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Kelvin transform and multi-harmonic polynomials

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En mémoire de ma mère

The classical *Kelvin transform* associates to a smooth function f on $\mathbf{R}^N \setminus \{0\}$ ($N \geqslant 2$) the function Kf (also defined on $\mathbf{R}^N \setminus \{0\}$) by the formula

$$Kf(\xi) = \|\xi\|^{2-N} f(\xi/\|\xi\|^2).$$

The main result is that if f is harmonic, then Kf is also harmonic. Although we shall not use this remark, this result reflects a covariance property of the Laplace operator under the action of the conformal group. The Kelvin transform is used to generate (all) harmonic polynomials on $\mathbf{R}^N$ by the following process (due to Maxwell, cf. [CH]): take p to be any polynomial on $\mathbf{R}^N$, form the usual constant-coefficient differential operator $\partial(p)$, apply it to the Green kernel $G(\xi) = \|\xi\|^{2-N}$ (to be replaced by $\log \|\xi\|$ in case $N=2$). The result $\partial(p)G$ is defined and harmonic on $\mathbf{R}^N \setminus \{0\}$, so that we may apply the Kelvin transform to get a harmonic function $K\partial(p)G$, which can be shown to extend to all of $\mathbf{R}^N$ as a harmonic polynomial. Moreover, all harmonic polynomials can be obtained by this process.

We generalize such a transformation in the context of analysis on matrix spaces. By this terminology is usually meant analysis that gives a special role to some subgroup of the linear or orthogonal group which can be interpreted as (say) a left action on a space of (rectangular) matrices. One classical case is the left action of $GL(n, \mathbf{R})$ on $\mathrm{Mat}_{n,m}(\mathbf{R})$, and related versions where $\mathbf{R}$ is replaced by $\mathbf{C}$ or the field of quaternions. Various notions of harmonic polynomials (under various names) were introduced in the literature (see the references at the end), related to invariance or covariance properties under the above-mentioned subgroup. Here we work in the context of representations of (Euclidean) Jordan algebras, where the appropriate theory of harmonic polynomials (called *Stiefel harmonics*) was developed in [C1]. It covers these classical cases, but also contains new cases, associated to the representation of the *Lorentzian* Jordan algebra $\mathbf{R}\oplus W$ on a Clifford module for the Clifford algebra of W.

In §3, we introduce a function which to some extent plays the role of the Green kernel in the classical potential theory. In his thesis (see [He]), C. Herz already introduced this function for the space $\mathrm{Mat}_{n,m}(\mathbf{R})$, stating incidentally some of its properties, but did not push the theory further, and, up to our knowledge, his remarks have staid unnoticed.

In §4, we define the (generalized) *Kelvin transform* and show in §5 that it is possible, under some mild conditions, to generate all these harmonic polynomials by a process similar to the one described above.

Let us mention some places were such harmonic polynomials have been used:

- harmonic analysis on Stiefel manifolds ([GM], [Ge], [C1]);
- decomposition of unitary representations ([KV], [C2]);
- construction of zeta functions and series ([M], [C3]).

1. Representation of a Jordan algebra

Most of the results needed in this section can be found in [FK] (see also [C1]). Let V be a simple Euclidean Jordan algebra over $\mathbf{R}$, with identity element e, of dimension n, rank r and characteristic number d, so that

$$n=r+d\cdot\tfrac{1}{2}r(r-1).$$

For any $x\in V$, denote by $L(x)\colon V\to V$ the endomorphism $y\mapsto xy$. Denote by tr and det respectively the *trace* and *norm function* (generalized determinant), and recall that, by assumption, $(x,y)=\operatorname{tr} xy$ defines an inner product on V, for which the operators $L(x)$ and $P(x)=2L(x)^2-L(x^2)$ are symmetric. The norm function is a polynomial, homogeneous of degree r. The set of invertible elements is denoted by $V^\times$, and $V^\times$ is exactly the set of elements $x\in V$ such that $\det x\neq 0$. The connected component of the identity in $V^\times$ is an open convex cone, denoted by Ω.

The group of linear transformations which preserve Ω is denoted by G. It is a reductive Lie group, and it acts transitively on Ω. The stabilizer in G of the identity element e is a maximal compact subgroup of G, denoted by K. The elements of K are automorphisms of the Jordan algebra structure, and they are isometries for the inner product on V.

Let $e=c_1+c_2+...+c_r$ be a Peirce decomposition of the identity. Let

$$\mathfrak{a}=\left\{\lambda=\sum_{i=1}^{r}\lambda_i c_i : \lambda_i\in\mathbf{R}\right\},$$

$$R=\left\{\lambda=\sum_{i=1}^{r}\lambda_i c_i : \lambda_1<\lambda_2<...<\lambda_r\right\},\quad R_+=\{\lambda\in R:0<\lambda_1\}.$$

Then, up to a set of Lebesgue measure 0, every element x of V can be written as $k\lambda$, where $k\in K$ and $\lambda\in R$ (polar decomposition). Up to a set of Lebesgue measure 0, every element in Ω can be written as $k\lambda$, where $k\in K$ and $\lambda\in R_+$. There is a corresponding integration formula for the Lebesgue measure on E, namely,

$$\int_V f(\xi)\,d\xi=c_0\int_K\int_R f(k\lambda)\prod_{i<j}(\lambda_j-\lambda_i)^d\,d\lambda_1\,d\lambda_2\,...\,d\lambda_r\,dk \tag{1}$$

where dk is the normalized Haar measure on K, and c_0 is a positive constant depending only on V.

Let E be a Euclidean vector space of dimension N, with an inner product denoted by $\langle\xi,\eta\rangle$. A *representation* of V on E is a linear map $\phi: V\to\mathrm{End}(E)$ satisfying the assumptions

$$\phi(xy)=\tfrac{1}{2}(\phi(x)\phi(y)+\phi(y)\phi(x)),$$
$$\phi(e)=\mathrm{Id},$$
$$\langle\phi(x)\xi,\eta\rangle=\langle\xi,\phi(x)\eta\rangle,$$

for all $x,y\in V$, $\xi,\eta\in E$.

There is an associated symmetric bilinear map $H: E\times E\to V$, defined for $\xi,\eta\in E$ by

$$\langle\phi(x)\xi,\eta\rangle=(x,H(\xi,\eta)),\quad \text{for all } x\in V.$$

Let Q be the associated quadratic map defined by

$$Q(\xi)=H(\xi,\xi).$$

The relation

$$\phi(P(x)y)=\phi(x)\phi(y)\phi(x),\quad \text{for all } x,y\in V,$$

implies that

$$Q(\phi(x)\xi)=P(x)Q(\xi),\quad \text{for all } x\in V,\ \xi\in E. \tag{2}$$

Let us also recall that the dimension of E is a multiple of the rank of V, and

$$\operatorname{Det}\phi(x) = (\det x)^{N/r}, \quad \text{for all } x \in V.$$

The representation is said to be *regular* if the set

$$E' = \{\xi \in E : \det Q(\xi) \neq 0\}$$

is nonempty. If this is the case, then E' is a dense open set in E. Let

$$\Sigma = \{\xi \in V : Q(\xi) = e\}.$$

It is a (nonempty) compact submanifold of V, called the (generalized) *Stiefel manifold*. Every element ξ in E' can be written

$$\xi = \phi(x^{1/2})\sigma, \quad x \in \Omega,\ \sigma \in \Sigma.$$

Moreover, $x = Q(\xi)$, $\sigma = \Phi(x^{-1/2})\xi$, and the map

$$(x, \sigma) \mapsto \phi(x^{1/2})\sigma$$

is a diffeomorphism of $\Omega \times \Sigma$ onto E' (polar coordinates on E).

There is a corresponding integration formula for the Lebesgue measure on E:

$$\int_E f(\xi)\, d\xi = c_1 \int_\Omega \int_\Sigma f(\phi(x^{1/2})\sigma) \det x^{N/2r - n/r}\, dx\, d\sigma \tag{3}$$

where $c_1 = \pi^{N/2}/\Gamma_\Omega(N/2r)$ and $d\sigma$ is the (normalized) Euclidean volume element on Σ.

2. The measure $d\mu_S$

This section is devoted to introduce a positive measure supported on the singular set $S = E \setminus E'$, and to give its expression in polar coordinates . It is related, through the map Q, to the Euclidean measure on the singular set $\partial\Omega = \bar{\Omega} \setminus \Omega$.

We first need an extension of the polar coordinates in V up to the boundary. For $x \in \bar{\Omega}$, let $\mu_1, \mu_2, \ldots, \mu_k$ be the (positive) distinct eigenvalues of x, and C_j, $1 \leqslant j \leqslant k$, be the corresponding idempotents (not necessarily primitive), such that $x = \sum_{j=1}^k \mu_j C_j$. Then the element $y = \sum_{j=1}^k \sqrt{\mu_j}\, C_j$ is in $\bar{\Omega}$ and satisfies $y^2 = x$. Conversely, if y is any element in $\bar{\Omega}$ such that $y^2 = x$, let $y = \sum_{j=1}^l \nu_j D_j$ be its spectral decomposition. Then $x = y^2 = \sum_{j=1}^l \nu_j^2 D_j$. By the uniqueness of the spectral decomposition, we get that $k = l$, that the sets of idempotents are the same, and after relabelling, that the eigenvalues are the same. So for each $x \in \bar{\Omega}$, there exists in $\bar{\Omega}$ a *unique* square root, which is denoted by $x^{1/2}$.

LEMMA 1. *The map $x \mapsto x^{1/2}$ from $\bar{\Omega}$ into itself is continuous.*

Let $(p_n(t))_{n \geqslant 0}$ be a sequence of real-valued polynomials which converges to $\sqrt{t}$ on $[0, +\infty)$, uniformly on any compact subset. If $x = \sum_{j=1}^k \mu_j C_j$ is the spectral decomposition of any element in $\bar{\Omega}$, then $p_n(x) = \sum_{j=1}^k p_n(\mu_j) C_j$, and hence $p_n(x) \to x^{1/2}$. The eigenvalues μ_j satisfy $|\mu_j| \leqslant \|x\|$, and so they are bounded when x stays in a bounded set. Hence, on any compact set of $\bar{\Omega}$, $p_n(x)$ converges *uniformly* to $x^{1/2}$, and the continuity of the mapping $x \mapsto x^{1/2}$ follows.

PROPOSITION 1. *The mapping $(x, \sigma) \mapsto \phi(x^{1/2})\sigma$ is a continuous, proper, surjective map from $\bar{\Omega} \times \Sigma$ onto E.*

The continuity is clear from the previous lemma. If $\xi = \phi(x^{1/2})\sigma$, then

$$Q(\xi) = Q(\phi(x^{1/2})\sigma) = P(x^{1/2})Q(\sigma) = (x^{1/2})^2 = x.$$

So, if ξ stays in a compact set of E, then x runs through a bounded set of $\bar{\Omega}$. So the map is proper. This implies that its image is closed in E. As the representation ϕ is assumed to be regular, we already know that the image contains a dense open set (namely E'). Hence the surjectivity.

Let $\mathcal{I}_c$ be the space of functions on E which can be written as $F \circ Q$, where $F \in \mathcal{C}_c(V)$. It can also be described as the space of continuous functions on E with compact support which are constant on each level set of the map Q. It is a closed subspace of $\mathcal{C}_c(E)$ when equipped with the topology of uniform convergence on compacta. For the inner product $(f, g) \mapsto \int_E f(\xi)\bar{g}(\xi)\, d\xi$, let us consider the orthogonal space $\mathcal{I}_c^{\perp}$. From the integration formula (3), we deduce the following characterization of $\mathcal{I}_c^{\perp}$:

$$g \in \mathcal{I}_c^{\perp} \iff \int_{\Sigma} g(\phi(x)\sigma)\, d\sigma = 0, \quad \text{for all } x \in \bar{\Omega}.$$

PROPOSITION 2. *Let μ be a (positive Radon) measure on E. Then the following properties are equivalent*:

(i) $\int_E g(\xi)\, d\mu(\xi) = 0$, *for all* $g \in \mathcal{I}_c^{\perp}$.

(ii) *There exists a measure ν on $\bar{\Omega}$ such that*

$$\int_E g(\xi)\, d\mu(\xi) = c_1 \int_{\bar{\Omega}} \int_{\Sigma} g(\phi(x^{1/2})\sigma)\, d\nu(x)\, d\sigma, \quad \textit{for all } g \in \mathcal{C}_c(E). \tag{$*$}$$

Conversely, given any measure ν on $\bar{\Omega}$, the formula $(*)$ defines a measure μ on E which satisfies either property.

Notice that the measure ν is uniquely determined by the formula

$$\int_{\bar{\Omega}} F(x)\, d\nu(x) = \int_E F(Q(\xi))\, d\mu(\xi), \quad \text{for all } F \in \mathcal{C}_c(\bar{\Omega}).$$

The proof follows a standard pattern, using mainly Proposition 1.

Our next step is to determine the (Euclidean) surface measure on the singular set $\partial\Omega=\bar{\Omega}\setminus\Omega=\{x:\det x=0\}\cap\bar{\Omega}$. The strategy is the following: We first compute the Euclidean length of the gradient of the function det on $\partial\Omega$. As we will see, it is nonzero on a dense open set of $\partial\Omega$. Near these points, $\partial\Omega$ is a hypersurface, and we determine the associated *Leray form* $\delta(\det)$ (see [GC]). This can be obtained as the residue of $\det x^s$ at the first pole $s=-1$, and it turns out to be a positive measure on $\partial\Omega$. Then the Euclidean surface measure on $\partial\Omega$ is

$$d\nu_{\partial\Omega}=\|\nabla\det\|\,\delta(\det).$$

Before stating the result, we need some more notation. For $\lambda=\sum_{i=1}^r\lambda_i c_i$ an element of $\mathfrak{a}$, write $\lambda=\lambda_1c_1+\lambda^0$ with $\lambda^0=(\lambda_2,...,\lambda_r)$. Let

$$R_+^0=\left\{\lambda^0=\sum_{i=2}^r\lambda_i c_i:0<\lambda_2<...<\lambda_r\right\}.$$

The elements of R_+^0 all have rank $r-1$. The elements of rank $r-1$ in $\partial\Omega$ form a dense open set in $\partial\Omega$, and, except for a subset of $d\nu_{\partial\Omega}$-measure 0, an element of rank $r-1$ in $\partial\Omega$ can always be written as $k\lambda^0$ with $\lambda^0\in R_+^0$.

THEOREM 1. *The Euclidean surface measure $d\nu_{\partial\Omega}$ on $\partial\Omega$ is given by*

$$\int_{\partial\Omega}g(x)\,d\nu_{\partial\Omega}(x)=c_0\int_{R_+^0}\int_K g(k\lambda^0)\prod_{j=2}^r\lambda_j^d\prod_{2\leqslant i<j\leqslant r}(\lambda_j-\lambda_i)^d\,d\lambda_2\,...\,d\lambda_r\,dk.$$

For the proof, we first compute $\nabla\det$ on $\partial\Omega$ (Step 1), then compute $\delta(\det)$ (Step 2).

Step 1. Inside Ω, there is a well-known formula for the gradient of the function det:

$$\nabla\det(x)=(\det x)x^{-1} \tag{4}$$

(see [FK, Proposition III.4.2]). Of course, $\nabla\det(x)$ is really a polynomial map, and it is possible to find its value on the boundary of Ω by continuity. For $\lambda=\sum_{i=1}^r\lambda_i c_i$ (with $0<\lambda_1\leqslant\lambda_2\leqslant...\leqslant\lambda_r$), then $\nabla\det(\lambda)=\sum_{i=1}^r\lambda_1\,...\,\check{\lambda}_i\,...\,\lambda_r c_i$, so that for $\lambda^0=\sum_{i=2}^r\lambda_i c_i$ one has, by continuity, $\nabla\det(\lambda^0)=\lambda_2\,...\,\lambda_r c_1$ and hence $\|\nabla\det(\lambda^0)\|=\lambda_2\,...\,\lambda_r$. As the norm of the gradient of the function det is invariant by K, one has

$$\|\nabla\det(k\lambda^0)\|=\lambda_2\,...\,\lambda_r$$

for $\lambda^0\in R_+^0$. This shows in particular that the gradient does not vanish on the set of elements of rank exactly $r-1$ in $\partial\Omega$.

Step 2. Assume now that $\operatorname{Re} s > -1$, and let g be any function in the Schwartz space $\mathcal{S}(V)$. Then the integral

$$\int_\Omega g(x)(\det x)^s \, dx$$

converges. As a tempered distribution, it has a meromorphic continuation in the variable s, with simple poles at $s=-1,-2,\dots$ (see [FK, Chapter VII]). The residue at $s=-1$ is a positive measure $\delta(\det)$ supported on $\partial\Omega$. We will now find its expression in terms of the polar decomposition.

LEMMA 2. *Let g be a K-invariant function in $\mathcal{S}(V)$. Then*

$$\begin{aligned}\int \delta(\det)(x) g(x) &= \lim_{s\to -1} \int_\Omega g(x) \det x^s \, dx \\ &= c_0 \int_{R_+^0} g(0,\lambda^0) \Big(\prod_{j=2}^r \lambda_j\Big)^{d-1} \prod_{2\leqslant i<j\leqslant r} (\lambda_j-\lambda_i)^d \, d\lambda^0.\end{aligned}$$

Proof. We use formula (1) to get

$$\int_\Omega g(x)\det x^s \, dx = c_0 \int_{R_+} \int_K g(k\lambda)\, dk \Big(\prod_{i=1}^r \lambda_i\Big)^s \prod_{i<j} (\lambda_j-\lambda_i)^d \, d\lambda_1 \dots d\lambda_r.$$

As $\det^s$ is K-invariant, we may assume without losing any generality that g is already K-invariant. Abusing somewhat notation, denote by $g(\lambda)$ the restriction of g to $\mathfrak{a}$. For $\lambda_1 > 0$ set

$$\begin{aligned} R_+(\lambda_1) &= \{\lambda^0 = (\lambda_2, \dots, \lambda_r) : \lambda_1 < \lambda_2 < \dots < \lambda_r\}, \\ \omega(\lambda^0)\, d\lambda^0 &= \prod_{2\leqslant i<j\leqslant r} (\lambda_j-\lambda_i)^d \, d\lambda_2 \dots d\lambda_r, \end{aligned}$$

and define for $\lambda_1 > 0$ and $\operatorname{Re} s > -1$,

$$G(\lambda_1, s) = \int_{R_+(\lambda_1)} g(\lambda_1, \lambda^0) \Big(\prod_{i=2}^r \lambda_i\Big)^s \prod_{2\leqslant j\leqslant r} (\lambda_j-\lambda_1)^d \omega(\lambda^0)\, d\lambda^0.$$

This integral converges and moreover can be extended continuously to the closed quadrant $\{\lambda_1 \geqslant 0, s \geqslant -1\}$. Moreover, G is a smooth function of both variables up to the boundary, and in particular its value at $(0,-1)$ is

$$G(0,-1) = \int_{R_+(\lambda_1)} g(0,\lambda^0) \Big(\prod_{2\leqslant j\leqslant r} \lambda_j\Big)^{d-1} \omega(\lambda^0)\, d\lambda^0.$$

LEMMA 3. *Let $G(\lambda_1, s)$ be a function on $\{\lambda_1 \geqslant 0, s \geqslant -1\}$, smooth up to the boundary, with compact support in the variable λ_1. Then*

$$\lim_{s\to -1}(s+1)\int_0^{+\infty} G(\lambda_1, s)\lambda_1^s\, d\lambda_1 = G(0,-1).$$

The proof is elementary.

Lemma 2 now is an easy consequence of Lemma 3. Finally, by combining the results of Step 1 and of Step 2, we get Theorem 1.

Now to the measure $d\nu_{\partial\Omega}$ we associate, as explained in Proposition 2, a measure $d\mu_S$ by the formula

$$\int_E f(\xi)\, d\mu_S(\xi) = c_1 \int_{\partial\Omega}\int_\Sigma f(\phi(x^{1/2})\sigma)\, d\sigma\, d\nu_{\partial\Omega}(x). \tag{5}$$

3. The Green function

This section is devoted to constructing the Green function G, which is a sort of substitute for the classical Green function adapted to the context of multi-harmonic functions (see §4). It depends only on $Q(\xi)$, behaves well under the action of the representation ϕ, and is harmonic outside of the singular set S. Being locally integrable near any point of S, it extends as a distribution on E, and the computation of ΔG as a (singular) distribution is of major importance. Some preliminaries are needed.

To state the first result, introduce an orthonormal basis $(a_i)_{1\leqslant i\leqslant n}$ of V. The next proposition gives the expression for the (analogue of) the *radial part* $\mathrm{rad}(\Delta)$ of the Euclidean Laplace operator on E.

PROPOSITION 3. *Let $F\in C^\infty(V)$ and set $f=F\circ Q$. Then, for $\xi\in E$,*

$$\begin{aligned}\Delta f(\xi) &= (\mathrm{rad}(\Delta)F)(Q(\xi))\\ &= 4\sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2 F}{\partial a_i\,\partial a_j}(Q(\xi))(a_ia_j, Q(\xi)) + \frac{2N}{r}\sum_{i=1}^n \frac{\partial F}{\partial a_i}(Q(\xi))(a_i, e).\end{aligned}$$

For the proof, let us introduce an orthonormal basis $(\xi_\alpha)_{1\leqslant\alpha\leqslant N}$ of E. For each i, $1\leqslant i\leqslant n$, set $q_i(\xi)=\langle\phi(a_i)\xi,\xi\rangle$, so that $Q(\xi)=\sum_{i=1}^n q_i(\xi)a_i$. With this notation,

$$\frac{\partial f}{\partial \xi_\alpha} = \sum_{i=1}^n \frac{\partial F}{\partial a_i}(Q(\xi))\frac{\partial q_i}{\partial \xi_\alpha},$$

$$\frac{\partial^2 f}{\partial \xi_\alpha^2} = \sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2 F}{\partial a_i\,\partial a_j}(Q(\xi))\frac{\partial q_i}{\partial \xi_\alpha}\frac{\partial q_j}{\partial \xi_\alpha} + \sum_{i=1}^n \frac{\partial F}{\partial a_i}(Q(\xi))\frac{\partial^2 q_i}{\partial \xi_\alpha^2},$$

so that

$$\sum_{\alpha=1}^{N}\frac{\partial^2 f}{\partial\xi_\alpha^2}=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 F}{\partial a_i\partial a_j}(Q(\xi))\sum_{\alpha=1}^{N}\frac{\partial q_i}{\partial\xi_\alpha}\frac{\partial q_j}{\partial\xi_\alpha}+\sum_{i=1}^{n}\frac{\partial F}{\partial a_i}(Q(\xi))\sum_{\alpha=1}^{N}\frac{\partial^2 q_i}{\partial\xi_\alpha^2}.$$

Now if q is *any* quadratic form on E, expressed as $\langle A\xi,\xi\rangle$ for some symmetric operator A, then $\operatorname{grad} q(\xi)=2A\xi$, and $\Delta q(\xi)=2\operatorname{tr} A$, so that

$$\sum_{\alpha=1}^{N}\frac{\partial q_i}{\partial\xi_\alpha}\frac{\partial q_j}{\partial\xi_\alpha}=4\langle\phi(a_i)\xi,\phi(a_j)\xi\rangle=4(a_ia_j,Q(\xi))$$

and

$$\sum_{\alpha=1}^{N}\frac{\partial^2 q_i}{\partial\xi_\alpha^2}=2\operatorname{tr}\phi(a_i)=\frac{2N}{r}(a_i,e).$$

Proposition 3 follows from these computations.

If the function F is invariant under K, then the value $F(x)$ depends only on the eigenvalues of x, and it is possible to express the result uniquely in terms of these eigenvalues. To be precise, denote by $(\lambda_i)_{1\leqslant i\leqslant r}$ the eigenvalues of $Q(\xi)$.

PROPOSITION 4. *Let F be a C^∞-function on V, invariant under K. Set $f=F\circ Q$. Then*

$$\Delta f(\xi)=4\Bigg(\sum_{i=1}^{r}\lambda_i\frac{\partial^2 F}{\partial\lambda_i^2}+\gamma\sum_{i=1}^{r}\frac{\partial F}{\partial\lambda_i}+\frac{1}{2}d\sum_{\substack{1\leqslant i,j\leqslant r\\ i\neq j}}\frac{1}{\lambda_i-\lambda_j}\left(\lambda_i\frac{\partial F}{\partial\lambda_i}-\lambda_j\frac{\partial F}{\partial\lambda_j}\right)\Bigg),$$

where

$$\gamma=\frac{N}{2r}-\frac{1}{2}d(r-1)=\frac{N}{2r}-\frac{n}{r}+1.$$

For the proof, we use computations originally due to Dib ([Di]), in the simplified version which appeared in [FK]. In the latter reference, the authors introduce, for any complex parameter ν, the differential operator $\mathcal{B}_\nu$ on $C^\infty(V)$ with values in $C^\infty(V,V^{\mathbf{C}})$, defined by

$$\mathcal{B}_\nu g(x)=\sum_{1\leqslant i,j\leqslant n}\frac{\partial^2 g}{\partial a_i\partial a_j}(x)P(a_i,a_j)x+\nu\sum_{1\leqslant i\leqslant n}\frac{\partial g}{\partial a_i}(x)a_i.$$

As $\operatorname{tr}P(a_i,a_j)x=(a_ia_j,x)$, the radial part of the Laplace operator, computed in Proposition 1, is easily seen to be $4\operatorname{tr}\mathcal{B}_\nu$, with $\nu=N/2r$. Our assertion is then just a consequence of Theorem XV.2.7 of [FK].

PROPOSITION 5. *For* $\nu_0 = n/r - N/2r$,

$$\Delta \det Q(\xi)^{\nu_0} = 0, \quad \textit{for all } \xi \in E'.$$

In case $N=2n$ *(which corresponds to* $\nu_0=0$*),* $\Delta \log \det Q(\xi) = 0$ *on* E'.

Using the previous notations, $\det Q(\xi) = \lambda_1 \lambda_2 \dots \lambda_r$, and the result follows by an easy computation.

In what follows, denote by $G(\xi)$ (*Green function*) the function $\det Q(\xi)^{\nu_0}$ when $N \neq 2n$, and $\log \det Q(\xi)$ in case $N=2n$. This function is well defined and smooth on E', locally integrable everywhere in E (and has slow growth at infinity), so it defines a (tempered) distribution. We now wish to compute the distribution ΔG, which by the previous result is supported on S.

THEOREM 2.

$$\Delta G = -4\left(\frac{N}{2r} - \frac{n}{r}\right)\mu_S \quad \textit{if } N \neq 2n, \tag{6}$$

$$\Delta G = -4\mu_S \quad \textit{if } N = 2n. \tag{6'}$$

Proof. Let $\mathcal{I}_\infty$ be the set of functions f on E which can be written as $F(Q(\xi))$, with $F \in \mathcal{C}_c^\infty(V)$. Because Q is a *proper* map from E to V, the functions in $\mathcal{I}_\infty$ are in fact in $\mathcal{C}_c^\infty(E)$. Let $\mathcal{I}_\infty^\perp$ be the orthogonal (in $\mathcal{C}_c^\infty(E)$) of $\mathcal{I}_\infty$ for the inner product $(f,g) = \int_E f(\xi)\overline{g(\xi)}\, d\xi$. As for $\mathcal{I}_c$, one sees that

$$\mathcal{I}_\infty^\perp = \left\{ f \in \mathcal{C}_c^\infty(E) : \int_\Sigma f(\phi(x)\sigma)\, d\sigma = 0 \text{ for all } x \in \Omega \right\}.$$

Now, if f belongs to $\mathcal{I}_\infty$, Δf also belongs to $\mathcal{I}_\infty$, as Proposition 3 shows. As Δ is self-adjoint, $\mathcal{I}_\infty^\perp$ is stable by Δ. Hence $(\Delta G, f) = (G, \Delta f) = 0$ for any function in $\mathcal{I}_\infty^\perp$. Clearly, as $\mathcal{I}_\infty + \mathcal{I}_\infty^\perp$ is dense in $\mathcal{C}_c^\infty(E)$, we only need to compute ΔG against functions in $\mathcal{I}_\infty$. Further, it is possible to take advantage of the action of K. Reflecting the fact that the Laplace operator is invariant under rotations, its radial part $\mathrm{rad}(\Delta)$ commutes with the action of K (more generally, the operator $\operatorname{tr} \mathcal{B}_\nu$ commutes with the action of K as a consequence of [FK, Proposition XV.2.3]). In order to compute the integral $\int_E G(\xi)\Delta f(\xi)\, d\xi$, we may use the expansion along K-types of the function F. For each K-type, the corresponding integral vanishes except for the K-invariant contribution. In other terms, it suffices to determine the value of ΔG against functions of the form $F \circ Q$ where F is a K-invariant function in $\mathcal{C}_c^\infty(V)$.

Let F be such a function. The integral we wish to evaluate is (at least in the case where $N\neq 2n$)

$$\begin{aligned}\int_E G(\xi)\,\Delta(F\circ Q)(\xi)\,d\xi &= c_1\int_\Omega \operatorname{rad}(\Delta)F(x)\,dx\\ &= 4c_0c_1\int_{R_+}\Bigg(\sum_{i=1}^r \lambda_i\frac{\partial^2 F}{\partial\lambda_i^2}+\gamma\sum_{i=1}^r\frac{\partial F}{\partial\lambda_i}+\frac{1}{2}d\sum_{\substack{1\leqslant i,j\leqslant r\\ i\neq j}}\frac{1}{\lambda_i-\lambda_j}\left(\lambda_i\frac{\partial F}{\partial\lambda_i}-\lambda_j\frac{\partial F}{\partial\lambda_j}\right)\Bigg)\\ &\quad\times\prod_{1\leqslant i<j\leqslant r}(\lambda_j-\lambda_i)^d\,d\lambda_1\,d\lambda_2\,\dots\,d\lambda_r.\end{aligned}$$

This can be rewritten as $4c_0c_1\sum_{i=1}^r I_i$, where

$$I_i=\int_{R_+}\left(\lambda_i\frac{\partial^2F}{\partial\lambda_i^2}+\gamma\frac{\partial F}{\partial\lambda_i}+d\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\lambda_i\frac{\partial F}{\partial\lambda_i}\right)\prod_{1\leqslant j<k\leqslant r}(\lambda_k-\lambda_j)^d\,d\lambda.$$

Now, for a fixed index i,

$$\begin{aligned}&\int_{R_+}\lambda_i\frac{\partial^2F}{\partial\lambda_i^2}\prod_{1\leqslant j<k\leqslant r}(\lambda_k-\lambda_j)^d\,d\lambda_1\,d\lambda_2\,\dots\,d\lambda_r\\ &=\int\left(\int_{\lambda_{i-1}}^{\lambda_{i+1}}\lambda_i\frac{\partial^2F}{\partial\lambda_i^2}\prod_{j\neq i}(\lambda_i-\lambda_j)^d\,d\lambda_i\right)\Bigg(\pm\prod_{\substack{k<l\\ k,l\neq i}}(\lambda_l-\lambda_k)^d\Bigg)\,d\lambda_1\,\dots\,\breve{d\lambda_i}\,\dots\,d\lambda_r.\end{aligned}$$

For the integral with respect to λ_i, we use integration by parts to get

$$\int_{\lambda_{i-1}}^{\lambda_{i+1}}\lambda_i\frac{\partial^2F}{\partial\lambda_i^2}\prod_{j\neq i}(\lambda_i-\lambda_j)^d\,d\lambda_i=-\int_{\lambda_{i-1}}^{\lambda_{i+1}}\frac{\partial F}{\partial\lambda_i}\left(\prod_{j\neq i}(\lambda_i-\lambda_j)^d\right)\left(1+d\lambda_i\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\right)d\lambda_i,$$

so that

$$I_i=\int_{R_+}(\gamma-1)\frac{\partial F}{\partial\lambda_i}\prod_{1\leqslant i<j\leqslant r}(\lambda_j-\lambda_i)^d\,d\lambda_1\,d\lambda_2\,\dots\,d\lambda_r.$$

We are left with the integral

$$4(\gamma-1)c_0c_1\int_{R_+}\sum_{i=1}^r\frac{\partial F}{\partial\lambda_i}\prod_{1\leqslant i<j\leqslant r}(\lambda_j-\lambda_i)^d\,d\lambda_1\,d\lambda_2\,\dots\,d\lambda_r.$$

Let $\mathcal{E}$ be the vector field defined by

$$\mathcal{E}F(\lambda)=\frac{d}{dt}F(\lambda+te)\Big|_{t=0}=\sum_{i=1}^r\frac{\partial F}{\partial\lambda_i}(\lambda).$$

Observe that $\mathcal{E}(\prod_{1\leqslant i<j\leqslant r}(\lambda_j-\lambda_i)^d)=0$, so that by an integration by parts there remains only the contribution of the boundary of the domain of integration. The boundary consists of several pieces, a typical one being

$$\{0<\lambda_1<\ldots<\lambda_i=\lambda_{i+1}<\ldots<\lambda_r\}.$$

But the density $\prod_{1\leqslant i<j\leqslant r}(\lambda_j-\lambda_i)^d$ vanishes on this piece of the boundary. Hence, the only remaining piece of the boundary which contributes to the integral is the domain $\{0=\lambda_1<\lambda_2<\ldots<\lambda_r\}=R_+^0$, for which the contribution is easily computed to be

$$-4(\gamma-1)c_0c_1\int_{R_+^0}F(0,\lambda^0)\prod_{2\leqslant j}\lambda_j^d\prod_{2\leqslant i<j\leqslant r}(\lambda_j-\lambda_i)^d\,d\lambda^0.$$

This finishes the proof of the theorem, at least in the case where $N\neq 2n$. If $N=2n$, the proof follows exactly the same pattern. Details are left to the reader.

The previous computation has a consequence that will be needed later.

PROPOSITION 6. *Assume that $N\geqslant 2n$. Then S is a polar set.*

Recall that a set $A\subset E$ is said to be *polar* if every point of A has an open connected neighborhood U such that there is a subharmonic function u, not identically equal to $-\infty$, but equal to $-\infty$ on $A\cap U$ (see [Do] or [Ho, p. 203]). But Theorem 2 implies that the Laplacian of $-G$ (as a distribution) is a positive measure (this is where the assumption $N\geqslant 2n$ is needed), and hence $-G$ is subharmonic. But the set where $-G$ takes the value $-\infty$ is exactly S, so that S is polar as stated.

4. The Kelvin transform

Before defining the Kelvin transform and developing some applications, we first need to recall some definitions and properties of multi-harmonic functions. We use this terminology rather than pluriharmonic used in [KV], or Stiefel harmonic used in [Ge] and [Cl].

An open set $\mathcal{O}$ of E is said to be ϕ-invariant if $\mathcal{O}$ is invariant under all the diffeomorphisms $(\phi(x))_{x\in V^\times}$. In practice, $\mathcal{O}$ will always be E or E'. A smooth function f defined on such a ϕ-invariant open set $\mathcal{O}$ is said to be *multi-harmonic* if, for each $x\in V^\times$, the function $f\circ\phi(x)$ is a harmonic function on $\mathcal{O}$ in the ordinary sense. There are several equivalent characterizations of multi-harmonic functions (see [Cl]).

We also need to recall the notion of ϕ-homogeneity (we slightly modify the usual definition for our purposes). A function f defined on an open ϕ-invariant set $\mathcal{O}$ is said to be ϕ-homogeneous of degree m (m an integer) if

$$f(\phi(x)\xi)=(\det x)^m f(\xi),\quad \text{for all } x\in V^\times,\ \xi\in\mathcal{O}. \tag{7}$$

Clearly, a function which is ϕ-homogeneous (of some degree) is multi-harmonic if and only if it is harmonic.

The *inversion* ι is the transform defined on E' by the formula

$$\iota(\xi) = \phi(Q(\xi)^{-1})\xi. \tag{8}$$

By (2),

$$Q(\iota(\xi)) = P(Q(\xi)^{-1})Q(\xi) = Q(\xi)^{-1},$$

so that $\iota(\iota(\xi)) = \phi(Q(\xi))(\phi(Q(\xi)^{-1})\xi) = \xi$. Hence ι is an involutive diffeomorphism of E'. The inversion satisfies another important property, namely,

$$\iota(\phi(x)\xi) = \phi(x)^{-1}\iota(\xi), \quad \text{for all } x \in V^{\times},\ \xi \in E'. \tag{9}$$

The *Kelvin transform* of a function f defined on E' is given by

$$Kf(\xi) = \det Q(\xi)^{\nu_0} f(\iota(\xi)). \tag{10}$$

THEOREM 3. *Let f be a smooth function on E' which is ϕ-homogeneous of degree m and multi-harmonic. Then Kf is ϕ-homogeneous of degree $2\nu_0 - m$ and multi-harmonic on E'.*

Proof. Thanks to the ϕ-homogeneity,

$$Kf(\xi) = \det Q(\xi)^{\nu_0 - m} f(\xi).$$

This shows that Kf is ϕ-homogeneous of the right degree (use (9)). So, it is enough to prove that $\Delta Kf = 0$ on E'.

We need to recall some elementary facts and prove a few lemmas. First recall the formula for the Laplacian of a product of two arbitrary (smooth) functions u and v on an open subset of E:

$$\Delta(uv) = v\Delta u + 2\langle \nabla u, \nabla v\rangle + u\Delta v. \tag{11}$$

Next, introduce for convenience the following notation: for any complex number ν, denote by P_ν the function on E' defined by

$$P_\nu(\xi) = \det Q(\xi)^{\nu}.$$

LEMMA 4. *For all $\xi \in E'$,*

$$\nabla(\log\det\circ Q)(\xi) = 2\iota(\xi), \tag{12}$$

$$\nabla P_\nu(\xi) = 2\nu P_\nu(\xi)\iota(\xi). \tag{13}$$

As (13) is an immediate consequence of (12), we prove (12). Using notation from §3,

$$\frac{\partial}{\partial\xi_\alpha}\det Q(\xi) = \sum_{j=1}^n \frac{\partial\det}{\partial a_j}(Q(\xi))\frac{\partial q_j}{\partial\xi_\alpha}.$$

Next,

$$\frac{\partial q_j}{\partial\xi_\alpha} = 2\langle\phi(a_j)\xi, \xi_\alpha\rangle \quad \text{and} \quad \frac{\partial\det}{\partial a_j}(x) = \det x\,(x^{-1}, a_j).$$

Hence

$$\det Q(\xi)^{-1}\frac{\partial}{\partial\xi_\alpha}\det Q(\xi) = 2\sum_{j=1}^n (Q(\xi)^{-1}, a_j)\langle\phi(a_j)\xi, \xi_\alpha\rangle$$
$$= 2\left\langle\phi\left(\sum_{j=1}^n (Q(\xi)^{-1}, a_j)a_j\right)\xi, \xi_\alpha\right\rangle = 2\langle\phi(Q(\xi)^{-1}\xi, \xi_\alpha\rangle,$$

and the lemma follows.

LEMMA 5. *Let ν be any real number. Then*

$$\Delta P_\nu(\xi) = 4\nu(\nu-\nu_0)P_\nu(\xi)(Q(\xi)^{-1}, e). \tag{14}$$

This is an easy computation using Proposition 4.

LEMMA 6. *Let f be a smooth function defined on a ϕ-invariant open set $\mathcal{O}$, ϕ-homogeneous of degree m . Then, for any $x \in V^\times$ and any $\xi \in \mathcal{O}$,*

$$\langle\phi(x)\xi, \nabla f(\xi)\rangle = m\operatorname{tr} x\, f(\xi). \tag{15}$$

From the homogeneity property, we get for any small real number t,

$$f(\phi(e+tx)\xi) = \det(e+tx)^m f(\xi).$$

Now differentiate both sides with respect to t, and use the fact that

$$\det(e+tx) = 1 + t\operatorname{tr} x + O(t^2) \quad \text{as } t\to 0$$

to get the result.

We are now ready to complete the proof of Theorem 3. Let $g=Kf$ so that $g(\xi)=P_{\nu_0-m}(\xi)f(\xi)$. Then, thanks to (11),

$$\Delta g=(\Delta P_{\nu_0-m})f+2\langle\nabla P_{\nu_0-m},\nabla f\rangle,$$

which by (13) and (14) gives

$$\begin{aligned}\Delta g(\xi)&=4(\nu_0-m)(\nu_0-m-\nu_0)P_{\nu_0-m}(\xi)(Q(\xi)^{-1},e)f(\xi)\\&\quad+4(\nu_0-m)P_{\nu_0-m}(\xi)\langle\iota(\xi),(\nabla f)(\xi)\rangle.\end{aligned}$$

Now use (15) with $x=Q(\xi)^{-1}$ to get $\langle\iota(\xi),(\nabla f)(\xi)\rangle=m\operatorname{tr}Q(\xi)^{-1}f(\xi)$. Putting all things together gives $\Delta g=0$, which finishes the proof of Theorem 3.

5. Multi-harmonic ϕ-homogeneous polynomials

One of the classical applications of the Kelvin transform is the generation of harmonic polynomials from the Green kernel (see [CH]). We imitate the process, but have to use more refined analytic arguments instead of algebraic computations ([Ko] may also be quoted as a source of inspiration for our results).

If p is any polynomial on E, associate the constant-coefficient differential operator $\partial(p)$ on E characterized by

$$\partial(p)e^{\langle\xi,\cdot\rangle}=p(\xi)e^{\langle\xi,\cdot\rangle},\quad\text{for all }\xi\in E.$$

THEOREM 3. *Assume that $N>2n$. Let m be a nonnegative integer, and let p be a polynomial on E, ϕ-homogeneous of degree m. Then*

$$K(\partial(p)G),$$

originally defined on E', extends to E as a polynomial, is ϕ-homogeneous of degree m, and is multi-harmonic.

In case $N=2n$, the same result is true, provided $m\geqslant 1$.

Let us assume first that $N>2n$. For the homogeneity, observe that if f and g are ϕ-homogeneous of degree k and l respectively, then fg is ϕ-homogeneous of degree $k+l$. If p is a ϕ-homogeneous polynomial of degree m then $\partial(p)f$ is ϕ-homogeneous of degree $k-m$, as is easily checked. Hence the homogeneity of $K(\partial(p)G)$ (at least in E') is clear. As Δ commutes with $\partial(p)$, it is clear that $\partial(p)G$ is harmonic in E'. By Theorem 3, $K(\partial(p)G)$ is harmonic on E'. The last easy observation is that as a consequence of the homogeneity, the function $g=K(\partial(p)G)$ remains locally bounded near any point of S.

By Proposition 6, S is a polar set. Any harmonic function on the complement of a polar set which is locally bounded near any point of the polar set can be continued (uniquely) as a harmonic function on the whole space (see [Do] or [Ho]). Still denote by g this extension. As g is harmonic, g is an analytic function on E. The ϕ-homogeneity of g implies homogeneity of degree mr in the ordinary sense. By considering the Taylor development of g at the origin, we may conclude that g is indeed a polynomial. This finishes the proof of the theorem, for the case $N>2n$.

For the remaining case, we observe that the Green function in this case is *quasi-homogeneous* of degree 0, in the sense that for any $x\in V^{\times}$,

$$G(\phi(x)\xi)=G(\xi)+2\log\det x, \quad \text{for all } \xi\in E',$$

which is a consequence of the formula $\det(P(x)y)=\det x^2\det y$. Now if p is a ϕ-homogeneous polynomial of degree $m\geqslant 1$, then $\partial(p)G$ is still ϕ-homogeneous of degree $-m$, and the rest of the proof is the same.

Denote by M the map that associates to any ϕ-homogeneous polynomial p the extension to E of $K(\partial(p)G)$. The map M produces many multi-harmonic ϕ-homogeneous polynomials. In fact, under appropriate assumptions it generates all multi-harmonic ϕ-homogeneous polynomials. To state the result, we need some more notation and some preliminary results.

Let $\mathcal{P}=\mathcal{P}(E)$ denote the space of all polynomials (with complex coefficients) on E. There is a standard inner product on $\mathcal{P}$ called the *Fischer inner product*, defined by

$$(r,s)_F=\partial(r)\bar{s}(0). \tag{16}$$

With the help of the Fischer inner product, it is possible to reinterpret the notion of multi-harmonic polynomial. We already introduced the polynomials q_j for $1\leqslant j\leqslant n$ as defined by

$$q_j(\xi)=\langle\phi(a_j)\xi,\xi\rangle=(a_j,Q(\xi)).$$

Let $\mathcal{J}=\mathcal{J}(E)$ be the ideal in $\mathcal{P}$ generated by the q_j, $1\leqslant j\leqslant n$. Then the space $\mathcal{H}=\mathcal{H}(E)$ is the orthogonal space of $\mathcal{J}$ in $\mathcal{P}$ (see [C1]).

Denote by $\mathcal{P}^{\det}$ the space generated by the ϕ-homogeneous polynomials of any degree, and let $\mathcal{H}^{\det}$ (resp. $\mathcal{J}^{\det}$) be the intersection of $\mathcal{P}^{\det}$ with $\mathcal{H}$ (resp. $\mathcal{J}$). For $x\in V^{\times}$, the map $p\mapsto p\circ\phi(x)$ maps $\mathcal{H}$ into itself. As it is self-adjoint for the Fischer inner product, it maps also $\mathcal{J}$ into itself. It obviously maps $\mathcal{P}^{\det}$ into itself, and hence $\mathcal{H}^{\det}$ (resp. $\mathcal{J}^{\det}$) into itself. We clearly have

$$\mathcal{P}^{\det}=\mathcal{H}^{\det}\oplus\mathcal{J}^{\det} \tag{17}$$

(orthogonal direct sum).

We also need an assumption already considered in [C1]. Denote by $V_{\mathbf{C}}$ the complexified Jordan algebra of V, and extend the inner product on V as a $\mathbf{C}$-bilinear form on $V_{\mathbf{C}}$. Similarly let $E_{\mathbf{C}}$ be the complexified space of E, and extend the inner product on E to a $\mathbf{C}$-bilinear form on $E_{\mathbf{C}}$. The map H has a $\mathbf{C}$-bilinear extension to $E_{\mathbf{C}}\times E_{\mathbf{C}}$, still denoted by H, and, with the same convention, the map Q is now regarded as a $\mathbf{C}$-quadratic map from $E_{\mathbf{C}}$ into $V_{\mathbf{C}}$. Consider the algebraic set

$$\mathcal{N}=\{\xi\in E_{\mathbf{C}}:Q(\xi)=0\}.$$

An element ξ in $E_{\mathbf{C}}$ is said to be *regular* if the differential of Q at ξ is surjective, which is tantamount to the fact that the map $\eta\mapsto H(\xi,\eta)$ is surjective. Then our assumption, denoted by (H), is:

There exists a regular element in $\mathcal{N}$. (H)

As a consequence, the set $\mathcal{N}'$ of all regular elements in $\mathcal{N}$ is dense in $\mathcal{N}$.

Now we can state our main result.

THEOREM 4. *Assume that the representation ϕ satisfies the condition* (H) *and that $N>4n-2r$. Then the map M is a surjective map from $\mathcal{P}^{\det}$ onto $\mathcal{H}^{\det}$.*

We need several lemmas before attacking the proof of Theorem 4.

LEMMA 7. *Let $p\in\mathcal{P}$ be ϕ-homogeneous of degree k, and let f be a smooth function defined on some open subset of V. Let $\zeta\in E_{\mathbf{C}}$. Then*

$$\partial(p)f(H(\cdot,\zeta))=p(\zeta)(\partial(\det{}^k)f)(H(\cdot,\zeta)).$$

In fact, let $A\colon E_{\mathbf{C}}\to V_{\mathbf{C}}$ be a linear map. Then $\partial(p)(f\circ A)=(\partial(\pi)f)\circ A$, where $\pi=p\circ A^t$. Apply this to the operator A defined by $A\xi=H(\xi,\zeta)$. Then, for $x\in V$, $A^tx=\phi(x)\zeta$, and so $\pi(x)=p(\phi(x)\zeta)=(\det x)^kp(\zeta)$, and the claim follows.

LEMMA 8. *Let k be any positive integer, and ν a complex number. Then*

$$\partial(\det{}^k)\det{}^\nu=b(\nu)b(\nu-1)\dots b(\nu-k+1)\det{}^{\nu-k} \tag{18}$$

with $b(\lambda)=\lambda\left(\lambda+\frac{1}{2}d\right)\dots\left(\lambda+\frac{1}{2}d(r-1)\right)$.

This is an obvious extension of the Bernstein identity for the polynomial $\det x$ (see [FK, Proposition VII.1.4]).

COROLLARY. *Assume condition* (H) *to be satisfied. Suppose also that $N>4n-2r$. Let $k\in\mathbf{N}$. Then*

$$\partial(\det{}^k)\det{}^{\nu_0}\not\equiv 0.$$

Recall that $\nu_0 = n/r - N/2r$, so that

$$b(\nu_0) = \left(\frac{n}{r} - \frac{N}{2r}\right) \cdots \left(\frac{n}{r} - \frac{N}{2r} + \frac{1}{2}d(r-1)\right).$$

The last factor in the product is equal to $(4n-N-2r)/2r$. Hence our condition implies that this factor (as well as the other factors) is <0. Thus $b(\nu_0) \neq 0$, and more generally, $b(\nu_0 - k) \neq 0$ for any k.

PROPOSITION 7. *Assume that* (H) *is satisfied and that* $N > 4n - 2r$. *Let* $p \in \mathcal{P}^{\det}$ *be such that* $\partial(p)G = 0$ *on* E'. *Then* p *vanishes on* $\mathcal{N}$.

Let $p \in \mathcal{P}^{\det}$ be such that $\partial(p)G = 0$. More precisely, assume that p is ϕ-homogeneous of degree k. The function G has an extension to the set $E'_{\mathbf{C}}$, at least locally, because one might have to choose a determination of the square root of $\det Q(\xi)$ in case ν_0 is half an integer. The operator $\partial(p)$ commutes with translations and is homogeneous (in the ordinary sense). Hence, for any $t > 0$ and any $\zeta \in E_{\mathbf{C}}$, the function $G(t\xi + \zeta)$ satisfies

$$\partial(p)(G(t \cdot + \zeta)) = 0$$

where defined. If $\zeta \in \mathcal{N}$, then $Q(t\xi + \zeta) = t^2 Q(\xi) + 2tH(\xi, \zeta)$. If moreover $\zeta \in \mathcal{N}'$, the map $\xi \mapsto H(\xi, \zeta)$ is surjective, so that $\det H(\xi, \zeta) \neq 0$ on a dense open set Ω_ζ of E. Let t tend to 0. Then, on Ω_ζ, $t^{-r\nu_0} G(t\xi + \zeta)$ tends to $\det(2H(\xi, \zeta))^{\nu_0}$, uniformly on any compact subset of Ω_ζ, and the same is true for any partial derivative. So, on Ω_ζ,

$$\partial(p)(\det H(\cdot, \zeta))^{\nu_0} = 0. \tag{19}$$

From Lemma 7 and the corollary to Lemma 8, we see that $p(\zeta) = 0$. But ζ was arbitrary in $\mathcal{N}'$, so p vanishes on $\mathcal{N}'$ and hence on $\mathcal{N}$ by continuity.

Now we are ready for the proof of Theorem 4.

Let us first determine the kernel of the map M. If p belongs to $\mathcal{J}^{\det}$, then $\partial(p)G = 0$, and hence $Mp = 0$. Conversely, let $p \in \mathcal{P}^{\det}$ and assume that it belongs to the kernel of the map M. Then $\partial(p)G = 0$. By Proposition 7, p vanishes on $\mathcal{N}$. But an important result of [C1] is, under the assumption (H), the equivalence (valid for *any* polynomial p on E):

$$p \text{ vanishes on } \mathcal{N} \iff p \in \mathcal{J}.$$

Hence the kernel of the map M is exactly $\mathcal{J}^{\det}$. So the map M induces an injective map from $\mathcal{P}^{\det} \bmod \mathcal{J}^{\det}$ into $\mathcal{H}^{\det}$. But $\mathcal{P}^{\det} \bmod \mathcal{J}^{\det} \simeq \mathcal{H}^{\det}$, and the map M preserves the degree of ϕ-homogeneity. Hence by a dimension count we get the surjectivity of M.

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