{9}\chi_3 & \frac{1}{9}\chi_3 & \frac{2}{9}\chi_2 & \frac{2}{9}\chi_2 & 0 \end{bmatrix}.$$

The constant term A_0 is nilpotent and the tactic in the theory of asymptotic expansions at an irregular singularity is to use shearing transforms $\operatorname{diag}[1, t^r, \ldots, t^{5r}]$ (cf. the appendix) to "separate" the eigenvalues, that is, to lower the multiplicities of eigenvalues of the constant matrix; then to reduce the linear system of differential equations to smaller systems and then to repeat this procedure until the eigenvalues are distinct. The type of shearing transform used in the general theory will take care of every possible case, but it seems inefficient and we want to use the particular features of our equation. Therefore we use a shearing transform of the form $\mathrm{diag}[t^{n_1},\dots,t^{n_6}]$ to gain more flexibility and try to find $n_1,\dots,n_6$ such that the resulting constant matrix is most tractable. The best and the only choice according to our judgement is $n_1=0$, $n_2=n_3=1$, $n_4=n_5=2$ and $n_6=3$. Then the resulting constant matrix B_0 is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \sigma & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 \\ \sigma & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & -e^{2\alpha_2}+\frac{1}{2}e^{2\alpha_1} & -\frac{1}{2}e^{2\alpha_2}+4e^{2\alpha_1} & 0 & 0 & \frac{3}{2} \\ 0 & 4e^{2\alpha_2}-\frac{1}{2}e^{2\alpha_1} & \frac{7}{2}e^{2\alpha_2}-e^{2\alpha_1} & 0 & 0 & \frac{3}{2} \\ 3\sigma^2 & 0 & 0 & \frac{5}{3}e^{2\alpha_2}-\frac{4}{3}e^{2\alpha_1} & \frac{4}{3}e^{2\alpha_2}+\frac{5}{3}e^{2\alpha_1} & 0 \end{bmatrix}.$$

Set $v=e^{2\alpha_2}-e^{2\alpha_1}$. The characteristic polynomial is then

$$p(x)=x^6-3\sigma x^4+\tfrac{3}{4}(9v^2-5\sigma^2)x^2-\sigma^3.$$

With $y=x^2$, it becomes

$$p(y)=y^3-3\sigma y^2+\tfrac{3}{4}(9v^2-5\sigma^2)y-\sigma^3,$$

a polynomial of degree 3, and can be handled by Cartan's method. Thus the eigenvalues of the constant matrix are

$$\pm(e^{2\alpha_1/3}+\lambda e^{2\alpha_2/3})^{3/2}, \quad \lambda=1,\zeta,\zeta^2,$$

where ζ is a primitive 3rd root of unity. The eigenvalues are distinct whenever

$$v=e^{2\alpha_2}-e^{2\alpha_1}\neq 0.$$

Now we look at the leading exponents predicted by Zuckerman's conjecture for $\mathrm{GL}(3,\mathbf{R})$ (see the introduction). They are $S=-(p_1+2p_2+3p_3)$, where (p_1,p_2,p_3) satisfies the algebraic equations

$$\begin{gathered}\sum p_i=0,\\ \sum p_ip_j+x_1^2+x_2^2=0,\\ p_1p_2p_3+x_1^2p_3+x_2^2p_1=0,\end{gathered}$$

or equivalently,

$$\begin{bmatrix} p_1 & -x_1^2 & 0 \\ 1 & p_2 & -x_2^2 \\ 0 & 1 & p_3 \end{bmatrix} \quad \text{is nilpotent.}$$

(Notice that this system of algebraic equations, when quantized, is the system $D_i=0$, $i=1,2,3$.) There are six branches of such S and they are exactly the same as the eigenvalues of the constant matrix B.

We make two observations. Firstly, the powers of t in the shearing transform we used are the same as the degrees of the basis of harmonics we chose. Secondly, we obtain $A(t_0;t)$ by replacing ω_i by certain expressions in D_i and it seems that those D_i hidden in $A(x_0;t)$ can be "unwound".

3. Shearing transforms and the constant matrix

In the general theory of asymptotic expansions of solutions of a system of ordinary differential equations at a singular point, the existence of an asymptotic expansion is proved by reducing the rank of the system and the degree of irregularity using shearing transforms. Motivated by calculations for $\mathrm{GL}(3,\mathbf{R})$ in the last section, we use a special shearing transform namely $\mathrm{Sh}(t)=\mathrm{diag}[t^{d_1},\dots,t^{d_w}]$, $d_i=\deg e_i$, $i=1,\dots,w$, instead of the shearing transforms suggested by the general theory.

Set $\Psi(x_0;t)=\mathrm{Sh}(t)^{-1}\Phi(x_0;t)$. Then

$$\begin{aligned}\frac{d\Psi(x_0;t)}{dt} &= \frac{d\,\mathrm{Sh}^{-1}(t)}{dt}\Phi(x_0;t)+\mathrm{Sh}^{-1}(t)\frac{d\Phi(x_0;t)}{dt} \\ &= \left[\frac{d\,\mathrm{Sh}^{-1}(t)}{dt}\,\mathrm{Sh}(t)+\mathrm{Sh}^{-1}(t)A(x_0;t)\,\mathrm{Sh}(t)\right]\Phi(x_0;t) \\ &= B(x_0;t)\Phi(x_0;t).\end{aligned} \tag{3.1}$$

Since

$$\frac{d\,\mathrm{Sh}^{-1}}{dt}=[-\delta_{ij}d_i t^{-d_i-1}]_{ij},$$

$$\begin{aligned}B(x_0;t)_{ij} &= \sum t^{s_{ij}^k+d_j-d_i-1}u_{ij}^k(x_0)\chi_\nu(\omega_k)-\delta_{ij}d_j t^{-1} \\ &= \sum t^{-\deg\omega_k}u_{ij}^k(x_0)\chi_\nu(\omega_k)-\delta_{ij}d_j t^{-1} \qquad (3.2a)\\ &= u_{ij}^0(x_0)-\delta_{ij}d_j t^{-1}+\text{lower order terms}. \qquad (3.2b)\end{aligned}$$

Thus the system (3.1) is regular at $t=+\infty$ and the general theory of asymptotic expansion tells us that if the constant term $B_0(x_0)=[u_{ij}^0(x_0)]_{i,j}$ of $B_0(x_0;t)$ is diagonalizable and has distinct eigenvalues then there exists a fundamental matrix of solutions

$$\Psi(x_0;t)=\widehat{\Psi}(x_0;t)t^\Lambda e^{tQ(x_0)}$$

with $\widehat{\Psi}(x_0;t)\sim\sum_{r=0}^{\infty}\widehat{\Psi}_r(x_0)t^{-r}$, $t\to\infty$, $\det\widehat{\Psi}_0(x_0)\neq 0$, Λ a constant diagonal matrix (which may depend on x_0), $Q(x_0)=\mathrm{diag}[S_1(x_0),\dots,S_w(x_0)]$, where $\{S_i(x_0)\}$ are eigenvalues of $B_0(x_0)$. We will diagonalize the constant matrix in the following section.

4. Diagonalization of the constant matrix

It is extremely difficult to calculate the constant matrix $B_0(x_0)$ explicitly. However, the information we want to extract is that its eigenvalues are distinct and can de described in a certain feasible way. Therefore we will approach this task using a method similar to the characteristic method in the theory of differential equations.

Let $\{h_1,\dots,h_l\}$ be a coordinate system on $\mathfrak{a}$. We use the standard multi-index notation:

$$\gamma=(\gamma_1,\dots,\gamma_l),\quad \gamma_i\in\mathbf{N},$$
$$|\gamma|=\sum\gamma_i,\quad \partial^{\gamma}=\frac{\partial^{|\gamma|}}{\partial h_1^{\gamma_1}\dots\partial h_l^{\gamma_l}}.$$

For $v\in\mathfrak{a}$, denote T_v the operator of translation on $C^{\infty}(\mathfrak{a})$ by v, i.e., $(T_vf)(x)=f(x+v)$ for $f\in C^{\infty}(\mathfrak{a})$, $x\in\mathfrak{a}$. Since, for all multi-indices, $\partial^{\beta}T_v=T_v\partial^{\beta}$ and $T_v\circ f\circ T_{-v}=T_v(f)$, if $D=\sum_{\beta}f_{\beta}\partial^{\beta}$ is a differential operator, then $T_vDT_{-v}=\sum_{\beta}T_v(f_{\beta})\partial^{\beta}$.

Let $D=\sum a_{\beta,\gamma}e^{\sum\beta_j\alpha_j}\partial^{\gamma}$ be a differential operator, we define $\deg D=\max\{|\beta|+|\gamma|: a_{\beta,\gamma}\neq 0\}$ and

$$\sigma_{\mathrm{tot}}(D,d\varphi)=\sum_{|\beta|+|\gamma|=\deg D}a_{\beta,\gamma}e^{\sum\beta_j\alpha_j}\left(\frac{\partial\varphi}{\partial h_1}\right)^{\gamma_1}\cdots\left(\frac{\partial\varphi}{\partial h_l}\right)^{\gamma_l}.$$

Consider the expression

$$E(D)=e^{-t\varphi}T_{\tau H_0}DT_{-\tau H_0}e^{t\varphi},$$

where $\varphi\in C^{\infty}(\mathfrak{a})$ and $t=e^{\tau}$. Then

$$\begin{aligned}E(D)&=\sum_{\beta,\gamma}a_{\beta,\gamma}t^{|\beta|}e^{\sum\beta_j\alpha_j}e^{-t\varphi}\partial^{\gamma}e^{t\varphi}\\&=\sum_{\beta,\gamma}a_{\beta,\gamma}t^{|\beta|}e^{\sum\beta_j\alpha_j}\sum_{j=0}^{|\gamma|}\frac{t^j(\mathrm{ad}\,\varphi)^j\partial^{\gamma}}{j!},\end{aligned}\tag{4.1}$$

where

$$(\mathrm{ad}\,f)^jL=\underbrace{[\dots[L,f],\dots,f]}_{j\text{ times}}$$

for L a differential operator. We refer the reader to [GS] for the last equality. The highest order term in $E(D)$ is

$$t^{\deg D}\sum_{|\beta|+|\gamma|=\deg D} a_{\beta,\gamma}e^{\sum\beta_j\alpha_j}\frac{(\operatorname{ad}\varphi)^{|\gamma|}\partial^\gamma}{|\gamma|!}.$$

Note that

$$\frac{(\operatorname{ad}\varphi)^{|\gamma|}\partial^\gamma}{|\gamma|!}=\left(\frac{\partial\varphi}{\partial h_1}\right)^{\gamma_1}\cdots\left(\frac{\partial\varphi}{\partial h_l}\right)^{\gamma_l}=(\partial^\gamma)(d\varphi)=\sigma_{\mathrm{tot}}(\partial^\gamma,d\varphi),$$

where $\gamma=(\gamma_1,\dots,\gamma_l)$.

Recall that $H_0e_i=\sum v_{ij}^k e_j\omega_k$ and $(\partial H_0)E_i=\sum u_{ij}^k E_j\pi_\eta(\omega_k)$ ((1.14) and (1.15), respectively). Let $p_k=\deg\omega_k$, $d_j=\deg e_j$. Then, by (4.1), we have

$$E(\pi_\eta H_0e_i)=t^{d_i+1}\frac{(\operatorname{ad}\varphi)^{d_i+1}(\partial H_0)E_i}{(d_i+1)!}+\text{lower order terms}\tag{4.2}$$

and

$$\begin{aligned}
&E\Big(\sum u_{ij}^k E_j\pi_\eta(\omega_k)\Big)\\
&\quad=\sum t^{s_{ij}^k}u_{ij}^k\left(\sum_{n=0}^{d_j}t^n\frac{(\operatorname{ad}\varphi)^nE_j}{n!}\right)(t^{\deg\omega_k}\sigma_{\mathrm{tot}}(\omega_k,d\varphi)+\text{lower order terms})\\
&\quad=\sum t^{s_{ij}^k+d_j+\deg\omega_k}u_{ij}^k\frac{(\operatorname{ad}\varphi)^{d_j}E_j}{d_j!}\sigma_{\mathrm{tot}}(\omega_k,d\varphi)+\text{lower order terms}.
\end{aligned}\tag{4.3}$$

Since $d_j+1=s_{ij}^k+d_j+\deg\omega_k$ and $\partial H_0E_i=\sum u_{ij}^kE_j\pi_\eta(\omega_k)$, we obtain

$$(\partial H_0E_i)(d\varphi)=\sum u_{ij}^kE_j(d\varphi)\sigma_{\mathrm{tot}}(\omega_k,d\varphi)\tag{4.4}$$

by comparing the highest order terms in (4.2) and (4.3). Since (4.4) is valid at any point, we may replace $d\varphi$ by a 1-form. Thus we have almost proved the following proposition.

PROPOSITION 4.1. *If β is a 1-form defined on an open subset O of $\mathfrak{a}$ and it satisfies*

$$\sigma_{\mathrm{tot}}(\omega,\beta)=0\quad\textit{for } \omega\in U(\mathfrak{b})_+^\Omega,$$

then for $x\in O$, $H_0(\beta)_x$ is an eigenvalue of $B_0(x)$ and $[E_1(\beta)_x,\dots,E_w(\beta)_x]^t$ is the corresponding eigenvector.

To finish the proof, we have to show that the algebra $\mathcal{B}$ generated by $\{\omega_k\,|\,u_{ij}^k\neq0$ for some $i,j\}$ is in fact $U(\mathfrak{b})^\Omega$. Before we prove this result, we will introduce some notation. As usual, take a basis of $U(\mathfrak{u})$, say $\{u_p\}_{p=0}^\infty$, with $u_0=1$, consisting of homogeneous elements. Let $\{e_k\}$ be a basis of the harmonics $\mathcal{H}$ such that $e_0=1$ and each e_k is homogeneous. For any $u\in U(\mathfrak{a})$, u can be written uniquely as the sum $\sum u_pe_k\omega_{pk}$ with $\omega_{pk}\in U(\mathfrak{b})^\Omega$ and we define $\Omega(u)$ to be ω_{00}.

LEMMA 4.2. $U(\mathfrak{b})^{\Omega}=\{\Omega(u)\,|\,u\in S(\mathfrak{a})\}$.

Proof. If ω is a homogeneous element in $U(\mathfrak{b})^{\Omega}$, then $\mu(\omega)=u\in U(\mathfrak{a})$ and $\omega-u\in U(\mathfrak{b})$, i.e., $u=\omega+\sum x_j v_j$ for some $X_j\in u$ and $v_j\in U(\mathfrak{b})$. Therefore $\omega=\Omega(u)$.

LEMMA 4.3. *If $\{H_i\}$ is a basis of $\mathfrak{a}$, then the algebra generated by $\{\Omega(H_i e_j)\}_{ij}$ is $U(\mathfrak{b})^{\Omega}$.*

Proof. If $u=\sum u_p e_k \omega_{pk}$ and $H_i e_j=\sum u_r e_t \omega_{ijrt}$, then

$$\begin{aligned} H_i u &= \sum H_i u_p e_k \omega_{pk} = \sum [H_i, u_p] e_k \omega_{pk} + \sum u_p H_i e_k \omega_{pk} \\ &= \sum [H_i, u_p] e_k \omega_{pk} + \sum u_p u_r e_t \omega_{ijrt} \omega_{pk}. \end{aligned}$$

Thus if $u=p(H_1,\dots,H_l)e_j$ for some $p\in\mathbf{C}[x_1,\dots,x_l]$ then $\Omega(u)$ is in the algebra $\mathcal{B}_0$ generated by $\{\Omega(H_i e_j)\}_{ij}$ but every $u\in U(\mathfrak{a})$ can be written as $\sum v_j e_j$ with v_i polynomials of $H_1,\dots,H_l$, so $\Omega(u)\in\mathcal{B}_0$ and our assertion follows from Lemma 4.2.

LEMMA 4.4. *If $u\in U(\mathfrak{a})$ and u is homogeneous, then $\Omega(s\cdot u)=\Omega(u)$ for any $s\in W$.*

Proof. There are $v_j\in S(\mathfrak{a})^W$ such that $u=\sum e_j v_j$. Since $v_j\in S(\mathfrak{a})^W$, there exists $\omega^j\in U(\mathfrak{b})^{\Omega}$ such that $\mu\omega^j=v_j$. Therefore $u=\sum e_j\omega^j+\sum e_j(v_j-\omega^j)$ and $v_j-\omega^j\in uU(\mathfrak{b})$. If $s\in W$, $s\cdot u=\sum(s\cdot e_j)v_j=v_0+\sum_{j\neq 0}(s\cdot e_j)v_j$. Since for $j>0$, $s\cdot e_j\in\operatorname{span}\{e_1,\dots,e_{w-1}\}$, $\Omega(s\cdot u)=\omega^0=\Omega(u)$.

PROPOSITION 4.5. $\mathcal{B}=U(\mathfrak{b})^{\Omega}$.

Proof. Let $\mathcal{C}=\{\Omega(H_{\tilde{\varrho}}e_j)\,|\,j=0,\dots,w-1\}$ and $\mathcal{C}'=\{\Omega(sH_{\tilde{\varrho}}\cdot e_j)\,|\,s\in W, j=0,\dots,w-1\}$. Since $\{s\cdot H_{\tilde{\varrho}}\}$ contains a basis of $\mathfrak{a}$, $\langle\mathcal{C}'\rangle=U(\mathfrak{b})^{\Omega}$. For $s\in W$, $\Omega(sH_{\tilde{\varrho}}\cdot e_j)=\Omega(s(H_{\tilde{\varrho}}\cdot s^{-1}e_j))=\Omega(H_{\tilde{\varrho}}\cdot s^{-1}e_j)$. But since $s^{-1}e_j\in\operatorname{span}\{e_0,\dots,e_{w-1}\}$, $\Omega(sH_{\tilde{\varrho}}\cdot e_j)\in\langle\mathcal{C}\rangle$. Therefore $\mathcal{B}=\langle\mathcal{C}\rangle\supseteq\langle\mathcal{C}'\rangle=U(\mathfrak{b})^{\Omega}$.

Let $J\colon U(\mathfrak{b})\to S(\mathfrak{b})$ be the inverse of the symmetrization map, then $J(U(\mathfrak{b})^{\Omega})=S(\mathfrak{b})^{J(\Omega)}$ [GW2]. Let $\{h_i\}$ be the basis of $\mathfrak{a}$ defined by $\langle\alpha_i,h_j\rangle=\delta_{ij}$. For any $\omega\in U(\mathfrak{b})^{\Omega}$, there are $q_\beta\in\mathbf{C}[y_1,\dots,y_l]$, $\beta=(\beta_1,\dots,\beta_l)\in\mathbf{N}^l$ such that

$$J(\omega)=\sum_\beta q_\beta(e^{\alpha_1},\dots,e^{\alpha_l})h^\beta\in S(\mathfrak{b}),\quad h^\beta=h_1^{\beta_1}\dots h_l^{\beta_l}.$$

If $J(\omega)$ is homogeneous, one has, for a 1-form γ,

$$\sigma_{\mathrm{tot}}(\omega,\gamma)=\sum_\beta q_\beta(-c_1e^{\alpha_1},\dots,-c_le^{\alpha_l})\partial(h^\beta)(\gamma),$$

where $c_j=\eta(e_{\alpha_j})$, $j=1,\dots,l$. If $\gamma=\sum p_i\,d\alpha_i$, $\partial(h^\beta)(\gamma)=p_1^{\beta_1}\dots p_l^{\beta_l}$. Without loss of generality, we may assume that $c_j=\pm\sqrt{-1}$ for $j=1,\dots,l$.

$J(\omega)\in S(\mathfrak{b})^{J(\Omega)}$ may be regarded as polynomial on $\mathfrak{b}^*$ and

$$J(\omega)\Big(\sum p_i\alpha_i+\sum y_i\bar{e}_i^*\Big)=\sum_{\beta} q_\beta(y_1,\dots,y_l)p_1^{\beta_1}\dots p_l^{\beta_l},$$

where $\bar{e}_i^*\in\mathfrak{b}^*$, $\bar{e}_i^*(e_{\alpha_k})=\delta_{ik}$ and $\bar{e}_i^*|_{\mathfrak{a}}=0$. Therefore finding a 1-form $\gamma=\sum p_i\,d\alpha_i$ defined on some open subset of $\mathfrak{a}$ such that $\sigma_{\text{tot}}(\omega,\gamma)=0$ is equivalent to solving for $(p_1,\dots,p_l)$ in the algebraic equations $\sum_\beta q_\beta(iy_1,\dots,iy_l)p_1^{\beta_1}\dots p_l^{\beta_l}=0$ for the given values $y_j=e^{\alpha_j}$, $j=1,\dots,l$. Since each q_β in the expression $J(\omega)$ is a polynomial of even degree in each variable ([GW2]) it does not matter whether we take iy_j or $-iy_i$. Therefore we can reformulate Proposition 4.1 as follows.

PROPOSITION 4.6. *If there exist an open subset O of $\mathfrak{a}$ and smooth functions defined on O, say $p_k=p_k(y_1,\dots,y_l)$, $y_j=e^{\alpha_j}$, $j,k=1,\dots,l$, such that for any $\omega\in U(\mathfrak{b})_+^{\Omega}$,*

$$J(\omega)\Big(\sum p_k\alpha_k+\sum\sqrt{-1}\,y_k\bar{e}_k^*\Big)=0$$

on O and if $\gamma=\sum p_k\,d\alpha_k$, then for $x\in O$, $H_0(\gamma)_x$ is an eigenvalue of $B_0(x)$ and $[E_1(\gamma)_x,\dots,E_w(\gamma)_x]^t$ is the corresponding eigenvector.

5. Non-vanishing of Jacobians

Let G be a connected semi-simple Lie group split over $\mathbf{R}$, with Lie algebra $\mathfrak{g}$ and Iwasawa decomposition $G=KAN$ ($\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$) as in §1. Let $\theta:\mathfrak{g}\to\mathfrak{g}$ be the Cartan involution associated with $\mathfrak{k}$. Set $\bar{\mathfrak{n}}=\theta(\mathfrak{n})$. Since $\mathfrak{g}$ is split, one has $\mathfrak{g}=\bar{\mathfrak{n}}+\mathfrak{a}+\mathfrak{n}$. If $X\in\mathfrak{g}$, then we write $X=X_++X_{\mathfrak{a}}+X_-$, where $X_+\in\mathfrak{n}$, $X_-\in\bar{\mathfrak{n}}$ and $X_{\mathfrak{a}}\in\mathfrak{a}$. Let $\mathfrak{p}=\{X\in\mathfrak{g}\,|\,\theta X=-X\}$. Let Δ^+ be the set of positive roots $\Delta^+(\mathfrak{g},\mathfrak{a})$ associated with $\mathfrak{n}$, $\Pi=\{\alpha_1,\dots,\alpha_l\}$ be the set of all positive simple roots and $\Delta^+=\{\alpha_1,\dots,\alpha_d\}$. Choose $e_i=e_{\alpha_i}\in\mathfrak{g}_{\alpha_i}$ such that $-B(e_i,\theta e_i)=\delta_{ij}$. Put $f_i=-\theta e_i$, $X_i=e_i+f_i$ and $Y_i=e_i-f_i$. Then

$$\mathfrak{k}=\sum_{i=1}^d\mathbf{R}Y_i,\quad \mathfrak{p}=\mathfrak{a}\oplus\sum_{i=1}^d\mathbf{R}X_i;$$
$$B(Y_i,Y_j)=-2\delta_{ij},\quad B(X_i,X_j)=2\delta_{ij}.$$

Recall that on $\mathfrak{b}$ we put the inner product $\langle\cdot,\cdot\rangle$, defined by

$$\langle\cdot,\cdot\rangle|_{\mathfrak{a}\times\mathfrak{a}}=B|_{\mathfrak{a}\times\mathfrak{a}},\quad \langle\bar{e}_i,\bar{e}_j\rangle=\tfrac{1}{2}\delta_{ij},$$

where $\bar{\ }:\mathfrak{s}\to\mathfrak{b}$ is the canonical quotient homomorphism. Let $\mathfrak{b}^*$ be the real dual of $\mathfrak{b}$, endowed with the dual inner product. For $X\in\mathfrak{b}$, define $X^{\#}\in\mathfrak{b}^*$ by $X^{\#}(Y)=\langle X,Y\rangle$, $Y\in\mathfrak{b}$. For $\lambda\in\mathfrak{b}^*$, define $\lambda^{\flat}\in\mathfrak{b}^*$ by $\langle\lambda^{\flat},X\rangle=\lambda(X)$, $X\in\mathfrak{b}$.

Let $\mathfrak{p}_1=\mathfrak{a}\oplus\sum_{k=1}^{l}\mathbf{R}X_k\subset\mathfrak{p}$. Let $\bar{e}_1^*\in\mathfrak{b}^*$ be such that $\bar{e}_1^*(H+\sum_{k=1}^{l}c_k\bar{e}_k)=c_i$, for $H\in\mathfrak{a}$. Now we introduce a linear map $F:\mathfrak{p}\to\mathfrak{b}^*$ defined by

$$F\left(H+\sum_{i=1}^{d}c_iX_i\right)=H^{\#}+\sum_{i=1}^{l}c_i\bar{e}_i^*.$$

Notice that $F|_{\mathfrak{p}_1}$ is an isometry since $\langle\bar{e}_i^*,\bar{e}_j^*\rangle=2\delta_{ij}$. If ϕ is a function on $\mathfrak{p}$, we define a function w_ϕ on $\mathfrak{b}^*$ by

$$w_\phi(F(X))=\phi(X),\quad X\in\mathfrak{p}_1.$$

In [GW2], it has been shown that if we take a set of algebraically independent generators for $S(\mathfrak{p}^*)^{\mathfrak{k}}$, say $\{\phi_1,\dots,\phi_l\}$, then $S(\mathfrak{b})^{J(\Omega)}$ is generated by $\{w_{\phi_1},\dots,w_{\phi_l}\}$. In particular, $\{w_\phi,w_\psi\}=0$ whenever $\phi,\psi\in S(\mathfrak{p}^*)^{\mathfrak{k}}$. Here $\{\cdot,\cdot\}$ is the Poisson structure on $S(\mathfrak{b})$ defined by

(i) $\{X,Y\}=[X,Y]$ for $X,Y\in\mathfrak{b}$;

(ii) $\{fg,h\}=\{f,h\}g+f\{g,h\}$ for $f,g,h\in S(\mathfrak{b})$.

Since $\mathrm{Res}_{\mathfrak{g}|\mathfrak{p}}:\mathcal{P}(\mathfrak{g})^G\to S(\mathfrak{p}^*)^{\mathfrak{k}}$, defined by $\mathrm{Res}_{\mathfrak{g}|\mathfrak{p}}(P)=P|_{\mathfrak{p}}$, is an algebra isomorphism, we have that, if $\{\psi_1,\dots,\psi_l\}$ is a set of algebraically independent generators for $\mathcal{P}(\mathfrak{g})^G$, then $\{\psi_1|_{\mathfrak{p}},\dots,\psi_l|_{\mathfrak{p}}\}$ is a set of algebraical generators for $S(\mathfrak{p}^*)^{\mathfrak{k}}$. For simplicity, we will drop $|_{\mathfrak{p}}$ when the context is clear.

Let $h_i=\alpha_i^\flat$, $i=1,\dots,l$. For $\phi\in\mathcal{P}(\mathfrak{g})^{\mathfrak{g}}$, define v_ϕ by

$$\begin{aligned}v_\phi(p_1,\dots,p_l,y_1,\dots,y_l)&=w_\phi\left(\sum_{i=1}^{l}p_i\alpha_i+\sum_{i=1}^{l}y_i\bar{e}_i^*\right)\\&=\phi\left(\sum_{i=1}^{l}p_ih_i+\sum_{i=1}^{l}y_i(e_i+f_i)\right).\end{aligned}\tag{5.1}$$

Note that if we choose $x\in\mathfrak{a}_{\mathbf{C}}$ so that $e^{\alpha_i(x)}=y_i$, $i=1,\dots,l$, then

$$e^{\mathrm{ad}\,x}\left(\sum_{i=1}^{l}p_ih_i+\sum_{i=1}^{l}y_i(e_i+f_i)\right)=f+\sum_{i=1}^{l}p_ih_i+\sum_{i=1}^{l}y_i^2e_i\tag{5.2}$$

and

$$v_\phi(p_1,\dots,p_l,y_1,\dots,y_l)=\phi\left(f+\sum_{i=1}^{l}p_ih_i+\sum_{i=1}^{l}y_i^2e_i\right),\tag{5.3}$$

where $f=\sum_{i=1}^{l}f_i$. Therefore, through (5.1) or (5.3), v_ϕ is defined for $p_i,y_i\in\mathbf{C}$. Nevertheless, we always assume $y_i\neq0$.

If $F\in\mathcal{P}(\mathfrak{g})$, the gradient of F, $\nabla F:\mathfrak{g}\to\mathfrak{g}$ is defined by $B(\nabla F(X),Y)=dF_X(Y)$ for $X,Y\in\mathfrak{g}$. Since f is nilpotent, there exist $e,h\in\mathfrak{g}$ such that $\{e,h,f\}$ forms a standard basis of a T.D.S., say $\mathfrak{g}_1$. Then $\mathfrak{g}$ can be decomposed into a direct sum of irreducible $\mathfrak{g}_1$-modules, say $\mathfrak{g}=\bigoplus_{i=1}^{l}\mathfrak{g}_i$ (see [K3]).

LEMMA 5.1. *If $F\in\mathcal{P}(\mathfrak{g})^G$, then $[\nabla F(X), X]=0$ for $X\in\mathfrak{g}$.*

Proof. We have $\nabla F(\mathrm{Ad}(g)X)=\mathrm{Ad}(g)\nabla F(X)$ for $g\in G$. Let $g(t)=\exp tX$. Then

$$\mathrm{Ad}(g(t))X=X.$$

Thus

$$[X,\nabla F(X)]=\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}\,g(t)\cdot\nabla F(X)=\frac{d}{dt}\Big|_{t=0}\nabla F(\mathrm{Ad}(g(t))X)=0.$$

In particular, $[\nabla F(f),f]=0$ for $F\in\mathcal{P}(\mathfrak{g})^G$. In other words, $\nabla F(f)\in\mathfrak{g}^f=\bigoplus_{i=1}^l\mathfrak{g}_i^f$. Since $\dim\operatorname{span}\{\nabla F_i'(f)\,|\,\{F_i'\}$ is a set of basic invariants$\}=l$ and $\dim\mathfrak{g}_i^f=1$ for each i, we can pick a set of basic invariants $\{\phi_1,\dots,\phi_l\}$ such that $\nabla\phi_i(f)\in\mathfrak{g}_i^f\setminus\{0\}$, $i=1,\dots,l$. Set $f_k(p,y)=v_{\phi_k}(p,y)$, $p=(p_1,\dots,p_l)$, $y=(y_1,\dots,y_l)$, $y_i\neq0$. We may regard f_k as a function defined on $\mathfrak{b}^*$ or $f+\mathfrak{b}$ via (5.1) or (5.3).

LEMMA 5.2. *The Jacobians $J_p=\|\partial f_i/\partial p_j\|$ and $J_y=\|\partial f_i/\partial y_j\|$ are non-zero at $z_0=e^{\mathrm{ad}\,e}f$.*

Proof. For each j, there are $d_j\in\mathbf{N}$, $h_j\in\mathfrak{a}$ such that $\mathbf{R}(\mathrm{ad}\,f)^{d_j}h_j=\mathfrak{g}_j^f$. Put $x_j=(\mathrm{ad}\,e)(h_j)\in\mathfrak{n}$. Then $\{\bar{h}_j,\bar{x}_j\}_{j=1,\dots,l}$ forms a basis of $\mathfrak{b}$. Let $h_j'=F(h_j)=h_j^\#$ and $x_j'=F(x_j-\theta x_j)$. We also use $\{h_j',x_j'\}$ to denote the corresponding coordinate system on $\mathfrak{b}^*$. For $\lambda\in\mathfrak{b}^*$, $\lambda=\sum p_i\alpha_i+\sum y_i\bar{e}_i^*$,

$$\begin{aligned}\frac{\partial f_i}{\partial h_j'}(\lambda)&=\frac{\partial w_{\phi_i}}{\partial h_j'}(\lambda)=\lim_{s\to0}\frac{1}{s}[w_{\phi_i}(\lambda+sh_j^\#)-w_{\phi_i}(\lambda)]\\&=\lim_{s\to0}\frac{1}{s}[\phi_i(F^{-1}\lambda+sh_j)-\phi_i(F^{-1}\lambda)]\\&=(d\phi_i)_{F^{-1}\lambda}(h_j)=B(\nabla_{\phi_i}(F^{-1}\lambda),h_j)\end{aligned}\tag{5.4}$$

and

$$\begin{aligned}\frac{\partial f_i}{\partial x_j'}(\lambda)&=\frac{\partial w_{\phi_i}}{\partial x_j'}(\lambda)=\lim_{s\to0}\frac{1}{s}[w_{\phi_i}(\lambda+sx_j')-w_{\phi_i}(\lambda)]\\&=\lim_{s\to0}\frac{1}{s}[\phi_i(F^{-1}\lambda+s(x_j-\theta x_j))-\phi_i(F^{-1}\lambda)]\\&=B(\nabla_{\phi_i}(F^{-1}\lambda),x_j-\theta x_j).\end{aligned}\tag{5.5}$$

For some $x\in a$, $e^{\mathrm{ad}\,x}z_0\in p_1$, put $\lambda_0=F(e^{\mathrm{ad}\,x}z_0)$. Therefore $F|_{\mathfrak{p}_1}^{-1}(\lambda_0)=e^{\mathrm{ad}\,x}z_0$ and

$$\nabla_{\phi_i}(F^{-1}\lambda_0)=e^{\mathrm{ad}\,x}\nabla_{\phi_i}(z_0).$$

Since $\nabla_{\phi_i}(f)\in\mathfrak{g}_i^f$, one has

$$\begin{aligned}B(\nabla_{\phi_i}(z_0),h_j)&=B(\nabla_{\phi_i}(f),e^{-\mathrm{ad}\,e}h_j)=B\left(\nabla_{\phi_i}(f),\frac{(-\mathrm{ad}\,e)^{d_j}}{d_j!}h_j\right)\\&=\frac{(-1)^{d_j}}{d_j!}B(\nabla_{\phi_i}(f),(\mathrm{ad}\,e)^{d_j}h_j),\end{aligned}$$

$$B(\nabla_{\phi_i}(z_0),\mathrm{ad}\,e\,h_j)=B(\nabla_{\phi_i}(f),e^{-\mathrm{ad}\,e}\,\mathrm{ad}\,e\,h_j)=\frac{(-1)^{d_j-1}}{(d_j-1)!}B(\nabla_{\phi_i}(f),(\mathrm{ad}\,e)^{d_j}h_j)$$

and

$$B(\nabla_{\phi_i}(e^{\operatorname{ad} e} f), x_j - \theta x_j) = \langle \nabla_{\phi_i}(e^{\operatorname{ad} e} f), x_j \rangle = \langle \nabla_{\phi_i}(f), e^{-\operatorname{ad} e} \operatorname{ad} e\, h_j \rangle = 0.$$

So if we choose ϕ_i such that $B(\nabla_{\phi_i}(f), (\operatorname{ad} e)^{d_j} h_j) = \delta_{ij}$, we have

$$\begin{aligned} B(\nabla_{\phi_i}(z_0), h_j) &= \frac{(-1)^{d_j}}{d_j!}\delta_{ij}, \\ B(\nabla_{\phi_i}(z_0), x_j) &= \frac{(-1)^{d_j-1}}{(d_j-1)!}\delta_{ij}, \\ B(\nabla_{\phi_i}(z_0), -\theta x_j) &= 0. \end{aligned} \tag{5.6}$$

Since $e^{-\operatorname{ad} x}|_{\mathfrak{u}}$ and $e^{-\operatorname{ad} x}|_{\theta\mathfrak{u}}$ are isomorphisms and $B(\nabla_{\phi_i}(z_0), -\theta x_j) = 0$, (5.4) and (5.5) imply the result.

Let $\mathfrak{b}_f = f + \mathfrak{b}$. We have identified $\mathfrak{b}$ with $\mathfrak{a} + \sum_{i=1}^{l} \mathbf{R} e_i \subset \mathfrak{g}$. Set

$$\mathfrak{d}_1 = \left\{ \sum_{i=1}^{l} a_i e_i \,\middle|\, a_i \in \mathbf{C}^* = \mathbf{C} \setminus \{0\},\ i = 1, \ldots, l \right\}$$

and $Z = f + \mathfrak{a} + \mathfrak{d}_1 \subseteq \mathfrak{b}_f$. Now we introduce two algebraic varieties

$$\begin{aligned} \mathcal{U} &= \left\{ x = f + \sum_{i=1}^{l} p_i h_i + \sum_{i=1}^{l} x_i e_i \in Z \,\middle|\, I(x) = 0 \text{ for } I \in \mathcal{P}(\mathfrak{g})^{\mathfrak{g}} \right\} \\ &= \{ x \in Z \mid \phi_k(x) = 0,\ k = 1, \ldots, l \} \end{aligned}$$

and

$$\begin{aligned} \mathcal{U}' &= \left\{ \lambda = \sum_{i=1}^{l} p_i \alpha_i + \sum_{i=1}^{l} y_i \bar{e}_i^* \in (\mathfrak{b}^*)_{\mathbf{C}} \,\middle|\, w_\phi(\lambda) = 0 \text{ for } \phi \in \mathcal{P}(\mathfrak{g})^{\mathfrak{g}} \text{ and } y_i \in \mathbf{C}^*,\ i = 1, \ldots, l \right\} \\ &= \{ (p, y) \in \mathbf{C}^l \times \mathbf{C}^{*l} \mid f_k(p, y) = 0,\ k = 1, \ldots, l \}. \end{aligned}$$

Following our previous discussion, especially (5.1)–(5.3), we have a regular morphism $\mathcal{F}\colon \mathcal{U}' \to \mathcal{U}$ defined by

$$F\left(\sum_{i=1}^{l} p_i \alpha_i + \sum_{i=1}^{l} y_i \bar{e}_i^* \right) = f + \sum_{i=1}^{l} p_i h_i + \sum_{i=1}^{l} y_i^2 e_i.$$

Recall that $h_i = \alpha_i^\flat$. It is clear that, for $\lambda \in \mathfrak{b}^*$,

$$\frac{\partial f_k}{\partial p_j}(\lambda) = \frac{\partial \phi_k}{\partial p_j}(\mathcal{F}\lambda)$$

and

$$\frac{\partial f_k}{\partial y_j}(\lambda)=\frac{\partial \phi_k}{\partial x_j}(\mathcal{F}\lambda)\cdot 2y_j.$$

Therefore $J_p(f_k)|_\lambda$ and $J_y(f_k)|_\lambda\neq 0$ if and only if $J_p(\phi_k)|_{\mathcal{F}\lambda}$ and $J_x(\phi_k)|_{\mathcal{F}\lambda}\neq 0$. Let $\lambda_0\in\mathcal{U}'$ be as defined in the proof of Lemma 5.2. Then we have shown that $J_p(f_k)|_{\lambda_0}$ and $J_y(f_k)|_{\lambda_0}\neq 0$. Hence, $J_p(\phi_k)$ and $J_x(\phi_k)$ are non-zero at the point $\mathcal{F}\lambda_0\in\mathcal{U}$ which has been shown to be irreducible in [K2, Theorem 2.4, pp. 224–225]. Since both $J_p(\phi_k)$ and $J_x(\phi_k)$ are non-zero polynomials on $\mathcal{U}$, the irreducibilty of $\mathcal{U}$ implies that $J_p(\phi_k)$ and $J_x(\phi_k)$ are non-vanishing on a Zariski dense open subset U of $\mathcal{U}$. It is clear that $\mathcal{F}$ is a two-fold covering map and $\mathcal{F}^{-1}(U)$ is a Zariski open dense subset of $\mathcal{U}'$. Therefore, $J_p(f_k)$ and $J_y(f_k)$ are non-vanishing on $U'=\mathcal{F}^{-1}(U)$. Summing up, we have:

PROPOSITION 5.3. *There exists a Zariski open dense subset U' of $\mathcal{U}'$ such that $J_p(f_k)$ and $J_y(f_k)$ are non-vanishing on U'.*

6. The order of the fibres

We use the notation from the end of the previous section. We now consider the projection $\pi_2\colon\mathcal{U}\to(\mathbf{C}^*)^l$ (or $\pi_2'\colon\mathcal{U}'\to(\mathbf{C}^*)^l$) from $\mathcal{U}$ (or $\mathcal{U}'$) to the x-plane (or y-plane), i.e., $\pi_2(p,x)=x$ (or $\pi_2'(p,y)=y$). By results of Kostant concerning principal nilpotents ([K2, Proposition 2.5.1]), π_2 is surjective and $1\leqslant|\pi_2^{-1}(x_0)|\leqslant w=|W|$ for $x_0\in(\mathbf{C}^*)^l$. We now give a finer result concerning the order of fibres.

PROPOSITION 6.1. *There exists a Zariski open dense subset $U\subseteq(\mathbf{C}^*)^l$ (or U') such that $|\pi_2^{-1}(x)|=w$ for $x\in U$ (or $|\pi_2'^{-1}(y)|=w$ for $y\in U'$).*

Proof. Since $\pi_2\circ\mathcal{F}=\pi_2'$, the result for π_2 follows from that for π_2', so we focus on the variety $\mathcal{U}'$. Since we will use Bezout's theorem which applies only to projective varieties, we introduce the following projective variety

$$\mathcal{W}'=\{[p,y]\in\mathbf{PC}^{2l-1}\mid f_k(p,y)=0\}.$$

Though the choice of f_k in the last section may not be homogeneous, here f_k can be chosen to be homogeneous if we set $f_k(p,y)=\phi_k(\sum y_if_i+\sum p_ih_i+\sum y_ie_i)$, where $\{\phi_k\}$ forms a set of homogeneous basic invariants of $P(g)^g$, and the non-vanishing of the Jacobians is true for any set of basic invariants.

Let D_i be the divisor corresponding to f_i and $D^{ij}_{y_0}$ be the hypersurface in $\mathbf{PC}^{2l-1}$ given by the equations

$$y_0^iy^j=y_0^jy^i,$$

where $y_0=(y_0^1, \ldots, y_0^l)$, $y=(y^1, \ldots, y^l)\in \mathbf{C}^l$. We shall use $(D_1, \ldots, D_n)_x$ to denote the intersection index of the effective divisors $D_1, \ldots, D_n$ at $x\in\bigcap_{i=1}^n \operatorname{supp} D_i$ (cf. [S]). We now make an assertion which we will prove later: There exists a Zariski open dense set U' such that, for $y_0\in U'$, one has

(i) $\{D_i, D_{y_0}^{ij}\}_{i=1,\ldots,l,j=1,\ldots,l,i\neq j}$ are in general position, that is,

$$\bigcap_i \operatorname{supp} D_i \cap \bigcap_{\substack{i,j\\ i\neq j}} \operatorname{supp} D_{y_0}^{ij} = {\pi_2'}^{-1}(y_0)$$

consists of isolated points;

(ii) If $(p_0, y_0)\in\mathcal{W}'$, then

(a) (p_0, y_0) is a simple point in each D_i and each $D_{y_0}^{ij}$;

(b) $\bigcap_i T_{(p_0,y_0)}D_i\cap\bigcap_{i,j,i\neq j} T_{(p_0,y_0)}D_{y_0}^{ij}=\{(p_0, y_0)\}$, where T_xD denotes the tangent space of $\operatorname{supp} D$ at x.

Assume this is true. We then have

(iii) $(D_1, \ldots, D_l, D_{y_0}^{12}, \ldots, D_{y_0}^{l-1,l})_{(p,y_0)}=1$ for $(p, y_0)\in\mathcal{W}'$ and $y_0\in U'$;

(iv) $\sum_{(p,y_0)\in{\pi_2'}^{-1}(y_0)}(D_1, \ldots, D_l, D_{y_0}^{12}, \ldots, D_{y_0}^{l-1,l})_{(p,y_0)}=\prod_{i=1}^l \deg D_i$ for $y_0\in U'$.

The statement (iv) follows from Bezout's theorem. For (iii) we refer to the result of Chapter IV, §1, Example 2 in [S]: if $D_1, \ldots, D_n$ are prime divisors and $x\in D_1\cap\ldots\cap D_n$, then $(D_1, \ldots, D_n)_x=1$ if $D_1, \ldots, D_n$ intersect at x transversally, so that x is a simple point on all the D_i and $\bigcap T_{x,D_i}=x$. The condition that $D_1, \ldots, D_n$ are prime is unnecessary in our case. Suppose that for each i, D_i has local equation f_i in some neighborhood of x. Then what we really need is that the germs of those polynomials f_i generate the maximal ideal at x, i.e., $(f_{1,x}, \ldots, f_{n,x})=m_x$. Those points being considered in our case are simple on all the D_i. Therefore, if $p_i(x)=0$, $p_i|f_i$ and p_i is prime, then $g_i(x)\neq 0$, where $g_i=f_i/p_i$. Thus $(f_{1,x}, \ldots, f_{n,x})=(p_{1,x}, \ldots, p_{n,x})$ and then we can apply the result in that reference to $p_1, \ldots, p_n$.

From (iii), (iv) we obtain $|{\pi_2'}^{-1}(y_0)|=\prod_{i=1}^l \deg D_i$ for $y_0\in U'$. But $\prod_{i=1}^l \deg D_i=\prod_{i=1}^l \deg f_i=\prod_{i=1}^l \deg w_{\phi_i}=w$. The last equality is due to the facts that $\{w_{\phi_i}\}_{i=1,\ldots,l}$ forms a set of generators of $S(\mathfrak{b})^{J(\Omega)}$ as a polynomial ring and that $S(\mathfrak{b})^{J(\Omega)}$ is isomorphic to $S(\mathfrak{a})^W$ [GW2] and standard facts about finite Coxeter groups [B].

Now it suffices then to prove our assertion. (i) follows from $|\pi_2^{-1}(y_0)|\leqslant|W|$. (ii) is equivalent to

(v)

(a) $\left(\dfrac{\partial f_k}{\partial p_1}, \ldots, \dfrac{\partial f_k}{\partial p_l}, \dfrac{\partial f_k}{\partial y_1}, \ldots, \dfrac{\partial f_k}{\partial y_l}\right)\neq 0$ for $y_0\in\mathbf{C}^l$, $y_0^iy_0^j\neq 0$;

(b) $J=\begin{bmatrix} \dot{J}_p(f) & \dot{J}_y(f) \\ 0 & B\end{bmatrix}$ has rank $2l-1$, where $\dot{J}_p(f)=\left(\dfrac{\partial f_k}{\partial p_j}\right)_{k,j}$, $\dot{J}_y(f)=\left(\dfrac{\partial f_k}{\partial y_j}\right)_{k,j}$

and

$$B = (b_{(i,j),k})_{i=1,\dots,l,\, j=1,\dots,l,\, i\neq j,\, k=1,\dots,l}$$

is a $\frac{1}{2}l(l-1)\times l$ matrix with $b_{(i,j),k}=\delta_{ik}y_0^j-\delta_{jk}y_0^i$.

By Proposition 5.3, $\dot{J}_p(f)$ and $\dot{J}_y(f)$ have full rank on some Zariski open dense subset of $\mathcal{W}'$, say $\mathcal{W}_0'$. Let $Z_0=\mathcal{W}'\backslash\mathcal{W}_0'$. Then Z_0 is Zariski closed and $\dim Z_0\leqslant l-2$ ($\dim\mathcal{W}'=\dim\mathcal{U}-1=l-1$). By Chevalley's theorem ([CC], [M]), the closure of $\pi_2(Z_0)$ in Zariski topology has dimension less than $l-2$. Then the set $U'=\mathbf{CP}^{l-1}\backslash\overline{\pi_2(Z_0)}$ is a Zariski open dense set for dimension reasons, and for $y_0\in U'$, ${\pi_2'}^{-1}(y_0)\in\mathcal{W}_0'$. This U' is what we want. For $y_0\in U'$, $J_p(f)_{(p_0,y_0)}\neq 0$ and $J_y(f)_{(p_0,y_0)}\neq 0$ for any $(p_0,y_0)\in{\pi_2'}^{-1}(y_0)$, so $(\partial f_k/\partial p_1,\dots,\partial f_k/\partial p_l,\partial f_k/\partial y_1,\dots,\partial f_k/\partial y_l)\neq 0$ for $k=1,\dots,l$.

Since the first l column vectors and the last l column vectors of J are linearly independent, to prove (ii), it suffices to show that if there are c_j, $j=1,\dots,l$, c_j not all zero, and d_j, $j=1,\dots,l$, such that

$$\sum c_j\frac{\partial f_k}{\partial y_j}\bigg|_{(p_0,y_0)}=\sum d_j\frac{\partial f_k}{\partial p_j}\bigg|_{(p_0,y_0)}\quad\text{for } k=1,\dots,l,$$

and

$$c_jy_0^i=c_iy_0^j\quad\text{for } i\neq j,$$

then $d_i=\lambda p_0^i$, $i=1,\dots,l$, for some non-zero $\lambda\in\mathbf{C}$. Suppose that such c_j exist, by multiplying by a constant $\lambda\in\mathbf{C}^*$, we may assume $c_j=y_0^j$. As f_k is chosen to be homogeneous, we have

$$\sum y_j\frac{\partial f_k}{\partial y_j}+\sum p_i\frac{\partial f_k}{\partial p_i}=(\deg f_k)f_k.$$

Therefore

$$\sum y_0^j\frac{\partial f_k}{\partial y_j}\bigg|_{(p_0,y_0)}+\sum p_0^i\frac{\partial f_k}{\partial p_i}\bigg|_{(p_0,y_0)}=(\deg f_k)f_k(p_0,y_0)=0$$

and then

$$\sum(d_i+p_0^i)\frac{\partial f_k}{\partial p_i}\bigg|_{(p_0,y_0)}=0.$$

But $J_p(f)|_{(p_0,y_0)}\neq 0$, hence $d_i+p_0^i=0$ for $i=1,\dots,l$.

PROPOSITION 6.2. *Let $\{\phi_k\}_{k=1,\dots,l}$ be a set of homogeneous basic invariants of $\mathcal{P}(\mathfrak{g})^{\mathfrak{g}}$. Set $f_k(p_1,\dots,p_l,y_1,\dots,y_l)=v_{\phi_k}(p_1,\dots,p_l,y_1,\dots,y_l)$, $p_i\in\mathbf{C}$, $y_i\in\mathbf{C}^*$. Then there exists a Zariski open dense subset $O\subseteq(\mathbf{C}^*)^l$ such that for any connected, simply connected open subset $V\subseteq O$, there exist w differentiable functions $p^{(m)}: V\to\mathbf{C}^l$, $m=1,\dots,w$, such that for $y\in V$,*

(i) *$f_k(p^{(m)}(y),y)=0$ for all k;*

(ii) $|\{(p^{(m)}(y),y)\,|\,m=1,\dots,w\}|=w$;

(iii) *If* $S^{(m)}(y)\sum_{j=1}^{l}\langle\tilde{\varrho},\beta_j\rangle p_j^{(m)}(y)$, *where* β_j *is defined by* $\langle\alpha_i,\beta_j\rangle=\delta_{ij}$, *then*

$$|\{S^{(m)}(y)\mid m=1,\dots,w\}|=w.$$

Proof. Let U' be the Zariski open dense set in the previous proposition. Let $y_0\in U'$. Then $J_p(f_k)|_{(p_0,y_0)}$, $J_y(f_k)|_{(p_0,y_0)}$ are non-zero, so by the implicit-function theorem, there exists an open neighborhood V_0 of y_0 and $p\colon V_0\to\mathbf{C}^l$ a differentiable function such that $f_k(p(y),y)=0$, $k=1,\dots,l$, whenever $y\in V_0$ and $p(y_0)=p_0$. For any $y\in U'$, $|\pi_l'^{-1}(y)|=w$, therefore there exists an open neighborhood $V\subseteq U'$ of y_0 such that there are w differentiable functions $p^{(m)}\colon V\to\mathbf{C}^l$, $m=1,\dots,w$, so that, for $y\in V$,

(i) $\{(p^{(m)}(y),y)\}$ consists of w distinct points;

(ii) $f_k(p^{(m)}(y),y)=0$.

On V, the sums $S^{(m)}(y)=\sum\langle\tilde{\varrho},\beta_j\rangle p_j^{(m)}(y)$ are defined. Set $h_i=\alpha_i^\flat$. Regard y_j as a function defined on $\mathfrak{a}_{\mathbf{C}}$ through $y_j=e^{\alpha_j}$. Let $\{q_j\}$ be the coordinates associated with the basis $\{h_j\}$ of $\mathfrak{a}_{\mathbf{C}}$. Since $\{w_{\phi_k}\}_{k=1,\dots,l}$ are mutually Poisson commutative [GW2], we have

$$\sum_{j=1}^{l}\frac{\partial f_m}{\partial p_j}\cdot\frac{\partial f_n}{\partial q_j}=\sum_{j=1}^{l}\frac{\partial f_m}{\partial q_j}\cdot\frac{\partial f_n}{\partial p_j}.$$

On V, $p_j^{(m)}$ is a smooth function of y_j, hence of q_j. So

$$\sum_{i,j}\frac{\partial f_n}{\partial p_j}\cdot\frac{\partial f_n}{\partial p_i}\cdot\frac{\partial p_i}{\partial q_i}=\sum_{i,j}\frac{\partial f_n}{\partial p_j}\cdot\frac{\partial f_n}{\partial p_i}\cdot\frac{\partial p_i}{\partial q_j}.$$

Arranging the indices we obtain

$$\sum_{i,j}\frac{\partial f_n}{\partial p_j}\cdot\frac{\partial f_n}{\partial p_i}\left(\frac{\partial p_i}{\partial q_j}-\frac{\partial p_j}{\partial q_i}\right)=0. \tag{6.1}$$

Set $\lambda_{ij}=\partial p_i/\partial q_j-\partial p_j/\partial q_i$ and $\Lambda=(\lambda_{ij})_{ij}$. Let $M=(\partial f_j/\partial p_i)_{i,j}$. Then (6.1) can be written as $M^t\Lambda M=0$. But $\det M=J_p\neq0$ on V. Therefore M is invertible and then $\Lambda=0$, that is,

$$\frac{\partial p_i}{\partial q_i}=\frac{\partial p_j}{\partial q_i}. \tag{6.2}$$

Then

$$\begin{aligned}\frac{\partial S^{(m)}}{\partial q_i}&=\sum_j\langle\tilde{\varrho},\beta_j\rangle\frac{\partial p_j^{(m)}(y)}{\partial q_i}=\sum_j\langle\tilde{\varrho},\beta_j\rangle\frac{\partial p_i^{(m)}(y)}{\partial q_j}\\&=\sum_{j,k}\langle\tilde{\varrho},\beta_j\rangle\frac{\partial p_i^{(m)}(y)}{\partial y_k}\cdot\frac{\partial y_k}{\partial q_j}=\sum_{j,k}\langle\tilde{\varrho},\beta_j\rangle\langle\alpha_k,h_j\rangle\frac{\partial p_i^{(m)}(y)}{\partial y_k}y_k\\&=\sum_k\left\langle\alpha_k,\sum_j\langle\tilde{\varrho},\beta_j\rangle h_j\right\rangle\frac{\partial p_i^{(m)}(y)}{\partial y_k}y_k=\sum_k y_k\frac{\partial p_i^{(m)}(y)}{\partial y_k}.\end{aligned}$$

But since f_k are homogeneous, $p_i^{(m)}$ is homogeneous of degree one in y. Hence

$$\frac{\partial S^{(m)}}{\partial q_i}=p_i^{(m)}. \tag{6.3}$$

Therefore suppose for some u, v, $S^{(u)}=S^{(v)}$ on V, then $\partial S^{(u)}/\partial q_i=\partial S^{(v)}/\partial q_i$, that is, $p_i^{(u)}=p_i^{(v)}$, $i=1, \ldots, l$. But then we must have $u=v$. Consequently, we conclude that for $y\in V$, $S^{(m)}(y)$, $m=1, \ldots, w$, are distinct.

Since each f_k is a polynomial, hence holomorphic, $p^{(m)}(y)$ is holomorphic on a small neighborhood of y_0. So locally there are w holomorphic functions satisfying $f_k(p^{(m)}(y), y)=0$, $k=1, \ldots, l$. Therefore on any connected, simply-connected open subset V of U', there exist holomorphic continuations of $p^{(m)}(y)$, $m=1, \ldots, w$, which are only defined on some neighborhood of $y_0\in V$. Hence the proposition follows if we take O to be U'.

We now set $Z(p_1, \ldots, p_l, y_1, \ldots, y_l)=f+\sum_{i=1}^{l} p_i h_i-\sum_{i=1}^{l} y_i^2 e_i$. We make the choice that $\mathcal{F}^{-1}(Z(p, y))=\sum_{i=1}^{l} p_i\alpha_i+\sum_{i=1}^{l}\sqrt{-1}\, y_i\bar{e}_i^*$. It will be clear later that this choice makes no real difference since any polynomial $J(\omega)$, $\omega\in U(\mathfrak{b})^{\Omega}$, has even degree in each variable y_i.

PROPOSITION 6.3. *There exists a Zariski open dense subset O of A such that for any connected, simply-connected open subset V of O, there exist differentiable functions $p^{(m)}: V\to\mathbf{C}^l$, $m=1, \ldots, w$, such that, for $x\in V$,*

(i) *for all $\omega\in U(\mathfrak{b})^{\Omega}$, $J(\omega)(\mathcal{F}^{-1}(Z(p^{(m)}, y)))=0$, where $p^{(m)}=(p_1^{(m)}(x), \ldots, p_l^{(m)}(x))$ and $y=(x^{\alpha_1}, \ldots, x^{\alpha_l})$;*

(ii) *$S^{(m)}(x)=\sum\langle\tilde{\varrho}, \beta_j\rangle p_j^{(m)}(x)$, $m=1, \ldots, w$, is the set of all eigenvalues of the constant matrix $B_0(\log x)$ and they are distinct. Furthermore, $[E_1(dS^{(m)}), \ldots, E_w(dS^{(m)})]$ is an eigenvector corresponding to the eigenvalue $S^{(m)}$.*

Proof. We extend the domain of B_0 to $\mathfrak{a}_{\mathbf{C}}$ in the usual manner. By Propositions 4.6 and 6.2, $\{S^{(m)}(x)\mid m=1, \ldots, w\}$ is a complete set of eigenvalues for $B_0(\log x)$ on a Zariski open dense set. The characteristic polynomial $Q(x; \lambda)$ of $B_0(\log x)$ has real coefficients when regarded as a polynomial in $x_k=e^{\alpha_k}(\log x)$ and λ. So the resultant $R(x)$ of $Q(x; \lambda)$ and $(dQ/d\lambda)(x; \lambda)$ is a polynomial in x_k with real coefficients. But $R(x)\neq 0$ on a Zariski open dense set, therefore, $R(x)\neq 0$ on a Zariski open dense subset of A.

7. The main theorem

Let G be a semi-simple Lie group split over $\mathbf{R}$. $G=NAK$ is an Iwasawa decomposition. For $\nu\in\mathfrak{a}_{\mathbf{C}}^*$, (π_ν, H) denotes the associated spherical principal series representation of G.

As in §1, $W(\nu)$ is the space of all Whittaker functions associated to π_ν. Let $\{u_1, \dots, u_l\}$ be a set of algebraically independent generators of $\mathcal{P}(\mathfrak{g})^{\mathfrak{g}}$ consisting of homogeneous elements. As in §5 and §6 we define $Z(p, y)$ and f_k, $k=1, \dots, l$, by

$$Z(p,y) = f + \sum_{i=1}^{l} p_i h_i - \sum_{i=1}^{l} y_i^2 e_i \in \mathfrak{g},$$
$$f_k(p,y) = u_k(Z(p,y)),$$

where $h_i = \alpha_i^\flat$. Let w be the order of the Weyl group $W = W(G, A)$.

Before we give the statement of our main result, we establish some notation and definitions. If ϕ is a vector-valued function from A into $\mathbf{C}^n$, for $v \in \mathfrak{a}$, one defines ϕ_v: $A \times \mathbf{R}_+ \to \mathbf{C}^n$ by $\phi_v(x;t) = \phi(x \exp(\log t) v)$ for $x \in A$, $t > 0$. Then ϕ is said to be homogeneous of degree k in the direction $v \in \mathfrak{a}$ if for all $x \in A$, $\phi_v(x;t)$ is homogeneous of degree k in t, i.e., $\phi_v(x; \lambda t) = \lambda^k \phi_v(x;t)$ for $\lambda > 0$, $t > 0$. If $\Omega \subseteq A$ and for any $x \in \Omega$, $t > 0$, $x \exp(\log t) v \in \Omega$ and if ϕ is a function defined on Ω then we define ϕ_v by the same formula. We call such a subset Ω of A v-conical.

Definition. Let Ω be a v-conical set and let ϕ be a function defined on Ω. A series $q(x) \sum_{k=-1}^{\infty} \phi_k(x) t^{-k-\mu(x)}$ on Ω is said to be an asymptotic expansion of ϕ with a shift of order $\mu = \mu(x)$ in the direction v, if

(i) ϕ_k is homogeneous of degree $-k$ in the direction v;

(ii) $q(x \exp(\log t) v) = t^{-\mu} q(x)$;

(iii) for all $x \in \Omega$,

$$\phi_v(x;t) \sim q(x) \sum_{k=-1}^{\infty} \phi_k(x) t^{-k-\mu}, \quad \text{as } t \to \infty.$$

We write $\phi \overset{\mu}{\underset{v}{\approx}} q \sum_{k=-1}^{\infty} \phi_k$.

MAIN THEOREM. *Let G be a semi-simple Lie group split over $\mathbf{R}$. Then there exists a Zariski open dense set O of A such that for any connected simply connected open subset Ω of O, there exist differentiable functions $p^{(m)}: \Omega \to \mathbf{C}^l$, $m=1, \dots, w$, so that*

(i) *$f_k(p^{(m)}(x), y(x)) = 0$, $y(x) = (x^{\alpha_1}, \dots, x^{\alpha_l})$, for $x \in \Omega$, $k = 1, \dots, l$;*

(ii) *$S = \{S^{(m)}(x) = \sum \langle \tilde{\varrho}, \beta_j \rangle p_j^{(m)}(x), m = 1, \dots, w\}$ has w distinct elements for $x \in \Omega$.*

(iii) *Suppose further that Ω is H_0-conical and there is an ordering of S such that $\operatorname{Re} S^{(w)} \leqslant \dots \leqslant \operatorname{Re} S^{(1)}$. Then there exists a basis $\{\phi^{(m)}\}_{m=1,\dots,w}$ of $W(\nu)$ such that for each m, there exist functions μ_m, q_m and $\phi_k^{(m)}$, $k = -1, 0, \dots$, such that*

$$e^{-(\varrho + S^{(m)})} \phi^{(m)} \overset{\mu_m}{\underset{H_0}{\approx}} q_m \sum_{k=-1}^{\infty} \phi_k^{(m)}$$

on Ω with $q_m \neq 0$, $\phi_{-1}^{(m)} \neq 0$.

Remark 1. The functions μ_m in the main theorem are homogeneous of degree 0 in the direction $H_{\tilde{\varrho}}$. In fact, we will see from (7.3) that they are rational functions of $S^{(1)},\dots,S^{(w)},x^{\alpha_1},\dots,x^{\alpha_l}$ with denominator $\prod_{i\neq j}(S^{(i)}-S^{(j)})$. Since $q_m(x_0+\tau H_{\tilde{\varrho}})=e^{-\tau\mu_m(x_0)}q_m(x_0)$, $x_0\in\log\Omega$, $e^{-(\varrho+S^{(m)})}\phi^{(m)}$ has growth of order t^{μ_m}.

Remark 2. Suppose that Ω is an open subset of $\mathfrak{a}_{\mathbf{C}}$ such that $\Omega_0=\Omega\cap\mathfrak{a}$ is non-empty and there exist w holomorphic branches $S^{(1)},\dots,S^{(m)}$ of eigenvalues of the constant matrix $B_0(x_0)$ on Ω. Since $S^{(i)}$ is holomorphic, $S^{(i)}|_{\Omega_0}$ is a real analytic function. For $i\neq j$, set

$$\begin{aligned}P_{ij}^{+}&=\{x\in\Omega_0\mid \operatorname{Re}S^{(i)}(x)>\operatorname{Re}S^{(j)}(x)\},\\ P_{ij}^{-}&=\{x\in\Omega_0\mid \operatorname{Re}S^{(i)}(x)<\operatorname{Re}S^{(j)}(x)\},\\ K_{ij}&=\{x\in\Omega_0\mid \operatorname{Re}S^{(i)}(x)=\operatorname{Re}S^{(j)}(x)\}.\end{aligned}$$

Let $P_{ij}^0=\operatorname{Int}K_{ij}$ be the interior subset of K_{ij}. Since $S^{(i)}|_{\Omega_0}$ is real analytic, if P_{ij}^0 is empty, then K_{ij} is of dimension less than $l-1$. If P_{ij}^0 is non-empty, the boundary of K_{ij}, ∂K_{ij}, is of lower dimension. We have $\Omega_0\backslash\partial K_{ij}=P_{ij}^{+}\cup P_{ij}^{0}\cup P_{ij}^{-}$. Set $\Sigma(\Omega)=\Omega_0\backslash\bigcup_{i>j}\partial K_{ij}=\bigcap_{i>j}(P_{ij}^{+}\cup P_{ij}^{0}\cup P_{ij}^{-})$. $\Sigma(\Omega)$ is a union of open subsets $P_m=\bigcap_{i>j}P_{ij}^{m_{ij}}$, $m=(m_{ij})_{i<j}$, $m_{ij}=+$, 0 or $-$. If $P_m\neq\varnothing$, let $x\in P_m$. One has some permutation $\sigma\in S_w$ so that $\operatorname{Re}S^{(\sigma w)}(x)\leqslant\dots\leqslant\operatorname{Re}S^{(\sigma 1)}(x)$. By the definition of P_m, if $\operatorname{Re}S^{(i)}(x)\gtreqless\operatorname{Re}S^{(j)}(x)$, then $\operatorname{Re}S^{(i)}(y)\gtreqless\operatorname{Re}S^{(j)}(y)$ for all $y\in P_m$. Therefore, $\operatorname{Re}S^{(\sigma w)}(y)\leqslant\dots\leqslant\operatorname{Re}S^{(\sigma 1)}(y)$ for all $y\in P_m$. In other words, for any $x\in\Sigma(\Omega)$, there exists a sufficiently small neighborhood V_x such that there is an ordering of those eigenvalues so that $\operatorname{Re}S^{(w)}(y)\leqslant\dots\leqslant\operatorname{Re}S^{(1)}(y)$ for all $y\in V_x$. The closed set $\bigcup_{i>j}\partial K_{ij}$ is of lower dimension. Therefore, if we throw away a certain closed subset of lower dimension, any sufficiently small open subset and the smallest H_0-conical set containing it will satisfy the condition in the statement (iii) of our main theorem.

The statement (i) and (ii) in the main theorem has been proved and stated in Proposition 6.3. Let $O\subseteq A$ be the Zariski open dense set described in that proposition. For $x\in O$, the constant matrix $B_0(\log x)$ of the system (1.16), i.e., the constant term in the expansion of $B(\log x;t)$ in t is diagonalizable and has distinct eigenvalues. On an open subset Ω_0 of O, if we have an ordering of those eigenvalues of $B_0(x)$, say $\{S^{(1)}(x),\dots,S^{(w)}(x)\}$, such that $S^{(i)}(x)$ are differentiable, then, by Proposition 6.3 (iii) there is a matrix-valued function $E=E(x)$ on Ω_0 such that $E^{-1}B_0E=\operatorname{diag}[S^{(1)},\dots,S^{(w)}]$.

Now we recall the linear system of differential equations (1.16),

$$\begin{aligned}&\frac{d\Psi}{dt}(x_0;t)=B(x_0;t)\Psi(x_0;t),\\ &B(x_0;t)=B_0(x_0)-Dt^{-1}+\text{lower order terms},\\ &\Phi(x_0;t)=t^{D}\Psi(x_0;t),\end{aligned}\tag{7.1}$$

where $x_0=\log x$, $D=\mathrm{diag}[d_1,\dots,d_w]$, $d_i=\deg e_i$, $i=1,\dots,w$. For $x\in O$, Theorem A.4 in the appendix asserts that there exists a fundamental matrix solution

$$\Psi(x_0;t)=E(x_0)\widehat{\Psi}(x_0;t)t^{-\Lambda(x_0)}e^{tS(x_0)} \tag{7.2}$$

such that

$$\widehat{\Psi}(x_0;t)\sim\sum_{k=0}^{\infty}\Psi_k(x_0)t^{-k},\quad \text{as } t\to\infty,$$

with $\Psi_0(x_0)=I$. Here

$$S(x_0)=\mathrm{diag}[S^{(1)}(x_0),\dots,S^{(w)}(x_0)]$$

and

$$\begin{aligned}\Lambda(x_0)&=\text{diagonal part of } E(x_0)^{-1}DE(x_0)\\ &=\mathrm{diag}[\lambda_1(x_0),\dots,\lambda_w(x_0)].\end{aligned}$$

Remark 3. We give here a description of Λ in terms of B_0 and its eigenvalues $S^{(i)}$, $i=1,\dots,w$. Since, for $x\in U$, $B=B_0(x)$ has w distinct eigenvalues, B is a regular element in $\mathfrak{gl}(w)$. Let $\mathfrak{h}$ be the centralizer of B. Then $\mathfrak{h}$ is a Cartan subalgebra and is spanned by $\{B^j\}_{j=1,\dots,w-1}$. Let $\{B_j\}$ be the dual basis with respect to the trace form on $\mathfrak{gl}(w)$. Then the $\mathfrak{h}$-component $D_{\mathfrak{h}}$ of D in the root decomposition of $\mathfrak{gl}(w)$ with respect to $\mathfrak{h}$ is $\sum_{j=0}^{w-1}\mathrm{tr}(DB^j)B_j$. One also has $B^j=\sum_{k=0}^{w-1}\mathrm{tr}(B^{j+k})B_k$. Let $M=(\mathrm{tr}\,B^{j+k-2})_{j,k}$ and $M^{-1}=(m^{jk})$. Then

$$B_j=\sum_{k=0}^{w-1}m^{j+1,k+1}B^k,\quad j=0,\dots,w-1,$$

and

$$D_{\mathfrak{h}}=\sum_{k,j=0}^{w-1}\mathrm{tr}(DB^j)m^{j+1,k+1}B^k.$$

Observe that

$$\mathrm{tr}\,B^{j+k-2}=\sum_{i=1}^{w}(S^{(i)})^{j+k-2}\quad\text{and}\quad M=H(S^{(1)},\dots,S^{(w)})^t,$$

where $H(x_1,\dots,x_n)=(x_j^{i-1})_{i,j=1,\dots,n}$.

Motivated by the above considerations, we set $\gamma_i=\mathrm{tr}\,DB_0^i$. Then

$$\gamma_i=\mathrm{tr}(E^{-1}DE)(\mathrm{diag}[S^{(1)},\dots,S^{(w)}])^i=\sum\lambda_j(S^{(j)})^i.$$

Write $\vec{\gamma}=[\gamma_1,\dots,\gamma_w]^t$, $\vec{\lambda}=[\lambda_1,\dots,\lambda_w]^t$. Then

$$\vec{\gamma}=H(S^{(1)},\dots,S^{(w)})\vec{\lambda}$$

or

$$\vec{\lambda} = H(S^{(1)}, \ldots, S^{(w)})^{-1}\vec{\gamma}. \tag{7.3}$$

$\det H(S^{(1)}, \ldots, S^{(w)}) = \prod_{i>j}(S^{(i)} - S^{(j)})$, so each λ_i is a rational function of $S^{(1)}, \ldots, S^{(w)}$ and $x^{\alpha_1}, \ldots, x^{\alpha_l}$ with denominator $\prod_{i>j}(S^{(i)} - S^{(j)})$. Note also that $\lambda_i(x_0 + \tau H_0) = \lambda_i(x_0)$.

Let $\{\tilde{\phi}_1, \ldots, \tilde{\phi}_w\}$ be a basis of $W(\nu)$. Set $\tilde{\phi}_i^{\varrho} = e^{-\varrho}\tilde{\phi}_i|_A$ and $\tilde{\Phi} = [e_i\tilde{\phi}_i^{\varrho}, \ldots, e_w\tilde{\phi}_i^{\varrho}]$, where $e_1, \ldots, e_w$ form a basis of the space of all the harmonics in $S(\mathfrak{a})$, $e_1 = 1$ and e_i are homogeneous. From the discussion in §1, $\tilde{\Phi}_i(x;t)$ is a solution of (7.1) and $\tilde{\Phi}(x;t) = [\tilde{\Phi}_1(x;t), \ldots, \tilde{\Phi}_w(x;t)]$ is a fundamental matrix solution. By (7.2), for $x \in O$, there exists $C(x)$, $\det C(x) \neq 0$, such that

$$\tilde{\Phi}(x;t) = t^D E(x)\hat{\Phi}(x;t)t^{-\Lambda(x)}e^{tS(x)}C(x). \tag{7.4}$$

For a fixed $x \in O$, there is a sufficiently small neighborhood V of x such that $E(x)$ is defined on V and $\hat{\Psi}(x';t) \to I$ as $t \to \infty$ uniformly for $x' \in V$ (see Lemma 7.3). Therefore $\hat{\Psi}(x;t)$ is invertible for large t and $C(x')$ can be written as a product of matrix-valued functions smooth in x'. Thus on any open subset Ω of O, if there is a unified ordering of eigenvalues of $B_0(x)$, $x \in \Omega_0$, then there exists a smooth $C(x)$ on Ω_0 such that (7.4) is satisfied.

Before we examine $C(x)$, we want to know more about the dependence on the parameter x_0 of the asymptotic expansion of $\hat{\Psi}(x_0;t)$. Though the differential equation (7.1) is defined for $x_0 \in \mathfrak{a}$, we may extend it to $\mathfrak{a}_{\mathbf{C}}$ and $B(x_0;t)$ is then holomorphic in both variables x_0 and t whenever it is defined. We will need the following theorem later.

THEOREM 7.1 [Wa]. *Let S be the closed sector $\{x \in \mathbf{C} \mid \alpha \leqslant \arg x \leqslant \beta,\ |x| \geqslant c\}$ and T a compact domain in $\mathbf{C}$. Let $f(x,y)$ be holomorphic in both variables in $S \times T$ and*

$$f(x,y) \sim \sum_{r=0}^{\infty} a_r(y)x^{-r} \quad \text{as } x \to \infty \text{ in } S,$$

uniformly for $y \in T$, i.e., for each k,

$$x^k\left[f(x,y) - \sum_{r=0}^{k} a_r(y)x^{-r}\right] \to 0$$

uniformly with respect to y. Then all of the $a_r(y)$ are holomorphic in T and

$$\frac{\partial f(x,y)}{\partial y} \sim \sum_{r=0}^{\infty} \frac{da_r(y)}{dy}x^{-r} \quad \text{as } x \to \infty \text{ in } S,$$

uniformly in every proper compact subset of T.

LEMMA 7.2. *For $x' \in U$, there exists an open neighborhood V of x' such that $\widehat{\Psi}(x;t)$ possesses a uniformly valid asymptotic series expansion as $t \to \infty$, for $x \in V$.*

Proof. It suffices to show that the analytic simplification described in the appendix can be done in a uniform manner once we know that $B(x;t)$ in (7.1) possesses a uniformly valid asymptotic expansion as $t \to \infty$. The first step in the analytic simplification is to reduce the problem to the case where the given formal power series solution is zero. Therefore, we need to show that, given a formal power series $\sum_{r=1}^{\infty} w_r(x)t^{-r}$, where $w_r(x)$ are holomorphic, there exists $\phi(x;t)$ holomorphic in both variables x and t such that $\phi(x;t) \sim \sum_{t=0}^{\infty} w_r(x)t^{-r}$ as $t \to \infty$ is uniformly valid. In fact, $\phi(x;t)$ can be chosen to be $\sum_{r=0}^{\infty} w_r(x)\alpha_r(x,t)t^{-r}$, where $\alpha_r(x,t) = \exp(-|w_r(x)|^{-1}t^{\beta})$, $0<\beta<1$. The verification of this is standard. The existence of such a ϕ enables us to conclude that every asymptotic expansion involved in the expression of $p(x;t,u)$ in (a.8) is uniformly valid for x. Since $\Lambda(x)$ in (a.9) is continuous where it is defined, we have that in a sufficiently small neighborhood V of x', $\Gamma(x;\xi)$ in (a.9) does not depend on $x \in V$. The equation (a.8) can be solved by successive approximations using the integral operator $\mathcal{P}(x)$ which is defined by the right hand side of (a.9). The following estimation (cf. [Wa]) is used to show that the successive approximations converge to the solution: if $\|\chi(\xi)\| \leqslant c|\xi|^{-m}$, there exists K which depends on m but not on c or χ such that

$$\left\| \int_{\Gamma(\xi)} e^{(\xi-t)\Lambda}\chi(t)\,dt \right\| \leqslant Kc|\xi|^{-m}.$$

The constant might depend on Λ, but this dependence on Λ may be eliminated by shrinking V. Another estimation needed is that for $\|Z^{(i)}\| \leqslant c_0$, c_0 small, $i=1,2$, there exists γ such that

$$\|(B(t)-\Lambda)(z^{(2)}-z^{(1)})+h(t;z^{(2)})-h(t;z^{(1)})\| \leqslant \gamma\|z^{(2)}-z^{(1)}\|.$$

(Recall that $p(t,z)=b(t)+(B(t)-\Lambda)z+h(t,z)$.) Since $B(t) \to \Lambda$ uniformly in t and as a polynomial in z the coefficients of $h(t;z)$ have uniformly valid asymptotic expansions, γ can be chosen to be independent of $x \in V$. Therefore the successive approximations can be carried out in a uniform manner.

LEMMA 7.3. *Let $\lambda, S \in \mathbf{C}$ and $h(t) \sim t^m \sum_{j=0}^{\infty} d_j t^{-j}$ as $t \to \infty$ with $d_0 \neq 0$. Suppose that $\lim_{t\to\infty} t^{\lambda}e^{tS}h(t)$ exists and is finite, then it is zero except when $S=0$ and $\lambda=-m$, and in this case the limit is d_0.*

Proof. It is clear that if $\operatorname{Re} S>0$, $t^{\lambda}e^{tS}h(t) \to \infty$ as $t \to \infty$ and if $\operatorname{Re} S<0$, $t^{\lambda}e^{tS}h(t) \to 0$ as $t \to \infty$. Therefore, we may assume that $\operatorname{Re} S=0$. If $\operatorname{Re}\lambda+m>0$, then $t^{\lambda}e^{tS}h(t) \to \infty$ as $t \to \infty$, and if $\operatorname{Re}\lambda+m<0$, then the limit is zero. So we may assume that $\operatorname{Re}\lambda=-m$.

If $\lim_{t\to\infty} t^{\lambda+m}e^{tS}(t^{-m}h(t))$ exists and is finite, then

$$\lim_{t\to\infty} t^{\lambda+m}e^{tS} = \lim_{t\to\infty} \exp i[\operatorname{Im}\lambda\log t+t\operatorname{Im}S]$$

exists, i.e., $\lim_{t\to\infty}(\operatorname{Im}\lambda\log t+t\operatorname{Im}S) \pmod{2\pi}$ exists. This happens only when $\operatorname{Im}\lambda=\operatorname{Im}S=0$, and in this case $\lim_{t\to\infty} t^{\lambda}e^{tS}h(t)=d_0$.

Remark 4. If $\operatorname{Re}S=0$, $h(t)\sim 0$ as $t\to\infty$, take $m>0$ such that $\operatorname{Re}\lambda-m<0$. Then $t^{\lambda-m}e^{tS}(t^mH(t))\to 0$ as $t\to\infty$. Therefore the condition that $h\not\sim 0$ is superfluous when $\operatorname{Re}S=0$.

LEMMA 7.4. *Suppose that on an open convex set of $\mathbf{C}^n$ or $\mathbf{R}^n$, $\mathcal{D}$: $(\partial H)F(x)=U_H(x)F(x)$, $H\in\mathbf{C}^n$, with $U_H(x)$ an upper triangular $n\times n$ matrix, defines an integrable system. Then there is an upper triangular fundamental matrix solution.*

Remark 5. If all of the U_H in the lemma are block upper triangular and all their diagonal blocks are diagonal matrices, then there exists a fundamental matrix of solutions of the same form.

Proof. Let $F=(f_1,\dots,f_n)^t$ and $F_j=(f_j,\dots,f_n)^t$, $j=1,\dots,n$. Observe that F_j is a solution of a similar system. Therefore we can prove the lemma by induction on j. But we first consider the case when all the U_H are diagonal. $\mathcal{D}$ can then be rewritten as

$$\frac{df_j}{dx_i}=u_{i,j}f_j,\quad i,j=1,\dots,n,$$

where $\{x_1,\dots,x_n\}$ is the standard coordinate system on $\mathbf{C}^n$ or $\mathbf{R}^n$. Then $d(\ln f_j)=\sum u_{i,j}\,dx_i$, if $f_j\not\equiv 0$, the integrability of the system implies that $\sum u_{i,j}\,dx_i$ is closed, hence there exists a unique function g_j such that $f_j=c_je^{g_j}$ for some constants $c_j\in\mathbf{C}$. Hence in this case, the system has a fundamental matrix of solutions $\operatorname{diag}[e^{g_1},\dots,e^{g_n}]$. In the general case, let $\Delta_H=$ the diagonal part of U_H. Since $[U_{H_1},U_{H_2}]=\partial H_1U_{H_2}-\partial H_2U_{H_1}$, $0=[\Delta_{H_1},\Delta_{H_2}]=\partial H_1\Delta_{H_2}-\partial H_2\Delta_{H_1}$. Therefore the system

$$(\partial H)G=\Delta_HG$$

is integrable. Let $\operatorname{diag}[e^{g_1},\dots,e^{g_n}]=M$ be a fundamental matrix of solutions of this system.

$$\begin{aligned}(\partial H)(M^{-1}F)&=-M^{-1}(\partial HM)M^{-1}F+M^{-1}U_HMM^{-1}F\\&=M^{-1}(U_H-\Delta_H)M\cdot M^{-1}F.\end{aligned}$$

Therefore, we may assume Δ_H is zero, i.e., U_H is nilpotent. Let $\mathcal{D}_m$ be the subsystem

$$\frac{\partial f_i}{\partial x_i}=\sum_{k\geqslant j+1}u_{i,j,k}f_k,\quad i=1,\dots,n,\ j=m,\dots,n,$$

of $\mathcal{D}$. Then $\mathcal{D}=\mathcal{D}_1$. We prove the lemma by induction on m. When $m=n$, it is clear. Suppose $\mathcal{D}_m$ has a fundamental matrix of solutions G_m which is upper triangular. Now we consider the system

$$\frac{\partial v}{\partial x_i}=[u_{i,m+1,m},\dots,u_{i,m+1,n}]G_m$$

with v a row vector. This system is integrable. Let g_{m-1} be the unique solution (up to scalar). Set

$$G_{m-1}=\begin{bmatrix}1 & g_{m-1}\\ 0 & G_m\end{bmatrix}.$$

Then

$$\frac{\partial G_{m-1}}{\partial x_i}=[u_{i,j,k}]_{m-1\leqslant j,k\leqslant n}G_{m-1},$$

and the column vectors of G_{m-1} are linearly independent, i.e., G_{m-1} is a fundamental matrix solution of $\mathcal{D}_{m-1}$. Therefore the subsystem $\mathcal{D}_{m-1}$ has an upper triangular fundamental matrix solution and the lemma follows.

LEMMA 7.5. *If V is an open convex subset of O such that there is an ordering of the eigenvalues of $B_0(x)$ so that $\operatorname{Re}S^{(j)}\leqslant\operatorname{Re}S^{(i)}$ if $i\leqslant j$, then there exists a constant matrix C such that $C(x)C^{-1}$ is upper triangular. Furthermore, the (i,j)-th entry of $C(x)C^{-1}$ is zero whenever $\operatorname{Re}S^{(i)}=\operatorname{Re}S^{(j)}$.*

Proof. For $H\in\mathfrak{a}$, one has

$$(\partial H)\widetilde{\Phi}(x;t)=\Gamma_H(x;t)\widetilde{\Phi}(x;t).$$

Since $\widetilde{\Phi}(x;t)=t^D\widehat{\Phi}(x;t)t^{-\Lambda(x)}e^{tS(x)}C(x)$ with $\widehat{\Phi}(x;t)=E(x)\widehat{\Psi}(x;t)$, we have

$$\begin{aligned}(\partial H)\widetilde{\Phi}(x;t)&=t^D\{(\partial H)\widehat{\Phi}(x;t)-\log t\,\widehat{\Phi}(x;t)(\partial H)\Lambda(x)+t\widehat{\Phi}(x;t)(\partial H)S(x)\}\\&\qquad\times t^{-\Lambda(x)}e^{tS(x)}C(x)+t^D\widehat{\Phi}(x;t)t^{-\lambda(x)}e^{tS(x)}(\partial H)C(x)\\&=\Gamma_H(x;t)t^D\widehat{\Phi}(x;t)t^{-\Lambda(x)}e^{tS(x)}C(x).\end{aligned}$$

Therefore

$$\begin{aligned}(\partial H)C(x)\cdot C(x)^{-1}&=\mathrm{Ad}(t^{\Lambda(x)}e^{-tS(x)})\\&\qquad\times\{\log t(\partial H)\Lambda(x)-t(\partial H)S(x)-\widehat{\Phi}(x;t)^{-1}(\partial H)\widehat{\Phi}(x;t)\\&\qquad+\widehat{\Phi}(x;t)^{-1}t^{-D}\Gamma_H(x;t)t^D\widehat{\Phi}(x;t)\}.\end{aligned}\tag{7.5}$$

$\widehat{\Phi}(x;t)$ is invertible for large t because $\lim_{t\to\infty}\widehat{\Phi}(x;t)=E(x)$. The left hand side does not depend on t, so as $t\to\infty$, the right hand side has a limit. By Lemma 7.3 each off diagonal entry of the term inside the bracket in (7.5) has an asymptotic expansion as

$t \to \infty$. Since we have $\operatorname{Re} S^{(j)} \leqslant \operatorname{Re} S^{(i)}$, for $i \leqslant j$, by Lemma 7.4 and the remark following it, $(\partial H)C(x) \cdot C(x)^{-1}$ is upper triangular. Furthermore, when $\operatorname{Re} S^{(j)} = \operatorname{Re} S^{(i)}$, $i \neq j$, the (i,j)th entry of $(\partial H)C(x) \cdot C(x)^{-1}$ is zero. Therefore, $C(x)$ forms a fundamental matrix of solutions of the system of differential equations

$$(\partial H)v(x) = U_H(x)v(x),$$

where $U_H(x)$ is the limit of the right hand side of equality (7.5) as $t \to \infty$. Therefore, there exists a constant matrix C such that $C(x)C^{-1}$ is upper triangular.

LEMMA 7.6. *Suppose $E(x)$, $C(x)$ and $\widehat{\Phi}(x;t)$ are defined on some open H_0-conical subset Ω_0 of O. Then, for $x \in \Omega$,*

(i) $C(x \exp \tau H_0) = e^{\tau \Lambda} C(x)$;

(ii) $\widehat{\Phi}(x;t)(x \exp \tau H_0; t) = \widehat{\Phi}(x; e^{\tau} t)$.

Proof. Let $t^{-1} t^{-D} \Gamma_{H_0}(x;t) t^{D} = \sum_{r=0}^{\infty} \mathring{A}_r(x) t^{-r}$ (in fact, the sum is finite). Let $A_r(x) = E(x)^{-1} \mathring{A}_r(x) E(x)$. Then

$$\widehat{\Phi}(x;t) = P(x;t) \exp(-D(x;t))$$

with

$$P(x;t) \sim \sum_{r=0}^{\infty} P_r(x) t^{-r} \quad \text{and} \quad D(x;t) \sim \sum_{r=1}^{\infty} \frac{D_{r+1} t^{-r}}{r} \quad \text{as } t \to \infty,$$

where $P_r(x)$, $D_r(x)$ are determined by procedures described in the appendix, especially, by equations (a.3) and (a.4). But here we use the notation D_r instead of B_r. Notice that B_r are diagonal matrices and $P_r(x)$ have zero diagonal entries. Since $S^{(i)}(x \exp \tau H_0) = e^{\tau} S^{(i)}(x)$, by Proposition 6.3 (iii), one has

$$E(x \exp \tau H_0) = e^{\tau D} E(x). \tag{7.6}$$

It is easy to see that $\mathring{A}_r(x) = \left(\sum_{\deg \omega_k = r} u_{ij}^k \chi(\omega_k) \right)_{i,j}$, $\deg u_{ij}^k = d_i - d_j - r + 1$. Hence

$$\mathring{A}_r(x \exp \tau H_0) = \left(e^{(d_i - d_j - r + 1)\tau} \sum_{\deg \omega_k = r} u_{ij}^k(x) \chi(\omega_k) \right)_{i,j} = e^{(-r+1)\tau} e^{\tau D} \mathring{A}_r(x) e^{-\tau d},$$

and then

$$A_r(x \exp \tau H_0) = e^{-(r-1)\tau} A_r(x). \tag{7.7}$$

From (a.3), (a.4), (7.7), we have

$$\begin{aligned} P_r(x \exp \tau H_0) &= e^{-r\tau} P_r(x), \\ B_r(x \exp \tau H_0) &= e^{-(r-1)\tau} B_r(x). \end{aligned} \tag{7.8}$$

Thus $\widehat{\Phi}(x\exp\tau H_0;t)$ and $\widehat{\Phi}(x;e^\tau t)$ have the same asymptotic expansion as $t\to\infty$. Since they, as functions in t, satisfy the same differential equation,

$$\widehat{\Phi}(x\exp\tau H_0;t)=\widehat{\Phi}(x;e^\tau t)C_1(\tau)$$

for some $C_1(\tau)$, $\det C_1(\tau)\neq 0$. Then

$$\widehat{\Phi}(x;e^\tau t)(C_1(\tau)-I)\sim 0 \quad \text{as } t\to\infty,$$

and $C_1(\tau)-I=\lim_{t\to\infty}\widehat{\Phi}(x;e^\tau t)(C_1(\tau)-I)=0$, i.e., $C_1(\tau)=I$. Thus (ii) follows. To prove (i), note that

$$\widetilde{\Phi}(x;e^\tau t)=e^{\tau D}t^D E(x)\widehat{\Phi}(x;e^\tau t)t^{-\Lambda(x)}e^{e^\tau tS(x)}C(x)$$

and

$$\widetilde{\Phi}(x\exp\tau H_0;t)=t^D E(x\exp\tau H_0)\widehat{\Phi}(x\exp\tau H_0;t)t^{-\Lambda(x)}e^{e^\tau tS(x)}C(x\exp\tau H_0).$$

Therefore, by (7.6) and (ii), $C(x\exp\tau H_0)=e^{-\tau\Lambda(x)}C(x)$.

Proof of the main theorem. Since $E_1=1$, $[\tilde{\phi}_1^\varrho,\dots,\tilde{\phi}_w^\varrho]$ is the first row of the fundamental matrix solution $\widetilde{\Phi}(x;t)$. As t^D is diagonal and its first diagonal entry is 1, $[\tilde{\phi}_1^\varrho,\dots,\tilde{\phi}_w^\varrho]$ is the first row of $\widehat{\Phi}(x;t)t^{-\Lambda(x)}e^{tS(x)}C(x)$ with $\widehat{\Phi}(x;t)=E(x)\widehat{\Phi}(x;t)=(\phi_{ij}(x;t))_{i,j}$. Let $E(x)=(e_{ij}(x))_{i,j}$. By Lemma 7.5, we may assume that $C(x)=(q_{ij}(x))_{i,j}$ is an upper triangular matrix with $q_{ij}(x)=0$ whenever $\operatorname{Re}S^{(i)}=\operatorname{Re}S^{(j)}$ and $i\neq j$. Then

$$\begin{aligned}\tilde{\phi}_i^\varrho(x;t)&=\sum_{i\geqslant j}\phi_{ij}(x;t)t^{-\lambda_j(x)}e^{tS^{(j)}(x)}q_{ij}(x)\\&=t^{\lambda_i(x)}e^{tS^{(j)}(x)}\Big\{\phi_{1i}(x;t)q_{ii}(x)+\sum_{i>j}\phi_{1j}(x;t)q_{ji}(x)\tau^{\lambda_i-\lambda_j}e^{t(S^{(j)}-S^{(i)})}\Big\}.\end{aligned}$$

But if $\operatorname{Re}S^{(j)}\leqslant\operatorname{Re}S^{(i)}$, either $q_{ji}=0$ or $\exp t(S^{(j)}-S^{(i)})\sim 0$ as $t\to\infty$, so $\phi_i'=\phi_{1i}q_{ii}+\sum_{i>j}\phi_{1j}q_{ji}t^{\lambda_1-\lambda_j}e^{t(S^{(j)}-S^{(i)})}$ and $\phi_{1i}q_{ii}$ have the same asymptotic expansion as $t\to\infty$. If $\phi_{1j}(x;t)\sim\sum_{r=0}^\infty\phi_{jr}(x)t^{-r}$, by Lemma 7.6, ϕ_{jr} is homogeneous of degree $-(r-1)$ in the direction H_0 and $q_{ii}(x\exp\tau H_0)=e^{-\lambda_i(x)\tau}q_i(x)$. Note that $\phi_{j0}=e_{1j}\not\equiv 0$ on Ω. This completes the proof.

8. The affine case

In this section, we show how one can generalize the main theorem in the last section to the affine Lie algebras. Let $\mathfrak{g}_0$ be a simple Lie algebra of type A, B, C, D or E_6,

$\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{a}_0\oplus\mathfrak{n}_0$ is an Iwasawa decomposition. Let $\Pi_0=\{\alpha_1,...,\alpha_l\}$ be the simple root system for the pair $(\mathfrak{n}_0,\mathfrak{a}_0)$. Let β be the longest positive root. Set $\alpha_{l+1}=-\beta$. By adjoining α_{l+1} to the Dynkin diagram associated to Π_0, we get an extended Dynkin diagram Π_β. Then there is an affine Lie algebra $\mathfrak{g}$ associated to Π_β. If X_{α_i} (or X_β) is a non-zero root vector of $\mathfrak{g}_0$ for the root α_i (or β), then $\mathfrak{g}$ is generated by $H\otimes 1$, $H\in\mathfrak{n}$, $X_{\alpha_i}\otimes 1$, $i=1,...,l$, $X_{\alpha_{l+1}}\otimes t$ and $X_\beta\otimes t^{-1}$. (For details, see, for example, [Ka].) For the Dynkin diagram associated to Π_0, one an define $\mathfrak{h}_0$, $\mathfrak{a}_0$, $\mathfrak{n}_0$, $\mathfrak{s}_0$ and $\mathfrak{u}_0$ as in §1. For the extended Dynkin diagram Π_β, we associate to it a finite-dimensional real Lie algebra $\mathfrak{b}$ equipped with a positive definite inner product $\langle\cdot,\cdot\rangle$, such that

(i) $\mathfrak{b}$ is the orthogonal direct sum of two abelian Lie algebras $\mathfrak{a}=\mathfrak{a}_0$ and $\mathfrak{u}$ such that $[\mathfrak{a},\mathfrak{u}]\subseteq\mathfrak{u}$;

(ii) for $H\in\mathfrak{a}$, $\mathrm{ad}(H)$ is symmetric relative to $\langle\cdot,\cdot\rangle$. Hence, there exists an orthonormal basis $\{Y_1,...,Y_{l+1}\}$ of $\mathfrak{u}$ and $\alpha_i\in\mathfrak{a}^*$ such that

$$[H,Y_i]=\alpha_i(H)Y_i,\quad H\in\mathfrak{a},\ i=1,...,l+1,$$

and

(iii) the α's are exactly those in the extended Dynkin diagram Π_β.

Remark 1. We may assume that $\langle\cdot,\cdot\rangle|_{\mathfrak{b}_0\times\mathfrak{b}_0}$ is the same inner product for $\mathfrak{b}_0$ we used earlier, and one might identify those X_i in the previous sections with Y_i, $i=1,...,l$, since we have an obvious injection from $\mathfrak{b}_0$ into $\mathfrak{b}$ which sends the α_i root space in $\mathfrak{b}_0$ to the α_i root space in $\mathfrak{b}$. In fact, Y_i can be taken to be $X_{\alpha_i}\otimes 1$, $i=1,...,l$, and Y_{l+1} to be $X_{\alpha_{l+1}}\otimes t$.

Let $\{Z_i\}$ be a basis for $\mathfrak{b}$ and $\{W^i\}$ another basis that is dual to $\{Z_i\}$, that is, $\langle W^i,Z_j\rangle=\delta_{ij}$. Then the Laplacian associated to $\mathfrak{b}$ is

$$\Omega=\sum Z_iW^i\in U(\mathfrak{b}).$$

It does not depend on our choice of basis. Let $\{h_j\}$ be any orthonormal basis for $\mathfrak{a}$ and $\{Y_i\}$ be the basis of $\mathfrak{u}$ given in (iii). Then

$$\Omega=\sum_{j=1}^{l}h_j^2+\sum_{i=1}^{l+1}Y_i^2.$$

The structure of $U(\mathfrak{b})^\Omega$ has been studied in [GW4]; here we quote a result from this paper.

THEOREM. *Let $u_1,...,u_l$ be a set of homogeneous algebraically independent generators of $U(\mathfrak{a})^W$. Then there exist elements $\Omega_1,...,\Omega_l$ in $U(\mathfrak{b})^\Omega$ such that*

(i) *the elements $\Omega_1,...,\Omega_l$ mutually commute and are algebraically independent*;

(ii) $\mu\Omega_i = u_i$ *and* $\deg \Omega_i = \deg u_i$;

(iii) Ω_i *is in the subalgebra of* $U(\mathfrak{b})$ *generated by* $\mathfrak{a}$ *and* $Y_1^2, \ldots, Y_l^2, Y_{l+1}^2$;

(iv) $U(\mathfrak{b})^{\Omega}$ *is generated as an algebra by* $\Omega_1, \ldots, \Omega_l$ *and* ξ.

Here μ is the projection from $\mathfrak{b}$ onto $\mathfrak{n}$ and ξ is $Y_{l+1} \prod_{i=1}^{l} Y_i^{n_i}$ if $\beta = \sum_{i=1}^{l} n_i \alpha_i$.

Remark. This theorem is also true when Π_0 is a Dynkin diagram of type B, C, and β is the short dominant root.

Let $\mathcal{J}$ be the subalgebra of $U(\mathfrak{b})$ generated by $\Omega_1, \ldots, \Omega_l$. The map

$$U(\mathfrak{u}) \otimes H \otimes \mathcal{J} \to U(\mathfrak{b})$$
$$z \otimes e \otimes w \mapsto zew$$

is a linear isomorphism. More precisely, if $\{U_j(\mathfrak{b})\}$ is the usual filtration,

$$U_j(\mathfrak{u}) = U(\mathfrak{u}) \cap U_j(\mathfrak{b}), \quad \mathcal{H}_j = \mathcal{H} \cap U_j(\mathfrak{b}) \quad \text{and} \quad \mathcal{J}_j = \mathcal{J} \cap U_j(\mathfrak{b}),$$

one has

$$U_j(\mathfrak{b}) = \sum_{r+s+t=j} U_r(\mathfrak{u}) \cdot \mathcal{H}_s \cdot \mathcal{J}_t. \tag{8.1}$$

Let η be a generic character on $\mathfrak{u}$, $\eta_0 = \eta|_{\mathfrak{u}_0}$. One can define a representation Π_η of $U(\mathfrak{b})$ on $C^\infty(\mathfrak{a})$ using (1.5a) and (1.5b). Also we have Π_{η_0}, a representation of $U(\mathfrak{b}_0)$ on $C^\infty(\mathfrak{a})$. If χ is a homomorphism from $\mathcal{J}$ into $\mathbf{C}$, analogous to (1.6), one might consider the following system of differential equations:

$$\Pi_\eta(\mathfrak{u})\phi = \chi(\mathfrak{u})\phi, \quad \mathfrak{u} \in g. \tag{8.2}$$

We will relate this system with the one associated with $\mathfrak{b}_0$ that we studied in the previous sections. To this end, we introduce a family of Lie algebra homomorphisms σ_s of $\mathfrak{b}$. σ_s when restricted to $\mathfrak{b}_0$ is the identity and $\sigma_s(Y_{l+1}) = sY_{l+1}$. It is clear that they are Lie algebra homomorphisms and they can be extended to homomorphisms of enveloping algebras. Furthermore, they are isomorphisms except when $s=0$. Note also that σ_s preserves the standard filtration $U_j(\mathfrak{b})$, that is,

$$\sigma_s(U_j(\mathfrak{b})) \subseteq U_j(\mathfrak{b}).$$

$U_j(\mathfrak{b})$ is finite-dimensional and the map

$$s \mapsto \sigma_s|_{U_j(\mathfrak{b})}$$

is continuous.

Let $\mu:\mathfrak{b}\to\mathfrak{a}$ and $\mu_0:\mathfrak{b}_0\to\mathfrak{a}_0$ be the canonical projections. Then it is clear that

$$\mu_0(\sigma_0(z))=\mu(z)$$

and

$$\mu(\sigma_s(z))=\mu(z),\quad z\in U(\mathfrak{b}). \tag{8.3}$$

Set $\Omega_s=\sigma_s\Omega$. Let $\Omega_1,...,\Omega_l$ be those invariants in the above theorem. Since σ_s is a Lie algebra homomorphism, $\sigma_s\Omega_1,...,\sigma_s\Omega_l$ are in $U(\mathfrak{b})^{\Omega_s}$, they mutually commute, and they are algebraically independent since $\mu(\sigma_s\Omega_i)=\mu(\Omega_i)=u_i$. If $s\neq 0$, together with ξ, they generate $U(\mathfrak{b})^{\Omega_s}$ as an algebra. Let $\mathcal{J}_s$ be the subalgebra of $U(\mathfrak{b})^{\Omega_s}$ generated by $\sigma_s\Omega_1,...,\sigma_s\Omega_l$. For $m=(m_1,...,m_l)\in\mathbf{N}^l$, set $\omega_m=\Omega_1^{m_1}...\Omega_l^{m_l}$. Then as vector spaces $\{\omega_m\}$ is a basis for $\mathcal{J}$ and $\{\sigma_s\omega_m\}$ is a basis for $\mathcal{J}_s$. One can identify $\mathcal{J}_0$ with $U(\mathfrak{b})^{\Omega_0}$ and the basis for $U(\mathfrak{b}_0)^{\Omega_0}$ in previous sections can be taken to be $\{\sigma_0\omega_m\}$. Note that

$$\lim_{s\to 0}\sigma_s(\omega_m)=\sigma_0(\omega_m). \tag{8.4}$$

If u is a homogeneous element in $U(\mathfrak{b})$, then one has the decomposition by using (8.1),

$$u=\sum u_{jm}e_j\omega_m, \tag{8.5}$$

where $u_{jm}\in U(\mathfrak{u})$ and $\{e_j\}$ is a basis of $\mathcal{H}$ which consists of homogeneous elements. If $u\in U(\mathfrak{a})$, apply σ_s to both sides of (8.5). Since $\sigma_s u=u$, we have

$$u=\sum\sigma_s(u_{jm})e_j\sigma_s(\omega_m). \tag{8.6}$$

One may assume that u_{jm} is homogeneous and that it can be written as

$$u_{jm}=\sum_{k=0}^{\deg u_{jm}}u_{jm,k}(Y_1,...,Y_l)Y_{l+1}^k, \tag{8.7}$$

where $u_{jm,k}$ is a polynomial of degree $\deg u_{jm}-k$. Then

$$\sigma_s u_{jm}=\sum s^k u_{jm,k}(Y_1,...,Y_l)Y_{l+1}^k \tag{8.8}$$

and

$$\lim_{s\to 0}\sigma_s u_{jm}=u_{jm,0}(Y_1,...,Y_l). \tag{8.9}$$

Therefore, letting s approach 0 in (8.6), it follows from (8.4) and (8.9) that

$$u=\sum u_{jm,0}e_j\sigma_0(\omega_m). \tag{8.10}$$

Let $\tilde{\varrho}$, $H_{\tilde{\varrho}}$ be as in previous sections, that is, $\langle\tilde{\varrho},\alpha_i\rangle=1$, $i=1,\dots,l$. One may assume $\eta(Y_1)=\pm\sqrt{-1}$ and $\eta(Y_{l+1})=-c$ by conjugating an element in A.

Applying the above discussion to $H_{\tilde{\varrho}}e_i$, one has

$$H_{\tilde{\varrho}}e_i=\sum u_{ijm}e_j\omega_m, \tag{8.11}$$

$$u_{ijm}=\sum u_{ijm,k}(Y_1,\dots,Y_l)Y_{l+1}^k \tag{8.12}$$

and

$$H_{\tilde{\varrho}}e_i=\sum u_{ijm,0}e_j\sigma_0\omega. \tag{8.13}$$

Set $v_{ijm}=\pi_\eta(u_{ijm})\in\mathbf{R}[e^{\alpha_1},\dots,e^{\alpha_l},e^{\alpha_{l+1}}]$ and identify $H_{\tilde{\varrho}}e_i,e_j$ with $\pi_\eta(H_{\tilde{\varrho}}e_i),\pi_\eta(e_j)$. Suppose ϕ is a solution to (8.2), then

$$(H_{\tilde{\varrho}}e_i)\phi=\sum v_{ijm}\chi(\omega_m)e_j\phi. \tag{8.14}$$

Set $F=[e_1\phi,\dots,e_w\phi]^t$ and $\Psi(x;\tau)=F(x+\tau H_{\tilde{\varrho}})$ for $x\in\mathfrak{a}$, $\tau\in\mathbf{R}$. From (8.14), one has

$$H_{\tilde{\varrho}}F=\Gamma F,$$

where Γ is a $w\times w$ matrix and

$$\Gamma_{ij}=\sum v_{ijm}\chi(\omega_m).$$

Applying π_η to both sides of (8.12) one has

$$v_{ijm}=\sum v_{ijm,k}c^ke^{k\alpha_{l+1}},$$

where $v_{ijm,k}=\pi_\eta(\mathrm{u}_{ijm})=\pi_{\eta_0}(\mathrm{u}_{ijm})\in\mathbf{R}[e^{\alpha_1},\dots,e^{\alpha_l}]$. Then

$$\Gamma_{ij}(x+\tau H_{\tilde{\varrho}})=\sum v_{ijm,k}(x)\chi(\Omega_m)c^ke^{k\alpha_{l+1}(x)}\cdot e^{\tau(\deg v_{ijm,k}-k|\beta|)},$$

where $|\beta|=\sum n_i$ if $\beta=\sum n_i\alpha_i$. Note that

$$d_{ijm,k}=\deg v_{ijm,k}=d_i+1-d_j-l_m-k,$$

where $d_i=\deg e_i$, $l_m=\deg\omega_m$. Therefore,

$$\frac{d\Psi}{d\tau}(X;\tau)=(H_{\tilde{\varrho}}F)(x+\tau H_{\tilde{\varrho}})=\Gamma(x+\tau H_{\tilde{\varrho}})F(x+\tau H_{\tilde{\varrho}})=\Gamma(x+xH_{\tilde{\varrho}})\Psi(x;\tau). \tag{8.15}$$

Set $t=e^{\tau}$, (8.15) becomes

$$\frac{d\Psi}{dt}=A\Psi, \tag{8.16}$$

where

$$A_{ij}=\sum v_{ijm,k}(x)\chi(\omega_m)c^k e^{k\alpha_{l+1}(x)}\cdot t^{d_{ijm,k}-k|\beta|-1}.$$

Use the shearing transform $\mathrm{Sh}(t)=\mathrm{diag}[t^{d_1},\ldots,t^{d_w}]$ to get a new linear system as we did in §3,

$$\frac{d\Psi}{dt}=B\Psi, \tag{8.17}$$

where

$$\begin{aligned}B_{ij}&=\sum v_{ijm,k}(x)\chi(\omega_m)c^k e^{k\alpha_{l+1}(x)}\cdot t^{d_{ijm,k}-k|\beta|-1-(d_i-d_j)}-\delta_{ij}d_j t^{-1}\\&=\sum v_{ijm,k}(x)\chi(\omega_m)c^k e^{k\alpha_{l+1}(x)}\cdot t^{-l_m-k(1+|\beta|)}-\delta_{ij}d_j t^{-1}\\&=-v_{ij0,0}(x)-\delta_{ij}d_j t^{-1}+\text{lower order terms,}\end{aligned}$$

because $l_m\geqslant 2$ when $m\neq(0,0,\ldots,0)$ and $k(1+|\beta|)\geqslant 2k\geqslant 2$ when $k\geqslant 1$. Now we compare (8.17) with the system (3.1). The (i,j)th entry in the constant matrix $B(x;t)$ of the system (3.1) is $\underline{v}^0_{ij}(x)=\Pi_{\eta_0}(\underline{u}^0_{ij})$, where $\underline{u}^0_{ij}$ is defined by

$$H_{\tilde{e}}e_i=\sum \underline{u}^m_{ij}e_j\sigma_0\omega_m \tag{8.18}$$

(cf. (1.14)–(1.16)). Compare (8.18) with (8.13). Since the decomposition is unique (for (8.1) is an isomorphism), $\underline{u}^m_{ij}=u_{ijm,0}$, in particular, $\underline{u}^0_{ij}=u_{ij0,0}$ and $\underline{v}^0_{ij}=v_{ij0,0}$. Therefore, the linear systems (3.1) and (8.17) are essentially the same, since the major terms in the asymptotic expansions of their solutions depend only on the constant term and the t^{-1} term. As a consequence, the main theorem is valid for the affine Lie algebras we considered here.

Appendix: Asymptotic expansions of solutions of ordinary differential equations at irregular singularities

A.1

In this appendix, we will give a brief and selective tour to the general theory of asymptotic expansions of solutions of an ordinary differential equation at an irregular singularity [Wa]. Since our primary interest is the linear system

$$x^{-q}\frac{dY}{dx}=A(x)Y \tag{a.1}$$

with q a non-negative integer and $A(x)$ having an asymptotic expansion as $x\to\infty$, we will formulate those results only in the case when the singular point is ∞.

Definition. Let $S\subseteq\mathbf{C}$ be a point-set having ∞ as as accumulation point. Let $f(x)$ be a function defined on S. Then the formal power series $\sum_{r=0}^{\infty} a_r x^{-r}$ is said to be an asymptotic expansion of $f(x)$ or to represent $f(x)$ asymptotically, as $x\to\infty$ in S, if for all $m\geqslant 0$,

$$\lim_{\substack{x\to\infty\\ x\in S}} x^m\left\{f(x)-\sum_{r=0}^{m} a_r x^{-r}\right\}=0.$$

And we will write

$$f(x)\sim\sum_{r=0}^{\infty} a_r x^{-r},\quad x\in S,\ x\to\infty.$$

Also we write

$$f(x)\sim g(x)\sum_{r=0}^{m} a_r x^{-r},\quad x\to\infty \text{ in } S,$$

if $f(x)/g(x)\sim\sum_{r=0}^{\infty} a_r x^{-r}$, $x\to\infty$ in S.

A.2. Formal simplification

We will assume that $A(x)$ in (a.1) is holomorphic and $A(x)\sim\sum_{r=0}^{\infty} A_r x^{-r}$, $x\to\infty$, on an open sector $S=\{x\in\mathbf{C}\,|\,|x|\geqslant x_0,\ \theta_0<\arg x<\theta_1\}$. We consider the case when

$$A_0=\begin{bmatrix} A_0^{11} & 0\\ 0 & A_0^{22}\end{bmatrix},$$

where A_0^{11} is a $p\times p$ matrix, A_0^{22} an $(n-p)\times(n-p)$ matrix and A_0^{11}, A_0^{22} have different sets of eigenvalues. Our goal here is to find a formal power series $P(x)=\sum_{r=0}^{\infty} P_r x^{-r}$ with $\det P_0\neq 0$, such that the formal substitution

$$Y=\left(\sum_{r=0}^{\infty} P_r x^{-r}\right)Z$$

changes the differential equation (a.1) into the formal differential equation

$$x^{-q}Z'=\left(\sum_{r=0}^{\infty} B_r x^{-r}\right)Z,$$

where all B_r are of the same block-diagonal form as A_0. Let $B(x)=\sum_{r=0}^{\infty} B_r x^{-r}$. Then one has

$$B(x)=P^{-1}(x)A(x)P(x)-x^{-q}P^{-1}(x)P'(x), \tag{a.2}$$

and more explicitly

$$\begin{cases} A_0P_0-P_0B_0=0,\\ A_0P_r-P_rB_0=\sum_{s=0}^{r-1}(P_sB_{r-s}-A_{r-s}P_s)-(r-q-1)P_{r-q-1}, \quad r>0.\end{cases} \tag{a.3}$$

We choose $B_0 = A_0$, $P_0 = I$. Then we have

$$A_0 P_r - P_r A_0 = B_r + H_r, \quad r > 0, \tag{a.4}$$

where H_r is a polynomial in P_j, B_j with $j<r$. For $r>0$, if we confine P_r to be of the form

$$\begin{bmatrix} 0 & P_r^{12} \\ P_r^{21} & 0 \end{bmatrix}$$

and let

$$B_r = \begin{bmatrix} B_r^{11} & 0 \\ 0 & B_r^{22} \end{bmatrix},$$

then (a.4) becomes

$$\begin{cases} B_r^{ii} = -H_r^{ii}, \\ A_0^{ii} P_r^{ij} - P_r^{ij} A_0^{ij} = H_r^{ij}, \quad i \neq j, \end{cases} \tag{a.5}$$

where

$$H_r = \begin{bmatrix} H_r^{11} & H_r^{12} \\ H_r^{21} & H_r^{22} \end{bmatrix}.$$

Therefore B_r and P_r can be found successively in light of the following result in linear algebra which is standard.

LEMMA A.1. *Suppose that $A \in M_n$, $B \in M_m$ have different sets of eigenvalues. Then for any $C \in M_{n\times m}$, the equation $AX - XB = C$ has a unique solution $X \in M_{n\times m}$.*

A.3. Analytic simplification

The formal simplification described above can be made rigorous by the following considerations. We introduce new unknowns $\widehat{P}(x)$, $\widehat{B}(x)$ by relations $P(x) = I + \widehat{P}(x)$ and $B(x) = A_0 + \widehat{B}(x)$ and both matrices are of the same form as P_r and B_r, respectively. Then (a.2) becomes

$$x^{-q} \frac{d\widehat{P}^{ij}}{dx} = A^{ij}(x) + A^{ii}(x)\widehat{P}^{ij} - \widehat{P}^{ij} A^{jj}(x) - \widehat{P}^{ij} A^{ji}(x) \widehat{P}^{ij}, \quad i \neq j. \tag{a.6}$$

Regard $\widehat{P}^{ij}$ as a vector in $\mathbf{C}^{p(n-p)}$, then (a.6) takes the form

$$x^{-q} w' = f(x, w), \tag{a.7}$$

where $f(x,w) = f_0(x) + F(x)w + \sum f_{ij}(x) w_i w_j$ and f_0, F, f_{ij} are holomorphic on S and have asymptotic expansions as $x \to \infty$ in S. Furthermore, $\lim_{x\to\infty} F(x)$ is non-singular because A_0^{11} and A_0^{22} have no common eigenvalues. We may also assume that (a.7) has a formal power series solution $w=\sum_{r=1}^{\infty} w_r x^{-r}$. Then on any subsector S' of S which has a positive central angle not exceeding $\pi/(q+1)$, there exists a solution $w=\phi(x)$ of (a.7) such that $\phi(x)\sim\sum_{r=1}^{\infty} w_r x^{-r}$, $x\to\infty$ in S'. The proof of this result is lengthy and we will only give a sketch of it.

The first step is to reduce the problem to the case when the given formal power series solution is zero. This can be done because there always exists a holomorphic function $\phi(x)$ with $\phi(x)\sim\sum_{r=1}^{\infty} w_r x^{-r}$. If we set $u=w-\phi(x)$, then

$$x^{-q}u' = \Lambda u + p(x,u), \tag{a.8}$$

where $p(x,u)=b(x)+(B(x)-\Lambda)u+h(x,u)$ with $b(x)\sim 0$, $\lim_{x\to\infty} B(x)=\Lambda$ and $h(x,u)$ is a polynomial in u without constant or linear terms.

The second step is to transform (a.8) into an equivalent integral equation which is

$$u(\xi) = \int_{\Gamma(\xi)} \exp\left[\frac{\xi^{q+1}-t^{q+1}}{q+1}\Lambda\right] t^q p(t,u(t))\,dt, \tag{a.9}$$

where $\Gamma(\xi)$ is a path toward ξ. The detailed description of $\Gamma(\xi)$ will not be discussed here. Consider the right hand side of (a.9) to be a non-linear operator $\mathcal{P}$ on u. Then (a.9) is equivalent to $u=\mathcal{P}u$. As usual, we solve (a.9) by successive approximations: A sequence of functions $(u_r(x))$, $r=0,1,\dots$, is defined by

$$u_0 \equiv 0, \quad u_{r+1} = \mathcal{P}u_r \quad \text{for } r \geqslant 0,$$

and the limit of this sequence will be the solution of (a.9) provided that we can get a nice estimation of the differences $u_{r+1}-u_r=\mathcal{P}u_r-\mathcal{P}u_{r-1}$. The details of the estimation can be found in [Wa].

A.4

So far we have given a reduction to the case when A_0 has a single eigenvalue and we also have the following theorem.

THEOREM A.4. *Let $A(x)$ be an $n\times n$ matrix function holomorphic in $S=\{x\in\mathbf{C}\mid |x|\geqslant x_0, \theta_0<\arg x<\theta_1\}$, $\theta_1-\theta_0<\pi/(q+1)$ with an asymptotic expansion*

$$A(x) \sim \sum_{r=0}^{\infty} A_r x^{-r}, \quad x\to\infty \text{ in } S,$$

such that A_0 is diagonalizable and has distinct eigenvalues. Then the differential equation

$$x^{-q}\frac{dY}{dx} = A(x)Y, \quad q\geqslant 0,$$

possesses a fundamental matrix solution of the form

$$Y(x)=\widehat{Y}(x)x^D e^{Q(x)}.$$

Here $Q(x)$ is a diagonal matrix whose entries are polynomials of degree $q+1$ and its leading term is

$$x^{q+1}\operatorname{diag}[\lambda_1,\dots,\lambda_n]/(q+1)$$

if A_0 conjugates to $\operatorname{diag}[\lambda_1,\dots,\lambda_n]$. *$D$ is a constant diagonal matrix and*

$$\widehat{Y}(x)\sim\sum_{r=0}^{\infty}\widehat{Y}_r x^{-r},\quad x\to\infty,$$

with $\det\widehat{Y}_0\neq 0$.

Remark. After applying the formal simplification to the differential equation, we obtain a new equation

$$x^{-q}Z'=B(x)Z,$$

where $B(x)=\sum_{r=0}^{\infty}B_r x^{-r}$ and B_r are diagonal matrices. Then

$$Q(x)=\sum_{j=0}^{q}B_j\frac{x^{q-j+1}}{q-j+1}$$

and $D=B_{q+1}$. When $q=0$, $Q(x)=B_0=A_0$ and D is the diagonal part of A_1.

A.5. The nilpotent case

Assume that A_0 has only one eigenvalue, say λ. If we set $Y=Z\exp[\lambda x^{q+1}/(q+1)]$, then (a.1) becomes $x^{-q}Z'=(A(x)-\lambda I)Z$. Therefore without loss of generality, we may assume that A_0 is nilpotent. In fact, we can further reduce to the case when A_0 is a direct sum of shift matrices $H_1\oplus\dots\oplus H_s$ and A_r, $r>0$, are block diagonal matrices with non-zero entries occurring in the last rows of blocks corresponding to H_k, $k=1,\dots,s$.

Assume this is the case, then a further reduction of the problem is possible by using shearing transforms. That is, we transform the equation (a.1) by $Y=S(x)Z$ with $S(x)=\operatorname{diag}[1,x^{-g},x^{-2g},\dots,x^{-(n-1)g}]$ and g is a positive number to be determined. The resulting equation is

$$x^{-q}Z'=B(x)Z$$

with $B(x)=S^{-1}(x)A(x)S(x)-x^{-q}S^{-1}(x)S'(x)$. A rational number g can be chosen such that $\lim_{x\to\infty}x^g B(x)=B_0^*$ exists and such that it equals A_0 above the main diagonal but has at least one non-zero entry on or below the main diagonal. The resulting equation is

$$x^{-(q-g)}Z'=x^g B(x)Z.$$

By the change of variable $x=\alpha t^p$, $\alpha=p^{1/(g-q-1)}$, where p is the smallest positive integer such that gp is a whole number, it becomes

$$t^{-h}\frac{dZ}{dt}=C(t)Z$$

with $h=p(q+1-g)-1$, $C(t)\sim\sum_{r=0}^{\infty}C_r t^{-r}$, $t\to\infty$. (Notice that the sector S will change accordingly.) C_0 may have only a single eigenvalue, but then g is an integer or C_0 is nilpotent. If g is an integer, then the problem has been reduced to one of lower rank. If C_0 is nilpotent, then one compares the invariant factors of C_0 and A_0 and it happens that successive application of shearing transforms will lower the degrees of invariant factors and finally arrive at the case when C_0 has only one Jordan block and, after applying one more shearing transform, we can always choose g in the shearing transform to be integer. Therefore we can lower either the rank or the order of the system and finally reduce to the regular singularity case or the one-dimensional case. Hence

THEOREM A.5. *In a sufficiently small subsector of S, the differential equation*

$$x^{-q}Y'=A(x)Y$$

has a fundamental matrix solution of the form

$$Y(x)=\widehat{Y}(x)x^{C}e^{Q(x)}.$$

Here $Q(x)$ is a diagonal matrix whose diagonal entries are polynomials in $x^{1/p}$, p a positive integer, C a constant matrix and $\widehat{Y}(x)\sim\sum_{r=0}^{\infty}\widehat{Y}_r x^{-r/p}$, $x\to\infty$.

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