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On Dyakonov's paper "Equivalent norms on Lipschitz-type spaces of holomorphic functions"

by

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A continuous, increasing function ω on the interval $[0,2]$ is called a *majorant* if $\omega(0)=0$ and the function $\omega(t)/t$ is decreasing. A majorant ω is said to be *regular* if there exists a constant C such that

$$\int_0^x \frac{\omega(t)}{t}\,dt + x\int_x^2 \frac{\omega(t)}{t^2}\,dt \leqslant C\omega(x), \quad 0<x<2.$$

Given a majorant ω we define $\Lambda_\omega(\mathbf{D})$, where $\mathbf{D}$ is the unit disk of the complex plane, to be the class of those complex-valued functions f for which there exists a constant C such that

$$|f(w)-f(z)| \leqslant C\omega(|w-z|), \quad z,w\in\mathbf{D}.$$

The class $\Lambda_\omega(\mathbf{T})$, where $\mathbf{T}$ is the unit circle, is defined similarly.

Recently, Dyakonov [1] gave some characterizations of the holomorphic functions of class Λ_ω in terms of their moduli. Here we state the main result of [1] as Theorems A and B (cf. Theorem 2 and Corollary 1 (ii) in [1]).

THEOREM A. *Let ω be a regular majorant. A function f holomorphic in $\mathbf{D}$ is in $\Lambda_\omega(\mathbf{D})$ if and only if so is its modulus $|f|$.*

Of course, the "only if" part of this theorem is trivial.

Let $A(\mathbf{D})$ denote the disk algebra, i.e., the class of holomorphic functions in $\mathbf{D}$ that are continuous up to the boundary. If f is in $A(\mathbf{D})$, then the function $|f|$ is subharmonic, and therefore the Poisson integral, $P|f|$, of the boundary function of $|f|$, is equal to the smallest harmonic majorant of $|f|$ in $\mathbf{D}$. In particular, $P|f|-|f|\geqslant 0$ in $\mathbf{D}$.

THEOREM B. *Let ω be a regular majorant, $f\in A(\mathbf{D})$, and let the boundary function of $|f|$ belong to $\Lambda_\omega(\mathbf{T})$. Then f is in $\Lambda_\omega(\mathbf{D})$ if and only if*

$$P|f|(z)-|f(z)| \leqslant C\omega(1-|z|)$$

for some constant C.

Dyakonov deduced Theorems A and B from some classical results, essentially due to Hardy and Littlewood, and Privaloff (see Lemmas 1 and 3 below), and a theorem of Dyn'kin on pseudoanalytic continuation. The main ingredient in Dyakonov's proof is a very complicated construction of a suitable pseudoanalytic continuation. In this paper we give a very simple proof of Theorems A and B. The proof uses only the basic Lemmas 2, 3, 4 and 6 of [1] and the Schwarz lemma, and is therefore considerably shorter than that in [1].

LEMMA 1. *Let* ω *be a regular majorant. A function* f *holomorphic in* $\mathbf{D}$ *belongs to* $\Lambda_\omega(\mathbf{D})$ *if and only if*

$$|f'(z)| \leqslant C \frac{\omega(1-|z|)}{1-|z|}$$

for some constant C *independent of* z.

For a proof see, for example, Lemma 6 of [1]. Besides this elementary fact we need a consequence of the Schwarz lemma.

LEMMA 2. *Let* $D_z = \{w : |w-z| \leqslant 1-|z|\}$, $f \in A(\mathbf{D})$ *and* $M_z = \sup\{|f(w)| : w \in D_z\}$. *Then*

$$\tfrac{1}{2}(1-|z|)|f'(z)| + |f(z)| \leqslant M_z, \quad z \in \mathbf{D}.$$

Proof. If $z=0$ and $M_0=1$, the Schwarz lemma gives

$$|f'(0)| \leqslant 1-|f(0)|^2 \leqslant 2(1-|f(0)|),$$

which is our inequality in this special case. The general case follows by applying the special case to the function F defined by

$$F(\zeta) = \frac{f(z+\zeta(1-|z|))}{M_z}, \quad \zeta \in \mathbf{D}.$$

Proof of Theorem A. The "only if" part is trivial. Assuming that $|f| \in \Lambda_\omega(\mathbf{D})$ we have

$$|f(w)| - |f(z)| \leqslant C\omega(|w-z|) \leqslant C\omega(1-|z|)$$

for every $z \in \mathbf{D}$ and $w \in D_z$. Taking the supremum over $w \in D_z$ and then using Lemma 2 we get

$$|f'(z)|(1-|z|) \leqslant 2C\omega(1-|z|).$$

Now the result follows from Lemma 1.

For the proof of Theorem B we need another classical result.

LEMMA 3. *Let ω be a regular majorant. A real-valued function g defined on* $\mathbf{T}$ *belongs to $\Lambda_\omega(\mathbf{T})$ if and only if Pg (= the Poisson integral of g) belongs to $\Lambda_\omega(\mathbf{D})$.*

See the proof of Lemma 4 of [1].

Proof of Theorem B. We begin with the "only if" part. Let $h(z)=P|f|(z)$ and assume that $f\in\Lambda_\omega(\mathbf{D})$. Then $h\in\Lambda_\omega(\mathbf{D})$ by Lemma 3, and $|f|$ will be in the same Lipschitz class. Hence

$$h(z)-|f(z)|=h(z)-|f(z/|z|)|+|f(z/|z|)|-|f(z)|\leqslant C\omega(1-|z|),\quad z\in\mathbf{D}\setminus\{0\},$$

which finishes this part of the proof.

To prove the converse, we have for z fixed in $\mathbf{D}$ and $w\in D_z$ that

$$|f(w)|-|f(z)|\leqslant h(w)-|f(z)|=h(w)-h(z)+h(z)-|f(z)|.$$

From the hypothesis $|f|\in\Lambda_\omega(\mathbf{T})$ and Lemma 3 it follows that

$$h(w)-h(z)\leqslant C\omega(|w-z|)\leqslant C\omega(1-|z|),\quad w\in D_z.$$

By assumption, $h(z)-|f(z)|\leqslant C\omega(1-|z|)$, and we get

$$|f(w)|-|f(z)|\leqslant C\omega(1-|z|),\quad w\in D_z.$$

Arguing as in the proof of Theorem A, we obtain Theorem B.

Remark. The assumption that the majorant ω is regular plays a role only in proving Lemmas 1 and 3 (see [1] for details).

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References

[1] DYAKONOV, K. M., Equivalent norms on Lipschitz-type spaces of holomorphic functions. *Acta Math.*, 178 (1997), 143–167.

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