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Ergodic properties of classical dissipative systems I

by

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1. Introduction

The occurrence of irreversible behavior in microscopically reversible systems is conceptually well understood (see [Le] and references therein). However, from a mathematical point of view, our understanding of dissipative phenomena is still incomplete, and the status of non-equilibrium statistical mechanics is far from being satisfactory. In this paper we investigate the ergodic properties of some classical, dissipative dynamical systems with a finite number of degrees of freedom, near thermal equilibrium. In our models, dissipation arises dynamically from the interaction with some "large" environment, conventionally called *the reservoir*. Under appropriate conditions on its initial state, this reservoir acts as a pool of energy and entropy. It plays a dual role: On one hand, its ability to absorb energy-momentum without substantial changes to its internal state gives the physical mechanism for dissipation. On the other hand, its large entropy content provides the fluctuations needed to prevent the small system from relaxing into some stationary state (see [KKS] for a study of the dynamics of finite-energy states in such coupled systems).

For a large set of physically relevant initial conditions, the expected asymptotic behavior of the coupled system is qualitatively described by the zeroth law of thermodynamics. This empirical statement asserts that a large system, left alone and under normal conditions, eventually approaches an equilibrium state characterized by a few macroscopic parameters such as temperature, density, etc. (see [UF, Chapter 1] and [RS1, §2.5]). Since the early days of statistical mechanics, the mathematical status of the zeroth law has been a much controversial subject and, starting with the famous

Fermi–Pasta–Ulam paper [FPU], the object of extensive numerical studies. We refer the interested reader to [C3P] and [P] for recent contributions.

The systems we will consider consist of a "small" subsystem $\mathcal{A}$, with a finite number of interacting degrees of freedom, coupled to an "infinite" reservoir $\mathcal{B}$. The reservoir is a thermodynamic limit of an assembly of harmonic oscillators, and its temperature $1/\beta$ is the average energy per oscillation mode. Let us suppose that the systems $\mathcal{A}$ and $\mathcal{B}$, initially isolated, start interacting. According to the zeroth law, the coupled system should evolve toward a joint equilibrium state. Since $\mathcal{B}$ is an infinite system, its temperature will remain constant and thermal equilibrium is achieved when the system $\mathcal{A}$ reaches the temperature $1/\beta$ of the reservoir. This phenomenon is not only a fundamental experimental fact—it also underlies the very *definition* of the notion of temperature for the "small" system $\mathcal{A}$.

Assume for definiteness that the system $\mathcal{A}$ is a finite collection of weakly interacting particles confined to a finite box. Then a continual energy-momentum exchange with the reservoir will turn the motion of the individual particles into a random walk, a phenomenon known as "Brownian motion". If the reservoir is initially in thermal equilibrium, its strong statistical properties allow for a reduced probabilistic description of the dynamics of the particles based on a random integro-differential equation, namely the Langevin equation. It departs from the original Newton equation by the addition of two terms, a random force describing the direct action of the reservoir on the particles, and a dissipative term arising from the reaction of the reservoir to the presence of the particles. Dissipation generally depends on the history of the particles, and is responsible for hysteresis effects. In the usual discussions of the Langevin equation, these effects are eliminated by making appropriate assumptions on the form of the coupling of the particles to the reservoir. Under these assumptions, and after a simple renormalization process, the Langevin equation turns into a stochastic differential equation, and the motion of the particles becomes a (degenerate) Markovian diffusion in phase space. This limiting form of the Langevin equation was first studied by Ornstein and Uhlenbeck ([UO], see also [Wx]), and the resulting stochastic process is called the Ornstein–Uhlenbeck process. The history of the Langevin equation is discussed in [LT] and [Ne]. By construction the OU process is Markovian. This brings the powerful Fokker–Planck equation into the game and reduces the ergodic theory of the Ornstein–Uhlenbeck process to the spectral analysis of some parabolic PDE (for a general discussion of these problems, see [T]).

Our main motivation is to overcome the difficulties related to the presence of memory in the Langevin equation, and to develop tools for studying the ergodic properties of the Ornstein–Uhlenbeck process when the usual Markovian techniques fail. More precisely, the goal of this paper is twofold:

(I) To develop a general framework for the models described above, in the spirit of the Ford–Kac–Mazur philosophy (see [FKM] and [LT]). This starts with the definition of the phase space $\mathcal{G}$ and of the Hamiltonian $H:\mathcal{G}\to\mathbf{R}$ of the system $\mathcal{A}+\mathcal{B}$. The corresponding thermal equilibrium state μ^β (a probability measure on $\mathcal{G}$) is then constructed. The associated *Koopman space* is the separable *complex* Hilbert space $L^2(\mathcal{G}, d\mu^\beta)$. Observables of the system are elements of the algebra $L^\infty(\mathcal{G}, d\mu^\beta)$. Admissible initial states which are "not too far" from thermal equilibrium are probability measures on $\mathcal{G}$ which are absolutely continuous with respect to μ^β. We denote this class of states by $\mathcal{S}^\beta$. The major problem centers around the existence and regularity properties of the Hamiltonian flow Ξ^t on $\mathcal{G}$ generated by the Hamiltonian H. We show that this flow induces, via the usual formula

$$\mathcal{U}^t F = F\circ\Xi^t,$$

a strongly continuous unitary group on Koopman's space. In particular, Ξ^t leaves the equilibrium measure μ^β invariant. A reduced description of the dynamics of the system $\mathcal{A}$ is obtained by integrating out the variables of the reservoir in $\mathcal{U}^t F$. If these variables are initially distributed according to a given probability law, then the reduced description is given by a random integro-differential equation: the "generalized" Langevin equation.

(II) Once part (I) is completed, we have a specific class of systems for which we can formalize the problem of return to equilibrium (the zeroth law) in the following way.

Definition 1.1. We say that the combined system $\mathcal{A}+\mathcal{B}$ returns to equilibrium if the dynamical system $(\mathcal{G}, \Xi^t, \mu^\beta)$ satisfies

$$\lim_{t\to\infty}\int F\circ\Xi^t\, d\mu = \int F\, d\mu^\beta, \tag{1.1}$$

for all $\mu\in\mathcal{S}^\beta$ and $F\in L^\infty(\mathcal{G}, d\mu^\beta)$.

The second goal of this paper is to find sufficient conditions to ensure that the system $\mathcal{A}+\mathcal{B}$ returns to equilibrium. To achieve this goal we invoke the spectral theory of dynamical systems (also known as "Koopmanism", we refer the reader to [CFS], [M], [RS1] and [Wa] for details). This theory relates ergodic properties of $(\mathcal{G}, \Xi^t, \mu^\beta)$ to the spectral properties of the Liouvillean $\mathcal{L}$: the skew-adjoint generator of Koopman's group $\mathcal{U}^t$. More specifically, it is known that if $\mathcal{L}$ has purely absolutely continuous spectrum except for the simple eigenvalue 0, then return to equilibrium (or equivalently the *strong mixing property*) holds. Thus part (II) of our program reduces to the investigation of the singular spectrum of $\mathcal{L}$.

In order to keep the size of this paper within reasonable limits, we will not discuss any applications of our results. However, in the forthcoming paper [JP1], we will give important physical examples where our general approach leads to the verification of the zeroth law. We also refer the interested reader to the letter [JP2], where we announced the results presented here for the simple model of a particle interacting with a phonon field at positive temperature.

The paper is organized as follows: In §2 we introduce the model and state our results. In §3 we give the proofs pertaining to part (I) of our program. Finally §4 is devoted to part (II).

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2. Model and results

The small system $\mathcal{A}$ is described as follows. Its configuration space is a finite-dimensional connected manifold $\mathcal{M}$. To avoid uninteresting complications we assume $\mathcal{M}$ to be of class C^∞ with a piecewise smooth boundary. Its phase space is the cotangent bundle $\mathbf{T}^*\mathcal{M}$ endowed with its natural symplectic structure $\boldsymbol{\Omega}_{\mathcal{A}}$. We denote the points of $\mathbf{T}^*\mathcal{M}$ by $\xi=(q,p)$ and its Liouville measure by $d\xi$. The Hamiltonian $H_{\mathcal{A}}$ of the system $\mathcal{A}$ is a C^∞-function on the interior of $\mathbf{T}^*\mathcal{M}$. We assume that $\exp(-\beta H_{\mathcal{A}}(\xi))\in L^1(\mathbf{T}^*\mathcal{M}, d\xi)$ for each $\beta>0$ and we denote by $\mu_{\mathcal{A}}^\beta$ the normalized Gibbs measure,

$$d\mu_{\mathcal{A}}^\beta = \frac{1}{Z_{\mathcal{A}}^\beta}\,\mathrm{e}^{-\beta H_{\mathcal{A}}(\xi)}\,d\xi.$$

For simplicity, we also assume that the boundary of phase space (including the points at infinity) is appropriately screened by a soft potential barrier:

(H1) *For any real E the set $K_E\equiv\{\xi : H_{\mathcal{A}}(\xi)\leqslant E\}$ satisfies:*
(i) *K_E is compact,*
(ii) *$K_E\cap\partial\mathbf{T}^*\mathcal{M}=\varnothing$.*

We now set up the heat reservoir $\mathcal{B}$. Let $\mathcal{H}$ be a *real* Hilbert space and B a positive self-adjoint operator on $\mathcal{H}$. We denote by $[D(B)]$ the completion of the domain $D(B)$

in the norm $\|Bu\|$. For simplicity, we use the same notation to denote both B and its extension to $[D(B)]$. Let

$$\mathcal{H}_{\mathcal{B}} \equiv [D(B)] \oplus \mathcal{H},$$

with the inner product

$$\left(\begin{pmatrix} \varphi \\ \pi \end{pmatrix}, \begin{pmatrix} \varphi' \\ \pi' \end{pmatrix} \right) \equiv (B\varphi, B\varphi') + (\pi, \pi').$$

We denote by ϕ the elements of $\mathcal{H}_{\mathcal{B}}$. The Hamilton function of the free reservoir is

$$H_{\mathcal{B}}(\phi) \equiv \tfrac{1}{2}\|\phi\|^2, \tag{2.1}$$

and therefore we will refer to $\mathcal{H}_{\mathcal{B}}$ as the phase space of finite-energy configurations of the reservoir. This space, as a Hilbert manifold, is endowed with a weak symplectic structure, i.e., a densely defined non-degenerate 2-form

$$\boldsymbol{\Omega}_{\mathcal{B}}(\phi, \phi') \equiv (L_{\mathcal{B}}^{-1}\phi, \phi'), \quad \text{with } L_{\mathcal{B}} \equiv \begin{pmatrix} 0 & 1 \\ -B^2 & 0 \end{pmatrix}. \tag{2.2}$$

The operator $L_{\mathcal{B}}$ is skew-adjoint on $\mathcal{H}_{\mathcal{B}}$ with domain

$$D(L_{\mathcal{B}}) \equiv \left\{ \begin{pmatrix} \varphi \\ \pi \end{pmatrix} : \varphi \in [D(B)],\, B\varphi \in D(B);\, \pi \in D(B) \right\}.$$

The Hamilton equation corresponding to (2.1) and (2.2) is

$$\dot{\phi} = L_{\mathcal{B}}\phi, \tag{2.3}$$

and the corresponding Hamiltonian flow is given by the strongly continuous unitary group (see [RS3, §10]),

$$e^{L_{\mathcal{B}}t} = \begin{pmatrix} \cos(Bt) & B^{-1}\sin(Bt) \\ -B\sin(Bt) & \cos(Bt) \end{pmatrix}. \tag{2.4}$$

Later in this section we will give a precise description of the phase space and of the thermal equilibrium states of this dynamical system. Its ergodic properties, with respect to thermal equilibrium, are well known (see [LL] for example): If the spectrum of B is purely absolutely continuous, the flow (2.4) is Bernoulli, i.e., very strongly mixing. As soon as B acquires some point spectrum, ergodicity (and hence mixing) is broken. Since we want to ensure good mixing properties of the reservoir we assume that B has purely absolutely continuous spectrum. The following argument, however, indicates that this may not be sufficient for our purposes: Let $G \subset \mathbf{R}_+$ be a spectral gap of B, an open interval such that $G \cap \sigma(B) = \varnothing$ and $\partial G \subset \partial\sigma(B)$. It is a simple exercise to show that a generic, self-adjoint, rank-one perturbation of B has an eigenvalue in G, and thus generates a non-ergodic dynamics. Along the same lines one can show that coupling such a reservoir to a finite collection $\mathcal{A}$ of harmonic oscillators results in a non-ergodic system. This is obvious if the frequency spectrum of $\mathcal{A}$ overlaps with the gap G. The previous argument shows that it remains generically true in the fully resonant case. Therefore, in order to enforce some stability of the mixing behavior of the reservoir, we shall also assume that the spectrum of B has no gap. Equivalently, we may assume that $L_{\mathcal{B}}$ has purely absolutely continuous spectrum filling the entire real line. A simple extension of the above argument leads us to assume that this spectrum also has uniform multiplicity.

A simple way to formulate the above requirements is to invoke the Lax–Phillips theory (see [LP]), and make the following assumption on the propagation properties of this group (see [LT, §9.7] for a related discussion):

(H2) *There is a closed subspace* $D_+ \subset \mathcal{H}_{\mathcal{B}}$ *such that*
(i) $e^{L_{\mathcal{B}}t} D_+ \subset D_+$ *for all* $t \geqslant 0$,
(ii) $\bigcap_{t\in\mathbf{R}} e^{L_{\mathcal{B}}t} D_+ = \{0\}$,
(iii) $\bigvee_{t\in\mathbf{R}} e^{L_{\mathcal{B}}t} D_+ = \mathcal{H}_{\mathcal{B}}$,
where $\bigvee$ *denotes the closed linear span of a set of vectors.*

In the terminology of the Lax–Phillips theory, D_+ is an outgoing subspace for the group $e^{L_{\mathcal{B}}t}$ (see [LP]). For the classical hyperbolic systems (the wave equation, Maxwell's equations, etc.), the existence of an outgoing subspace is a well-known fact.

A consequence of the Lax–Phillips theory is the existence of an auxiliary complex Hilbert space $\mathfrak{h}$, endowed with a conjugation C, and such that $\mathcal{H}_{\mathcal{B}}$ has the representation

$$\mathcal{H}_{\mathcal{B}} \cong \hat{L}^2(\mathbf{R}, d\omega; \mathfrak{h}). \tag{2.5}$$

Here, $\hat{L}^2(\mathbf{R}, d\omega; \mathfrak{h})$ denotes the real Hilbert space of square integrable, $\mathfrak{h}$-valued functions of $\omega \in \mathbf{R}$ satisfying $f(-\omega) = Cf(\omega)$ almost everywhere. In this new representation, the unitary group (2.4) acts as a multiplication operator

$$(e^{L_{\mathcal{B}}t}\phi)(\omega) = e^{i\omega t}\phi(\omega), \tag{2.6}$$

and the symplectic form (2.2) becomes

$$\boldsymbol{\Omega}_{\mathcal{B}}(\phi, \phi') = ((i\omega)^{-1}\phi, \phi').$$

From now on we shall always work in the outgoing spectral representation (2.5).

Remark. It is evident from (2.4) that the reservoir is reversible:

$$J_{\mathcal{B}} e^{L_{\mathcal{B}}t} = e^{-L_{\mathcal{B}}t} J_{\mathcal{B}},$$

with a natural time reversal

$$J_{\mathcal{B}}\colon \begin{pmatrix} \varphi \\ \pi \end{pmatrix} \mapsto \begin{pmatrix} \varphi \\ -\pi \end{pmatrix}.$$

In the spectral representation (2.5), the time-reversal operator becomes

$$J_{\mathcal{B}}\colon \phi(\omega) \mapsto j_{\mathcal{B}}\phi(-\omega), \tag{2.7}$$

where $j_{\mathcal{B}}$ is some unitary involution of $\mathfrak{h}$.

We now construct the phase space of the reservoir at positive temperature. Let Λ_0 be a positive, real, self-adjoint operator on the Hilbert space $\mathfrak{h}$ such that Λ_0^{-s} is Hilbert–Schmidt for all $s>1$. For the time being, the choice of this operator is arbitrary. Later, it will affect the class of allowed couplings. The operator

$$\Lambda \equiv (-\partial_\omega^2 + \omega^2)^{1/2} \otimes \Lambda_0 \tag{2.8}$$

is self-adjoint and positive on $\mathcal{H}_{\mathcal{B}}$, and Λ^{-s} is again Hilbert–Schmidt for $s>1$. For $s>0$ we define the scale of spaces

$$\mathcal{H}_{\mathcal{B}}^s \equiv D(\Lambda^s),$$

equipped with the graph norm $\|f\|_s \equiv \|\Lambda^s f\|$. We further denote the dual of $\mathcal{H}_{\mathcal{B}}^s$ by $\mathcal{H}_{\mathcal{B}}^{-s}$. The space

$$\mathcal{N} \equiv \bigcap_s \mathcal{H}_{\mathcal{B}}^s,$$

with its natural locally convex topology, is nuclear. Its dual is given by

$$\mathcal{N}' = \bigcup_s \mathcal{H}_{\mathcal{B}}^s,$$

and is endowed with the weak∗-topology. As usual, we denote this duality by

$$\phi(f) = (\phi, f), \quad \phi \in \mathcal{N}', f \in \mathcal{N},$$

and we have

$$\mathcal{N} \subset \mathcal{H}_{\mathcal{B}} \subset \mathcal{N}',$$

with dense and continuous inclusions. A simple calculation shows that, for $f \in \mathcal{N}$, one has the estimates

$$\begin{aligned} \|e^{L_{\mathcal{B}}t} f\|_s &\leqslant C_s \|f\|_s \langle t\rangle^{|s|}, \\ \|(e^{L_{\mathcal{B}}t} - 1) f\|_s &\leqslant C_s \|f\|_{s+\alpha} \langle t\rangle^{|s|} |t|^\alpha, \end{aligned} \tag{2.9}$$

for any s, t and $0 \leqslant \alpha \leqslant 1$. Therefore, the unitary evolution $e^{L_{\mathcal{B}}t}$ extends to a continuous group of continuous transformations of $\mathcal{N}'$,

$$(\Xi_{\mathcal{B}}^t \phi)(f) \equiv \phi(e^{-L_{\mathcal{B}}t} f), \tag{2.10}$$

which defines the free dynamics on the full phase space $\mathcal{N}'$ of the reservoir.

A state of the reservoir is a Radon probability measure μ on its phase space. Since any cylinder measure uniquely extends to a Radon measure on $\mathcal{N}'$, Minlos' theorem gives a one to one correspondence between such measures and functions $S:\mathcal{N}\to\mathbf{C}$ satisfying the three conditions:

(i) S is continuous.

(ii) S is normalized by $S(0)=1$.

(iii) S is of positive type, namely

$$\sum_{i,j=1}^{n} S(f_i-f_j)\bar{z}_i z_j \geqslant 0,$$

for any $n\geqslant 1$, arbitrary $f_1,\dots,f_n\in\mathcal{N}$, and $z_1,\dots,z_n\in\mathbf{C}$.

The function S, the so-called *characteristic function*, is related to the measure μ by

$$S(f)\equiv\int \mathrm{e}^{i\phi(f)}\,d\mu(\phi).$$

The thermal equilibrium state of the reservoir $\mathcal{B}$ at inverse temperature β is the Gaussian measure $\mu_{\mathcal{B}}^{\beta}$ corresponding to

$$S^{\beta}(f)=\mathrm{e}^{-\|f\|^2/2\beta}. \tag{2.11}$$

This formula can be established from the thermodynamic limit of microcanonical or canonical ensembles associated to finite-dimensional approximations of the reservoir.

A simple calculation shows that for a Hilbert–Schmidt operator T on $\mathcal{H}_{\mathcal{B}}$, the following holds:

$$\int \|T\phi\|^2\,d\mu_{\mathcal{B}}^{\beta}(\phi)=\beta^{-1}\operatorname{Tr}(T^*T)<\infty.$$

Applying this formula to Λ^{-s} we conclude that for $s>1$ the norm $\|\phi\|_{-s}$ is finite with probability 1. Thus

$$\operatorname{supp}(\mu_{\mathcal{B}}^{\beta})\subset\mathcal{H}_{\mathcal{B}}^{-s}\quad\text{for } s>1, \tag{2.12}$$

a fact which will be used in the sequel. The Koopman space of the reservoir is the *separable complex Hilbert space* $L^2(\mathcal{N}',d\mu_{\mathcal{B}}^{\beta})$, on which the dynamics is implemented by

$$\mathcal{U}_{\mathcal{B}}^{t}F\equiv F\circ\Xi_{\mathcal{B}}^{t}. \tag{2.13}$$

It follows from the Fock representation of Koopman space (to be described in §4) that (2.13) defines a strongly continuous unitary group. Its skew-adjoint generator has a simple eigenvalue zero and absolutely continuous spectrum filling the imaginary axis. As already noted, hypothesis (H2) implies that $\Xi_{\mathcal{B}}^{t}$ is a Bernoulli flow on the measure space $(\mathcal{N}',\mu_{\mathcal{B}}^{\beta})$.

We now describe the coupling of the system $\mathcal{A}$ to the reservoir. Its main feature is the linearity of the interaction energy in the field ϕ, which allows us to write a Langevin equation for the evolution of $\mathcal{A}$. The finite-energy phase space of the coupled system is $\mathcal{G}^0 \equiv \mathbf{T}^*\mathcal{M} \times \mathcal{H}_\mathcal{B}$ and its full phase space is

$$\mathcal{G} \equiv \mathbf{T}^*\mathcal{M} \times \mathcal{N}'.$$

In the sequel we will also make extensive use of the (trivial) vector bundles

$$\mathcal{G}^s \equiv \mathbf{T}^*\mathcal{M} \times \mathcal{H}^s_\mathcal{B}.$$

The total Hamiltonian is given by

$$H(\xi, \phi) \equiv H_{\mathcal{A},\mathrm{ren}}(\xi) + H_\mathcal{B}(\phi) + \lambda\phi(\alpha(\xi)), \tag{2.14}$$

where λ is a real coupling constant, $\alpha: \mathbf{T}^*\mathcal{M} \to \mathcal{H}^{s_c}_\mathcal{B}$ a C^∞-section of $\mathcal{G}^{s_c}$ ($s_c > 0$ will be specified later), and

$$H_{\mathcal{A},\mathrm{ren}}(\xi) \equiv H_\mathcal{A}(\xi) + \tfrac{1}{2}\lambda^2 \|\alpha(\xi)\|^2. \tag{2.15}$$

This is a convenient renormalization of the Hamiltonian of $\mathcal{A}$, obtained by Wick ordering the Gibbs–Boltzmann factor $\mathrm{e}^{-\beta(H_{\mathcal{A},\mathrm{ren}}+\lambda\phi(\alpha))} = :\mathrm{e}^{-\beta(H_\mathcal{A}+\lambda\phi(\alpha))}:$ with respect to $\mu^\beta_\mathcal{B}$. The renormalization of $H_\mathcal{A}$ is not necessary, but it ensures that the stability of the system $\mathcal{A}+\mathcal{B}$ is not spoiled by the interaction, which greatly simplifies the discussion.

The symplectic structure of the phase space $\mathcal{G}^0$ is given by $\mathbf{\Omega}_\mathcal{A} \oplus \mathbf{\Omega}_\mathcal{B}$, and the equations of motion are

$$\begin{aligned} \dot\phi &= L_\mathcal{B}(\phi + \lambda\alpha(\xi)), \\ \dot\xi &= Z_{H_{\mathcal{A},\mathrm{ren}}}(\xi) + \lambda Z_{\phi(\alpha)}(\xi). \end{aligned} \tag{2.16}$$

Here Z_F stands for the Hamiltonian vector field on $\mathbf{T}^*\mathcal{M}$ generated by the Hamiltonian F. Denoting by (ξ, ϕ) the initial condition for (2.16), one easily obtains

$$\phi_t(f) = \phi(\mathrm{e}^{-L_\mathcal{B}t} f) + \lambda \int_0^t (L_\mathcal{B}\alpha(\xi_\tau), \mathrm{e}^{-L_\mathcal{B}(t-\tau)} f)\, d\tau \tag{2.17}$$

for the evolution of the reservoir, and

$$\dot\xi_t = Z_{H_{\mathcal{A},\mathrm{ren}}}(\xi_t) - \lambda^2 \int_0^t D(t-\tau; \xi_\tau, \xi_t)\, d\tau + \lambda F(t; \xi_t) \tag{2.18}$$

for the small system. Here

$$F(t; \xi) \equiv Z_{\phi(\mathrm{e}^{-L_\mathcal{B}t}\alpha)}(\xi) \tag{2.19}$$

is the time-dependent force field generated by the reservoir, and the kernel

$$D(t;\xi,\xi') \equiv -(L_{\mathcal{B}}\alpha(\xi), \mathrm{e}^{-L_{\mathcal{B}}t} Z_\alpha(\xi')) \tag{2.20}$$

describes the forces due to the reaction of the reservoir to the system $\mathcal{A}$. If the initial state ϕ of the reservoir is distributed according to a given probability law, then $F(t;\xi)$ becomes a random noise and the solution $\xi_t = \xi_t(\xi,\phi)$ of (2.18) defines a family of stochastic processes on $\mathbf{T}^*\mathcal{M}$, indexed by the initial data $\xi \in \mathbf{T}^*\mathcal{M}$. Obviously, (2.18) is a generalization of the usual *Langevin equation* (see [Ne], [LT] and [JP2]). For simplicity, we will call it *the Langevin equation.*

The following hypothesis is essential for the stability of the coupled system $\mathcal{A}+\mathcal{B}$:

(H3) *There exist constants* C *and* D *such that*

$$\|\alpha(\xi)\|_{s_c}^2 \leqslant C(H_{\mathcal{A}}(\xi)+D),$$

for all $\xi \in \mathbf{T}^*\mathcal{M}$*, and some* $s_c > 2$.

Remark 1. Here we see how the choice of the operator Λ_0 in (2.8) determines the class of allowed couplings. Typically, the above condition imposes some regularity on the functions $\alpha(\xi)$.

Remark 2. In the following, we shall absorb the constant D in the definition of $H_{\mathcal{A}}$, and consequently assume that hypothesis (H3) holds with $D=0$.

Our first result is an existence theorem for the solutions of the Langevin equation. We recall that $s_c > 2$ by hypothesis (H3), and that $\mu_{\mathcal{B}}^\beta$ is the equilibrium state of the reservoir defined by (2.11).

THEOREM 2.1. *Suppose that hypotheses* (H1)–(H3) *hold, and let* s *be such that* $0 \leqslant s \leqslant s_c - 1$. *Then the Hamilton equation* (2.16) *defines a flow* Ξ^t *on* $\mathcal{G}^{-s}$. *For fixed* $t \in \mathbf{R}$, *the map* $(\xi,\phi) \mapsto \Xi^t(\xi,\phi)$ *is of class* $C^1(\mathcal{G}^{-s})$. *Moreover,* $(t,\xi,\phi) \mapsto \Xi^t(\xi,\phi)$ *is of class* $C^1(\mathbf{R}\times\mathcal{G}^{-s};\mathcal{G}^{-s-1})$. *In particular, by* (2.12), *this flow is well defined on* $\mu_{\mathcal{B}}^\beta$-*almost all initial configurations of the reservoir. The* C^1-*map* $t \mapsto \xi_t(\xi,\phi)$ *defines a family of stochastic processes on* $\mathbf{T}^*\mathcal{M}$ *(indexed by* $\xi \in \mathbf{T}^*\mathcal{M}$*), which we collectively call the Ornstein–Uhlenbeck process at inverse temperature* β.

The Gibbs measure corresponding to the Hamiltonian (2.14) is given by

$$d\mu^\beta = \frac{1}{Z^\beta} \mathrm{e}^{-\beta(\lambda\phi(\alpha(\xi))+\lambda^2\|\alpha(\xi)\|^2/2)}\, d\mu_{\mathcal{A}}^\beta(\xi)\, d\mu_{\mathcal{B}}^\beta(\phi),$$

and the associated Koopman space is

$$\mathfrak{F}^\beta \equiv L^2(\mathcal{G}, d\mu^\beta).$$

Remark. A consequence of the renormalization (2.15) is that the equilibrium measure of $\mathcal{A}$ is not perturbed by the interaction, i.e., for any observable F depending only on ξ, we have

$$\int F\,d\mu^\beta = \int F\,d\mu^\beta_{\mathcal{A}}.$$

Our second result states the fundamental property of the map

$$\mathcal{U}^t F \equiv F \circ \Xi^t, \tag{2.21}$$

on the Koopman space.

THEOREM 2.2. *If hypotheses* (H1)–(H3) *hold, then $\mathcal{U}^t$ is a strongly continuous unitary group on $\mathfrak{F}^\beta$. In particular, the measure μ^β is invariant under the Hamiltonian flow Ξ_t.*

Remark. Let us denote by $\mathcal{C}^l_b(\mathcal{G}^{-s})$ the set of C^l-functions on $\mathcal{G}^{-s}$ which have bounded support with respect to the pseudo-norm

$$\mathcal{E}_s(\xi,\phi) \equiv H_{\mathcal{A}}(\xi) + \tfrac{1}{2}\|\phi\|^2_{-s}, \tag{2.22}$$

and uniformly bounded derivatives. Then, for any $s>0$, the flow Ξ^t leaves $\mathcal{C}^1_b(\mathcal{G}^{-s})$ invariant. Furthermore, if $s_c>3$, the generator of the group $\mathcal{U}^t$ is essentially skew-adjoint on $\mathcal{C}^1_b(\mathcal{G}^{-s})$, provided $2<s\leqslant s_c-1$. We will explicitly identify this generator in Proposition 3.5.

Theorems 2.1 and 2.2 complete part (I) of our program. We now turn to part (II): The question of return to equilibrium formulated in Definition 1.1. We are not able to resolve this problem at the current level of generality, and we have to restrict ourselves to a special class of couplings which have *finite rank* in the following sense.

Definition 2.3. The coupling α is called "simple" if, for some integer M, there exists a linear injection $A: \mathbf{R}^M \to \mathcal{H}_{\mathcal{B}}$ such that

$$\alpha(\xi) = Au(\xi), \tag{2.23}$$

for some function $u \in C^\infty(\mathbf{T}^*\mathcal{M}; \mathbf{R}^M)$.

Given the spectral representation (2.5) of $\mathcal{H}_{\mathcal{B}}$, the operator A extends by linearity to a map from $\mathbf{C}^M$ to the complex Hilbert space $L^2(\mathbf{R}, d\omega)\otimes\mathfrak{h}$. Then the formula

$$A(\omega)u = (Au)(\omega), \quad u \in \mathbf{C}^M,$$

induces a family of operators $A(\omega): \mathbf{C}^M \to \mathfrak{h}$. They satisfy the reality relation

$$CA(\omega)C_0 = A(-\omega), \tag{2.24}$$

where C is the conjugation on $\mathfrak{h}$ introduced after (2.5), and C_0 is the usual complex conjugation on $\mathbf{C}^M$.

Definition 2.4. We define the spectral strength of the simple coupling α to be the $(M\times M)$-matrix-valued function

$$T(\omega)=(A(\omega)^*A(\omega))^{1/2}. \tag{2.25}$$

By the previous discussion this is a positive, self-adjoint matrix which satisfies the reality relation $T(-\omega)=\bar{T}(\omega)$. Moreover, $\|T(\omega)\|\in L^2(\mathbf{R},d\omega)$ holds by construction.

Points $\omega\in\mathbf{R}$ at which the matrix $T(\omega)$ becomes singular are bad since, at such frequencies, some modes of the reservoir decouple from the system $\mathcal{A}$. Clearly, we cannot allow such singularities to occur on an open set, since this would have the same effect as the existence of a gap in the spectrum of the reservoir. In many physically interesting situations, however, one cannot avoid isolated singularities. To keep these bad frequencies under control we need some hypothesis.

Definition 2.5. An isolated singularity $\omega_0\in\mathbf{R}$ of the matrix $T(\omega)$ is admissible if it has a regularizer, a matrix-valued rational function $G_0(\omega)$ satisfying $G_0(-\omega)=\bar{G}_0(\bar{\omega})$, and such that

(i) $\pm\omega_0$ are the only poles of G_0,

(ii) G_0 is non-singular in the closed lower half-plane, i.e.,

$$\det(G_0(\omega))\neq 0 \quad \text{for } \operatorname{Im}(\omega)\leqslant 0,$$

(iii) $\|TG_0\|\in L^2(\mathbf{R},d\omega)$,

(iv) $\|(TG_0)^{-1}\|^2$ is locally integrable near $\pm\omega_0$,

(v) $\int_{-\infty}^{\infty}\log\|(TG_0)^{-1}\|\,d\omega/(1+\omega^2)<\infty$.

Let us set our hypotheses on the coupling.

(H4) *The coupling is simple and its spectral strength $T(\omega)$ is non-singular, except for a finite set of admissible singularities $\Omega\equiv\{\pm\omega_1,...,\pm\omega_L\}\subset\mathbf{R}$. Outside of Ω, the function $\|T^{-1}(\omega)\|$ is locally integrable, and*

$$\sup_{|\omega|>R}\frac{\|T(\omega)^{-1}\|}{|\omega|^{\nu}}<\infty, \tag{2.26}$$

for some $R>0$ and $\nu>0$.

Roughly speaking, the last condition ensures that the Langevin process ξ_t is not "too smooth" as a function of t, see e.g. [DM], or equivalently, that the system $\mathcal{A}$ is "strongly coupled" to the high-frequency modes of the reservoir.

Remark. The requirement (2.26) is probably too strong. We conjecture that our result still holds as long as $\|T^{-1}(\omega)\|$ remains locally integrable and the "finite-entropy" condition

$$\int_{-\infty}^{\infty} \frac{\log\det T(\omega)}{1+\omega^2}\, d\omega > -\infty \tag{2.27}$$

is satisfied. Note that violation of (2.27) leads to a deterministic noise in the Langevin equation, a circumstance that radically changes the nature of the model. On the other hand, if the matrix T is a rational function of ω, then the model becomes essentially Markovian and the techniques of [T] apply.

The next hypothesis is a micro-reversibility assumption for the system $\mathcal{A}+\mathcal{B}$. Recall that $\mathcal{B}$ is reversible, with a time reversal $J_{\mathcal{B}}$ given by (2.7).

(H5) *There exists an anti-symplectic involution τ of $\mathbf{T}^*\mathcal{M}$ such that*

$$H_{\mathcal{A}}\circ\tau = H_{\mathcal{A}}. \tag{2.28}$$

Moreover, the coupling (2.23) *is time-reversal invariant:*

$$u\circ\tau = J_{\mathcal{A}}u \quad \textit{and} \quad AJ_{\mathcal{A}} = J_{\mathcal{B}}A, \tag{2.29}$$

for some involution $J_{\mathcal{A}}$ of $\mathbf{R}^M$.

Our last assumption deals with the kinematical structure of the coupling. To formulate this hypothesis we need some further notation. Let us denote by $\{\cdot\,,\cdot\}$ the Poisson bracket on $\mathbf{T}^*\mathcal{M}$, and by $\mathcal{P}$ the corresponding Lie algebra of smooth functions, with the locally convex topology of uniform convergence of arbitrary derivatives on compact sets. For $\mathcal{Q}_i\subset\mathcal{P}$, we denote by $\bigvee_i \mathcal{Q}_i$ the smallest closed sub-algebra containing all $\mathcal{Q}_i$. The Hamiltonian vector field generated by $F\in\mathcal{P}$ is written Z_F. We also use the standard notation $\mathrm{ad}_F\equiv\{\cdot\,,F\}$ for the adjoint action of $F\in\mathcal{P}$. Finally we shall say that a sub-algebra $\mathcal{P}_0\subset\mathcal{P}$ has *full rank* if, at each point $\xi\in\mathbf{T}^*\mathcal{M}$, the set $\{Z_F(\xi): F\in\mathcal{P}_0\}$ spans the tangent space.

(H6) *Let $\mathcal{P}_\alpha$ be the sub-algebra generated by the set $\{(\alpha,f): f\in\mathcal{H}_{\mathcal{B}}\}$. The Lie algebra*

$$\bigvee_{n\geqslant 0} \mathrm{ad}^n_{H_{\mathcal{A}}}\mathcal{P}_\alpha \tag{2.30}$$

has full rank.

Intuitively, this means that the random force in the Langevin equation can push the system $\mathcal{A}$ in all available directions of its phase space. In particular, hypothesis (H6) ensures that the flows generated by $H_{\mathcal{A}}$ and the coupling $\phi(\alpha)$ do not have common non-trivial invariant subspaces. If they do, then of course one cannot expect (1.1) to hold.

The principal result of this paper is

THEOREM 2.6. *Suppose that hypotheses* (H1)–(H6) *hold. Then, for any* $\lambda\neq 0$, *the Liouvillean* $\mathcal{L}$ (*the skew-adjoint generator of the group* $\mathcal{U}^t$) *has purely absolutely continuous spectrum, except for the simple eigenvalue* 0.

It is a well-known fact of abstract ergodic theory that return to equilibrium is equivalent to the strong mixing property, which is in turn a direct consequence of Theorem 2.6 (see [CFS], [M] or [Wa]).

THEOREM 2.7. *Suppose that hypotheses* (H1)–(H6) *hold. Then, for any* $\lambda\neq 0$, *the system* $\mathcal{A}+\mathcal{B}$ *returns to equilibrium.*

Remark 1. We emphasize that these results are non-perturbative: They do not require the coupling λ to be small.

Remark 2. In a recent series of papers [JP3]–[JP5], we have obtained similar results in the framework of quantum mechanics and for small coupling.

3. Dynamical theory of Brownian motion

This section is devoted to the proofs of Theorem 2.1 and Theorem 2.2. We will use freely concepts and notation related to infinite-dimensional manifolds, as discussed for example in [Ru]. In particular, if E and F are Banach spaces, and U is an open subset of F, we denote by $\mathcal{C}^l(U;F)$ the Banach space of C^l-functions $f: U\to F$ which have uniformly bounded derivatives:

$$\|f\|_{\mathcal{C}^l(U;F)} \equiv \sup\{\|D^k f(x)\| : x\in U,\, k=0,\dots,l\} < \infty.$$

With this definition, the class $\mathcal{C}^l_b(\mathcal{G}^{-s})$ mentioned in the remark following Theorem 2.2 can be written as the union of the spaces

$$\{f\in\mathcal{C}^l(U;\mathbf{C}) : \operatorname{supp} f\subset U\},$$

as U runs over all open sets of $\mathcal{G}^{-s}$ which are bounded in the pseudo-norm $\mathcal{E}_s$.

Proof of Theorem 2.1. If $\dim(\mathcal{M})=n$, then the set $\mathcal{G}^s=\mathbf{T}^*\mathcal{M}\times\mathcal{H}^s_{\mathcal{B}}$ is clearly a C^∞-manifold modeled on the Banach space $\mathbf{R}^{2n}\times\mathcal{H}^s_{\mathcal{B}}$. We start by proving the existence of a local flow Ξ^t. The only technical difficulty is the unboundedness of the operator $L_{\mathcal{B}}$. To circumvent this problem we make the ansatz

$$\Xi^t = \Xi^t_{\mathcal{B}}\circ\Theta(0,t). \tag{3.1}$$

Here, $\Xi_{\mathcal{B}}$ is the evolution of the free reservoir, see (2.10). From (2.16), we get the non-autonomous equation of motion

$$\frac{d}{dt}\Theta(0,t) = X(t)\circ\Theta(0,t),$$

where the time-dependent vector field X is given by

$$X(t;\xi,\phi) \equiv \begin{pmatrix} Z_{H_{\mathcal{A},\mathrm{ren}}+\lambda\phi(\mathrm{e}^{-L_{\mathcal{B}}t}\alpha)}(\xi) \\ \lambda\mathrm{e}^{-L_{\mathcal{B}}t}L_{\mathcal{B}}\alpha(\xi) \end{pmatrix}.$$

A well-known trick transforms this equation into the autonomous system

$$\frac{d}{d\tau}\mathbf{\Theta}^{\tau} = \mathbf{X}\circ\mathbf{\Theta}^{\tau}, \tag{3.2}$$

where $\mathbf{X}$ is the vector field on $\mathbf{R}\times\mathcal{G}^{-s}$ given by

$$\mathbf{X}(t,\xi,\phi) \equiv \begin{pmatrix} 1 \\ X(t;\xi,\phi) \end{pmatrix}.$$

One easily checks that, for $0\leqslant s\leqslant s_c-1$, this vector field is of class $C^1(\mathbf{R}\times\mathcal{G}^{-s})$. It follows that (3.2) has a local solution of the form

$$\mathbf{\Theta}^{\tau}\colon (t,\xi,\phi)\mapsto (t+\tau,\Theta(t,t+\tau,\xi,\phi)),$$

which is of class $C^1(]-T,T[\times U)$ for some neighborhood $\mathbf{R}\times\mathcal{G}^{-s}\supset U\ni(t,\xi,\phi)$ and for some $T=T(t,\xi,\phi)>0$. Using the estimates (2.9), one finally shows that (3.1) defines a local flow for the original equation (2.16).

To prove that these local solutions can be globally extended, we derive an "energy" estimate. Starting from the fact that

$$0=\mathbf{\Omega}_{\mathcal{A}}(\dot{\xi}_t,\dot{\xi}_t) = dH_{\mathcal{A},\mathrm{ren}}\cdot\dot{\xi}_t+\lambda(\phi_t,d\alpha\cdot\dot{\xi}_t) = \frac{d}{dt}H_{\mathcal{A},\mathrm{ren}}+\lambda\left(\phi_t,\frac{d}{dt}\alpha\right),$$

and using (2.17), a first integration gives

$$\begin{aligned} H_{\mathcal{A},\mathrm{ren}}(\xi_t)-H_{\mathcal{A},\mathrm{ren}}(\xi) = {}&-\lambda\int_0^t d\tau\,\phi\left(\mathrm{e}^{-L_{\mathcal{B}}\tau}\frac{d}{d\tau}\alpha(\xi_\tau)\right) \\ &+\lambda^2\int_0^t d\tau\int_0^\tau d\sigma\left(\left[\frac{d}{d\sigma}\mathrm{e}^{-L_{\mathcal{B}}\sigma}\right]\alpha(\xi_\sigma),\mathrm{e}^{-L_{\mathcal{B}}\tau}\frac{d}{d\tau}\alpha(\xi_\tau)\right). \end{aligned}$$

Then a few integrations by parts with respect to the variables σ and τ yield

$$H_{\mathcal{A}}(\xi_t)-H_{\mathcal{A}}(\xi) = -\lambda(\phi+\lambda\alpha(\xi),\tilde{\phi}_t)-\tfrac{1}{2}\lambda^2\|\tilde{\phi}_t\|^2, \tag{3.3}$$

where we used the definition (2.15) of the renormalized Hamiltonian, and introduced the auxiliary field

$$\tilde{\phi}_t \equiv \int_0^t d\tau\, \mathrm{e}^{-L_B\tau} \frac{d}{d\tau}\alpha(\xi_\tau). \tag{3.4}$$

Integrating (3.4) by parts and using the estimate (2.9), hypothesis (H3) and the fact that $0 \leqslant s \leqslant s_c - 1$, we get the bound

$$\|\tilde{\phi}_t\|_s^2 \leqslant C\langle t\rangle^{2(s+1)} \sup_{|\tau|\leqslant|t|} H_{\mathcal{A}}(\xi_\tau).$$

Using the last inequality in (3.3), we obtain

$$\sup_{|\tau|\leqslant|t|} H_{\mathcal{A}}(\xi_\tau) \leqslant C\big(H_{\mathcal{A}}(\xi) + \tfrac{1}{2}\|\phi\|_{-s}^2\big)\langle t\rangle^{2(s+1)}. \tag{3.5}$$

In a very similar way one gets, from (2.17),

$$\|\phi_t\|_{-s}^2 \leqslant C\Big(\sup_{|\tau|\leqslant|t|} H_{\mathcal{A}}(\xi_\tau) + \tfrac{1}{2}\|\phi\|_{-s}^2\Big)\langle t\rangle^{2(s+1)}. \tag{3.6}$$

Recall the definition (2.22) of the pseudo-norm $\mathcal{E}_s$. Combining equations (3.5)–(3.6), we finally get that, for $0 \leqslant s \leqslant s_c - 1$,

$$\mathcal{E}_s(\xi_t, \phi_t) \leqslant C\mathcal{E}_s(\xi, \phi)\langle t\rangle^{2s+4}. \tag{3.7}$$

By hypothesis (H1), we conclude that solutions of the Hamilton equation (2.16) cannot reach the boundary of the phase space in a finite amount of time. Therefore, these solutions can be extended to arbitrarily large times, and the flow $\Xi^t(\xi,\phi) = (\xi_t, \phi_t)$ is well defined for any $t \in \mathbf{R}$. Its regularity properties follow from standard estimates. In particular, we shall need the fact that the first derivative $D\Xi^t$ is uniformly bounded on any $\mathcal{E}_s$-bounded open subset of $\mathcal{G}^{-s}$. □

Proof of Theorem 2.2. We start with a simple change of variables which will play an important role in the sequel. To this end, let us define the map

$$T\colon (\xi, \phi) \mapsto (\xi, \psi) \equiv (\xi, \phi + \lambda\alpha(\xi)). \tag{3.8}$$

Note that the new quantity ψ is nothing but the total field, namely the sum of the reservoir field and of the particles' *self-field.* It is easy to show that, for any $s \geqslant 0$, there exists a constant $C > 0$ such that

$$\frac{1}{C}\mathcal{E}_s \leqslant \mathcal{E}_s \circ T^{-1} \leqslant C\mathcal{E}_s.$$

Therefore, T is a C^∞-diffeomorphism of $\mathcal{G}^{-s}$ which preserves boundedness with respect to the pseudo-norm $\mathcal{E}_s$. A simple calculation shows that, in the new variables (ξ,ψ), the Hamiltonian becomes

$$H\circ T^{-1}(\xi,\psi)=H_{\mathcal{A}}(\xi)+\tfrac{1}{2}\|\psi\|^2.$$

Accordingly, the Gibbs measure μ^β factorises as

$$\int F\circ T\,d\mu^\beta=\int F\,d\mu^\beta_{\mathcal{A}}\,d\mu^\beta_{\mathcal{B}},$$

see for example [GJ, §9.1]. Thus, in the new dynamical variables, we can write the Koopman space as

$$\mathfrak{F}^\beta=L^2(\mathbf{T}^*\mathcal{M},d\mu^\beta_{\mathcal{A}})\otimes L^2(\mathcal{N}',d\mu^\beta_{\mathcal{B}}). \tag{3.9}$$

From formula (2.17) and definition (3.8) we get the following expression for the time evolution of the field:

$$\psi_t(f)=\psi(\mathrm{e}^{-L_{\mathcal{B}}t}f)+\lambda\int_0^t\left(\frac{d}{d\tau}\alpha(\xi_\tau),\mathrm{e}^{-L_{\mathcal{B}}(t-\tau)}f\right)d\tau, \tag{3.10}$$

while the motion of the system $\mathcal{A}$ is governed by

$$\dot{\xi}_t=Z_{H_{\mathcal{A}}+\lambda\psi(\alpha)}(\xi_t), \tag{3.11}$$

as one easily verifies from (2.18). In the sequel we will exclusively work in this new representation and, whenever there is no danger of confusion, we will not distinguish between a quantity and the same quantity transformed by T. For example, the flow Ξ^t gets transformed into $T\circ\Xi^t\circ T^{-1}$ and inherits all the properties of the original Hamiltonian flow. We denote again the transformed flow and the corresponding Koopman group by Ξ^t and $\mathcal{U}^t$ respectively.

We decompose the proof of Theorem 2.2 into the sequence of lemmas.

LEMMA 3.1. *If $s>1$, then the class $C^1_b(\mathcal{G}^{-s})$ is dense in $\mathfrak{F}^\beta$.*

Proof. Since $C^1_0(\mathbf{T}^*\mathcal{M})$ is clearly dense in $L^2(\mathbf{T}^*\mathcal{M},d\mu^\beta_{\mathcal{A}})$, it suffices to prove that the class $C^1_b(\mathcal{H}^{-s}_{\mathcal{B}})$ is dense in $L^2(\mathcal{N}',d\mu^\beta_{\mathcal{B}})$. Consider functions of the form

$$G(\psi)\equiv\chi(\|\psi\|^2_{-s})\mathrm{e}^{i\psi(f)},$$

where $\chi\in C_0^\infty(\mathbf{R})$ is non-negative and $f\in\mathcal{H}^s_{\mathcal{B}}$. One easily checks that $G\in C^1_b(\mathcal{H}^{-s}_{\mathcal{B}})$. If the function $F\in L^2(\mathcal{N}',d\mu^\beta_{\mathcal{B}})$ is orthogonal to all finite linear combinations of such functions then

$$\int\mathrm{e}^{i\psi(f)}F(\psi)\chi(\|\psi\|^2_{-s})\,d\mu^\beta_{\mathcal{B}}(\psi)=0,$$

for all $f \in \mathcal{H}_{\mathcal{B}}^{s}$. Without loss of generality, we may assume that F is real-valued. Decomposing F into the sum of its positive and negative parts, and applying Minlos' theorem to the two resulting integrals, we get that $F(\psi)\chi(\|\psi\|_{-s}^{2})=0$ for all χ. Since for $s>1$ the vector ψ $\mu_{\mathcal{B}}^{\beta}$-almost surely belongs to $\mathcal{H}_{\mathcal{B}}^{-s}$, we conclude that $F=0$ almost everywhere. The result follows. □

LEMMA 3.2. *If* $0 \leqslant s \leqslant s_c - 1$, *then the class* $C_b^1(\mathcal{G}^{-s})$ *is invariant under* $\mathcal{U}^t$.

Proof. Pick $F \in C_b^1(\mathcal{G}^{-s})$. Since the derivative $D\Xi^t$ is locally uniformly bounded, we only have to show that $F \circ \Xi^t$ has bounded support. This is an immediate consequence of (3.7). □

LEMMA 3.3. *If* $1 \leqslant s \leqslant s_c - 1$, *then* $\mathcal{U}^t$ *is isometric on* $C_b^1(\mathcal{G}^{-s})$.

Proof. By the group property and Lemma 3.2, it suffices to show that

$$\frac{d}{dt}\|\mathcal{U}^{-t}F\|^2\bigg|_{t=0}=0,$$

for any $F \in C_b^1(\mathcal{G}^{-s})$. Since the evolution of the free reservoir $\mathcal{U}_{\mathcal{B}}^t$ is unitary (see (2.13), and the remarks that follow it), this is equivalent to

$$\frac{d}{dt}\|\mathcal{U}_{\mathcal{B}}^{t}\mathcal{U}^{-t}F\|^2\bigg|_{t=0}=\frac{d}{dt}\|F\circ\Theta(t,0)\|^2\bigg|_{t=0}=0.$$

Finally, since C_b^1 is an algebra, it suffices to show that

$$\mathcal{I} \equiv \frac{d}{dt}\int F\circ\Theta(t,0)\,d\mu^{\beta}\bigg|_{t=0}=0. \tag{3.12}$$

Note that $\Theta(t,0)$ is the inverse of the C^1-map $\Theta(0,t)$. A simple application of the inverse function theorem shows that $t \mapsto \Theta(t,0)$ is C^1 near $t=0$, and that

$$\frac{d}{dt}\Theta(t,0;\xi,\psi)\bigg|_{t=0}=-\frac{d}{dt}\Theta(0,t;\xi,\psi)\bigg|_{t=0}=-\begin{pmatrix} Z_{H_{\mathcal{A}}+\lambda\psi(\alpha)}(\xi) \\ \lambda\, d\alpha(\xi)\cdot Z_{H_{\mathcal{A}}+\lambda\psi(\alpha)}(\xi)\end{pmatrix}.$$

Thus differentiation can be brought into the integral in (3.12). The resulting expression splits according to $d=d_{\mathcal{A}}\oplus d_{\mathcal{B}}$, and we get

$$\mathcal{I}=\mathcal{I}_{\mathcal{A}}+\mathcal{I}_{\mathcal{B}},$$

where

$$\mathcal{I}_{\mathcal{A}}=\int d_{\mathcal{A}}F\cdot Z_{H_{\mathcal{A}}+\lambda\psi(\alpha)}\,d\mu^{\beta},$$

$$\mathcal{I}_{\mathcal{B}}=\lambda\int (d_{\mathcal{B}}F, d_{\mathcal{A}}\alpha\cdot Z_{H_{\mathcal{A}}+\lambda\psi(\alpha)})\,d\mu^{\beta}.$$

To handle the first integral, we use the integration by parts formula

$$\int d_{\mathcal{A}}F\cdot Z_G\, d\mu_{\mathcal{A}}^{\beta} = -\beta\int F\, d_{\mathcal{A}}G\cdot Z_{H_{\mathcal{A}}}\, d\mu_{\mathcal{A}}^{\beta},$$

which is easily proved for F, G in C^1 and F compactly supported. Taking into account the identity $dH_{\mathcal{A}}\cdot Z_{H_{\mathcal{A}}}=0$, we obtain

$$\mathcal{I}_{\mathcal{A}} = -\beta\lambda\int(\psi, d\alpha\cdot Z_{H_{\mathcal{A}}})\, d\mu^{\beta}.$$

To handle $\mathcal{I}_{\mathcal{B}}$, we use

$$\int(d_{\mathcal{B}}F, f)G\, d\mu_{\mathcal{B}}^{\beta} + \int(d_{\mathcal{B}}G, f)F\, d\mu_{\mathcal{B}}^{\beta} = \beta\int GF\psi(f)\, d\mu_{\mathcal{B}}^{\beta},$$

which follows from the usual integration by parts formula for Gaussian measures (see [GJ], for example). Using the identity $d_{\mathcal{A}}\psi(\alpha)\cdot Z_{\psi(\alpha)}=0$, we get

$$\mathcal{I}_{\mathcal{B}} = \beta\lambda\int(\psi, d\alpha\cdot Z_{H_{\mathcal{A}}})\, d\mu^{\beta},$$

and (3.12) follows. □

LEMMA 3.4. *If $s_c>2$, then $\mathcal{U}^t$ extends to a strongly continuous unitary group on $\mathfrak{F}^{\beta}$.*

Proof. Fix s such that $1<s\leqslant s_c-1$. By Lemma 3.1 and Lemma 3.3, the operator $\mathcal{U}^t$ extends to an isometry of $\mathfrak{F}^{\beta}$. Since $\mathcal{U}^t$ inherits the group property from Ξ^t, it is actually a unitary group. Thus we have only to prove strong continuity on a total subset of $\mathfrak{F}^{\beta}$. This is most easily done using functions of the type

$$F = \chi(\xi)\,\mathrm{e}^{i\psi(f)},$$

with $\chi\in C_0^{\infty}(\mathbf{T}^*\mathcal{M})$ and $f\in\mathcal{N}$. □

The proof of Theorem 2.2 is now complete. In the following proposition we explicitly identify the generator of the group $\mathcal{U}^t$.

PROPOSITION 3.5. *If $2<s\leqslant s_c-1$, then the Liouvillean $\mathcal{L}$ is essentially skew-adjoint on $\mathcal{C}_b^1(\mathcal{G}^{-s})$, where it is given by the formula*

$$\mathcal{L}F(\xi,\psi) = (d_{\mathcal{B}}F(\xi,\psi), L_{\mathcal{B}}\psi) + (d_{\mathcal{A}}F(\xi,\psi) + \lambda(d_{\mathcal{B}}F(\xi,\psi), d_{\mathcal{A}}\alpha(\xi)))\cdot Z_{H_{\mathcal{A}}+\lambda\psi(\alpha)}(\xi).$$

Proof. By Theorem 2.1 and the chain rule, if $F\in\mathcal{C}_b^1(\mathcal{G}^{-s})$ then $F\circ\Xi^t$ belongs to

$$\mathcal{C}_b^1(\mathcal{G}^{-s})\cap C^1(\mathbf{R}\times\mathcal{G}^{-s+1}).$$

Since $\mathcal{G}^{-s+1}$ is of full measure, $\mathcal{U}^t F$ is μ^β-almost everywhere differentiable with respect to t. Its derivative is given by

$$\frac{d}{dt}\mathcal{U}^t F(\xi,\psi)=(\mathcal{L}F)\circ\Xi^t(\xi,\psi),$$

where $\mathcal{L}$ is given by the above formula. In particular, if $F\in\mathcal{C}_b^1(\mathcal{G}^{-s})$, then $\mathcal{L}F\in\mathfrak{F}^\beta$. On the other hand, if $\mathcal{L}F\in\mathfrak{F}^\beta$, then Taylor's formula gives the estimate

$$\frac{1}{t}\|\mathcal{U}^t F-F-t\mathcal{L}F\|\leqslant\int_0^1\|(\mathcal{U}^{st}-I)\mathcal{L}F\|\,ds. \tag{3.13}$$

Since the right-hand side of (3.13) vanishes as $t\to 0$ by strong continuity and the Lebesgue dominated convergence theorem, so does the left-hand side. We conclude that $\mathcal{C}_b^1(\mathcal{G}^{-s})$ is a dense subspace of $\mathfrak{F}^\beta$ which is invariant under $\mathcal{U}^t$, and on which $\mathcal{U}^t$ is strongly differentiable. The result follows from Theorem VIII.10 in [RS1]. □

We finish this section with a description of micro-reversibility, a property of the model which will play an important role in the sequel. By hypothesis (H5), the system $\mathcal{A}$ has a time reversal τ (see (2.28)). Since the involution $J_\mathcal{B}$ given by (2.7) clearly extends to $\mathcal{N}'$, the map

$$j\colon(\xi,\psi)\mapsto(\tau(\xi),J_\mathcal{B}\psi)$$

defines an involution of $\mathcal{G}$, which is easily seen to be anti-symplectic:

$$j^*\boldsymbol{\Omega}_A\oplus\boldsymbol{\Omega}_B=-\boldsymbol{\Omega}_A\oplus\boldsymbol{\Omega}_B.$$

By construction $H\circ j=H$, and it follows (see e.g. Proposition 4.3.13 in [AM]) that

$$\Xi^t\circ j=j\circ\Xi^{-t}$$

holds on $\mathcal{G}^{-s}$ for $0\leqslant s\leqslant s_c-1$. Time reversal is readily lifted to the Koopman space: The map

$$\mathcal{J}\colon F\mapsto F\circ j$$

defines a unitary involution of $\mathfrak{F}^\beta$, which intertwines the forward and the backward evolution

$$\mathcal{J}\mathcal{U}^t=\mathcal{U}^{-t}\mathcal{J}.$$

4. Spectral theory of the Liouvillean

In this section we complete the proof of our main result, Theorem 2.6: We show that the only vectors in the spectral subspace $\mathfrak{F}_{\rm sing}$ associated to the singular spectrum of the Liouvillean $\mathcal{L}$ are the constant functions. Our argument splits into the following conceptually and technically distinct parts:

Dynamical reduction. We exploit the hypotheses (H2) (the Lax–Phillips structure), (H4) (simplicity of the coupling, and in particular the bound (2.26) on the spectral strength) and (H5) (micro-reversibility) to show that a vector $\Psi\in\mathfrak{F}_{\rm sing}$ can only depend on ξ and finitely many field "coordinates" $\zeta_1=\psi(e_1),\dots,\zeta_N=\psi(e_N)$. We obtain an explicit description of the subspace $\mathcal{H}_0\subset\mathcal{H}_{\mathcal{B}}$ spanned by $e_1,\dots,e_N$.

Elimination of the reservoir. We show that the reservoir completely dominates the small-time dynamics on the subspace of functions $\Psi(\zeta_1,\dots,\zeta_N,\xi)$. Using the fact that the free evolution $\mathrm{e}^{-i\omega t}$ has no invariant subspace in $\mathcal{H}_0$, we inductively eliminate the field variables ζ. This is the weaker point in our proof: A more sophisticated argument should be able to eliminate infinitely many field modes ζ (see our conjecture in the remark following hypothesis (H4)).

Kinematic reduction. The last step in the previous elimination process yields that $\mathfrak{F}_{\rm sing}$ contains only functions of ξ. We invoke hypothesis (H6) (kinematic completeness) to show that it consists entirely of constant functions.

To set up our notation, we start with a brief review of some basic facts of the theory of Gaussian random fields. We refer the reader to [GJ], [CFS] and [S] for details and additional informations. The complex Hilbert space $\mathfrak{F}_{\mathcal{B}}^{\beta}\equiv L^2(\mathcal{N}',d\mu_{\mathcal{B}}^{\beta})$ is isomorphic to the bosonic Fock space over $\mathcal{H}_{\mathcal{B}}$:

$$\mathfrak{F}_{\mathcal{B}}^{\beta}\cong\Gamma(\mathcal{H}_{\mathcal{B}})\equiv\bigoplus_{N=0}^{\infty}\Gamma^N(\mathcal{H}_{\mathcal{B}})=\mathbf{C}\oplus\mathcal{H}_{\mathcal{B}}^{\mathbf{C}}\oplus(\mathcal{H}_{\mathcal{B}}^{\mathbf{C}}\otimes_{\rm S}\mathcal{H}_{\mathcal{B}}^{\mathbf{C}})\oplus\dots, \tag{4.1}$$

where $\otimes_{\rm S}$ denotes the completely symmetrized tensor product, and $\mathcal{H}_{\mathcal{B}}^{\mathbf{C}}$ the complexification of $\mathcal{H}_{\mathcal{B}}$. This isomorphism is obtained by identifying the Wick monomial $:\psi(f_1)\dots\psi(f_n):$ with $f_1\otimes_{\rm S}\dots\otimes_{\rm S}f_n\in\Gamma^N(\mathcal{H}_{\mathcal{B}})$ for any $f_1,\dots,f_n\in\mathcal{H}_{\mathcal{B}}$. Recall that the second quantization $\Gamma(k)$ of a contraction k of $\mathcal{H}_{\mathcal{B}}$ is the real contraction of $\Gamma(\mathcal{H}_{\mathcal{B}})$ which acts on real elements of $\Gamma^N(\mathcal{H}_{\mathcal{B}})$ as $k\otimes\dots\otimes k$. If e^{-At} is a strongly continuous contraction semi-group, so is its second quantization. The generators of these two semi-groups are related by $\Gamma(\mathrm{e}^{-At})=\mathrm{e}^{-d\Gamma(A)t}$, where $d\Gamma(A)$ is the real operator acting on real elements of $\Gamma^N(\mathcal{H}_{\mathcal{B}})$ as $A\otimes I\otimes\dots\otimes I+I\otimes A\otimes\dots\otimes I+\dots+I\otimes I\otimes\dots\otimes A$. For example, the Koopman group of the free reservoir is given by the second quantization of its Hamiltonian flow

$$\mathcal{U}_{\mathcal{B}}^{t}=\Gamma(\mathrm{e}^{-L_{\mathcal{B}}t}),$$

which is the unitary group of Bogoliubov transformations generated by $d\Gamma(L_{\mathcal{B}})$. It immediately follows from this representation and hypothesis (H2) that the evolution of the free reservoir is a Lebesgue automorphism.

If $\mathcal{K}$ is a closed subspace of $\mathcal{H}_\mathcal{B}$, we denote by $\mathbf{G}_\mathcal{K}$ the minimal σ-field generated by $\{\psi(f): f\in\mathcal{K}\}$. The conditional expectation with respect to $\mathbf{G}_\mathcal{K}$ is the orthogonal projection of $\mathfrak{F}_\mathcal{B}^\beta$ onto the subspace of $\mathbf{G}_\mathcal{K}$-measurable functions. It is related to the orthogonal projection p from $\mathcal{H}_\mathcal{B}$ onto $\mathcal{K}$ by second quantization, i.e., one has

$$\mathbf{E}(\,\cdot\mid\mathbf{G}_\mathcal{K})=\Gamma(p). \tag{4.2}$$

On the other hand, the operator $d\Gamma(p)$ is the number operator associated to the subspace $\mathcal{K}$. It is a simple exercise to show that

$$\operatorname{Ker}(d\Gamma(p))=\operatorname{Ran}(\Gamma(1-p))\subset\operatorname{Ran}(\Gamma(p))^\perp, \tag{4.3}$$

a fact that will be useful later. Finally we recall that there exists a unitary map $\Gamma(\mathcal{H}_\mathcal{B})\to\Gamma(\mathcal{K})\otimes\Gamma(\mathcal{K}^\perp)$ which, under the identification (4.1), translates into an isomorphism

$$\mathfrak{F}_\mathcal{B}^\beta\cong L^2(\mathcal{N}',\mathbf{G}_\mathcal{K},d\mu_\mathcal{B}^\beta)\otimes L^2(\mathcal{N}',\mathbf{G}_{\mathcal{K}^\perp},d\mu_\mathcal{B}^\beta), \tag{4.4}$$

reflecting the Gaussian nature of the measure $\mu_\mathcal{B}^\beta$.

4.1. Dynamical reduction

We now turn to the proof of Theorem 2.6. Taking into account the fact that the coupling α is simple (recall Definition 2.3), we can rewrite the ψ-dependent part of the driving force in the Langevin equation (3.11) as

$$F(t,\xi_t)=\sum_{j=1}^{M}\psi(\mathrm{e}^{-L_\mathcal{B}t}Au_j)Z_{Q_j}(\xi_t), \tag{4.5}$$

where the u_j form a basis of $\mathbf{R}^M$, and the $Q_j(\xi)$ are smooth coefficients. Let us introduce the notation $I(s,t)\equiv[\min(s,t),\max(s,t)]$. It is apparent from formula (4.5) that, at time $t\in\mathbf{R}$, the position of the Ornstein–Uhlenbeck process $\xi_t(\xi,\psi)$ depends only on its starting position ξ and on the field values $\psi(f)$ for $f\in\{\mathrm{e}^{-L_\mathcal{B}\tau}Au: u\in\mathbf{R}^M, \tau\in I(0,t)\}$.

Remark. By the last statement, we really mean that $\xi_t(\xi,\psi)$ is measurable with respect to the minimal σ-field generated by ξ and the $\psi(f)$. We shall continue to use this abuse of language, leaving the simple measurability arguments to the reader.

The previous observation is the starting point of the first part of the proof. In order to formulate its deep consequences we make the following definition.

Definition 4.1. To any closed interval $I\subset\mathbf{R}$ we associate the subspace

$$\mathcal{H}_I\equiv\bigvee_{t\in I}\mathrm{e}^{-L_{\mathcal{B}}t}A\mathbf{R}^M\subset\mathcal{H}_{\mathcal{B}}.$$

Two immediate consequences of this definition are

$$\mathrm{e}^{-L_{\mathcal{B}}t}\mathcal{H}_I=\mathcal{H}_{I+t}, \tag{4.6}$$

for any $t\in\mathbf{R}$, and the time-reversal covariance

$$J_{\mathcal{B}}\mathcal{H}_I=\mathcal{H}_{-I}. \tag{4.7}$$

The space $\mathcal{H}_{\mathbf{R}}$, and hence $\mathcal{H}_{\mathbf{R}}^{\perp}$, are both invariant subspaces of $\mathrm{e}^{-L_{\mathcal{B}}t}$. For $f\in\mathcal{H}_{\mathbf{R}}^{\perp}$, the equation of motion (3.10) immediately reduces to $\psi_t(f)=\psi(\mathrm{e}^{-L_{\mathcal{B}}t}f)$. If $f\in\mathcal{H}_I$, on the other hand, the above considerations show that ξ_t and $\psi_t(f)$ depend only on the starting point ξ and on $\mathbf{G}_{\mathcal{H}_{I(0,t)\cup I+t}}$. Thus, according to (4.4), the Koopman space further factorises as

$$\mathfrak{F}^{\beta}\equiv(L^2(\mathbf{T}^*\mathcal{M},d\mu_{\mathcal{A}}^{\beta})\otimes L^2(\mathcal{N}',\mathbf{G}_{\mathcal{H}_{\mathbf{R}}},d\mu_{\mathcal{B}}^{\beta}))\otimes L^2(\mathcal{N}',\mathbf{G}_{\mathcal{H}_{\mathbf{R}}^{\perp}},d\mu_{\mathcal{B}}^{\beta}),$$

with a corresponding factorization of the Koopman group

$$\mathcal{U}^t=\mathcal{U}^t|_{L^2(\mathbf{T}^*\mathcal{M},d\mu_{\mathcal{A}}^{\beta})\otimes\Gamma(\mathcal{H}_{\mathbf{R}})}\otimes\mathcal{U}_{\mathcal{B}}^t|_{\Gamma(\mathcal{H}_{\mathbf{R}}^{\perp})}. \tag{4.8}$$

Setting $I=[0,\infty[$ in the previous argument shows that $\Gamma(\mathcal{H}_{[0,\infty[})$ is invariant under the forward evolution, i.e.,

$$\mathcal{U}^tL^2(\mathbf{T}^*\mathcal{M},d\mu_{\mathcal{A}}^{\beta})\otimes\Gamma(\mathcal{H}_{[0,\infty[})\subset L^2(\mathbf{T}^*\mathcal{M},d\mu_{\mathcal{A}}^{\beta})\otimes\Gamma(\mathcal{H}_{[0,\infty[})\quad\text{for } t\geqslant 0. \tag{4.9}$$

The second factor in (4.8) describes the part of the reservoir which is left unperturbed by the interaction with the system $\mathcal{A}$. Corresponding to (4.8), the dynamical system $(\mathbf{T}^*\mathcal{M}\times\mathcal{N}',\Xi^t,\mu^{\beta})$ decomposes into a direct product, the second factor of which is a Bernoulli flow. Therefore, we can concentrate on the first factor, which can be brought into a more explicit form with the help of the following result.

LEMMA 4.2. *Let us denote by $\mathbf{M}_M(\mathbf{C})$ the set of $(M\times M)$-matrices, by $\mathbf{C}^-$ the lower complex half-plane and by $H^2(\mathbf{C}^-)$ the Hardy space of analytic functions on $\mathbf{C}^-$. Then there is a factorization*

$$A(\omega)=W(\omega)O(\omega), \tag{4.10}$$

with the following properties:

(i) *The operator* $W(\omega)\colon \mathbf{C}^M \to \mathfrak{h}$ *is an isometry, and the map*

$$W\colon \hat{L}^2(\mathbf{R}, d\omega; \mathbf{C}^M) \to \mathcal{H}_{\mathbf{R}},$$

defined by $(Wu)(\omega) = W(\omega)u(\omega)$, *is an isomorphism. Here* $\hat{L}^2(\mathbf{R}, d\omega; \mathbf{C}^M)$ *denotes the real Hilbert space of square integrable,* $\mathbf{C}^M$*-valued functions* u *satisfying* $u(-\omega) = \bar{u}(\omega)$.

(ii) $O \in H^2(\mathbf{C}^-) \otimes \mathbf{M}_M(\mathbf{C})$ *is an outer function, i.e.,*

$$\bigvee_{t \geqslant 0} \mathrm{e}^{-i\omega t} O(\omega)\mathbf{C}^M = H^2(\mathbf{C}^-) \otimes \mathbf{C}^M. \tag{4.11}$$

Moreover, it satisfies $O(-\omega) = \bar{O}(\omega)$.

Proof. The proof of this lemma is a simple application of Wiener's factorization theorem [Wi] (see also [De] and [He, Lecture XI]). Recall that $\|T(\omega)\| \in L^2(\mathbf{R})$ by construction. Furthermore, hypothesis (H4) ensures that

$$\int_{-\infty}^{\infty} \frac{\ln \det T(\omega)}{1+\omega^2}\, d\omega > -\infty. \tag{4.12}$$

To see this, we break the above integral into several pieces. The integration over any finite interval disjoint from the singular set Ω gives a finite integral. The integration near infinity is controlled by the bound (2.26). Near an admissible singularity $\omega_0 \in \Omega$, we use the fact that $\log\det(T) = \log\det(TG_0) - \log\det(G_0)$. The first term is controlled by property (v) of the regularizer G_0 (see Definition 2.5). The second term gives a finite integral since $\det(G_0)$ is a rational function. Thus (4.12) holds, and the hypotheses of Wiener's theorem are satisfied.

A first factorization is obtained by the polar decomposition $A(\omega) = W_1(\omega)T(\omega)$, where $W_1(\omega)$ is an isometry from $\mathbf{C}^M$ to $\mathfrak{h}$, and $T(\omega)$ the spectral strength (2.25). By Wiener's theorem, we further have $T^2(\omega) = O^*(\omega)O(\omega)$, where O is an outer function belonging to $H^2(\mathbf{C}^-) \otimes \mathbf{M}_M(\mathbf{C})$. Applying the polar decomposition again we obtain $T(\omega) = W_2(\omega)O(\omega)$, where $W_2(\omega)$ is unitary.

We claim that it is possible to choose O in such a way that

$$\bar{O}(-\omega) = O(\omega), \tag{4.13}$$

holds. To prove this, let us introduce the conjugation $\widehat{C}_0\colon u(\omega) \mapsto \bar{u}(-\omega)$ in the complex Hilbert space $L^2(\mathbf{R}, d\omega) \otimes \mathbf{C}^M$. Since $T(\omega) = \bar{T}(-\omega)$, we can write

$$\widehat{C}_0 W_2 O \widehat{C}_0 = W_2 O, \tag{4.14}$$

from which we deduce, using the characterization (4.11) of outer functions, the relation

$$\widehat{C}_0 W_2 \widehat{C}_0 H^2(\mathbf{C}^-)\otimes\mathbf{C}^M = W_2 H^2(\mathbf{C}^-)\otimes\mathbf{C}^M.$$

By a well-known uniqueness result for invariant subspaces (see the last lemma of Lecture VI in [He]), there exists a unitary R, independent of ω, such that

$$\widehat{C}_0 W_2 \widehat{C}_0 = W_2 R. \tag{4.15}$$

Inserting this relation in (4.14), we conclude that $R\overline{O}(-\omega)=O(\omega)$. On the other hand, multiplying (4.15) on both sides by $\widehat{C}_0$, we obtain $W_2=\widehat{C}_0 W_2\widehat{C}_0\overline{R}$ which, together with (4.15), gives $\overline{R}=R^*$. Now it is easy to verify that the outer function $R^{-1/2}O(\omega)$ has the property (4.13).

Defining $W(\omega)\equiv W_1(\omega)W_2(\omega)$, we obtain the desired factorization. It remains only to show that $W\colon \hat{L}^2(\mathbf{R},d\omega;\mathbf{C}^M)\to\mathcal{H}_\mathbf{R}$ is surjective. To this end, we introduce the conjugation $\widehat{C}\colon f(\omega)\mapsto Cf(-\omega)$ in the complex Hilbert space $\mathcal{H}_B^\mathbf{C}=L^2(\mathbf{R},d\omega)\otimes\mathfrak{h}$, and remark that (2.24) translates into $\widehat{C}A=A\widehat{C}_0$. Since (4.13) gives $\widehat{C}_0 O\widehat{C}_0=O$, we conclude that $\widehat{C}W=W\widehat{C}_0$. Clearly $\mathcal{H}_\mathbf{R}$ is nothing but the subspace of real elements of

$$\mathcal{H}_\mathbf{R}^\mathbf{C}\equiv\bigvee_{t\in\mathbf{R}} \mathrm{e}^{-i\omega t}A\mathbf{C}^M,$$

with respect to $\widehat{C}$. By our factorization we have

$$\mathcal{H}_\mathbf{R}^\mathbf{C}=W\bigvee_{t\in\mathbf{R}} \mathrm{e}^{-i\omega t}O(\omega)\mathbf{C}^M=WL^2(\mathbf{R},d\omega)\otimes\mathbf{C}^M,$$

and therefore

$$\begin{aligned}\mathcal{H}_\mathbf{R}&=(1+\widehat{C})\mathcal{H}_\mathbf{R}^\mathbf{C}=(1+\widehat{C})WL^2(\mathbf{R},d\omega)\otimes\mathbf{C}^M\\&=W(1+\widehat{C}_0)L^2(\mathbf{R},d\omega)\otimes\mathbf{C}^M=W\hat{L}^2(\mathbf{R},d\omega;\mathbf{C}^M),\end{aligned}$$

as required. □

A characteristic property of outer functions which will be useful later is

LEMMA 4.3. *The function $O\in H^2(\mathbf{C}^-)\otimes\mathbf{M}_M(\mathbf{C})$ is outer if and only if, for any $F\in H^2(\mathbf{C}^-)\otimes\mathbf{M}_M(\mathbf{C})$, the condition*

$$\int_{-\infty}^{\infty}\|O^{-1}(\omega)F(\omega)\|^2\,d\omega<\infty$$

implies $O^{-1}F\in H^2(\mathbf{C}^-)\otimes\mathbf{M}_M(\mathbf{C})$.

For a proof of Lemma 4.3, see for example [Ni, §I]. From now on we shall consider only the reduced system in the representation induced by W. Equivalently, we set

$$\begin{aligned}\mathcal{H}_{\mathcal{B}} &= \hat{L}^2(\mathbf{R}, d\omega; \mathbf{C}^M),\\ L_{\mathcal{B}} &= i\omega,\\ \alpha(\xi) &= Ou(\xi).\end{aligned}$$

The only point requiring some special care is the form of the time-reversal operator $J_{\mathcal{B}}$ in this new representation. A simple calculation using equations (2.7) and (2.29) leads to

$$(J_{\mathcal{B}}f)(\omega) = O(\omega) J_{\mathcal{A}} O(-\omega)^{-1} f(-\omega). \tag{4.16}$$

The next result is a sharpening of the observation following (4.5).

LEMMA 4.4. *Let us denote by p_I the orthogonal projection on $\mathcal{H}_I$, and by $E_I = I \otimes \Gamma(p_I)$ the associated conditional expectation. Then the relation*

$$\mathcal{U}^{-t} E_{[0,\infty[} \mathcal{U}^t = E_{[-t,\infty[} \tag{4.17}$$

holds for any $t \geqslant 0$.

Proof. Using (4.6), one easily sees that (4.17) is equivalent to the commutation relation

$$[\mathcal{V}(0,t), E_{[0,\infty[}] = 0, \tag{4.18}$$

for $t \geqslant 0$, where $\mathcal{V}(s,t) = \mathcal{U}_{\mathcal{B}}^s \mathcal{U}^{t-s} \mathcal{U}_{\mathcal{B}}^{-t}$. This family of unitary operators satisfies $\mathcal{V}(t,t) = I$, $\mathcal{V}(s,t)\mathcal{V}(t,u) = \mathcal{V}(s,u)$ and, by definition, $(\mathcal{V}(s,t)F)(\xi,\psi) = F(\eta,\chi)$, with

$$\begin{aligned}\eta &= \xi_{t-s}(\xi, \mathrm{e}^{L_{\mathcal{B}}s}\psi),\\ \chi(f) &= \psi(f) + \lambda \int_0^{t-s} \left(\frac{d}{d\tau} \alpha \circ \xi_\tau(\xi, \mathrm{e}^{L_{\mathcal{B}}s}\psi), \mathrm{e}^{L_{\mathcal{B}}(s+\tau)} f \right) d\tau.\end{aligned}$$

It follows from the previous discussion that if F depends only on ξ and $\psi(f)$, with $f \in \mathcal{H}_I$, then $\mathcal{V}(s,t)F$ depends only on ξ and $\psi(f)$ with $f \in \mathcal{H}_{I \cup I(s,t)}$. We can reformulate this statement as

$$(1 - E_{I \cup I(s,t)}) \mathcal{V}(s,t) E_I = 0.$$

Setting $I = [0,\infty[$ in the last identity and combining it with its adjoint leads to (4.18). □

We are now ready to prove the main result in the first part of our argument.

PROPOSITION 4.5. *If $\mathfrak{F}_{\mathrm{sing}} \subset \mathfrak{F}^{\beta}$ is the spectral subspace associated to the singular spectrum of the Liouvillean $\mathcal{L}$, then*

$$\mathfrak{F}_{\mathrm{sing}} \subset L^2(\mathbf{T}^*\mathcal{M}, d\mu_{\mathcal{A}}^{\beta}) \otimes \Gamma(\mathcal{H}_0), \tag{4.19}$$

where

$$\mathcal{H}_0 \equiv \mathcal{H}_{[0,\infty[} \cap J_{\mathcal{B}} \mathcal{H}_{[0,\infty[}. \tag{4.20}$$

Proof. We set $\mathfrak{F}_I \equiv \mathrm{Ran}(E_I)$. Using the definition of E_I (in Lemma 4.4) and the fact that $\Gamma(\mathcal{H}) \cap \Gamma(\mathcal{H}') = \Gamma(\mathcal{H} \cap \mathcal{H}')$, we get

$$L^2(\mathbf{T}^*\mathcal{M}, d\mu_{\mathcal{A}}^{\beta}) \otimes \Gamma(\mathcal{H}_0) = \mathfrak{F}_{]-\infty,0]} \cap \mathfrak{F}_{[0,\infty[}.$$

Since $\mathcal{J}\mathfrak{F}_{\mathrm{sing}} = \mathfrak{F}_{\mathrm{sing}}$, Relation (4.7) reduces the claim to $\mathfrak{F}_{\mathrm{sing}} \subset \mathfrak{F}_{[0,\infty[}$. We shall prove this by constructing an $\mathcal{L}$-invariant subspace containing $\mathfrak{F}_{[0,\infty[}^{\perp}$, on which $\mathcal{L}$ has purely absolutely continuous spectrum.

Let $\mathfrak{F}_{+\infty}$ denote the maximal $\mathcal{L}$-invariant subspace of $\mathfrak{F}_{[0,\infty[}$:

$$\mathfrak{F}_{+\infty} \equiv \bigcap_{t \in \mathbf{R}} \mathcal{U}^t \mathfrak{F}_{[0,\infty[}.$$

By (4.9), the subspace

$$\widetilde{\mathfrak{F}}_{[0,\infty[} \equiv \mathfrak{F}_{[0,\infty[} \ominus \mathfrak{F}_{+\infty}$$

is simply $\mathcal{L}$-invariant, i.e.,

$$\mathcal{U}^t \widetilde{\mathfrak{F}}_{[0,\infty[} \subset \widetilde{\mathfrak{F}}_{[0,\infty[} \quad \text{for } t \geqslant 0, \tag{4.21}$$

but has no non-trivial $\mathcal{L}$-invariant subspace

$$\bigcap_{t \in \mathbf{R}} \mathcal{U}^t \widetilde{\mathfrak{F}}_{[0,\infty[} = \{0\}. \tag{4.22}$$

Moreover, applying Lemma 4.4, we get

$$\bigvee_{t \in \mathbf{R}} \mathcal{U}^t \widetilde{\mathfrak{F}}_{[0,\infty[} = \mathfrak{F}_{+\infty}^{\perp}. \tag{4.23}$$

Equations (4.21)–(4.23) show that $\widetilde{\mathfrak{F}}_{[0,\infty[}$ is an outgoing subspace for the unitary group

$$\mathcal{U}^t|_{\mathfrak{F}_{+\infty}^{\perp}},$$

and the Lax–Phillips theorem allows us to conclude that its generator

$$\mathcal{L}|_{\mathfrak{F}_{+\infty}^{\perp}}$$

has absolutely continuous spectrum, as announced. □

We finish the first part of our argument by deriving an explicit representation of the space $\mathcal{H}_0$. Let $f\in\mathcal{H}_0$ be given, and set $r\equiv O^{-1}f$. Then by (4.20) and Lemma 4.2, we have $f\in\mathcal{H}_{[0,\infty[}\subset H^2(\mathbf{C}^-)\otimes\mathbf{C}^M$. Since $O(\omega)$ is an outer function, its determinant is outer and thus cannot vanish in $\mathbf{C}^-$ (see [He, Theorem 5 and Chapter 11]). Therefore, $O(\omega)$ has an analytic inverse there, and

$$r(\omega)\in O(\omega)^{-1}H^2(\mathbf{C}^-)\otimes\mathbf{C}^M \tag{4.24}$$

is analytic in $\mathbf{C}^-$. Using again (4.20) and the explicit form of $J_{\mathcal{B}}$ given in (4.16), we further obtain

$$r(-\omega)\in J_{\mathcal{A}}O(\omega)^{-1}H^2(\mathbf{C}^-)\otimes\mathbf{C}^M, \tag{4.25}$$

from which we conclude that r is also analytic in the upper half-plane $\mathbf{C}^+$. The following result shows that r can be continued across the real axis as a meromorphic function.

LEMMA 4.6. *Assume that hypothesis* (H4) *holds. If $f\in\mathcal{H}_0$, then the function $r\equiv O^{-1}f$ is meromorphic. Its poles belong to the singular set Ω. Moreover, the order of a pole ω_0 of r does not exceed the order of the pole of the corresponding regularizer G_0.*

Proof. The proof is a simple adaptation of the argument of §6.c in [LM]. Fix a point $\omega_0\in\mathbf{R}$. If $\omega_0\in\Omega$, then let G_0 be a regularizer of T at ω_0. In the other case, set $G_0\equiv I$. We claim that $q\equiv G_0^{-1}r$ is continuous across the real line near ω_0. Postponing the proof of this claim, let us complete the argument leading to Lemma 4.6. Since the rational function G_0^{-1} has all its poles in the open upper half-plane (Definition 2.5 (ii)), it follows that q is analytic in a complex neighborhood U of ω_0. If $\omega_0\notin\Omega$, the same is true for r. In the other case, by condition (i) of Definition 2.5, the only possible singularity of $r=G_0q$ in U is a pole at ω_0. Since $\omega_0\in\mathbf{R}$ was arbitrary, the proof of Lemma 4.6 is complete.

We now turn to the proof of our claim. By condition (iii) of Definition 2.5, we have

$$\|(TG_0)^{-1}\|^{-1}\leqslant\|TG_0\|\in L^2(\mathbf{R},d\omega),$$

whereas condition (v) implies

$$\int_{-\infty}^{\infty}\frac{\log\|(TG_0)^{-1}\|^{-1}}{1+\omega^2}\,d\omega>-\infty.$$

Therefore, by Szegö's theorem (the scalar version of Wiener's factorization theorem), there exists an outer function $h\in H^2(\mathbf{C}^-)$ such that

$$|h(\omega)|\cdot\|(TG_0)^{-1}(\omega)\|=|h(\omega)|\cdot\|G_0^{-1}(\omega)O^{-1}(\omega)\|=1 \tag{4.26}$$

holds almost everywhere. By construction, $g \equiv hq = hG_0^{-1}O^{-1}f$ is square integrable and $\|g\| \leqslant \|f\|$. Let us show that $g \in H^2(\mathbf{C}^-) \otimes \mathbf{C}^M$. Let us assume first that $f \in H^\infty(\mathbf{C}^-) \otimes \mathbf{C}^M$. Then $hf \in H^2(\mathbf{C}^-) \otimes \mathbf{C}^M$ and $g = (OG_0)^{-1}hf$ is square integrable. Since the rational function $\det(G_0)$ has no zeros or poles in the lower half-plane (conditions (i), (ii) of Definition 2.5), it is outer. Thus G_0, and hence OG_0, are outer, and Lemma 4.3 shows that, indeed, $g \in H^2(\mathbf{C}^-) \otimes \mathbf{C}^M$. A density argument extends this result to arbitrary $f \in H^2(\mathbf{C}^-) \otimes \mathbf{C}^M$.

To summarize, we have established that

$$q = \frac{g}{h},$$

where $g \in H^2(\mathbf{C}^-) \otimes \mathbf{C}^M$ and $h \in H^2(\mathbf{C}^-)$. Moreover, h is outer and, by (4.26) and condition (iv) of Definition 2.5, $|h|^{-2}$ is locally integrable near ω_0. We complete the proof of our claim by invoking the argument of [LM] mentioned above. $\square$

We are now ready to write down the promised representation of $\mathcal{H}_0$.

PROPOSITION 4.7. *Assume that hypothesis* (H4) *holds. Then*

$$\mathcal{H}_0 = \{Or \in \mathcal{H}_\mathcal{B} : r \text{ is a real rational function of } i\omega\}.$$

In particular, it follows from Lemma 4.6 *that this space is finite-dimensional.*

Proof. The idea is simple: We prove that r is polynomially bounded in a complex neighborhood of infinity, and invoke Liouville's theorem.

For $\omega \in \Omega$, let $\mu(\omega)$ be the order of the pole of the regularizer at ω. By Lemma 4.6, we can find a polynomial p, of order $\mu \equiv \sum_{\omega \in \Omega} \mu(\omega)$, such that

$$pr \equiv pO^{-1}f$$

is entire for any $f \in \mathcal{H}_0$. We first claim that

$$q^\pm(\omega) \equiv (1+i\omega)^{-\nu-\mu} p(\pm\omega) r(\pm\omega) \in H^2(\mathbf{C}^-) \otimes \mathbf{C}^M. \tag{4.27}$$

Indeed, $q^\pm$ is analytic near the real axis and both $(1+i\omega)^{-\nu}O(\pm\omega)^{-1}$ and $(1+i\omega)^{-\mu}r(\pm\omega)$ are uniformly bounded in a real neighborhood of infinity by hypothesis (H4). It follows that $q^\pm$ is square integrable. Using (4.24), (4.25) and the fact that $(1+i\omega)^{-\nu-\mu}p(\pm\omega) \in H^\infty(\mathbf{C}^-)$, we can write

$$\begin{aligned} q^+ &= O^{-1}h^+, \\ J_\mathcal{A} q^- &= O^{-1}h^-, \end{aligned}$$

with $h^{\pm}\in H^2(\mathbf{C}^-)\otimes\mathbf{C}^M$. Since O is outer, the claim now follows from Lemma 4.3.

Next we show that

$$q^{\pm}\in H^2\big(\{\mathrm{Im}(\omega)<\tfrac{1}{2}\}\big)\otimes\mathbf{C}^M. \tag{4.28}$$

It follows from Lemma 4.6 that these functions are analytic in the half-plane $\{\mathrm{Im}(\omega)<1\}$, and from the previous paragraph that they belong to the Hardy space of the lower half-plane. Thus it suffices to show that they also belong to the Hardy space of the strip $\{0<\mathrm{Im}(\omega)<\frac{1}{2}\}$. In this strip, a simple calculation shows that

$$q^{\pm}(\omega)=\left(\frac{1-i\omega}{1+i\omega}\right)^{\nu+\mu}q^{\mp}(-\omega).$$

Since

$$\left(\frac{1-i\omega}{1+i\omega}\right)^{\nu+\mu}\in H^\infty\big(\{0<\mathrm{Im}(\omega)<\tfrac{1}{2}\}\big),$$

the claim follows from (4.27).

Finally it follows from (4.27) and (4.28) that

$$p(\pm\omega)r(\pm\omega)=(1+i\omega)^{\nu+\mu}g^{\pm}(\omega-\tfrac{1}{2}i),$$

with $g^{\pm}\in H^2(\mathbf{C}^-)\otimes\mathbf{C}^M$. Thus the Cauchy integral representation

$$p(\pm\omega)r(\pm\omega)=(1+i\omega)^{\nu+\mu}\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\omega'\,\frac{g^{\pm}(\omega')}{\omega-\omega'-\frac{1}{2}i}$$

holds for $\omega\in\mathbf{C}^-$, and Cauchy's inequality yields

$$|r(\pm\omega)p(\pm\omega)|\leqslant C\|g^{\pm}\|\cdot|1+i\omega|^{\nu+\mu}.$$

We conclude that pr is a polynomial of degree less than or equal to $\nu+\mu$, as required. $\square$

Remark. It is only in the proof of Proposition 4.7 that we really need the full strength of the bound (2.26) in hypothesis (H4): The finite-entropy condition (4.12) and local integrability of $\|T(\omega)^{-1}\|$ are sufficient for the other steps in our argument.

Notation. Let us denote the degree of a polynomial p by $\deg(p)$. We define the degree of a rational function r to be $\deg(r)\equiv\deg(p)-\deg(q)$, where p and q are two polynomials such that $r=p/q$. Since $|r(\omega)|\simeq|\omega|^{\deg(r)}$ at infinity, this definition is independent of the representation of r. The usual rules apply, e.g., $\deg(r_1+r_2)\leqslant\max(\deg(r_1),\deg(r_2))$ and $\deg(r_1r_2)=\deg(r_1)+\deg(r_2)$.

Proposition 4.7 concludes the first part of the proof of Theorem 2.6: We have shown that the singular spectrum of the Liouvillean $\mathcal{L}$ is entirely localized within the subspace $L^2(\mathbf{T}^*\mathcal{M},d\mu_{\mathcal{A}}^{\beta})\otimes\Gamma(\mathcal{H}_0)$, where $\mathcal{H}_0$ is a finite-dimensional space.

4.2. Elimination of the reservoir

In the second part of the proof, we show that an invariant subspace of $\mathcal{U}^t$ lying in $L^2(\mathbf{T}^*\mathcal{M}, d\mu_{\mathcal{A}}^{\beta})\otimes\Gamma(\mathcal{H}_0)$ must be entirely contained in $L^2(\mathbf{T}^*\mathcal{M}, d\mu_{\mathcal{A}}^{\beta})$. The basic idea is that the dynamics of the free reservoir, which has no non-trivial invariant subspace in $\Gamma(\mathcal{H}_0)$, completely dominates the evolution over very short time periods.

We proceed by induction over the degree of the rational functions associated to the elements of $\mathcal{H}_0$. To formulate our induction step we introduce the subspaces

$$\mathfrak{R}_l \equiv \{Or \in \mathcal{H}_0 : \deg(r) \leqslant l\}.$$

Clearly these subspaces form an increasing sequence

$$\{0\} = \ldots = \mathfrak{R}_{-\mu-1} \subset \mathfrak{R}_{-\mu} \subset \ldots \subset \mathfrak{R}_{\nu} = \ldots = \mathcal{H}_0. \tag{4.29}$$

Here is the main result of this subsection:

PROPOSITION 4.8. *If a subspace of $L^2(\mathbf{T}^*\mathcal{M}, d\mu_{\mathcal{A}}^{\beta})\otimes\Gamma(\mathfrak{R}_l)$ is invariant under the group $\mathcal{U}^t$, then it is contained in $L^2(\mathbf{T}^*\mathcal{M}, d\mu_{\mathcal{A}}^{\beta})\otimes\Gamma(\mathfrak{R}_{l-1})$.*

To prove it, we shall need the following algebraic "synthesis" lemma.

LEMMA 4.9. *For any $f\in\mathfrak{S}_l\equiv\mathfrak{R}_l\ominus\mathfrak{R}_{l-1}$, there exists $g\in\mathfrak{R}_l^{\perp}\cap D(i\omega)$ such that*

$$f = r_l i\omega g, \tag{4.30}$$

where r_l denotes the orthogonal projection on $\mathfrak{R}_l$.

Proof. We can rewrite (4.30) for the unknown function g as

$$(i\omega-1)g = f-(1-r_l)h,$$
$$r_l g = 0.$$

Thus a solution can be written as

$$g = (i\omega-1)^{-1}(f-(1-r_l)h),$$

provided h can be found such that

$$r_l(i\omega-1)^{-1}(1-r_l)h = r_l(i\omega-1)^{-1}f.$$

We show that this is indeed possible by proving

$$r_l(i\omega-1)^{-1}f \in \operatorname{Ran}(r_l(i\omega-1)^{-1}(1-r_l)) = \operatorname{Ker}((1-r_l)(i\omega+1)^{-1}r_l)^{\perp}.$$

Let $u\in\operatorname{Ker}((1-r_l)(i\omega+1)^{-1}r_l)$. Then $v\equiv(i\omega+1)^{-1}r_l u\in\mathfrak{R}_l$ and $i\omega v = r_l u - v\in\mathfrak{R}_l$. Thus $v\in\mathfrak{R}_{l-1}\perp\mathfrak{S}_l$ and

$$(u, r_l(i\omega-1)^{-1}f) = -(v, f) = 0,$$

as required. □

The next result provides the necessary control over the small-time dynamics.

LEMMA 4.10. *Let q be an orthogonal projection in $\mathcal{H}_{\mathcal{B}}$ and $G\equiv\chi(\xi)e^{i\psi(f)}$, with $\chi\in C_0^\infty(\mathbf{T}^*\mathcal{M})$ and $f\in\mathcal{H}_{\mathcal{B}}$. Then, if $(1-q)f\in D(i\omega)$, the function $t\mapsto\Gamma(q)\mathcal{U}^t(1-\Gamma(q))G$ is differentiable at $t=0$. Its derivative is given by*

$$
\begin{aligned}
-\frac{d}{dt}\Gamma(q)\mathcal{U}^t(1-\Gamma(q))G\Big|_{t=0} &= e^{-\|(1-q)f\|^2/2\beta}\chi(i\psi(h)+\beta^{-1}(qf,h))e^{i\psi(qf)} \\
&\quad + i\lambda\beta^{-1}\{\chi,(\alpha,(1-q)f)\} \\
&\quad + i\lambda\chi\{H_{\mathcal{A}}+\lambda\psi(q\alpha)+i\lambda\beta^{-1}(\alpha,qf),(\alpha,(1-q)f)\},
\end{aligned}
\tag{4.31}
$$

where $h\equiv qi\omega(1-q)f$.

We postpone the proof of Lemma 4.10 to the end of this section. Formally, this proof reduces to a simple computation using the equations of motion (3.10), (3.11). The technical difficulty is to show that $\Gamma(q)\mathcal{U}^t(1-\Gamma(q))G$ is differentiable in $\mathfrak{F}^\beta$ even if $(1-\Gamma(q))G\notin D(\mathcal{L})$.

Proof of Proposition 4.8. Let Ψ be an element of the invariant subspace. By hypothesis we have

$$(1-\Gamma(r_l))\mathcal{U}^t\Psi=0,$$

for any $t\in\mathbf{R}$. By duality,

$$(\Gamma(r_l)\mathcal{U}^t(1-\Gamma(r_l))G,\Psi)=0 \tag{4.32}$$

holds for any $G\in\mathfrak{F}^\beta$ and in particular for $G=\chi(\xi)e^{i\psi(f)}$, with $\chi\in C_0^\infty(\mathbf{T}^*\mathcal{M})$ and $f\in\mathcal{H}_{\mathcal{B}}$. We shall be more specific in the choice of f. Fix $h\in\mathfrak{S}_l$ with $\|h\|=1$. Then by Lemma 4.9 there exists $f_0\in\mathfrak{R}_l^\perp\cap D(i\omega)$ such that $h=r_l i\omega f_0$. We set $f=k\oplus\theta f_0$, with $k\in\mathfrak{R}_l$ and $\theta\in\mathbf{R}$. In the first part of our argument, f_0 is fixed and θ,k are variables. We may invoke Lemma 4.10 with $q=r_l$ to derive (4.32) with respect to t at $t=0$. Formula (4.31) generates several terms which we rearrange according to their dependence in θ and k. This leads to the equation

$$e^{-\theta^2\|f_0\|^2/2\beta}\int(\beta\psi(h)+i(h,k))e^{-i\psi(k)}F(\psi)\,d\mu_{\mathcal{B}}^\beta(\psi)=a+(b,k), \tag{4.33}$$

where

$$F(\psi)\equiv i\beta^{-1}\int\bar{\chi}(\xi)\Psi(\xi,\psi)\,d\mu_{\mathcal{A}}^\beta(\xi),$$

and $a\in\mathbf{C}$, $b\in\mathfrak{R}_l$ are independent of θ and k. Letting $\theta\to\infty$ in (4.33) leads to $a=0$ and $b=0$. Therefore, (4.33) turns into

$$((i\psi(h)+\beta^{-1}(h,k))\widetilde{G},\Psi)=0, \tag{4.34}$$

where $\widetilde{G}\equiv\chi e^{i\psi(k)}$. Next we use the fact that, for an orthogonal projection q, one has

$$d\Gamma(q)e^{i\psi(f)}=(i\psi(qf)+\beta^{-1}\|qf\|^2)e^{i\psi(f)}. \tag{4.35}$$

Multiplying (4.34) with (k,h), we immediately obtain

$$(d\Gamma(p_h)\widetilde{G},\Psi)=0, \tag{4.36}$$

where p_h denotes the orthogonal projection on the subspace spanned by h.

Since $d\Gamma(p_h)$ is self-adjoint on the space $L^2(\mathbf{T}^*\mathcal{M},d\mu_{\mathcal{A}}^{\beta})\otimes\Gamma(\mathfrak{R}_l)$, and since the set of $\widetilde{G}$ for which (4.36) holds is a core of $d\Gamma(p_h)$ (see [RS2, Theorem X.49]), we conclude that

$$d\Gamma(p_h)\Psi=0.$$

By Lemma 4.9, this holds for any $h\in\mathfrak{S}_l=\mathfrak{R}_l\ominus\mathfrak{R}_{l-1}$. Therefore, if $\{h_n\}$ denotes an orthonormal basis of $\mathfrak{S}_l$, we obtain

$$d\Gamma(s_l)\Psi=\sum_n d\Gamma(p_{h_n})\Psi=0,$$

where s_l is the orthogonal projection on $\mathfrak{S}_l$. Applying (4.3) to the projection s_l yields $\mathrm{Ker}(d\Gamma(s_l))=\mathrm{Ran}(\Gamma(r_{l-1}))$, which leads to the conclusion

$$(I-\Gamma(r_{l-1}))\Psi=0.$$

The proof is complete. □

As an immediate consequence of Proposition 4.8 we obtain the announced result:

COROLLARY 4.11. *The spectral subspace associated to the singular spectrum of the Liouvillean $\mathcal{L}$ satisfies*

$$\mathfrak{F}_{\mathrm{sing}}\subset L^2(\mathbf{T}^*\mathcal{M},d\mu_{\mathcal{A}}^{\beta}).$$

We conclude this section with the proof of our main dynamical estimate.

Proof of Lemma 4.10. We start by deriving an expansion of $\mathcal{U}^t\chi e^{i\psi(f)}$ for small t. If $\chi\in C_0^\infty(\mathbf{T}^*\mathcal{M})$ then, by Theorem 2.1, $\mathcal{U}^t\chi(\xi,\psi)$ is differentiable for any fixed $(\xi,\psi)\in\mathcal{G}^{-s}$ $(1<s<s_c-1)$. Equation (3.11) further yields

$$\frac{d}{dt}\mathcal{U}^t\chi(\xi,\psi)=\mathcal{U}^t\{\chi,H_{\mathcal{A}}+\lambda\psi(\alpha)\}(\xi,\psi),$$

which clearly belongs to $\mathfrak{F}^{\beta}$. The same argument as in the proof of Proposition 3.5 (see (3.13)) yields that $\chi\in D(\mathcal{L})$. Therefore, the formula

$$\mathcal{U}^t\chi=\chi+\{\chi,H_{\mathcal{A}}+\lambda\psi(\alpha)\}\,t+o(t) \tag{4.37}$$

holds in $\mathfrak{F}^{\beta}$.

Next we turn to the field ψ_t. Integration by parts in (3.10) leads to

$$\psi_t(f)=\psi(\mathrm{e}^{-i\omega t}f)+\lambda(\alpha(\xi_t)-\alpha(\xi),f)+\lambda\int_0^t(F_{t-\tau}(\xi_\tau)-F_{t-\tau}(\xi))\,d\tau, \tag{4.38}$$

where the function F is given by

$$F_t(\xi)\equiv\lambda(i\omega\alpha(\xi),\mathrm{e}^{-i\omega t}f).$$

Let us denote by Δ_t the last term on the right-hand side of (4.38). We claim that $\|\Delta_t\|=o(t)$. Since

$$\frac{\Delta_t}{t}=\int_0^1(\mathcal{U}^{\tau t}-I)F_{(1-\tau)t}\,d\tau,$$

we immediately get the estimate

$$\frac{\|\Delta_t\|}{t}\leqslant\int_0^1\|(\mathcal{U}^{\tau t}-I)F_0\|\,d\tau+\int_0^1\|(\mathcal{U}^{\tau t}-I)(F_{(1-\tau)t}-F_0)\|\,d\tau. \tag{4.39}$$

By Lebesgue's dominated convergence theorem, the first term on the right-hand side of inequality (4.39) vanishes as $t\to0$. To handle the second term, we use estimate (2.9) to get

$$|F_{(1-\tau)t}-F_0|\leqslant C\|\alpha(\xi)\|_2 t.$$

Finally, hypothesis (H3) and the fact that $H_{\mathcal{A}}$ is integrable lead to

$$\|(\mathcal{U}^{\tau t}-I)(F_{(1-\tau)t}-F_0)\|=O(t),$$

which concludes the proof of our claim.

We are now ready to combine our two estimates. It follows from (4.38) and the above claim that

$$\mathcal{U}^t\chi\mathrm{e}^{i\psi(f)}=\mathrm{e}^{-i\lambda(\alpha,f)}(\chi\mathrm{e}^{i\lambda(\alpha,f)})(\xi_t)\mathrm{e}^{i\psi(\mathrm{e}^{-i\omega t}f)}+o(t).$$

Since $\chi\mathrm{e}^{i\lambda(\alpha,f)}\in C_0^\infty(\mathbf{T}^*\mathcal{M})$, (4.37) further gives

$$\mathcal{U}^t\chi\mathrm{e}^{i\psi(f)}=(\chi+t\{\chi,H_{\mathcal{A}}+\lambda\psi(\alpha)\}+i\lambda t\chi\{(\alpha,f),H_{\mathcal{A}}+\lambda\psi(\alpha)\})\mathrm{e}^{i\psi(\mathrm{e}^{-i\omega t}f)}+o(t). \tag{4.40}$$

Using the fact that, for an orthogonal projection q, one has

$$\begin{aligned}\Gamma(q)\mathrm{e}^{i\psi(f)}&=\mathrm{e}^{-\|(1-q)f\|^2/2\beta}\mathrm{e}^{i\psi(qf)},\\ \Gamma(q)\mathrm{e}^{i\psi(f)}\psi(g)&=\mathrm{e}^{-\|(1-q)f\|^2/2\beta}(\psi(qg)-i\beta^{-1}((1-q)g,f))\mathrm{e}^{i\psi(qf)},\end{aligned} \tag{4.41}$$

we obtain from (4.40)

$$\Gamma(q)\mathcal{U}^t\chi e^{i\psi(f)}=e^{-\|(1-q)f\|^2/2\beta}(\chi+t\{\chi,H_{\mathcal{A}}+\lambda\psi(q\alpha)-i\lambda\beta^{-1}((1-q)\alpha,f)\}$$
$$+i\lambda t\chi\{(\alpha,f),H_{\mathcal{A}}+\lambda\psi(q\alpha)-i\lambda\beta^{-1}((1-q)\alpha,f)\})e^{i\psi(qe^{-i\omega t}f)}+o(t).$$

From this formula and (4.41) we get, after some tedious algebra,

$$\begin{aligned}&\Gamma(q)\mathcal{U}^t(I-\Gamma(q))\chi e^{i\psi(f)}\\&\quad=e^{-(\|f\|^2-\|qe^{-i\omega t}qf\|^2)/2\beta}[(\chi+o(1))(N-1)-i\lambda t(\beta^{-1}\{\chi,((1-q)\alpha,f)\}\\&\qquad+\chi\{H_{\mathcal{A}}+\lambda\psi(q\alpha)+i\lambda\beta^{-1}(\alpha,f),((1-q)\alpha,f)\})N]+o(t),\end{aligned}\tag{4.42}$$

where we have set

$$N\equiv e^{i\psi(qe^{-i\omega t}(1-q)f)}e^{\|qe^{-i\omega t}(1-q)f\|^2/2\beta}e^{(qe^{-i\omega t}qf,qe^{-i\omega t}(1-q)f))/\beta}.$$

Since $(1-q)f\in D(i\omega)$, we have

$$qe^{-i\omega t}(1-q)f=-th+o(t),$$

where $h=qi\omega(1-q)f$. Therefore,

$$e^{i\psi(qe^{-i\omega t}(1-q)f)}=1-it\psi(h)+o(t)$$

holds in $\mathfrak{F}^\beta$. From this we conclude that

$$N=1-it\psi(h)-t\beta^{-1}(h,f)+o(t).$$

Inserting the last estimate in (4.42) leads to the desired result. □

4.3. Kinematic reduction

In the third and last part of the proof, we show that the only invariant subspace of $\mathcal{U}^t$ in $L^2(\mathbf{T}^*\mathcal{M},d\mu_{\mathcal{A}}^\beta)$ consists of the constant functions. We start by repeating the argument leading to (4.36), this time using $q=0$ in Lemma 4.10. For $\lambda\neq0$, this yields the formula

$$(\{\chi,(\alpha,f)\}-\beta\chi\{H_{\mathcal{A}},(\alpha,f)\},\Psi)=0,\tag{4.43}$$

valid for any Ψ in the invariant subspace, $\chi\in C_0^\infty(\mathbf{T}^*\mathcal{M})$ and $f\in\mathcal{H}_{\mathcal{B}}$. To clarify the meaning of (4.43) we associate, to each $F\in\mathcal{P}$, the operator

$$L_F:\chi\mapsto e^{\beta H_{\mathcal{A}}}\{e^{-\beta H_{\mathcal{A}}}\chi,F\}=\{\chi,F\}-\beta\chi\{H_{\mathcal{A}},F\}.$$

A simple calculation shows that

$$(L_F\chi, \Psi) = (\chi, \{F, \Psi\}),$$

so that (4.43) becomes

$$\{F, \Psi\} = 0, \tag{4.44}$$

in distributional sense, with $F=(\alpha, f)$. By a well-known result ([Hö, Theorem 8.3.1]), the wave front set of Ψ satisfies

$$\mathrm{WF}(\Psi) \subset \{(\xi, \eta) \in \mathbf{T}^*(\mathbf{T}^*\mathcal{M})\backslash\{0\} : \eta \cdot Z_F(\xi) = 0\}. \tag{4.45}$$

Since L_F maps C_0^∞ into itself, and $[L_F, L_G] = L_{\{F,G\}}$, we conclude that (4.44), and hence (4.45), hold for any $F \in \mathcal{P}_\alpha$. On the other hand, since we are dealing with an invariant subspace, we can replace Ψ by $\mathcal{U}^t\Psi$ to obtain

$$\begin{aligned} 0 &= (L_F\chi, \mathcal{U}^t\Psi) = (\mathcal{U}^{-t}L_F\chi, \Psi) = (L_F\chi - t\{L_F\chi, H_{\mathcal{A}} + \lambda\psi(\alpha)\} + o(t), \Psi) \\ &= -t(L_{H_{\mathcal{A}}}L_F\chi, \Psi) + o(t), \end{aligned}$$

from which we conclude

$$(L_{\{F,H_{\mathcal{A}}\}}\chi, \Psi) = (L_F L_{H_{\mathcal{A}}}\chi, \Psi) - (L_{H_{\mathcal{A}}}L_F\chi, \Psi) = 0.$$

Iterating the last argument, we finally extend (4.45) to arbitrary F in the full-rank algebra of hypothesis (H6). It immediately follows that $\mathrm{WF}(\Psi) = \varnothing$, and therefore $\Psi \in C^\infty(\mathbf{T}^*\mathcal{M})$. Going back to (4.44) we have, for each $\xi \in \mathbf{T}^*\mathcal{M}$,

$$d\Psi(\xi) \cdot Z_F(\xi) = \{\Psi, F\}(\xi) = 0.$$

Hypothesis (H6) yields now that $d\Psi = 0$, and hence, since $\mathcal{M}$ is connected, Ψ must be constant. The proof of Theorem 2.6 is complete.

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