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The colored Jones polynomials and the simplicial volume of a knot

by

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In [13], R. M. Kashaev defined a family of complex-valued link invariants indexed by integers $N \geqslant 2$ using the quantum dilogarithm. Later he calculated the asymptotic behavior of his invariant, and observed that for the three simplest hyperbolic knots it grows as $\exp(\mathrm{Vol}(K)N/2\pi)$ when N goes to infinity, where $\mathrm{Vol}(K)$ is the hyperbolic volume of the complement of a knot K [14]. This amazing result and his conjecture that the same also holds for any hyperbolic knot have been almost ignored by mathematicians since his definition of the invariant is too complicated (though it uses only elementary tools).

The aim of this paper is to reveal his mysterious definition and to show that his invariant is nothing but a specialization of the colored Jones polynomial. The colored Jones polynomial is defined for colored links (each component is decorated with an irreducible representation of the Lie algebra $\mathrm{sl}(2, \mathbf{C})$). The original Jones polynomial corresponds to the case that all the colors are identical to the 2-dimensional fundamental representation. We show that Kashaev's invariant with parameter N coincides with the colored Jones polynomial in a certain normalization with every color the N-dimensional representation, evaluated at the primitive Nth root of unity. (We have to normalize the colored Jones polynomial so that the value for the trivial knot is one, for otherwise it always vanishes.)

On the other hand, there are other colored polynomial invariants, such as the generalized multivariable Alexander polynomial defined by Y. Akutsu, T. Deguchi and T. Ohtsuki [1]. They used the same Lie algebra $\mathrm{sl}(2, \mathbf{C})$ but a different hierarchy of representations. Their invariants are parameterized by $c+1$ parameters: an integer N

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and complex numbers p_i $(i=1,2,\ldots,c)$ decorating the components, where c is the number of components of the link. In the case where $N=2$, their invariant coincides with the multivariable Alexander polynomial, and their definition is the same as that of the second author [22]. Using the Akutsu–Deguchi–Ohtsuki invariants we have another coincidence. We will show that if all the colors are $\frac{1}{2}(N-1)$ then the generalized Alexander polynomial is the same as Kashaev's invariant since it coincides with the specialization of the colored Jones polynomial as stated above. Therefore the set of colored Jones polynomials and the set of generalized Alexander polynomials of Akutsu–Deguchi–Ohtsuki intersect at Kashaev's invariants.

The paper is organized as follows. In the first section we recall the definition of the link invariant defined by Yang–Baxter operators. In §2 we show that the Akutsu–Deguchi–Ohtsuki invariant coincides with the colored Jones polynomial when the colors are all $\frac{1}{2}(N-1)$ by showing that their representation becomes the usual representation corresponding to the irreducible N-dimensional representation of $\mathrm{sl}(2,\mathbf{C})$. In §3 we show that if we transform the $\breve{R}$-matrix used in the colored Jones polynomial by a Vandermonde matrix then it has a form very similar to Kashaev's $\breve{R}$-matrix. In fact, it is proved in §4 that these two $\breve{R}$-matrices differ only by a constant. We also confirm the well-definedness of Kashaev's invariant by using this fact.

In the final section we propose our "dröm i Djursholm". We use M. Gromov's simplicial volume for a knot to generalize Kashaev's conjecture. Observing that the simplicial volume is additive and unchanged by mutation, we conjecture that Kashaev's invariants (= specializations of the colored Jones polynomial = specializations of the Akutsu–Deguchi–Ohtsuki invariant) determine the simplicial volume. If our dream comes true, then we can show that a knot is trivial if and only if all of its Vassiliev invariants are trivial.

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1. Preliminaries

In this section we recall the definitions of Yang–Baxter operators and associated link invariants. If an invertible linear map $R: \mathbf{C}^N \otimes \mathbf{C}^N \to \mathbf{C}^N \otimes \mathbf{C}^N$ satisfies the following Yang–Baxter equation, it is called a Yang–Baxter operator:

$$(R\otimes \mathrm{id})(\mathrm{id}\otimes R)(R\otimes \mathrm{id}) = (\mathrm{id}\otimes R)(R\otimes \mathrm{id})(\mathrm{id}\otimes R),$$

where $\mathrm{id}: \mathbf{C}^N \to \mathbf{C}^N$ is the identity. If there exists a homomorphism $\mu: \mathbf{C}^N \to \mathbf{C}^N$ and scalars α, β satisfying the following two equations, the quadruple $S=(R,\mu,\alpha,\beta)$ is called an enhanced Yang–Baxter operator [27]:

$$(\mu\otimes\mu)R = R(\mu\otimes\mu),$$
$$\mathrm{Sp}_2(R^{\pm 1}(\mathrm{id}\otimes\mu)) = \alpha^{\pm 1}\beta\,\mathrm{id},$$

where $\mathrm{Sp}_k: \mathrm{End}(\mathbf{C}^{\otimes k}) \to \mathrm{End}(\mathbf{C}^{\otimes k-1})$ is the operator trace defined as

$$\mathrm{Sp}_k(f)(v_{i_1}\otimes v_{i_2}\otimes \ldots \otimes v_{i_{k-1}}) = \sum_{j_1,j_2,\ldots,j_{k-1},j=0}^{N-1} f^{j_1,j_2,\ldots,j_{k-1},j}_{i_1,i_2,\ldots,i_{k-1},j}(v_{j_1}\otimes v_{j_2}\otimes \ldots \otimes v_{j_{k-1}}),$$

where

$$f(v_{i_1}\otimes v_{i_2}\otimes \ldots \otimes v_{i_k}) = \sum_{j_1,j_2,\ldots,j_k=0}^{N-1} f^{j_1,j_2,\ldots,j_k}_{i_1,i_2,\ldots,i_k}(v_{j_1}\otimes v_{j_2}\otimes \ldots \otimes v_{j_k})$$

for a basis $\{v_0, v_1, \ldots, v_{N-1}\}$ of $\mathbf{C}^N$.

For an enhanced Yang–Baxter operator one can define a link invariant as follows [27]. First we represent a given link L as the closure of a braid ξ with n strings. Consider the n-fold tensor product $(\mathbf{C}^N)^{\otimes n}$, and associate the homomorphism $b_R(B): (\mathbf{C}^N)^{\otimes n} \to (\mathbf{C}^N)^{\otimes n}$ by replacing $\sigma_i^{\pm 1}$ (the usual ith generator of the braid group) in ξ with

$$\underbrace{\mathrm{id}\otimes \ldots \otimes \mathrm{id}}_{i-1} \otimes R^{\pm 1} \otimes \underbrace{\mathrm{id}\otimes \ldots \otimes \mathrm{id}}_{n-i-1}.$$

Then taking the operator trace n times we define

$$T_S(\xi) = \alpha^{-w(\xi)}\beta^{-n}\mathrm{Sp}_1(\mathrm{Sp}_2(\ldots(\mathrm{Sp}_n(b_R(\xi)\mu^{\otimes n})))),$$

where $w(\xi)$ is the sum of the exponents. Then $T_S(\xi)$ defines a link invariant, and we denote it by $T_S(L)$.

To define the (generalized) Alexander polynomial from an enhanced Yang–Baxter operator we have to be more careful, since T_S always vanishes in this case. If the homomorphism

$$T_{S,1}(\xi) = \alpha^{-w(\xi)}\beta^{-n}\mathrm{Sp}_2(\mathrm{Sp}_3(\ldots(\mathrm{Sp}_n(b_S(\xi)(\mathrm{id}\otimes\mu^{\otimes(n-1)}))))) \in \mathrm{End}(\mathbf{C}^N)$$

is a scalar multiple of id and

$$\mathrm{Sp}_1(\mu)T_{S,1}(\xi)=T_S(\xi)$$

for any ξ, then the scalar defined by $T_{S,1}(\xi)$ becomes a link invariant (even if $\mathrm{Sp}_1(\mu)=0$) and is denoted by $T_{S,1}(L)$. Note that this invariant can be regarded as an invariant of $(1,1)$-tangles, where a $(1,1)$-tangle is a link minus an open interval.

2. The intersection of the generalized Alexander polynomials and the colored Jones polynomials

In [1] Akutsu, Deguchi and Ohtsuki defined a generalization of the multivariable Alexander polynomial for colored links. First we will briefly describe their construction only for the case where all the colors are the same according to [6].

Fix an integer $N\geqslant 2$ and a complex number p. Put $s=\exp(\pi\sqrt{-1}/N)$ and $[k]=(s^k-s^{-k})/(s-s^{-1})$ for a complex number k. Note that $[N]=0$ and $[N-k]=[k]$.

Let $U_q(\mathrm{sl}(2,\mathbf{C}))$ be the quantum group generated by X,Y,K with the relations

$$KX=sXK,\quad KY=s^{-1}YK,\quad XY-YX=\frac{K^2-K^{-2}}{s-s^{-1}}.$$

Let $F(p)$ be the N-dimensional vector space over $\mathbf{C}$ with basis $\{f_0,f_1,\dots,f_{N-1}\}$. We give an action of $U_q(\mathrm{sl}(2,\mathbf{C}))$ on $F(p)$ by

$$\begin{aligned}X(f_i)&=\sqrt{[2p-i+1][i]}\,f_{i-1},\\ Y(f_i)&=\sqrt{[2p-i][i+1]}\,f_{i+1},\\ K(f_i)&=s^{p-i}f_i.\end{aligned}$$

Using Drinfel'd's universal R-matrix given in [7], we can define a set of enhanced Yang–Baxter operators $S_A(p)$ with complex parameter p. Then Akutsu–Deguchi–Ohtsuki's generalized Alexander polynomial is defined to be $T_{S_A(p),1}$ by using the notation in the previous section. We denote it by $\Phi_N(L,p)$ for a link L. Note that if $N=2$ the invariant $\Phi_2(L,p)$ is the same as the multivariable Alexander polynomial [22].

Next we review the colored Jones polynomial at the Nth root of unity. There is another N-dimensional representation of $U_q(\mathrm{sl}(2,\mathbf{C}))$, corresponding to the usual N-dimensional irreducible representation of $\mathrm{sl}(2,\mathbf{C})$. Let E be the N-dimensional complex vector space with basis $\{e_0,e_1,\dots,e_{N-1}\}$, and define the action of $U_q(\mathrm{sl}(2,\mathbf{C}))$ by

$$\begin{aligned}X(e_i)&=[i+1]e_{i+1},\\ Y(e_i)&=[i]e_{i-1},\\ K(e_i)&=s^{i-(N-1)/2}e_i.\end{aligned}$$

(See for example [18, (2.8)].) By using Drinfel'd's universal R-matrix again we have another enhanced Yang–Baxter operator S_J. Then the invariant $T_{S_J,1}$ coincides with the colored Jones polynomial of a link each of whose component is decorated by the N-dimensional irreducible representation evaluated at $t=s^2=\exp(2\pi\sqrt{-1}/N)$. It is clear from the actions of X and Y that the representation remains irreducible after the evaluation. Note that before evaluating at s^2, we have to normalize the colored Jones polynomial so that its value of the trivial knot is one, for otherwise the invariant would be identically zero. This is well-defined since the colored Jones polynomial defines a well-defined $(1,1)$-tangle invariant ([18, Lemma (3.9)]). We will denote $T_{S_J,1}$ by J_N.

Now we put $p=\frac{1}{2}(N-1)$ in the Akutsu–Deguchi–Ohtsuki invariant. Then since $[N-k]=[k]$, we have

$$\begin{aligned} X(f_i) &= [i] f_{i-1}, \\ Y(f_i) &= [i+1] f_{i+1}, \\ K(f_i) &= s^{(N-1)/2-i} f_i, \end{aligned}$$

and so the two representations $F\left(\frac{1}{2}(N-1)\right)$ and E are quite similar. In fact, if we exchange X and Y, and replace K with K^{-1}, then these two coincide. (This automorphism is known as the Cartan automorphism [16, Lemma VI.1.2].) Therefore they determine the same Yang–Baxter operator and the same link invariant, that is, we have the following theorem.

THEOREM 2.1. *The Akutsu–Deguchi–Ohtsuki invariant with all the colors* $p=\frac{1}{2}(N-1)$ *coincides with the colored Jones polynomial corresponding to the* N*-dimensional irreducible representation evaluated at* $\exp(2\pi\sqrt{-1}/N)$. *More precisely, we have*

$$\Phi_N\left(L, \tfrac{1}{2}(N-1)\right) = J_N(L)$$

for every link L.

Remark 2.2. After finishing this work we were informed by Deguchi that it was already observed in [5] that the R-matrices given by $F\left(\frac{1}{2}(N-1)\right)$ and E coincide.

3. $\check{R}$-matrix for the colored Jones polynomial at the Nth root of unity

Let R_J be the $\check{R}$-matrix shown in [18, Corollary (2.32)], which is the $(N^2 \times N^2)$-matrix with $((i,j),(k,l))$th entry

$$(R_J)^{ij}_{kl} = \sum_{n=0}^{\min(N-1-i,j)} \delta_{l,i+n}\delta_{k,j-n} \frac{(s-s^{-1})^n}{[n]!} \frac{[i+n]!}{[i]!} \frac{[N-1+n-j]!}{[N-1-j]!} \times s^{2(i-(N-1)/2)(j-(N-1)/2)-n(i-j)-n(n+1)/2},$$

where $[k]!=[k][k-1]\dots[2][1]$. Note that our matrix R_J corresponds to $\breve{R}$ in [18, Definition (2.35)]. This matrix is used to define an enhanced Yang–Baxter operator and the link invariant J_N described in the previous section.

The aim of this section is to transform it to a matrix similar to Kashaev's $\breve{R}$-matrix.

Since R_J corresponds to the intersection of two link invariants, the colored Jones polynomials and the generalized Alexander polynomials, the authors have been looking for its special property. On the other hand, Kashaev's $\breve{R}$-matrix and R_J have the same Jordan canonical form for $N=2$ and 3. So we expected that in fact they are congruent for any N after several calculations using Maple V.

Let W and D be the $(N\times N)$-matrices with (i,j)th entries $W^i_j=s^{2ij}$ and $D^i_j=\delta_{i,j}s^{(N-1)i}$ respectively, where $\delta_{i,j}$ is Kronecker's delta. We will calculate the product $\widetilde{R}_J=(W\otimes W)(\mathrm{id}\otimes D)R_J(\mathrm{id}\otimes D^{-1})(W^{-1}\otimes W^{-1})$, with id the identity $(N\times N)$-matrix, and show the following proposition.

PROPOSITION 3.1.

$$(\widetilde{R}_J)^{cd}_{ab}=\begin{cases}\varrho(a,b,c,d)(-1)^{a+b+1}\dfrac{[d-c-1]!\,[N-1+c-a]!}{[d-b]!\,[b-a-1]!} & \text{if } d\geqslant b>a\geqslant c,\\[2mm] \varrho(a,b,c,d)(-1)^{a+c}\dfrac{[b-d-1]!\,[N-1+c-a]!}{[c-d]!\,[b-a-1]!} & \text{if } b>a\geqslant c\geqslant d,\\[2mm] \varrho(a,b,c,d)(-1)^{b+d}\dfrac{[N-1+b-d]!\,[c-a-1]!}{[c-d]!\,[b-a-1]!} & \text{if } c\geqslant d\geqslant b>a,\\[2mm] \varrho(a,b,c,d)(-1)^{c+d}\dfrac{[N-1+b-d]!\,[a-b]!}{[c-d]!\,[a-c]!} & \text{if } a\geqslant c\geqslant d\geqslant b,\\[2mm] 0 & \text{otherwise,}\end{cases}$$

where $\varrho(a,b,c,d)=s^{-N^2/2+1/2+c+d-2b+(a-d)(c-b)}[N-1]!\,(s-s^{-1})^{2(N-1)}/N^2$.

We will prepare a lemma to prove the proposition.

For integers α and β with $0\leqslant\alpha\leqslant N-1$ and $\alpha+\beta$ even, put

$$S(\alpha,\beta)=\sum_{i=0}^{N-1}s^{\beta i}\begin{bmatrix}\alpha+i\\ i\end{bmatrix},$$

where

$$\begin{bmatrix}x\\ y\end{bmatrix}=\frac{[x]!}{[y]!\,[x-y]!}.$$

Note that the summation in $S(\alpha,\beta)$ is essentially from 0 to $N-1-\alpha$. Then we have

LEMMA 3.2.

$$S(\alpha,\beta)=s^{-(N-1-\alpha)(N-\alpha+2\,\mathrm{res}((\alpha-\beta)/2))/2}(s-s^{-1})^{N-1-\alpha}\frac{[N-1-\alpha+\mathrm{res}((\alpha-\beta)/2)]!}{[\mathrm{res}((\alpha-\beta)/2)]!},$$

where $\mathrm{res}(x)\in\{0,1,2,\dots,N-1\}$ *is the residue modulo* N.

Proof. We will show

$$S(\alpha,\beta)=\prod_{j=1}^{N-\alpha-1}(1-s^{\beta-\alpha-2j})=(1-s^{\beta-\alpha-2})(1-s^{\beta-\alpha-4})\ldots(1-s^{\beta+\alpha-2N+2}).$$

Then the required formula follows immediately.

We use the quantized Pascal relation

$$\begin{bmatrix}\alpha+i\\ i\end{bmatrix}=s^{-\alpha}\begin{bmatrix}\alpha+i-1\\ i-1\end{bmatrix}+s^{i}\begin{bmatrix}\alpha+i-1\\ i\end{bmatrix}.$$

Then since

$$S(\alpha,\beta)=s^{-\alpha}\sum_{i=0}^{N-1}s^{\beta i}\begin{bmatrix}\alpha+i-1\\ i-1\end{bmatrix}+\sum_{i=0}^{N-1}s^{(\beta+1)i}\begin{bmatrix}\alpha+i-1\\ i\end{bmatrix}$$

(putting $k=i-1$ in the first term)

$$=s^{\beta-\alpha}\sum_{k=-1}^{N-2}s^{k}\begin{bmatrix}\alpha+k\\ k\end{bmatrix}+S(\alpha-1,\beta+1)=s^{\beta-\alpha}S(\alpha,\beta)+S(\alpha-1,\beta+1),$$

we have the recursive formula

$$S(\alpha-1,\beta+1)=(1-s^{\beta-\alpha})S(\alpha,\beta).$$

Now the required formula follows since $S(N-1,\gamma)=1$ for any integer γ. □

Proof of Proposition 3.1. Since $(W\otimes W)^{ef}_{ab}=s^{2ae+2bf}$, $(\mathrm{id}\otimes D)^{kl}_{ef}=\delta_{e,k}\delta_{f,l}s^{(N-1)l}$, $(\mathrm{id}\otimes D^{-1})^{gh}_{ij}=\delta_{g,i}\delta_{h,j}s^{-(N-1)j}$, $(W^{-1}\otimes W^{-1})^{cd}_{gh}=s^{-2cg-2dh}/N^2$, we have

$$\begin{aligned}N^2(\widetilde{R}_J)^{cd}_{ab}=&\sum_{i,j,k,l,e,f,g,h=0}^{N-1}\ \sum_{n=0}^{\min(N-1-i,j)}\delta_{e,k}\delta_{f,l}\delta_{g,i}\delta_{h,j}\delta_{l,i+n}\delta_{k,j-n}\\ &\quad\times s^{2ae+2bf-2cg-2dh+(N-1)(l-j)}\frac{(s-s^{-1})^n}{[n]!}\,\frac{[i+n]!}{[i]!}\,\frac{[N-1+n-j]!}{[N-1-j]!}\\ &\quad\times s^{2(i-(N-1)/2)(j-(N-1)/2)-n(i-j)-n(n+1)/2}\\ =&\sum_{i,j=0}^{N-1}\ \sum_{n=0}^{\min(N-1-i,j)}s^{(N-1)^2/2}s^{(2b-2a+N)n-n^2/2-3n/2+(2a-2d+n+2)j+(2b-2c+2j-n)i}\\ &\quad\times(s-s^{-1})^n\frac{[N-1+n-j]!}{[N-1-j]!}\begin{bmatrix}n+i\\ i\end{bmatrix}.\end{aligned}$$

Since the summation $\sum_{i,j=0}^{N-1}\sum_{n=0}^{\min(N-1-i,j)}$ is the same as $\sum_{n=0}^{N-1}\sum_{j=n}^{N-1}\sum_{i=0}^{N-1-n}$, we have

$$N^2(\widetilde{R}_J)_{ab}^{cd} = s^{(N-1)^2/2}\sum_{n=0}^{N-1}\sum_{j=n}^{N-1} s^{(2b-2a+N)n-n^2/2-3n/2+(2a-2d+n+2)j}(s-s^{-1})^n \times \frac{[N-1+n-j]!}{[N-1-j]!}S(n,2(b-c+j)-n)$$

with

$$S(\alpha,\beta)=\sum_{i=0}^{N-1}s^{\beta i}\begin{bmatrix}\alpha+i\\ i\end{bmatrix}.$$

Replacing $j-n$ with k, the summation turns out to be $\sum_{k=0}^{N-1}\sum_{n=0}^{N-1-k}$, and we have

$$N^2(\widetilde{R}_J)_{ab}^{cd} = s^{(N-1)^2/2}\sum_{k=0}^{N-1}s^{2(a-d+1)k}[N-1-k]!\,X(k),$$

where

$$X(k)=\sum_{n=0}^{N-1-k}(-1)^n s^{2(b-d)n+kn+n(n+1)/2}\frac{(s-s^{-1})^n}{[N-1-k-n]!}S(n,2(b-c+k)+n).$$

From Lemma 3.2, we have

$$X(k)=(s-s^{-1})^{N-1}s^{-N(N-1)/2-\mathrm{res}(c-b-k)(N-1)} \times \sum_{n=0}^{N-1-k}s^{(2b-2d+k+\mathrm{res}(c-b-k))n}\frac{[N-1-n+\mathrm{res}(c-b-k)]!}{[N-1-k-n]!\,[\mathrm{res}(c-b-k)]!}.$$

Putting $i=N-1-k-n$ we have

$$\begin{aligned}X(k) &= (s-s^{-1})^{N-1}s^{-N(N-1)/2+2(b-d)(N-1)-k^2-k\,\mathrm{res}(c-b-k)+(N-1-2b+2d)k}\\ &\quad\times\frac{[k+\mathrm{res}(c-b-k)]!}{[\mathrm{res}(c-b-k)]!}S(k+\mathrm{res}(c-b-k),2d-2b-k-\mathrm{res}(c-b-k))\\ &=(s-s^{-1})^{N-1}s^{-N(N-1)/2+2(b-d)(N-1)-k^2-k\,\mathrm{res}(c-b-k)+(N-1-2b+2d)k}\\ &\quad\times\frac{[k+\mathrm{res}(c-b-k)]!}{[\mathrm{res}(c-b-k)]!}\\ &\quad\times s^{-(N-1-k-\mathrm{res}(c-b-k))(N-k-\mathrm{res}(c-b-k)+2\,\mathrm{res}(k+b-d+\mathrm{res}(c-b-k)))/2}\\ &\quad\times(s-s^{-1})^{N-1-k-\mathrm{res}(c-b-k)}\\ &\quad\times\frac{[N-1-k-\mathrm{res}(c-b-k)+\mathrm{res}(k-d+b+\mathrm{res}(c-b-k))]!}{[\mathrm{res}(k-d+b+\mathrm{res}(c-b-k))]!}.\end{aligned}$$

Therefore $X(k)$ vanishes unless $\mathrm{res}(c-b-k)+k=\mathrm{res}(c-b)$, and in this case

$$\begin{aligned} X(k) = & (s-s^{-1})^{2(N-1)-\mathrm{res}(c-b)}(-1)^{\mathrm{res}(c-b)+\mathrm{res}(c-d)+1} \\ & \times s^{-N^2-2b+2d+(\mathrm{res}(c-b)+1)(-\mathrm{res}(c-b)+2\,\mathrm{res}(c-d))/2} \\ & \times \frac{[N-1-\mathrm{res}(c-b)+\mathrm{res}(c-d)]!}{[\mathrm{res}(c-d)]!} \\ & \times (-1)^k s^{(2d-2b-\mathrm{res}(c-b)-1)k} \frac{[\mathrm{res}(c-b)]!}{[\mathrm{res}(c-b)-k]!}, \end{aligned}$$

noting that $\mathrm{res}(x+\mathrm{res}(y))=\mathrm{res}(x+y)$ for any x and y.

Then we have

$$\begin{aligned} N^2(\widetilde{R}_J)^{cd}_{ab} = & s^{(N-1)^2/2} \sum_{k=0}^{\mathrm{res}(c-b)} s^{2(a-d+1)k}[N-1-k]! \\ & \times (s-s^{-1})^{2(N-1)-\mathrm{res}(c-b)}(-1)^{\mathrm{res}(c-b)+\mathrm{res}(c-d)+1} \\ & \times s^{-N^2-2b+2d+\{\mathrm{res}(c-b)+1\}\{-\mathrm{res}(c-b)+2\,\mathrm{res}(c-d)\}/2} \\ & \times \frac{[N-1-\mathrm{res}(c-b)+\mathrm{res}(c-d)]!}{[\mathrm{res}(c-d)]!} \\ & \times (-1)^k s^{\{2d-2b-\mathrm{res}(c-b)-1\}k} \frac{[\mathrm{res}(c-b)]!}{[\mathrm{res}(c-b)-k]!} \\ = & (-1)^{\mathrm{res}(c-b)+\mathrm{res}(c-d)} \\ & \times s^{-N^2/2+1/2-2b+2d+\{\mathrm{res}(c-b)+1\}\{-\mathrm{res}(c-b)+2\,\mathrm{res}(c-d)\}/2} \\ & \times (s-s^{-1})^{2(N-1)-\mathrm{res}(c-b)} \frac{[N-1-\mathrm{res}(c-b)+\mathrm{res}(c-d)]!}{[\mathrm{res}(c-d)]!}[N-1]! \\ & \times \sum_{k=0}^{\mathrm{res}(c-b)} s^{\{2a-2b-\mathrm{res}(c-b)+1+N\}k} \begin{bmatrix} N-1-\mathrm{res}(c-b)+k \\ k \end{bmatrix} \\ = & (-1)^{\mathrm{res}(c-b)+\mathrm{res}(c-d)} \\ & \times s^{-N^2/2+1/2-2b+2d+\{\mathrm{res}(c-b)+1\}\{-\mathrm{res}(c-b)+2\,\mathrm{res}(c-d)\}/2} \\ & \times (s-s^{-1})^{2(N-1)-\mathrm{res}(c-b)} \frac{[N-1-\mathrm{res}(c-b)+\mathrm{res}(c-d)]!}{[\mathrm{res}(c-d)]!}[N-1]! \\ & \times S(N-1-\mathrm{res}(c-b), N+1+2a-2b-\mathrm{res}(c-b)) \\ = & (-1)^{\mathrm{res}(c-b)+\mathrm{res}(c-d)} \\ & \times s^{-N^2/2+1/2-2b+2d+\mathrm{res}(c-d)\{\mathrm{res}(c-b)+1\}-\mathrm{res}(c-b)\{\mathrm{res}(c-b)+\mathrm{res}(b-a-1)+1\}} \\ & \times (s-s^{-1})^{2(N-1)}[N-1]! \\ & \times \frac{[\mathrm{res}(c-b)+\mathrm{res}(b-a-1)]!\,[N-1-\mathrm{res}(c-b)+\mathrm{res}(c-d)]!}{[\mathrm{res}(c-d)]!\,[\mathrm{res}(b-a-1)]!}. \end{aligned}$$

Now suppose that $c \geqslant b$. In this case $\operatorname{res}(c-b)=c-b$, and so

$$\begin{aligned}\frac{N^2(\widetilde{R}_J)^{cd}_{ab}}{\varrho(a,b,c,d)} &= (-1)^{c-b+\operatorname{res}(c-d)} \\ &\quad \times s^{-c+d-(a-d)(c-b)+\operatorname{res}(c-d)(c-b+1)-(c-b)\{c-b+\operatorname{res}(b-a-1)+1\}} \\ &\quad \times \frac{[c-b+\operatorname{res}(b-a-1)]!\,[N-1-c+b+\operatorname{res}(c-d)]!}{[\operatorname{res}(c-d)]!\,[\operatorname{res}(b-a-1)]!}.\end{aligned}$$

If $c-d<0$ then $[N-1-c+b+\operatorname{res}(c-d)]!$ vanishes, and so we will assume $c \geqslant d$. Then

$$\frac{N^2(\widetilde{R}_J)^{cd}_{ab}}{\varrho(a,b,c,d)} = (-1)^{b+d} s^{(c-b)\{b-a-1-\operatorname{res}(b-a-1)\}} \frac{[c-b+\operatorname{res}(b-a-1)]!\,[N-1+b-d]!}{[c-d]!\,[\operatorname{res}(b-a-1)]!},$$

and the conclusion follows immediately in this case noting that this vanishes unless $d \geqslant b$.

Next we assume that $b>c$. In this case $\operatorname{res}(c-b)=N+c-b$, and so

$$\begin{aligned}\frac{N^2(\widetilde{R}_J)^{cd}_{ab}}{\varrho(a,b,c,d)} &= (-1)^{N+c-b+\operatorname{res}(c-d)} \\ &\quad \times s^{-c+d-(a-d)(c-b)+\operatorname{res}(c-d)(N+c-b+1)-(N+c-b)\{N+c-b+\operatorname{res}(b-a-1)+1\}} \\ &\quad \times \frac{[N+c-b+\operatorname{res}(b-a-1)]!\,[-1-c+b+\operatorname{res}(c-d)]!}{[\operatorname{res}(c-d)]!\,[\operatorname{res}(b-a-1)]!}.\end{aligned}$$

This vanishes unless $b-a-1 \geqslant 0$, and so we assume $b-a-1 \geqslant 0$. Then

$$\frac{N^2(\widetilde{R}_J)^{cd}_{ab}}{\varrho(a,b,c,d)} = (-1)^{a+c} s^{(c-b+1)\{d-c+\operatorname{res}(c-d)\}} \frac{[N+c-a-1]!\,[-1-c+b+\operatorname{res}(c-d)]!}{[\operatorname{res}(c-d)]!\,[b-a-1]!}.$$

Then the conclusion follows immediately noting that this vanishes unless $a \geqslant c$.

This completes the proof. $\square$

4. Kashaev's $\breve{R}$-matrix and his invariant

In this section we will calculate Kashaev's $\breve{R}$-matrix given in [13], and prove that it coincides with the matrix $\widetilde{R}_J$ up to a constant given in the previous section.

We prepare notations following [13]. Fix an integer $N \geqslant 2$. Put $(x)_n = \prod_{i=1}^{n}(1-x^i)$ for $n \geqslant 0$. Define $\theta\colon \mathbf{Z} \to \{0,1\}$ by

$$\theta(n) = \begin{cases} 1 & \text{if } N > n \geqslant 0, \\ 0 & \text{otherwise.}\end{cases}$$

For an integer x, we denote by $\operatorname{res}(x) \in \{0,1,2,\dots,N-1\}$ the residue modulo N.

Now Kashaev's $\breve{R}$-matrix R_K is given by

$$(R_K)_{ab}^{cd} = Nq^{1+c-b+(a-d)(c-b)} \frac{\theta(\mathrm{res}(b-a-1)+\mathrm{res}(c-d))\theta(\mathrm{res}(a-c)+\mathrm{res}(d-b))}{(q)_{\mathrm{res}(b-a-1)}(q^{-1})_{\mathrm{res}(a-c)}(q)_{\mathrm{res}(c-d)}(q^{-1})_{\mathrm{res}(d-b)}}$$

with $q=s^2$. Note that we are using $P\circ R$ with R defined in [13, 2.12] rather than R itself, where P is the homomorphism from $\mathbf{C}^N\otimes\mathbf{C}^N$ to $\mathbf{C}^N\otimes\mathbf{C}^N$ sending $x\otimes y$ to $y\otimes x$.

We will show the following proposition.

PROPOSITION 4.1.

$$(R_K)_{ab}^{cd} = \begin{cases} \lambda(a,b,c,d)(-1)^{a+b+1}\dfrac{[d-c-1]!\,[N-1+c-a]!}{[d-b]!\,[b-a-1]!} & \textit{if } d\geqslant b>a\geqslant c,\\ \lambda(a,b,c,d)(-1)^{a+c}\dfrac{[b-d-1]!\,[N-1+c-a]!}{[c-d]!\,[b-a-1]!} & \textit{if } b>a\geqslant c\geqslant d,\\ \lambda(a,b,c,d)(-1)^{b+d}\dfrac{[N-1+b-d]!\,[c-a-1]!}{[c-d]!\,[b-a-1]!} & \textit{if } c\geqslant d\geqslant b>a,\\ \lambda(a,b,c,d)(-1)^{c+d}\dfrac{[N-1+b-d]!\,[a-b]!}{[c-d]!\,[a-c]!} & \textit{if } a\geqslant c\geqslant d\geqslant b,\\ 0 & \textit{otherwise,} \end{cases}$$

where $\lambda(a,b,c,d)=s^{N^2/2-N/2+2+c+d-2b+(a-d)(c-b)}(s-s^{-1})^{1-N}N/([N-1]!)^2$.

Proof. From the definitions of θ and res, there are 16 cases to be considered according to the signs of $b-a-1$, $c-d$, $a-c$ and $d-b$:

(A0) $b-a-1\geqslant 0$, (A1) $0>b-a-1$,
(B0) $c-d\geqslant 0$, (B1) $0>c-d$,
(C0) $a-c\geqslant 0$, (C1) $0>a-c$,
(D0) $d-b\geqslant 0$, (D1) $0>d-b$.

For the case (Ai) & (Bj) & (Ck) & (Dl), we see that

$$\mathrm{res}(b-a-1)+\mathrm{res}(c-d)+\mathrm{res}(a-c)+\mathrm{res}(d-b)=(i+j+k+l)N-1.$$

Therefore if $i+j+k+l\geqslant 2$, then $(R_K)_{ab}^{cd}$ vanishes since if two integers x and y satisfies $x+y\geqslant 2N-1$, then one of them is bigger than N. The case (A0) & (B0) & (C0) & (D0) is empty, since we have $b>a\geqslant c\geqslant d$ from (A0), (B0) and (C0), which contradicts (D0).

Therefore we see that $(R_K)_{ab}^{cd}$ vanishes except for the following four cases, which have already appeared in Proposition 3.1: (i) $d\geqslant b>a\geqslant c$, (ii) $b>a\geqslant c\geqslant d$, (iii) $c\geqslant d\geqslant b>a$ and (iv) $a\geqslant c\geqslant d\geqslant b$.

We will only prove the first case because the other cases are similar. Noting that

$$(q)_n = (-1)^n s^{n(n+1)/2}(s-s^{-1})^n [n]!,$$
$$(q^{-1})_n = s^{-n(n+1)/2}(s-s^{-1})^n [n]!,$$

we have

$$(R_K)_{ab}^{cd} = (-1)^{a+b+1} N s^{2+c-2b+d+(a-d)(c-b)+N^2/2-N/2}(s-s^{-1})^{-N+1} \times \frac{1}{[d-b]!\,[b-a-1]!\,[N+c-d]!\,[a-c]!}$$

since $\operatorname{res}(b-a-1)=b-a-1$, $\operatorname{res}(a-c)=a-c$, $\operatorname{res}(c-d)=N+c-d$ and $\operatorname{res}(d-b)=d-b$. Now since $[N-n]=[n]$, we see that

$$\frac{1}{[d-b]!\,[b-a-1]!\,[N+c-d]!\,[a-c]!} = \frac{1}{([N-1])^2}\,\frac{[d-c-1]!\,[N+c-a-1]!}{[b-a-1]!\,[d-b]!}.$$

Therefore

$$(R_K)_{ab}^{cd} = \lambda(a,b,c,d)(-1)^{a+b+1}\frac{[d-c-1]!\,[N-1+c-a]!}{[d-b]!\,[b-a-1]!}$$

as required. □

Therefore R_K and R_J are equal up to a constant depending only on N. More precisely we have

PROPOSITION 4.2. *Let R_K and R_J be the $\check{R}$-matrices defined as above. Then we have*

$$R_K = s^{-(N+1)(N-3)/2}(W\otimes W)(\mathrm{id}\otimes D)R_J(\mathrm{id}\otimes D^{-1})(W^{-1}\otimes W^{-1})$$

for any $N\geqslant 2$.

Proof. From Propositions 3.1 and 4.1, we only have to check that

$$\frac{\varrho(a,b,c,d)}{\lambda(a,b,c,d)} = s^{(N+1)(N-3)/2}.$$

We have

$$\frac{\varrho(a,b,c,d)}{\lambda(a,b,c,d)} = (-1)^N s^{(N-3)/2}\left(\frac{(s-s^{-1})^{N-1}[N-1]!}{N}\right)^3,$$

but this coincides with $s^{(N+1)(N-3)/2}$ as shown below.

We have

$$(s-s^{-1})^{N-1}[N-1]! = \prod_{k=1}^{N-1}\left(2\sqrt{-1}\,\sin(k\pi/N)\right) = \sqrt{-1}^{\,N-1}\prod_{k=1}^{N-1}(2\sin(k\pi/N)).$$

On the other hand, from [9, I.392-1, p. 33], we have

$$\sin(Nx) = 2^{N-1}\prod_{k=0}^{N-1}\sin(x+k\pi/N).$$

Dividing by $\sin x$ and taking the limit $x\to 0$, we have

$$N = \prod_{k=1}^{N-1}(2\sin(k\pi/N)).$$

Therefore we have

$$\begin{aligned}(-1)^N s^{(N-3)/2}\left(\frac{(s-s^{-1})^{N-1}[N-1]!}{N}\right)^3 &= (-1)^N s^{(N-3)/2}\sqrt{-1}^{\,3(N-1)}\\ &= s^{N^2+(N-3)/2+3(N-1)N/2}\\ &= s^{(N+1)(N-3)/2},\end{aligned}$$

completing the proof. □

We will show that the matrix R_K also satisfies the Yang–Baxter equation. To do that we prepare a lemma.

LEMMA 4.3. *The matrices D and D^{-1} can go through R_J in pair, that is, the following equality holds*:

$$(\mathrm{id}\otimes D)R_J(\mathrm{id}\otimes D^{-1}) = (D^{-1}\otimes\mathrm{id})R_J(D\otimes\mathrm{id}).$$

Proof. It is sufficient to show that $(D\otimes D)R_J = R_J(D\otimes D)$. Since $D^i_j = \delta_{i,j}s^{(N-1)j}$,

$$((D\otimes D)R_J)^{ij}_{kl} = \sum_{a,b}\delta_{a,k}\delta_{b,l}s^{(N-1)k}s^{(N-1)l}(R_J)^{ij}_{ab} = s^{(N-1)(k+l)}(R_J)^{ij}_{kl}$$

and

$$(R_J(D\otimes D))^{ij}_{kl} = \sum_{a,b}\delta_{a,i}\delta_{b,j}s^{(N-1)i}s^{(N-1)j}(R_J)^{ab}_{kl} = s^{(N-1)(i+j)}(R_J)^{ij}_{kl}.$$

But these two coincide since $(R_J)^{ij}_{kl}$ vanishes unless $k+l=i+j$ (the charge conservation law), completing the proof. □

Using Lemma 4.3 we can give another proof of

PROPOSITION 4.4 (Kashaev). *Kashaev's $\breve{R}$-matrix R_K satisfies the Yang–Baxter equation, that is,*

$$(R_K\otimes\mathrm{id})(\mathrm{id}\otimes R_K)(R_K\otimes\mathrm{id})=(\mathrm{id}\otimes R_K)(R_K\otimes\mathrm{id})(\mathrm{id}\otimes R_K).$$

Proof. From Proposition 4.2, we have

$$\begin{aligned}
&s^{3(N+1)(N-3)/2}(R_K\otimes\mathrm{id})(\mathrm{id}\otimes R_K)(R_K\otimes\mathrm{id})\\
&\quad=(W\otimes W\otimes\mathrm{id})(\mathrm{id}\otimes D\otimes\mathrm{id})(R_J\otimes\mathrm{id})(\mathrm{id}\otimes D^{-1}\otimes\mathrm{id})(W^{-1}\otimes W^{-1}\otimes\mathrm{id})\\
&\qquad\times(\mathrm{id}\otimes W\otimes W)(\mathrm{id}\otimes\mathrm{id}\otimes D)(\mathrm{id}\otimes R_J)(\mathrm{id}\otimes\mathrm{id}\otimes D^{-1})(\mathrm{id}\otimes W^{-1}\otimes W^{-1})\\
&\qquad\times(W\otimes W\otimes\mathrm{id})(\mathrm{id}\otimes D\otimes\mathrm{id})(R_J\otimes\mathrm{id})(\mathrm{id}\otimes D^{-1}\otimes\mathrm{id})(W^{-1}\otimes W^{-1}\otimes\mathrm{id})\\
&\quad=(W\otimes W\otimes W)(\mathrm{id}\otimes D\otimes D)(R_J\otimes\mathrm{id})\\
&\qquad\times(\mathrm{id}\otimes D^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes R_J)(\mathrm{id}\otimes D\otimes\mathrm{id})\\
&\qquad\times(R_J\otimes\mathrm{id})(\mathrm{id}\otimes D^{-1}\otimes D^{-1})(W^{-1}\otimes W^{-1}\otimes W^{-1})\\
&\quad=(W\otimes W\otimes W)(\mathrm{id}\otimes D\otimes D)(R_J\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes D)(\mathrm{id}\otimes R_J)(\mathrm{id}\otimes\mathrm{id}\otimes D^{-1})\\
&\qquad\times(R_J\otimes\mathrm{id})(\mathrm{id}\otimes D^{-1}\otimes D^{-1})(W^{-1}\otimes W^{-1}\otimes W^{-1})\\
&\quad=(W\otimes W\otimes W)(\mathrm{id}\otimes D\otimes D^{2})(R_J\otimes\mathrm{id})(\mathrm{id}\otimes R_J)(R_J\otimes\mathrm{id})\\
&\qquad\times(\mathrm{id}\otimes D^{-1}\otimes D^{-2})(W^{-1}\otimes W^{-1}\otimes W^{-1}).
\end{aligned}$$

Similar calculations show

$$\begin{aligned}
s^{3(N+1)(N-3)/2}(\mathrm{id}\otimes R_K)(R_K\otimes\mathrm{id})(\mathrm{id}\otimes R_K)&=(W\otimes W\otimes W)(\mathrm{id}\otimes D\otimes D^{2})\\
&\quad\times(\mathrm{id}\otimes R_J)(R_J\otimes\mathrm{id})(\mathrm{id}\otimes R_J)\\
&\quad\times(\mathrm{id}\otimes D^{-1}\otimes D^{-2})(W^{-1}\otimes W^{-1}\otimes W^{-1}).
\end{aligned}$$

From the Yang–Baxter equation for R_J these two coincide, completing the proof. □

To show that R_J and R_K define the same link invariant, we will construct enhanced Yang–Baxter operators by using R_J and R_K.

Let μ_J be the $(N\times N)$-matrix with (i,j)-entry $(\mu_J)^i_j=\delta_{i,j}s^{2i-N+1}$. Then the quadruple $S_J=(R_J,\mu_J,s^{(N^2-1)/2},1)$ is an enhanced Yang–Baxter operator since the following lemma holds.

LEMMA 4.5.

$$(\mu_J\otimes\mu_J)R_J=R_J(\mu_J\otimes\mu_J),$$
$$\sum_{j=0}^{N-1}((R_J)^{\pm1}(\mathrm{id}\otimes\mu_J))^{ij}_{kj}=(s^{(N^2-1)/2})^{\pm1}\delta_{i,k}.$$

Proof. The proof for the first equality is similar to that of Lemma 4.3.

For the second equality we first note that R_J defines an invariant of $(1,1)$-tangles. Therefore the right-hand side equals a scalar times $\delta_{i,k}$, and it suffices to show

$$\sum_{j=0}^{N-1}((R_J)^{\pm 1}(\mathrm{id}\otimes\mu_J))_{N-1,j}^{N-1,j}=(s^{(N^2-1)/2})^{\pm 1}.$$

The equality for R_J follows from

$$\begin{aligned}(R_J(\mathrm{id}\otimes\mu_J))_{N-1,j}^{N-1,j}&=\sum_{a,b}(R_J)_{ab}^{N-1,j}\delta_{a,N-1}\delta_{b,j}s^{2j-N+1}\\&=(R_J)_{N-1,j}^{N-1,j}s^{2j-N+1}=\delta_{j,N-1}s^{(N-1)^2/2}s^{2j-N+1}=\delta_{j,N-1}s^{(N^2-1)/2}.\end{aligned}$$

To show the equality for R_J^{-1} we use the explicit formula

$$\begin{aligned}(R_J^{-1})_{kl}^{ij}=&\sum_{n=0}^{\min(N-1-j,i)}\delta_{l,i-n}\delta_{k,j+n}\frac{(s-s^{-1})^n}{[n]!}\frac{[j+n]!}{[j]!}\frac{[N-1+n-i]!}{[N-1-i]!}\\&\times(-1)^n s^{-2(i-(N-1)/2)(j-(N-1)/2)-n(i-j)+n(n+1)/2},\end{aligned}$$

which can be checked by direct calculation. A similar calculation shows

$$(R_J^{-1}(\mathrm{id}\otimes\mu_J))_{0,j}^{0,j}=\delta_{j,0}s^{-(N^2-1)/2},$$

and the proof is complete. □

Next we will give a Yang–Baxter operator using R_K. Let μ_K be the $(N\times N)$-matrix with (i,j)-entry $(\mu_K)_j^i=-s\delta_{i,j+1}$. Then we have

LEMMA 4.6.

$$WD\mu_J D^{-1}W^{-1}=\mu_K.$$

Proof. Since $W_j^a=s^{2aj}$, $D_a^b=\delta_{a,b}s^{(N-1)b}$, $(\mu_J)_b^c=\delta_{b,c}s^{2c-N+1}$, $(D^{-1})_c^d=\delta_{c,d}s^{-(N-1)d}$ and $(W^{-1})_d^i=s^{-2di}/N$, we have

$$(WD\mu_J D^{-1}W^{-1})_j^i=\frac{1}{N}(-s)\sum_{a=0}^{N-1}s^{2(j-i+1)a}=-s\delta_{i,j+1},$$

completing the proof. □

Combining Lemmas 4.5 and 4.6, we show that $S_K=(R_K,\mu_K,-s,1)$ is also an enhanced Yang–Baxter operator.

LEMMA 4.7.

$$(\mu_K\otimes\mu_K)R_K=R_K(\mu_K\otimes\mu_K),$$
$$\sum_{j=0}^{N-1}((R_K)^{\pm 1}(\mathrm{id}\otimes\mu_K))_{kj}^{ij}=(-s)^{\pm 1}\delta_{i,k}.$$

Proof. Noting that μ_J and D commute since they are diagonal, the first equality follows immediately from that in Lemma 4.5.

The second equality follows from

$$\begin{aligned}(R_K)^{\pm 1}&(\mathrm{id}\otimes\mu_K)\\ &=s^{\mp(N+1)(N-3)/2}(W\otimes W)(\mathrm{id}\otimes D)(R_J)^{\pm}(\mathrm{id}\otimes\mu_J)(\mathrm{id}\otimes D^{-1})(W^{-1}\otimes W^{-1}),\end{aligned}$$

completing the proof.

Note that the lemma can also be proved by using [13, (2.8) and (2.17)]. □

Now we see that S_J and S_K define the same link invariant by using

LEMMA 4.8. *Let ξ be an n-braid. Then*

$$\begin{aligned}b_{R_K}(\xi)=s^{-w(\xi)(N+1)(N-3)/2}&(W^{\otimes n})(\mathrm{id}\otimes D\otimes D^2\otimes\ldots\otimes D^{n-1})b_{R_J}(\xi)\\ &\times(\mathrm{id}\otimes D^{-1}\otimes D^{-2}\otimes\ldots\otimes D^{-(n-1)})((W^{-1})^{\otimes n}).\end{aligned}$$

Proof. First note that

$$R_J^{\pm 1}=(D^k\otimes D^k)R_J^{\pm 1}(D^{-k}\otimes D^{-k})$$

since

$$\begin{aligned}R_J(D\otimes D)&=(D\otimes\mathrm{id})(D^{-1}\otimes\mathrm{id})R_J(D\otimes\mathrm{id})(\mathrm{id}\otimes D)\\ &=(D\otimes\mathrm{id})(\mathrm{id}\otimes D)R_J(\mathrm{id}\otimes D^{-1})(\mathrm{id}\otimes D)=(D\otimes D)R_J\end{aligned}$$

from Lemma 4.3.

Therefore

$$\begin{aligned}R_K^{\pm 1}&=s^{\mp(N+1)(N-3)/2}(W\otimes W)(\mathrm{id}\otimes D)R_J^{\pm 1}(\mathrm{id}\otimes D^{-1})(W^{-1}\otimes W^{-1})\\ &=s^{\mp(N+1)(N-3)/2}(W\otimes W)(D^{i-1}\otimes D^i)R_J^{\pm 1}(D^{-i+1}\otimes D^{-i})(W^{-1}\otimes W^{-1}),\end{aligned}$$

and the required formula follows immediately since $b_{R_K}(\sigma_i^{\pm 1})$ can be written as

$$\begin{aligned}s^{\mp(N+1)(N-3)/2}(W^{\otimes n})&(\mathrm{id}\otimes D\otimes D^2\otimes\ldots\otimes D^{n-1})b_{R_J}(\sigma_i^{\pm 1})\\ &\times(\mathrm{id}\otimes D^{-1}\otimes D^{-2}\otimes\ldots\otimes D^{-(n-1)})((W^{-1})^{\otimes n}).\end{aligned}$$ □

Since we know that $J_N = T_{S_J,1}$ is well-defined as described in §1, from the previous lemma $T_{S_K,1}(L)$ is also a link invariant, which we denote by $\langle L\rangle_N$. Note that it is implicitly stated in [13] that the invariant can be regarded as an invariant for $(1,1)$-tangles. Note also that though the invariant was defined only up to a multiple of s in [13], we can now define it without ambiguity.

Since

$$b_{R_K}(\xi)(\mathrm{id}\otimes\mu_K{}^{\otimes(n-1)}) = (W^{\otimes n})(D^{k_1}\otimes D^{k_2}\otimes\ldots\otimes D^{k_n})b_{R_J}(\xi) \\ \times(\mathrm{id}\otimes\mu_J{}^{\otimes(n-1)})(D^{-k_1}\otimes D^{-k_2}\otimes\ldots\otimes D^{-k_n})((W^{-1})^{\otimes n})$$

from Lemma 4.8, we conclude that S_J and S_K define the same link invariant.

THEOREM 4.9. *For any link L and any integer $N\geqslant 2$, $\langle L\rangle_N$ and $J_N(L)$ coincide.*

5. Relation between the simplicial volume and the colored Jones polynomials

Let K be one of the three simplest hyperbolic knots 4_1, 5_2 and 6_1. Kashaev found in [14] that the hyperbolic volume of $S^3\backslash K$, denoted by $\mathrm{Vol}(K)$, coincides numerically with the growth rate of the absolute value of $\langle K\rangle_N$ with respect to N. More precisely,

$$\mathrm{Vol}(K) = 2\pi \lim_{N\to\infty} \frac{\log|\langle K\rangle_N|}{N}.$$

We would like to modify his conjecture taking Gromov's simplicial volume (or the Gromov norm) [10] into account. Let us consider the torus decomposition of the complement of a knot K [11], [12]. Then the simplicial volume of K, denoted by $\|K\|$, is equal to the sum of the hyperbolic volumes of the hyperbolic pieces of the decomposition divided by v_3, the volume of the ideal regular tetrahedron in $\mathbf{H}^3$, the 3-dimensional hyperbolic space. Recall that it is additive under the connect sum [26],

$$\|K_1\sharp K_2\| = \|K_1\| + \|K_2\|,$$

and that it does not alter by mutation [25, Theorem 1.5].

Noting that J_N is multiplicative under connect sum, that is,

$$J_N(K_1\sharp K_2) = J_N(K)J_N(K_2),$$

and that it does not alter by mutation [25, Corollary 6.2.5], we propose the following conjecture.

CONJECTURE 5.1 (volume conjecture). *For any knot K,*

$$\|K\| = \frac{2\pi}{v_3} \lim_{N\to\infty} \frac{\log|J_N(K)|}{N}. \tag{5.1}$$

Remark 5.2. First note that if Kashaev's conjecture is true then our conjecture holds for hyperbolic knots and their connect sums. It is also true for torus knots since Kashaev and O. Tirkkonen [15] showed that the right-hand side of (5.1) vanishes in this case by using H. Morton's formula [20] (see also [23]).

Remark 5.3. Note however that the volume conjecture does not hold for links since J_N of the split union of two links vanishes.

As a consequence of the volume conjecture, we can deduce

CONJECTURE 5.4. *The union of all the colored Jones polynomials is an unknot detector.*

We show that the volume conjecture implies Conjecture 5.4 by using the following two lemmas.

LEMMA 5.5 ([8, Corollary 4.2]). *If $\|K\|=0$ then K is obtained from the trivial knot by applying a finite number (possibly zero) of the two operations:*

(1) *making a connect sum,*

(2) *making a cable.*

LEMMA 5.6. *If $\|K\|=0$ then the Alexander polynomial $\Delta(K;t)$ of K is trivial if and only if K is the trivial knot.*

Proof. This lemma comes from Lemma 5.5 and the following three facts [3, §2] (see also [4, Proposition 8.23]):

(i) The Alexander polynomial of a non-trivial torus knot is not trivial.

(ii) The Alexander polynomial is multiplicative under connect sum. Therefore if $\Delta(K_1;t)$ and $\Delta(K_2;t)$ are non-trivial, then $\Delta(K_1\sharp K_2;t)$ is also non-trivial.

(iii) If K' is a knot obtained from K by a cabling operation, then $\Delta(K';t)$ is $\Delta(K;t^n)f(t)$ with some Laurent polynomial $f(t)$, where n is a non-zero integer. Hence if $\Delta(K;t)$ is non-trivial, so is $\Delta(K';t)$. □

Proof that the volume conjecture implies Conjecture 5.4. Suppose that a knot K has trivial colored Jones polynomial for every color. Then $\Delta(K;t)$ is also trivial since the Alexander polynomial can be determined by the colored Jones polynomials from the Melvin–Morton–Rozansky conjecture [19], [24] proved by Bar-Natan and S. Garoufalidis [2].

Therefore from Lemma 5.6 we conclude that K is trivial since the volume conjecture implies $\|K\|=0$. □

Remark 5.7. The above argument using the Melvin–Morton–Rozansky conjecture was due to Bar-Natan and Vaintrob. Our original conjecture was a weaker one; see below.

We also anticipate the following simplest case of V. Vassiliev's conjecture [28, Stabilization Conjecture 6.1] (see also [17, Chapter 1, Part V(L), Conjecture]), which follows from Conjecture 5.4 since every coefficient of the colored Jones polynomial as a power series in $h=\log t$ is a Vassiliev invariant.

CONJECTURE 5.8 (Vassiliev). *Assume that every Vassiliev (finite-type) invariant of a knot is identical to that of the trivial knot. Then it is unknotted.*

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