Acta Math., 197 (2006), 145–166
DOI: 10.1007/s11511-006-0010-8

Proof of a conjecture of Pólya on the zeros of successive derivatives of real entire functions

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1. Introduction

The theory of entire functions began as a field of research in the work of Laguerre [21], soon after the Weierstrass product representation became available. Laguerre introduced the first important classification of entire functions, according to their genera. We recall this notion. Let f be an entire function and $\{z_k\}_{k=1}^{\omega}$, $\omega\leqslant\infty$, the sequence of its zeros in $\mathbf{C}\backslash\{0\}$, repeated according to their multiplicities. If s is the smallest non-negative integer such that the series

$$\sum_{k=1}^{\omega}|z_k|^{-s-1}$$

converges, then f has the Weierstrass representation

$$f(z)=z^m e^{P(z)}\prod_{k=1}^{\omega}\left(1-\frac{z}{z_k}\right)\exp\left(\frac{z}{z_k}+...+\frac{1}{s}\left(\frac{z}{z_k}\right)^s\right).$$

If P in this representation is a polynomial, then the genus is defined as $g:=\max\{s,\deg P\}$. If P is a transcendental entire function, or if the integer s does not exist, then f is said to have infinite genus. A finer classification of entire functions by their order

$$\varrho=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r},\quad \text{where } M(r,f)=\max_{|z|\leqslant r}|f(z)|,$$

was later introduced by Borel, based on the work of Poincaré and Hadamard. For functions of non-integral order we have $g=[\varrho]$, and if ϱ is a positive integer, then $g=\varrho-1$ or $g=\varrho$. So the order and the genus are simultaneously finite or infinite.

The first author is supported by the Alexander von Humboldt Foundation, by the NSF, and by the German-Israeli Foundation for Scientific Research and Development, grant G-809-234.6/2003. The second author is supported by the NSF, grants DMS-0555279 and DMS-0244547.

The state of the theory of entire functions in 1900 is described in the survey of Borel [5]. One of the first main problems of the theory was finding relations between the zeros of a real entire function and the zeros of its derivatives. (An entire function is called real if it maps the real axis into itself.) If f is a real polynomial with all zeros real, then all derivatives $f^{(n)}$ have the same property. The proof based on Rolle's theorem uses the finiteness of the set of zeros of f in an essential way. Laguerre discovered that this result still holds for entire functions of genus 0 or 1, but in general fails for functions of genus 2 or higher [21]. Pólya [26] refined this result as follows. Consider the class of all entire functions which can be approximated uniformly on compact subsets of the plane by real polynomials with all zeros real. Pólya proved that this class coincides with the set of functions of the form

$$f(z)=e^{az^2+bz+c}w(z),$$

where $a\leqslant 0$, b and c are real, and w is a canonical product of genus at most 1, with all zeros real. This class of functions is called the *Laguerre–Pólya class* and denoted by LP. Evidently, the class LP is closed with respect to differentiation, so all derivatives of a function in LP have only real zeros. For a function of class LP, the order and the genus do not exceed 2.

The class LP has several other interesting characterizations, and it plays an important role in many parts of analysis [15], [18].

The results of Laguerre and Pólya on zeros of successive derivatives inspired much research in the 20th century. In his survey article [29], Pólya writes: "*The real axis seems to exert an influence on the non-real zeros of* $f^{(n)}$; *it seems to attract these zeros when the order is less than* 2, *and it seems to repel them when the order is greater than* 2." In the original text Pólya wrote "complex" instead of "non-real". We replaced this in accordance with the modern usage to avoid confusion. Pólya then put this in a precise form by making the following two conjectures.

CONJECTURE A. *If the order of the real entire function f is less than* 2, *and f has only a finite number of non-real zeros*, *then its derivatives*, *from a certain one onwards*, *will have no non-real zeros at all.*

CONJECTURE B. *If the order of the real entire function f is greater than* 2, *and f has only a finite number of non-real zeros*, *then the number of non-real zeros of* $f^{(n)}$ *tends to infinity*, *as* $n\to\infty$.

Conjecture A was proved by Craven, Csordas and Smith [7]. Later they proved in [6] that the conclusion holds if $\log M(r,f)=o(r^2)$. The result was refined by Ki and Kim [19], [20], who proved that it holds if $f=Ph$, where P is a real polynomial and $h\in\mathrm{LP}$.

In this paper we establish Conjecture B. Actually, we prove that its conclusion holds if $f=Ph$, with a real polynomial P and a real entire function h with real zeros that does not belong to LP.

THEOREM 1. *Let f be a real entire function of finite order with finitely many non-real zeros. Suppose that f is not a product of a real polynomial and a function in the class* LP. *Then the number $N(f^{(n)})$ of non-real zeros of $f^{(n)}$ satisfies*

$$\liminf_{n\to\infty} \frac{N(f^{(n)})}{n} > 0. \tag{1}$$

For real entire functions of infinite order with finitely many non-real zeros, we have the recent result by Langley [22] that $N(f^{(n)})=\infty$ for $n\geqslant 2$. Theorem 1 and Langley's result together prove Conjecture B. Combining Theorem 1 with the results of Langley and Ki and Kim, we conclude the following:

For every real entire function f, either $f^{(n)}$ has only real zeros for all sufficiently large n, or the number of non-real zeros of $f^{(n)}$ tends to infinity with n.

We give a short survey of the previous results concerning Conjecture B. These results can be divided into two groups: asymptotic results on the zeros of $f^{(n)}$, as $n\to\infty$, and non-asymptotic results for fixed n.

The asymptotic results begin with the papers of Pólya [27], [28]. In the first paper, Pólya proved that if

$$f = Se^T, \tag{2}$$

where S and T are real polynomials, and $f^{(n)}$ has only real zeros for all $n\geqslant 0$, then $f\in\text{LP}$. In the second paper, he introduced the *final set*, that is the set of limit points of zeros of successive derivatives, and he found this limit set for all meromorphic functions with at least two poles, as well as for entire functions of the form (2). It turns out that for a function of the form (2), the final set consists of equally spaced rays in the complex plane. Namely, if $T(z)=az^d+bz^{d-1}+\dots$, then these rays emanate from the point $-b/(da)$ and have directions $\arg z=(\arg a+(2k+1)\pi)/d$, with $k=0,\dots,d-1$. It follows that the number of non-real zeros of $f^{(n)}$ for such a function f tends to infinity, as $n\to\infty$, unless $d=2$ and $a<0$, or $d\leqslant 1$, that is, unless $e^T\in\text{LP}$. Notice that the final set of a function of the form (2) is independent of the polynomial S.

This result of Pólya was generalized by McLeod [25] to the case where the S in (2) is an entire function satisfying $\log M(r,S)=o(r^{d-1})$, as $r\to\infty$. This growth restriction seems natural: a function S of faster growth will influence the final set. This is seen from Pólya's result, where the final set depends on b.

All papers on final sets use some form of the saddle point method to obtain an asymptotic expression for $f^{(n)}$ when n is large. This leads to conclusions about the zeros of $f^{(n)}$. McLeod used a very general and powerful version of the saddle point method which is due to Hayman [11]. We do not survey here many other interesting results on the final sets of entire functions, as these results have no direct bearing on Conjecture B, and we refer the interested reader to [4], [10], [31] and references therein.

Passing to the non-asymptotic results, we need the following definition. For a non-negative integer p we define V_{2p} as the class of entire functions of the form

$$e^{az^{2p+2}}w(z),$$

where $a\leqslant 0$, and w is a real entire function of genus at most $2p+1$ with all zeros real. Then we define $U_0:=V_0$ and $U_{2p}:=V_{2p}\setminus V_{2p-2}$. Thus $\mathrm{LP}=U_0$.

Many of the non-asymptotic results were motivated by an old conjecture, attributed to Wiman (1911) by his student Ålander [1], [2], that every real entire function f such that ff'' has only real zeros belongs to the class LP. For functions of finite order, Wiman made the more precise conjecture that f'' has at least $2p$ non-real zeros if $f\in U_{2p}$.

Levin and Ostrovskii [24] proved that if f is a real entire function which satisfies

$$\limsup_{r\to\infty}\frac{\log\log M(r,f)}{r\log r}=\infty, \tag{3}$$

then ff'' has infinitely many non-real zeros. Hellerstein and Yang [14] proved that, under the same assumptions, $ff^{(n)}$ has infinitely many non-real zeros if $n\geqslant 2$. Thus Conjecture B was established for functions satisfying (3).

A major step towards Conjecture B was then made by Hellerstein and Williamson [12], [13]. They proved that for a real entire function f, the condition that $ff'f''$ has only real zeros implies that $f\in\mathrm{LP}$. It follows that for a real entire function f with only real zeros, such that $f\notin\mathrm{LP}$, the number $N(f^{(n)})$ of non-real zeros of $f^{(n)}$ satisfies

$$\limsup_{n\to\infty}N(f^{(n)})>0.$$

The next breakthrough was made by Sheil-Small [30] by proving Wiman's conjecture for functions of finite order: f'' has at least $2p$ non-real zeros if $f\in U_{2p}$. This was a new and deep result even for the case $f=e^T$ with a polynomial T. By refining the arguments of Sheil-Small, Edwards and Hellerstein [8] were able to prove that

$$N(f^{(n)})\geqslant 2p \quad \text{for } n\geqslant 2 \text{ and } f=Ph,$$

where P is a real polynomial and $h\in U_{2p}$.

In the paper [3], the result of Sheil-Small was extended to functions of infinite order, thus establishing Wiman's conjecture in full generality. As we already mentioned, Langley [22] extended the result of [3] to higher derivatives. He proved the following: Let f be a real entire function of infinite order with finitely many non-real zeros. Then $N(f^{(n)})=\infty$ for all $n\geqslant 2$. Thus, Conjecture B was established for functions of infinite order.

To summarize the previous results, we can say that all asymptotic results were proved under strong a priori assumptions on the asymptotic behavior of f, while the non-asymptotic results estimated $N(f^{(n)})$ from below, for $f\in U_{2p}$, in terms of p rather than n.

In order to state a refined version of our result, we denote by $N_{\gamma,\delta}(f^{(n)})$ the number of zeros of $f^{(n)}$ in

$$\{z:|\operatorname{Im} z|>\gamma|z|,|z|>n^{1/\varrho-\delta}\},$$

where ϱ is the order of f, and γ and δ are arbitrary positive numbers.

THEOREM 2. *Suppose that $f=Ph$, where P is a real polynomial and $h\in U_{2p}$ with $p\geqslant 1$. Then there exist positive numbers α and γ depending only on p such that*

$$\liminf_{n\to\infty} \frac{N_{\gamma,\delta}(f^{(n)})}{n} \geqslant \alpha>0 \tag{4}$$

for every $\delta>0$.

Estimates (1) and (4) seem to be new even for functions of the form (2) with polynomials S and T. We recall that for a real entire function f of genus g with only finitely many non-real zeros, we have $N(f')\leqslant N(f)+g$, by a theorem of Laguerre and Borel [5], [9]. Thus, (1) and (4) give the right order of magnitude. The question remains on how α and γ in (4) depend on p.

We conclude this introduction with a sketch of the proof of Theorem 1, which combines the saddle point method used in the asymptotic results with potential theory and the theory of analytic functions with positive imaginary part in the upper half-plane. The latter theory was a common tool in all non-asymptotic results, since the discovery of Levin and Ostrovskii [24] that for a real entire function f with real zeros, the logarithmic derivative is a product of a real entire function and a function which maps the upper half-plane into itself (Lemma 4 below).

Our proof consists of three steps.

1. *Rescaling.* We assume for simplicity that all zeros of f are real. The Levin–Ostrovskii representation gives $L:=f'/f=P_0\psi_0$, where P_0 is a real polynomial of degree at least 2, and ψ_0 has non-negative imaginary part in the upper half-plane (Lemma 4). So we have a good control of the behavior of f'/f in the upper half-plane (Lemma 1). For any sequence σ of positive integers, we can find a subsequence σ' and positive numbers a_k such that for $k\to\infty$, $k\in\sigma'$, the following limits exist in the upper half-plane (§3):

$$q(z)=\lim_{k\to\infty}\frac{a_kL(a_kz)}{k}\quad\text{and}\quad u(z)=\lim_{k\to\infty}\frac{\log|f^{(k)}(a_kz)|-c_k}{k},$$

with an appropriate choice of real constants c_k. The second limit makes sense in L^1_{loc}, and u is a subharmonic function in the upper half-plane (Lemma 5). If the condition (1) is not satisfied, that is the $f^{(k)}$'s have few zeros in the upper half-plane, then u will be a harmonic function. Our goal is to show that this is impossible.

2. *Application of the saddle point method to find a functional equation for u.* We express the $f^{(k)}$'s as Cauchy integrals over appropriately chosen circles in the upper half-plane, and apply the saddle point method to find asymptotics of these integrals, as $k\to\infty$ (§4). The Levin–Ostrovskii representation and properties of analytic functions with positive imaginary part in the the upper half-plane give enough information for the estimates needed in the saddle point method. As a result, we obtain an expression of u in terms of q in a *Stolz angle* at infinity, by which we mean a region of the form $\{z:|z|>R,\varepsilon<\arg z<\pi-\varepsilon\}$ with $R>0$ and $\varepsilon>0$. The expression we obtain for u is

$$u\left(z-\frac{1}{q(z)}\right)=\mathrm{Re}\int_i^z q(\tau)\,d\tau+\log|q(z)|. \tag{5}$$

This equation, which plays a fundamental role in our proof, can be derived heuristically as follows. When applying Cauchy's formula to obtain an expression for $f^{(k)}(a_kw)$, it seems reasonable to write the radius of the circle in the form a_kr_k, so that

$$\begin{aligned}f^{(k)}(a_kw)&=\frac{k!}{2\pi i}\int_{|\xi|=a_kr_k}\frac{f(a_kw+\xi)}{\xi^k}\,\frac{d\xi}{\xi}\\&=\frac{k!}{2\pi i}\int_{|\zeta|=r_k}\exp(\log f(a_k(w+\zeta))-k\log a_k\zeta)\,\frac{d\zeta}{\zeta}.\end{aligned}$$

The saddle point method of finding the asymptotic behavior of such an integral as $k\to\infty$ involves a stationary point of the function in the exponent, that is a solution of the equation

$$0=\frac{d}{d\zeta}\left(\frac{\log f(a_k(w+\zeta))}{k}-\log a_k\zeta\right)=\frac{a_kL(a_k(w+\zeta))}{k}-\frac{1}{\zeta}=:q_k(w+\zeta)-\frac{1}{\zeta}.$$

This suggests to take $r_k=|\zeta|$, where ζ is a solution of the last equation. Setting $z=w+\zeta$, we obtain $w=z-1/q_k(z)$, and the saddle point method gives

$$\frac{1}{k}\left(\log f^{(k)}\left(a_k\left(z-\frac{1}{q_k(z)}\right)\right)-c_k\right)\sim\int_i^z q_k(\tau)\,d\tau+\log q_k(z),$$

with some constants c_k. From this we derive (5).

3. *Study of the functional equation* (5). If the $f^{(k)}$'s have few zeros in the upper half-plane, then the function u should be harmonic in the upper half-plane. On the other hand, we show that a harmonic function u satisfying (5) in a Stolz angle cannot have a harmonic continuation into the whole upper half-plane (§5). Here again we use the properties of analytic functions with positive imaginary part in the upper half-plane.

Acknowledgment. The authors thank David Drasin, Victor Katsnelson, Jim Langley, Iossif Ostrovskii and the referees for useful comments on this work.

2. Preliminaries

In this section we collect, for the reader's convenience, all necessary facts on potential theory and on functions with positive imaginary part in the upper half-plane

$$H:=\{z:\operatorname{Im} z>0\}.$$

LEMMA 1. *Let $\psi\not\equiv 0$ be an analytic function in H with non-negative imaginary part. Then*

$$|\psi(i)|\frac{\operatorname{Im} z}{(1+|z|)^2}\leqslant|\psi(z)|\leqslant|\psi(i)|\frac{(1+|z|)^2}{\operatorname{Im} z}, \tag{6}$$

$$\left|\frac{\psi'(z)}{\psi(z)}\right|\leqslant\frac{1}{\operatorname{Im} z}, \tag{7}$$

$$\left|\log\frac{\psi(z+\zeta)}{\psi(z)}\right|\leqslant 1 \quad \textit{for } |\zeta|<\frac{1}{2}\operatorname{Im} z. \tag{8}$$

Inequality (6) is a well-known estimate due to Carathéodory; see, for example, [32, §26]. Inequality (7) is the Schwarz lemma: it says that the derivative of ψ with respect to the hyperbolic metric in H is at most 1. Finally, (8) follows from (7) by integration.

LEMMA 2. *A holomorphic function ψ in H has non-negative imaginary part if and only if it has the form*

$$\psi(z)=a+\lambda z+\int_{-\infty}^{\infty}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)d\nu(t),$$

where $\lambda\geqslant 0$, a is real, and ν is a non-decreasing function of finite variation on the real line.

In the case where ψ is a meromorphic function in the whole plane, ν is piecewise constant with jumps at the poles of ψ. Then, the representation in Lemma 2 is similar to the familiar Mittag–Leffler representation. Lemma 2 can be found in [17], where it is derived from the similar Riesz–Herglotz representation of functions with positive imaginary part in the unit disc.

LEMMA 3. *Let G be a holomorphic function in H with non-negative imaginary part. Then there exists $\lambda\geqslant 0$ such that*

$$G(z)=\lambda z+G_1(z),$$

where $\operatorname{Im} G_1(z)\geqslant 0$, $z\in H$, *and* $G_1(z)=o(z)$, *as* $z\to\infty$, *in every Stolz angle.*

This is a theorem by Wolff; see, for example, [32, §26], where it is derived from the Schwarz lemma. Another proof can be easily obtained from Lemma 2. The number λ is called the *angular derivative* of G at infinity.

LEMMA 4. *Let h be a function of class U_{2p}. Then the logarithmic derivative of h has a representation*

$$\frac{h'}{h}=P_0\psi_0, \tag{9}$$

where P_0 is a real polynomial, $\deg P_0=2p$, *the leading coefficient of P_0 is negative*, *and $\psi_0\not\equiv 0$ is a function with non-negative imaginary part in H.*

Equation (9) is the Levin–Ostrovskii representation already mentioned, which was used in many papers on the subject [3], [8], [12], [14], [22], [24], [30]. Levin and Ostrovskii [24] did not assume that h has finite order, and showed that (9) holds with some real entire function P_0. Hellerstein and Williamson [12, Lemma 2] showed that P_0 is a polynomial if h has finite order, and they gave estimates of the degree of P_0 in [12, Lemma 8]. As they noted, an upper bound for the degree of P_0 follows from an old result of Laguerre [21, p. 172].

The factorization (9) is not unique, and the degree of P_0 is not uniquely determined by h. As our version of this factorization is different from [12, Lemma 8], we include a proof.

Proof. We first consider the simple case where $h=e^T$.

If $\deg T=2p+2$, then the leading coefficient of T is negative, by the definition of the class U_{2p}. As the degree of T' is odd, T' has a real root c. So we can put

$$P_0(z):=\frac{T'(z)}{z-c} \quad \text{and} \quad \psi_0(z):=z-c.$$

If $\deg T = 2p+1$, then we set $P_0 := \pm T'$ and $\psi_0 := \pm 1$, where the sign is chosen in such a way that the leading coefficient of P_0 is negative.

Finally, if $\deg T = 2p$, then the condition $h \in U_{2p}$ implies that the leading coefficient of T is positive, and we set $P_0(z) := -zT'(z)$ and $\psi_0(z) := -1/z$.

From now on, we assume that h has at least one real zero a_0. If h has only finitely many negative zeros and $h(x) \to 0$ as $x \to -\infty$, then we also consider $-\infty$ as a zero of h. Similarly, we consider $+\infty$ as a zero of h, if h has only finitely many positive zeros and $h(x) \to 0$ as $x \to +\infty$. We arrange the zeros of h into an increasing sequence $\{a_j\}$, where each zero occurs once, disregarding multiplicity. The range of the subscript j will be $M < j < N$, where $-\infty \leqslant M < 0 \leqslant N \leqslant \infty$, with $a_{M+1} = -\infty$ and $a_{N-1} = +\infty$ in the cases described above.

By Rolle's theorem, each open interval (a_j, a_{j+1}) contains a zero b_j of h'. To make a definite choice, we take for b_j the largest or the smallest zero in this interval. Each b_j occurs in this sequence only once, and we disregard multiplicity. We define

$$\psi(z) := \frac{1}{z - a_{N-1}} \prod_{M < j < N-1} \frac{1 - z/b_j}{1 - z/a_j},$$

where the factor $z - a_{N-1}$ is omitted if $a_{N-1} = +\infty$ or $N = \infty$, and the factor $1 - z/a_{M+1}$ is omitted if $a_{M+1} = -\infty$. If for some $j \in (M, N-1)$ we have $a_j = 0$ or $b_j = 0$, then the jth factor has to be replaced by $(z - b_j)/(z - a_j)$. As in [23], [24], we see that the product converges and is real only on the real axis. We define $P := h'/(h\psi)$. Then standard estimates, using the lemma on the logarithmic derivative and Lemma 1, imply that P is a polynomial [12], [24]. In particular, P has only finitely many zeros. The zeros of P are precisely the zeros that h'/h has in addition to the b_j's. These additional zeros were called *extraordinary* zeros by Ålander [1]. An extraordinary zero may be real or not, it can be real but different from all of the b_j's, or may be one of the b_j's: if b_j is a zero of h' of multiplicity $n \geqslant 2$, then b_j is considered an extraordinary zero of multiplicity $n-1$. Since the number of zeros of h' in every interval (a_j, a_{j+1}) is odd (counted with multiplicity), the number of extraordinary zeros of h' in this interval is even. As the non-real extraordinary zeros come in complex conjugate pairs, their number is also even. Overall, h' has an even number of extraordinary zeros, counted with multiplicity. Thus, the degree of P is even.

We choose $\varepsilon = \pm 1$ so that $P_0 := \varepsilon P$ has negative leading coefficient and define $\psi_0 := \varepsilon \psi$. Then equation (9) holds. Since the number of extraordinary zeros in each interval (a_j, a_{j+1}) is even, P_0 has an even number of zeros to the left of a_0, and, since $P_0(x) \to -\infty$ as $x \to -\infty$, we conclude that $P_0(a_0) < 0$. Since h'/h has a simple pole with positive residue at a_0, and $P_0(a_0) < 0$, we obtain that ψ_0 has a pole with negative residue at a_0. Since ψ_0

is real only on the real axis, this implies that $\psi_0(H)\subset H$. We note that

$$\psi_1 := -\frac{1}{\psi_0}$$

also maps H into itself.

Next we show that $\deg P_0 \leqslant 2p$. We recall that h has the form

$$h(z) = e^{az^{2p+2}} w(z), \tag{10}$$

where $a\leqslant 0$ and w is a real entire function of genus at most $2p+1$. Then

$$\log M(r,w) = o(r^{2p+2}), \quad \text{as } r\to\infty,$$

(see, for example, [23, I.4]). Schwarz's formula [23, I.6] shows that

$$\left|\frac{w'(z)}{w(z)}\right| = o(z^{2p+1}), \quad \text{as } z\to\infty,$$

in every Stolz angle. We thus have $h'(z)/h(z)=bz^{2p+1}+o(z^{2p+1})$, as $z\to\infty$, in every Stolz angle, with $b:=(2p+2)a\leqslant 0$. Denote by λ_1 the angular derivative of ψ_1 (see Lemma 3). Then $\psi_1(z)=\lambda_1 z+o(z)$, as $z\to\infty$, in every Stolz angle. Altogether we find that

$$P_0(z) = -\frac{h'(z)}{h(z)}\psi_1(z) = -b\lambda_1 z^{2p+2} + o(z^{2p+2}), \quad \text{as } z\to\infty,$$

in every Stolz angle. If $b<0$ and $\lambda_1>0$, then P_0 has the positive leading coefficient $-b\lambda_1$, a contradiction. Thus $b=0$ or $\lambda_1=0$, so that $|P_0(z)|=o(z^{2p+2})$, and hence $\deg P_0<2p+2$. Since $\deg P_0$ is even we thus obtain that $\deg P_0\leqslant 2p$.

Now we show that $\deg P_0\geqslant 2p$. We have

$$P_0(z) = cz^d + \dots, \quad \text{where } c<0.$$

Let $\lambda_0\geqslant 0$ be the angular derivative of ψ_0. Then in every Stolz angle

$$\frac{h'(z)}{h(z)} = c\lambda_0 z^{d+1} + o(z^{d+1}), \quad \text{as } z\to\infty.$$

Integrating this along straight lines, we conclude that

$$\log h(z) = \frac{c\lambda_0}{d+2} z^{d+2} + o(z^{d+2}), \quad \text{as } z\to\infty.$$

If $c\lambda_0<0$, we compare this with (10) and obtain that $d=2p$. If $\lambda_0=0$, we obtain that $a=0$ in (10), so the genus g of h is at most $2p+1$.

Now we follow [12, Lemma 8]. The logarithmic derivative of h has the form

$$\frac{h'(z)}{h(z)} = Q(z) + z^g \sum_j \frac{m_j}{a_j^g (z-a_j)}, \tag{11}$$

where $g \leqslant 2p+1$ is the genus of h, m_j is the multiplicity of the zero a_j and Q is a polynomial. On the other hand, (9) combined with Lemma 2 gives

$$\frac{h'(z)}{h(z)} = P_0(z)\left(\lambda_0 z + c_0 + \sum_j A_j \left(\frac{1}{a_j - z} - \frac{a_j}{1+a_j^2}\right)\right), \tag{12}$$

where $\lambda_0 \geqslant 0$, $A_j \geqslant 0$ and c_0 is real. We also have

$$\sum_j \frac{A_j}{a_j^2} < \infty. \tag{13}$$

Equating the residues at each pole in the expressions (11) and (12), we obtain

$$P_0(a_j) = -\frac{m_j}{A_j} < 0. \tag{14}$$

Now we choose $C>0$ such that $0 < -P_0(a_j) \leqslant C|a_j|^d$. Then, (13) and (14) imply

$$\frac{1}{C} \sum_j \frac{m_j}{|a_j|^{d+2}} \leqslant \sum_j \frac{-m_j}{a_j^2 P_0(a_j)} = \sum_j \frac{A_j}{a_j^2} < \infty,$$

which shows that $d+2 \geqslant g+1$, that is $d \geqslant g-1$. If $g=2p+1$, we have $d \geqslant 2p$. If $g=2p$, then also $d \geqslant 2p$ because d is even. It remains to notice that one cannot have $g \leqslant 2p-1$, because h is of genus g and belongs to U_{2p}. This completes the proof of Lemma 4. □

LEMMA 5. *Let $\{u_k\}_{k=1}^{\infty}$ be a sequence of subharmonic functions in a region D, and suppose that the u_k's are uniformly bounded from above on every compact subset of D. Then one can choose a subsequence of $\{u_k\}_{k=1}^{\infty}$, which either converges to $-\infty$ uniformly on compact subsets of D, or converges in L^1_{loc} (with respect to the Lebesgue measure in the plane) to a subharmonic function u. In the latter case, the Riesz measures of the u_k's converge weakly to the Riesz measure of u.*

This result can be found in [16, Theorem 4.1.9]. We recall that the Riesz measure of a subharmonic function u is $(2\pi)^{-1}\Delta u$ in the sense of distributions.

3. Beginning of the proof: rescaling

We write $f=Ph$, where P is a real polynomial and h is a real entire function of finite order, with all zeros real, and $h\notin\mathrm{LP}$. By Lemma 4, we have

$$\frac{h'}{h}=P_0\psi_0,$$

where ψ_0 is as in Lemma 4, and P_0 is a polynomial of degree $2p$ whose leading coefficient is negative. Note that $p\geqslant 1$ since $h\notin\mathrm{LP}$. We have

$$\frac{f'}{f}=P_0\psi_0+\frac{P'}{P}. \tag{15}$$

Using (6), we obtain

$$\frac{rf'(ir)}{f(ir)}\to\infty, \quad \text{as } r\to\infty,\ r>0. \tag{16}$$

Fix $\gamma,\delta>0$, and let σ be a sequence of positive integers along which the lower limit in (4) is attained; that is,

$$\beta:=\liminf_{k\to\infty}\frac{N_{\gamma,\delta}(f^{(k)})}{k}=\lim_{\substack{k\to\infty\\ k\in\sigma}}\frac{N_{\gamma,\delta}(f^{(k)})}{k}.$$

In the course of the proof we will choose subsequences of σ, and will continue to denote them by the same letter σ.

By (16), for every large $k\in\sigma$ there exist $a_k>0$, with $a_k\to\infty$, such that

$$\left|\frac{a_kf'(ia_k)}{f(ia_k)}\right|=k.$$

An estimate of the logarithmic derivative using Schwarz's formula implies that

$$\left|\frac{a_kf'(ia_k)}{f(ia_k)}\right|\leqslant|a_k|^{\varrho+o(1)}, \quad \text{as } k\to\infty,$$

where ϱ is the order of f. We may thus assume that

$$|a_k|\geqslant k^{1/\varrho-\delta/2} \tag{17}$$

for $k\in\sigma$. We define

$$q_k(z):=\frac{a_kf'(a_kz)}{kf(a_kz)}. \tag{18}$$

Then $|q_k(i)|=1$. From (15) and (6) we deduce that the q_k's are uniformly bounded on compact subsets of H, and thus the q_k's form a normal family in H. Passing to a subsequence, we may assume that

$$q_k\to q, \quad \text{as } k\to\infty,\ k\in\sigma, \tag{19}$$

uniformly on compact subsets in H. We choose a branch of the logarithm in a neighborhood of $f(ia_k)$, put $b_k{:=}\log f(ia_k)$, and define

$$Q(z) := \int_i^z q(\zeta)\,d\zeta \tag{20}$$

and

$$Q_k(z) := \int_i^z q_k(\zeta)\,d\zeta.$$

It follows from (19) that

$$Q_k(z) = \frac{\log f(a_k z) - b_k}{k} \to Q(z), \quad \text{as } k\to\infty,\ k\in\sigma,$$

uniformly on compact subsets of H, and that $Q_k(i)=0$. The chosen branches of $\log f$ are well defined on every compact subset of H, if k is large enough, because f has only finitely many zeros in the upper half-plane, and $a_k\to\infty$.

Let z be a point in H, and $0<t<\operatorname{Im} z$. Then the disc $\{\zeta:|\zeta-a_k z|<ta_k\}$ is contained in H and does not contain any zeros of f if k is large enough. Thus, by Cauchy's formula,

$$f^{(k)}(a_k z) = \frac{k!}{2\pi i}\int_{|\zeta|=a_k t} \frac{f(a_k z+\zeta)}{\zeta^k}\,\frac{d\zeta}{\zeta} = \frac{k!}{2\pi i}\int_{|\zeta|=a_k t} \frac{\exp(kQ_k(z+\zeta/a_k)+b_k)}{\zeta^k}\,\frac{d\zeta}{\zeta}. \tag{21}$$

So

$$|f^{(k)}(a_k z)| \leqslant \frac{k!}{(a_k t)^k}\exp\Bigl(\operatorname{Re} b_k + k\max_{|\zeta|=t}\operatorname{Re} Q_k(z+\zeta)\Bigr),$$

and with

$$u_k(z) := \frac{\log|f^{(k)}(a_k z)| - \operatorname{Re} b_k - \log k!}{k} + \log a_k \tag{22}$$

we obtain

$$u_k(z) \leqslant \max_{|\zeta|=t}\operatorname{Re} Q_k(z+\zeta) - \log t. \tag{23}$$

Since $Q_k\to Q$, we deduce that the u_k's are uniformly bounded from above on compact subsets of H. By Lemma 5, after choosing a subsequence, we obtain

$$u_k \to u, \tag{24}$$

where u is a subharmonic function in H or $u\equiv-\infty$. The convergence in (24) holds in H in the sense described in Lemma 5. We will later see that $u\not\equiv-\infty$. We will then show that u cannot be harmonic in H. This will prove Theorem 1.

Moreover, we will see that there exist positive constants γ and α, depending only on p, such that the total Riesz measure of u in the region $\{z:\operatorname{Im} z>\gamma|z|\}$ is at least $\alpha/2$.

On the other hand, it follows from the definition of u_k and $N_{\gamma,\delta}(f^{(k)})$ that

$$\frac{N_{\gamma,\delta}(f^{(k)})}{2k}$$

is the total Riesz measure of u_k in the region $\{z:\operatorname{Im} z>\gamma|z|, |z|>k^{1/\varrho-\delta}/a_k\}$. Note that $k^{1/\varrho-\delta}/a_k\to0$, by (17). Passing to the limit as $k\to\infty$, $k\in\sigma$, we obtain $\beta\geqslant\alpha>0$, which will complete the proof of Theorem 2.

4. Application of the saddle point method

It follows from (15) and the definitions of q_k in (18) and q in (19) that

$$q(z) = -z^{2p}\psi(z), \tag{25}$$

where $\psi\colon H\to\overline{H}\setminus\{0\}$ is defined by

$$\psi(z) := \lim_{k\to\infty} \frac{\psi_0(a_k z)}{|\psi_0(ia_k)|}.$$

Here ψ_0 is the real meromorphic function in $\mathbf{C}$ from (15). Note that

$$|q(i)| = |\psi(i)| = 1. \tag{26}$$

We now define

$$F(z) := z - \frac{1}{q(z)}.$$

From (25) and (6), it follows that $1/q(z)\to 0$, as $z\to\infty$, in every Stolz angle. So

$$F(z) \sim z, \quad \text{as } z\to\infty,$$

in every Stolz angle. We fix $\delta_0\in(0,\pi/2)$, for example $\delta_0=\pi/4$. It follows that there exists $R>0$ such that $\operatorname{Im} F(z)>0$ in the region

$$S_R := \{z : |z| > R, \delta_0 < \arg z < \pi-\delta_0\}. \tag{27}$$

Moreover, if $\delta_1\in(0,\delta_0)$, then

$$F(z)\in S' := \{w : \delta_1 < \arg w < \pi-\delta_1\}, \quad \text{when } z\in S_R,$$

provided that R is large enough, say $R>R_0$. We notice that R_0 is independent of f; this follows from (6), (25) and (26). We will enlarge R_0 in the course of the proof, but it will always be a constant independent of f. We will obtain, for $R>R_0$, the identity

$$u(F(z)) = \operatorname{Re} Q(z) + \log|q(z)|, \quad z\in S_R. \tag{28}$$

In order to do this, we use Cauchy's formula (21) with z replaced by $F_k(z):=z-r_k(z)$, and $t=|r_k(z)|$, where we have set $r_k:=1/q_k$ to simplify our formulas.

We obtain

$$\begin{aligned} f^{(k)}(a_k F_k(z)) &= \frac{k!}{2\pi}\int_{-\pi}^{\pi} \frac{\exp(kQ_k(F_k(z)+r_k(z)e^{i\theta})+b_k)}{(a_k r_k(z)e^{i\theta})^k}\,d\theta \\ &= \frac{k!\,q_k^k(z)}{2\pi a_k^k} e^{b_k}\int_{-\pi}^{\pi} \exp(kQ_k(F_k(z)+r_k(z)e^{i\theta})-ik\theta)\,d\theta. \end{aligned} \tag{29}$$

To determine the asymptotic behavior of this integral as $k\to\infty$, we expand

$$L_k(\theta):=Q_k(F_k(z)+r_k(z)e^{i\theta})$$

into a Taylor series:

$$L_k(\theta)=L_k(0)+L_k'(0)\theta+\tfrac{1}{2}L_k''(0)\theta^2+\tfrac{1}{6}E_k(\theta)\theta^3,$$

where

$$|E_k(\theta)|\leqslant \max_{t\in[0,\theta]}|L_k'''(t)|.$$

We notice that

$$L_k(0)=Q_k(F_k(z)+r_k(z))=Q_k(z),$$

and

$$L_k'(\theta)=iQ_k'(F_k(z)+r_k(z)e^{i\theta})r_k(z)e^{i\theta}=iq_k(F_k(z)+r_k(z)e^{i\theta})r_k(z)e^{i\theta},$$

so that $L_k'(0)=i$. Moreover,

$$L_k''(\theta)=-q_k'(F_k(z)+r_k(z)e^{i\theta})r_k^2(z)e^{2i\theta}-q_k(F_k(z)+r_k(z)e^{i\theta})r_k(z)e^{i\theta},$$

so that

$$L_k''(0)=-\frac{q_k'(z)}{q_k^2(z)}-1. \tag{30}$$

Finally, we have

$$\begin{aligned}L_k'''(\theta)=&-iq_k''(F_k(z)+r_k(z)e^{i\theta})r_k^3(z)e^{3i\theta}\\&-3iq_k'(F_k(z)+r_k(z)e^{i\theta})r_k^2(z)e^{2i\theta}\\&-iq_k(F_k(z)+r_k(z)e^{i\theta})r_k(z)e^{i\theta}.\end{aligned}$$

To estimate $L_k''(0)$ and $L_k'''(\theta)$ we notice that

$$\frac{q'(\zeta)}{q(\zeta)}=\frac{2p}{\zeta}+\frac{\psi'(\zeta)}{\psi(\zeta)}.$$

It follows from (7) that

$$\left|\frac{q'(\zeta)}{q(\zeta)}\right|\leqslant\frac{2p+1}{\operatorname{Im}\zeta}.$$

From this, we deduce that

$$\left|\frac{d}{d\zeta}\left(\frac{q'(\zeta)}{q(\zeta)}\right)\right|=\frac{1}{2\pi}\left|\int_{|z-\zeta|=\frac{1}{2}\operatorname{Im}\zeta}\frac{q'(z)}{q(z)(z-\zeta)^2}\,dz\right|\leqslant\frac{2}{\operatorname{Im}\zeta}\max_{|z-\zeta|=\frac{1}{2}\operatorname{Im}\zeta}\left|\frac{q'(z)}{q(z)}\right|\leqslant\frac{4(2p+1)}{(\operatorname{Im}\zeta)^2},$$

so that

$$\left|\frac{q''(\zeta)}{q(\zeta)}\right|=\left|\frac{d}{d\zeta}\left(\frac{q'(\zeta)}{q(\zeta)}\right)+\left(\frac{q'(\zeta)}{q(\zeta)}\right)^2\right|\leqslant\frac{(2p+5)(2p+1)}{(\operatorname{Im}\zeta)^2}.$$

Since $q_k \to q$, we deduce from the above estimates that

$$\left|\frac{q_k'(\zeta)}{q_k(\zeta)}\right| \leqslant \frac{2p+2}{\operatorname{Im}\zeta} \quad \text{and} \quad \left|\frac{q_k''(\zeta)}{q_k(\zeta)}\right| \leqslant \frac{(2p+5)^2}{(\operatorname{Im}\zeta)^2} \tag{31}$$

on any compact subset of H, provided k is sufficiently large, $k \in \sigma$.

Fix $\eta > 0$. It follows from (30) and (31) that if $z \in S_R$, where $R > R_0$ and k is large enough, then

$$|L_k''(0)+1| < \eta.$$

In particular,

$$\operatorname{Re} L_k''(0) \leqslant -1+\eta. \tag{32}$$

Moreover, if $z \in S_R$ with $R > R_0$, then

$$\zeta := F_k(z)+r_k(z)e^{i\theta} \in S' \tag{33}$$

and $\zeta \to \infty$, as $z \to \infty$. For large $|z|$ we have

$$|L_k'''(\theta)| \leqslant \left|\frac{q_k(\zeta)}{q_k(z)}\right| \left(\frac{|q_k''(\zeta)|}{|q_k(\zeta)||q_k(z)|^2} + 3\left|\frac{q_k'(\zeta)}{q_k(\zeta)q_k(z)}\right| + 1 \right).$$

By (8) and (25), we have

$$\frac{q_k(\zeta)}{q_k(z)} \to 1,$$

where ζ is defined in (33), uniformly with respect to $z \in S_R$, and $\theta \in [-\pi, \pi]$, as $R \to \infty$. Combining these estimates with (31), we find that $|L_k'''(\theta)| \leqslant 1+\eta$ and hence

$$|E_k(\theta)| \leqslant 1+\eta \tag{34}$$

for $z \in S_R$ and $|\theta| \leqslant \pi$, provided $R > R_0$. Altogether, we have the Taylor expansion

$$L_k(\theta) = Q_k(z)+i\theta+\tfrac{1}{2}L_k''(0)\theta^2+\tfrac{1}{6}E_k(\theta)\theta^3, \tag{35}$$

with $L_k''(0)$ and $E_k(\theta)$ satisfying (32) and (34), respectively. We define $C_k(z) := -\frac{1}{2}L_k''(0)$, and notice that, in view of (30),

$$C_k(z) \to C(z) := \frac{1}{2}\left(1+\frac{q'(z)}{q^2(z)}\right), \quad \text{as } k \to \infty, \ k \in \sigma,$$

uniformly on any compact subset in H.

We note that if

$$|\theta| \leqslant \theta_0 := \frac{3(1-3\eta)}{1+\eta},$$

then

$$\begin{aligned}\operatorname{Re}\bigl(-C_k(z)\theta^2+\tfrac{1}{6}E_k(\theta)\theta^3\bigr) &= \operatorname{Re}\bigl(\tfrac{1}{2}L_k''(0)\theta^2+\tfrac{1}{6}E_k(\theta)\theta^3\bigr)\\ &\leqslant \theta^2\bigl(\tfrac{1}{2}(-1+\eta)+\tfrac{1}{6}(1+\eta)|\theta|\bigr)\\ &= \theta^2\bigl(\tfrac{1}{2}(-1+\eta)+\tfrac{1}{2}(1-3\eta)\bigr)\\ &\leqslant -\eta\theta^2.\end{aligned} \tag{36}$$

Using (35), we obtain

$$\begin{aligned}&\int_{-\theta_0}^{\theta_0} \exp(kQ_k(F_k(z)+r_k(z)e^{i\theta})-ik\theta)\,d\theta\\ &\qquad= e^{kQ_k(z)}\int_{-\theta_0}^{\theta_0}\exp\biggl(k\biggl(-C_k(z)\theta^2+\frac{1}{6}E_k(\theta)\theta^3\biggr)\biggr)d\theta\\ &\qquad= \frac{e^{kQ_k(z)}}{\sqrt{k}}\int_{-\theta_0\sqrt{k}}^{\theta_0\sqrt{k}}\exp\biggl(-C_k(z)t^2+\frac{1}{6}E_k\biggl(\frac{t}{\sqrt{k}}\biggr)\frac{t^3}{\sqrt{k}}\biggr)\,dt.\end{aligned}$$

Combining this with (36) and the theorem on dominated convergence, we obtain

$$\int_{-\theta_0}^{\theta_0} \exp(kQ_k(F_k(z)+r_k(z)e^{i\theta})-ik\theta)\,d\theta \sim \frac{e^{kQ_k(z)}}{\sqrt{k}}\sqrt{\frac{\pi}{C(z)}}, \tag{37}$$

as $k\to\infty$, $k\in\sigma$, uniformly on compact subsets of S_R, for $R>R_0$.

In order to estimate the rest of the integral (29), we will show that

$$\operatorname{Re}Q_k(F_k(z)+r_k(z)e^{i\theta}) \leqslant \operatorname{Re}Q_k(z)-\frac{1-\cos\theta_0}{2} \tag{38}$$

for $\theta_0\leqslant\theta\leqslant\pi$. We have

$$\begin{aligned}Q_k(z)-Q_k(F_k(z)+r_k(z)e^{i\theta}) &= Q_k(F_k(z)+r_k(z))-Q_k(F_k(z)+r_k(z)e^{i\theta})\\ &= \int_{r_k(z)e^{i\theta}}^{r_k(z)} q_k(F_k(z)+\zeta)\,d\zeta\\ &= \int_0^1 q_k(F_k(z)+r_k(z)e^{i\theta}+tr_k(z)(1-e^{i\theta}))(1-e^{i\theta})r_k(z)\,dt\\ &= (1-e^{i\theta})\int_0^1 \frac{q_k(\zeta_t)}{q_k(z)}\,dt,\end{aligned}$$

where

$$\zeta_t := F_k(z)+r_k(z)e^{i\theta}+tr_k(z)(1-e^{i\theta}).$$

Now $\zeta_t/z\to 1$, as $z\to\infty$, $z\in S_R$, uniformly with respect to $t\in[0,1]$. Using (8), we see that $q_k(\zeta_t)/q_k(z)\to 1$ for $z\in S_R$, as $R\to\infty$. In particular, we have

$$\mathrm{Re}\left((1-e^{i\theta})\frac{q_k(\zeta_t)}{q_k(z)}\right)\geqslant\frac{1-\cos\theta}{2}\geqslant\frac{1-\cos\theta_0}{2}$$

for $\theta_0\leqslant\theta\leqslant\pi$, $z\in S_R$ and $R>R_0$, and this yields (38).

It follows from (38) that

$$\begin{aligned}\left|\int_{\theta_0\leqslant|\theta|\leqslant\pi}\exp(kQ_k(F_k(z)+r_k(z)e^{i\theta})-ik\theta)\,d\theta\right|&\\ \leqslant\int_{\theta_0\leqslant|\theta|\leqslant\pi}&\exp(\mathrm{Re}\,(kQ_k(F_k(z)+r_k(z)e^{i\theta})))\,d\theta\\ \leqslant 2(\pi-\theta_0)&\exp\left(k\left(\mathrm{Re}\,Q_k(z)-\frac{1-\cos\theta_0}{2}\right)\right)\\ =o&\left(\frac{e^{kQ_k(z)}}{\sqrt{k}}\sqrt{\frac{\pi}{C(z)}}\right).\end{aligned}$$

Combining this with (37), we see that

$$\int_{|\theta|\leqslant\pi}\exp(kQ_k(F_k(z)+r_k(z)e^{i\theta})-ik\theta)\,d\theta\sim\frac{e^{kQ_k(z)}}{\sqrt{k}}\sqrt{\frac{\pi}{C(z)}},\quad\text{as }\ k\to\infty,$$

uniformly on compact subsets in S_R, where $R>R_0$. Together with (29), this gives

$$f^{(k)}(a_kF_k(z))\sim\frac{k!\,q_k^k(z)e^{b_k}e^{kQ_k(z)}}{2a_k^k\sqrt{\pi kC(z)}}.$$

Taking logarithms, dividing by k and passing to the limit as $k\to\infty$ we obtain, using (22),

$$u(F(z))=\mathrm{Re}\,Q(z)+\log|q(z)|,\quad z\in S_R,$$

for $R>R_0$. This is the same as (28). One consequence of (28) is that $u\not\equiv-\infty$.

5. Analytic continuation of u and conclusion of the proof

As pointed out at the end of §3, in order to prove Theorem 1 it suffices to show that the function u obtained is not harmonic.

LEMMA 6. *Let*

$$q(z) = -z^{2p}\psi(z), \tag{39}$$

where $p \geqslant 1$ *and* ψ *is an analytic function mapping* H *to* $\overline{H}\setminus\{0\}$. *Define*

$$Q(z) := \int_i^z q(\zeta)\,d\zeta$$

and

$$F(z) := z - \frac{1}{q(z)}. \tag{40}$$

Let u *be a subharmonic function in* H *satisfying* (28), *in a region* S_R *defined in* (27). *Then* u *is not harmonic in* H.

Proof. Suppose that u is harmonic. Then there exists a holomorphic function h in H such that $u = \operatorname{Re} h$, and

$$h(F(z)) = Q(z) + \log q(z), \quad z \in S_R. \tag{41}$$

Differentiating (41) and using $q = Q'$ and (40), we obtain

$$h'(F(z)) = q(z) = \frac{1}{z - F(z)}.$$

From (39), (40) and (6), it follows that there is a branch G of the inverse $F^{-1}(w)$ which is defined in a Stolz angle S and satisfies

$$G(w) \sim w, \quad \text{as } w \to \infty,\ w \in S. \tag{42}$$

In particular, $G(w) \in H$ for $w \in S$ and $|w|$ large enough. We have

$$h'(w) = q(G(w)) = \frac{1}{G(w) - w} \tag{43}$$

for $w \in S$ and $|w|$ large enough. Since h is holomorphic in H, we see that G has a meromorphic continuation to H. Using (39), the second equation in (43) can be rewritten as

$$\psi(G(w)) = \frac{1}{G^{2p}(w)(w - G(w))}. \tag{44}$$

We will derive from (44) that G maps H into itself. To show this, we establish first that G never takes a real value in H. In view of (42), there exists a point $w_0 \in H$ such that $G(w_0) \in H$. If G takes a real value in H, then there exists a curve $\phi\colon [0,1] \to H$, beginning at w_0 and ending at some point $w_1 \in H$, such that $G(\phi(t)) \in H$ for $0 \leqslant t < 1$, but $G(w_1) = G(\phi(1)) \in \mathbf{R}$. We may assume that $G(w_1) \neq 0$; this can be achieved by a small perturbation of the curve ϕ and the point w_1. Using (44), we obtain an analytic continuation of ψ to the real point $G(w_1)$ along the curve $G(\phi)$. We have

$$\lim_{t\to 1} \operatorname{Im} \psi(G(\phi(t))) \geqslant 0,$$

because the imaginary part of ψ is non-negative in H.

It follows that, as $w\to w_1$, the right-hand side of (44) has negative imaginary part, while the left-hand side has non-negative imaginary part, which is a contradiction. Thus we have proved that G never takes real values in H.

Since $G(w_0)\in H$, we see that G maps H into itself. Then Lemma 3 and (42) imply that $\operatorname{Im}(G(w)-w)>0$ for $w\in H$. Combining this with the second equation of (43), we obtain $\operatorname{Im} q(G(w))<0$ for $w\in H$. Using (42), we find in particular that $\operatorname{Im} q(e^{i\pi/(2p)}y)<0$ for large $y>0$. On the other hand, we have, by (39), that

$$\operatorname{Im} q(e^{i\pi/(2p)}y) = y^{2p} \operatorname{Im} \psi(e^{i\pi/(2p)}y) \geqslant 0.$$

This contradiction proves Lemma 6, and it also completes the proof of Theorem 1. □

Proof of Theorem 2. It remains to show that there exist positive constants γ and α, depending only on p, such that the total Riesz measure of u in the region $\{z:\operatorname{Im} z>\gamma|z|\}$ is bounded below by $\alpha/2$, with u and p as in §3 and §4.

First we note that

$$u(z) \leqslant \max_{|\zeta|=t} \operatorname{Re} Q(z+\zeta) - \log t, \quad t=\tfrac{1}{2}\operatorname{Im} z, \tag{45}$$

for $z\in H$, by (23). Using (20), (25), (26) and (6), we see that for every compact subset K of H, the right-hand side of (45) is bounded from above by a constant that depends only on p and K.

Thus, for fixed p, the functions u, Q and q under consideration belong to normal families. Arguing by contradiction, we assume that α and γ as above do not exist. Then there exists a sequence $\{u_k\}_{k=1}^{\infty}$ of subharmonic functions in H, satisfying equations of the form

$$u_k(F_k) = \operatorname{Re} Q_k + \log|q_k|$$

similar to (28), such that the Riesz measure of u_k in $\{z:\operatorname{Im} z>|z|/k\}$ tends to 0, as $k\to\infty$. (The functions u_k, F_k, q_k and Q_k are not the functions introduced in §3 and §4.) It is important that all these equations hold in the same region S_R. Using normality, we can take convergent subsequences, and we obtain a limit equation of the same form, satisfied by a function u harmonic in H. But this contradicts Lemma 6, and thus the proof of Theorem 2 is completed. □

Remark. In [30] and in most subsequent work on the subject, the auxiliary function $F_0(z)=z-f(z)/f'(z)$ plays an important role. Note that our function F of (40) is of a similar nature, except that f'/f is replaced by q, which is obtained from f'/f by rescaling.

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Received December 6, 2005