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Classical area minimizing surfaces with real-analytic boundaries

by

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Let C be a smooth embedded closed curve in $\mathbf{R}^n$ (or more generally in an n-manifold with a real-analytic Riemannian metric) and let S be an area minimizing disk with boundary C. Then S can be parametrized by an almost-conformal map F from the closed unit disk $\mathbf{D}$ to $\mathbf{R}^n$. Almost-conformality of F means that F is conformal except for finitely many points at which DF vanishes. Such exceptional points are called branch points. Even though DF vanishes at a branch point p, there may be a neighborhood U of p such that $F(U)$ is a smooth embedded 2-manifold; that is, the image surface may be smooth even though the parametrization is not. If so, the branch point is called a false branch point. Otherwise it is called a true branch point. In [G2], R. Gulliver proved that F cannot have any false branch points. In this paper, we show that F cannot have true branch points along any real-analytic portion of the portion of the boundary curve C. This is somewhat surprising for the following reason. There are many examples of area minimizing disks in $\mathbf{R}^n$ (if $n\geqslant 4$) with interior true branch points, such as

$$z\in\mathbf{D}\subset\mathbf{C}\mapsto(z^3,z^{3k+1})\in\mathbf{C}^2\cong\mathbf{R}^4,$$

which is area minimizing by the Wirtinger inequality [F]. If S is such a surface and C' is a closed curve in S that passes through one of the branch points, then the portion of S bounded by C' will be an area minimizing disk with a true boundary branch point. In this way one can make, for any $k<\infty$, a $\mathcal{C}^k$-curve in $\mathbf{R}^4$ that bounds an area minimizing disk with a true boundary branch point. Moreover, R. Gulliver has pointed out that the example in [G3] is the real part of a holomorphic curve S in $\mathbf{C}^3\cong\mathbf{R}^6$; this surface S is area minimizing and has a $\mathcal{C}^\infty$-boundary curve with a true boundary branch point. However, according to our theorem, no such boundary curve can be real-analytic.

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In case the ambient manifold is 3-dimensional, this theorem was proved by Gulliver and Lesley [GL]. Whether true branch points can occur along C^∞-boundaries of area minimizing disks in $\mathbf{R}^3$ (or other 3-manifolds) is perhaps the major open question about regularity of classical minimal surfaces. (Such surfaces cannot have true interior branch points by the work of Osserman [O], or false branch points by the work of Alt [Alt] and Gulliver [G1]; see also [GOR].) Gulliver [G3] gave a very interesting example of a C^∞-curve in $\mathbf{R}^3$ bounding a minimal disk with a boundary branch point; it is not known whether that example is area minimizing. In any case, it is impossible to prove the kind of local curvature estimates that would, in general, exclude boundary branch points [W].

Aside from the result in this paper, there are two situations in which true boundary branch points can be excluded. First, if a smooth simple closed curve C in $\mathbf{R}^n$ (or, more generally, in a manifold with nonpositive sectional curvatures) has total curvature less than or equal to 4π, then C does not bound any minimal disk with branch points (interior or boundary) [N, §380]. Second, if S is a minimal surface lying in a uniformly convex subset Ω of a Riemannian manifold and if ∂S is a smooth embedded curve in $\partial\Omega$, then S has no boundary branch points (see the proof of the corollary to Theorem 4.5 in [MW]).

The result of this paper (absence of true boundary branch points) also holds for surfaces that minimize area plus the integral of a differential form; see the discussion in [MW, §5]. Furthermore, since the arguments are local, the result is valid not only for disk-type solutions, but also for classical (Douglas) Plateau problem solutions of any finite topological type. (False branch points, on the other hand, cannot be ruled out by local arguments, and for non-disk surfaces or for surfaces minimizing area plus the integral of a differential form, additional hypotheses are required. See [G2].)

Sheldon Chang [C], building on work of Almgren [Alm], proved that the 2-dimensional integral current solutions to Plateau's problem are classical branched minimal surfaces away from the boundary curve. But very little is known about boundary regularity (except in 3-manifolds, where the integral current solution is known to be a smooth embedded manifold with boundary [HS]). For analytic boundary curves, one could perhaps prove boundary regularity theorems using the methods of Almgren and Chang. Of course if one could exclude a sequence of handles accumulating at the boundary, then the result in this paper would give full boundary regularity.

Proving partial regularity for integral currents at C^∞-boundaries seems to be much harder. Much work on interior singularities (for example that of Almgren [Alm] and Chang [C] for integral currents and that of Gulliver [G1], [G2], Alt [Alt], and Micallef and White [MW] for classical minimal surfaces) depends on the fact that a minimal surface cannot be flat to infinite order at a point unless the surface is flat. (More generally, different sheets of a minimal surface cannot make infinite-order contact with each other.) This enables one to get nontrivial surfaces as limits of suitable blow-up sequences. However, in the Gulliver example mentioned above, the surface is flat to infinite order at a true boundary branch point. Thus it seems unlikely that any blow-up methods will work for C^∞-boundary curves.

Reformulation of theorem

Let N be a manifold with a real-analytic Riemannian metric and let

$$F\colon \mathbf{D}\subset\mathbf{C}\to N$$

be a classical minimal surface parametrization, i.e. a continuous map such that

(1) $F|\partial\mathbf{D}$ is one-one,

(2) F is harmonic, and

(3) F is almost conformal on the interior of $\mathbf{D}$ (i.e., conformal except at isolated points where $\mathrm{D}F$ vanishes).

Let $p\in\partial\mathbf{D}$ belong to an arc α of $\partial\mathbf{D}$ such that $F(\alpha)$ is a real-analytic curve (i.e., an embedded real-analytic 1-dimensional submanifold) in N. Then (by [HH]), F is real-analytic near p and in fact extends analytically to a minimal map F^* on a domain of the form

$$\mathbf{D}\cup\mathbf{B}(p,\varepsilon).$$

Thus it suffices to show that if F^* has a true branch point at p, then F does not minimize area. Indeed, we will show that for $r<\varepsilon$, there is a map

$$G\colon \overline{\mathbf{B}(p,r)}\to N$$

such that

$$G(z)=F^*(z) \quad \text{if } z\in\partial\mathbf{B}(p,r) \text{ or if } z\notin\mathbf{D},$$

and such that

$$\operatorname{area}(G)<\operatorname{area}(F^*|\mathbf{B}(p,r)).$$

It is convenient here to replace F by $F\circ u$, where u is a linear fractional transformation that takes 0 to p and that maps the upper half plane $\{z:\operatorname{Im}(z)>0\}$ to the interior of $\mathbf{D}$. Then, by the above discussion, it suffices to prove the following theorem. (In this theorem, the branch point is now at the origin, the disk $\mathbf{D}$ corresponds to $\mathbf{B}(p,r)$ above, and we drop the notational distinction between F and F^*.)

THEOREM 1. *Let*

$$F\colon \mathbf{D}\subset \mathbf{C}\to W\subset \mathbf{R}^n$$

be a smooth map that is almost conformal and harmonic with respect to a real-analytic Riemannian metric g on W. Suppose that F maps $\mathbf{R}\cap\mathbf{D}$ homeomorphically to a real-analytic arc Γ in W, and that F has a true branch point at 0. Then there is a map $G\colon \mathbf{D}\to W$ such that

$$G(z)=F(z) \quad \textit{if } z\in\partial\mathbf{D} \textit{ or if } \operatorname{Im}(z)\leqslant 0,$$

and such that $\operatorname{area}(G)<\operatorname{area}(F)$.

The rest of the paper is devoted to proving Theorem 1.

Structure of boundary branch points

We may assume without loss of generality that

$$\begin{aligned} F(0)&=0,\\ g_{ij}(0)&=\delta_{ij},\\ \mathrm{D}g_{ij}(0)&=0. \end{aligned} \tag{c}$$

As in [MW], we say that a function $\phi(z)$ is $O_k(|z|^p)$ if $\mathrm{D}^j\phi(z)=O(|z|^{p-j})$ for $j=0,\dots,k$, and we define $o_k(|z|^p)$ similarly.

PROPOSITION 2. *Let*

$$F\colon \mathbf{D}\subset \mathbf{C}\to W\subset \mathbf{R}^n$$

be an almost-conformal harmonic map with respect to a real-analytic Riemannian metric g on W satisfying the conditions (c). *Suppose that F has a true branch point at 0. Then there is an integer $Q\geqslant 2$, a linear similitude $L\colon \mathbf{R}^n\to\mathbf{R}^n$, and a C^2-diffeomorphism $w\colon \mathbf{C}\to\mathbf{C}$ (with $w(z)=z+o(|z|)$) such that for all z sufficiently near 0,*

$$(L\circ F\circ w)(z)=(z^Q, f(z))\in \mathbf{C}\times\mathbf{R}^{n-2}\cong\mathbf{R}^n,$$

where $f(z)$ is $O_2(|z|^{Q+1})$ and has the form

$$f(z)=\phi(z^Q)+h(z)+o_1(|z|^\mu), \tag{$*$}$$

and where $h(\cdot)$ is a nonzero homogeneous harmonic polynomial of a degree μ that is greater than Q and not divisible by Q.

Furthermore, if F maps $(-\varepsilon,\varepsilon)\subset\mathbf{R}\cap\mathbf{D}$ (for some $\varepsilon>0$) homeomorphically to a real-analytic arc Γ, then Q is odd and $h(z)=0$ for all real z.

Proof. Except for the last sentence, this theorem was proved (even when the metric is only $\mathcal{C}^2$) in [MW, Theorem 1.4 and Corollary 1.5]. (When the metric is analytic, one can replace O_2 and o_1 above by O_∞ and o_∞; see [MW, §8].) The oddness of Q follows easily (and is also true in much greater generality [HH]).

Thus it remains only to show that $h(z)=0$ for all real z.

Without loss of generality, we assume that $L(X)\equiv X$. (Otherwise replace F by $L\circ F$.)

Let γ be the projection of Γ onto the (x^1,x^2)-plane. Then γ is a real-analytic curve tangent to the real axis in $\mathbf{C}\cong\mathbf{R}^2$ and there is a real-analytic function

$$u\colon\gamma\to\mathbf{R}^{n-2}$$

such that

$$z\mapsto(z,u(z)),\quad z\in\gamma,$$

parametrizes Γ. Note that $u(z)=o(|z|)$ (since $f(z)=o(|z|^Q)$).

CLAIM. *For every degree $d\leqslant\mu$, there exists a real-analytic function*

$$\psi=\psi_d\colon\mathbf{B}(0,r)\subset\mathbf{R}^2\to\mathbf{R}^{n-2}$$

(for some $r=r_d>0$) and a homogeneous harmonic polynomial

$$P=P_d\colon\mathbf{R}^2\to\mathbf{R}^{n-2}$$

of degree d such that

(1) *the graph of ψ is a minimal surface,*

(2) *$\psi(z)=u(z)$ for $z\in\gamma$, and*

(3) *$f(z)-\psi(z^Q)=P(z)+o(|z|^d)$.*

Furthermore, $P(z)=0$ for all real z.

Proof of claim. Let us first prove that the last assertion is implied by (1)–(3). Note that there is a curve γ_* tangent to the real axis such that the map $z\mapsto z^Q$ maps γ_* homeomorphically to γ. By (2), $f(z)=\psi(z^Q)$ when $z\in\gamma_*$. Thus by (3), $P(z)$ must vanish when z is real.

Now we prove that there exist ψ and P satisfying (1)–(3). By the Cauchy–Kowalevski theorem, there is a real-analytic function

$$\psi\colon\mathbf{B}(0,r)\to\mathbf{R}^{n-2}$$

satisfying (1), (2), and such that

$$D_2\psi(z) \equiv 0 \quad \text{for } z \in \gamma$$

(here $D_2 = \partial/\partial y$, where $z = x + iy$). Since $u(z) = o(|z|)$, this means $D\psi(0) = 0$. Since ψ is analytic, this implies $\psi(z) = O(|z|^2)$ or, equivalently, $\psi(z^Q) = O(|z|^{2Q})$. Thus we have (1)–(3) with $d = Q$ by letting $P(z) \equiv 0$. That is, we have proved the claim for $d = Q$ (and therefore also for any $d < Q$).

Now suppose that there exist a ψ and a P as claimed for a certain value of $d < \mu$. If we can prove there exist a ψ' and a P' as claimed corresponding to $d+1$, then by induction we will have established the claim. Combining (∗) and (3), and using the fact that $d < \mu$, we see that $P(z)$ must be a function of z^Q:

$$P(z) = p(z^Q).$$

We already saw that $P(z) = 0$ when z is real. Hence $p(z) = 0$ when z is real. Now by the Cauchy–Kowalevski theorem, there is a ψ' satisfying (1), (2), and

$$D_2\psi'(z) = D_2\psi(z) + D_2 p(z) \quad \text{for } z \in \gamma. \tag{4}$$

Since the graphs of ψ' and ψ are minimal surfaces,

$$\psi'(z) - \psi(z) = H(z) + o_1(|z|^{\deg H}) \tag{5}$$

for some homogeneous harmonic polynomial H [MW, Remark 1.6]. Since ψ' and ψ coincide along γ, $H(z)$ must vanish when z is real. Thus $H(z) = p(z)$ when z is real. Likewise, by (4) and (5), $D_2 H = D_2 p$ along the real axis. Thus $H \equiv p$. Thus by (5),

$$\psi'(z) = \psi(z) + p(z) + o(|z|^{\deg p})$$

so

$$\psi'(z^Q) = \psi(z^Q) + P(z) + o(|z|^d)$$

and so (3) becomes

$$f(z) - \psi'(z^Q) = o(|z|^d).$$

By [MW, Remark 1.6], there is a homogeneous harmonic polynomial $P'(z)$ of degree $d' > d$ such that

$$f(z) - \psi'(z^Q) = P'(z) + o(|z|^{d'}).$$

This completes the proof of the claim. □

To complete the proof of Proposition 2, note that, by (∗) and (3), $h(z) \equiv P_\mu(z)$, which, according to the claim, vanishes for real z.

COROLLARY. *The harmonic function h in Proposition 2 has the form*

$$h(re^{i\theta}) = (r^{\mu} \sin \mu\theta)\mathbf{v}$$

for some vector $\mathbf{v} \in \mathbf{R}^{n-2}$. In particular, h takes values in a 1-dimensional subspace of $\mathbf{R}^{n-2}$.

Proof. Since h is a homogeneous harmonic polynomial, it can be written as

$$h(z) = \mathbf{a}z^{\mu} + \bar{\mathbf{a}}\bar{z}^{\mu}$$

for some $\mathbf{a} \in \mathbf{C}^{n-2}$. Thus

$$0 = h(1) = \mathbf{a} + \bar{\mathbf{a}}$$

so $\mathbf{a} = \mathbf{b}i$ for some $\mathbf{b} \in \mathbf{R}^{n-2}$. Thus

$$h(z) = \mathbf{b}iz^{\mu} - \mathbf{b}i\bar{z}^{\mu} = \mathbf{b}i(2r^{\mu} \sin \mu\theta\, i) = (r^{\mu} \sin \mu\theta)(-2\mathbf{b}). \qquad \square$$

The graph-Dirichlet functional

Let P be an oriented 2-plane in $\mathbf{R}^n \cong \mathbf{R}^2 \times \mathbf{R}^{n-2}$. If P is the graph of a linear function from $\mathbf{R}^2 \to \mathbf{R}^{n-2}$, denote the linear function by L_P. Let

$$\pi: \mathbf{R}^n \cong \mathbf{R}^2 \times \mathbf{R}^{n-2} \to \mathbf{R}^2$$

be the orthogonal projection, and let

$$\mathcal{D}(P) = |L_p|^2 J_P,$$

where $|L_P|^2$ is the sum of the squares of the entries of the matrix for L_P and J_P is the Jacobian determinant of the map

$$\pi|P: P \to \mathbf{R}^2.$$

If P is not the graph of a linear map L_P, let $\mathcal{D}(P)$ and $|L_P|^2$ be ∞.

Let M be a compact oriented 2-manifold with piecewise smooth boundary and let F be a Lipschitz map from M into $\mathbf{R}^2 \times \mathbf{R}^{n-2}$. We say that F is *graph-like* if the map

$$\pi \circ F: M \to \mathbf{R}^2$$

is an orientation-preserving branched immersion, and if for almost every $z \in M$, the image of $DF(z)$ is a plane $P = P(z)$ with $|L_P|^2$ bounded (independently of z).

(Thus for almost every $z \in M$, there is a neighborhood of z whose image under F is the graph of a Lipschitz function from an open subset of $\mathbf{R}^2$ to $\mathbf{R}^{n-2}$, and the Lipschitz constant is uniformly bounded. In other words, at most points, the image of F looks like the graph of a Lipschitz function. In [MW, §2], two such maps were regarded as equivalent if they were related by a homeomorphism of M, and the equivalence class of such an F was called a "Lipschitz graph-like M-surface".)

We define a functional on graph-like maps as follows. If we identify oriented 2-planes with simple unit 2-vectors in the usual way, we can define $\mathcal{D}(v)$ for all simple 2-vectors $v \in \Lambda_2(\mathbf{R}^n)$ by requiring that $\mathcal{D}(\lambda v) \equiv \lambda \mathcal{D}(v)$ for $\lambda \geqslant 0$. If F is a graph-like map, we let

$$\mathcal{D}(F) = \int_M \mathcal{D}\left(\frac{\partial F}{\partial x} \wedge \frac{\partial F}{\partial y}\right) dx\, dy.$$

Note by the area formula that if $S = F(M)$, then

$$\mathcal{D}(F) = \int_{p \in S} \mathcal{D}(\operatorname{Tan}_p S)\, \mathcal{H}^0(F^{-1}(p))\, d\mathcal{H}^2 p. \tag{6}$$

In other words, $\mathcal{D}(F)$ depends only on the image of F (counting multiplicity), and not on F itself.

We call this the *graph-Dirichlet* functional for the following reason. If $\pi \circ F$ is a one-one map from M to a region $U \in \mathbf{R}^2$, then the image of F is the graph of a function $u: U \to \mathbf{R}^{n-2}$ and $\mathcal{D}(F)$ is just the usual Dirichlet energy of u:

$$\mathcal{D}(F) = \int_U |\mathrm{D}u|^2. \tag{7}$$

(In [MW, §2], this $\mathcal{D}(F)$ is written $\mathrm{Dir}[F]$. Note that it is *not* the usual Dirichlet energy of F.)

We say that a graph-like map $F: M \to \mathbf{R}^n$ minimizes the graph-Dirichlet functional if

$$\mathcal{D}(H) \leqslant \mathcal{D}(H')$$

whenever $H': M \to \mathbf{R}^n$ is a graph-like map with

$$H' | \partial M = H | \partial M.$$

The graph-Dirichlet functional arises from the area functional in the following way.

PROPOSITION 3. *Let F and h be as in Proposition 2. Let $\mathbf{D}^+ = \{z = x + iy \in \mathbf{D} : y \geqslant 0\}$ be the closed upper half-disk.*

(1) *If $F|\mathbf{D}$ is area minimizing, then the map*

$$H: \mathbf{D} \to \mathbf{R}^n,$$
$$H(z) = (z^Q, h(z))$$

minimizes the graph-Dirichlet functional.

(2) *If $F|\mathbf{D}^+$ is area minimizing, then the map*

$$H^+ : \mathbf{D}^+ \to \mathbf{R}^n,$$
$$H^+(z) = (z^Q, h(z))$$

minimizes the graph-Dirichlet functional.

Proof. Conclusion (1) was proved in [MW, Theorem 2.1].

To prove (2), suppose that $F|\mathbf{D}^+$ minimizes area. Then clearly

$$\text{area}(F) \leqslant \text{area}(G)$$

whenever

$$G: \mathbf{D} \to \mathbf{R}^n,$$
$$G(z) = F(z) \quad \text{for } z \in \mathbf{D}^- \cup \partial\mathbf{D}$$

(where $\mathbf{D}^- = \{x+iy \in \mathbf{D} : y \leqslant 0\}$ is the lower half-disk). The proof of (1) in [MW, Theorem 2.1] then shows that

$$\mathcal{D}(H) \leqslant \mathcal{D}(H')$$

whenever

$$H': \mathbf{D} \to \mathbf{R}^n,$$
$$H'(z) = H(z) \quad \text{for } z \in \mathbf{D}^- \cup \partial\mathbf{D}.$$

But this clearly implies that $H|\mathbf{D}^+$ minimizes the graph-Dirichlet functional. □

Maps that minimize the graph-Dirichlet functional have the following regularity:

PROPOSITION 4. *Suppose that $H: M \to \mathbf{R}^n$ is a graph-like map that minimizes the graph-Dirichlet functional. Let U be a simply-connected open subset of $\mathbf{R}^2 \setminus \pi(H(\partial M))$ that does not contain the image of any branch point of $\pi \circ H$. Then*

$$(\pi \circ H)^{-1}(U)$$

consists of a finite union $\bigcup_j U_j$ of disjoint open subsets of the interior of M, and H maps each U_j homeomorphically onto the graph of a harmonic function

$$u_j : U \to \mathbf{R}^{n-2}.$$

Proof. By the definition of graph-like surface, $\pi \circ H$ is a branched immersion and therefore $(\pi \circ H)^{-1}(U)$ consists of a finite union $\bigcup_j U_j$ of open subsets of M, each of which gets mapped homeomorphically by $\pi \circ H$ onto U. Hence F maps U_j homeomorphically to the graph of some function

$$u_j : U \to \mathbf{R}^{n-2}.$$

But then by (7), u_j must minimize $\int |\mathrm{D}u_j|^2$. That is, u_j is harmonic. □

Conclusion of proof

By Proposition 2 and its corollary and by Proposition 3, the main result (Theorem 1) reduces to

PROPOSTION 5. *Let*

$$H^+ : \mathbf{D}^+ \to \mathbf{R}^3 \cong \mathbf{C} \times \mathbf{R},$$
$$H^+(z) = (z^Q, r^\mu \sin \mu\theta),$$

where Q is odd, $\mu > Q$ is not divisible by Q, and $z = re^{i\theta}$. Then H^+ does not minimize the graph-Dirichlet functional.

Proof. This follows from exactly the same cut-and-paste argument as used in [O] and [GL], so we simply sketch the argument. (The argument is in fact easier to carry out here, since we only need to apply it to the explicit and rather simple map H^+ above.)

Note that the portion of the image of H^+ above $\mathbf{D}^+$ (in other words, the intersection of the image $H^+(\mathbf{D}^+)$ with $\pi^{-1}(\mathbf{D}^+)$) consists of a union of graphs of harmonic functions

$$u_j : \mathbf{D}^+ \to \mathbf{R}$$

(namely functions $\zeta \mapsto \mathrm{Re}((\zeta^{1/Q})^\mu)$ corresponding to different choices of the Qth root). These functions do not all coincide since μ is not divisible by Q. Thus suppose that u_1 and u_2 do not coincide. Now for every j,

$$u_j(\zeta) = o(|\zeta|).$$

Hence by the Hopf boundary point lemma, there must be some point p in the interior of $\mathbf{D}^+$ such that $u_1(p) = u_2(p)$. By the homogeneity of h,

$$u_1(sp) \equiv u_2(sp).$$

These two graphs must cross transversely along $I = \{sp : s \in (0,1)\}$ since they do not coincide.

Let I' be the graph of $u_1|I$ (or, equivalently, of $u_2|I$). Now cut $H^+(\mathbf{D}^+)$ along I' and re-glue to get a new surface. It is an exercise in elementary topology to check that the new surface is still topologically a disk (or half-disk), and hence may be parametrized by a graph-like map

$$H' : \mathbf{D}^+ \to \mathbf{R}^n$$

with $H' = H^+$ on $\partial \mathbf{D}^+$.

Note that H' is not a branched immersion, but has two creases where we did the cutting and pasting along I'. Hence by Proposition 4, H' does not minimize the graph-Dirichlet functional. (Let $U = \{z \in \operatorname{int} \mathbf{D}^+ : |z| < |p|\}$.) By (6), $\mathcal{D}(H') = \mathcal{D}(H^+)$, so H^+ does not minimize the graph-Dirichlet functional either. $\square$

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