$L^p$ Carleman inequalities and uniqueness of solutions of nonlinear Schrödinger equations

by

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1. Introduction

The purpose of this paper is twofold. First we prove a delicate Carleman inequality, involving nonconvex weights, for the operator $i\partial_t + \Delta_x$ acting on functions on $\mathbb{R}^n \times [-T,T]$. Then we use this inequality to study uniqueness properties of solutions of nonlinear Schrödinger equations of the form

$$(i\partial_t + \Delta_x)u = Vu + F(u),$$

where $V$ is a potential and $F$ is a nonlinear term. We are concerned with the following type of question:

**Question Q.** Assume that $u_1$ and $u_2$ are solutions in $\mathbb{R}^n \times [0,1]$ to (1.1) (in a suitable function space) with the property that for some domain $D \subseteq \mathbb{R}^n$ we have $u_1(x,0) = u_2(x,0)$ and $u_1(x,1) = u_2(x,1)$ for a.e. $x \in D$. Can we then conclude that $u_1 \equiv u_2$ in $D' \times [0,1]$ for some domain $D'$?

In our theorems the domain $D$ will be a half-space.$^{(1)}$ Under suitable assumptions on the potential $V$, the function $F$ and the solutions $u_1$ and $u_2$, we answer Question Q in the affirmative, with the domain $D'$ equal to the entire $\mathbb{R}^n$.

This type of uniqueness question seems to originate in control theory. Zhang [21] used inverse scattering theory to answer Question Q in the affirmative in the special case.

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$^{(1)}$ We are not aware of any positive results for domains $D$ that do not contain a half-space.
case $n=1$, $V \equiv 0$, $F = \alpha |u|^2 u$, $\alpha \in \mathbb{R}$, $u_2 \equiv 0$, $D = (a, \infty)$, with $D' = \mathbb{R}$. Bourgain [1] proved uniqueness under analyticity assumptions on the nonlinear term $F = F(u, \bar{u})$, with $u_2 \equiv 0$, $V \equiv 0$ and the stronger assumption that $u_1$ is compactly supported for all $t \in [0, 1]$. Kenig, Ponce and Vega [9] answered Question Q in the affirmative for sufficiently smooth functions $u_1$ and $u_2$, when the domain $D$ is the complement of a convex cone, $V \equiv 0$ and $F = F(u, \bar{u})$ satisfies bounds of the form

$$|\nabla F(u, \bar{u})| \leq C(|u|^{p_1-1} + |u|^{p_2-1}), \quad p_1, p_2 > 1.$$ 

We remark that the Carleman estimates of [9] could also have been used to include potentials $V \in L^\infty_{\text{loc}}(\mathbb{R}^n \times [0, 1]) \cap L^1 L^\infty_{\text{loc}}(\mathbb{R}^n \times [0, 1])$, with

$$\lim_{R \to \infty} \|V\|_{L^1 L^\infty_{\text{loc}}(x : |x| \geq R)} = 0.$$ 

See also the remark following Theorem 2.1. Local unique continuation theorems were proved by Isakov [6].

A question similar to Question Q was considered in the setting of the generalized Korteweg-de Vries equation on $\mathbb{R} \times \mathbb{R}$,

$$\left(\partial_t + \partial_x^2\right)u + F(x, t, u, \partial_x u, \partial_x^2 u) = 0.$$ 

Zhang [20] proved uniqueness if $u_2 \equiv 0$, in the cases $F = u \partial_x u$ and $F = -6u^2 \partial_x u$. This was extended by Kenig, Ponce and Vega [8], [10] to include a large family of functions $F$, as well as two nonzero solutions $u_1$ and $u_2$. Bourgain [1] proved uniqueness of solutions for the more general nonlinear equation

$$\left(i^{s-1} \partial_t + \partial_x^2\right)u + F(u, \partial_x u, \ldots, \partial_x^{s-2} u) = 0, \quad s \geq 2,$$

under analyticity assumptions on $F$, with $u_2 \equiv 0$ and $u_1$ compactly supported at each time $t \in [0, 1]$. This last equation was also considered by Kenig, Ponce and Vega [11], who proved uniqueness under more general assumptions on $F$, as well as for two nonzero solutions $u_1$ and $u_2$. Local unique continuation theorems were proved by Saut and Scheurer [18].

Our results in this paper (and the methods used) mostly resemble those of Kenig, Ponce and Vega [9]. However, we prove theorems under weaker regularity assumptions on the potential $V$ and the function $F$; in particular, we allow locally unbounded potentials $V$. We also improve the space of solutions $u$ for which we have uniqueness, and reduce the domain $D$ on which we require the solutions to agree. To explain our theorems, consider the simplest assumption on the potential $V$ and the function $F$, namely
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VEL(n+2)/2(Rn×[0,1]) and F≡0. Let H denote the operator \( i\partial_t + \Delta_x \). The relevant Carleman inequality to use in this case is

\[
\|e^{\beta \phi_\lambda(x_1)} u\|_{L^{(2n+4)/(n+4)}(\mathbb{R}^n \times [0,1])} \leq C \|e^{\beta \phi_\lambda(x_1)} H u\|_{L^{(2n+4)/(n+4)}(\mathbb{R}^n \times [0,1])} + C \|e^{\beta \phi_\lambda(x_1)} u(\cdot, 0)\|_{L^2(\mathbb{R}^n)} + \|e^{\beta \phi_\lambda(x_1)} u(\cdot, 1)\|_{L^2(\mathbb{R}^n)}.
\]

In Theorem 2.1 we prove a stronger estimate for functions \( u \in C([0,1];L^2(\mathbb{R}^n)) \) with \( Hu \in L^{(2n+4)/(n+4)}(\mathbb{R}^n \times [0,1]) \), any \( \beta \geq 0 \) and any \( \lambda \geq \Lambda(\beta) \). The function \( \phi_\lambda \) is defined by \( \phi_\lambda(r) = \lambda \phi(r/\lambda) \), where \( \phi \) is a fixed smooth function on \( \mathbb{R} \) with the properties \( \phi(0)=0 \), \( \phi' \) nonincreasing, \( \phi'(r)=1 \) if \( r \leq 1 \), and \( \phi'(r)=0 \) if \( r \geq 2 \). The main point of this Carleman inequality is uniformity: the constant \( C \) should not depend on \( \beta \), \( \lambda \) or the function \( u \).

We would like to apply the inequality (1.2) to the function \( u = u_1 - u_2 \), where \( u_1 \) and \( u_2 \) are the two solutions in Question Q, and let \( \beta, \lambda \to \infty \). For the Carleman argument to go through (i.e. to be able to absorb the main term in the right-hand side) we need to have

\[
\|e^{\beta \phi_\lambda(x_1)} u\|_{L^{(2n+4)/(n+4)}(\mathbb{R}^n \times [0,1])} < \infty.
\]

This condition explains why it is important to prove a Carleman inequality like (1.2) with a bounded weight \( e^{\beta \phi_\lambda(x_1)} \). Such a Carleman inequality can be applied to all solutions \( u_1 \) and \( u_2 \) in \( C([0,1];L^2(\mathbb{R}^n)) \) with \( Hu_1, Hu_2 \in L^{(2n+4)/(n+4)}(\mathbb{R}^n \times [0,1]) \). For contrast, by (1.3), the easier Carleman inequality with the weight \( e^{\beta \phi_\lambda(x_1)} \) replaced by the exponential weight \( e^{\beta x_1} \) can only be applied to solutions \( u_1 \) and \( u_2 \) that have faster-than-exponential decay at infinity. This was already noticed by Kenig, Ponce and Vega [9], who proved \( L^2 \) Carleman inequalities with the bounded weight \( e^{\beta \phi_\lambda(x_1)} \). It is also similar to the situation in the proof of Ionescu and Jerison [4] of the absence of positive eigenvalues of Schrödinger operators \(-\Delta + V\) on \( \mathbb{R}^n\) to eliminate the possibility of all \( L^2 \) solutions one needs a Carleman inequality with nonconvex weights.

Assume as before that \( V \in L^{(n+2)/2}(\mathbb{R}^n \times [0,1]) \) and \( F \equiv 0 \). Assume that \( u_1, u_2 \in C([0,1];L^2(\mathbb{R}^n)) \) are solutions to (1.1), with \( Hu_1, Hu_2 \in L^{(2n+4)/(n+4)}(\mathbb{R}^n \times [0,1]) \). For any \( w_0 \in \mathbb{R}^n \), \( |w_0|=1 \), let \( D(w_0) = \{ x : x \cdot w_0 > 0 \} \) denote a half-space. The Carleman inequality (1.2) and an additional local argument can be used to prove that

\[
\text{if } u_1 \equiv u_2 \text{ in } D(w_0) \times \{0, 1\} \text{ then } u_1 \equiv u_2 \text{ in } \mathbb{R}^n \times [0,1].
\]

In Theorems 2.4 and 2.5 we prove uniqueness statements of this type under more general assumptions on the potential \( V \) and the function \( F \). We also have an existence theorem: If \( u(\cdot, 0) \in L^2(\mathbb{R}^n) \), \( V \in L^{(n+2)/2}(\mathbb{R}^n \times [0,1]) \) and \( F \equiv 0 \), the equation (1.1) admits a unique solution \( u \in C([0,1];L^2(\mathbb{R}^n)) \), with \( Hu \in L^{(2n+4)/(n+4)}(\mathbb{R}^n \times [0,1]) \) (see Theorem 2.7).
The rest of the paper is organized as follows: In §2 we set up the notation and state
the main theorems. The first of our main theorems is the Carleman inequality in Theo-
rem 2.1. We prove this inequality in §§3–8: First we construct suitable parametrices of
the conjugated operator \( e^{i\varphi(x)}(iD_t + \Delta_x)e^{-i\varphi(x)} \) (§3). To construct the parametrices
at a given frequency, we think of the equation as either an evolution in time, or a reverse
evolution in time, or an evolution in the variable \( x_1 \). Then we prove that these paramet-
rices are represented by operators which are bounded between Strichartz spaces (§§4–7).
The key technical ingredient we need is a theorem of Keel and Tao [7] that gives a simple
criterion for checking this boundedness. In §8 we prove that the remainder terms in the
parametrices are small. In §9 we apply the Carleman inequality to prove the uniqueness
theorems described above.

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of this paper.

2. The main theorems

We define the set \( \mathcal{A} \) of acceptable Strichartz exponents \((p, q)\) by the conditions
\[
\frac{2}{p} + \frac{n}{q} = \frac{n+4}{2}, \quad p \in [1, 2], \quad q \in [1, 2], \quad (p, q) \neq (2, 1).
\] (2.1)

For any \((p, q) \in \mathcal{A}\) let \((p', q')\) denote the dual exponent, i.e. \(1/p + 1/p' = 1/q + 1/q' = 1\).
Clearly \(2/p' + n/q' = n/2\); \(p' \in [2, \infty]\), \(q' \in [2, \infty]\) and \((p', q') \neq (2, \infty)\); let \(\mathcal{A}'\) denote the set
of such exponents \((p', q')\). The basic Strichartz spaces we will work with are
\[
L_t^p L_x^q = L_t^p L_x^q(\mathbb{R}^n \times \mathbb{R}) = \{ f \in L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}) : \|f\|_{L_t^p L_x^q} < \infty \},
\]
where \((p, q) \in \mathcal{A}\) or \((p, q) \in \mathcal{A}'\).

We define two Banach spaces of functions \(X\) and \(X'\) on \(\mathbb{R}^n \times \mathbb{R}\): if \(n=1\) then \(X =
L_t^1 L_\infty 2 + L_t^{1/3} L_\infty^1\) and \(X' = L_t^\infty L_\infty^2 \cap L_t^4 L_\infty^1\), i.e.
\[
\|f\|_{X} = \inf_{f_1 + f_2 = f} \{ \|f_1\|_{L_t^1 L_\infty^2(\mathbb{R}^n \times \mathbb{R})} + \|f_2\|_{L_t^{4/3} L_\infty^1(\mathbb{R}^n \times \mathbb{R})} \}
\]
and
\[
\|f\|_{X'} = \max\{ \|f\|_{L_t^{\infty} L_\infty^2(\mathbb{R}^n \times \mathbb{R})}, \|f\|_{L_t^4 L_\infty^1(\mathbb{R}^n \times \mathbb{R})} \}.
\]
If \(n \geq 3\) we define \(X = L_t^1 L_\infty^2 + L_t^2 L_\infty^{2n/(n+2)}\) and \(X' = L_t^\infty L_\infty^2 \cap L_t^{2n/(n-2)} L_\infty^2\). In dimension
\(n=2\) we have to exclude the endpoint spaces \(L_t^2 L_\infty^2\) and \(L_t^2 L_\infty^\infty\) for which the Strichartz
estimates fail (cf. [16]). For this purpose we fix an acceptable pair \((p_0, q_0)\), \(1 < p_0 < 2\), and
define \(X = X_{p_0} = L_t^1 L_\infty^2 + L_t^{p_0} L_\infty^{q_0}\) and \(X' = X'_{p_0} = L_t^\infty L_\infty^2 \cap L_t^{p_0} L_\infty^{q_0}\). Spaces of this type were
used in recent work by Koch and Tataru [14], [15]. They are often more suitable for Carleman inequalities than the spaces $L^p_t L^q_x$, since they allow better control of the error terms. We notice that

$$L^p_t L^q_x \subseteq X \quad \text{and} \quad L^p_t L^q_x \supseteq X'$$

if $(p, q) \in A$ (and $p \leq p_0$ if $n=2$), and

$$\int_{\mathbb{R}^n \times \mathbb{R}} fg \, dx \, dt \leq \|f\|_X \|g\|_{X'},$$

for any locally integrable functions $f$ and $g$.

For any interval $[a, b]$, we define the Banach space $X([a, b])$ as the space of locally integrable functions $f: \mathbb{R}^n \times [a, b] \to \mathbb{C}$ with

$$\|f\|_{X([a, b])} := \|\tilde{f}\|_X < \infty,$$  \hspace{1cm} (2.2)

where $\tilde{f}(x, t) = f(x, t)$ if $t \in [a, b]$ and $\tilde{f}(x, t) = 0$ if $t \notin [a, b]$. We define the space $X'([a, b])$ in a similar way. For a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, we set, by a slight abuse of notation,

$$\|f\|_{X([a, b])} := \sup_{\eta} \|\eta(t)f\|_{X([a, b])},$$  \hspace{1cm} (2.3)

where the supremum is taken over all smooth functions $\eta: \mathbb{R} \to [0, 1]$ supported in $(a, b)$. Clearly, $\|f\|_{X([a, b])}$ can be finite only if $f$ agrees with a locally integrable function in $\mathbb{R}^n \times (a, b)$. Also, the definitions (2.2) and (2.3) clearly agree for functions $f \in X([a, b])$.

Let $H$ denote the operator $i \partial_t + \Delta_x$ acting (in the sense of distributions) on functions in $L^2(\mathbb{R}^n \times \mathbb{R})$. For any interval $[a, b]$, we define the space $Z([a, b])$ as the space of locally integrable functions $u: \mathbb{R}^n \times [a, b] \to \mathbb{C}$ with the properties

$$u \in C([a, b]; L^2(\mathbb{R}^n)) \quad \text{and} \quad \|Hu\|_{X([a, b])} < \infty.$$  \hspace{1cm} (2.4)

The meaning of the first condition is that $u$ is a continuous mapping from the interval $[a, b]$ to $L^2(\mathbb{R}^n)$. The second condition is to be interpreted as in (2.3). Corollary 1.4 in [7] shows that

$$\|u\|_{X([a, b])} \leq C \|Hu\|_{X([a, b])} + C\|u(\cdot, a)\|_{L^2(\mathbb{R}^n)}$$

if $u \in Z([a, b])$. In particular, $Z([a, b]) \subseteq X'([a, b])$.

Let $\varphi$ denote a fixed smooth function on $\mathbb{R}$ with the following properties: $\varphi(0) = 0$, $\varphi'$ nonincreasing, $\varphi'(r) = 1$ if $r \leq 1$, and $\varphi'(r) = 0$ if $r \geq 2$. For any $\lambda \geq 1$ let $\varphi_\lambda(r) = \lambda \varphi(r/\lambda)$. Clearly $\varphi_\lambda(r) = r$ if $r \leq \lambda$, and the function $r \mapsto \varphi_\lambda(r)$ is increasing and bounded.

In this section and in the rest of the paper, we will use the letters $C$ and $c$ to denote constants that may depend only on the dimension $n$ if $n \neq 2$, and on the exponent $p_0$ if $n = 2$. For any set $E$, $\chi_E$ will denote its characteristic function. Our first main theorem is a Carleman inequality.
THEOREM 2.1. There is an increasing function \( \Lambda: [0, \infty) \to [0, \infty) \) and a constant \( C \) such that
\[
\| e^{\beta \varphi(x_1)} u \|_{X^\prime([-T,T])} \leq C \| e^{\beta \varphi(x_1)} Hu \|_{X([-T,T])} + \| e^{\beta \varphi(x_1)} u (\cdot, -T) \|_{L^2(\mathbb{R}^n)} + \| e^{\beta \varphi(x_1)} u (\cdot, T) \|_{L^2(\mathbb{R}^n)}
\tag{2.5}
\]
for any \( u \in C([-T,T], X) \), any \( \beta \in [0, \infty) \) and any \( \lambda \geq T^{1/2} \Lambda(T^{1/2} \beta) \).

The norm \( \| e^{\beta \varphi(x_1)} Hu \|_{X([-T,T])} \) is defined as in (2.3). A weaker form of the Carleman inequality (2.5) was implicitly proved by Kenig, Ponce and Vega [9] by the use of energy methods. This implicit result in [9] corresponds to the inequality (2.5) with the spaces \( X \) and \( X^\prime \) replaced by \( L^1_t L^2_x \) and \( L^\infty_t L^2_x \), respectively. Most likely, however, the energy methods of [9] cannot be used to prove the \( L^p \) estimates in Theorem 2.1. Our proof of Theorem 2.1 is based on constructing suitable parametrices.

The estimates in §8 show that we may take
\[
\Lambda(\beta) = C(1+\beta)^6
\]
for some large constant \( C \). In our applications it is important to have a Carleman inequality like (2.5) with bounded weights \( e^{\beta \varphi(x_1)} \) (which is equivalent to \( \lambda < \infty \)). Such a Carleman inequality can be applied to a large class of functions \( u \), not just to those \( u \) that have faster-than-exponential decay. We remark that a Carleman inequality like (2.5) with nonconvex weights can only hold for functions \( u \) with bounded support in \( t \), i.e. if \( T < \infty \). Without this support restriction it is only possible to prove a Carleman inequality with linear weights.

COROLLARY 2.2. For any \( u \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}) \) and any \( \beta \in \mathbb{R} \),
\[
\| e^{\beta x_1} u \|_{X^\prime} \leq C \| e^{\beta x_1} Hu \|_X.
\]

Corollary 2.2 follows from Theorem 2.1 with \( T \) larger than the time support of \( u \) and \( \lambda \to \infty \). Since \( L^{2(n+2)/(n+1)}_{t,x} \subseteq X \) and \( L^\infty_t L^{2(n+2)/n}_{x} \supseteq X^\prime \), this improves the Carleman inequality of Kenig and Sogge [13]. Notice that the case \( \beta = 0 \) is equivalent to the Strichartz estimates for the Schrödinger operator, including the endpoint estimate of Keel and Tao [7]. Such estimates have a long history, starting with the fundamental paper of Strichartz [19]; for more references on the development of Strichartz-type estimates for the wave equation and the Schrödinger equation, we refer the reader to the recent work of Keel and Tao [7], where a nontrivial endpoint estimate is proved.

Our main applications concern quantitative and qualitative properties of solutions of nonlinear Schrödinger equations of the form
\[
Hu = Vu + F(u),
\tag{2.6}
\]
where \( V \) is a potential and \( F: \mathbb{C} \to \mathbb{C} \) is a continuous function. We define the Banach space \( Y \) in such a way that

\[
\| V \|_X < \| V \|_Y \| u \|_X, \tag{2.7}
\]

Thus, if \( n=1 \) then \( Y = L^4_t L^\infty_x + L^1_x L^1_t \). If \( n \geq 3 \) then \( Y = L^1_t L^\infty_x + L^{2(p_0/(2-p_0))}_x / (2-p_0) L^\infty_x \). If \( n=2 \) then \( Y = L^1_t L^\infty_x + L^{p_0/(2-p_0)}_x / (2-p_0) \). Notice that \( L^2_{x,t} \subseteq Y \) in any dimension \( n \), if \( p_0 \) is sufficiently close to 2. For any interval \([a, b]\), we also define the space \( Y[a, b] \) as the Banach space of functions \( V: \mathbb{R}^n \times [a, b] \to \mathbb{C} \), with \( \| V \|_{Y[a, b]} = \| V \|_Y \) if \( t \in [a, b] \), and \( \| V \|_Y \equiv 0 \) if \( t \notin [a, b] \).

Let \( \bar{C} \) denote the constant in Theorem 2.1 and \( \bar{c} = 1/2 \bar{C} \). We have the following quantitative estimate:

**Theorem 2.3.** Assume that \( V: \mathbb{R}^n \times [0, 1] \to \mathbb{C} \) is a potential with the property that

\[
\| V \|_{Y((0,1))} \leq \bar{c}. \tag{2.8}
\]

Assume that \( u \in Z([0, 1]) \) and

\[Hu = Vu \text{ in } X([0, 1]). \]

Then

\[
\sup_{t \in [0,1]} \| e^{\beta z t} u(\cdot, t) \|_{L^2(\mathbb{R}^n)} \leq C \| e^{\beta z t} u(\cdot, 0) \|_{L^2(\mathbb{R}^n)} + \| e^{\beta z t} u(\cdot, 1) \|_{L^2(\mathbb{R}^n)}
\]

uniformly in \( \beta \in \mathbb{R} \).

We consider now uniqueness properties of solutions of the Schrödinger equation (2.6). We are concerned with the following question: Assume that \( u_1, u_2 \in C([0, 1]:L^2(\mathbb{R}^n)) \) are (weak) solutions to (2.6) with the property that for some domain \( D \subseteq \mathbb{R}^n \) we have \( u_1 = u_2 \) in \( D \times \{0, 1\} \). Under what assumptions on \( F, V, u_1 \) and \( u_2 \) can we then conclude that \( u_1 = u_2 \) (or \( u_1 = u_2 \) in \( D' \times [0, 1] \) for some domain \( D' \))?

For the nonlinear term \( F \), we make the assumption that there is a function \( G: \mathbb{C} \to [0, \infty) \) with the property that

\[
|F(z_1) - F(z_2)| \leq |z_1 - z_2| (|G(z_1)| + |G(z_2)|) \tag{2.9}
\]

for any \( z_1, z_2 \in \mathbb{C} \). For any unit vector \( w_0 \), let \( D(w_0) = \{ x : x \cdot w_0 > 0 \} \) denote a half-space.

**Theorem 2.4.** Assume that \( u_1, u_2 \in C([0, 1]:L^2(\mathbb{R}^n)) \cap X'([0, 1]) \) are (weak) solutions of the nonlinear Schrödinger equation

\[Hu = Vu + F(u) \text{ in } S'(\mathbb{R}^n \times [0, 1]).\]
Let \( W = |V| + G(u_1) + G(u_2) \); assume that

\[
W \in \mathcal{Y}([0, 1]) \quad \text{and} \quad \|W \chi_{b + D(w_0)}(x)\|_{\mathcal{Y}([0,1])} \leq \tilde{c} \quad \text{for some } b \in \mathbb{R}. \tag{2.10}
\]

If \( u_1 = u_2 \) in \([b w_0 + D(w_0)] \times [0, 1]\), then \( u_1 \equiv u_2 \) in \([b w_0 + D(w_0)] \times [0, 1]\).

We notice that \(|F(u)| \leq |F(0)| + |u_1(G(u_1) + G(0))|\). Since \( u_1, u_2 \in X'([0, 1])\), it follows from (2.10) that \( V u_1 + F(u_1), V u_2 + F(u_2) \in X([0, 1]) + L^\infty(\mathbb{R}^n \times [0, 1]) \subseteq S'(\mathbb{R}^n \times \mathbb{R})\).

Using a local unique continuation argument we also prove a global vanishing theorem. Our local unique continuation argument is sharper than the one used by Kenig, Ponce and Vega [9], who assumed that the functions \( u_1 \) and \( u_2 \) agree in the complement of a convex cone at times 0 and 1.

**Theorem 2.5.** Assume that \( u_1, u_2 \in C([0, 1]; L^2(\mathbb{R}^n)) \cap X'([0, 1]) \) are (weak) solutions of the nonlinear Schrödinger equation

\[
Hu = V u + F(u) \quad \text{in } S'(\mathbb{R}^n \times (0, 1)).
\]

Let \( W = |V| + G(u_1) + G(u_2) \); assume that

\[
W \in L^{p_1}_{t, x} L^{q_1}(\mathbb{R}^n \times [0, 1]) + L^{p_2}_{t, x} L^{q_2}(\mathbb{R}^n \times [0, 1]) \tag{2.11}
\]

for some \( p_1, q_1, p_2, q_2 \in [1, \infty) \) with \( 2/p_1 + n/q_1 \leq 2 \) and \( 2/p_2 + n/q_2 \leq 2 \). If \( u_1 = u_2 \) in \([b w_0 + D(w_0)] \times \{0, 1\} \) for some \( b \in \mathbb{R} \), then \( u_1 \equiv u_2 \) in \( \mathbb{R}^n \times [0, 1] \).

We remark that (2.10) and (2.11) are, in fact, conditions on the potential \( V \), the function \( F \), and the space of solutions \( u_1 \) and \( u_2 \). For technical reasons, (2.11) is slightly more restrictive than (2.10). In fact, Theorem 2.5 holds if the assumption (2.11) is replaced by the less restrictive assumptions (2.10) and (9.3), see the proof in §9.

**Example 2.6.** Assume that \( V \in L^{p_1}_{t, x} L^{q_1} \) with \( p_1 \) and \( q_1 \) as in (2.11), that \( G(z) = C(|z|^{a_1} + |z|^{a_2}) \), \( a_1, a_2 \in (0, \infty) \), and that \( u_1, u_2 \in C([0, 1]; L^2 \cap L^\infty) \). Then (2.11) holds (with \( p_2 \) and \( q_2 \) large) and Theorem 2.5 applies (compare with [9, Theorem 1.1]).

We conclude with a theorem concerning well-posedness in \( Z([0, T]) \).

**Theorem 2.7.** Assume that \( V: \mathbb{R}^n \times [0, T] \) is a potential with the property that there is \( \varepsilon > 0 \) such that

\[
\|V\|_{Y_{\{a, a+\varepsilon\}}} \leq \tilde{c} \quad \text{for any } a \in [0, T - \varepsilon] \tag{2.12}
\]

Then the initial value problem

\[
\begin{cases}
(i \partial_t + \Delta_x) u = Vu, \\
u(\cdot, 0) = u_0,
\end{cases}
\]
$u_0 \in L^2(\mathbb{R}^n)$, admits a unique solution $u \in Z([0, T])$ with

$$\|u\|_{X'([0, T])} \leq C(T)\|u_0\|_{L^2}.$$ 

The proof of this theorem is routine and probably known: it follows from the Strichartz estimates, Duhamel's formula and a fixed-point argument (for details, see [5]). The counterexample in [5, §3] shows that the space of potentials $Y$ (see (2.12)) is optimal for local well-posedness.

3. Proof of Theorem 2.1: construction of parametrices

Notice first that it suffices to prove the following simplified version of Theorem 2.1:

**Lemma 3.1.** With the same notation as in Theorem 2.1 we have

$$\|e^{\beta \varphi_\lambda(x)}u(x, t)\|_{X'} \leq C\|e^{\beta \varphi_{\lambda_1}(x)}(Hu)(x, t)\|_X$$

for any function $u \in C^\infty_0(\mathbb{R}^n \times \mathbb{R})$ supported in $\mathbb{R}^n \times [-T, T]$, any $\beta \in [0, \infty)$ and any $\lambda \geq T^{1/2}\Lambda(T^{1/2}\beta)$.

To deduce Theorem 2.1 from Lemma 3.1 we show first that for any $\varepsilon \in (0, \frac{1}{10}T]$ the bound (3.1) holds uniformly for any $v \in Z([-T, T])$ supported in $\mathbb{R}^n \times [-T+\varepsilon, T-\varepsilon]$. Let $\psi: \mathbb{R}^n \times \mathbb{R} \to [0, \infty)$ denote a smooth function supported in the set $\{(x, t): |x|, |t| \leq 1\}$ with $\int_{\mathbb{R}^n \times \mathbb{R}} \psi(x, t) \, dx \, dt = 1$, and for $0 < \delta < \min\{\frac{1}{10} \varepsilon^{1/2}, 1\}$ let $\psi_\delta(x, t) = \delta^{-(n+2)}\psi(x/\delta, t/\delta^2)$. Let $\bar{\psi}: \mathbb{R}^n \to [0, 1]$ denote a smooth function equal to 1 in the set $\{x: |x| \leq 1\}$ and equal to 0 in the set $\{x: |x| \geq 2\}$, and for $R \geq 1$ let $\bar{\psi}_R(x) = \psi(x/R)$. We apply (3.1) to the smooth, compactly supported function

$$u(x, t) = (v \ast \psi_\delta)(x, t) \bar{\psi}_R(x).$$

Then

$$\|e^{\beta \varphi_{\lambda_1}(x)}(v \ast \psi_\delta)(x, t) \bar{\psi}_R(x)\|_{X'} \leq C\|e^{\beta \varphi_{\lambda_1}(x)}(Hv \ast \psi_\delta)(x, t) \bar{\psi}_R(x)\|_X$$

$$+ C\|e^{\beta \varphi_{\lambda_1}(x)}(\nabla_x (v \ast \psi_\delta)(x, t)) \cdot |\nabla_x \bar{\psi}_R(x)| + C\|((v \ast \psi_\delta)(x, t)) \cdot |\nabla_x \bar{\psi}_R(x)|\|_{L^1 L^2_2}.$$ (3.2)

For the term in the second line of (3.2) notice that $|\nabla_x (v \ast \psi_\delta)(x, t)| \leq C\delta^{-1}(|v| \ast \chi_\delta)(x, t)$, where $\chi$ is the characteristic function of the set $\{(x, t): |x|, |t| \leq 1\}$ and $\chi_\delta(x, t) = \delta^{-(n+2)}\chi(x/\delta, t/\delta^2)$. Also, for the term in the third line, $|((v \ast \psi_\delta)(x, t))| \leq C(|v| \ast \chi_\delta)(x, t)$.
Since \( v \in C([-T, T]; L^2(\mathbb{R}^n)) \) and the weight \( e^{\beta \varphi_\lambda(x_1)} \) is bounded, the term in the second and third lines of (3.2) converges to 0 as \( R \to \infty \). Then we let \( \delta \to 0 \) to conclude that

\[
\|e^{\beta \varphi_\lambda(x_1)}u\|_{X'} \lesssim C\|e^{\beta \varphi_\lambda(x_1)}Hv\|_{X},
\]

if \( v \in Z([-T, T]) \) is supported in \( \mathbb{R}^n \times [-T + \varepsilon, T - \varepsilon] \).

To deduce Theorem 2.1 apply the inequality (3.3) to the function

\[
v(x, t) = u(x, t) \eta_\varepsilon(t),
\]

where \( \varepsilon \leq \frac{1}{10} T \), and the smooth cutoff functions \( \eta_\varepsilon: [-T, T] \to [0, 1] \) have the properties \( \eta_\varepsilon(t) = 1 \) if \( t \in [-T + 2\varepsilon, T - 2\varepsilon] \), \( \eta_\varepsilon(t) = 0 \) if \( t \notin [-T + \varepsilon, T - \varepsilon] \), and \( \int_{\mathbb{R}} \eta_\varepsilon(t)^2 = 2 \). Clearly

\[
Hv(x, t) = Hu(x, t) \eta_\varepsilon(t) + iu(x, t) \partial_t \eta_\varepsilon(t).
\]

By (3.3),

\[
\|e^{\beta \varphi_\lambda(x_1)}u(x, t)\eta_\varepsilon(t)\|_{X'} \lesssim C\|e^{\beta \varphi_\lambda(x_1)}Hu(x, t)\eta_\varepsilon(t)\|_{X'} + C\|e^{\beta \varphi_\lambda(x_1)}u(x, t)\eta_\varepsilon(t)\|_{L^2}.
\]

Recall that the weight \( e^{\beta \varphi_\lambda(x_1)} \) is bounded. By (2.4) we may let \( \varepsilon \) tend to 0; the Carleman inequality (2.5) follows.

We turn now to the proof of Lemma 3.1. By rescaling (using the anisotropic dilations \((x, t) \to (bx, \delta^2 t)\)) we can assume that \( T = 1 \). Let

\[
f = (i\partial_t + \Delta_x)u.
\]

We have to prove that

\[
\|e^{\beta \varphi_\lambda(x_1)}u\|_{X'} \lesssim C\|e^{\beta \varphi_\lambda(x_1)}f\|_{X}
\]

for \( \lambda \geq \Lambda(\beta) \) and \( u \in C_0^\infty(\mathbb{R}^n \times (-1, 1)) \). We will assume from now on that \( \lambda \geq (\beta + 1)^2 \).

Let \( U = e^{\beta \varphi_\lambda(x_1)}u \) and \( F = e^{\beta \varphi_\lambda(x_1)}f \). The estimate (3.5) is equivalent to

\[
\|U\|_{X'} \lesssim C\|F\|_{X}.
\]

The equation (3.4) is equivalent to

\[
(i\partial_t + \Delta_x - a_{\beta, \lambda}(x_1) \partial_{x_1} + b_{\beta, \lambda}(x_1))U = F,
\]

where \( a_{\beta, \lambda} = 2\beta \varphi_\lambda' \) and \( b_{\beta, \lambda} = \beta^2 [\varphi_\lambda']^2 - \beta \varphi_\lambda'' \). We have \( a_{\beta, \lambda}(x_1) \in [0, 2\beta] \); more importantly, for any integer \( j \geq 0 \) and \( x_1 \in [\lambda, 2\lambda] \),

\[
(\beta + 1)^{-1} |\partial^j a_{\beta, \lambda}(x_1)| + (\beta + 1)^{-2} |\partial^j b_{\beta, \lambda}(x_1)| \lesssim C_j \lambda^{-j}.
\]
The term in the left-hand side of (3.8) vanishes if \( x_1 \notin [\lambda, 2\lambda] \) and \( j \geq 1 \).

Let \( \psi: \mathbb{R} \to [0, 1] \) denote a smooth, even cutoff function supported in the interval \([-2, 2]\) and equal to 1 in the interval \([-1, 1]\). Let \( \chi_+, \chi_- \) and \( \chi_1 \) denote the characteristic functions of the intervals \([0, \infty), (-\infty, 0]\) and \([-1, 1]\), respectively. For numbers \( \gamma \geq 1 \) let \( \psi_\gamma(r) = \psi(r/\gamma) \). We fix \( \gamma = C(\beta + 1) \), where \( C \) is a large constant. We define the operators \( A_+, A_-, \hat{A} \) and \( B \) (acting on Schwartz functions on \( \mathbb{R}^n \times \mathbb{R} \)) by Fourier multipliers:

- \( A_- \) defined by the Fourier multiplier \( \chi_-(\xi) \psi_\gamma(\xi) \),
- \( A_+ \) defined by the Fourier multiplier \( \chi_+(\xi) \psi_\gamma(\xi) \),
- \( \hat{A} \) defined by the Fourier multiplier \( [1 - \psi_\gamma(\xi_1)][1 - \psi(10(\tau + |\xi_2|)/\xi_1^2)] \),
- \( B \) defined by the Fourier multiplier \( [1 - \psi_\gamma(\xi_1)] \psi(10(\tau + |\xi_2|)/\xi_1^2) \).

The variables \( \tau, \xi_1 \), etc., are the dual variables to \( t, x_1 \), etc., and clearly

\[ A_- + A_+ + \hat{A} + B = \text{Id}. \]

For \( \varepsilon > 0 \), let \( P_\varepsilon \) denote the operator defined by the Fourier multiplier \( (\xi, \tau) \mapsto e^{-\varepsilon|\xi|^2} \), and \( Q_\varepsilon \) the operator defined by the Fourier multiplier \( (\xi, \tau) \mapsto e^{-\varepsilon^2(\tau + |\xi|^2)^2} \). We will prove the estimates

\[
\|x_1(t)P_\varepsilon A_-(U)\|_{X^r} \leq C\|F\|_X + C_1(\gamma, \lambda)\|U\|_{L^p_t L^2_x}, \tag{3.9}
\]

\[
\|x_1(t)P_\varepsilon A_+(U)\|_{X^r} \leq C\|F\|_X + C_1(\gamma, \lambda)\|U\|_{L^p_t L^2_x}, \tag{3.10}
\]

\[
\|x_1(t)Q_\varepsilon P_\varepsilon \hat{A}(U)\|_{X^r} \leq C\|F\|_X + C_1(\gamma, \lambda)\|U\|_{L^p_t L^2_x}, \tag{3.11}
\]

\[
\|x_1(t)Q_\varepsilon B(U)\|_{X^r} \leq C\|F\|_X. \tag{3.12}
\]

The constant \( C_1(\gamma, \lambda) \) is small if \( \lambda \) is sufficiently large compared to \( \beta \). Thus the estimates (3.9)-(3.12) would suffice to prove (3.6).

The parametrices for \( A_- \) and \( A_+ \). In this case the variable \( \xi_1 \) is much smaller than \( \lambda \). We construct the parametrices starting from the equation (3.7), as if the functions \( a_{\beta, \lambda} \) and \( b_{\beta, \lambda} \) were constant. Consider the integral

\[
I_-(F)(x, t) = \int_{\mathbb{R}^n} \int_{-\infty}^{t_-} F(y, s) \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} e^{-i(t-s)|\xi|^2} \times \psi_\gamma(\xi_1) \chi_-(\xi_1) e^{-\varepsilon^2|\xi|^2} e^{a_{\beta, \lambda}(y_1) (t-s)} e^{b_{\beta, \lambda}(y_1)(t-s)} d\xi ds dy.
\]

Recall that \( F(y, s) = (i \partial_s + D_y) U(y, s) \), where \( D_y = \Delta_y - a_{\beta, \lambda}(y_1) \partial_{y_1} + b_{\beta, \lambda}(y_1) \). We sub-
Substitute this into the formula of $I_-(F)(x, t)$ and integrate by parts in $s$ and $y$. The result is

$$I_-(F)(x, t) = \int_{\mathbb{R}^n} iU(y, t) \int_{\mathbb{R}^n} e^{i(x-y)\xi} \psi_\gamma(\xi_1) \chi_-(\xi_1) e^{-\varepsilon^2|\xi|^2} d\xi dy$$

$$+ \int_{\mathbb{R}^n} \int_{-\infty}^t U(y, s)[-i\partial_s + D^*_y]$$

$$\times \int_{\mathbb{R}^n} e^{i(x-y)\xi} e^{-i(t-s)|\xi|^2} \psi_\gamma(\xi_1) \chi_-(\xi_1) e^{-\varepsilon^2|\xi|^2}$$

$$\times e^{\partial_{\beta, \lambda}(y_1)\xi_1(t-s)} e^{\partial_{\beta, \lambda}(y_1)(t-s)} d\xi ds dy$$

$$(3.13)$$

$$= cP_x A_-(U)(x, t) + c\tilde{R}_1(U)(x, t),$$

where

$$\tilde{R}_1(U)(x, t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} U(y, s) \chi_+(t-s) \int_{\mathbb{R}^n} e^{i(x-y)\xi} e^{-i(t-s)|\xi|^2}$$

$$\times \psi_\gamma(\xi_1) \chi_-(\xi_1) e^{-\varepsilon^2|\xi|^2} e^{\partial_{\beta, \lambda}(y_1)\xi_1(t-s)} e^{\partial_{\beta, \lambda}(y_1)(t-s)} d\xi ds dy.$$

The function $q_1(y_1, \xi_1, t, s)$ can be written explicitly by inspecting the identity above; the important fact is that when we compute $-i\partial_s + D^*_y$, all the terms that are not small cancel out. The remaining terms have either a derivative of $a_{\beta, \lambda}$ or a derivative of $b_{\beta, \lambda}$. By (3.8), if $|t-s|<2$ and $1 + |\xi_1| \leq C\gamma$, we have

$$|q_1(y_1, \xi_1, t, s)| + |\partial_s q_1(y_1, \xi_1, t, s)| + |\partial_{\beta, \lambda} q_1(y_1, \xi_1, t, s)| \leq C_{\gamma}^3$$

$$(3.14)$$

for $y_1 \in [\lambda, 2\lambda]$, and the left-hand side of (3.14) vanishes if $y_1 \not\in [\lambda, 2\lambda]$. From (3.13) we get

$$P_x A_-(U) = cI_-(F) + c\tilde{R}_1(U).$$

$$(3.15)$$

To summarize, for (3.9), we have to prove first that the operator

$$T_1(g)(x, t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(y, s) \int_{\mathbb{R}^n} e^{i(x-y)\xi} e^{-i(t-s)|\xi|^2} e^{-\varepsilon^2|\xi|^2}$$

$$\mu_1(y_1, \xi_1, t, s) d\xi ds dy$$

is bounded from $X$ to $X'$, where

$$\mu_1(y_1, \xi_1, t, s) = \chi_1(t) \chi_1(s) \chi_+(t-s) \psi_\gamma(\xi_1) \chi_-(\xi_1) e^{\partial_{\beta, \lambda}(y_1)\xi_1(t-s)} e^{\partial_{\beta, \lambda}(y_1)(t-s)}. $$

$$(3.16)$$

In addition we have to prove that the operator

$$R_1(g)(x, t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(y, s) \int_{\mathbb{R}^n} e^{i(x-y)\xi} e^{-i(t-s)|\xi|^2} e^{-\varepsilon^2|\xi|^2}$$

$$s_1(y_1, \xi_1, t, s) d\xi ds dy$$
is bounded from $L^p_s L^2_y$ to $X'$ with small norm, where

$$s_1(y_1, \xi_1, t, s) = \chi_1(t) \chi_1(s) \chi_-(t-s) \psi_\gamma(\xi_1) \chi_-(\xi_1)$$

$$\times e^{a_{\beta, \lambda}(y_1) \xi_1(t-s)} e^{b_{\beta, \lambda}(y_1)(t-s)} q_1(y_1, \xi_1, t, s). \tag{3.17}$$

(Note that the role of the various signs is clear: because of the exponential term we need that $a_{\beta, \lambda}(y_1) \xi_1(t-s) \leq 0$, and this is achieved because $t-s \geq 0$, $\xi_1 \leq 0$ and $a_{\beta, \lambda}(y_1) \geq 0$.)

The construction for $A_+$ is similar; the only changes are to replace the function $\mu_1$ with

$$\mu_2(y_1, \xi_1, t, s) = \chi_1(t) \chi_1(s) \chi_-(t-s) \psi_\gamma(\xi_1) \chi_+(\xi_1) e^{a_{\beta, \lambda}(y_1) \xi_1(t-s)} e^{b_{\beta, \lambda}(y_1)(t-s)} q_1(y_1, \xi_1, t, s), \tag{3.18}$$

and the error function $s_1$ with

$$s_2(y_1, \xi_1, t, s) = \chi_1(t) \chi_1(s) \chi_-(t-s) \psi_\gamma(\xi_1) \chi_+(\xi_1)$$

$$\times e^{a_{\beta, \lambda}(y_1) \xi_1(t-s)} e^{b_{\beta, \lambda}(y_1)(t-s)} q_1(y_1, \xi_1, t, s). \tag{3.19}$$

We then construct the operators $T_2$ and $R_2$ in the same way as the operators $T_1$ and $R_1$.

The parametrix for $A$. We start from the integral

$$\tilde{I}(F)(x, t) = \int_{R^n} \int_{R^n} F(y, s) \int_{R^n} e^{i(x-y)\xi} e^{i(t-s)\tau} e^{-\varepsilon^2|\xi|^2} e^{-\varepsilon^2(\tau+|\xi|^2)^2}$$

$$\times \left[1 - \psi_\gamma(\xi_1) \right] \left[1 - \psi(10(\tau + |\xi|^2)/\xi_1^2) \right]$$

$$\times \left(-\tau - |\xi|^2 - i \xi_1 a_{\beta, \lambda}(y_1) + b_{\beta, \lambda}(y_1) \right) d\tau ds d\xi dy. \tag{3.20}$$

We substitute the formula (3.7) and integrate by parts. Let

$$q_3(y_1, \xi, \tau) = \frac{1}{-\tau - |\xi|^2 - i \xi_1 a_{\beta, \lambda}(y_1) + b_{\beta, \lambda}(y_1)}.$$ 

The result is

$$\tilde{I}(F)(x, t) = c \int_{R^n} \int_{R^n} U(y, s) \int_{R^n} e^{i(x-y)\xi} e^{i(t-s)\tau} e^{-\varepsilon^2|\xi|^2}$$

$$\times e^{-\varepsilon^2(\tau+|\xi|^2)^2} \left[1 - \psi_\gamma(\xi_1) \right] \left[1 - \psi(10(\tau + |\xi|^2)/\xi_1^2) \right] d\tau d\xi ds dy$$

$$+ c \int_{R^n} \int_{R^n} U(y, s) \int_{R^n} e^{i(x-y)\xi} e^{i(t-s)\tau} e^{-\varepsilon^2|\xi|^2} e^{-\varepsilon^2(\tau+|\xi|^2)^2}$$

$$\times \left[1 - \psi_\gamma(\xi_1) \right] \left[1 - \psi(10(\tau + |\xi|^2)/\xi_1^2) \right] \left[\partial_{y_1} a_{\beta, \lambda}(y_1) q_3(y_1, \xi, \tau) + (a_{\beta, \lambda}(y_1) - 2i\xi_1) \partial_{\xi_1} q_3(y_1, \xi, \tau) + \partial_{\xi_1}^2 q_3(y_1, \xi, \tau) \right] d\tau d\xi ds dy \tag{3.21}$$

$$= c Q_{\varepsilon} P_{\varepsilon} A(U)(x, t) + c R_3(U)(x, t).$$
Thus, for (3.11), we have to prove first that the operator
\[ T_3(g)(x,t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(y,s) e^{i(x-y) \xi} e^{-i(t-s)\xi^2} e^{-\xi^2/2} \mu_3(y_1, \xi_1, t, s) d\xi ds dy \]
is bounded from \( X \) to \( X' \), where
\[ \mu_3(y_1, \xi_1, t, s) = \frac{1 - \psi(10\tau/\xi_1^2)}{-\tau - i\xi_1 a_{\beta,\lambda}(y_1) + b_{\beta,\lambda}(y_1)} \]
(3.22)
In addition, we have to prove that the operator
\[ R_3(g)(x,t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(y,s) e^{i(x-y) \xi} e^{-i(t-s)\xi^2} e^{-\xi^2/2} s_3(y_1, \xi_1, t, s) d\xi ds dy \]
is bounded from \( L^2_X \) to \( L^2_{X'} \) with small norm, where
\[ s_3(y_1, \xi_1, t, s) = \frac{1 - \psi(10\tau/\xi_1^2)}{-\tau - i\xi_1 a_{\beta,\lambda}(y_1) + b_{\beta,\lambda}(y_1)} \]
(3.23)
The notation in (3.23) is \( \tilde{q}_3(y_1, \xi_1, \tau) = \frac{1 - \psi(10\tau/\xi_1^2)}{-\tau - i\xi_1 a_{\beta,\lambda}(y_1) + b_{\beta,\lambda}(y_1)} \), and the primes denote differentiation with respect to \( y_1 \).

The parametrix for \( B \). This is the more delicate case. We think of the equation as an evolution in \( x_1 \) rather than \( t \), and start from (3.4) rather than (3.7). Let \( \tilde{u}(x_1, \xi', \tau) \), \( \tilde{f}(x_1, \xi', \tau) \), etc., denote the partial Fourier transforms of the functions \( u, f \), etc., in the variables \( x' \) and \( t \). By taking this partial Fourier transform the equation (3.4) becomes
\[ \partial_2 \tilde{u}(z_1, \xi', \tau) = \tilde{f}(z_1, \xi', \tau). \]
By using this equation and integrating by parts we have
\[ \int_{z_1}^{\infty} \tilde{f}(y_1, \xi', \tau) \sin\left(\frac{(z_1 - y_1)\sqrt{(\tau + |\xi'|^2)}}{\sqrt{-(\tau + |\xi'|^2)}}\right) dy_1 = -\tilde{u}(z_1, \xi', \tau) \]
whenever \( \tau + |\xi'|^2 \leq 0 \). Let
\[ L(z_1 - y_1, \sqrt{-(\tau + |\xi'|^2)}) = \chi_+(y_1 - z_1) \frac{\sin\left(\frac{(z_1 - y_1)\sqrt{-(\tau + |\xi'|^2)}}{\sqrt{-(\tau + |\xi'|^2)}}\right)}{\sqrt{-(\tau + |\xi'|^2)}}. \]
(3.25)
We multiply the equation (3.24) by \( e^{2\phi_\lambda(z_1)} \) to obtain
\[ \tilde{U}(z_1, \xi', \tau) = -\int_{\mathbb{R}} \tilde{F}(y_1, \xi', \tau) e^{2\phi_\lambda(z_1) - \beta(x_1)} L(z_1 - y_1, \sqrt{-(\tau + |\xi'|^2)}) dy_1, \]
and take the Fourier transform in $z_1$ to obtain
\[
\hat{U}(\xi_1, \xi', \tau) = -\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-iz_1\xi_1} \hat{F}(y_1, \xi', \tau) e^{\beta \phi_\lambda(z_1) - \beta \phi_\lambda(y_1)} L(z_1 - y_1, \sqrt{-(\tau + |\xi'|^2)}) \, dz_1 \, dy_1.
\]
We multiply this by $[1 - \psi_7(\xi_1)] \psi(10(\tau + |\xi'|^2)/\xi_1^2) e^{-\varepsilon^2 |\xi'|^2}$ and notice that
\[
\psi(10(\tau + |\xi'|^2)/\xi_1^2) = 0
\]
unless $\tau + |\xi'|^2 \in [-\frac{6}{5} \xi_1^2, -\frac{3}{5} \xi_1^2]$. We use the fact that
\[
\hat{F}(y_1, \xi', \tau) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} F(y_1, y', s) e^{-i(y' - y) \xi'} \, ds \, dy'
\]
and take the inverse Fourier transform. The result is
\[
P_B(U)(x, x', t) = c \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} F(y_1, y', s) K(x_1, y_1, x', y', t, s) \, dy_1 \, dy' \, ds,
\]
where
\[
K(x_1, y_1, x', y', t, s) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(x_1 - z_1)\xi_1} e^{i(x' - y') \xi'} e^{(t-s)\tau} \times e^{\beta \phi_\lambda(z_1) - \beta \phi_\lambda(y_1)} [1 - \psi_7(\xi_1)] \psi(10(\tau + |\xi'|^2)/\xi_1^2) \times e^{-\varepsilon^2 |\xi'|^2} L(z_1 - y_1, \sqrt{-(\tau + |\xi'|^2)}) \, d\xi' \, d\tau \, dz_1.
\]
We make the change of variables $z_1 = y_1 - \alpha$ and $\tau = -w - |\xi'|^2$. The integral for $K$ becomes
\[
K(x_1, y_1, x', y', t, s) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(x - y) \xi_1} e^{-i(t-s)w} [1 - \psi_7(\xi_1)] e^{-\varepsilon^2 |\xi'|^2} \times \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\alpha \xi_1} e^{-(t-s)w} e^{\beta \phi_\lambda(y_1 - \alpha) - \beta \phi_\lambda(y_1)} \times \psi(10(\xi_1^2 - w)/\xi_1^2) L(-\alpha, \sqrt{w}) \, dw \, da \, d\xi' \, d\xi_1.
\]
The change of variable $w = \xi_1^2 r^2$ in the inner integral together with the fact that $L(-\alpha, r) = -\chi_+ (\alpha) \sin (\alpha r)/r$ shows that
\[
K(x_1, y_1, x', y', t, s) = -2 \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i(x - y) \xi} e^{-i(t-s)\xi_1^2} [1 - \psi_7(\xi_1)] e^{-\varepsilon^2 |\xi'|^2} \times \int_{0}^{\infty} \int_{0}^{\infty} e^{i\alpha \xi_1} e^{-i(t-s)\xi_1^2 r^2} e^{\beta \phi_\lambda(y_1 - \alpha) - \beta \phi_\lambda(y_1)} \times \psi(10(1 - r^2)) \sin (\xi_1 \alpha r) \xi_1 \, d\alpha \, d\xi' \, d\xi_1.
\]
For $r > 0$ let $\tilde{\psi}(r) = \psi(10(1-r^2))$; clearly $\tilde{\psi}$ is smooth and supported in the interval $[(\frac{1}{2})^{1/2}, (\frac{3}{2})^{1/2}]$. The formula for $K$ becomes

$$K(x_1,y_1,x',y',t,s) = c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} e^{-i(t-s)|\xi|^2} [1 - \tilde{\psi}_r(\xi_1)] e^{-\epsilon^2|\xi|^2}$$

$$\times \int_0^\infty \int_{\mathbb{R}} e^{i\alpha_1 \xi_1} e^{-i(t-s)\xi_1^2} e^{\beta_1 \psi(y_1-a) - \beta_1 \psi(y_1)} \tilde{\psi}(r) \sin(\xi_1 |\alpha r|) \xi_1 \, dr \, da \, d\xi_1.$$

By (3.26) it is clear that (3.12) follows if we can prove that the operator

$$T_4(g)(x,t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(y,s) \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} e^{-i(t-s)|\xi|^2} e^{-\epsilon^2|\xi|^2} \mu_4(y_1, \xi_1, t, s) \, d\xi \, ds \, dy$$

is bounded from $X$ to $X'$, where

$$\mu_4(y_1, \xi_1, t, s) = \chi_1(t) \chi_1(s) [1 - \tilde{\psi}_r(\xi_1)]$$

$$\times \int_0^\infty \int_{\mathbb{R}} e^{i\alpha_1 \xi_1} e^{-i(t-s)\xi_1^2} e^{\beta_1 \psi(y_1-a) - \beta_1 \psi(y_1)} \tilde{\psi}(r) \sin(\xi_1 |\alpha r|) \xi_1 \, dr \, da.$$

To summarize, it remains to prove that the operators $T_j$, $j = 1, 2, 3, 4$, are bounded from $X$ to $X'$, and that the operators $R_j$, $j = 1, 2, 3$, are bounded from $L^2_x L^2_t$ to $X'$ with small norm. The estimates for the operators $T_j$ are proved in §§5–7, and the estimates for the operators $R_j$ are proved in §8. We first prove some preliminary symbol-type estimates for the multiplier $\mu_4$ and the associated kernel.

### 4. Preliminary estimates

We start by defining two spaces of symbols on $\mathbb{R}$. For functions $m \in C^1(\mathbb{R})$ we define the bounded-variation norm

$$\|m\|_{BV} = \sup_{\eta \in \mathbb{R}} |m(\eta)| + \int_{\mathbb{R}} |m'(\eta)| \, d\eta,$$

(4.1)

and define the space $BV(\mathbb{R}) = \{m \in C^1(\mathbb{R}) : \|m\|_{BV} < \infty\}$. Also, for $b \in \mathbb{R}$ and functions $m \in C^1(\mathbb{R} \setminus \{b\})$ we define the Hörmander–Mikhlin norm

$$\|m\|_{HM^b} = \sup_{\eta \in \mathbb{R} \setminus \{b\}} |m(\eta)| + \sup_{\eta \in \mathbb{R} \setminus \{b\}} |(\eta - b)m'(\eta)|,$$

(4.2)

and define the space $HM^b(\mathbb{R}) = \{m \in C^1(\mathbb{R} \setminus \{b\}) : \|m\|_{HM^b} < \infty\}$. Notice that

$$\|\eta \mapsto m(\eta)\|_{BV} = \|m\|_{BV},$$

$$\|\eta \mapsto m(\eta + b)\|_{BV} = \|m\|_{BV}$$

(4.3)
for any \(a \in (0, \infty)\) and \(b \in \mathbb{R}\), and
\[
\| \eta \mapsto m(a \eta) \|_{H^a} = \| m \|_{H^a},
\]
\(\| \eta \mapsto m(b_1 + \eta) \|_{H^{a_2}} = \| m \|_{H^{a_1 + a_2}}\) (4.4)
for any \(a \in (0, \infty)\) and \(b_1, b_2 \in \mathbb{R}\). Also we have
\[
\| m_1 m_2 \|_{B^V} \leq 3 \| m_1 \|_{B^V} \| m_2 \|_{B^V},
\]
(4.5)
\[
\| m_1 m_2 \|_{H^{a}} \leq 3 \| m_1 \|_{H^{a}} \| m_2 \|_{H^{a}}
\]
(4.6)
for any \(b \in \mathbb{R}\).

**Lemma 4.1.** Assume that \(\| m \|_{B^V} \leq 1\). Then we have the uniform bound
\[
\left| \int_R e^{i \delta \xi^2} e^{i a \xi} e^{- \epsilon^2 (\xi - b)^2} m(\xi) \, d\xi \right| \leq C |\delta|^{-1/2}
\]
for any \(\delta \in \mathbb{R} \setminus \{0\}\), \(a, b \in \mathbb{R}\) and \(\epsilon \in (0, \infty)\).

**Proof.** By a linear change of variable using (4.3) we can assume that \(\delta = \pm 1\) and \(a = 0\). Then we break up the integral into two parts, corresponding to \(|\xi|\) small and \(|\xi|\) large, and integrate by parts when \(|\xi| \gg 1\). The estimate follows easily. \(\square\)

**Lemma 4.2.** Assume that \(a_1, \ldots, a_k\) are real numbers and that the functions \(m_j \in C^1(\mathbb{R} \setminus \{a_j\})\) have the property that \(\| m_j \|_{H^{a_j}} \leq 1\) for \(j = 1, 2, \ldots, k\). Then we have the uniform bound
\[
\left| \int_R e^{i \delta \xi^2} e^{i a \xi} e^{- \epsilon^2 (\xi - b)^2} m_1(\xi) \ldots m_k(\xi) \, d\xi \right| \leq C_k |\delta|^{-1/2}
\]
(4.7)
for any \(\delta \in \mathbb{R} \setminus \{0\}\), \(a, b \in \mathbb{R}\) and \(\epsilon \in (0, \infty)\).

**Proof.** By a linear change of variable using (4.4) we can assume that \(\delta = \pm 1\) and \(a = 0\). Let \(\tilde{m}(\xi) = e^{- \epsilon^2 (\xi - b)^2} m_1(\xi) \ldots m_k(\xi)\) and \(B\) denote the set of numbers \(b, a_1, \ldots, a_k\). Clearly \(\tilde{m} \in L^1(\mathbb{R})\) and
\[
|\tilde{m}(\xi) + \text{dist}(\xi, B)| |\tilde{m}'(\xi)| \leq C_k
\]
for any \(\xi \in \mathbb{R} \setminus B\). By breaking up the integral in (4.7) into at most \(2k + 2\) integrals we see that it suffices to prove that
\[
\left| \int_A e^{i \delta \xi^2} m(\xi) \, d\xi \right| \leq C
\]
uniformly in \(A, \tilde{A} \in \mathbb{R}\), provided that \(\delta = \pm 1\) and
\[
|\tilde{m}(\xi) | + |(\xi - A) \tilde{m}'(\xi)| \leq 1.
\]
This follows by a routine integration-by-parts argument. \(\square\)

The first main lemma in this section concerns the multiplier \(\mu_4\):
**Lemma 4.3.** The multiplier $\mu_4$ in (3.27) satisfies the bound

$$\|\mu_4(\cdot, \xi_1, t, s)\|_{BV_y} \leq C \quad (4.8)$$

uniformly in $\xi_1$, $t$ and $s$.

**Proof.** By taking limits we can assume that $t \neq s$. We will assume that $t-s>0$ (the case $t-s<0$ then follows since $\mu_4(y_1, \xi_1, t, s) = \mu_4(y_1, -\xi_1, -t, -s)$). Let $A=2(t-s)^{1/2}$. Thus $A \in (0, \sqrt{2}]$. In the integral in (3.27) that defines the multiplier $\mu_4$ we make the change of variable $\alpha=2(t-s)^{1/2}\theta = A\theta$. We then have

$$\mu_4(y_1, \xi_1, t, s) = 2\chi_1(t)\chi_1(s)[1-\psi_1(\xi_1)]I(y_1, (t-s)^{1/2}\xi_1), \quad (4.9)$$

where

$$I(y_1, \eta_1) = \int_0^\infty \int_{\mathbb{R}} e^{2i\eta_1 \theta} e^{-\eta_1^2 (r^2-1)} e^{\beta \varphi_3(y_1 - A\theta)-\beta \varphi_3(y_1)} \psi(r) \sin(2\eta_1 \theta r) \eta_1 \, dr \, d\theta. \quad (4.10)$$

It suffices to prove that the function $I$ has bounded variation in $y_1$, i.e.

$$\|I(\cdot, \eta_1)\|_{BV_y} \leq C$$

for any $\eta_1 \in \mathbb{R}$, provided that $A \in (0, C]$. Assume first that $|\eta_1| \leq 2$. In this case we write the integral for the function $I$ in the form

$$I(y_1, \eta_1) = \int_{\mathbb{R}} \chi_+(\theta) e^{2i\eta_1 \theta} e^{-\eta_1^2 \alpha^2} e^{\beta \varphi_3(y_1 - A\theta)-\beta \varphi_3(y_1)} H(\eta_1, \theta) \, d\theta, \quad (4.11)$$

where

$$H(\eta_1, \theta) = \int_{\mathbb{R}} \eta_1 e^{-\eta_1^2 \alpha^2} \sin(2\eta_1 \theta r) \psi(r) \, dr. \quad (4.12)$$

Notice that

$$|H(\eta_1, \theta)| \leq C|\eta_1|(1+|\eta_1\theta|)^{-2}$$

if $|\eta_1| \leq 2$. Thus

$$\|I(\cdot, \eta_1)\|_{BV_y} \leq C \int_{\mathbb{R}} \chi_+(\theta) \|y_1 \mapsto e^{\beta \varphi_3(y_1 - A\theta)-\beta \varphi_3(y_1)} \|_{BV_y} |H(\eta_1, \theta)| \, d\theta$$

$$\leq C \int_{\mathbb{R}} |\eta_1|(1+|\eta_1\theta|)^{-2} \, d\theta \leq C,$$

as desired (we used the fact that the function $y_1 \mapsto e^{\beta \varphi_3(y_1 - A\theta)-\beta \varphi_3(y_1)}$ takes values in the interval $[0, 1]$ for any $A\theta \geq 0$ and is nondecreasing in $y_1$, and thus has bounded variation).
It remains to prove the same estimate in the case \(|\eta_1| \geq 2\). We start from (4.11) and (4.12). Recall that the function \(\tilde{\psi}\) is smooth and supported in the interval \([\left(\frac{4}{3}\right)^{1/2}, \left(\frac{6}{3}\right)^{1/2}]\).

Let \(\psi_1 : \mathbb{R} \to [0, 1]\) be a smooth function supported in the set \(\{\eta : |\eta| \in \left[\frac{4}{3}, \frac{6}{3}\right]\}\), and equal to 1 in the set \(\{\eta : |\eta| \in \left[\frac{4}{3}, \frac{6}{3}\right]\}\). We have

\[
H(\eta_1, \theta) = \frac{1}{2i} e^{i\theta^2} \int_{\mathbb{R}} \eta_1 e^{-i\theta^2} e^{-i|\eta_1|^2} e^{2i\eta_1 \cdot \theta - e^{-2i\eta_1 \cdot \theta}} \tilde{\psi}(r) \, dr
\]

(4.13)

where

\[
H_0(\eta_1, \theta) = [1 - \psi_1(\theta/\eta_1)] \frac{1}{2i} \int_{\mathbb{R}} e^{-i\theta^2} [\tilde{\psi}((r+\theta)/\eta_1) - \tilde{\psi}((r-\theta)/\eta_1)] \, dr
\]

and

\[
H_1(\eta_1, \theta) = \psi_1(\theta/\eta_1) \frac{1}{2i} \int_{\mathbb{R}} e^{-i\theta^2} [\tilde{\psi}((r+\theta)/\eta_1) - \tilde{\psi}((r-\theta)/\eta_1)] \, dr.
\]

By the support properties of the functions \(\tilde{\psi}\) and \(\psi_1\) we can integrate by parts in the integral defining \(H_0(\eta_1, \theta)\) to obtain

\[
|H_0(\eta_1, \theta)| \leq C(1 + |\theta|)^{-2}
\]

(4.14)

if \(|\eta_1| \geq 1\). Also, the function \(H_1(\eta_1, \theta)\) is supported in the set \(\{\eta_1 : |\theta/\eta_1| \in \left[\frac{4}{3}, \frac{6}{3}\right]\}\). We substitute the formula (4.13) into the definition (4.11) of the function \(I\), and decompose \(I(y_1, \eta_1) = I_0(y_1, \eta_1) + I_1(y_1, \eta_1)\) corresponding to the terms \(e^{i\theta^2} H_0\) and \(e^{i\theta^2} H_1\). By (4.14) and an argument similar to the one used in the case \(|\eta_1| \leq 2\) we have

\[
\|I_0(\cdot, \eta_1)\|_{B_{BV_1}} \leq C.
\]

It remains to prove a similar estimate for the function \(I_1\). We have

\[
I_1(y_1, \eta_1) = \int_{\mathbb{R}} \chi_3(\theta) e^{2i\eta_1 \cdot \theta} e^{i\eta_1^2} e^{-\beta \phi_\lambda(y_1 - A\theta) - \beta \phi_\lambda(y_1)} e^{i\theta^2} H_1(\eta_1, \theta) \, d\theta
\]

\[
= \int_{\mathbb{R}} \chi_3(\alpha - \eta_1) e^{-i\alpha^2} e^{\beta \phi_\lambda(y_1 - A(\alpha - \eta_1)) - \beta \phi_\lambda(y_1)} H_1(\eta_1, \alpha - \eta_1) \, d\alpha.
\]

We consider two cases depending on the sign of \(\eta_1\). It is somewhat harder to prove estimates if \(\eta_1\) is negative, so we will concentrate on this case. Since \(|\eta_1| \geq 2\) we can
assume that $\eta_1 \leq -2$. By the support property of the function $H_1$ and because of the factor $\chi_+ (\alpha - \eta_1)$, the variable $\alpha$ in the integral representing $I_1$ runs over the interval $\alpha \in [-\frac{1}{2} \vert \eta_1 \vert, \frac{1}{2} \vert \eta_1 \vert]$. Thus

$$I_1(y_1, \eta_1) = \int_{\mathbb{R}} e^{i \alpha^2} e^{i \nu (y_1 - A (\alpha - \eta_1)) - \psi (y_1)} \tilde{\psi}_1 (\alpha/\vert \eta_1 \vert) H_1 (\eta_1, \alpha - \eta_1) \, d\alpha,$$

where $\tilde{\psi}_1$ is a smooth function supported in the interval $[-\frac{2}{5}, \frac{2}{5}]$ and equal to 1 in the interval $[-\frac{1}{5}, \frac{1}{5}]$. Let $\delta_0, \delta : \mathbb{R} \to [0, 1]$ denote two smooth functions with the property that

$$1 = \delta_0 (\alpha) + \sum_{j \geq 1} \delta (2^{-j} \alpha)$$

for any $\alpha \in \mathbb{R}$. We can also assume that $\delta_0$ is supported in the interval $[-2, 2]$ and $\delta$ is supported in the set $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$. We insert this partition of unity into the integral formula defining $I_1$; the result is

$$I_1(y_1, \eta_1) = \sum_{j \geq 0} I_j^1 (y_1, \eta_1),$$

where, with $\delta_j (\alpha) = \delta (2^{-j} \alpha)$ for any $j \geq 1$,

$$I_j^1 (y_1, \eta_1) = \int_{\mathbb{R}} e^{i \alpha^2} e^{i \nu (y_1 - A (\alpha - \eta_1)) - \psi (y_1)} \delta_j (\alpha) \tilde{\psi}_1 (\alpha/\vert \eta_1 \vert) H_1 (\eta_1, \alpha - \eta_1) \, d\alpha.$$

The main estimate we will prove is

$$\|I_j^1 (\cdot, \eta_1)\|_{BV_{\nu_1}} \leq C 2^{-j}$$  \hspace{1cm} (4.15)

for any integer $j \geq 0$. Notice that for $j = 0$ this follows by the same argument as in the case $\vert \eta_1 \vert \leq 2$. We only need to notice that by Lemma 4.1,

$$\vert H_1 (\eta_1, \alpha - \eta_1) \vert \leq C$$

uniformly in $\eta_1$ and $\alpha$.

We turn to the proof of (4.15) in the case $j \geq 1$. By a change of variable, the integral for $I_j^1$ becomes

$$I_j^1 (y_1, \eta_1) = 2^j \int_{\mathbb{R}} e^{i 2^j \alpha^2} e^{i \nu (y_1 - A (2^j \alpha - \eta_1)) - \psi (y_1)} \delta (\alpha, \eta_1) \, d\alpha,$$

where $\delta (\alpha, \eta_1) = \delta (\alpha) \tilde{\psi}_1 (2^j \alpha/\vert \eta_1 \vert) H_1 (\eta_1, 2^j \alpha - \eta_1)$. The function $\delta (\alpha, \eta_1)$ is smooth and supported in the set $\{ \alpha : \vert \alpha \vert \in [\frac{1}{2}, 2] \}$. By integrating by parts in the formula of $H_1$ it is easy to see that

$$\vert H_1 (\eta_1, \theta) \vert + \vert \eta_1 \partial_\theta H_1 (\eta_1, \theta) \vert \leq C$$
if $\eta_1 \leq -1$. Thus if $9 \cdot 2^{j-2} \leq |\eta_1|$ and $|\alpha| \in \left[\frac{1}{2}, 2\right]$, we have

$$|\delta(\alpha, \eta_1)| + |\partial_\alpha \delta(\alpha, \eta_1)| \leq C.$$  \hspace{1cm} (4.17)

Clearly $\delta(\alpha, \eta_1) \equiv 0$ if $9 \cdot 2^{j-2} > |\eta_1|$.

In (4.16) we integrate by parts in $\alpha$ to obtain

$$I_1^j(y_1, \eta_1) = -2^j \int e^{i2^j\alpha^2 + \beta \phi_{x}(y_1 - A(2^j\alpha - \eta_1)) - \beta \phi_{x}(y_1)}$$

$$\times \partial_\alpha \frac{\delta(\alpha, \eta_1)}{i2^{2j+1}\alpha - \beta A2^j \phi_{x}'(y_1 - A(2^j\alpha - \eta_1))} \, d\alpha.$$  \hspace{1cm} (4.18)

Since $A \leq C$ and $|\phi_{x}'(r)| \leq C/\lambda \leq C/\beta$ for any $r \in \mathbb{R}$, we have by (4.17),

$$|\partial_\alpha \frac{\delta(\alpha, \eta_1)}{i2^{2j+1}\alpha - \beta A2^j \phi_{x}'(y_1 - A(2^j\alpha - \eta_1))}| \leq C2^{-2j}.$$  \hspace{1cm} (4.19)

Thus

$$|I_1^j(y_1, \eta_1)| \leq C2^{-j}$$  \hspace{1cm} (4.20)

uniformly in $y_1$ and $\eta_1$, as desired.

By taking the $y_1$-derivative in (4.18) we have

$$|\partial_{y_1} I_1^j(y_1)| \leq 2^j \int_{|\alpha| \in \left[\frac{1}{2}, 2\right]} \left[|\beta \phi_{x}'(y_1 - A(2^j\alpha - \eta_1)) - \beta \phi_{x}'(y_1)|ight]$$

$$\times e^{i2^j\alpha^2 + \beta \phi_{x}(y_1 - A(2^j\alpha - \eta_1)) - \beta \phi_{x}(y_1)}$$

$$\times \left|\partial_\alpha \frac{\delta(\alpha, \eta_1)}{i2^{2j+1}\alpha - \beta A2^j \phi_{x}'(y_1 - A(2^j\alpha - \eta_1))}\right| \, d\alpha$$

$$+ 2^j \int_{|\alpha| \in \left[\frac{1}{2}, 2\right]} e^{i2^j\alpha^2 + \beta \phi_{x}(y_1 - A(2^j\alpha - \eta_1)) - \beta \phi_{x}(y_1)}$$

$$\times \left|\partial_{y_1} \partial_\alpha \frac{\delta(\alpha, \eta_1)}{i2^{2j+1}\alpha - \beta A2^j \phi_{x}'(y_1 - A(2^j\alpha - \eta_1))}\right| \, d\alpha$$

$$= J_1(y_1) + J_2(y_1).$$

By (4.19)

$$\|J_1\|_{L^1_{y_1}} \leq C2^{-j},$$  \hspace{1cm} (4.22)

as desired. For $J_2(y_1)$ we estimate the $\partial_\alpha \partial_{y_1}$-derivative. By (4.17),

$$\left|\partial_\alpha \partial_{y_1} \frac{\delta(\alpha, \eta_1)}{i2^{2j+1}\alpha - \beta A2^j \phi_{x}'(y_1 - A(2^j\alpha - \eta_1))}\right| \leq C2^{-3j} \beta A |\phi_{x}''(y_1 - A(2^j\alpha - \eta_1))| + C2^{-2j} \beta A^2 |\phi_{x}'''(y_1 - A(2^j\alpha - \eta_1))|.$$  \hspace{1cm} (4.23)
Notice that the second term in the right-hand side of (4.23) is dominated by

\[ C 2^{-2j} \beta \lambda^{-2} \chi_{\lambda, 2\lambda}(y_1 - A(2^j \alpha - \eta_1)). \]

Since \( \beta \leq \lambda \) this suffices to control the second term. For the first term we recall that

\[ 9 \cdot 2^{j-2} \leq |\eta_1| \quad \text{and} \quad \eta_1 \leq -1. \]

Thus \( 2^j \alpha - \eta_1 \geq c 2^j \) if \( \alpha \in [\frac{1}{2}, 2] \), and so, to prove that \( \|J_2\|_{L^1_{\lambda_1}} \leq C 2^{-j} \), it suffices to prove that

\[ 2^{-j} \beta A \int |\varphi''_\lambda(y_1)| e^{\beta \varphi_\lambda(y_1) - \beta \varphi_\lambda(y_1 + c A 2^j)} dy_1 \leq C. \]  

(4.24)

The function \( \varphi'_\lambda \) is nonincreasing and nonnegative. Thus

\[ \beta \varphi_\lambda(y_1) - \beta \varphi_\lambda(y_1 + c A 2^j) \leq -c A 2^j \varphi'_\lambda(y_1 + c A 2^j). \]

Therefore, the expression in the left-hand side of (4.24) can be dominated by

\[ 2^{-j} \beta A \int |\varphi''_\lambda(y_1)| e^{-c A 2^j \varphi_\lambda(y_1 + c A 2^j)} dy_1, \]

(4.25)

\[ + 2^{-j} \beta A \int |\varphi''_\lambda(y_1)| - \varphi''_\lambda(y_1 + c A 2^j)| dy_1. \]

The first term in (4.25) can be dominated by \( C 2^{-2j} \), and the second term can be dominated by \( C \beta A^2 \lambda^{-1} \leq C. \) Thus (4.24) follows. The main estimate (4.15) follows by (4.20) and (4.22). This completes the proof of the lemma. \( \square \)

We will now prove an estimate for the kernel of the operator \( T_4 \). Recall that the operators \( T_j \) are of the form

\[ T_j(g)(x,t) = \int_{\mathbb{R}^n} \int g(y,s) \int_{\mathbb{R}^n} e^{i(x-y) \xi} e^{-i(t-s) \xi^2} e^{-\varepsilon \xi^2} |\mu_j(y_1, \xi_1, t, s)| d\xi d\xi_1 dy. \]

where the multipliers \( \mu_j \) are defined in (3.16), (3.18), (3.22) and (3.27). Let

\[ K_j(x, y, t, s) = \int_{\mathbb{R}^n} e^{i(x-y) \xi} e^{-i(t-s) \xi^2} e^{-\varepsilon \xi^2} |\mu_j(y_1, \xi_1, t, s)| d\xi \]  

(4.26)

and

\[ k_j(x_1, y_1, t, s) = \int_{\mathbb{R}^n} e^{i(x_1-y_1) \xi_1} e^{-i(t-s) \xi_1^2} e^{-\varepsilon \xi_1^2} |\mu_j(y_1, \xi_1, t, s)| d\xi_1. \]  

(4.27)

Note that the integral representing \( K_j \) splits as a product of \( n \) integrals, the first of which is the integral representing \( k_j \). In this section we prove estimates for the kernel \( k_4 \).
Assume $t-s>0$ and, as in Lemma 4.3, let $A=2(t-s)^{1/2}$. By (4.9), (4.27) and the change of variable $\xi=2\eta/A$, we have

$$k_{4}(x_{1},y_{1},t,s)=\chi_{1}(t)\chi_{1}(s)\frac{C}{A}\int_{\mathbb{R}}e^{2\eta_{1}(1-x_{1}/y_{1})/A}e^{-i\eta_{1}^{2}[1-\psi_{1}(2\eta_{1}/A)]}I(y_{1},\eta_{1})\,d\eta_{1}$$

with $\varepsilon=2\varepsilon/A$. For the function $I$ we use the integral formula (4.10). Then

$$k_{4}(x_{1},y_{1},t,s)=\chi_{1}(t)\chi_{1}(s)\frac{C}{A}\int_{0}^{\infty}J_{0}(r)e^{i\varphi_{1}(y_{1}-A\theta)}e^{i\varphi_{1}(y_{1})}$$

$$\times\int_{\mathbb{R}}e^{-i\eta_{1}^{2}/2\eta_{1}^{2}}e^{i2\eta_{1}[(x_{1}-y_{1})/A+\theta]}\sin(2\eta_{1}\theta r)\eta_{1}e^{-i\eta_{1}^{2}/2\eta_{1}^{2}}[1-\psi_{1}(2\eta_{1}/A)]\,d\eta_{1}\,dr\,d\theta.$$

To compute the $\eta_{1}$-integral notice that

$$\int_{\mathbb{R}}e^{-a\eta_{1}^{2}+b\eta_{1}}\,d\eta_{1}=Ca^{-1/2}e^{b^{2}/4a}$$

for any $a,b\in\mathbb{R}$, $a>0$. By taking a derivative with respect to $b$ we have

$$\int_{\mathbb{R}}e^{-a\eta_{1}^{2}+b\eta_{1}}\,d\eta_{1}=Ca^{-3/2}be^{b^{2}/4a}$$

for any $a,b\in\mathbb{R}$, $a>0$. By analytic continuation, (4.30) holds for any $a,b\in\mathbb{C}$ with $\text{Re}a>0$.

Let $H_{A}=H_{A,\varepsilon,\gamma}$ denote the inverse Fourier transform of the function

$$\eta_{1}\mapsto e^{-i\eta_{1}^{2}/2\eta_{1}^{2}}[1-\psi_{1}(2\eta_{1}/A)],$$

so that

$$e^{-i\eta_{1}^{2}/2\eta_{1}^{2}}[1-\psi_{1}(2\eta_{1}/A)]\stackrel{d}{=}\int_{\mathbb{R}}H_{A}(\alpha)e^{-i\eta_{1}\alpha}\,d\alpha.$$

We also have $\|H_{A}\|_{L^{1}}\leq C$ uniformly. By (4.30) the $\eta_{1}$-integral in (4.28) is equal to

$$C\rho^{-3}\int_{\mathbb{R}}H_{A}(\alpha)\left(e^{i[(x_{1}-y_{1})/A+\theta+\theta r-\alpha/2]}\right)^{2/\rho^{2}}[(x_{1}-y_{1})/A+\theta+\theta r-\alpha/2]$$

$$\times e^{i[(x_{1}-y_{1})/A+\theta+\theta r-\alpha/2]}[(x_{1}-y_{1})/A+\theta+\theta r-\alpha/2]\,d\alpha.$$

To rewrite the integral in (4.28) let

$$F_{A,\pm}(\tilde{x}_{1},\tilde{y}_{1})=\int_{0}^{\infty}J_{0}(r)e^{i\varphi_{1}(A\tilde{y}_{1}/A)}e^{i\varphi_{1}(A\tilde{y}_{1})}$$

$$\times e^{i(\tilde{x}_{1}-\tilde{y}_{1}+\omega+\theta r)^{2}/2\rho^{2}}(\tilde{x}_{1}-\tilde{y}_{1}+\theta+\theta r)\,dr\,d\theta.$$

Then, by (4.28)

$$k_{4}(x_{1},y_{1},t,s)=\chi_{1}(t)\chi_{1}(s)\frac{C}{A}\int_{\mathbb{R}}H_{A}(\alpha)$$

$$\times\left[F_{A,+}(x_{1}/A-\frac{1}{2}\alpha,y_{1}/A)-F_{A,-}(x_{1}/A-\frac{1}{2}\alpha,y_{1}/A)\right]\,d\alpha.$$
Lemma 4.4. We have

\[ F_{A,\pm}(\tilde{x}_1, \tilde{y}_1) = e^{i(x_1 - y_1)^2} m_{A,\pm}(\tilde{x}_1, \tilde{y}_1) + J_{A,\pm}(\tilde{x}_1, \tilde{y}_1), \tag{4.32} \]

where

\[ ||m_{A,\pm}(\tilde{x}_1, \cdot)||_{BV^t} + ||m_{A,\pm}(\cdot, \tilde{y}_1)||_{BV^t} \leq C \tag{4.33} \]

uniformly in \( \tilde{x}_1 \) and \( \tilde{y}_1 \), and

\[ (1 + |\tilde{x}_1 - \tilde{y}_1|) |J_{A,\pm}(\tilde{x}_1, \tilde{y}_1)| \leq C. \tag{4.34} \]

Proof. By a change of variable we have

\[ F_{A,\pm}(\tilde{x}_1, \tilde{y}_1) = \int_0^\infty e^{i((x_1 - \tilde{x}_1) - \theta)^2 r^2} G_{\pm}(\tilde{x}_1 - \tilde{y}_1, \theta) d\theta, \tag{4.35} \]

where

\[ G_{\pm}(\tilde{x}_1 - \tilde{y}_1, \theta) = \int_{\mathbb{R}^3} \psi_0(r) e^{i((x_1 - \tilde{x}_1 + \theta)^2 - \theta^2 r^2)} \frac{1}{(\tilde{x}_1 + \theta)^2} dr. \tag{4.36} \]

In (4.36), \( \psi_0(r) = \tilde{\psi}(1/r) \) is a smooth function supported in the interval \([\frac{5}{4}, \frac{5}{3}] \). Recall that we fixed \( \psi: [0, 1] \to \mathbb{R} \), a smooth cutoff function supported in the interval \([-2, 2] \) and equal to 1 in the interval \([-1, 1] \). Let \( \chi: [0, 1] \to \mathbb{R} \) denote a smooth function supported in the interval \((-\infty, -10] \) and equal to 1 in the interval \((-\infty, -20] \). Let

\[ m_{A,\pm}(\tilde{x}_1, \tilde{y}_1) = e^{-i((x_1 - y_1)^2 \chi_-(\tilde{x}_1 - \tilde{y}_1))} \times \int_0^\infty \psi(x_1 - \tilde{y}_1 + \theta) e^{i\theta(x_1 - \tilde{x}_1 - \theta) - \theta x_1} G_{\pm}(\tilde{x}_1 - \tilde{y}_1, \theta) d\theta \]

and

\[ J_{A,\pm}(\tilde{x}_1, \tilde{y}_1) = [1 - \chi_-(\tilde{x}_1 - \tilde{y}_1)] \int_0^\infty e^{i\theta(x_1 - \tilde{x}_1 - \theta) - \theta x_1} G_{\pm}(\tilde{x}_1 - \tilde{y}_1, \theta) d\theta \]

\[ + \chi_-(\tilde{x}_1 - \tilde{y}_1) \int_0^\infty [1 - \psi(x_1 - \tilde{x}_1 + \theta)] e^{i\theta(x_1 - \tilde{x}_1 - \theta) - \theta x_1} G_{\pm}(\tilde{x}_1 - \tilde{y}_1, \theta) d\theta. \]

The identity (4.32) is clear; it remains to prove the bounds (4.33) and (4.34).

For the bound (4.33) we may assume that \( \tilde{x}_1 - \tilde{y}_1 \geq 10 \) and make the change of variable \( \theta = \tilde{y}_1 - \tilde{x}_1 + u/(\tilde{y}_1 - \tilde{x}_1) \). The formula (4.36) shows that

\[ G_{\pm}(\tilde{x}_1 - \tilde{y}_1, \tilde{y}_1 + u/(\tilde{y}_1 - \tilde{x}_1)) = e^{i((\tilde{y}_1 - \tilde{x}_1)^2 + u/(\tilde{y}_1 - \tilde{x}_1) H_{\pm}(u, \tilde{y}_1 - \tilde{x}_1), \tag{4.37} \]

where

\[ H_{\pm}(u, \eta) = \int_{\mathbb{R}^3} \psi_0(r) e^{i(\pm 2u r \pm 1)} + u r^2 \eta^2 [\pm 1 + u/r^2 / \eta^2] dr. \]
A routine integration-by-parts argument shows that
\[ |H_{\pm}(u, \eta)| + |\partial_\eta H_{\pm}(u, \eta)||\eta|^3 \leq C(1 + |u|)^{-2} \quad (4.38) \]
if $|\eta| \geq 1$ and $|u| \leq 2|\eta|$. We substitute the formula (4.37) into the definition of the function $m_{A, \pm}$. Thus
\[ m_{A, \pm}(\tilde{x}_1, \tilde{y}_1) = \tilde{X}_-(\tilde{x}_1 - \tilde{y}_1) \]
\[ \times \int_{\mathbb{R}} \psi(u/(\tilde{y}_1 - \tilde{x}_1)) H_{\pm}(u, \tilde{y}_1 - \tilde{x}_1) e^{\beta \phi_\lambda(A\tilde{x}_1 - Au/\tilde{y}_1) - \beta \phi_\lambda(A\tilde{y}_1)} \, du. \]
Recall that the function $\phi_\lambda$ is nondecreasing and that $|H_{\pm}(u, \eta)| \leq C(1 + |u|)^{-2}$. Thus $|m_{A, \pm}(\tilde{x}_1, \tilde{y}_1)| \leq C$. To estimate the derivatives of $m_{A, \pm}$ notice that
\[ |\partial_{\tilde{x}_1} m_{A, \pm}(\tilde{x}_1, \tilde{y}_1)| \leq C(1 + |\tilde{y}_1 - \tilde{x}_1|)^{-2} + C \chi_+(\tilde{y}_1 - \tilde{x}_1 - 10) \]
\[ \times \int_{|u| \leq 2(\tilde{y}_1 - \tilde{x}_1)} (1 + |u|)^{-2} |\partial_{\tilde{x}_1} e^{\beta \phi_\lambda(A\tilde{x}_1 - Au/\tilde{y}_1) - \beta \phi_\lambda(A\tilde{y}_1)}| \, du \]
and
\[ |\partial_{\tilde{y}_1} m_{A, \pm}(\tilde{x}_1, \tilde{y}_1)| \leq C(1 + |\tilde{y}_1 - \tilde{x}_1|)^{-2} + C \chi_+(\tilde{y}_1 - \tilde{x}_1 - 10) \]
\[ \times \int_{|u| \leq 2(\tilde{y}_1 - \tilde{x}_1)} (1 + |u|)^{-2} |\partial_{\tilde{y}_1} e^{\beta \phi_\lambda(A\tilde{x}_1 - Au/\tilde{y}_1) - \beta \phi_\lambda(A\tilde{y}_1)}| \, du. \]
These estimates follow easily by inspecting the formula (4.39) and using (4.38). The term $(1 + |\tilde{y}_1 - \tilde{x}_1|)^{-2}$ in these two estimates is integrable, thus harmless. For (4.33) it remains to prove that for any $u \in \mathbb{R}$,
\[ \|\chi_+(\tilde{y}_1 - \tilde{x}_1 - 10) \chi_+(2(\tilde{y}_1 - \tilde{x}_1) - |u|) \partial_{\tilde{x}_1} e^{\beta \phi_\lambda(A\tilde{x}_1 - Au/\tilde{y}_1) - \beta \phi_\lambda(A\tilde{y}_1)}\|_{L^1_{\tilde{x}_1}} \leq C \quad (4.40) \]
and
\[ \|\chi_+(\tilde{y}_1 - \tilde{x}_1 - 10) \chi_+(2(\tilde{y}_1 - \tilde{x}_1) - |u|) \partial_{\tilde{y}_1} e^{\beta \phi_\lambda(A\tilde{x}_1 - Au/\tilde{y}_1) - \beta \phi_\lambda(A\tilde{y}_1)}\|_{L^1_{\tilde{y}_1}} \]
\[ \leq C(1 + |u|)^{1/2} \quad (4.41) \]
For (4.40) we notice that the function $\tilde{x}_1 \mapsto A\tilde{x}_1 - Au/(\tilde{y}_1 - \tilde{x}_1)$ is increasing in the interval $\tilde{x}_1 \in (-\infty, \min\{\tilde{y}_1 - 10, \tilde{y}_1 - \frac{1}{2}|u|\})$. Since $\phi_\lambda$ is a nondecreasing function it follows that $\tilde{x}_1 \mapsto e^{\beta \phi_\lambda(A\tilde{x}_1 - Au/\tilde{y}_1) - \beta \phi_\lambda(A\tilde{y}_1)}$ is a nondecreasing function in the relevant interval,
which proves (4.40). To prove (4.41) we notice that if \( \bar{y}_1 - \bar{x}_1 \geq \max \{10, \text{\frac{1}{2}}|u|\} \) and \( A \leq C \), then
\[
|\partial_{\bar{y}_1} e^{\beta \varphi_0(A \bar{x}_1 - A u/(\bar{y}_1 - \bar{x}_1))} - \beta \varphi_0(A \bar{y}_1)| \\
\leq e^{\beta \varphi_0(A \bar{x}_1 - A u/(\bar{y}_1 - \bar{x}_1))} \beta A |u| \left( A \bar{x}_1 - \frac{A u}{\bar{y}_1 - \bar{x}_1} \right) \\
\leq 2e^{\beta \varphi_0(A \bar{x}_1 + 2A) - \beta \varphi_0(A \bar{y}_1)} \\
\times \left[ \beta A \varphi_0'(A \bar{y}_1) + \frac{\beta A |u|}{(\bar{y}_1 - \bar{x}_1)^2} \left( A \bar{x}_1 - \frac{A u}{\bar{y}_1 - \bar{x}_1} \right) \right] \\
\leq C e^{\frac{\beta |u|}{(\bar{y}_1 - \bar{x}_1)^2} \min \{1, \frac{\bar{y}_1 - \bar{x}_1}{\lambda}\}}.
\]

The estimate (4.41) follows easily by integrating the two terms in the last line of the above estimate and recalling that \( 1 + \beta^2 \leq \lambda \) (the first term is equal to the derivative of a non-increasing function). This completes the proof of (4.33).

To estimate the function \( J_{A, \pm} \) notice first that
\[
|G_{\pm}(\bar{x}_1 - \bar{y}_1, \theta)| \leq C (1 + |\bar{x}_1 - \bar{y}_1 + \theta|)^{-2}
\]
if \( \bar{x}_1 - \bar{y}_1 \geq -20 \) and \( \theta \geq 0 \). These estimates follow easily by integrating by parts in (4.36) and using standard bounds for oscillatory integrals. The estimate (4.34) for the functions \( J_{A, \pm} \) follows in the range \( \bar{x}_1 - \bar{y}_1 \geq -20 \). In the range \( \bar{x}_1 - \bar{y}_1 \leq -20 \), only the integral in the second line of the formula of \( J_{A, \pm} \) does not vanish. If, in addition, \( |\bar{x}_1 - \bar{y}_1 + \theta| \geq 1 \) then we integrate by parts in (4.36). Recall that the function \( \psi_0 \) in (4.36) is supported in a small interval around 1. By checking the cases \( \theta \leq \frac{9}{10} |\bar{x}_1 - \bar{y}_1| \), \( \theta \in \left[ \frac{9}{10} |\bar{x}_1 - \bar{y}_1|, \frac{11}{10} |\bar{x}_1 - \bar{y}_1| \right] \) and \( \theta \geq \frac{11}{10} |\bar{x}_1 - \bar{y}_1| \), it is not hard to see that
\[
|G_{\pm}(\bar{x}_1 - \bar{y}_1, \theta)| \leq C (\theta + |\bar{x}_1 - \bar{y}_1|)^{-2}
\]
if \( \bar{x}_1 - \bar{y}_1 \leq -20 \) and \( |\bar{x}_1 - \bar{y}_1 + \theta| \geq 1 \). The estimate (4.34) in the range \( \bar{x}_1 - \bar{y}_1 \leq -20 \) follows.

This completes our analysis in the case \( t - s > 0 \). If \( t - s < 0 \) then we let \( A = 2(s - t)^{1/2} \) and argue as before. Notice also that the function \( (2/A)H_A(2\alpha/A) = H(\alpha) \) does not depend on \( A \). By rewriting (4.31) and using Lemma 4.4 we have
\[
k_4(x_1, y_1, t, s) \\
= \frac{1}{|t-s|^{1/2}} \chi_1(t) \chi_1(s) \int_\mathbb{R} H(\alpha) e^{i(x_1 - \alpha - y_1)^2/4(t-s)} m_4(t, s, x_1 - \alpha, y_1) d\alpha \\
+ \frac{1}{|t-s|^{1/2}} \chi_1(t) \chi_1(s) \int_\mathbb{R} H(\alpha) J_4(t, s, (x_1 - \alpha)/(2|t-s|^{1/2}), y_1/(2|t-s|^{1/2})) d\alpha \\
= k_4^1(x_1, y_1, t, s) + k_4^2(x_1, y_1, t, s),
\]
where \( \|H\|_{L^{1}(\mathbb{R})} \leq C \),
\[
\|m_{4}(t, s, \tilde{x}_{1}, \cdot)\|_{\text{BV}_{\tilde{x}_{1}}} + \|m_{4}(t, s, \cdot, \tilde{y}_{1})\|_{\text{BV}_{\tilde{y}_{1}}} \leq C
\]
(4.43)
uniformly in \( t, s, \tilde{x}_{1} \) and \( \tilde{y}_{1} \), and
\[
(1 + |\tilde{x}_{1} - \tilde{y}_{1}|) |J_{4}(t, s, \tilde{x}_{1}, \tilde{y}_{1})| \leq C.
\]
(4.44)

5. Boundedness of the operators \( T_{j} \), I

In this section, we start proving that the operators \( T_1, T_2, T_3 \) and \( T_4 \) are bounded from \( \mathcal{X} \) to \( \mathcal{X}' \). To cover all dimensions, fix an acceptable pair \( (p, q) \), with \( p \leq \frac{3}{2} \) if \( n = 1 \), \( p \leq p_{0} \) if \( n = 2 \), and \( p \leq 2 \) if \( n \geq 3 \). Clearly, an operator is bounded from \( \mathcal{X} \) to \( \mathcal{X}' \) if it is bounded from \( L_{q}^{1}L_{p}^{2} \) to \( L_{q}^{\infty}L_{p}^{2} \), from \( L_{q}^{2}L_{p}^{2} \) to \( L_{q}^{\infty}L_{p}^{2} \), from \( L_{q}^{1}L_{p}^{2} \) to \( L_{q}^{r_{0}}L_{p}^{r_{0}} \), and from \( L_{q}^{r_{0}}L_{p}^{r_{0}} \) to \( L_{q}^{r}L_{p}^{r} \), with bounds that depend only on the dimension \( n \) (or \( p_{0} \) if \( n = 2 \)). Recall that the operators \( T_{j} \) are of the form

\[
T_{j}(g)(x, t) = \int_{\mathbb{R}^{n}} \int_{\mathbb{R}} g(y, s) \int_{\mathbb{R}^{n}} e^{i(x-y)\xi} e^{-i(t-s)|\xi|^{2}} e^{-\varepsilon|\xi|^{2}} \mu_{j}(y, \xi_{1}, t, s) \, d\xi \, ds
\]
where the multipliers \( \mu_{j} \) are defined in (3.16), (3.18), (3.22) and (3.27).

PROPOSITION 5.1. The operators \( T_{j}, j = 1, 2, 3, 4 \), are bounded from \( L_{q}^{1}L_{p}^{2} \) to \( L_{q}^{\infty}L_{p}^{2} \).

Proof. A simple condition for \( L_{q}^{1}L_{p}^{2} \to L_{q}^{\infty}L_{p}^{2} \) boundedness of an operator of the form

\[
T(g)(x, t) = \int_{\mathbb{R}^{n}} \int_{\mathbb{R}} g(y, s) \int_{\mathbb{R}^{n}} e^{i(x-y)\xi} e^{-i(t-s)|\xi|^{2}} e^{-\varepsilon|\xi|^{2}} \mu(y, \xi_{1}, t, s) \, d\xi \, ds
\]
is that the operator

\[
S_{t,s}(h)(x) = \int_{\mathbb{R}^{n}} h(y) \int_{\mathbb{R}^{n}} e^{i(x-y)\xi} e^{-i(t-s)|\xi|^{2}} e^{-\varepsilon|\xi|^{2}} \mu(y, \xi_{1}, t, s) \, d\xi
\]
is bounded on \( L^{2}(\mathbb{R}^{n}) \) uniformly in \( t \) and \( s \). The fact that this condition is sufficient follows easily by the Minkowski inequality for integrals. By Plancherel’s theorem it suffices to prove that for any \( h \in \mathcal{S}(\mathbb{R}) \),

\[
\left\| \int_{\mathbb{R}} h(y_{1}) \mu(y_{1}, \xi_{1}, t, s) e^{-iy_{1}\xi_{1}} \, dy_{1} \right\|_{L_{\xi_{1}}^{2}} \leq C \|h\|_{L^{2}}
\]
(5.1)
uniformly in \( t \) and \( s \). A simple criterion for this to hold is that \( \mu \) has bounded variation in \( y_{1} \):

\[
\|\mu(\cdot, \xi_{1}, t, s)\|_{\text{BV}_{\xi_{1}}} \leq C
\]
(5.2)
uniformly in $\xi_1$, $t$, and $s$. The BV-norm was defined in (4.1). To see that (5.2) implies (5.1) we can use Carleson’s theorem [2]: the operator

$$C(h)(\xi_1) = \sup_N \left| \int_{-\infty}^N h(y_1) e^{-iy_1 \xi_1} dy_1 \right|$$

is bounded from $L^2_{y_1}$ to $L^2_{\xi_1}$. Thus, for any $\xi_1$ we have

$$\left| \int_{R} h(y_1) \mu(y_1, \xi_1, t, s) e^{-iy_1 \xi_1} dy_1 \right| = \left| \int_{R} \int_{-\infty}^{y_1} h(z) e^{-iz \xi_1} dz \mu(y_1, \xi_1, t, s) dy_1 \right|$$

$$\leq |\mu(\infty, \xi_1, t, s)| \cdot h(\xi_1)$$

$$+ \left| \int_{R} C(h)(\xi_1) |\mu'(y_1, \xi_1, t, s)| dy_1 \right|$$

$$\leq C(h)(\xi_1) \left\| \mu(\cdot, \xi_1, t, s) \right\|_{BV_{y_1}}.$$

By Carleson’s theorem this proves (5.1).

For the multiplier $\mu_1$ in (3.16) notice first that the factor $e^{i\theta y_1(y_1)(t-s)}$ is bounded and depends only on $y_1$ (and not on $\xi_1$), so it can be incorporated into $h$. In addition, the function $e^{-\delta y_1(y_1)}$ is nondecreasing and bounded for any $\delta \geq 0$. Thus the bounded-variation condition (5.2) is clearly verified. The same argument applies for the multiplier $\mu_2$ in (3.18).

For the multiplier $\mu_3$ in (3.22) we make the change of variable $\tau = \xi_1^2 u$ and write

$$\mu_3(y_1, \xi_1, t, s) = \chi_1(t) \chi_1(s) [1 - \psi_\gamma(\xi_1)]$$

$$\times \int_{R} e^{i(t-s)\xi_1^2 u} e^{-e^2 \xi_1^2 u^2} \frac{1-\psi(10u)}{u-i\theta(y_1) / \xi_1 + b_{y_1}(y_1) / \xi_1^2} du. \quad (5.3)$$

Notice that the variable $u$ in the integral has the property $|u| \geq \frac{1}{10}$ and $|\xi_1| \geq \gamma \geq C(1+\beta)$. Therefore the integral in (5.3) is the inverse Fourier transform of a Hörmander–Mikhlin multiplier evaluated at $(t-s)\xi_1^2$, and is thus bounded. By differentiating with respect to $y_1$ we have

$$|\partial_{y_1} \mu_3(y_1, \xi_1, t, s)| \leq C[1 - \psi_\gamma(\xi_1)] \left( \frac{|a_{\beta, \lambda}(y_1)|}{|\xi_1|} + \frac{|b_{\beta, \lambda}(y_1)|}{|\xi_1|^2} \right).$$

By (3.8) this suffices to prove the estimate (5.2) for the multiplier $\mu_3$. Finally, for the multiplier $\mu_4$ the condition (5.2) is proved in Lemma 4.3. \qed

Using the decomposition $k_4 = k_4^1 + k_4^2$ in (4.42), we decompose the kernel $K_4$ into $K_4^1 + K_4^2$, where

$$K_4^m(x,y,t,s) = k_4^m(x_1, y_1, t, s) \int_{R_n-1} e^{i(x' - y') \cdot \xi'} e^{-i(t-s) \xi' \cdot \xi} e^{-e^2 |\xi'|^2 \xi_1^2} d\xi', \quad m = 1, 2. \quad (5.4)$$
and then decompose the operator $T_4$ as $T_4^1 + T_4^2$. We can use (4.43) and the criterion (5.2) to prove that the kernel $K_1(x_1, y_1, t, s)$ defines a bounded operator on $L^2(\mathbb{R})$: since the function $H$ in (4.42) is in $L^1(\mathbb{R})$, it suffices to prove that

$$\left| \int_{\mathbb{R}} h(y_1) e^{i(x_1 - y_1)^2/4(t-s)} m_4(t,s,x_1-\alpha, y_1) dy_1 \right| \leq C \|h\|_{L^2}.$$  

This follows from (5.2) and scaling. By Proposition 5.1 we know that the kernel $K_1(x_1, y_1, t, s)$ defines a bounded operator from $L^2_{y_1}(\mathbb{R})$ to $L^2_{y_1}(\mathbb{R})$. Thus the kernel $K_2(x_1, y_1, t, s)$ defines a bounded operator from $L^2_{y_1}(\mathbb{R})$ to $L^2_{y_1}(\mathbb{R})$ as well. To summarize, both kernels $K_1(x_1, y_1, t, s)$ and $K_2(x_1, y_1, t, s)$ define bounded operators on $L^2(\mathbb{R})$, both kernels $K_3(x, y, t, s)$ and $K_4(x, y, t, s)$ define bounded operators from $L^2_y(\mathbb{R}^n)$ to $L^2_x(\mathbb{R}^n)$, and both operators $T_4^1$ and $T_4^2$ are bounded from $L^2_y L^2_x$ to $L^\infty_y L^2_x$.

6. Boundedness of the operators $T_j$, II

In this section we prove that the operators $T_j$ are bounded from $L^p_y L^q_x$ to $L^\infty_y L^2_x$ and from $L^p_y L^q_x$ to $L^p_y L^q_x$. For this, we use a theorem of Keel and Tao [7, Theorem 1.2]:

**Lemma 6.1.** (Keel and Tao [7]) Assume that $U(t): L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ denotes a family of operators indexed over $t \in \mathbb{R}$ with the properties

$$\|U(t)f\|_{L^2(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)}$$

for any $t \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^n)$, and

$$\|U(s)U(t)^*f\|_{L^\infty(\mathbb{R}^n)} \leq C |t-s|^{-n/2} \|f\|_{L^1(\mathbb{R}^n)}$$

for any $t, s \in \mathbb{R}^n$ and $f \in S(\mathbb{R}^n)$. Then

$$\|U(t)f\|_{L^p_y L^q_x} \leq C \|f\|_{L^2}.$$  

**Proposition 6.2.** The operators $T_1$, $T_2$, $T_3$ and $T_4$ are bounded from $L^p_y L^q_x$ to $L^\infty_y L^2_x$.

**Proof.** An operator of the form

$$T(g)(x,t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}} g(y,s) K(x,y,t,s) ds dy$$

is bounded from $L^p_y L^q_x$ to $L^\infty_y L^2_x$ if the operators

$$S_{t_0,\pm}(g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}} g(y,s) e^{-(\epsilon')^2 |x|} K(x,y,t_0,s) \chi_{\pm}(t_0-s) ds dy$$
are bounded from $L^p_x L^q_y$ to $L^2_z$ uniformly in $t_0$ and $\varepsilon'>0$. This is equivalent to the fact that the operators $S^*_p$ are bounded from $L^2_z$ to $L^p_x L^q_y$ uniformly in $t_0$ and $\varepsilon'>0$. For this we apply Lemma 6.1. The $L^2$-condition was already verified in Proposition 5.1 (for the operators $T_1$, $T_2$ and $T_3$) and the remark at the end of §5 (for the operator $T_4^1$): the kernels $K_1(x,y,t,s)$, $K_2(x,y,t,s)$, $K_3(x,y,t,s)$ and $K_4(x,y,t,s)$ define bounded operators from $L^2_z$ to $L^2_y$ uniformly in $t$ and $s$. It remains to check the $L^1\to L^\infty$ bound, i.e.

$$\int e^{-(\varepsilon')^2|z|^2} K(z,y,t_0,s)\chi_{\pm}(t_0-s)e^{-(\varepsilon')^2|z|^2} \bar{K}(z,x,t_0,t)\chi_{\pm}(t_0-t)\,dz \leq C|t-s|^{-n/2}$$

(6.1)

uniformly in $x$, $y$, $t$ and $s$, where $K$ stands for $K_1$, $K_2$, $K_3$ or $K_4$. For the kernels $K_j$, $j=1,2,3$, we substitute the formula (4.26) and integrate first the variable $z$. Notice that all the integrals converge absolutely because of the exponentially decaying factors. It remains to prove that for any $v=(v_1, \ldots, v_n)\in \mathbb{R}^n$ the absolute value of the integral

$$\int e^{i(x-y)\cdot \xi} e^{-i(t-s)|\xi|^2} e^{it/2} e^{-i|\xi+v|^2/2} \mu_j(y_1, \xi_1, t_0, s) \bar{\mu}_j(x_1, \xi_1+v_1, t_0, t)\,d\xi$$

(6.2)

is dominated by $C|t-s|^{-n/2}$, provided that $(t_0-s)(t_0-t)>0$ and $j=1,2,3$. For this we use Lemma 4.2. Notice that the integral in (6.2) splits as a product of $n$ integrals in $\xi_1, \xi_2, \ldots, \xi_n$. The integrals in $\xi_2, \ldots, \xi_n$ are each bounded by $C|t-s|^{-1/2}$ by Lemma 4.2. It remains to prove the same bound for the integral in $\xi_1$. Let $w_1=x_1-y_1+2v_1(t_0-t)$. We need to prove that

$$\int e^{i(x-y)\cdot \xi} e^{-i(t-s)|\xi|^2} e^{-i|\xi+v|^2/2} \mu_j(y_1, \xi_1, t_0, s) \bar{\mu}_j(x_1, \xi_1+v_1, t_0, t)\,d\xi_1 \leq C|t-s|^{-1/2}$$

(6.3)

uniformly in all the variables, where $j=1,2,3$.

The estimate (6.3) for $j=1,2,3$ would follow from Lemma 4.2 with $k=2$, provided that we could verify that the multipliers $\mu_j$, $j=1,2,3$, belong to $HM_{\xi_1}^0$. This is clear if $j=1$ or $j=2$, simply by inspecting the formulas (3.16) and (3.18) and noticing that the functions $\chi_+(\xi_1)e^{-\delta \xi_1}$ and $\chi_-(\xi_1)e^{\delta \xi_1}$ belong to $HM_{\xi_1}^0$ uniformly in $\delta \geq 0$. If $j=3$, we examine the formula (5.3). We already noticed in the proof of Lemma 5.1 that the function $\mu_3$ is bounded; an elementary estimate using the fact that $|\xi_1|\geq \gamma\geq C(1+\beta)$ shows that it is actually in the symbol class $HM_{\xi_1}^0$ (see (6.22) below for a more precise estimate).

To prove (6.1) for the kernel $K_4^1$ we substitute the formula (5.4) into (6.1) and notice that the integral splits as a product of $n$ integrals. By the same argument as before, the integrals in $z_2, \ldots, z_n$ are each bounded by $C|t-s|^{-1/2}$. It remains to prove that

$$\int e^{-(\varepsilon')^2|z|^2} k_4^1(z_1,y_1,t_0,s)\chi_{\pm}(t_0-s)e^{-(\varepsilon')^2|z|^2} \bar{k}_4^1(z_1,x_1,t_0,t)\chi_{\pm}(t_0-t)\,dz_1 \leq C|t-s|^{-1/2}$$
uniformly in all the variables. For this we substitute the formula (4.42) and integrate the variable $z_1$ first. The estimate follows from Lemma 4.1 with $\delta = (s-t)/4(t_0-s)(t_0-t)$. \(\square\)

**Proposition 6.3.** The operator $T_2^4$ is bounded from $L^p L^{q_2} L^{y_2}$ to $L^p L^{q_2} L^{y_2}$.

**Proof.** With the same notation as in Proposition 6.2, we have to prove that the operators $S_{t_0, \pm}$ are bounded from $L^p L^{q_2} L^{y_2}$ to $L^p L^{q_2} L^{y_2}$ uniformly in $t_0$. This is equivalent to the fact that the operators $S^*_{t_0, \pm} S_{t_0, \pm}$ are bounded from $L^p L^{q_2} L^{y_2}$ to $L^p L^{q_2} L^{y_2}$. The kernels of the operators $S^*_{t_0, \pm} S_{t_0, \pm}$ are

$$L^2_{t_0, t, s}(x, y, t, s) = \int_{\mathbb{R}^n} e^{-\langle \xi \rangle^2 |z|^2/4} K_4^2(z, y, t_0, s) \chi_{\pm}(t_0-s) e^{-\langle \xi \rangle^2 |t|^2/4} K_4^2(z, x, t_0, t) \chi_{\pm}(t_0-t) dz.$$

Let

$$U^2_{t_0, t, s}(h)(x) = \int_{\mathbb{R}^n} L^2_{t_0, t, s}(x, y, t, s) h(y) dy.$$

We claim that

$$\|U^2_{t_0, t, s}\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C$$

(6.4)

and

$$\|U^2_{t_0, t, s}\|_{L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)} \leq C |t-s|^{-(n-1)/2} (|t-t_0| + |s-t_0|)^{-1/2} \log \frac{|t-t_0|^2 + |s-t_0|^2}{|t-t_0| |s-t_0|}$$

(6.5)

uniformly in $t$, $s$ and $t_0$. Assuming (6.4) and (6.5) we would have by interpolation

$$\|U^2_{t_0, t, s}\|_{L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \leq C \left[ |t-s|^{-(n-1)/2} (|t-t_0| + |s-t_0|)^{-1/2} \log \frac{|t-t_0|^2 + |s-t_0|^2}{|t-t_0| |s-t_0|} \right]^{2/q} \cdot$$

By the Minkowski inequality for integrals we would have

$$\|S^*_{t_0, \pm} S_{t_0, \pm}(g)(\cdot, t)\|_{L^q_x} \leq \int_{\mathbb{R}^n} \left\| \int_{\mathbb{R}^n} g(y, s) L^2_{t_0, t, s}(x, y, t, s) dy \right\|_{L^q_x} ds$$

$$\leq C \int_{\mathbb{R}^n} \left\| g(\cdot, s) \right\|_{L^q} \left[ |t-s|^{-(n-1)/2} (|t-t_0| + |s-t_0|)^{-1/2} \log \frac{|t-t_0|^2 + |s-t_0|^2}{|t-t_0| |s-t_0|} \right]^{2/q-1} ds.$$

We apply Lemma 6.4 below with $\delta = 1/n$ to conclude that the kernel

$$\left[ |t-s|^{-(n-1)/2} (|t-t_0| + |s-t_0|)^{-1/2} \log \frac{|t-t_0|^2 + |s-t_0|^2}{|t-t_0| |s-t_0|} \right]^{2/q-1}$$
defines a bounded operator from $L^p_t$ to $L^p_t$. Thus

$$
\|S^*_{t_0, \pm} S_{t_0, \pm}(g)\|_{L^p_t L^p_t} \leq C \|g\|_{L^p_t L^p_t}
$$
as desired. It remains to prove (6.4) and (6.5). The $L^2$-bound (6.4) was already proved in the remark at the end of §5. For (6.5) we need to control the absolute value of the kernels $L_{2, t_0, \pm}$. These kernels split as products of $n$ integrals; as in Proposition 6.2 the integrals in $z_2, ..., z_n$ are each bounded by $C |t-s|^{-1/2}$. Thus it remains to prove that

$$
\left| \int_{\mathbb{R}} e^{-\left(e^2\right)^{2z^2}} \frac{\partial^2}{\partial t_0 \partial t} \chi_\pm(t_0) e^{-\left(e^2\right)^{2z^2}} \chi_\pm(t_0-t) \psi_{t_0, \pm} \psi_{t_0, \pm} \frac{dz}{
}
\leq C(t_0-t)^{-1/2} \log \frac{|t-t_0| + |s-t_0|}{|t-t_0| \cdot |s-t_0|}
$$

uniformly in all the variables. We substitute the formula (4.42) and integrate the variable $z_1$ first. As before let $A = 2|t-t_0|^{1/2}$ and $B = 2|t-t_0|^{1/2}$. It suffices to prove that

$$
\int_{\mathbb{R}} \frac{1}{A} |J(t_0, s, (z_1-\alpha_1)/A, y_1/A) e^{-\left(e^2\right)^{2z^2}} | \frac{1}{B} |J(t_0, t, (z_1-\alpha_2)/B, x_1/B)| dz_1 \leq C(A^2 + B^2)^{-1/2} \log \frac{A^2 + B^2}{AB}.
$$

This follows easily from (4.44). \[ \square \]

In the proof of Proposition 6.3 we used the following lemma:

**Lemma 6.4.** For any $t, s > 0$, $\delta \in (0, 1]$ and $p \in [1, 2]$ let

$$
L_\delta(t, s) = |t-s|^{-2+2\delta} |t+s|^{-2\delta/p'} \left[ \log t^2 + s^2 \right]^{\gamma_{45}/p'}.
$$

For any continuous compactly supported function $f: (0, \infty) \to \mathbb{C}$ let

$$
S_\delta f(t) = \int_0^\infty f(s) L_\delta(t, s) ds.
$$

Then

$$
\|S_\delta f\|_{L^p((0, \infty))} \leq C_\delta \|f\|_{L^p((0, \infty))}.
$$

**Proof.** By analytic interpolation, using the family of kernels

$$
L_\delta(t, s) = |t-s|^{-2+2\delta} |t+s|^{-2\delta} \left[ \log t^2 + s^2 \right]^{\gamma_{45}/\delta},
$$

...
defined for $\text{Re} \ z \in \left[0, \frac{1}{2}\right]$, we see that it suffices to prove the lemma for $p=p'=2$. In this case, (6.7) is equivalent to a Hardy inequality. Let $(Y, d\mu=((0, \infty), dt/t)$, and for any $t, s > 0$ let $\tilde{f}(s)=s^{1/2}f(s)$ and $\tilde{F}(t)=t^{1/2}S_0f(t)$. Then

$$\tilde{F}(t)=t^{1/2}\int_0^\infty |t-s|^{1-\delta}(t+s)^{-\delta}\left[\log\frac{t^2+s^2}{ts}\right]^{2\delta}f(s)\,ds = \int_Y \tilde{L}_\delta(t, s)\tilde{f}(s)\,d\mu(s),$$

(6.8)

where $\tilde{L}_\delta(t, s)=t^{1/2}s^{1/2}|t-s|^{-(1-\delta)}(t+s)^{-\delta}\left[\log(t^2+s^2)/ts\right]^{2\delta}$. The inequality (6.7) with $p=p'=2$ is equivalent to

$$\|\tilde{F}\|_{L^2(Y, d\mu)} \leq C \|\tilde{f}\|_{L^2(Y, d\mu)}.$$

This follows from (6.8) and the observation that $\|\tilde{L}_\delta(\cdot, s)\|_{L^1(Y, d\mu(t))} \leq C_\delta$ uniformly in $s$, and $\|\tilde{L}_\delta(t, \cdot)\|_{L^1(Y, d\mu(s))} \leq C_\delta$ uniformly in $t$, provided that $\delta \in (0, 1]$. □

This completes the proof of the $L^p_t \rightarrow L^q_t$ boundedness of the operators $T_j$. We now turn to the question of $L^p_t L^q_y$ boundedness.

**Proposition 6.5.** The operators $T_1, T_2$ and $T_3$ are bounded from $L^p_t L^q_y$ to $L^p_t L^q_y$.

**Proof.** As in Proposition 6.2, by using Lemma 6.1 it suffices to prove the uniform bound

$$\left| \int_{\mathbb{R}^n} e^{-(e')^2|z|^2}K(x, z, t, s) e^{-(e')^2|z|^2}K(y, z, s, s_0) \,dz \right| \leq C|t-s|^{n/2}$$

(6.9)

for the kernels $K=K_1$, $K=K_2$ and $K=K_3$, under the assumption that $(t-s_0)(s-s_0) > 0$. The integrals in (6.9) split as products of $n$ integrals. By the same argument as in Proposition 6.2 the integrals in $z_2, \ldots, z_n$ in (6.9) are bounded by $C|t-s|^{-1/2}$ as desired. It remains to prove a similar bound for the integral in $z_1$. To summarize, it suffices to prove that

$$\left| \int_{\mathbb{R}} e^{-(e')^2z_1^2}k(x_1, z_1, t, s) e^{-(e')^2z_1^2}k(y_1, z_1, s, s_0) \,dz_1 \right| \leq C|t-s|^{-1/2}$$

(6.10)

for $k=k_1$, $k=k_2$ and $k=k_3$, where $t, s, s_0 \in [-1, 1]$ and $(t-s_0)(s-s_0) > 0$. Assume that $t-s_0 > 0$ and $s-s_0 > 0$ (the case $t-s_0 < 0$ and $s-s_0 < 0$ is similar). Notice that the bound (6.10) is trivial for $j=2$, since $\mu_2(\cdot, t, s, s_0) \equiv 0$ if $t-s_0 > 0$. Also, for the kernel $k=k_3$ the estimate (6.10) can be obtained as in Proposition 6.2. It remains to consider the case $k=k_1$.

Recall that $\gamma \geq \beta + 1$. Fix $h=h_{z_1, y_1}: \mathbb{R} \rightarrow [0, 1]$, a smooth function with the following properties:

$$h(z_1) = 1 \quad \text{if } \min\{|z_1-x_1|, |z_1-y_1|\} \leq 10\gamma,$$

$$h(z_1) = 0 \quad \text{if } \min\{|z_1-x_1|, |z_1-y_1|\} \geq 20\gamma,$$

$$|\partial_z^l h(z_1)| \leq C \gamma^{-l} \quad \text{for any } z_1 \in \mathbb{R} \text{ and } l = 0, 1, 2.$$
We use this function to break up the integral in the left-hand side of (6.10) into two parts. For the term that contains the function $1 - h$, i.e., when $z_1$ is far from $x_1$ and $y_1$, we integrate by parts in (4.27) and use the fact that $|(x_1 - z_1) - 2(t - s_0)\xi_1| > \frac{1}{10}|x_1 - z_1|$ if $|\xi_1| < 2\gamma$ and $t, s_0 \in [-1, 1]$. The result is

$$|k_1(x_1, z_1, t, s_0)| \leq C|x_1 - z_1|^{-1}$$

and

$$|\tilde{k}_1(y_1, z_1, s, s_0)| \leq C|y_1 - z_1|^{-1}$$

if $\min\{|z_1 - x_1|, |z_1 - y_1|\} > 10\gamma$. Thus

$$|k_1(x_1, z_1, t, s_0)\tilde{k}_1(y_1, z_1, s, s_0)| \leq C(|x_1 - z_1|^{-2} + |y_1 - z_1|^{-2})$$

if $\min\{|z_1 - x_1|, |z_1 - y_1|\} > 10\gamma$. It follows that

$$\left|\int_{\mathbb{R}} e^{-(\xi_1^2 + \eta_1^2)} k_1(x_1, z_1, t, s_0) e^{-i(t-s_0)\xi_1^2} \tilde{k}_1(y_1, z_1, s, s_0)(1 - h(z_1)) \, dz_1\right| \leq C\gamma^{-1} \leq C|t - s|^{-1/2}$$

(6.11)

if $|t - s| \leq 2$, as desired.

To estimate the term that contains the function $h$, assume first that $|x_1 - y_1| \leq 100\gamma$. Let

$$\tilde{k}_1(x_1, z_1, t, s_0) = \int_{\mathbb{R}} e^{i(x_1 - z_1)\xi_1} e^{-i(t-s_0)\xi_1^2} e^{-\xi_1^2} \mu_1(x_1, \xi_1, t, s_0) \, d\xi_1$$

(6.12)

and

$$\tilde{k}_1(y_1, z_1, s, s_0) = \int_{\mathbb{R}} e^{i(y_1 - z_1)\eta_1} e^{-i(s-s_0)\eta_1^2} e^{-\eta_1^2} \mu_1(y_1, \eta_1, s, s_0) \, d\eta_1.$$  

(6.13)

Since $|x_1 - z_1| + |y_1 - z_1| \leq C\gamma$, we can use (3.8) and the formula (3.16) to see that

$$|\mu_1(z_1, \xi_1, t, s_0) - \mu_1(x_1, \xi_1, t, s_0)| + |\mu_1(z_1, \eta_1, s, s_0) - \mu_1(y_1, \eta_1, s, s_0)| \leq C\frac{\gamma^4}{\lambda}.$$

By integrating we have

$$|k_1(x_1, z_1, t, s_0) - \tilde{k}_1(x_1, z_1, t, s_0)| + |k_1(y_1, z_1, s, s_0) - \tilde{k}_1(y_1, z_1, s, s_0)| \leq C\frac{\gamma^4}{\lambda}$$

(6.14)

if $|x_1 - z_1| + |y_1 - z_1| \leq C\gamma$. Also

$$|k_1(x_1, z_1, t, s_0)| + |\tilde{k}_1(x_1, z_1, t, s_0)| + |k_1(y_1, z_1, s, s_0)| + |\tilde{k}_1(y_1, z_1, s, s_0)| \leq C\gamma.$$  

(6.15)
By (6.14) and (6.15) and the fact that $|t-s| \leq 2$,}

$$
\left| \int_{\mathbb{R}} h(z_1) e^{-(c')^2 z_1^2} \tilde{k}_1(x_1, z_1, t, s_0) e^{-(c')^2 z_1^2} \tilde{k}_1(y_1, z_1, s, s_0) \, dz_1 \right| \\
\leq \left| \int_{\mathbb{R}} h(z_1) e^{-(c')^2 z_1^2} \tilde{k}_1(x_1, z_1, t, s_0) e^{-(c')^2 z_1^2} \tilde{k}_1(y_1, z_1, s, s_0) \, dz_1 \right| + C \frac{\gamma^6}{\lambda} 
$$

(6.16)

provided that $|x_1 - y_1| \leq 100 \gamma$ and

$$
\gamma^6 \leq \lambda. 
$$

(6.17)

It remains to estimate the first integral in the right-hand side of (6.16). For this we substitute the formulas (6.12) and (6.13), and integrate the variable $z_1$ first, as in Proposition 6.2. Let $\tilde{H} = \tilde{H}_{x_1, y_1}$ denote the Fourier transform of the function $z_1 \mapsto h(z_1) e^{-(c')^2 z_1^2}$.

The properties of the cutoff function $h$ guarantee that

$$
\|\tilde{H}\|_{L^1(\mathbb{R})} \leq C. 
$$

(6.18)

We have

$$
\int_{\mathbb{R}} h(z_1) e^{-(c')^2 z_1^2} \tilde{k}_1(x_1, z_1, t, s_0) e^{-(c')^2 z_1^2} \tilde{k}_1(y_1, z_1, s, s_0) \, dz_1 \\
= \int_{\mathbb{R}} \tilde{H}(\theta) e^{i x_1 \theta - i (t-s_0) \theta^2} \int_{\mathbb{R}} e^{i (x_1 - y_1) \theta + i ((x_1 + (t-s_0) \theta) \eta_1)} e^{-2 (\eta_1 + \theta)^2} \mu_1(x_1, \eta_1 + \theta, s, s_0) \, d\eta_1 \, d\theta. 
$$

(6.19)

The multiplier $\mu_1$ belongs to the symbol class $\text{HM}_0^0$. This was checked in Proposition 6.2. By Lemma 4.2, the $\eta_1$-integral in (6.19) is bounded by $C|t-s|^{-1/2}$. By (6.16) and (6.18) we have

$$
\left| \int_{\mathbb{R}} h(z_1) e^{-(c')^2 z_1^2} \tilde{k}_1(x_1, z_1, t, s_0) e^{-(c')^2 z_1^2} \tilde{k}_1(y_1, z_1, s, s_0) \, dz_1 \right| \leq C|t-s|^{-1/2} 
$$

(6.20)

if $|x_1 - y_1| \leq 100 \gamma$, as desired.

If $|x_1 - y_1| \geq 100 \gamma$, we break up the integral in (6.20) into two parts, depending on whether $z_1$ is close to $x_1$, or $z_1$ is close to $y_1$. Assume that we are looking to estimate the integral over $|z_1 - x_1| \leq 20 \gamma$. We argue as before: the only difference is that we replace the kernels $k_1(x_1, z_1, t, s_0)$ and $k_1(y_1, z_1, s, s_0)$ with the kernels

$$
\tilde{k}_1(x_1, y_1, z_1, t, s_0) = \int_{\mathbb{R}} e^{i (x_1 - z_1) \xi_1} e^{-i (t-s_0) \xi_1^2} e^{-2 \xi_1^2} \mu_1(x_1, \xi_1, t, s_0) \, d\xi_1.
$$
and
\[
\hat{k}_1(x_1,y_1,z_1,s,s_0) = \int_{\mathbb{R}} e^{i(y_1-z_1)m} e^{-i(s-s_0)n_1} e^{-\varepsilon^2 n_1^2} \mu_1(x_1,\eta_1,s,s_0) \, d\eta_1.
\]

The only difference compared to (6.12) and (6.13) is that we replace the multipliers \(\mu_1(z_1,\cdots)\) with \(\mu_1(x_1,\cdots)\) in both integrals. Since \(|z_1-x_1| \leq 20\gamma\), all the previous estimates apply, so the integral in the left-hand side of (6.20) over the set \(|z_1-x_1| \leq 20\gamma\) is bounded by \(|t-s|^{-1/2}\), as desired. The integral over the set \(|z_1-y_1| \leq 20\gamma\) is similar, the only difference being that we replace the multipliers \(\mu_1(z_1,\cdots)\) with \(\mu_1(y_1,\cdots)\) in both integrals. Together with (6.11) this completes the proof of (6.10) for the kernel \(k_1\).

**Proposition 6.6.** The operators \(T_3\) and \(T_4^2\) are bounded from \(L^p_t L^q_x\) to \(L^p_t L^q_x\).

**Proof.** By the same argument as in Proposition 6.3 it suffices to prove that
\[
\left| \int_{\mathbb{R}} e^{-(\varepsilon^2)^2} k(x_1,z_1,t,s_0) e^{-(\varepsilon^2)^2} \hat{k}(y_1,z_1,s,s_0) \, dz_1 \right| \leq C(t-s_0^2 + |s-s_0|^2)^{-1/2} \log \left( t-s_0^2 + |s-s_0|^2 \right) \frac{1}{|t-s_0| + |s-s_0|},
\]

for \(k=k_3\) and \(k=k_4^2\), provided that \((t-s_0)(s-s_0)>0\). For the kernel \(k_4^2\) this follows as in Proposition 6.3—see the proof of (6.6). For the kernel \(k_3\) we prove a bound similar to (4.42) and (4.44): by examining (5.3) and integrating by parts we see easily that
\[
|k_3(x_1,y_1,t,s)| \leq C(1 + |t-s|\xi_1^2)^{-2}
\]

for \(l=0,1,2\). We use this in (4.27) and integrate by parts (because of the decay in (6.22) as \(|t-s|\xi_1^2 \to \infty\), the factor \(e^{-i(t-s)\xi_1^2}\) may be absorbed in \(\mu_3(y_1,\xi_1,t,s)\)). It follows easily that
\[
|k_3(x_1,y_1,t,s)| \leq \frac{C}{|t-s|^{1/2} + |x_1-y_1|}.
\]

This estimate can be used to prove (6.21) for the kernel \(k_3\), as in Proposition 6.3.

**Proposition 7.1.** The operators \(T_j\), \(j=1,2,3,4\), are bounded from \(L^p_t L^q_x\) to \(L^p_t L^q_x\).

In dimensions \(n \geq 3\) we need an interpolation lemma of Keel and Tao [7] (see pp. 964–967 for the proof):

### 7. Boundedness of the operators \(T_j\)

It remains to prove the following result:

**Proposition 7.1.** The operators \(T_j\), \(j=1,2,3,4\), are bounded from \(L^p_t L^q_x\) to \(L^p_t L^q_x\).
LEMMA 7.2. (Keel and Tao [7]) Assume that \( n \geq 3 \) and

\[
U(f)(x, t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}} f(y, s) K(x, y, t, s) \, ds \, dy
\]
is an operator with a locally integrable kernel \( K \). Let

\[
U_{1}(f)(x, t) = \int_{\mathbb{R}^n} \int_{|t-s| \in [2^l, 2^{l+1}]} f(y, s) K(x, y, t, s) \, ds \, dy.
\]

Let

\[
\beta(a, b) = \frac{n}{2} - 1 - \frac{n}{2} \left( \frac{1}{a'} + \frac{1}{b'} \right)
\]

and assume that for any \( f \in S(\mathbb{R}^n \times \mathbb{R}) \) the estimate

\[
\|U_{1}(f)\|_{L^2 L^p} \leq C 2^{-\beta(a, b)} \|f\|_{L^2 L^p}
\]

holds for the exponents

(i) \( a = b = 1 \);

(ii) \( 2n/(n+2) \leq a \leq 2 \) and \( b = 2 \);

(iii) \( 2n/(n+2) \leq b \leq 2 \) and \( a = 2 \).

Then

\[
\|U(f)\|_{L^2 L^{2n/(n-2)}} \leq C \|f\|_{L^2 L^{2n/(n+2)}}.
\]

Proof of Proposition 7.1. We claim first that an operator of the form

\[
T(g)(x, t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(y, s) \int_{\mathbb{R}^n} e^{i(x-y) \xi - i(t-s)\xi^2} e^{-\xi^2/4} \mu(y_1, \xi_1, t, s) \, d\xi \, dy
\]
is bounded from \( L^p L^q \) to \( L^p L^q \) if \( p \in [1, 2) \), and the operator

\[
S_{t,s}(h)(x) = \int_{\mathbb{R}^n} h(y) \int_{\mathbb{R}^n} e^{i(x-y) \xi - i(t-s)\xi^2} e^{-\xi^2/4} \mu(y_1, \xi_1, t, s) \, d\xi \, dy
\]
satisfies the bounds

\[
\|S_{t,s}\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C
\]

and

\[
\|S_{t,s}\|_{L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)} \leq C |t-s|^{-n/2}
\]
uniformly in \( t \) and \( s \). Assuming (7.2) and (7.3) we would have by interpolation

\[
\|S_{t,s}\|_{L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \leq C |t-s|^{-n(1/q - 1/2)}.
\]
By the Minkowski inequality for integrals we would have
\[\|T(g)(\cdot, t)\|_{L^{p'}} \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(y, s) \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} e^{-i(t-s) |\xi|^2} e^{-\varepsilon |\xi|^2} \mu(y_1, \xi_1, t, s) \, d\xi \, dy \, ds\]
\[\leq \int_{\mathbb{R}^n} \|g(\cdot, s)\|_{L^p} |t-s|^{-n(1/p-1/2)} \, ds.\]

Since \(1/p-1/p'=1-n(1/q-1/2)\) and \(p<p'\), by fractional integration it would follow that
\[\|T(g)\|_{L^{p'}_x L^q_t} \leq C_p \|g\|_{L^p_x L^q_t},\]
as desired.

It is easy to check the estimates (7.2) and (7.3) for our multipliers \(\mu_j\). Notice that the \(L^2\)-bounds (7.2) were proved in Proposition 5.1. For the \(L^1\to L^\infty\) bounds, it suffices to prove that for \(j=1, 2, 3, 4,\)
\[\left| \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} e^{-i(t-s) |\xi|^2} e^{-\varepsilon |\xi|^2} \mu_j(y_1, \xi_1, t, s) \, d\xi \right| \leq C |t-s|^{-n/2} \tag{7.4}\]
uniformly in \(t, s, x\) and \(y\). For this we use Lemma 4.1 for \(j=1, 2, 3\) (the Hörmander–Mikhlin bounds for \(\mu_j\) were verified in Proposition 6.2), and the formula (4.42) for \(j=4\).

This completes the proof if \((p, q)\in A\) and \(p<2\). It remains to prove the endpoint estimate \((p, q)=(2, 2n/(n+2))\) in dimensions \(n\geq 3\). For this we use Lemma 7.2; we have to verify the estimate (7.1) for our operators \(T_{j, 1}, j=1, 2, 3, 4\). Notice that we can assume \(t\leq 0\); in addition we can assume that \(f\) is supported in a time interval of length \(2^{l+1}\), say \(\mathbb{R}^n \times [s_0-2^l, s_0+2^l]\). Then \(T_{j, 1}(f)\) is supported in \(\mathbb{R}^n \times [s_0-3\cdot 2^l, s_0+3\cdot 2^l]\). For the bound in the case \(a=b=1\) we have
\[\|T_{j, 1}(f)\|_{L^2_t L^2_x} \leq C 2^{l/2} \|T_{j, 1}(f)\|_{L^2_t L^{\infty}_x} \leq C 2^{l/2} \sup_{|t-s|=2^l} |K_{j}(x, y, t, s)| \|f\|_{L^1_t L^1_x} \]
\[\leq C 2^{l/2} 2^{-ln/2}\|f\|_{L^2_t L^2_x} \leq C 2^{-ln/2} \|f\|_{L^2_t L^2_x},\]
as desired (we used the bound (7.4)). In the case \(a\in [2n/(n+2), 2]\) and \(b=2\), let \(p(a)\in [1, 2]\) be the exponent with the property that \((p(a), a)\in A\). We use Propositions 6.2 and 6.3 to get
\[\|T_{j, 1}(f)\|_{L^2_t L^2_x} \leq C 2^{l/2} \sup_{t_0} \|T_{j, 1}(f)(t_0, \cdot)\|_{L^2_x} \leq C 2^{l/2} \|f\|_{L^{p(a)}_t L^{q(a)}_x} \]
\[\leq C 2^{l/2} 2^{(1/p(a)-1/2)} \|f\|_{L^2_t L^2_x} = C 2^{-ln/2} \|f\|_{L^2_t L^2_x},\]
as desired. The estimate in the case \(a=2\) and \(b\in [2n/(n+2), 2]\) is similar, by using Propositions 6.5 and 6.6 instead of Propositions 6.2 and 6.3. This completes the proof of the proposition. \(\square\)
8. Boundedness of the operators $R_j$

In this section we prove that the operators $R_j$, $j=1, 2, 3$, are bounded from $L_+^\infty L^2_\sigma$ to $X'$ with small norm. Recall that the operators $R_j$ are of the form

$$R_j(g)(x, t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(y, s) \int_{\mathbb{R}^n} e^{i(x-y) \xi} e^{-i(t-s)\xi^2} e^{-\xi^2 t} \xi^2 s_j(y, \xi, t, s) d\xi ds dy. \quad (8.1)$$

The multipliers $s_j$ are defined in (3.17), (3.19) and (3.23). The following proposition gives the main estimate in this section:

**Proposition 8.1.** If $(p,q) \in A$ is as in §5 then

$$\|R_j g\|_{L^p_+ L^q_\sigma} \leq C_5 \frac{\gamma^5}{\lambda} \|g\|_{L^\infty_+ L^2_\sigma}$$

for $j=1, 2, 3$.

**Proof.** Notice that it suffices to prove the stronger bound

$$\|R_j g\|_{L^p_+ L^q_\sigma} \leq C_5 \frac{\gamma^5}{\lambda} \|g\|_{L^1_+ L^2_\sigma}.$$

For $j=1, 2, 3$ let

$$m_j(x_1, y_1, t, s) = \int_{\mathbb{R}^n} e^{i(x_1-y_1)\xi} e^{-i(t-s)\xi^2} e^{-\xi^2 t} \xi^2 s_j(y_1, \xi_1, t, s) d\xi_1.$$

As in Propositions 5.1 and 6.6 it suffices to prove that for $j=1, 2, 3$,

$$\|s_j(\cdot, \xi, t, s)\|_{\text{BV}_x} \leq C_5 \frac{\gamma^5}{\lambda} \quad (8.2)$$

for the $L^1_+ L^2_\sigma \rightarrow L^\infty_+ L^2_\sigma$ bound, and

$$\left| \int_{\mathbb{R}} e^{-(\tau')^2 \xi^2} m_j(x_1, z_1, t, s_0) e^{-(\tau')^2 \xi^2} m_j(y_1, z_1, s, s_0) dz_1 \right| \leq C \left( \frac{\gamma^5}{\lambda} \right)^2 \left[ |t-s_0|^2 + |s-s_0|^2 \right]^{-1/2} \log \frac{|t-s_0|^2 + |s-s_0|^2}{|t-s_0|^2 + |s-s_0|^2} \quad (8.3)$$

for any $t, s, s_0 \in [-1, 1]$.

Assume first that $j=1$ or $j=2$. The bound (8.2) follows easily from (3.14) and the formulas (3.17) and (3.19). Also, by (3.14) we have

$$|m_{1,2}(x_1, z_1, t, s_0)| \leq C \frac{\gamma^4}{\lambda},$$
and, by integrating by parts, we have

\[ |m_{1,2}(x_1,z_1,t,s_0)| \leq C|x_1-z_1|^{-\frac{\gamma^5}{\lambda}}. \]

Thus

\[ \left| \int_{\mathbb{R}} e^{-\gamma^5 z_1^2} m_{1,2}(x_1,z_1,t,s_0) e^{-\gamma^5 z_1^2} \overline{m}_{1,2}(y_1,z_1,s,s_0) \, dz_1 \right| \leq C \left( \frac{\gamma^5}{\lambda} \right)^2, \]

which is better than (8.3). The proposition follows for \( j=1,2 \).

Assume now that \( j=3 \). We examine the formula (3.23). Recall that in this formula \( |\xi_1| \geq \gamma \) and \( |\tau| \geq \frac{1}{10} \xi_1^2 \). The symbol in the second line of (3.23) can be written in the form

\[ -a'_{3,\xi}(y_1) \tau^{-1} + Q_3(y_1,\xi_1,\tau), \]

where

\[ \lambda^4 |\xi_1|^l_2 |\tau|^{l_3} \partial_{y_1}^{l_2} \partial_{\xi_1}^{l_3} Q_3(y_1,\xi_1,\tau) \leq C_{l_1,l_2,l_3} \frac{\gamma}{\lambda} \xi_1^2 \frac{\xi_1^2}{\tau^2} \]

for any nonnegative integers \( l_1, l_2 \) and \( l_3 \) (using (3.8)). In addition, \( Q_3(y_1,\cdot,\cdot) \) is supported in the set \( y_1 \in [\lambda,2\lambda] \). It follows easily that \( \|s_3(\cdot,\xi_1,t,s)\|_{BV_{x_1}} \leq C\gamma/\lambda \), which is better than (8.2). Also, by integrating by parts as in (6.22) we have

\[ |\xi_1^l_2 \partial_{\xi_1}^{l_3} s_3(y_1,\xi_1,t,s)| \leq C \frac{\gamma}{\lambda} \frac{1}{(1+|t-s|\xi_1^2)^2}. \]

Thus, as in Proposition 6.6,

\[ |m_3(x_1,y_1,t,s)| \leq \frac{C\gamma/\lambda}{|t-s|^{1/2} + |x_1-y_1|}, \]

which suffices to prove (8.3) in the case \( j=3 \). This completes the proof of the proposition. \( \square \)

We can now establish the precise condition on \( \lambda \) and \( \beta \). The two relevant estimates are (6.17) and Proposition 8.1. Thus we need to assume that

\[ \lambda \geq \Lambda(\beta) = C(1+\beta)^{6} \]

for some large constant \( C \).
9. Applications

In this section we prove Theorems 2.3, 2.4 and 2.5. To simplify the notation, we write \( X \) for \( X([0,1]) \), \( X' \) for \( X'([0,1]) \), and \( Y \) for \( Y([0,1]) \). Recall that \( \bar{C} \) is the constant in Theorem 2.1. For Theorem 2.3 we simply apply Theorem 2.1:

\[
\|e^{\beta\varphi(x)}u(x,t)\|_{X'} \leq \bar{C}\left[\|e^{\beta\varphi(x)}Hu(x,t)\|_{X} + \|e^{\beta\varphi(x)}u(\cdot,0)\|_{L^2(\mathbb{R}^n)}\right] \\
\leq \bar{C}\|V\|_{Y}\|e^{\beta\varphi(x)}u(x,t)\|_{X'} \\
+ \bar{C}\left[\|e^{\beta\varphi(x)}u(\cdot,0)\|_{L^2(\mathbb{R}^n)} + \|e^{\beta\varphi(x)}u(\cdot,1)\|_{L^2(\mathbb{R}^n)}\right].
\]

If \( \|V\|_{Y} \leq 1/2\bar{C} \), the first term of the right-hand side of the inequality above can be absorbed into the left-hand side (this term is finite since \( u \in X' \) by Theorem 2.1 and \( e^{\beta\varphi(x)} \) is bounded). Theorem 2.3 follows by letting \( \lambda=\infty \).

For Theorem 2.4 we use a variant of the Carleman argument. Let \( u=u_1-u_2 \); we have

\[
Hu = \hat{W}u,
\]

where

\[
\hat{W}(x) = \begin{cases} 
V(x) + \frac{F(u_1(x)) - F(u_2(x))}{u_1(x) - u_2(x)} & \text{if } u_1(x) \neq u_2(x), \\
V(x) & \text{if } u_1(x) = u_2(x).
\end{cases}
\]

Since \( \hat{W} \in Y \), we have \( \hat{W}u \in X \), and thus the identity (9.1) holds in \( X \) and \( u \in Z([0,1]) \). By (2.9) and (2.10),

\[
\|X_{\hat{W}u_{w_0} + D(w_0)}(x)\|_{Y} \leq \bar{C}.
\]

By rotation we may assume without loss of generality that \( w_0=(1,0,\ldots,0) \). Let \( E = \|u(\cdot,0)\|_{L^2} + \|u(\cdot,1)\|_{L^2} \). By Theorem 2.1 with \( \lambda = \max\{b+1, A(\beta)\} \), the identity (9.1) and the support property of the functions \( u(\cdot,0) \) and \( u(\cdot,1) \), we have

\[
\|e^{\beta\varphi(x)}\chi_{\{x:x_1>b\}}u\|_{X'} \leq \|e^{\beta\varphi(x)}u\|_{X'} \\
\leq \bar{C}\|e^{\beta\varphi(x)}Hu\|_{X} + \bar{C}e^{\beta b}E \\
\leq \bar{C}\|e^{\beta\varphi(x)}(\chi_{\{x:x_1>b\}}\hat{W})(\chi_{\{x:x_1>b\}}u)\|_{X} \\
+ \bar{C}\|e^{\beta\varphi(x)}\chi_{\{x:x_1\leq b\}}Hu\|_{X} + \bar{C}e^{\beta b}E \\
\leq \frac{1}{2}\|e^{\beta\varphi(x)}\chi_{\{x:x_1>b\}}u\|_{X'} + \bar{C}e^{\beta b}\|Hu\|_{X} + \bar{C}e^{\beta b}E.
\]

We can absorb the first term of the right-hand side (which is clearly finite) into the left-hand side. The theorem follows by letting \( \beta, \lambda \to \infty \).

To prove Theorem 2.5 we define the functions \( u \) and \( \hat{W} \) as before. We have to show that if \( u \in Z([0,1]) \) vanishes in a half-space and \( Hu=\hat{W}u \), then \( u \equiv 0 \). This is similar to
the type of unique continuation theorems proved by Kenig, Ruiz and Sogge [12] for wave equations. As in [12, p. 331], we should remark that the classical examples of smooth solutions of the equation $Hu=0$ which vanish in a half-space (as in [3]) are not decaying, and are thus not in $C([0,1]:L^2(\mathbb{R}^n))$. Our argument is somewhat similar to the proofs of Theorem 1.3 and Corollary 6.1 in the work of Isakov [6]. The difference is that we use $L^p$ Carleman inequalities to cover rough potentials (in the space $Y$), as opposed to only bounded potentials as in [6].

The identity (9.1) holds and, by Theorem 2.4, $u \equiv 0$ in $[bw_0+D(w_0) \times [0,1]$. We will prove now that $u \equiv 0$ in $\mathbb{R}^n \times [0,1]$. By (2.11), there is $\varepsilon_0 \in (0,1]$ with the property that

$$
(9.3) \quad \|W(x \in [b',b+\varepsilon_0])\|_Y \leq \frac{1}{4C_1} \quad \text{for any } b' \in (-\infty, b-\varepsilon_0],
$$

where $C_1$ is the constant $C$ in Lemma 9.1 below. This is the only assumption we need on $W$ to carry out the proof. By rotation we may assume that $w_0=(1,0,...,0)$. Let $b_j=b-j\varepsilon_0$. By induction it suffices to prove that $u \equiv 0$ in $\{x : x_1 > b_j+1\} \times [0,1]$ assuming that $u \equiv 0$ in $\{x : x_1 > b_j\} \times [0,1]$ and $j \geq 0$. The Carleman inequality in Theorem 2.1 (or Corollary 2.2) does not apply directly, mainly because we do not have good control over the boundary terms $u(-,0)$ and $u(-,1)$. To avoid these boundary terms we make a change of variables (as in [6]). For any $\delta \in (0, \frac{1}{10}]$ fix a smooth function $w=\omega_{\delta,j} : [0,1] \to [b_j+1, b_j]$ with the property that $\omega(t) = b_{j+1}$ if $t \in [2\delta, 1-2\delta]$, $\omega(t) = b_j$ if $t \in [0,\delta] \cup [1-\delta, 1]$, and $|\omega(t)| + \delta^2 |\omega''(t)| \leq C$. We will show that

$$
(9.4) \quad u \equiv 0 \quad \text{in } \{(x,t) : x_1 > \omega(t), t \in [0,1]\}.
$$

Since $\delta$ is arbitrary and $u \in C([0,1]:L^2(\mathbb{R}^n))$ this would suffice to complete the proof of the induction step. Let

$$
v(y_1, y', s) = e^{-i\omega(s)y_1/2} u(y_1 + \omega(s), y', s).
$$

An elementary calculation using (9.1) shows that

$$
Hv(y_1, y', s) = v(y_1, y', s) \left[ \overline{W}(y_1 + \omega(s), y', s) + \frac{1}{2} \omega''(s)y_1 - \frac{1}{2} \omega'(s)^2 \right].
$$

The role of the exponential $e^{-i\omega(s)y_1/2}$ is to cancel the $\partial_{y_1}u$-term in the commutator. By the support properties of the function $u$, we know that $v \equiv 0$ in the sets

$$
\{(y_1, y', s) : y_1 > \varepsilon_0\} \quad \text{and} \quad \{(y_1, y', s) : y_1 > 0 \text{ and } s \in [0, \delta] \cup [1-\delta, 1]\}.
$$

It remains to prove that $v \equiv 0$ in the set $\{(y_1, y', s) : y_1 > 0\}$. The equation (9.5) may be written in the form

$$
Hv = (W_0 + M) \cdot v,
$$

(9.6)
where
\[ W_0(y_1, y', s) = \tilde{W}(y_1 + \omega(s), y', s) \]
and
\[ M(y_1, y', s) = \frac{1}{2} \omega''(s)y_1 - \frac{1}{4} \omega'(s)^2. \]

Theorem 2.1 cannot be applied in this case because the potential \( M \) does not belong to the space \( Y \). We use instead the following Carleman inequality:

**Lemma 9.1.** Assume that \( u \in C([0, 1]; L^2(\mathbb{R}^n)) \), \( u \equiv 0 \) in the set \( \{(x_1, x', t): x_1 > 1\} \), and \( (1 + |x_1|)^{-N}Hu \in X(\mathbb{R}^n \times (0, 1)) \) for some \( N \geq 0 \). Then, for some constant \( c_0 > 0 \),
\[
\|e^{\beta s(x_1)}x_{[0,1]}(x_1)u(x, t)\|_{X([0,1])} + \beta^c_0 \|e^{\beta s(x_1)}x_{[0,1]}(x_1)u(x, t)\|_{L^1L^2(\mathbb{R}^n \times [0,1])} \leq C\|e^{\beta s(x_1)}Hu(x, t)\|_{X([0,1])} + \|e^{\beta s(x_1)}u(\cdot, 0)\|_{L^2} + \|e^{\beta s(x_1)}u(\cdot, 1)\|_{L^2}
\]
for any \( \beta \in [2, \infty) \). The function \( \phi_\beta: \mathbb{R} \rightarrow \mathbb{R} \) is given by \( \phi_\beta(x_1) = \beta x_1 \) if \( x_1 \in [-1, 0) \), and \( \phi_\beta(x_1) = x_1 - \beta + 1 \) if \( x_1 \in (-\infty, -1] \).

The same argument as in the proof of Theorem 2.4 (see (9.2)), using (9.3) and the identity (9.6), shows that Lemma 9.1 suffices to prove that \( v \equiv 0 \) in the set \( \{(y_1, y', s): y_1 > 0\} \), which gives (9.4).

**Proof.** By the same argument as in §3 we may assume that \( u \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \) is supported in \( \mathbb{R}^n \times [0, 1] \). The bound for the first term in the left-hand side of (9.7) follows from the stronger inequality
\[
\|e^{\beta s(x_1)}u\|_{X([0,1])} \leq C\|e^{\beta s(x_1)}Hu\|_{X([0,1])}
\]
for any \( \beta \geq 2 \) and any \( u \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \), which is a consequence of Theorem 2.1 with \( \lambda = \infty \).

To control the second term in the left-hand side of (9.7) it suffices to prove that for some \( c_0 > 0 \),
\[
\beta^c_0 \|e^{\beta s(x_1)}x_{[0,1]}(x_1)u(x, t)\|_{L^1L^2(\mathbb{R}^n \times [0,1])} \leq C\|e^{\beta s(x_1)}Hu(x, t)\|_{X([0,1])}
\]
for any \( \beta \geq 2 \) and any \( u \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \) supported in \( \mathbb{R}^n \times [0, 1] \). Let \( f = Hu \). Let \( P_\epsilon^1 \) denote the operator defined by the Fourier multiplier \( (\xi_1, \xi', \tau) \rightarrow e^{-\epsilon^2|x'|^2}e^{-\epsilon^2(\tau + |\xi'|^2)^2} \). It suffices to prove that
\[
\|x_{[0,1]}(t)e^{\beta s(x_1)}x_{[0,1]}(x_1)P_\epsilon^1(u)\|_{L^1L^2(\mathbb{R}^n \times [0,1])} \leq C\beta^{-c_0}\|e^{\beta s(x_1)}f(x, t)\|_{X([0,1])}
\]
As in §3, let \( \tilde{u}(x_1, \xi', \tau) \) and \( \tilde{f}(x_1, \xi', \tau) \) denote the partial Fourier transforms of the functions \( u \) and \( f \) in the variables \( x' \) and \( t \). The equation \( (i\partial_t + \Delta_x)u = f \) becomes
\[
|\partial_{x_1}^2 - (\tau + |\xi'|^2)|\tilde{u}(x_1, \xi', \tau) = \tilde{f}(x_1, \xi', \tau).
\]
By integration by parts we have for any $x_1 \in [-1, 0]$,

$$\hat{u}(x_1, \xi', \tau) = \int_{\mathbb{R}} \hat{f}(y_1, \xi', \tau) G(x_1, y_1, \tau + |\xi'|^2) \, dy_1,$$

where

$$G(x_1, y_1, \mu) = \begin{cases} 
-\chi_+(y_1 - x_1) \frac{\sin [(x_1 - y_1) \sqrt{-\mu}]}{\sqrt{-\mu}} & \text{if } \mu \leq 0, \\
-\chi_+(y_1 - x_1) \frac{\sinh [(x_1 - y_1) \sqrt{\mu}]}{\sqrt{\mu}} & \text{if } 0 \leq \mu \leq \beta^2, \\
-\frac{e^{-|x_1 - y_1| \sqrt{\mu}}}{2\sqrt{\mu}} & \text{if } \beta^2 < \mu.
\end{cases} \quad (9.10)$$

By taking the inverse Fourier transform,

$$\mathcal{P}_\xi^t u(x_1, x', t) = C \int_{\mathbb{R}} \int_{\mathbb{R}} f(y, s) \int_{\mathbb{R}^{n-1}} e^{i(x' - y') \cdot \xi'} e^{i(t-s)\tau} e^{-\varepsilon^2 |\xi'|^2}$$

$$\times e^{\varepsilon^2 (\tau + |\xi'|^2)^2} G(x_1, y_1, \tau + |\xi'|^2) \, d\tau \, d\xi' \, ds \, dy.$$

Thus, to prove (9.9), it suffices to prove that the operator

$$\tilde{T}(g)(x, t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(y, s) K(x, y, t, s) \, ds \, dy$$

has the property that

$$\|\tilde{T}(g)\|_{L^1 L^2} \leq C\beta^{-\infty} \|g\|_X \quad (9.11)$$

for any bounded compactly supported function $g$, where

$$K(x, y, t, s) = \chi_{[0,1]}(t) \chi_{[0,1]}(s) \chi_{[0,1]}(x_1) \chi_{(-\infty,1]}(y_1) e^{\phi_\beta(x_1) - \phi_\beta(y_1)}$$

$$\times \int_{\mathbb{R}^{n-1}} e^{i(x' - y') \cdot \xi'} e^{i(t-s)\tau} e^{-\varepsilon^2 |\xi'|^2} e^{\varepsilon^2 (\tau + |\xi'|^2)^2} G(x_1, y_1, \tau + |\xi'|^2) \, d\tau \, d\xi'.$$

Let

$$\tilde{k}(x_1, y_1, t, s) = \chi_{[0,1]}(x_1) \chi_{(-\infty,1]}(y_1) e^{\phi_\beta(x_1) - \phi_\beta(y_1)} \int_{\mathbb{R}} e^{i(t-s)\mu} e^{-\gamma^2 \mu^2} G(x_1, y_1, \mu) \, d\mu.$$

For $l \leq 0$, let

$$\tilde{K}_l(x, y, t, s) = \chi_{[2^l, 2^{l+1}]}(|t-s|) \tilde{K}(x, y, t, s),$$

$$\tilde{h}_l(x_1, y_1, t, s) = \chi_{[2^l, 2^{l+1}]}(|t-s|) \tilde{h}(x_1, y_1, t, s).$$
and $\tilde{T}_t$ be the operator defined by the kernel $\tilde{k}_t$. By (9.10),

$$|\tilde{k}_t(x_1,y_1,t,s)| \leq C 2^{-1/2}[e^{-\beta|x_1-y_1|/10} + (1 + \beta 2^{1/2})^{-1} \chi_{[0,2]}(|x_1-y_1|)].$$

(9.12)

To see this, we substitute the formula (9.10) and break up the integral into three parts. To control the integral over $\mu \leq 0$ we make the change of variable $\mu = -\eta^2$ and use Lemma 4.2. To control the integral over $\mu \in [0, \beta^2]$ we make the change of variable $\mu = \eta^2$, use Lemma 4.2 for the integral over $\eta \in [0, \frac{1}{2}\beta]$, use again Lemma 4.2 for the integral over $\eta \in \left[\frac{1}{2}\beta, \beta\right]$ if $\beta^{2/2} \leq 1$, and integrate by parts if $\beta^{2/2} > 1$. To control the integral over $\mu > \beta^2$ we make the change of variable $\mu = \eta^2$, use Lemma 4.2 if $\beta^{2/2} \leq 1$, and integrate by parts if $\beta^{2/2} > 1$. The estimate (9.12) is the only estimate we need for the kernel $\tilde{k}_t$.

As in §6 we fix an acceptable pair $(p,q)$, with $p \leq \frac{4}{3}$ if $n=1$, $p \leq p_0$ if $n=2$, and $p \leq 2$ if $n \geq 3$. For (9.11) it suffices to prove that

$$\sum_{t \leq 0} \|\tilde{T}_t\|_{L^p\to L^2} \leq C \beta^{-c_0}$$

for some $c_0 > 0$. Since

$$\|\tilde{T}_t\|_{L^p\to L^2} \leq C \|\tilde{T}_t\|_{L^p\to L^2} \leq C 2^{1/p} \|\tilde{T}_t\|_{L^p\to L^2},$$

it suffices to prove that

$$\sum_{t \leq 0} 2^{1/p} \|\tilde{T}_t\|_{L^p\to L^2} \leq C \beta^{-c_0}. \quad (9.13)$$

To estimate $\|\tilde{T}_t\|_{L^p\to L^2}$ we argue as in Proposition 6.3. For any $t_0 \in [0,1]$ let $\tilde{T}_{t_0}$ denote the operator defined by the kernel $e^{-(c')^2 |z|^2} K_t(z,y,t_0,s)$. As in Proposition 6.3,

$$\|\tilde{T}_t\|_{L^p\to L^2} \leq \sup_{t_0, t_0'} \|\tilde{T}_{t_0, t_0'}\|_{L^p\to L^2} \leq \sup_{t_0, t_0'} \|\tilde{T}_{t_0, t_0'}\|_{L^p\to L^2}^{1/2} \|\tilde{T}_{t_0, t_0'}\|_{L^p\to L^2}^{1/2} \quad (9.14)$$

The kernel of the operator $\tilde{T}_{t_0}$ is

$$\tilde{L}_{t_0}(x,y,t,s) = \int_{R^n} e^{-(c')^2 |z|^2} K_t(z,y,t_0,s) e^{-(c')^2 |z|^2} \tilde{K}_t(z,x,t_0,t) \, dz.$$

Let $\tilde{U}_{t_0, t_0, t}$ denote the operator defined by the kernel $\tilde{L}_{t_0, t_0, t}$. By (9.12), the $L^1$-norm in both the variables $x_1$ and $y_1$ of the kernel $\tilde{k}_t(\cdot, \cdot, t, s)$ is bounded by $C 2^{-1}(1 + \beta^{2/2})^{-1}$. Thus

$$\|\tilde{U}_{t_0, t_0, t}\|_{L^2(R^n) \to L^2(R^n)} \leq C 2^{-1}(1 + \beta^{2/2})^{-1}.$$
The kernel $\tilde{L}_{i,t_0} (x, y, t, s)$ splits as a product of $n$ integrals. As in Propositions 6.2 and 6.3, the integrals over the variables $z_2, ..., z_n$ are each bounded by $|t-s|^{-1/2}$. For the integral over the variable $z_1$ we use (9.12). The result is

$$\|\tilde{U}_{i,t_0, t,s}\|_{L^p(R^n)} \leq C |t-s|^{-(n-1)/2} \chi_{[0,2^{i+2}](|t-s|)} 2^{-i} (1+\beta^2 2^i)^{-1} (1+\beta 2^i).$$

By interpolation and the Minkowski inequality for integrals, as in Proposition 6.3,

$$\|\tilde{T}_{i,t_0} \tilde{T}_{i,t_0}\|_{L^p L^q \rightarrow L^p L^q} \leq C 2^{-i} (1+\beta^2 2^i)^{-1} (1+\beta 2^i)^{2/q-1/2} (2/q-1).$$

By (9.14) and the fact that $1/p + 1/2q - 3/4 \geq 1/2n$ if $(p,q) \in A$,

$$\sum_{|i|\geq 0} 2^{i/p} \|\tilde{T}_i\|_{L^p L^q \rightarrow L^p L^q} \leq \sum_{|i|\geq 0} C 2^{i(1/p+1/2q-3/4)} \frac{1+\beta^2 2^i}{1+\beta 2^i/2} \leq C \beta^{1/2n}.$$

The main estimate (9.13) follows with $c_0 = 1/2n$. 

References


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