Acta Math., 187 (2001), 213–236

Escaping geodesics of Riemannian surfaces

by

JOSÉ L. FERNÁNDEZ

Universidad Autónoma de Madrid
Madrid, Spain

and

MARÍA V. MELIÁN

Universidad Autónoma de Madrid
Madrid, Spain

How many geodesics starting from a given point of a surface escape to infinity? In this paper, by a surface we shall mean a complete oriented non-compact Riemannian manifold $\mathcal{M}$ of dimension 2.

Let $\mathcal{M}$ be a surface and let p be a point of $\mathcal{M}$, denote by $\mathcal{S}(p)=\mathcal{S}(\mathcal{M},p)$ the unit circle of directions in the tangent plane of $\mathcal{M}$ at p; we are interested in the size of the set

$$\mathcal{E}(p)=\mathcal{E}(\mathcal{M},p)$$

of directions $v\in\mathcal{E}(p)$ so that the unit-speed geodesic γ emanating from p in the direction of v ($\gamma'(0)=v$) *escapes* to ∞, i.e. $\lim_{t\to\infty}\operatorname{dist}(\gamma(t),p)=+\infty$, where dist means geodesic distance in $\mathcal{M}$.

We shall denote by $\mathcal{R}(p)=\mathcal{R}(\mathcal{M},p)$ the set of directions at p which determine rays. A ray is a geodesic which minimizes the distance between any two of its points. Of course, $\mathcal{R}(p)\subset\mathcal{E}(p)$. It is easy to see that in any surface $\mathcal{M}$, there are at least as many different rays from a given point p as different ends of $\mathcal{M}$.

We shall be dealing with surfaces of negative curvature. A surface of constant negative curvature shall be termed a *hyperbolic* Riemann (on account of its canonically attached complex structure) surface. For some related results in the cases of positive Gaussian curvature and of integrable curvature, we refer the reader to [CE], [Mae], [Shioh], [SST], [Shioy], [HT], [Ba] and [Wo].

From now on, $\mathcal{M}$ denotes a hyperbolic Riemann surface. Our main result is the following tricothomy:

The first author was supported by Grant PB96-0032 from CICYT, Ministerio de Educación y Ciencia, Spain, and a grant from the HCM programme of the European Union. The second author was supported by a grant from Ministerio de Educación y Ciencia, Spain, and Grant PB96-0032.

THEOREM 1. *There are three possibilities*:

(i) $\mathcal{M}$ *has* finite area. *Then for every* $p \in \mathcal{M}$ *there is exactly a countable collection of directions in* $\mathcal{E}(p)$.

(ii) $\mathcal{M}$ *is* transient. *Then for every* $p \in \mathcal{M}$, $\mathcal{E}(p)$ *has full measure.*

(iii) $\mathcal{M}$ *is* recurrent *and of* infinite area. *Then* $\mathcal{E}(p)$ *has length zero, but its Hausdorff dimension is* 1.

We are calling a surface *transient* (resp. *recurrent*) if Brownian motion on $\mathcal{M}$ is transient (resp. recurrent). Notice that hyperbolic surfaces of finite area are recurrent. Therefore the cases above do not overlap and cover all possibilities. The Hausdorff dimension in (iii) is Hausdorff dimension with respect to the intrinsic Riemannian distance in $\mathcal{S}(p)$.

The cases (i) and (ii) are well known, likewise the zero-measure statement in case (iii). Some partial results concerning this last case were obtained in [FL2].

There is also a version of Theorem 1 for a single recurrent end. A *recurrent end* $\mathcal{F}$ of $\mathcal{M}$ is an end such that the extremal length of the family of curves in $\mathcal{F}$ from the boundary of $\mathcal{F}$ and escaping to infinity is infinite. The proof of Theorem 1 applies: from any point p of $\mathcal{M}$ there is a set of dimension 1 of geodesics emanating from p and escaping to infinity through the end $\mathcal{F}$. Besides, there is a version of Theorem 1 where the geodesics escape to infinity at a uniform speed, see §6.

Some closely related results concerning *bounded geodesics* of hyperbolic surfaces have been obtained recently. Denote by $\mathcal{B}(p)=\mathcal{B}(\mathcal{M},p)$ the collection of directions $v \in \mathcal{S}(p)$ such that for the geodesic γ from p in the direction v one has

$$\sup_{0 \leqslant t < +\infty} \operatorname{dist}(\gamma(t), p) < +\infty.$$

We shall denote by $\delta(\mathcal{M})$ the so-called *exponent of convergence* of $\mathcal{M}$, i.e. the infimum of the positive numbers $s>0$ for which

$$\sum_{[\omega]} e^{-s \cdot \operatorname{length}([\omega])} < +\infty,$$

where $[w]$ runs on the fundamental group of $\mathcal{M}$ at p, and length$([\omega])$ denotes the minimum length within the class $[w]$ of the loop w. The exponent of convergence does not depend on p.

The dimension of the set $\mathcal{B}(p)$ is determined by $\delta(\mathcal{M})$:

THEOREM A. *For every* $p \in \mathcal{M}$, *the Hausdorff dimension of* $\mathcal{B}(p)$ *is* $\delta(\mathcal{M})$.

Theorem A has a long history. It has its roots in results on diophantine approximation due to V. Jarník in the 1920's. In the present context, Theorem A, but for finite-area Riemann surfaces, was proved by Patterson, [Pa], and in full generality in [FM1] and [BJ]. It also holds in higher dimensions, see [BJ], [FM1] and [St]. The proof in [BJ] is particularly simple and general, it applies to non-elementary underlying groups, while the two others require the groups to be geometrically finite.

Observe that in Theorem 1 there is no scale of different possibilities.

We would like to remark that Theorem 1, as well as Theorem A, has interesting applications in function theory, see e.g. [FN] and [FP], and §7.

The proof of the main result of the paper, Theorem 1, is in §6, while §§ 4 and 5 contain the main ingredients of the actual construction of the large set of geodesics whose existence the theorem claims. §§2–4 collect some preliminary material on the geometry of Riemann surfaces and on Hausdorff dimension.

In a nutshell, the proof of Theorem 1, part (iii), goes as follows: first we decompose the Riemann surface into a sequence of (bordered) geometrically finite Riemann surfaces which tend to infinity with exponents converging to 1 (§4); for each one of these pieces one then has to locate a large number of long geodesics connecting appropriate boundary geodesics (§5); and finally we must join together these geodesics to form a network of geodesics that, when lifted to the universal cover, forms a tree whose "rim" has a large dimension (§3).

A word about notation. There are many estimates in this paper involving absolute constants. These are usually denoted by capital letters like C. Occasionally, we shall indicate a constant C depending on some parameter λ as $C(\lambda)$. The symbol $\#A$ denotes the number of elements of the set A.

Acknowledgements. We would like to thank Kari Astala for a very insightful remark and Antonio Ros for supplying some useful references. We are particularly grateful to Artur Nicolau for sharing many ideas with us, to Chris Bishop for some helpful correspondence and for detecting some inaccuracies in an earlier draft, and to Lennart Carleson and Peter Jones for encouragement.

We are greatly indebted to the referee whose excellent report, full with suggestions and simplifications, has allowed us to improve both the mathematical content and the presentation of this paper. He/she deserves our sincere thanks.

1. Hyperbolic surfaces and Fuchsian groups

In this section we shall assume throughout that $\mathcal{M}$ is a hyperbolic surface.

The surface $\mathcal{M}$ may be described as a quotient $\mathbf{P}/\Gamma$, where $\mathbf{P}$ is the hyperbolic plane and Γ is a group of orientation-preserving isometries which has no torsion and acts discontinuously on $\mathbf{P}$. We shall only use this kind of representation of surfaces. If we use for $\mathbf{P}$ the Poincaré disk model, then $\mathbf{P}\cong\mathbf{D}=\{z\in\mathbf{C}:|z|<1\}$. The group Γ is then a Fuchsian group, i.e. a discrete subgroup of the group Möb($\mathbf{D}$) of orientation-preserving Möbius transformations of the unit disk $\mathbf{D}$ onto itself. We shall denote the natural projection from $\mathbf{P}$ onto $\mathcal{M}=\mathbf{P}/\Gamma$ by Π. This projection Π is a local isometry. The hyperbolic distance in $\mathbf{P}$ between p and q will be denoted by $d(p,q)$.

The orbit of 0, $\Gamma(0)$, or the orbit of any other point for that matter, accumulates on a certain closed subset of $\partial\mathbf{D}$, $\Lambda(\Gamma)$, called the *limit set* of Γ. A particularly relevant subset of $\Lambda(\Gamma)$ is the so-called *conical limit set*, $\Lambda_c(\Gamma)$, which may be described as the set of points $\xi\in\partial\mathbf{D}$ such that there is a sequence of points in $\Gamma(0)$ tending to ξ inside a cone in $\mathbf{D}$ with vertex ξ. The geometric meaning of $\Lambda_c(\Gamma)$ is simple: it represents the set of directions of geodesics emanating from 0, which *do not escape* to ∞.

Let p be a point of $\mathcal{M}$. If we assume, as we may, that $\Pi(0)=p$, and identify $\partial\mathbf{D}$ with the circle of directions at 0, and also, with $\mathcal{S}(\mathcal{M},p)$, then $\mathcal{E}(p)$ gets identified with $\partial\mathbf{D}\setminus\Lambda_c(\Gamma)$. Moreover, if $\mathcal{P}$ is the Dirichlet fundamental polygon (see e.g. [Bea, p. 227]) of Γ then the set of rays $\mathcal{R}(p)$ may be identified with $\partial\mathcal{P}\cap\partial\mathbf{D}$.

Observe that as a consequence of these identifications the sets $\mathcal{E}(p)$ of different p's are diffeomorphic; and, similarly, for the $\mathcal{R}(p)$'s. Thus the dimensions of these sets (or whether they have full measure or measure zero) are conformal invariants of the surface. We will systematically identify $\mathcal{S}(\mathcal{M},p)$ with $\partial\mathbf{D}$.

If the Laplace–Beltrami operator of $\mathcal{M}$ has a *Green function* then

$$\sum_{\gamma\in\Gamma}(1-|\gamma(0)|)<+\infty,$$

see [Ts, p. 522], and then the Borel–Cantelli lemma tells us that $\Lambda_c(\Gamma)$ has length zero. In particular, if $\mathcal{M}$ has a Green function, or equivalently if $\mathcal{M}$ is transient, then $\mathcal{E}(p)$ has full measure. The converse, namely, that if $\Lambda_c(\Gamma)$ has length zero then $\mathcal{M}$ has a Green function, is also well known, see e.g. [Ga].

In other terms, the set $\mathcal{E}$ has either length zero or full length; the first case occurs when $\mathcal{M}$ is recurrent, the second if $\mathcal{M}$ is transient.

If $\mathcal{M}$ is written as $\mathbf{P}/\Gamma$, then the *exponent of convergence*, $\delta(\mathcal{M})$, can be expressed as the infimum of all positive numbers $s>0$ for which

$$\sum_{\gamma\in\Gamma}e^{-s\cdot d(0,\gamma(0))}<+\infty.$$

The *bottom of the spectrum* of the Laplace–Beltrami operator of $\mathcal{M}$ is denoted by $\beta(\mathcal{M})$. In terms of Rayleigh's quotients, $\beta(\mathcal{M})$ can be defined as

$$\beta(\mathcal{M})=\inf\left\{\frac{\int\|\nabla\Phi\|^2\,dA}{\int\Phi^2\,dA}:\Phi\in C_c^\infty(\mathcal{M})\right\},$$

where $\|\cdot\|$, ∇ and dA refer to the Poincaré metric of $\mathcal{M}$. The following theorem of Elstrodt–Patterson–Sullivan (see e.g. [Su2]) gives a relation between $\delta(\mathcal{M})$ and $\beta(\mathcal{M})$.

THEOREM B.

$$\beta(\mathcal{M})=\begin{cases}\frac{1}{4}, & \text{if } 0\leqslant\delta(\mathcal{M})\leqslant\frac{1}{2},\\ \delta(\mathcal{M})(1-\delta(\mathcal{M})), & \text{if } \frac{1}{2}\leqslant\delta(\mathcal{M})\leqslant 1.\end{cases}$$

In particular, if $\beta(\mathcal{M})<\varepsilon<\frac{1}{4}$, *then* $\delta(\mathcal{M})>1-2\varepsilon$.

Assume, finally, that $\mathcal{M}$ has *finite area*. In this case, $\mathcal{M}$ may be decomposed into a compact set Q and a finite union of pseudospheres: $\mathcal{M}=Q\cup\bigcup_{i=1}^{n}B_i$. Each B_i is isometric with $\{z\in\mathbf{C}:0<|z|<1\}$ endowed with the metric

$$ds=\frac{|dz|}{|z|\log(1/|z|)}$$

(see e.g. [Ber], [CdV]). One sees readily that $\mathcal{E}(p)$ is countable. As a matter of fact, for each end, say $i\in\{1,2,...,n\}$, and each homotopy class of curves joining p with ∂B_i (the extreme in ∂B_i is allowed to move freely within ∂B_i), there is a unique shortest curve γ; the direction of γ at p belongs to $\mathcal{E}(p)$, and, conversely, every $v\in\mathcal{E}(p)$ is obtained in this manner.

The above can be seen directly using the group description. One may assume, [Bea, §10.4], that the fundamental polygon $\mathcal{P}$ at 0 has finitely many sides and that its vertices all lie on $\partial\mathbf{D}$. We may identify $\mathcal{E}(p)\cong\Gamma(\partial\mathcal{P}\cap\partial\mathbf{D})$.

It is not true, in general, that $\mathcal{E}(p)\cong\Gamma(\partial\mathcal{P}\cap\partial\mathbf{D})$. For instance, if $\mathcal{M}=\mathbf{C}\setminus\mathbf{Z}$ then $\mathcal{E}(p)$ has dimension 1 while $\partial\mathcal{P}\cap\partial\mathbf{D}$ is countable. (See e.g. [FL2].)

2. Some basic facts about Riemann surfaces

Throughout this section $\mathcal{N}$ denotes a hyperbolic surface, non-compact, as always. It will be represented as a quotient $\mathcal{N}=\mathbf{P}/\Gamma$ with $\mathbf{P}=\mathbf{H}^2$ or $\mathbf{P}=\mathbf{D}$, whatever is more convenient.

2.1. *Cusps, funnels and collars.* If $\mathcal{Y}\subset\mathcal{N}$ is a domain isometric to $S^1\times[\log 2\pi,+\infty)$ with the metric $dr^2+e^{-2r}d\theta^2$, then we call the domain $\mathcal{Y}$ a *cusp* or *cuspidal end*. If $\mathcal{Z}\subset\mathcal{N}$ is a domain isometric to $S^1\times[a_i,+\infty)$, for some $a_i>0$, with the metric $dr^2+\cosh^2 r\,d\theta^2$, then we will refer to $\mathcal{Z}$ as a *funnel*.

A Riemann surface $\mathcal{N}$ of *finite type* may be split into a disjoint union of a compact set and a finite number of cusps and funnels, [Pa], [Ber], [CdV]; besides, if it has not funnels it has finite area. We remark that any Riemann surface with a funnel is transient.

As a bordered Riemann surface, a funnel with boundary length l will be denoted by $\mathcal{Z}_l$.

Let G be a simple closed geodesic in $\mathcal{N}$ of length l. The *collar of* G, denoted by collar(G), is the $f(l)$-neighborhood of G in $\mathcal{N}$, where $f(t)$ is the positive continuous decreasing function

$$f(t) = \operatorname{arcsinh} \frac{1}{\sinh\left(\frac{1}{2}t\right)}.$$

The well-known collar's lemma, see e.g. [Ber], asserts that collar(G) is topologically a cylinder, and that if G_1 and G_2 are two disjoint simple closed geodesics in $\mathcal{N}$, their corresponding collars are disjoint:

$$\operatorname{collar}(G_1) \cap \operatorname{collar}(G_2) = \varnothing.$$

If $\widetilde{G}$ is a lifting of G, then by collar$(\widetilde{G})$ we mean the $f(l)$-neighborhood of $\widetilde{G}$ in $\mathbf{P}$. Of course, collar$(\widetilde{G})$ projects onto collar(G).

Let $\gamma \in \Gamma$ be a primitive (i.e. without roots in Γ) hyperbolic transformation whose axis projects onto the closed geodesic G. Then the collar's lemma also claims that

$$\operatorname{collar}(\widetilde{G}) \cap \operatorname{collar}(\gamma_1(\widetilde{G})) = \varnothing \quad \text{for all } \gamma_1 \in \Gamma \setminus \{\gamma^m : m \in \mathbf{Z}\}.$$

The hyperbolic transformation γ is unique up to conjugation in Γ (see [Ra, p. 401]), and hereafter we will refer to γ as *a hyperbolic transformation associated to* G.

We shall need to express the collar's lemma in terms of Euclidean quantities for later use. This we do next and we will use $\mathbf{P} = \mathbf{D}$.

Let $\widetilde{G}$ be a geodesic in $\mathbf{D}$. The *diameter of* $\widetilde{G}$, denoted by diam$(\widetilde{G})$, is defined as the Euclidean diameter of the whole (Euclidean) circle which contains $\widetilde{G}$ as an arc, if the Euclidean distance between the origin and $\widetilde{G}$ is at least $\log(2+\sqrt{5})$. Otherwise, it is defined as 1. Thus, for instance, a geodesic through 0 has diameter 1. (The awkward constant $\log(2+\sqrt{5})$ is there simply to have continuity of this diameter.)

The next lemma gives us an estimate on separation of liftings of simple closed geodesics. The proof follows from the disjointness given by the collar's lemma.

LEMMA 2.1. *Let* $\widetilde{G}_1, \widetilde{G}_2$ *be two distinct liftings of* G *such that*

$$\operatorname{diam}(\widetilde{G}_1) + \operatorname{diam}(\widetilde{G}_2) < \tfrac{1}{8}.$$

Then

$$d_{\mathrm{Euc}}(\widetilde{G}_1, \widetilde{G}_2) > c \min\{\operatorname{diam}(\widetilde{G}_1), \operatorname{diam}(\widetilde{G}_2)\},$$

where d_{Euc} *denotes Euclidean distance, and* $c>0$ *depends only on the length of* G. *In particular, if* $\operatorname{diam}(\widetilde{G}_i)>\varrho$ $(i=1,2)$, *then*

$$d_{\mathrm{Euc}}(\widetilde{G}_1,\widetilde{G}_2)>c\varrho.$$

2.2. *Liftings of closed geodesics.* Let G be an oriented simple closed geodesic in $\mathcal{N}=\mathbf{D}/\Gamma$, and assume that Γ is non-elementary (recall that this only rules out the cyclic groups).

The next proposition gives us a local estimate of the number of liftings of G of (approximately) the same given diameter.

PROPOSITION 2.2. *Let* I *be an arc in* $\partial\mathbf{D}$ *which contains a hyperbolic fixed point of* Γ. *Consider the collection* $\mathcal{U}_n$ *of those liftings* $\widetilde{G}$ *of* G *with*

$$e^{-(n+1)}\leqslant \operatorname{diam}(\widetilde{G})<e^{-n}$$

and with final endpoint in the interval I. *Then*

$$\sum_{n=1}^{\infty}\#\mathcal{U}_n\left(\frac{1}{e^n}\right)^{\sigma}=\infty$$

for any $0<\sigma<\delta(\Gamma)$.

Moreover, for each $0<\sigma<\delta(\Gamma)$ *there is an increasing sequence of integers* n *such that for each one of those* n *there exists a subcollection* $\mathcal{T}_n$ *of* $\mathcal{U}_n$ *which satisfies*

$$\#\mathcal{T}_n>e^{n\sigma}$$

and has the additional property that if $\widetilde{G}_1,\widetilde{G}_2\in\mathcal{T}_n$ *then* $\widetilde{G}_1$ *does not separate* $\widetilde{G}_2$ *from* 0.

If the group Γ is geometrically finite, a more precise estimate is available: For ϱ small enough, the number of liftings of G with diameter approximately ϱ^n, and final endpoint in the interval I, is *comparable* (for all n large enough) to

$$\left(\frac{1}{\varrho^n}\right)^{\delta}\mu(I),$$

where μ denotes the Patterson measure. But we do not need this sharper result in this paper.

In our applications, the group Γ is always non-elementary, and the fixed points of hyperbolic transformations in Γ are dense in $\partial\mathbf{D}$.

The proof of the proposition rests on the following simple recollection lemma.

LEMMA 2.2. *Let G be a simple closed geodesic in $\mathcal{N}=\mathbf{D}/\Gamma$, and let $\mathcal{G}$ denote the set of all liftings of G in $\mathbf{D}$. If*

$$\sum_{\gamma\in\Gamma} e^{-\sigma\cdot d(0,\gamma(0))}=\infty,$$

then

$$\sum_{\widetilde{G}\in\mathcal{G}} e^{-\sigma\cdot d(0,\widetilde{G})}=\infty.$$

This is elementary, recall that the $\widetilde{G}$'s are disjoint, see [Su1] for the analogous case of a cusp.

Proof of Proposition 2.2. Fix $0<\sigma<\delta(\Gamma)$. Let $\mathcal{V}_n$ denote the set of liftings $\widetilde{G}$ of G such that

$$e^{-(n+1)}\leqslant \operatorname{diam}(\widetilde{G})<e^{-n}.$$

Notice that there is no reference to I, yet.

From Lemma 2.2 it follows that

$$\sum_{n=1}^{\infty} \#\mathcal{V}_n\left(\frac{1}{e^n}\right)^{\sigma}=\infty. \tag{2.2.1}$$

Let $g_1\in\Gamma$ denote the hyperbolic transformation which fixes the hyperbolic fixed point $\xi\in I$. It is geometrically clear that $\bigcup_n g_1^n(I)$ covers the set $\partial\mathbf{D}\setminus\{\xi\}$. Moreover, since Γ is non-elementary there exists $g_2\in\Gamma$ and n such that $\{\xi\}\subset g_2\circ g_1^n(I)$.

Hence, by compactness, the set $\partial\mathbf{D}$ is covered by a finite number of images, by elements of Γ, of the arc I. We obtain that the subcollection $\mathcal{U}_n$ of $\mathcal{V}_n$ which contains the liftings with final endpoint in the interval I satisfies

$$\#\mathcal{V}_n\leqslant C\cdot\#\mathcal{U}_n \tag{2.2.2}$$

with $C>0$.

Therefore, from (2.2.1) and (2.2.2) we obtain that

$$\sum_{n=1}^{\infty} \#\mathcal{U}_n\left(\frac{1}{e^n}\right)^{\sigma}=\infty. \tag{2.2.3}$$

If $\widetilde{G}\in\mathcal{U}_n$, then $\operatorname{diam}(\widetilde{G})\in[e^{-(n+1)},e^{-n})$. Hence, using disjointness of collars (see Lemma 2.1) it is clear that if $\widetilde{G}\in\mathcal{U}_n$, then the number of geodesics in $\mathcal{U}_n$ which are separated from 0 by $\widetilde{G}$ is bounded by a constant depending on the length of G. The existence of the subcollection $\mathcal{T}_n$ of $\mathcal{U}_n$ for n large follows easily from (2.2.3) and this last remark.

2.3. *Pasting handles and funnels.* A construction that will be particularly useful is the *pasting* of hyperbolic surfaces with boundary. If $\mathcal{N}_1, \mathcal{N}_2$ are two hyperbolic surfaces with boundary, and G_1 on $\mathcal{N}_1$ and G_2 on $\mathcal{N}_2$ are simple closed boundary geodesics with the same length, then we can construct from $\mathcal{N}_1 \cup \mathcal{N}_2$ a new hyperbolic surface by identifying $G_1(t)$ with $G_2(a-t)$ for a fixed $a \in \mathbf{R}$ (see e.g. [Bu]).

Frequently, we shall be interested in attaching simple bordered compact Riemann surfaces to some specified components of the boundary of a given hyperbolic surface.

Given $l>0$, let $0<t_l<1$ such that

$$\frac{4t_l^2}{(1-t_l^2)^2} = \cosh l. \tag{2.3}$$

Then we take $v_1 := -t_l$, $v_2 := t_l$, $v_3 := -t_l\sqrt{-1}$ and $v_4 := t_l\sqrt{-1}$ in $\mathbf{D}$.

Let G_j $(j=1,2,3,4)$ be the geodesics in $\mathbf{D}$ such that $v_j \in G_j$ and $d(0, G_j) = d(0, v_j)$. Moreover, let g be the hyperbolic transformation which fixes $1, -1$, and maps G_1 to G_2. And let h be the hyperbolic transformation which fixes $\sqrt{-1}$, $-\sqrt{-1}$, and maps G_3 to G_4.

We define Γ as the group generated by g and h, and we denote by $\mathcal{S}_l$ the Riemann surface of genus 1, $\mathcal{S}_l = \mathbf{D}/\Gamma$. We remark that $\mathcal{S}_l$ can be split into the disjoint union of a compact region and a funnel $\mathcal{Z}_l$. Moreover, since $\sinh \frac{1}{2} d(0, g(0)) \sinh \frac{1}{2} d(0, h(0)) = \cosh d(G_i, G_j)$ (see e.g. [Bea, p. 192]), it follows from (2.3) that for $i=1,2$ and $j=3,4$,

$$d(G_i, G_j) = l.$$

Therefore, the geodesic bounding the funnel $\mathcal{Z}_l$ has length l.

We will use $\mathcal{U}_l$ to denote the hyperbolic surface (with boundary) $\mathcal{S}_l \setminus \mathcal{Z}_l$. We remark that $\mathcal{U}_l$ has genus 1 and that its boundary is a simple closed geodesic of length l. Hereafter, we will refer to $\mathcal{U}_l$ as an *l-handle*.

We will use several times the following cutting and pasting operations:

(1) Given a Riemann surface $\mathcal{N}$ with a funnel $\mathcal{Z} = \mathcal{Z}_l$ whose boundary is a simple closed geodesic G of length l, we construct a new Riemann surface by cutting $\mathcal{N}$ along the closed geodesic G, removing the funnel and pasting there an l-handle.

(2) Given a Riemann surface $\mathcal{N}$ and a simple closed geodesic G (of length l) in $\mathcal{N}$, we construct a new Riemann surface by cutting $\mathcal{N}$ along G to obtain one or two bordered Riemann surfaces and pasting to one of them along the geodesic a funnel Z_l.

2.4. *Some hyperbolic trigonometry.* Let u and v be two geodesic arcs in the Riemann surface $\mathcal{N}$, and let $\gamma: [a,b] \to \mathcal{N}$ and $\eta: [c,d] \to \mathcal{N}$ be parameterizations such that $u = \gamma([a,b])$, $v = \eta([c,d])$. If $\gamma(b) = \eta(c)$, then by *the angle between u and v* we mean the angle from $\gamma'(b)$ to $\eta'(c)$. On the other hand, if $\gamma(a)=\eta(c)$, then by *the angle between* u *and* v we mean the angle from $\gamma'(a)$ to $\eta'(c)$. Angles are given mod 2π and between $-\pi$ and π. The case $\mathcal{N}=\mathbf{D}$ shall be very frequent.

Let E be a closed subset of $\overline{\mathbf{D}}$, and let $z\in\mathbf{D}\setminus E$. We will denote by $\omega(z,E)$ the *harmonic measure* from the point z of the set E in the component of $\overline{\mathbf{D}}\setminus E$ which contains z. The next lemma gives an estimate of the harmonic measure of a geodesic arc. This result appears in [FM2, Lemma 1.1.2] and its proof is simple.

LEMMA 2.4.1. *Let* $z\in\mathbf{D}$, *and let* $\widetilde{G}$ *be a geodesic arc in* $\mathbf{D}$. *Then*

$$e^{d(z,\widetilde{G})}=\operatorname{cotan}\left(\tfrac{1}{4}\pi\omega(z,\widetilde{G})\right).$$

Moreover, there exists $C>1$ *such that if* $d(z,\widetilde{G})\geqslant 1$, *then for all* $u\in\widetilde{G}$,

$$\frac{1}{C}\,\omega(z,\widetilde{G})\sin\theta_u\leqslant e^{-d(z,u)}\leqslant C\omega(z,\widetilde{G})\sin\theta_u,$$

where θ_u *denotes the absolute value of the smallest angle at* u *between* $\widetilde{G}$ *and the geodesic through* z *and* u.

The next lemma will allow us to compare piecewise geodesics with proper geodesics. The proof is not difficult, and it appears in [FM2, Lemma 1.3.1].

LEMMA 2.4.2. *Let* $\{z_n\}_{n=0}^{\infty}$ *be a sequence of points in* $\mathbf{D}$. *Let* γ_n, $n\geqslant 1$, *denote the oriented geodesic arc from* z_{n-1} *to* z_n. *Assume that, for each* $n\geqslant 1$, *the (absolute value of the) angle at* z_n *between* γ_n *and* γ_{n+1} *is at most* $\frac{1}{4}\pi$.

There exists a constant Λ *such that if*

$$\{d(z_{n-1},z_n)\}=\operatorname{length}(\gamma_n)\geqslant\Lambda,\quad \textit{for each } n\geqslant 1,$$

then the following conclusions hold:

(i) $d(z_0,z_n)\to\infty$ *and, moreover,* z_n *converges (in the Euclidean metric) to a single point,* ξ, *say, in* $\partial\mathbf{D}$.

(ii) *There is an absolute constant* $C>0$ *such that if* γ *denotes the whole geodesic going from* z_0 *to* ξ *then for each* n, *and each* $z\in\gamma_n$,

$$d(z,\gamma)<C.$$

3. A bound on Hausdorff dimension

Patterns. A *pattern* $\mathcal{P}$ is given by a positive integer $N>1$ and two real numbers r and R such that $0<r\leqslant R<1$. We will refer to (N, r, R) as the parameters of $\mathcal{P}$.

Given an interval $J\subset\mathbf{R}$, by applying the pattern $\mathcal{P}$ to the interval J we simply mean the operation of choosing N disjoint open subintervals $\{J_j\}$ of J satisfying

$$r<\frac{|J_j|}{|J|}<R.$$

Aside from the restrictions given by the parameters, the intervals can be chosen arbitrarily.

We may apply the pattern $\mathcal{P}$ again to each one of the intervals J_j, and we say that we have applied twice the pattern $\mathcal{P}$ to the original interval J. This does not mean reproducing at a different scale the same intervals; but, simply, that we choose the same number of intervals with the same bounds. We define to apply K times the pattern $\mathcal{P}$ to the interval J in a similar way.

Also, given a real number $0<s\leqslant 1$ we say that we *reduce* the interval J with reduction bound s when we choose a subinterval J' of J such that

$$|J'|\geqslant s\cdot|J|.$$

Again, the subinterval can be chosen arbitrarily, as long as the restriction above, which is just an inequality, is fulfilled.

Sequences of patterns. Let $\{\mathcal{P}_i\}_{i=1}^{\infty}$ be a sequence of patterns, with respective parameters (N_i, r_i, R_i), and let $\{s_i\}_{i=1}^{\infty}$ be a sequence of reduction bounds.

Given a sequence $\{K_i\}_{i=1}^{\infty}$ of *number of repetitions* we construct a Cantor-like set as follows:

We start with $I=[0,1]$. The interval I is the only interval of the 0th generation, $\mathcal{A}_0$.

Now we apply the pattern $\mathcal{P}_1$ to I, to obtain a first generation of intervals $\mathcal{A}_1$. To each of these intervals we again apply $\mathcal{P}_1$, to get $\mathcal{A}_2$. We continue to apply $\mathcal{P}_1$ a total of K_1 times obtaining generations $\mathcal{A}_3, ..., \mathcal{A}_{K_1}$.

We now reduce each one of the intervals in $\mathcal{A}_{K_1}$ with bound s_1, to get the next generation $\mathcal{A}_{K_1+1}$.

We start again, with these last intervals, apply $\mathcal{P}_2$ a total of K_2 times, and perform a final reduction with bound s_2. Thus reaching generation $\mathcal{A}_{(K_1+1)+(K_2+1)}$.

And so on.

The Cantor set $\mathcal{C}$ is given by

$$\mathcal{C}=\bigcap_{n=0}^{\infty}\bigcup_{J\in\mathcal{A}_n} J.$$

It is convenient to write

$$n_i := K_1+K_2+\dots+K_i+i-1, \quad i\geqslant 1.$$

These n_i codify the generations just before the reductions. Then given $\mathcal{A}_n$ with $n\geqslant K_1$ there is an i such that

$$n_i \leqslant n < n_{i+1},$$

and there is an $l\in\{0,1,\dots,K_{i+1}\}$ such that $n=n_i+l$. (Notice that $K_{i+1}=n_{i+1}-n_i-1$.) If $l=0$, then we get $\mathcal{A}_{n+1}$ by reducing each one of the intervals in $\mathcal{A}_n$ with bound s_i. Otherwise, we obtain $\mathcal{A}_{n+1}$ by applying the pattern $\mathcal{P}_{i+1}$ to each one of the intervals in $\mathcal{A}_n$.

We will need the following two bounds on the intervals in the nth generation $\mathcal{A}_n$, with $n=n_i+l$ and $l\in\{0,1,\dots,K_{i+1}\}$:

(3.1) An upper bound M_n on the size of the intervals in $\mathcal{A}_n$. We can take

$$M_n := \begin{cases} R_1^{K_1}\dots R_i^{K_i}, & \text{if } l=0,\\ R_1^{K_1}\dots R_i^{K_i}R_{i+1}^{l-1}, & \text{if } l\neq 0.\end{cases}$$

(3.2) A lower bound Γ_n on the size of the union of all the intervals in generation $\mathcal{A}_n$. We can write Γ_n as

$$\Gamma_n = \gamma_1\dots\gamma_n,$$

where $\gamma_j=N_ir_i$ when we obtain $\mathcal{A}_j$ from $\mathcal{A}_{j-1}$ by applying the pattern $\mathcal{P}_i$, and $\gamma_j=s$ when we obtain $\mathcal{A}_j$ from $\mathcal{A}_{j-1}$ by reducing each interval by s. Therefore,

$$\Gamma_n = \begin{cases} s_1\dots s_{i-1}(N_1r_1)^{K_1}\dots(N_ir_i)^{K_i} & \text{if } l=0,\\ s_1\dots s_i(N_1r_1)^{K_1}\dots(N_ir_i)^{K_i}(N_{i+1}r_{i+1})^{l-1} & \text{if } l\neq 0.\end{cases}$$

THEOREM 3.1. *Let there be given a sequence of patterns $\{\mathcal{P}_i\}$ with parameters (N_i,r_i,R_i), and a sequence of reduction bounds $\{s_i\}$. Then there exists a sequence $\{K_i\}$ of repetitions such that the associated Cantor-like set $\mathcal{C}$ satisfies*

$$\text{Hausdorff dimension}(\mathcal{C}) \geqslant \liminf_{i\to\infty}\frac{\log(N_ir_i/R_i)}{\log(1/R_i)}.$$

If there are no repetitions, i.e. $K_1=K_2=\dots=1$, the result is false even with $r_i=R_i$ for each i. For instance, let $H_0=2$, $H_i=2^{H_{i-1}}$, and consider the Cantor set where each interval of the $(i-1)$-generation splits into H_i subintervals of the same length; we select (among them) a total of $H_i^{1/2}$ consecutive subintervals for the i-generation. This Cantor set has Hausdorff dimension 0.

On the other hand, if we simply iterate a unique pattern, then the result is well known, see [Hu]. Our proof is modeled upon his.

We shall need the following elementary but crucial estimate.

LEMMA 3.1. *If the K_i's grow sufficiently fast, then*

$$\limsup_{n\to\infty}\frac{\log(1/\Gamma_{n+1})}{\log(1/M_n)}\leqslant 1-\liminf_{i\to\infty}\frac{\log(N_i r_i/R_i)}{\log(1/R_i)}.$$

This is a simple estimate, but we should remark that the choice of each K_i depends not only on the previous K's, on s_1 up to s_i, and on the parameters of the patterns $\mathcal{P}_1$ to $\mathcal{P}_i$, but also on the parameters of $\mathcal{P}_{i+1}$.

Proof of Theorem 3.1. We let

$$\alpha := \liminf_{i\to\infty}\frac{\log(N_i r_i/R_i)}{\log(1/R_i)}.$$

We construct a *probability* measure ν with support $\mathcal{C}$ in the following way: We define $\nu(I)=1$. Then for each interval I_n in $\mathcal{A}_n$ we define

$$\nu(I_n)=\frac{|I_n|}{\sum_{J\in\mathcal{A}_n,\,J\subset I_{n-1}}|J|}\,\nu(I_{n-1}),$$

where I_{n-1} denotes the unique interval in $\mathcal{A}_{n-1}$ such that $I_n\subset I_{n-1}$. Next, for any set $L\subset\mathbf{R}$,

$$\nu(L):=\inf\sum_{U\in\mathcal{U}}\nu(U),$$

where the infimum is taken over all the coverings $\mathcal{U}$ of L with intervals in $\bigcup\mathcal{A}_n$.

An easy calculation shows that if $I_j\in\mathcal{A}_j$ then

$$\sum_{\substack{J\in\mathcal{A}_{j+1}\\ J\subset I_j}}|J|\geqslant\gamma_{j+1}|I_j|.$$

Hence,

$$\nu(I_n)\leqslant\frac{|I_n|}{\gamma_1\dots\gamma_n}=\frac{|I_n|}{\Gamma_n}. \tag{3.3}$$

Let U be an interval with length

$$M_{n+1}\leqslant|U|<M_n,$$

and let $\mathcal{G}_{n+1}$ denote the set of intervals in $\mathcal{A}_{n+1}$ which intersect the interval U. Notice that

$$\bigcup_{J\in\mathcal{G}_{n+1}}J\subset 3U.$$

Here by $3U$ we denote the interval with the same center as U, and radius 3 times as large.

Therefore using (3.3) we obtain that

$$\nu(U) \leqslant \sum_{J\in\mathcal{G}_{n+1}} \nu(J) \leqslant \frac{\sum_{J\in\mathcal{G}_{n+1}}|J|}{\Gamma_{n+1}} \leqslant \frac{3|U|}{\Gamma_{n+1}}.$$

From this estimate and Lemma 3.1, we have that, for $\beta<\alpha$, if $|U|$ is small enough (depending on β), then

$$\nu(U) \leqslant C(\beta)|U|^{\beta}.$$

Then, by the standard Frostman argument, we get that

$$\text{Hausdorff dimension}(\mathcal{C}) \geqslant \alpha.$$

4. Chains of geodesic domains in a Riemann surface

Let $\mathcal{M}$ be a recurrent hyperbolic surface with infinite area.

By a *geodesic domain* in $\mathcal{M}$, we mean a domain $D\subset\mathcal{M}$ whose relative boundary consists of finitely many non-intersecting closed simple geodesics and whose area is finite. For instance, if $\mathcal{M}=\widehat{\mathbf{C}}\setminus\{0,\frac{1}{2},2,\infty\}$, then the region $\mathbf{D}\setminus\{0,\frac{1}{2}\}$ is a geodesic domain. It is sometimes convenient to consider the punctures as geodesics of length zero; we shall adhere to that convention.

The aim of this section is to prove Theorem 4.1 below. This theorem will allow us to find in $\mathcal{M}$ a chain of escaping geodesic domains, which, when completed by pasting funnels along its boundary geodesics, become surfaces with exponents arbitrarily close to 1.

THEOREM 4.1. *Given a point $p\in\mathcal{M}$, there exists a family $\mathcal{D}=\{D_i\}$ of geodesic domains in $\mathcal{M}$ satisfying:*

(i) *The D_i's are pairwise disjoint.*

(ii) *The boundaries of D_i and D_{i+1} have at least a simple closed geodesic G_{i+1} in common.*

(iii) *The D_i's escape to infinity:*

$$\lim_{i\to\infty} \operatorname{dist}(p, D_i) = \infty.$$

(iv) *If $\mathcal{M}_i$ denotes the Riemann surface obtained from D_i by pasting a funnel along each one of the simple closed geodesics of its boundary, then*

$$\lim_{i\to\infty} \delta(\mathcal{M}_i) = 1.$$

Observe that D_i is the convex core of $\mathcal{M}_i$ (see e.g. [Ra]).

In the proof of this theorem we will use several times the following statement, which appears in [AR]:

THEOREM C. *If $\mathcal{N}$ is a recurrent hyperbolic surface, and $\bar{B}$ is a closed ball in $\mathcal{N}$, then there exists a geodesic domain D in $\mathcal{N}$ such that $\bar{B} \subset D$.*

Proof of Theorem 4.1. The construction of the sequence of geodesic domains D_i in $\mathcal{M}$ proceeds inductively. As a matter of fact the D_i's will satisfy

(iii′) $\operatorname{dist}(p, D_i) \geqslant i$,

(iv′) $\delta(\mathcal{M}_i) \geqslant 1 - 1/i$.

First, we choose a geodesic domain D_0 such that $p \in D_0$. Let $\mathcal{E}_1$ be a component of $\mathcal{M} \setminus D_0$ of infinite area. (Recall that $\operatorname{area}(\mathcal{M}) = \infty$.) One of the boundary components of D_0, say G_1, is also a component of the boundary of $\mathcal{E}_1$. The component G_1 is a simple closed geodesic.

Now, suppose that we have already determined a family $\{D_j\}_{0 \leqslant j \leqslant k}$ of geodesic domains verifying the conditions (i), (ii) and (iv′) above, and satisfying the following additional property, for $j \leqslant k$:

There is a closed geodesic G_{j+1} in the boundary of D_j such that the component $\mathcal{E}_{j+1}$ of $\mathcal{M} \setminus \bigcup_{l=0}^{j} D_l$ which contains G_{j+1} on its boundary satisfies

(4.1) $\operatorname{area}(\mathcal{E}_{j+1}) = \infty$,

(4.2) $\mathcal{E}_{j+1} \subset \mathcal{E}_j$ and $\operatorname{dist}(p, \mathcal{E}_j) \geqslant j$.

Let G_{k+1} and $\mathcal{E}_{k+1}$ be, respectively, the simple closed geodesic and the component of $\mathcal{M} \setminus \bigcup_{l=0}^{k} D_l$ with infinite area given by the property (4.1) for $j = k$.

There are several steps to determine D_{k+1}.

First, we construct a recurrent hyperbolic surface $\widetilde{\mathcal{E}}_{k+1}$ containing $\mathcal{E}_{k+1}$ isometrically. $\widetilde{\mathcal{E}}_{k+1}$ is obtained from $\mathcal{E}_{k+1}$ by pasting handles along each one of the simple closed geodesics which are the components of the relative boundary of $\mathcal{E}_{k+1}$. Let us denote the union of these handles by $\mathcal{H}$:

$$\mathcal{E}_{k+1} = \widetilde{\mathcal{E}}_{k+1} \setminus \mathcal{H}.$$

Observe that $\widetilde{\mathcal{E}}_{k+1}$ is recurrent and of infinite area.

Next, we take a ball U in $\widetilde{\mathcal{E}}_{k+1}$ big enough so that

(4.3) all the added handles are contained in U,

(4.4) $\operatorname{dist}(\widetilde{\mathcal{E}}_{k+1} \setminus U, \partial \mathcal{E}_{k+1}) \geqslant 1$,

(4.5) $\operatorname{area}(U \setminus \mathcal{H}) \geqslant R$, where R is an appropriate constant (which depends only on D_k) which shall be fixed later.

Let P be a geodesic domain in $\widetilde{\mathcal{E}}_{k+1}$ which contains U. We take now an even bigger ball V such that

(4.6) $P \subset V$,

(4.7) the family of curves Γ in $V \setminus P$ joining ∂P with ∂V has extremal length (see [Ah]) at least 1.

(V can be choosen so that this extremal length is as large as desired, but the bound 1 is enough.)

Now, let Q be a geodesic domain in $\widetilde{\mathcal{E}}_{k+1}$ which contains V. Repeatedly we have used that $\widetilde{\mathcal{E}}_{k+1}$ is of infinite area and recurrent, and have applied Theorem C. Since $\widetilde{\mathcal{E}}_{k+1}$ has infinite area there is a simple closed geodesic G in the boundary of Q such that the component of $\widetilde{\mathcal{E}}_{k+1}\setminus Q$ which contains G on its boundary has infinite area. Observe that

$$U\subset P\subset V\subset Q\subset \widetilde{\mathcal{E}}_{k+1}.$$

Finally, we define the domain D_{k+1} as the geodesic domain $Q\setminus\mathcal{H}$, the geodesic G_{k+2} as the closed geodesic G, and, of course, the end $\widetilde{\mathcal{E}}_{k+2}$ as the component of $\mathcal{E}_{k+1}\setminus D_{k+1}$ which contains G_{k+2} on its boundary.

Properties (4.1) and (4.2) for $k+1$ follow from the construction and properties (4.3) and (4.4). Also it is easy to see (by construction) that D_{k+1} satisfies the conditions (i) and (ii) of the statement. The condition (iii′) follows from (4.2).

To finish the proof, we have to verify that the Riemann surface $\mathcal{M}_{k+1}$, obtained by pasting to the geodesic domain D_{k+1} a funnel along each one of the simple closed geodesics of its boundary, has exponent of convergence greater than or equal to $1-1/(k+1)$. To do this we shall exhibit an appropriate test function to verify that $\beta(\mathcal{M}_{k+1})\leqslant 1/2(k+1)$. Then, by Theorem B of §1, we conclude that $\delta(\mathcal{M}_{k+1})\geqslant 1-1/(k+1)$.

Observe that $\mathcal{M}_{k+1}$ is the union of D_{k+1} with $\mathcal{J}$ and $\mathcal{K}$, where $\mathcal{J}$ is the union of the funnels attached to D_{k+1} on its boundary with D_k, and $\mathcal{K}$ is the union of the rest of the attached funnels.

A test function Φ is defined as follows:

- On D_{k+1}: We define Φ on $P\setminus\mathcal{H}$ as 1, and on $Q\setminus P$ to be harmonic with boundary values 1 on ∂P and 0 on ∂Q (these two are boundaries relative to $\widetilde{\mathcal{E}}_{k+1}$).
- On $\mathcal{K}$: Φ is 0.
- On $\mathcal{J}$: $\Phi(q)=(1-\operatorname{dist}(q,\partial\mathcal{J}))^+$.

We use (an approximation of) Φ to estimate $\beta(\mathcal{M}_{k+1})$; from (4.5) and (4.7) we obtain

$$\beta(\mathcal{M}_{k+1})\leqslant \frac{\int_{\mathcal{J}}\|\nabla\Phi\|^2+1/\lambda(\Gamma)}{\operatorname{area}(P\setminus\mathcal{H})}\leqslant \frac{\int_{\mathcal{J}}\|\nabla\Phi\|^2+1}{\operatorname{area}(U\setminus\mathcal{H})}\leqslant \frac{\int_{\mathcal{J}}\|\nabla\Phi\|^2+1}{R}.$$

Hence, choosing R large enough it follows that $\beta(\mathcal{M}_{k+1})\leqslant 1/2(k+1)$.

5. Geodesics connecting two closed geodesics

Throughout this section, $\mathcal{N}=\mathbf{D}/\Gamma$ is a non-elementary Riemann surface with exponent of convergence δ.

Fix two oriented simple closed geodesics, G_1 and G_2, in $\mathcal{N}$, and two points q_1, q_2 such that $q_i \in G_i$ $(i=1,2)$. The case $G_1=G_2$ is allowed. The next theorem shows that we can find a large collection of long geodesics from q_1 to q_2 with precise control on the angles of intersection with G_1 and G_2.

THEOREM 5.1. *For any $\eta>0$, $\Psi\in(0,\frac{1}{2}\pi)$, and for L large enough (depending on η and Ψ), there is a collection $\mathcal{S}$ of geodesic arcs in $\mathcal{N}$ from q_1 to q_2 such that:*

(i) *For all $\gamma\in\mathcal{S}$,*

$$L \leqslant \operatorname{length}\gamma \leqslant L+\Delta(G_2),$$

where $\Delta(G_2)$ is a positive constant which depends only on the length of G_2.

(ii) *For all $\gamma\in\mathcal{S}$, both the absolute value of the angle between γ and G_1 at q_1, and the absolute value of the angle between γ and G_2 at q_2, are less than or equal to Ψ.*

(iii) *The angle at q_1 between any two geodesic arcs of $\mathcal{S}$ is (in absolute value) at least*

$$\frac{c}{\sin\Psi}\, e^{-L},$$

with $c>0$ an absolute constant.

(iv) *The number of geodesic arcs in $\mathcal{S}$ is at least*

$$e^{L(\delta-\eta)}.$$

It is important to remark that if G_1 and G_2 are closed geodesics limiting funnels of $\mathcal{N}$, then every geodesic arc $\gamma\in\mathcal{S}$ is contained in the convex core of $\mathcal{N}$.

Proof. By conjugation, we may assume that the interval $(-1,1)$, oriented from -1 to 1, projects onto the oriented geodesic G_1, and that 0 projects onto q_1.

Applying Proposition 2.2 to the interval I from $-\frac{1}{2}\Psi$ to $\frac{1}{2}\Psi$, and with $\sigma<\delta$, we can get, for $n>0$ large enough, a set $\mathcal{T}$ of liftings of the geodesic G_2 verifying:

(5.1) Each $\widetilde{G}_2\in\mathcal{T}$ has

$$e^{-(n+1)} \leqslant \operatorname{diam}(\widetilde{G}_2) < e^{-n}.$$

(5.2) Each $\widetilde{G}_2$ in $\mathcal{T}$ has both endpoints in $2I=\{e^{i\theta}: -\Psi\leqslant\theta\leqslant\Psi\}$.

(5.3) If $\widetilde{G}_2, \widetilde{G}'_2\in\mathcal{T}$, $z\in\widetilde{G}_2$ and $z'\in\widetilde{G}'_2$, then the absolute value of the angle at 0 between the radius through z and the radius through z' is at least $e^{-(n+1)}$. (One has to get rid of at most half of $\mathcal{T}$ to obtain this separation property.)

(5.4) $\#\mathcal{T}\geqslant e^{n\sigma}$.

To each *oriented* geodesic $\widetilde{G}_2 \in \mathcal{T}$ we associate a point z_2 in $\widetilde{G}_2$ as follows: z_2 is the *first* preimage (under the projection Π) of q_2 so that the radius from 0 to z_2 intersects $\widetilde{G}_2$ with angle at most Ψ. From (5.1) and Lemma 2.4.1 we have that

$$c_1 e^{-n} \sin \Psi \leqslant e^{-d(0,z_2)} \leqslant c_2 e^{-n} \sin \Psi, \tag{5.5}$$

with c_1, c_2 positive constants; c_2 is absolute, while c_1 depends only on the length of G_2.

The family of geodesics $\mathcal{S}$ consists of the projections (onto $\mathcal{N}$) of the radial segments from 0 to the points z_2.

Property (ii) is immediate.

If γ is an arc in $\mathcal{S}$, then from (5.5) we have that

$$L \leqslant \operatorname{length}(\gamma) \leqslant L + \Delta(G_2),$$

with

$$L = \log \frac{e^n}{c_2 \sin \Psi} \quad \text{and} \quad \Delta(G_2) = \log \frac{c_2}{c_1}.$$

Observe that $\Delta(G_2)$ only depends on the length of G_2. This proves condition (i), if n is large enough.

Writing

$$e^{-(n+1)} = \frac{e^{-(L+1)}}{c_2 \sin \Psi}$$

we see that (5.3) implies condition (iii).

Finally, the definition of L and (5.4) imply that

$$\# \mathcal{S} \geqslant \tfrac{1}{2} (c_2 \sin \Psi)^{\sigma} e^{L\sigma} = c_3 e^{L\sigma} > e^{L(\delta-\eta)}$$

for n large enough and $\sigma > \delta - \eta$.

6. Proof of Theorem 1

In this section we shall assume that $\mathcal{M}$ is a recurrent hyperbolic surface with infinite area.

A cuspidal end $\mathcal{F}$ of $\mathcal{M}$ may be represented as a tube T isometric to $\{z \in \mathbf{C} : 0 < |z| < a\}$ endowed with the metric

$$ds = \frac{|dz|}{|z| \log(1/|z|)},$$

for some positive number a. A geodesic escaping from a point p escapes to ∞ through $\mathcal{F}$ only if it intersects ∂T orthogonally. Therefore there is a unique escaping geodesic in every homotopy class of curves which starts at p and ends on ∂T; in particular, there are exactly countably many geodesics from p escaping to ∞ through $\mathcal{F}$.

To prove that $\mathcal{E}(p)$ has Hausdorff dimension 1, we have to look for geodesics escaping from p to infinity through ends of infinite area.

Proof of Theorem 1. We first get a family $\{D_i\}$ of geodesic domains in $\mathcal{M}$, with the corresponding collections $\{G_i\}$ of closed geodesics, and $\{\mathcal{M}_i\}$ of Riemann surfaces, satisfying the conclusions of Theorem 4.1. Next, we fix a sequence of points $\{p_i\}_{i=0}^{\infty}$ in $\mathcal{M}$ such that $p_0{:=}p$ and $p_i \in G_i$ for $i \geqslant 1$.

For each $i>0$, we get a collection $\mathcal{S}_i$ of geodesics in $\mathcal{M}_i$, with initial and final endpoint $p_i \in G_i$, and satisfying the conclusions of Theorem 5.1 with L_i (instead of L) appropriately large, a fixed Ψ appropriately small, and σ_i, which tends to 1, instead of $\delta-\eta$. (Here we are using that $\delta(\mathcal{M}_i) \to 1$.)

Finally, for each i we also get one (sic!) geodesic ω_i in $\mathcal{M}_i$ from $p_i \in G_i$ to $p_{i+1} \in G_{i+1}$ satisfying the conclusions (i) and (ii) of Theorem 5.1 (with the same L_i and Ψ as above).

Observe that the curve ω_i and all the curves in $\mathcal{S}_i$ are contained in D_i (see the remark after the statement of Theorem 5.1).

Let ω_0 be a geodesic arc in $\mathcal{M}$ from p_0 to p_1 of length larger than, say, L_0, and such that the intersection angle at p_1 between ω_0 and G_1 is at most Ψ. We may assume that 0 projects on p_0. So, if $\widetilde{G}_1$ denotes a lifting of G_1 with endpoint ξ, then a radial segment ending at a preimage of p_1 in $\widetilde{G}_1$ which is close enough to ξ will project onto such an ω_0.

The sequence of L_i's and the value of Ψ shall be determined later.

Now we are going to construct a tree $\mathcal{T}$ consisting of oriented geodesic arcs in $\mathbf{D}$.

First, lift ω_0 starting at 0.

From the endpoints of the lifting of ω_0 (which project onto p_1), lift the family $\mathcal{S}_1$; from each of the endpoints of these liftings (which still project onto p_1), lift again the family $\mathcal{S}_1$. We keep lifting $\mathcal{S}_i$ in this way a total of K_1 times.

Next, from each one of the endpoints obtained in the process above, we lift ω_1, and then from the endpoints of each one of those liftings of ω_1 (which project onto p_2), we perform K_2 sucessive liftings of the family $\mathcal{S}_2$, in the same way as above. And so on.

The sequence $\{K_i\}$ of repetitions shall be determinated later.

The tree $\mathcal{T}$ contains an uncountable collection $\mathcal{B}$ of (infinite) branches which are piecewise geodesics starting at 0. Clearly, the projections of these branches escape to infinity in $\mathcal{M}$, since the geodesic domains D_i do so. This means, in particular, that each branch in $\mathcal{B}$ converges to $\partial\mathbf{D}$. But more is true.

LEMMA 6.1. *Each branch b of $\mathcal{T}$ converges non-tangentially to a point (its* tip*) in $\partial\mathbf{D}$. More precisely, if ξ is the tip of b, then every point in b is within distance C from the radius from 0 to ξ, where C is some absolute positive constant.*

Given our freedom to choose Ψ and the L_i's, the above lemma follows directly from Lemma 2.4.2.

The set of all the tips of the branches of the tree $\mathcal{T}$ is called the *rim of $\mathcal{T}$*.

As a consequence of Lemma 6.1, the projection of a radius ending at a point in the rim of $\mathcal{T}$ also escapes to infinity. But those projections are geodesics in $\mathcal{M}$ emanating from p_0. To finish the proof, all we have to do is to show that the rim of $\mathcal{T}$ has Hausdorff dimension 1.

The vertices $\mathcal{V}$ of the tree $\mathcal{T}$ are classified into generations according to its graph distance from the root 0:

$$V_n = \{v \in \mathcal{V} : \text{graph distance}(v, v_0) = n\}.$$

The vertices in V_n are called the vertices of the generation n. Of course, the 0th generation is $V_0 = \{0\}$. Let u be a vertex in V_n, and v be a vertex in V_{n+1}. If u and v are connected by an arc in $\mathcal{T}$, then we will say that u is *the mother of* v, and consequently that v is *a daughter of* u. Moreover, we define the shadow of v, $S(v)$, as the set of points $w \in \partial\mathbf{D}$ such that the angle at v between the geodesic emanating from v with endpoint w and the geodesic emanating from u and going through v is less than or equal to $\frac{1}{4}\pi$.

If the L_i's are large enough, and Ψ is small, then the shadow of a daughter is contained in the shadow of its mother, and the shadows of daughters of a given mother are disjoint. To guarantee this, one needs Ψ small so that the bound in Theorem 5.1 (iii) becomes $C_0 e^{-L}$, where C_0 is some quite small absolute constant. We refer to Lemma 2.1.1 and Corollary 2.1.1 in [FM2] for details. (The shadows in [FM2] are a bit more general.)

The rim of $\mathcal{T}$ can now be described as

$$\text{rim of } \mathcal{T} = \bigcap_{n=1}^{\infty} \bigcup_{v \in V_n} S(v).$$

To see that the rim of $\mathcal{T}$ has Hausdorff dimension 1 we will use Theorem 3.1 on patterns.

If v is a daughter of u, then

$$\frac{1}{C} e^{-d(u,v)} \leqslant \frac{|S(v)|}{|S(u)|} \leqslant C e^{-d(u,v)}, \tag{6.1}$$

where $C > 0$ is an absolute constant. Again this require large L_i's and $\Psi \leqslant \frac{1}{4}\pi$; for details see [FM2, Lemma 2.1.2].

With the notations of §3, we see that the rim of $\mathcal{T}$ is a Cantor-like set obtained by a sequence $\{\mathcal{P}_i\}$ of patterns with parameters

$$r_i \geqslant \frac{1}{C} e^{-L_i - \Delta(G_{i+1})}, \quad R_i \leqslant C e^{-L_i} \quad \text{and} \quad N_i \geqslant e^{\sigma_i L_i}.$$

Here $\Delta(G_{i+1})$ is a constant depending on the length of the closed geodesic G_{i+1}, and C is the constant in (6.1). We are using Theorem 5.1 (i) and (iv), and (6.1).

We can take the s_i's in §3 as

$$s_i = \frac{1}{C} e^{-\text{length}(\omega_i)}.$$

(We are using again (6.1).)

The chain of domains D_i is our starting point. We remark that the sequences of constants, $\Delta(G_i)$ and s_i, are known altogether from the very beginning.

Therefore we may choose the L_i's satisfying

$$L_i \to \infty \quad \text{and} \quad \frac{\Delta(G_{i+1})}{L_i} \to 0$$

when $i \to \infty$.

As a consequence we obtain that

$$\liminf_{i\to\infty} \frac{\log(N_i r_i / R_i)}{\log(1/R_i)} = 1. \tag{6.2}$$

Here we are using that $\sigma_i \to 1$.

Hence, by Theorem 3.1 and (6.2) we have that there exists a sequence of repetitions K_i so that the rim of $\mathcal{T}$ has Hausdorff dimension 1.

As it was mentioned in the introduction, there is a stronger version of Theorem 1:

THEOREM 1′. *Given an interval I on $\mathcal{S}(p)$, and a recurrent end $\mathcal{F}$, there exists a closed set $A \subset I \cap \mathcal{E}(p)$ and a homeomorphism $\Phi: [0,\infty) \to [0,\infty)$ such that:*

(i) *If $v \in A$ and γ_v is the geodesic emanating from p with direction v, then*

$$\text{dist}(p, \gamma_v(t)) \geqslant \Phi(t),$$

and for $t \geqslant t_0$,

$$\gamma_v(t) \in \mathcal{F}.$$

(ii) *The Hausdorff dimension of A is* 1.

The proof of Theorem 1′ is a minor modification of the proof of Theorem 1.

7. Some applications and questions

(A) *Function theory.* As we mentioned above, Theorem 1 has applications to classical function theory.

Let f be a holomorphic function defined in the unit disk $\mathbf{D}$, and let us denote its range by $\Omega=f(\mathbf{D})$. A classical result of Nevanlinna claims:

THEOREM F. *If the logarithmic capacity of $\partial\Omega$ is not zero, then f has radial boundary values a.e. in $\partial\mathbf{D}$, i.e.*

$$\lim_{r\to 1} f(re^{i\theta}) \text{ exists for a.e. } \theta\in[0,2\pi].$$

Recall that the domain Ω is transient if and only if its boundary has positive logarithmic capacity, and that it has finite hyperbolic area if and only if has a finite complement in $\mathbf{C}$.

On the other hand, if E is a compact subset of $\mathbf{C}$ of logarithmic capacity zero, then the covering maps from $\mathbf{D}$ onto $\Omega=\mathbf{C}\setminus E$ have radial boundary values almost nowhere in $\partial\mathbf{D}$.

This result may be complemented in the following way:

THEOREM G [FN]. *If the logarithmic capacity of $\partial\Omega$ is zero, but $\partial\Omega$ is an infinite set, then the radial limits exist for all θ belonging to a set of Hausdorff dimension* 1.

If f omits only finitely many points then all one can assure is that there are countably many θ where the radial boundary value exists.

With no restrictions on the range of f, and no further assumptions on f, nothing can be assured along these lines. There are positive results if the functions involved are *Bloch functions*, i.e. holomorphic functions in $\mathbf{D}$ which are Lipschitz from the hyperbolic metric of $\mathbf{D}$ to the Euclidean metric of $\mathbf{C}$, see [Mak], [Ro], or have a restriction on its growth, [FL1].

If $f:\mathbf{D}\to\mathcal{M}$ is a holomorphic mapping with values in a Riemann surface $\mathcal{M}$, we say that f is *inner* if and only if the set

$$\{\theta\in[0,2\pi): \exists \lim_{r\to 1} f(re^{i\theta})\in\mathcal{M}\}$$

has measure zero. In other words, if we factorize f as $f=F\circ b$ with $F:\mathbf{D}\to\mathcal{M}$ the covering mapping, then f is inner if and only if b is inner in the usual way.

Using Theorem 1′ and arguing as in [FP], we get the following general result:

THEOREM 7.1. *Let $\mathcal{M}$ be a recurrent hyperbolic surface of infinite area, and let $f:\mathbf{D}\to\mathcal{M}$ be an inner function. Then, for all points $p\in\mathcal{M}$, the set*

$$\left\{\theta : \lim_{r\to 1} \operatorname{dist}(f(re^{i\theta}), p) = +\infty\right\}$$

has Hausdorff dimension 1.

(B) *Variable curvature.* One would expect that some version of the results above should hold for complete surfaces with pinched Gaussian curvature K, $-b^2 \leqslant K \leqslant -a^2 < 0$, or even further, for general Riemannian manifolds with pinched Gaussian curvature.

References

[Ah] AHLFORS, L. V., *Conformal Invariants.* McGraw-Hill, New York, 1973.

[AR] ALVAREZ, V. & RODRÍGUEZ, J., Structure theorems for topological and Riemann surfaces. Preprint.

[Ba] BANGERT, V., On the existence of escaping geodesics. *Comment. Math. Helv.*, 56 (1981), 59–65.

[Bea] BEARDON, A., *The Geometry of Discrete Groups.* Graduate Texts in Math., 91. Springer-Verlag, New York, 1983.

[Ber] BERS, L., An inequality for Riemann surfaces, in *Differential Geometry and Complex Analysis*, pp. 87–93. Springer-Verlag, Berlin, 1985.

[BJ] BISHOP, C. J. & JONES, P. W., Hausdorff dimension and Kleinian groups. *Acta Math.*, 179 (1997), 1–39.

[Bu] BUSER, P., *Geometry and Spectra of Compact Riemann Surfaces.* Progr. Math., 106. Birkhäuser-Boston, Boston, MA, 1992.

[CdV] COLIN DE VERDIÈRE, Y., Théorie spectrale des surfaces de Riemann d'aire infinie, in *Colloquium in Honor of Laurent Schwartz*, Vol. 2 (Palaiseau, 1983), pp. 259–275. Asterisque, 132. Soc. Math. France, Paris, 1985.

[CE] CHEEGER, J. & EBIN, D. G., *Comparison Theorems in Riemannian Geometry.* North-Holland Math. Library, 9. Elsevier, New York, 1975.

[FL1] FERNÁNDEZ, J. L. & LLORENTE, J. G., A note on the boundary behaviour of harmonic functions. *J. London Math. Soc.* (*2*), 46 (1992), 295–300.

[FL2] — Geodesic excursions to infinity in Riemann surfaces. *J. Anal. Math.*, 64 (1994), 97–119.

[FM1] FERNÁNDEZ, J. L. & MELIÁN, M. V., Bounded geodesics of Riemann surfaces and hyperbolic manifolds. *Trans. Amer. Math. Soc.*, 347 (1995), 3533–3549.

[FM2] — Rims of Cantor trees. *Illinois J. Math.*, 44 (2000), 329–348.

[FN] FERNÁNDEZ, J. L. & NICOLAU, A., Boundary behaviour of inner functions and holomorphic mappings. *Math. Ann.*, 310 (1998), 423–445.

[FP] FERNÁNDEZ, J. L. & PESTANA, D., Radial images by holomorphic mappings. *Proc. Amer. Math. Soc.*, 124 (1996), 429–435.

[Ga] GARNETT, J., *Applications of Harmonic Measure.* Univ. Arkansas Lecture Notes Math. Sci., 8. Wiley, New York, 1986.

[HT] HULIN, D. & TROYANOV, M., Prescribing curvature on open surfaces. *Math. Ann.*, 293 (1992), 277–315.

[Hu] HUNGERFORD, G., Boundaries of smooth sets and singular sets of Blaschke products in the little Bloch class. Thesis, California Institute of Technology, Pasadena, 1988.

[Mae] MAEDA, M., A geometric significance of total curvature on complete open surfaces, in *Geometry of Geodesics and Related Topics* (Tokyo, 1982), pp. 451–458. Adv. Stud. Pure Math., 3. North-Holland, Amsterdam, 1984.

[Mak] MAKAROV, N. G., Smooth measures and the law of the iterated logarithm. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53 (1989), 439–446 (Russian); English translation in *Math. USSR-Izv.*, 34 (1990), 455–463.

[Pa] PATTERSON, S. J., Diophantine approximation and Fuchsian groups. *Philos. Trans. Roy. Soc. London Ser. A*, 282 (1976), 527–563.

[Ra] RATCLIFFE, J. G., *Foundations of Hyperbolic Manifolds.* Graduate Texts in Math., 149. Springer-Verlag, New York, 1994.

[Ro] ROHDE, S., The boundary behavior of Bloch functions. *J. London Math. Soc.* (*2*), 48 (1993), 488–499.

[Shioh] SHIOHAMA, K., An integral formula for the measure of rays on complete open surfaces. *J. Differential Geom.*, 23 (1986), 197–205.

[Shioy] SHIOYA, T., Mass of rays in Alexandrov spaces of nonnegative curvature. *Comment. Math. Helv.*, 69 (1994), 208–228.

[SST] SHIOHAMA, K., SHIOYA, T. & TANAKA, M., Mass of rays on complete open surfaces. *Pacific J. Math.*, 143 (1990), 349–358.

[St] STRATMANN, B., The Hausdorff dimension of bounded geodesics on geometrically finite manifolds. *Ergodic Theory Dynam. Systems*, 17 (1997), 227–246.

[Su1] SULLIVAN, D., Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics. *Acta Math.*, 149 (1982), 215–237.

[Su2] — Related aspects of positivity in Riemannian geometry. *J. Differential Geom.*, 25 (1987), 327–351.

[Ts] TSUJI, M., *Potential Theory in Modern Function Theory.* Chelsea, New York, 1975.

[Wo] WOJTKOWSKI, M., Oscillating geodesics on 2-dimensional manifolds, in *Dynamical Systems*, Vol. III (Warsaw, 1977), pp. 443–456. Asterisque, 51. Soc. Math. France, Paris, 1978.

JOSÉ L. FERNÁNDEZ
Departamento de Matemáticas
Universidad Autónoma de Madrid
ES-28049 Madrid
Spain
joseluis.fernandez@uam.es

MARÍA V. MELIÁN
Departamento de Matemáticas
Universidad Autónoma de Madrid
ES-28049 Madrid
Spain
mavi.melian@uam.es

Received January 21, 1997
Received in revised form July 10, 2000