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Correction to "Witt vectors of non-commutative rings and topological cyclic homology"

by

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This note is to announce and correct two related mistakes. The first mistake is at the bottom of p. 114, where it is claimed that the elements ε_i defined recursively using (1.3.5) are commutators. Indeed, this is not always the case and, in general, a surjective ring homomorphism $A'\to A$ does not induce a surjective map $N(A')\to N(A)$. The second mistake is the claim in the proof of Lemma 2.2.1 that the abelian group $B/[B,B]$ is torsion free. This is not true and, in fact, the statement of Lemma 2.2.1 is false. The main consequence of these two mistakes is that, for a general associative ring A, the Verschiebung map $V\colon W_{n-1}(A)\to W_n(A)$ is not injective as it is falsely stated in Propositions 1.6.3 and 2.2.3. However, Theorems A, B and C of the introduction and Theorems 1.7.10 and 2.2.9 all remain true as stated. We briefly clarify the construction of the abelian group $W_n(A)$ and the proof of the central Theorem 2.2.9.

We define a homomorphism between two sets with unital composition laws to be a map that preserves the composition law and the unit. In general, we do not require composition laws to be associative or commutative or to have an inverse. Let A be a unital associative ring. The integral non-commutative polynomials

$$s_i(X_0,X_1,...,X_i,Y_0,Y_1,...,Y_i)$$

from (1.4.1) define a unital composition law on the set A^n by the rule

$$(a_0,a_1,...,a_{n-1})*(b_0,b_1,...,b_{n-1})=(s_0,s_1,...,s_{n-1}),$$

where $s_i=s_i(a_0,...,a_i,b_0,...,b_i)$. The unit element is the n-tuple $(0,0,...,0)$. We recall the ghost map $w\colon A^n\to A^n$ that to the n-tuple $(a_0,a_1,...,a_{n-1})$ associates the n-tuple

$(w_0, w_1, \dots, w_{n-1})$, where

$$w_s = a_0^{p^s} + p a_1^{p^{s-1}} + \dots + p^s a_s,$$

and let $q\colon A \to A/[A,A]$ be the canonical projection. Then the composite map

$$f_n = q^n \circ w\colon A^n \longrightarrow (A/[A,A])^n$$

is a homomorphism with the composition law $*$ on the left-hand side and componentwise addition on the right-hand side. We recursively define the abelian group $W_n(A)$ and a natural factorization of f_n as the composition of two homomorphisms

$$A^n \xrightarrow{q_n} W_n(A) \xrightarrow{\bar{w}} (A/[A,A])^n,$$

where q_n is surjective, and where $\bar{w}$ is injective if $A/[A,A]$ is p-torsion free. It follows from the proof of Theorem 2.2.9 that the map (2.2.6) is a homomorphism

$$\tilde{I}\colon A^n \longrightarrow \mathrm{TR}_0^n(A;p) = \pi_0 T(A)^{C_{p^{n-1}}}.$$

We show, inductively, that $\tilde{I}$ admits a natural factorization

$$A^n \xrightarrow{q_n} W_n(A) \xrightarrow{I} \mathrm{TR}_0^n(A;p)$$

and that the homomorphism I is an isomorphism.

We define $W_1(A)$ to be $A/[A,A]$, q_1 to be q, and $\bar{w}$ to be the identity map. The factorization $\tilde{I} = I \circ q_1$, with I an isomorphism, follows immediately from the definition of $T(A)$. So assume that the abelian group $W_{n-1}(A)$ with the properties above has been defined. We first define an abelian group structure on the set

$$\widetilde{W}_n(A) = A \times W_{n-1}(A)$$

together with a factorization of the homomorphism

$$\tilde{f}_n = (\mathrm{id} \times q^{n-1}) \circ w\colon A^n \longrightarrow A \times (A/[A,A])^{n-1}$$

as the composition of two homomorphisms

$$A^n \xrightarrow{\tilde{q}_n} \widetilde{W}_n(A) \xrightarrow{\tilde{w}} A \times (A/[A,A]).$$

We define $\tilde{q}_n$ to be the product map $\mathrm{id} \times q_{n-1}$. The ghost map $w\colon A^n \to A^n$ admits a factorization as the composite map

$$A \times A^{n-1} \xrightarrow{\mathrm{id} \times pw} A \times A^{n-1} \xrightarrow{\tau} A \times A^{n-1},$$

where the left-hand map is the bijection given by the formula

$$\tau(x_0, x_1, \dots, x_{n-1}) = (x_0, x_0^p + x_1, \dots, x_0^{p^{n-1}} + x_{n-1}),$$

and we then define the map $\widetilde{w}$ to be the composite map

$$A \times W_{n-1}(A) \xrightarrow{\mathrm{id} \times p\widetilde{w}} A \times (A/[A,A])^{n-1} \xrightarrow{\bar{\tau}} A \times (A/[A,A])^{n-1},$$

where $\bar{\tau}$ is the bijection induced by τ. Since $\tilde{I}$ is equal to the composite map

$$A^n \xrightarrow{\tilde{q}_n} \widetilde{W}_n(A) \xrightarrow{I'} \mathrm{TR}_0^n(A;p),$$

where $I'(a_0, a) = \Delta_{p^{n-1}}(a_0) + V(I(a))$, and since $\tilde{I}$ is a homomorphism, there exists a unique unital composition law on $\widetilde{W}_n(A)$ such that $\tilde{q}_n$ is a homomorphism. Moreover, since $\tilde{f}_n$ is a homomorphism, so is $\widetilde{w}$. If $A/[A,A]$ is p-torsion free, then $\widetilde{w}$ is injective by induction, and Lemma 1.3.2 implies that the image of $\widetilde{w}$ is a subgroup of the abelian group $A \times (A/[A,A])^{n-1}$. So the composition law on $\widetilde{W}_n(A)$ is an abelian group structure in this case. For a general ring A, we choose a surjective ring homomorphism $\phi: A' \to A$ from a ring A' such that $A'/[A',A']$ is p-torsion free. In the diagram

$$\begin{array}{ccc} (A')^n & \xrightarrow{\tilde{q}_n} & \widetilde{W}_n(A') \\ \downarrow{\scriptstyle \phi^n} & & \downarrow{\scriptstyle \widetilde{W}_n(\phi)} \\ A^n & \xrightarrow{\tilde{q}_n} & \widetilde{W}_n(A) \end{array}$$

the two horizontal maps and the left-hand vertical map are surjective homomorphisms, and hence, so is the right-hand vertical map. Since the composition law on $\widetilde{W}_n(A')$ is an abelian group structure, so is the composition law on $\widetilde{W}_n(A)$.

We define the abelian group $W_n(A)$ to be the cokernel of the homomorphism

$$\tilde{d}: \mathbf{Z}\langle A \times A \rangle \longrightarrow \widetilde{W}_n(A)$$

from the free abelian group generated by the set $A \times A$ that to the generator (a,b) associates the element $(ab, 0) - (ba, 0)$. Since the composition

$$\mathbf{Z}\langle A \times A \rangle \xrightarrow{\tilde{d}} \widetilde{W}_n(A) \xrightarrow{\widetilde{w}} A \times (A/[A,A])^n \xrightarrow{q \times \mathrm{id}} (A/[A,A])^n$$

is equal to zero, we obtain the factorization $f_n = \overline{w} \circ q_n$ as the composition of two homomorphisms. Similarly, the composition $I' \circ \tilde{d}$ is zero, since the Teichmüller map (2.2.4) satisfies $\Delta_{p^{n-1}}(ab) = \Delta_{p^{n-1}}(ba)$. This gives the factorization $\tilde{I} = I \circ q_n$ as the composition of two homomorphisms. To show that I is an isomorphism, we first show that there is a natural short exact sequence

$$0 \longrightarrow W_{n-1}(A) \xrightarrow{\widetilde{V}} \widetilde{W}_n(A) \xrightarrow{\mathrm{pr}_1} A \longrightarrow 0,$$

where $\widetilde{V}(a) = (0, a)$. We only need to show that $\widetilde{V}$ is a homomorphism and may assume that $A/[A, A]$ is p-torsion free. There is a commutative diagram

$$\begin{array}{ccc} W_{n-1}(A) & \xrightarrow{\widetilde{V}} & \widetilde{W}_n(A) \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \tilde{w}} \\ (A/[A,A])^{n-1} & \xrightarrow{\widetilde{V}^w} & A \times (A/[A,A])^{n-1} \end{array}$$

with the lower horizontal map given by

$$\widetilde{V}^w(x_0, x_1, \dots, x_{n-2}) = (0, px_0, px_1, \dots, px_{n-2}).$$

The maps $\widetilde{V}^w$, w and $\tilde{w}$ are homomorphisms, and since $A/[A, A]$ is p-torsion free, the maps w and $\tilde{w}$ are injective. It follows that $\widetilde{V}$ is a homomorphism. Moreover, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_{n-1}(A) & \xrightarrow{\widetilde{V}} & \widetilde{W}_n(A) & \xrightarrow{\mathrm{pr}_1} & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle q_1} & & \\ & & W_{n-1}(A) & \xrightarrow{V} & W_n(A) & \xrightarrow{R^{n-1}} & W_1(A) & \longrightarrow & 0 \end{array}$$

since the composition $\mathrm{pr}_1 \circ \tilde{d}$ is a surjective homomorphism onto the kernel of q_1. We now proceed to show that I is an isomorphism. In the diagram

$$\begin{array}{ccccccc} W_{n-1}(A) & \xrightarrow{V} & W_n(A) & \xrightarrow{R^{n-1}} & W_1(A) & \longrightarrow & 0 \\ \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} & & \\ \mathrm{TR}_0^{n-1}(A;p) & \xrightarrow{V} & \mathrm{TR}_0^n(A;p) & \xrightarrow{R^{n-1}} & \mathrm{TR}_0^1(A;p) & \longrightarrow & 0 \end{array}$$

the rows are exact and, inductively, the right and left-hand vertical maps I are isomorphisms. Suppose first that the abelian group $A/[A, A]$ is p-torsion free. Then the lower left-hand horizontal map V is injective. Indeed, the composite map

$$\mathrm{TR}_0^{n-1}(A;p) \xrightarrow{V} \mathrm{TR}_0^n(A;p) \xrightarrow{F} \mathrm{TR}_0^{n-1}(A;p)$$

is given by multiplication by p, and one proves by induction on n that $\mathrm{TR}_0^{n-1}(A;p)$ is p-torsion free. It follows that the upper left-hand horizontal map V is injective and that the middle vertical map I is an isomorphism as desired. To prove the general case, it suffices to show that if

$$P[1] \underset{d_1}{\overset{d_0}{\rightrightarrows}} P[0] \xrightarrow{\varepsilon} A$$

is a coequalizer diagram of rings, then the rows in the diagram

$$\begin{array}{ccccccc} W_n(P[1]) & \xrightarrow{d_0-d_1} & W_n(P[0]) & \xrightarrow{\varepsilon} & W_n(A) & \longrightarrow & 0 \\ \downarrow I & & \downarrow I & & \downarrow I & & \\ \mathrm{TR}_0^n(P[1];p) & \xrightarrow{d_0-d_1} & \mathrm{TR}_0^n(P[0];p) & \xrightarrow{\varepsilon} & \mathrm{TR}_0^n(A;p) & \longrightarrow & 0 \end{array}$$

are exact. Indeed, we can choose $P[q]$ such that $P[q]/[P[q],P[q]]$ is p-torsion free. The exactness of the lower row is well known, and the exactness of the upper row follows by an induction argument based on the exact sequences

$$0 \longrightarrow W_{n-1}(A) \xrightarrow{\tilde{V}} \widetilde{W}_n(A) \xrightarrow{\mathrm{pr}_1} A \longrightarrow 0$$

and

$$\mathbf{Z}\langle A\times A\rangle \xrightarrow{\tilde{d}} \widetilde{W}_n(A) \longrightarrow W_n(A) \longrightarrow 0.$$

This completes the recursive definition of $W_n(A)$.

We remark that $\tilde{d}$ factors through the projection $u\colon \mathbf{Z}\langle A\times A\rangle \to A\otimes A$. Indeed, this is true if $A/[A,A]$ is p-torsion free, since the kernels of the homomorphisms $\tilde{d}$ and $\mathrm{pr}_1\circ\tilde{d}$ are equal and contain the kernel of u. The general case follows since a surjective ring homomorphism induces a surjection of the kernels of the homomorphisms u. By a similar argument we conclude that there exists a natural exact sequence

$$\mathrm{HH}_1(A) \xrightarrow{\partial} W_{n-1}(A) \xrightarrow{V} W_n(A).$$

The value of the homomorphism ∂ on a Hochschild 1-cycle $\zeta = \sum_{1\leqslant i\leqslant m} a_i\otimes b_i$ is given as follows. Let P be the free associative ring on generators $x_1, y_1, \dots, x_m, y_m$. There exists a unique class $\theta(x_1, y_1, \dots, x_m, y_m)$ in $W_{n-1}(P)$ such that

$$V(\theta(x_1, y_1, \dots, x_m, y_m)) = [x_1y_1 + \dots + x_my_m]_n - [y_1x_1 + \dots + y_mx_m]_n,$$

where $[a]_n = q_n(a, 0, \dots, 0)$. Then $\partial(\zeta) = \theta(a_1, b_1, \dots, a_m, b_m)$. Using this description one can show that, in general, the map ∂ is non-zero. For example, define A to be the quotient of the free associative ring generated by x, y, z and w by the two-sided ideal generated by $xy-yx+zw-wz$, and let $n=2$ and $p=2$. Then $\partial(x\otimes y+z\otimes w)$ is equal to the class of $xyzw-wzyx$ in $W_1(A)$, which is a non-zero 2-torsion class. The map ∂ is zero, however, for any pointed monoid algebra over a commutative ring.

Finally, I would like to acknowledge Kåre Nielsen for discovering the two mistakes and for providing me with the example above. He found the mistakes during his thesis work at Aarhus University. I would also like to apologize for the mistakes and for the delay in the publishing of this correction.

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