

# The localization sequence for the algebraic $K$ -theory of topological $K$ -theory

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## Introduction

The algebraic  $K$ -theory of ring spectra is intimately related to the geometry of high-dimensional manifolds. In a series of papers in the 1970’s and 1980’s, Waldhausen established the deep connection between the  $K$ -theory of the sphere spectrum and stable pseudo-isotopy theory. The sphere spectrum has an infinite filtration called the chromatic filtration that forms a tower at each prime  $p$ . The layers in the chromatic tower capture periodic phenomena in stable homotopy theory, corresponding to the Morava  $K$ -theory “fields”. The chromatic viewpoint has organized the understanding of stable homotopy theory over the last thirty years.

Applying algebraic  $K$ -theory to the chromatic tower of the sphere spectrum leads to an analogous “chromatic tower” of algebraic  $K$ -theory spectra. Waldhausen conjectured that this tower converges in the homotopy inverse limit to the algebraic  $K$ -theory of the  $p$ -local sphere spectrum [10]. McClure and Staffeldt verified a closely related connective variant of this conjecture [6]. The bottom layer of the chromatic tower is  $K(\mathbb{Q})$ , the algebraic  $K$ -theory of the rational numbers, which is intimately related to questions in arithmetic. Thus, the  $K$ -theory chromatic tower can be viewed as an interpolation from arithmetic to geometry.

In [2], Ausoni and Rognes describe an ambitious program for analyzing the layers in the  $p$ -complete version of Waldhausen’s  $K$ -theory chromatic tower. The analysis is based on descent conjectures coming from Rognes’ Galois theory of  $S$ -algebras [8], which relate the layers of the tower to the  $K$ -theory of the Morava  $E$ -theory ring spectra  $E_n$  and to the  $K$ -theory of the  $p$ -completed Johnson–Wilson ring spectra  $E(n)_p^\wedge$ . The spectrum

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$E(n)$  is not connective, and no tools exist for computing the  $K$ -theory of non-connective ring spectra. However,  $E(n)$  is formed from the connective spectrum  $BP\langle n \rangle$  by inverting the element  $v_n$  in  $\pi_*(BP\langle n \rangle)$ , and Rognes conjectured the following localization cofiber sequence.

**ROGNES CONJECTURE.** *The transfer map  $K(BP\langle n-1 \rangle_p^\wedge) \rightarrow K(BP\langle n \rangle_p^\wedge)$  and the canonical map  $K(BP\langle n \rangle_p^\wedge) \rightarrow K(E(n)_p^\wedge)$  fit into a cofiber sequence in the stable category*

$$K(BP\langle n-1 \rangle_p^\wedge) \longrightarrow K(BP\langle n \rangle_p^\wedge) \longrightarrow K(E(n)_p^\wedge) \longrightarrow \Sigma K(BP\langle n-1 \rangle_p^\wedge).$$

In the case  $n=0$ , the statement is an old theorem of Quillen [7], the localization sequence  $K(\mathbb{Z}/p) \rightarrow K(\mathbb{Z}_p^\wedge) \rightarrow K(\mathbb{Q}_p^\wedge)$ . The conjecture implies a long exact sequence of homotopy groups:

$$\begin{aligned} \dots &\longrightarrow K_q(BP\langle n-1 \rangle_p^\wedge) \longrightarrow K_q(BP\langle n \rangle_p^\wedge) \longrightarrow K_q(E(n)_p^\wedge) \longrightarrow \dots \\ &\dots \longrightarrow K_0(BP\langle n-1 \rangle_p^\wedge) \longrightarrow K_0(BP\langle n \rangle_p^\wedge) \longrightarrow K_0(E(n)_p^\wedge) \longrightarrow 0. \end{aligned}$$

This in principle allows the computation of the algebraic  $K$ -groups of  $E(n)_p^\wedge$  from those of  $BP\langle n \rangle_p^\wedge$  and  $BP\langle n-1 \rangle_p^\wedge$ . Because the  $BP\langle n \rangle$  are connective spectra, their  $K$ -theory can be studied using TC, and the main purpose of [2] is to provide tools suitable for computing  $TC(BP\langle n \rangle_p^\wedge)$ , and in particular to obtain an explicit computation of the  $V(1)$ -homotopy in the case  $n=1$ .

The case  $n=1$  is the first non-classical case of the conjecture and is of particular interest. The spectra  $E(1)$  and  $BP\langle 1 \rangle$  are often denoted by  $L$  and  $\ell$  and are closely related to topological  $K$ -theory. Here  $L=E(1)$  is the  $p$ -local Adams summand of (complex periodic) topological  $K$ -theory,  $KU$ , and  $\ell=BP\langle 1 \rangle$  is the  $p$ -local Adams summand of connective topological  $K$ -theory,  $ku$ . The Rognes conjecture then admits a “global” version in this case, namely that the transfer map  $K(\mathbb{Z}) \rightarrow K(ku)$  and canonical map  $K(ku) \rightarrow K(KU)$  fit into a cofiber sequence in the stable category

$$K(\mathbb{Z}) \longrightarrow K(ku) \longrightarrow K(KU) \longrightarrow \Sigma K(\mathbb{Z}).$$

This version of the conjecture appears in [3], where the relationship between the algebraic  $K$ -theory of topological  $K$ -theory, elliptic cohomology, and the category of 2-vector bundles is discussed.

The purpose of this paper is to prove both the local and global versions of the Rognes conjecture in the case  $n=1$ . Specifically, we prove the following theorem.

**LOCALIZATION THEOREM.** *There are connecting maps, which together with the transfer maps and the canonical maps, make the sequences*

$$\begin{aligned} K(\mathbb{Z}_p^\wedge) &\longrightarrow K(\ell_p^\wedge) \longrightarrow K(L_p^\wedge) \longrightarrow \Sigma K(\mathbb{Z}_p^\wedge), \\ K(\mathbb{Z}_{(p)}) &\longrightarrow K(\ell) \longrightarrow K(L) \longrightarrow \Sigma K(\mathbb{Z}_{(p)}), \\ K(\mathbb{Z}) &\longrightarrow K(ku) \longrightarrow K(KU) \longrightarrow \Sigma K(\mathbb{Z}) \end{aligned}$$

*cofiber sequences in the stable category.*

Hesselholt observed that the Ausoni–Rognes and Ausoni calculations of  $\mathrm{THH}(\ell)$  and  $\mathrm{THH}(ku)$  in [2] and [1] are explained by the existence of a  $\mathrm{THH}$  version of this localization sequence along with a conjecture about the behavior of  $\mathrm{THH}$  for “tamely ramified” extensions of ring spectra. A precise formulation requires a construction of  $\mathrm{THH}$  for Waldhausen categories. We will explore this more fully in a forthcoming paper.

The localization theorem above is actually a consequence of a “dévissage” theorem for finitely generated finite stage Postnikov towers. For an  $S$ -algebra ( $A_\infty$  ring spectrum)  $R$ , let  $\mathcal{P}_R$  denote the full subcategory of left  $R$ -modules that have only finitely many non-zero homotopy groups, all of which are finitely generated over  $\pi_0 R$ . When  $R$  is connective and  $\pi_0 R$  is left Noetherian, this category has an associated Waldhausen  $K$ -theory spectrum. Restricting to the subcategory of  $S$ -algebras with morphisms the maps  $R \rightarrow R'$  for which  $\pi_0 R'$  is finitely generated as a left  $\pi_0 R$ -module, we regard  $K(\mathcal{P}_{(-)})$  as a contravariant functor  $K'$  to the stable category.

We use the notation  $K'$  because of the close connection with Quillen’s  $K'$ -theory, the  $K$ -theory of the exact category of finitely generated left modules over a left Noetherian ring. The analogous  $K$ -theory spectrum for chain complexes over the ring  $\pi_0 R$  is well known to be equivalent to  $K'(\pi_0 R)$  [9, Theorem 1.11.7]. The following theorem is the main result of this paper.

**DÉVISSAGE THEOREM.** *Let  $R$  be a connective  $S$ -algebra ( $A_\infty$  ring spectrum) with  $\pi_0 R$  left Noetherian. Then there is a natural isomorphism in the stable category*

$$K'(\pi_0 R) \longrightarrow K'(R),$$

*where  $K'(\pi_0 R)$  is Quillen’s  $K$ -theory of the exact category of finitely generated left  $\pi_0 R$ -modules, and  $K'(R)$  is the Waldhausen  $K$ -theory of the category of finitely generated finite stage Postnikov towers of left  $R$ -modules,  $\mathcal{P}_R$ .*

A longstanding open problem first posed explicitly by Thomason and Trobaugh [9, §1.11.1] is to develop a general dévissage theorem for Waldhausen categories that

specializes to Quillen's dévissage theorem when applied to the category of bounded chain complexes on an abelian category. We regard the theorem described in this paper as a step towards a solution to this problem.

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### 1. The main argument

In this section, we outline the proof of the dévissage theorem in terms of a number of easily stated results proved in later sections; we then deduce the localization theorem from the dévissage theorem. Although we assume some familiarity with the basics of Waldhausen  $K$ -theory, we review the standard definitions and constructions of Waldhausen [11] as needed. We begin with some technical conventions and a precise description of the Waldhausen categories we use.

Throughout this paper,  $R$  denotes a connective  $S$ -algebra with  $\pi_0 R$  left Noetherian. We work in the context of Elmendorf–Kriz–Mandell–May (EKMM)  $S$ -modules,  $S$ -algebras, and  $R$ -modules [4]. Since other contexts for the foundations of a modern category of spectra lead to equivalent  $K$ -theory spectra, presumably the arguments presented here could be adjusted to these contexts, but the EKMM categories have certain technical advantages that we exploit (see for example the proof of Lemma 4.2) and which affect the precise form of the statements below.

The input for Waldhausen  $K$ -theory is a (small) category together with a subcategory of “weak equivalences” and a subcategory of “w-cofibrations” satisfying certain properties. Let  $\mathcal{P}_R$  denote the full subcategory of left  $R$ -modules that have only finitely many non-zero homotopy groups, all of which are finitely generated over  $\pi_0 R$ . Although  $\mathcal{P}_R$  is not a small category, we can still construct a  $K$ -theory spectrum from it that is “homotopically small”, and we can find a small category with equivalent  $K$ -theory by restricting the sets allowed in the underlying spaces of the underlying prespectra; see Remark 1.7 below for details. In what follows, let  $\mathcal{P}$  denote  $\mathcal{P}_R$  or, at the reader's preference, the small category  $\mathcal{P}_k$  (for  $k$  large) discussed in Remark 1.7.

We make  $\mathcal{P}$  a Waldhausen category by taking the weak equivalences to be the usual weak equivalences (the maps that induce isomorphisms on all homotopy groups) and the w-cofibrations to be the Hurewicz cofibrations (the maps satisfying the homotopy extension property in the category of left  $R$ -modules). Specifically, a map  $i: A \rightarrow X$  is a Hurewicz cofibration if and only if the inclusion of the mapping cylinder

$$Mi = X \cup_i (A \wedge I_+)$$

in the cylinder  $X \wedge I_+$  has a retraction in the category of  $R$ -modules. Some easy consequences of this definition are:

- The initial map  $* \rightarrow X$  is a Hurewicz cofibration for any  $R$ -module  $X$ ;
- Hurewicz cofibrations are preserved by cobase change (pushout);
- for a map of  $S$ -algebras  $R \rightarrow R'$ , the forgetful functor from  $R'$ -modules to  $R$ -modules preserves Hurewicz cofibrations.

Use of this type of cofibration was a key tool in [4] for keeping homotopical control; a key fact is that pushouts along Hurewicz cofibrations preserve weak equivalences [4, Corollary I.6.5], and in particular Waldhausen's gluing lemma [11, §1.2] holds.

Since we are thinking of  $\mathcal{P}$  in terms of finite Postnikov stages, it makes more sense philosophically to work with fibrations:  $\mathcal{P}^{\text{op}}$  forms a Waldhausen category with weak equivalences the maps opposite to the usual weak equivalences, and with w-cofibrations the maps opposite to the Hurewicz fibrations (maps satisfying the covering homotopy property). Although not strictly necessary for the dévissage theorem, the following theorem proved in §2 straightens out this discrepancy.

**THEOREM 1.1.** *The spectra  $K(\mathcal{P})$  and  $K(\mathcal{P}^{\text{op}})$  are weakly equivalent.*

This result is essentially a consequence of the fact that  $\mathcal{P}$  is a stable category, and in particular follows from the observation that homotopy cocartesian and homotopy cartesian squares coincide in  $\mathcal{P}$ . As part of the technical machinery employed in the proof of the dévissage theorem, we introduce a variant of Waldhausen's  $S_{\bullet}$  construction, using homotopy cocartesian squares where pushout squares along w-cofibrations are used in the  $S_{\bullet}$  construction. We denote this construction by  $S'_{\bullet}$ . Theorem 1.1 follows from the comparison of the  $S'_{\bullet}$  construction with the  $S_{\bullet}$  construction in §2.

Let  $\mathcal{P}_m^n$  for  $m \leq n$  denote the full subcategory of  $\mathcal{P}$  consisting of those  $R$ -modules whose homotopy groups  $\pi_q$  are zero for  $q > n$  or  $q < m$ . In this notation, we permit  $m = -\infty$  and/or  $n = \infty$ , so  $\mathcal{P} = \mathcal{P}_{-\infty}^{\infty}$ . Define a w-cofibration in  $\mathcal{P}_m^n$  to be a Hurewicz cofibration whose cofiber is still in  $\mathcal{P}_m^n$ , or, equivalently, a Hurewicz cofibration inducing an injection on  $\pi_n$ . This definition makes  $\mathcal{P}_m^n$  into a Waldhausen category with the usual weak equivalences; it is a "Waldhausen subcategory" of  $\mathcal{P}$  [11, §1.2]. We prove the following theorem in §3 and §4.

**THEOREM 1.2.** *The inclusion  $\mathcal{P}_0^0 \rightarrow \mathcal{P}$  induces a weak equivalence of  $K$ -theory spectra.*

Let  $\mathcal{E}$  denote the exact category of finitely generated left  $\pi_0 R$ -modules (or a skeleton, to obtain a small category). This becomes a Waldhausen category with weak equivalences the isomorphisms and w-cofibrations the injections. Waldhausen's " $S_{\bullet} = Q$ " theorem [11, §1.9] identifies the Waldhausen  $K$ -theory  $K(\mathcal{E})$  as  $K'(\pi_0 R)$ . The functor

$\pi_0: \mathcal{P}_0^0 \rightarrow \mathcal{E}$  is an “exact functor” of Waldhausen categories: it preserves weak equivalences, w-cofibrations and pushouts along w-cofibrations. It follows that  $\pi_0$  induces a map from  $K(\mathcal{P}_0^0)$  to  $K(\mathcal{E}) \simeq K'(\pi_0 R)$ . We prove the following theorem in §4.

**THEOREM 1.3.** *The functor  $\pi_0$  induces a weak equivalence  $K(\mathcal{P}_0^0) \rightarrow K(\mathcal{E})$ .*

Theorems 1.2 and 1.3 together imply the dévissage theorem. For  $K$ -theoretic reasons, we should regard  $K'(\pi_0 R) \rightarrow K(\mathcal{P})$  to be the natural direction of the composite zigzag, as this is compatible with the forgetful functor  $\mathcal{P}_{H\pi_0 R} \rightarrow \mathcal{P}_R$  (induced by pullback along the map  $R \rightarrow H\pi_0 R$ ). Although it appears feasible to construct directly a map of spectra  $K'(\pi_0 R) \rightarrow K(\mathcal{P})$  using a version of the Eilenberg–MacLane bar construction, the technical work required would be unrelated to the arguments in the rest of this paper, and so we have not pursued it.

Next we deduce the localization theorem from the dévissage theorem. Let  $R$  be one of  $ku$ ,  $\ell$ , or  $\ell_p^\wedge$ , and let  $\beta$  denote the appropriate Bott element in  $\pi_* R$  in degree 2 or  $2p-2$ . Then  $R[\beta^{-1}]$  is  $KU$ ,  $L$ , or  $L_p^\wedge$ , respectively. For convenience, let  $Z$  denote  $\pi_0 R$ ; so  $Z = \mathbb{Z}$ ,  $\mathbb{Z}_{(p)}$ , or  $\mathbb{Z}_p^\wedge$ , in the respective cases. Then for  $A = HZ$ ,  $R$ , or  $R[\beta^{-1}]$ , let  $\mathcal{C}_A$  be the category of finite cell  $A$ -modules, which we regard as a Waldhausen category with w-cofibrations the maps that are isomorphic to inclusions of cell subcomplexes, and with weak equivalences the usual weak equivalences, that is, the maps that induce isomorphisms on homotopy groups. As explained in [4], the Waldhausen  $K$ -theory of  $\mathcal{C}_A$  is the algebraic  $K$ -theory of  $A$ . When  $A = HZ$ , the  $K$ -theory spectrum  $K(\mathcal{C}_A)$  is equivalent to Quillen’s  $K$ -theory of the ring  $Z$  [4, Theorem VI.4.3]. In the case of  $A = HZ$  or  $A = R$ , the  $K$ -theory spectrum  $K(\mathcal{C}_A)$  is equivalent to two other reasonable versions of the  $K$ -theory of  $A$ : the  $K$ -theory spectrum defined via the plus construction of “ $BGL(A)$ ” [4, §VI.7] and the  $K$ -theory spectrum defined via the permutative category of wedges of sphere  $A$ -modules [4, §VI.6]. However, when  $A = R[\beta^{-1}]$ , the construction  $K(\mathcal{C}_A)$  and its variants are essentially the only known ways to define  $K(A)$ .

Let  $\mathcal{C}_R[\beta^{-1}]$  denote the Waldhausen category whose underlying category is  $\mathcal{C}_R$  and whose w-cofibrations are the maps that are isomorphic to inclusions of cell subcomplexes, but whose weak equivalences are the maps that induce isomorphisms on homotopy groups after inverting  $\beta$ . The identity functor  $\mathcal{C}_R \rightarrow \mathcal{C}_R[\beta^{-1}]$  is then an exact functor, because the weak equivalences in  $\mathcal{C}_R$  are in particular weak equivalences in  $\mathcal{C}_R[\beta^{-1}]$ . Let  $\mathcal{C}_R^\beta$  be the full subcategory of  $\mathcal{C}_R$  of objects that are weakly equivalent in  $\mathcal{C}_R[\beta^{-1}]$  to the trivial object  $*$ , that is, the finite cell  $R$ -modules whose homotopy groups become zero after inverting  $\beta$ . In other words, we consider  $\mathcal{C}_R$  with two subcategories of weak equivalences, one of which is coarser than the other, and the category of acyclic objects for the coarse weak equivalences. This is the situation in which Waldhausen’s “fibration theorem”

[11, Theorem 1.6.4] applies. The conclusion in this case, restated in terms of the stable category, is the following proposition.

**PROPOSITION 1.4.** *There is a connecting map (of spectra)  $K(\mathcal{C}_R[\beta^{-1}]) \rightarrow \Sigma K(\mathcal{C}_R^\beta)$ , which together with the canonical maps makes the sequence*

$$K(\mathcal{C}_R^\beta) \longrightarrow K(R) \longrightarrow K(\mathcal{C}_R[\beta^{-1}]) \longrightarrow \Sigma K(\mathcal{C}_R^\beta)$$

*a cofiber sequence in the stable category.*

For the proof of the localization theorem, we need to identify  $K(\mathcal{C}_R^\beta)$  as  $K(\pi_0 R)$ ,  $K(\mathcal{C}_R[\beta^{-1}])$  as  $K(R[\beta^{-1}])$ , and the maps as the transfer map and the canonical map.

We begin with the comparison of  $K(\mathcal{C}_R[\beta^{-1}])$  with  $K(R[\beta^{-1}])$ . The localization functor  $R[\beta^{-1}] \wedge_R (-)$  restricts to an exact functor of Waldhausen categories

$$\mathcal{C}_R[\beta^{-1}] \longrightarrow \mathcal{C}_{R[\beta^{-1}]}.$$

Waldhausen developed a general tool for showing that an exact functor induces a weak equivalence of  $K$ -theory spectra, called the “approximation theorem” [11, Theorem 1.6.7]. In the case of a telescopic localization functor like this one, the argument of [10] adapts to show that the hypotheses of Waldhausen’s approximation theorem are satisfied. This then proves the following proposition.

**PROPOSITION 1.5.** *The canonical map  $K(R) \rightarrow K(R[\beta^{-1}])$  factors as the map*

$$K(R) \longrightarrow K(\mathcal{C}_R[\beta^{-1}])$$

*in Proposition 1.4 and a weak equivalence*

$$K(\mathcal{C}_R[\beta^{-1}]) \longrightarrow K(R[\beta^{-1}]).$$

Next we move on to the comparison of  $K(\mathcal{C}_R^\beta)$  and  $K(\pi_0 R)$ . Since finite cell  $R$ -modules that are  $\beta$ -torsion have finitely generated homotopy groups concentrated in a finite range,  $\mathcal{C}_R^\beta$  is a subcategory of  $\mathcal{P}_R$ . Cellular inclusions are Hurewicz cofibrations, so the inclusion  $\mathcal{C}_R^\beta \rightarrow \mathcal{P}_R$  is an exact functor of Waldhausen categories. On the other hand, since the cofiber of  $\beta$  is a model for the Eilenberg–MacLane  $R$ -module  $H\pi_0 R$ , an induction over Postnikov sections implies that every object in  $\mathcal{P}_R$  is weakly equivalent to an object in  $\mathcal{C}_R^\beta$ . Standard arguments (e.g., [4, Theorem VI.2.5] and the Whitehead theorem for cell  $R$ -modules) apply to verify that the inclusion  $\mathcal{C}_R^\beta \rightarrow \mathcal{P}_R$  satisfies the hypotheses of Waldhausen’s approximation theorem [11, Theorem 1.6.7], and thus induces a weak equivalence of  $K$ -theory spectra.

The cofiber sequence of Proposition 1.4 is therefore equivalent to one of the form (cf. [3, Proposition 6.8ff])

$$K(\mathcal{P}) \longrightarrow K(R) \longrightarrow K(R[\beta^{-1}]) \longrightarrow \Sigma K(\mathcal{P}).$$

Applying the dévissage theorem, we get a weak equivalence

$$K(HZ) \simeq K'(Z) \simeq K(\mathcal{P}) \simeq K(\mathcal{C}_R^\beta).$$

To complete the proof of the localization theorem, we need to identify the composite map  $K(HZ) \rightarrow K(R)$  with the transfer map  $K(HZ) \rightarrow K(R)$  induced from the forgetful functor. We begin by reviewing this transfer map.

The transfer map arises because  $HZ$  is weakly equivalent to a finite cell  $R$ -module; as a consequence all finite cell  $HZ$ -modules are weakly equivalent to finite cell  $R$ -modules. To put this into the context of Waldhausen categories and exact functors, let  $\mathcal{M}_R$  (resp.  $\mathcal{M}_{HZ}$ ) be the full subcategory of  $R$ -modules (resp.  $HZ$ -modules) that are weakly equivalent to finite cell modules. We make  $\mathcal{M}_R$  and  $\mathcal{M}_{HZ}$  Waldhausen categories with w-cofibrations the Hurewicz cofibrations and weak equivalences the usual weak equivalences. Then the forgetful functor from  $HZ$ -modules to  $R$ -modules restricts to an exact functor  $\mathcal{M}_{HZ} \rightarrow \mathcal{M}_R$ . The inclusions  $\mathcal{C}_R \rightarrow \mathcal{M}_R$  and  $\mathcal{C}_{HZ} \rightarrow \mathcal{M}_{HZ}$  are exact functors. Again, a standard argument [4, Proposition VI.3.5] with Waldhausen's approximation theorem shows that these inclusions induce weak equivalences of  $K$ -theory spectra. The transfer map  $K(HZ) \rightarrow K(R)$  is the induced map

$$K(HZ) \simeq K(\mathcal{M}_{HZ}) \longrightarrow K(\mathcal{M}_R) \simeq K(R).$$

The Waldhausen category  $\mathcal{P}_R$  is a Waldhausen subcategory of  $\mathcal{M}_R$ , and the exact functor  $\mathcal{C}_R^\beta \rightarrow \mathcal{M}_R$  lands in  $\mathcal{P}_R$  as does the exact functor  $\mathcal{M}_{HZ} \rightarrow \mathcal{M}_R$ . We therefore obtain the following commutative diagram of exact functors:

$$\begin{array}{ccc} \mathcal{C}_{HZ} & \xrightarrow{\quad} & \mathcal{C}_R^\beta \xrightarrow{\quad} \mathcal{C}_R \\ \downarrow & & \downarrow \\ \mathcal{M}_{HZ} & \longrightarrow & \mathcal{P}_R \longrightarrow \mathcal{M}_R. \end{array}$$

This induces the following commutative diagram of  $K$ -theory spectra, with the arrows marked “ $\simeq$ ” weak equivalences (see also Remark 1.7):

$$\begin{array}{ccccc} K(HZ) & & K(\mathcal{C}_R^\beta) & \longrightarrow & K(R) \\ \simeq \downarrow & & \simeq \downarrow & & \downarrow \simeq \\ K(\mathcal{M}_{HZ}) & \xrightarrow{\quad \simeq \quad} & K(\mathcal{P}_R) & \longrightarrow & K(\mathcal{M}_R). \end{array}$$

Here the lower left map  $K(\mathcal{M}_{HZ}) \rightarrow K(\mathcal{P}_R)$  is a weak equivalence by the dévissage theorem above, since  $\mathcal{M}_{HZ} = \mathcal{P}_{HZ}$ . This proves the following proposition.

**PROPOSITION 1.6.** *The transfer map  $K(HZ) \rightarrow K(R)$  factors in the stable category as a weak equivalence  $K(HZ) \rightarrow K(\mathcal{C}_R^\beta)$  and the map  $K(\mathcal{C}_R^\beta) \rightarrow K(R)$  in Proposition 1.4.*

Finally, the localization theorem of the introduction follows immediately from Propositions 1.4, 1.5 and 1.6. We close this section with a remark on smallness.

*Remark 1.7.* The Waldhausen categories  $\mathcal{P}_R$  and  $\mathcal{M}_R$  discussed above can be replaced by small Waldhausen categories with equivalent  $K$ -theory spectra. For any cardinal  $k$ , let  $\mathcal{P}_k$  and  $\mathcal{M}_k$  be the full subcategories of  $\mathcal{P}_R$  and  $\mathcal{M}_R$  (respectively) consisting of those objects  $M$  such that the underlying sets of the underlying spaces of the underlying prespectrum of  $M$  are subsets of the power-set of  $k$ . Then  $\mathcal{P}_k$  is a small category. When  $k$  is bigger than the continuum and the cardinality of the underlying sets of  $R$ , then  $\mathcal{P}_k$  and  $\mathcal{M}_k$  contain a representative of every weak equivalence class in  $\mathcal{P}_R$  and  $\mathcal{M}_R$ , respectively. Furthermore,  $\mathcal{P}_R$  and  $\mathcal{M}_R$  are closed under pushouts along Hurewicz cofibrations and pullbacks along Hurewicz fibrations, and closed up to natural isomorphism under the functors  $(-) \wedge X$  and  $F(X, -)$  for finite cell complexes  $X$ .

Let  $\mathcal{CP}_k$  be the Waldhausen category whose objects are the objects  $M$  of  $\mathcal{P}_k$  together with the structure of a cell complex on  $M$ , whose morphisms are the maps of  $R$ -modules, whose w-cofibrations are the maps that are isomorphic to inclusions of cell subcomplexes, and whose weak equivalences are the usual weak equivalences. Let  $\mathcal{CM}_k$  be the analogous category for  $\mathcal{M}_k$ . When  $k$  is bigger than the continuum and the cardinality of the underlying sets of  $R$ , then, for any cardinal  $\lambda \geq k$ , the functors  $\mathcal{CP}_k \rightarrow \mathcal{P}_\lambda$  and  $\mathcal{CM}_k \rightarrow \mathcal{M}_\lambda$  are exact and satisfy the hypotheses of Waldhausen's approximation theorem. It follows that these functors induce equivalences of  $K$ -theory spectra. The reader unwilling to consider the  $K$ -theory of Waldhausen categories that are not small can therefore use  $K(\mathcal{P}_k)$  in place of  $K(\mathcal{P}_R)$ ,  $K(\mathcal{M}_k)$  in place of  $K(\mathcal{M}_R)$ , etc.

## 2. The $S'_\bullet$ construction and the proof of Theorem 1.1

For the Waldhausen categories  $\mathcal{P}_m^n$ , the condition of being a w-cofibration consists of both a point-set requirement (HEP condition) and a homotopical requirement (injectivity on  $\pi_n$ ). It is convenient for the arguments in the next section to separate these two requirements. We do that in this section by describing a variant  $S'_\bullet$  of the  $S_\bullet$  construction defined in terms of “homotopy cocartesian” squares instead of pushouts of w-cofibrations. For a large class of Waldhausen categories including  $\mathcal{P}$ ,  $\mathcal{P}^{\text{op}}$ , and the categories  $\mathcal{P}_m^n$ , the  $S'_\bullet$  construction is equivalent to the  $S_\bullet$  construction and can therefore be used in its place

to construct algebraic  $K$ -theory. In fact, since the notions of homotopy cocartesian and homotopy cartesian agree in  $\mathcal{P}$ , Theorem 1.1, which compares the algebraic  $K$ -theory of  $\mathcal{P}$  (defined in terms of cofibrations) with that of  $\mathcal{P}^{\text{op}}$  (defined in terms of fibrations), then follows as an easy consequence.

The hypothesis on a Waldhausen category we use is a weak version of “functorial factorization”. The Waldhausen category  $\mathcal{C}$  admits *functorial factorization* when any map  $f: A \rightarrow B$  in  $\mathcal{C}$  factors as a w-cofibration followed by a weak equivalence

$$A \xrightarrow{\quad f \quad} Tf \xrightarrow{\quad \simeq \quad} B,$$

functorially in  $f$  in the category  $\text{Ar } \mathcal{C}$  of arrows in  $\mathcal{C}$ . In other words, given the map  $\phi$  of arrows on the left (i.e., the commuting diagram),

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ a \downarrow & \phi & \downarrow b \\ A' & \xrightarrow{f'} & B' \end{array} \quad \begin{array}{ccc} A & \xrightarrow{\quad} Tf & \xrightarrow{\quad \simeq \quad} B \\ a \downarrow & T\phi \downarrow & \downarrow b \\ A' & \xrightarrow{\quad} Tf' & \xrightarrow{\quad \simeq \quad} B' \end{array}$$

we have a map  $T\phi$  that makes the diagram on the right commute and that satisfies the usual identity and composition relations,  $T \text{id}_f = \text{id}_{Tf}$  and  $T(\phi' \circ \phi) = T\phi' \circ T\phi$ . For example, the Waldhausen categories  $\mathcal{P}$  and  $\mathcal{P}^{\text{op}}$  admit functorial factorizations using the usual mapping cylinder and mapping path-space constructions

$$Mf = (A \wedge I_+) \cup_A B \quad \text{for } \mathcal{P}, \quad Nf = B^{I_+} \times_B A \quad \text{for } \mathcal{P}^{\text{op}}. \quad (2.1)$$

Functorial factorization generalizes Waldhausen’s notion of “cylinder functor satisfying the cylinder axiom”, but is still not quite general enough to apply to the Waldhausen categories  $\mathcal{P}_m^n$  for  $n < \infty$ , where a map is weakly equivalent to a w-cofibration only when it is injective on  $\pi_n$ . This leads to the following definition.

*Definition 2.2.* Let  $\mathcal{C}$  be a Waldhausen category. Define a map  $f: A \rightarrow B$  in  $\mathcal{C}$  to be a *weak w-cofibration* if it is weakly equivalent in  $\text{Ar } \mathcal{C}$  (by a zigzag) to a w-cofibration. We say that  $\mathcal{C}$  admits *functorial factorization of weak w-cofibrations* (FFWC) when weak w-cofibrations can be factored functorially (in  $\text{Ar } \mathcal{C}$ ) as a w-cofibration followed by a weak equivalence.

Recall that a full subcategory  $\mathcal{B}$  of a Waldhausen category  $\mathcal{C}$  is called a *Waldhausen subcategory* when it forms a Waldhausen category with weak equivalences the weak equivalences of  $\mathcal{C}$ , and with w-cofibrations the w-cofibrations  $A \rightarrow B$  in  $\mathcal{C}$  (between objects  $A$

and  $B$  of  $\mathcal{B}$ ) whose quotient  $B/A = B \cup_A *$  is in  $\mathcal{B}$ . In particular, it is straightforward to check that a full subcategory of a Waldhausen category that is closed under extensions is a Waldhausen subcategory; examples include the subcategories  $\mathcal{P}_m^n$  of  $\mathcal{P}$ . We say that the Waldhausen subcategory  $\mathcal{B}$  is *closed* if every object of  $\mathcal{C}$  weakly equivalent to an object of  $\mathcal{B}$  is an object of  $\mathcal{B}$ . The advantage of the hypothesis of FFWC over the hypothesis of functorial factorization is the following proposition.

**PROPOSITION 2.3.** *If  $\mathcal{B}$  is a closed Waldhausen subcategory of a Waldhausen category  $\mathcal{C}$  that admits FFWC, then  $\mathcal{B}$  admits FFWC. Moreover, a weak w-cofibration  $f: A \rightarrow B$  in  $\mathcal{C}$  between objects in  $\mathcal{B}$  is a weak w-cofibration in  $\mathcal{B}$  if and only if the map  $A \rightarrow Tf$  in its factorization in  $\mathcal{C}$  is a w-cofibration in  $\mathcal{B}$ .*

*Proof.* Let  $f: A \rightarrow B$  be a weak w-cofibration in  $\mathcal{C}$  between objects in  $\mathcal{B}$ . Then  $f$  is weakly equivalent by a zigzag in  $\mathcal{B}$  to a w-cofibration  $f': A' \rightarrow B'$  in  $\mathcal{C}$ ,

$$\begin{array}{ccccccc} A' & \xrightarrow{\simeq} & A_1 & \leftarrow \simeq & \cdots & \leftarrow \simeq & A_n & \xrightarrow{\simeq} & A \\ f' \downarrow & & f_1 \downarrow & & & & f_n \downarrow & & f \downarrow \\ B' & \xrightarrow{\simeq} & B_1 & \leftarrow \simeq & \cdots & \leftarrow \simeq & B_n & \xrightarrow{\simeq} & B. \end{array}$$

Applying functorial factorization in  $\mathcal{C}$ , we get weak equivalences

$$B'/A' \xleftarrow{\simeq} Tf'/A' \xrightarrow{\simeq} Tf_1/A_1 \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} Tf_n/A_n \xrightarrow{\simeq} Tf/A,$$

which imply that the map  $A \rightarrow Tf$  is a w-cofibration in  $\mathcal{B}$  if and only if  $f: A \rightarrow B$  is a weak w-cofibration in  $\mathcal{B}$ .  $\square$

In general, the weak w-cofibrations do not necessarily form a subcategory of  $\mathcal{C}$ ; the composition of two weak w-cofibrations might not be a weak w-cofibration. However, in the presence of FFWC the weak w-cofibrations are well-behaved.

**PROPOSITION 2.4.** *Let  $\mathcal{C}$  be a Waldhausen category that admits FFWC. If  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are weak w-cofibrations in  $\mathcal{C}$ , then  $g \circ f: A \rightarrow C$  is a weak w-cofibration in  $\mathcal{C}$ .*

*Proof.* Applying functorial factorization in  $\mathcal{C}$ , we can factor  $f$  and  $g$  as the composites

$$A \longrightarrow Tf \xrightarrow{\simeq} B \quad \text{and} \quad B \longrightarrow Tg \xrightarrow{\simeq} C.$$

The composite map  $h: Tf \rightarrow B \rightarrow Tg$  is a weak w-cofibration, as it is weakly equivalent in  $\text{Ar } \mathcal{C}$  to  $g$ . We can therefore apply the factorization functor to  $h$  to obtain a w-cofibration  $Tf \rightarrow Th$  and a weak equivalence  $Th \rightarrow Tg$ . The composite w-cofibration  $A \rightarrow Tf \rightarrow Th$  is then weakly equivalent to  $g \circ f$ .  $\square$

A square diagram in a Waldhausen category (for example) is called *homotopy cocartesian* if it is weakly equivalent (by a zigzag) to a pushout square where one of the parallel sets of arrows consists of w-cofibrations. The corresponding set of parallel arrows in the original square then consists of weak w-cofibrations. When a Waldhausen category admits FFWC, there is a good criterion in terms of the factorization functor  $T$  for detecting homotopy cocartesian squares. We state it for a Waldhausen category that is *saturated*, i.e., one whose weak equivalences satisfy the “two-out-of-three property” (see, for example, [11, §1.2]). The proof is similar to that of Proposition 2.3.

**PROPOSITION 2.5.** *Let  $\mathcal{C}$  be a saturated Waldhausen category admitting FFWC with functor  $T$ . A commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

*with  $f$  a weak w-cofibration is homotopy cocartesian if and only if the map  $Tf \cup_A C \rightarrow D$  is a weak equivalence.*

Likewise, for saturated Waldhausen categories admitting FFWC, homotopy cocartesian squares have many of the usual expected properties. We summarize the ones we need in the following proposition; its proof is a straightforward application of the previous proposition.

**PROPOSITION 2.6.** *Let  $\mathcal{C}$  be a saturated Waldhausen category admitting FFWC.*

(i) *Given a commutative cube*

$$\begin{array}{ccccc} A' & \longrightarrow & B' & & \\ \downarrow & \searrow & \downarrow & \nearrow & \\ A & \longrightarrow & B & & \\ \downarrow & & \downarrow & & \\ C' & \longrightarrow & D' & \longrightarrow & \\ \downarrow & \searrow & \downarrow & \nearrow & \\ C & \longrightarrow & D & & \end{array}$$

*with the faces  $ABCD$  and  $A'B'C'D'$  homotopy cocartesian, if the maps  $A' \rightarrow A$ ,  $B' \rightarrow B$ , and  $C' \rightarrow C$  are weak equivalences, then the map  $D' \rightarrow D$  is a weak equivalence.*

(ii) *Given a commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & D & \longrightarrow & Y \end{array}$$

with the square  $ABCD$  homotopy cocartesian, if either  $A \rightarrow C$  is a weak w-cofibration or both  $A \rightarrow B$  and  $B \rightarrow X$  are weak w-cofibrations, then the square  $AXCY$  is homotopy cocartesian if and only if the square  $BXDY$  is homotopy cocartesian.

We use the concept of weak w-cofibration and the previous propositions to build a homotopical variant of the  $S_\bullet$  construction. First, we recall the  $S_\bullet$  construction in detail. Waldhausen's  $S_\bullet$  construction produces a simplicial Waldhausen category  $S_\bullet \mathcal{C}$  from a Waldhausen category  $\mathcal{C}$  and is defined as follows. Let  $\text{Ar}[n]$  denote the category with objects  $(i, j)$ , for  $0 \leq i \leq j \leq n$ , and a unique map  $(i, j) \rightarrow (i', j')$ , for  $i \leq i'$  and  $j \leq j'$ . Then  $S_n \mathcal{C}$  is defined to be the full subcategory of the category of functors  $A: \text{Ar}[n] \rightarrow \mathcal{C}$  (where we write  $A_{i,j}$  for  $A(i, j)$ ) such that the following conditions are satisfied:

- $A_{i,i} = *$  for all  $i$ ;
- the map  $A_{i,j} \rightarrow A_{i,k}$  is a w-cofibration for all  $i \leq j \leq k$ ;
- the diagram

$$\begin{array}{ccc} A_{i,j} & \longrightarrow & A_{i,k} \\ \downarrow & & \downarrow \\ A_{j,j} & \longrightarrow & A_{j,k} \end{array}$$

is a pushout square for all  $i \leq j \leq k$ .

The last two conditions can be simplified to the hypothesis that each map

$$A_{0,j} \longrightarrow A_{0,j+1}$$

is a w-cofibration and that the induced maps  $A_{0,j}/A_{0,i} \rightarrow A_{i,j}$  are isomorphisms. This becomes a Waldhausen category by defining a map  $A \rightarrow B$  to be a weak equivalence when each  $A_{i,j} \rightarrow B_{i,j}$  is a weak equivalence in  $\mathcal{C}$ , and to be a w-cofibration when each  $A_{i,j} \rightarrow B_{i,j}$  and each induced map  $A_{i,k} \cup_{A_{i,j}} B_{i,j} \rightarrow B_{i,k}$  is a w-cofibration in  $\mathcal{C}$ .

*Definition 2.7.* Let  $\mathcal{C}$  be a saturated Waldhausen category that admits FFWC. Define  $S'_n \mathcal{C}$  to be the full subcategory of functors  $A: \text{Ar}[n] \rightarrow \mathcal{C}$  such that the following conditions are satisfied:

- the initial map  $* \rightarrow A_{i,i}$  is a weak equivalence for all  $i$ ;
- the map  $A_{i,j} \rightarrow A_{i,k}$  is a weak w-cofibration for all  $i \leq j \leq k$ ;
- the diagram

$$\begin{array}{ccc} A_{i,j} & \longrightarrow & A_{i,k} \\ \downarrow & & \downarrow \\ A_{j,j} & \longrightarrow & A_{j,k} \end{array}$$

is a homotopy cocartesian square for all  $i \leq j \leq k$ .

We define a map  $A \rightarrow B$  to be a weak equivalence when each  $A_{i,j} \rightarrow B_{i,j}$  is a weak equivalence in  $\mathcal{C}$ , and to be a w-cofibration when each  $A_{i,j} \rightarrow B_{i,j}$  is a w-cofibration in  $\mathcal{C}$  and each induced map  $A_{i,k} \cup_{A_{i,j}} B_{i,j} \rightarrow B_{i,k}$  is a weak w-cofibration in  $\mathcal{C}$ .

The following theorem in particular allows us to iterate the  $S'_\bullet$  construction.

**THEOREM 2.8.** *Let  $\mathcal{C}$  be a saturated Waldhausen category that admits FFWC. Then  $S'_n \mathcal{C}$  is also a saturated Waldhausen category that admits FFWC.*

*Proof.* The proof that  $S'_n \mathcal{C}$  is a saturated Waldhausen category follows the same outline as the proof that  $S_n \mathcal{C}$  is a saturated Waldhausen category, substituting Propositions 2.5 and 2.6 for the homotopy cartesian squares in place of the usual arguments for pushout squares. To show that  $S'_n \mathcal{C}$  admits FFWC, we construct a factorization functor  $T$  as follows. Given a weak w-cofibration  $f: A \rightarrow B$  in  $S'_n \mathcal{C}$ , let  $(Tf)_{i,j} = T(f_{i,j})$ . This then factors  $f$  as a map  $A \rightarrow Tf$  followed by a weak equivalence  $Tf \rightarrow B$ ; we need to check that the map  $A \rightarrow Tf$  is a w-cofibration. Since, by construction, the maps  $A_{i,j} \rightarrow Tf_{i,j}$  are w-cofibrations, we just need to check that the maps  $A_{i,k} \cup_{A_{i,j}} Tf_{i,j} \rightarrow Tf_{i,k}$  are weak w-cofibrations. Since, by hypothesis,  $f$  is a weak w-cofibration in  $S'_n \mathcal{C}$ , it is weakly equivalent (by a zigzag) to a w-cofibration  $f': A' \rightarrow B'$  in  $S'_n \mathcal{C}$ . It follows that the maps

$$A_{i,k} \cup_{A_{i,j}} Tf_{i,j} \longrightarrow Tf_{i,k} \quad \text{and} \quad A'_{i,k} \cup_{A'_{i,j}} Tf'_{i,j} \longrightarrow Tf'_{i,k}$$

are weakly equivalent (by the corresponding zigzag). Since  $A'_{i,j} \rightarrow B'_{i,j}$  is a w-cofibration, the canonical map

$$\begin{array}{ccc} A'_{i,k} \cup_{A'_{i,j}} Tf'_{i,j} & \xrightarrow{\cong} & A'_{i,k} \cup_{A'_{i,j}} B'_{i,j} \\ \downarrow & & \downarrow \\ Tf'_{i,k} & \xrightarrow{\cong} & B'_{i,k} \end{array}$$

is a weak equivalence. The map  $A'_{i,k} \cup_{A'_{i,j}} B'_{i,j} \rightarrow B'_{i,k}$  is a weak w-cofibration by hypothesis, and so the maps  $A'_{i,k} \cup_{A'_{i,j}} Tf'_{i,j} \rightarrow Tf'_{i,k}$  and  $A_{i,k} \cup_{A_{i,j}} Tf_{i,j} \rightarrow Tf_{i,k}$  are therefore weak w-cofibrations.  $\square$

Both  $S_\bullet \mathcal{C}$  and  $S'_\bullet \mathcal{C}$  become simplicial Waldhausen categories with face map  $\partial_i$  deleting the  $i$ th row and column, and degeneracy map  $s_i$  repeating the  $i$ th row and column. For each  $n$ , we denote the nerve of the category of weak equivalences in  $S_n \mathcal{C}$  by  $w_* S_n \mathcal{C}$ . As  $n$  varies,  $w_* S_\bullet \mathcal{C}$  assembles into a bisimplicial set, and we denote the geometric realization by  $|w_* S_\bullet \mathcal{C}|$ . By definition, the algebraic  $K$ -theory space of  $\mathcal{C}$  is  $\Omega |w_* S_\bullet \mathcal{C}|$ . The algebraic  $K$ -theory spectrum is obtained by iterating the  $S_\bullet$  construction: its  $n$ th space is  $|w_* S^{(n)} \mathcal{C}|$ . The following theorem therefore implies that algebraic  $K$ -theory can be constructed from the  $S'_\bullet$  construction.

**THEOREM 2.9.** *Let  $\mathcal{C}$  be a saturated Waldhausen category that admits FFWC. The inclusion of  $S_\bullet \mathcal{C}$  in  $S'_\bullet \mathcal{C}$  is an exact simplicial functor and the induced map of bisimplicial sets  $w_\bullet S_\bullet \mathcal{C} \rightarrow w_\bullet S'_\bullet \mathcal{C}$  is a weak equivalence.*

*Proof.* The fact that  $S_\bullet \mathcal{C} \rightarrow S'_\bullet \mathcal{C}$  is an exact simplicial functor is clear from the definition of the simplicial structure and the weak equivalences and w-cofibrations in each category. To see that it induces a weak equivalence on nerves, it suffices to show that for each  $n$ , the map  $w_\bullet S_n \mathcal{C} \rightarrow w_\bullet S'_n \mathcal{C}$  is a weak equivalence. We do this by constructing a functor  $\Phi: S'_n \mathcal{C} \rightarrow S_n \mathcal{C}$  and natural weak equivalences relating  $\Phi$  and the identity in both  $S'_n \mathcal{C}$  and  $S_n \mathcal{C}$ . In  $S'_n \mathcal{C}$ , we construct a zigzag of natural weak equivalences of the form

$$\Phi \xleftarrow[\simeq]{q} \Theta \xrightarrow[\simeq]{\varepsilon} \text{Id},$$

and in  $S_n \mathcal{C}$ , we construct a weak equivalence  $\Phi \rightarrow \text{Id}$ .

We define the functor  $\Theta$  inductively as follows. Given an object  $A$  in  $S'_n \mathcal{C}$ , let  $\Theta_{0,0}A = A_{0,0}$  and let  $\varepsilon_{0,0}: \Theta_{0,0}A \rightarrow A_{0,0}$  be the identity map. Given  $\Theta_{0,j}A$  and

$$\varepsilon_{0,j}: \Theta_{0,j}A \longrightarrow A_{0,j},$$

define  $\Theta_{0,j+1}A$  to be  $Tf$  and  $\varepsilon_{0,j+1}: Tf \rightarrow A_{0,j+1}$  to be the canonical weak equivalence, where  $f: \Theta_{0,j}A \rightarrow A_{0,j+1}$  is the composite of  $\varepsilon_{0,j}$  with the map  $A_{0,j} \rightarrow A_{0,j+1}$  in the structure of the functor  $A$ . As a consequence of the construction, we have w-cofibrations  $\Theta_{0,j}A \rightarrow \Theta_{0,j+1}A$ . Next we construct the diagonal objects  $\Theta_{i,i}A$ ; given  $\Theta_{j,j}A$  and

$$\varepsilon_{j,j}: \Theta_{j,j}A \longrightarrow A_{j,j},$$

define  $\Theta_{j+1,j+1}A$  to be  $Tf$  and  $\varepsilon_{j+1,j+1}: Tf \rightarrow A_{j+1,j+1}$  to be the canonical weak equivalence, for  $f: \Theta_{j,j}A \rightarrow A_{j,j} \rightarrow A_{j+1,j+1}$ . Finally, for  $0 < i < j$ , let

$$\Theta_{i,j}A = \Theta_{0,j}A \cup_{\Theta_{0,i}A} \Theta_{i,i}A$$

and let  $\varepsilon_{i,j}: \Theta_{i,j}A \rightarrow A_{i,j}$  be the map induced by the universal property of the pushout. By Proposition 2.6, the maps  $\varepsilon_{i,j}$  are weak equivalences. Thus,  $\Theta$  defines a functor from  $S'_n \mathcal{C}$  to itself, and  $\varepsilon$  defines a natural transformation from  $\Theta$  to the identity.

We define  $\Phi$  and  $q$  by setting  $\Phi_{i,i}A = *$ ,

$$\Phi_{i,j}A = \Theta_{i,j}A / \Theta A_{i,i},$$

and letting  $q_{i,j}: \Theta_{i,j}A \rightarrow \Phi_{i,j}A$  be the quotient map. Since the initial map  $* \rightarrow \Theta_{i,i}A$  is a weak equivalence, so is the final map  $\Theta_{i,i}A \rightarrow *$ , and thus each  $q_{i,j}$  is a weak equivalence.

The construction of  $\Theta_{i,j}A$  as a pushout for  $0 < i < j$  implies that the induced map from the quotient  $\Phi_{0,j}A/\Phi_{0,i}A$  to  $\Phi_{i,j}A$  is an isomorphism. It follows that  $\Phi$  defines a functor from  $S'_n\mathcal{C}$  to  $S_n\mathcal{C}$  and a natural weak equivalence  $q$  in  $S'_n\mathcal{C}$  from  $\Theta$  to  $\Phi$ .

Finally, if  $A$  is an object of  $S_n\mathcal{C}$ , then  $A_{i,i} = *$  for all  $i$ , and the map  $\varepsilon: \Theta A \rightarrow A$  factors through the quotient  $\Phi A$ . This defines a natural weak equivalence from  $\Phi$  to the identity in  $S_n\mathcal{C}$ .  $\square$

Theorem 1.1 is an easy consequence of Theorem 2.9: because a square in  $\mathcal{P}$  is homotopy cocartesian if and only if it is homotopy cartesian (homotopy cocartesian in  $\mathcal{P}^{\text{op}}$ ), the categories  $S'_n\mathcal{P}$  and  $S'_n\mathcal{P}^{\text{op}}$  are (contravariantly) isomorphic, for each  $n$ , by the isomorphism that renames the diagram by the involution  $(i, j) \mapsto (n-j, n-i)$ . The bisimplicial sets  $w_*S'_*\mathcal{P}$  and  $w_*S'_*\mathcal{P}^{\text{op}}$  then only differ in the orientation of the simplices, and the spaces  $\Omega|w_*S'_*\mathcal{P}|$  and  $\Omega|w_*S'_*\mathcal{P}^{\text{op}}|$  are homeomorphic. Comparing iterates of the  $S'_*$  construction and the suspension maps, we obtain an equivalence of  $K$ -theory spectra.

### 3. A reduction of Theorem 1.2

The purpose of this section is to reduce Theorem 1.2 to Theorem 3.10 below. That theorem is similar in spirit to Theorem 1.3, and both are proved with closely related arguments in the next section.

We begin by reducing to the subcategory of connective objects.

**LEMMA 3.1.** *The inclusion  $\mathcal{P}_0^\infty \rightarrow \mathcal{P}$  induces an equivalence of  $K$ -theory.*

*Proof.* Since  $\mathcal{P} = \text{colim}_m \mathcal{P}_{-m}^\infty$  and  $S_*\mathcal{P} = \text{colim}_m S_*\mathcal{P}_{-m}^\infty$ , it suffices to show that the inclusions  $\mathcal{P}_{-m}^\infty \rightarrow \mathcal{P}_{-m-1}^\infty$  induce equivalences of  $K$ -theory. Suspension gives an exact functor  $\mathcal{P}_{-m-1}^\infty \rightarrow \mathcal{P}_{-m}^\infty$ . The composite endomorphisms on  $\mathcal{P}_{-m}^\infty$  and  $\mathcal{P}_{-m-1}^\infty$  are the suspension functors, which on the  $K$ -theory spectra induce multiplication by  $-1$  [11, Proposition 1.6.2], and in particular are weak equivalences.  $\square$

We also have  $\mathcal{P}_0^\infty = \text{colim}_n \mathcal{P}_0^n$  and  $S_*\mathcal{P}_0^\infty = \text{colim}_n S_*\mathcal{P}_0^n$ , which give the following proposition.

**PROPOSITION 3.2.** *The spectrum  $K\mathcal{P}_0^\infty$  is equivalent to the telescope of the sequence of maps  $K\mathcal{P}_0^0 \rightarrow \dots \rightarrow K\mathcal{P}_0^n \rightarrow K\mathcal{P}_0^{n+1} \rightarrow \dots$ .*

The proof of Theorem 1.2 will then be completed by showing that the maps

$$\mathcal{P}_0^n \longrightarrow \mathcal{P}_0^{n+1}$$

induce weak equivalences of  $K$ -theory spectra, which we do by studying the cofiber. According to Waldhausen [11, Corollary 1.5.6], for a Waldhausen category  $\mathcal{C}$  and a

Waldhausen subcategory  $\mathcal{B}$ , the spectrum-level cofiber of  $K\mathcal{B} \rightarrow K\mathcal{C}$  is the  $K$ -theory spectrum of the simplicial Waldhausen category  $F_\bullet(\mathcal{C}, \mathcal{B})$ , defined as follows. The Waldhausen category  $F_q(\mathcal{C}, \mathcal{B})$  has as objects the sequences of  $q$  composable w-cofibrations in  $\mathcal{C}$

$$C_0 \longrightarrow \cdots \longrightarrow C_i \longrightarrow \cdots \longrightarrow C_q$$

such that the quotients  $C_j/C_i$  are in  $\mathcal{B}$  for all  $0 \leq i < j \leq q$ , and has commutative diagrams as morphisms. A map  $C \rightarrow D$  is a weak equivalence when the maps  $C_i \rightarrow D_i$  are weak equivalences in  $\mathcal{C}$  for all  $i$ , and is a w-cofibration when the maps  $C_i \rightarrow D_i$  are w-cofibrations for all  $i$  and the maps  $C_j \cup_{C_i} D_i \rightarrow D_j$  are w-cofibrations for all  $i < j$ . The face map  $\partial_i$  deletes the object  $C_i$  and uses the composite w-cofibration  $C_{i-1} \rightarrow C_{i+1}$ , and the degeneracy map  $s_i$  repeats the object  $C_i$ , inserting the identity map. We use the following variant of this construction.

*Definition 3.3.* For a saturated Waldhausen category  $\mathcal{C}$  that admits FFWC and a closed Waldhausen subcategory  $\mathcal{B}$ , let  $F'_q(\mathcal{C}, \mathcal{B})$  denote the category that has as objects the sequences of  $q$  composable weak w-cofibrations in  $\mathcal{C}$ ,

$$C_0 \longrightarrow \cdots \longrightarrow C_i \longrightarrow \cdots \longrightarrow C_q,$$

such that for all  $0 \leq i < j \leq q$  there exist an object  $B_{i,j}$  in  $\mathcal{B}$  and a map  $C_j \rightarrow B_{i,j}$  for which the square

$$\begin{array}{ccc} C_i & \longrightarrow & C_j \\ \downarrow & & \downarrow \\ * & \longrightarrow & B_{i,j} \end{array}$$

commutes and is homotopy cocartesian. A morphism  $C \rightarrow D$  in  $F'_q(\mathcal{C}, \mathcal{B})$  is a commutative diagram

$$\begin{array}{ccccccc} C_0 & \longrightarrow & \cdots & \longrightarrow & C_i & \longrightarrow & \cdots \longrightarrow C_q \\ \downarrow & & & & \downarrow & & \downarrow \\ D_0 & \longrightarrow & \cdots & \longrightarrow & D_i & \longrightarrow & \cdots \longrightarrow D_q. \end{array}$$

We make  $F'_q(\mathcal{C}, \mathcal{B})$  into a Waldhausen category by declaring a map  $C \rightarrow D$  to be a weak equivalence when the maps  $C_i \rightarrow D_i$  are weak equivalences in  $\mathcal{C}$  for all  $i$ , and to be a w-cofibration when the maps  $C_i \rightarrow D_i$  are w-cofibrations for all  $i$  and the maps

$$C_j \cup_{C_i} D_i \longrightarrow D_j$$

are weak w-cofibrations for all  $i < j$ .

Note that in the above definition, the choice of “ $B_{i,j}$ ” in  $\mathcal{B}$  is not part of the structure of the object  $C$ . The hypothesis involving the existence of the objects  $B_{i,j}$  is a precise way of making sense of the condition that the “homotopy cofiber” of  $C_i \rightarrow C_j$  lies in  $\mathcal{B}$  for an arbitrary Waldhausen category  $\mathcal{C}$ . We assemble  $F'_*(\mathcal{C}, \mathcal{B})$  into a simplicial Waldhausen category just as  $F_*(\mathcal{C}, \mathcal{B})$  above: the face map  $\partial_i$  deletes the object  $C_i$  and uses the composite weak w-cofibration  $C_{i-1} \rightarrow C_{i+1}$ ; the degeneracy map  $s_i$  repeats the object  $C_i$ , inserting the identity map. The inclusion  $F_*(\mathcal{C}, \mathcal{B}) \rightarrow F'_*(\mathcal{C}, \mathcal{B})$  is an exact simplicial functor.

We have stated the definitions of  $F$  and  $F'$  for general  $\mathcal{C}$  and  $\mathcal{B}$ , rather than just for  $\mathcal{P}_0^{n+1}$  and  $\mathcal{P}_0^n$ , in order to apply a standard trick of Waldhausen [11] of commuting these constructions past another construction, in this case the  $S'_*$  construction. When  $\mathcal{C}$  is a saturated Waldhausen category that admits FFWC, the Waldhausen category  $S'_p F_q(\mathcal{C}, \mathcal{B})$  is isomorphic to  $F_q(S'_p \mathcal{C}, S'_p \mathcal{B})$  and the Waldhausen category  $S'_p F'_q(\mathcal{C}, \mathcal{B})$  is isomorphic to  $F'_q(S'_p \mathcal{C}, S'_p \mathcal{B})$ . The argument for Theorem 2.9 also shows that the inclusion  $w_* F_q(\mathcal{C}, \mathcal{B}) \rightarrow w_* F'_q(\mathcal{C}, \mathcal{B})$  is a homotopy equivalence of simplicial sets for all  $q$ . Since  $S'_p \mathcal{C}$  is also a saturated Waldhausen category that admits FFWC and  $S'_p \mathcal{B}$  is a closed Waldhausen subcategory of  $S'_p \mathcal{C}$ , it follows that the inclusion  $w_* F_q(S'_p \mathcal{C}, S'_p \mathcal{B}) \rightarrow w_* F'_q(S'_p \mathcal{C}, S'_p \mathcal{B})$  is a homotopy equivalence of simplicial sets for all  $p$  and  $q$ . This proves the following proposition.

**PROPOSITION 3.4.** *Let  $\mathcal{C}$  be a saturated Waldhausen category that admits FFWC and  $\mathcal{B}$  be a closed Waldhausen subcategory. Then the inclusion  $F_*(\mathcal{C}, \mathcal{B}) \rightarrow F'_*(\mathcal{C}, \mathcal{B})$  induces a weak equivalence of K-theory spectra. Thus, we have a cofibration sequence*

$$K\mathcal{B} \longrightarrow K\mathcal{C} \longrightarrow KF'_*(\mathcal{C}, \mathcal{B}) \longrightarrow \Sigma K\mathcal{B}$$

*in the stable category.*

The remainder of the proof of Theorem 1.2 is to show that  $w_* S'_* F'_*(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n)$  is weakly contractible for all  $n \geq 0$ . The first step is to eliminate one of the simplicial directions.

**Definition 3.5.** Let  $\mathcal{U}_{n+1}$  be the subcategory of  $\mathcal{P}_0^{n+1}$  consisting of all objects and those maps that induce an isomorphism on  $\pi_{n+1}$  and an injection in  $\pi_n$ . Let  $uS'_p \mathcal{P}_0^{n+1}$  denote the subcategory of  $S'_p \mathcal{P}_0^{n+1}$  consisting of all objects and those maps  $f$  in  $S'_p \mathcal{P}_0^{n+1}$  such that  $f_{i,j}$  is in  $\mathcal{U}_{n+1}$  for all  $0 \leq i < j \leq p$ .

To avoid a possible point of confusion, note that  $\mathcal{U}_{n+1}$  and  $uS'_p \mathcal{P}_0^{n+1}$  are not categories of weak equivalences in the sense of Waldhausen. Rather, the point of the category  $\mathcal{U}_{n+1}$  is that a map  $f: A \rightarrow B$  in  $\mathcal{P}_0^{n+1}$  has homotopy cofiber in  $\mathcal{P}_0^n$  if and only if  $f$  is a map

in  $\mathcal{U}_{n+1}$ . Writing  $u_* S'_p \mathcal{P}_0^{n+1}$  for the nerve of  $u_* S'_p \mathcal{P}_0^{n+1}$ , the set of  $q$ -simplices  $u_q S'_p \mathcal{P}_0^{n+1}$  is precisely the set of objects of  $F'_q(S'_p \mathcal{P}_0^{n+1}, S'_p \mathcal{P}_0^n)$ . The following lemma is a special case of Waldhausen's "swallowing lemma" [11, Lemma 1.6.5].

LEMMA 3.6. *The inclusion*

$$u_* S'_p \mathcal{P}_0^{n+1} = w_0 F'_*(S'_p \mathcal{P}_0^{n+1}, S'_p \mathcal{P}_0^n) \longrightarrow w_* F'_*(S'_p \mathcal{P}_0^{n+1}, S'_p \mathcal{P}_0^n)$$

is a weak equivalence for all  $p$ .

*Proof.* Let  $uw_r S'_p$  denote the category whose objects are the sequences of  $r$  composable weak equivalences between objects in  $S'_p \mathcal{P}_0^{n+1}$ ,

$$A_0 \xrightarrow{\cong} \cdots \xrightarrow{\cong} A_i \xrightarrow{\cong} \cdots \xrightarrow{\cong} A_r,$$

and whose maps  $A \rightarrow B$  are the commutative diagrams

$$\begin{array}{ccccccc} A_0 & \xrightarrow{\cong} & \cdots & \xrightarrow{\cong} & A_i & \xrightarrow{\cong} & \cdots \xrightarrow{\cong} A_r \\ u \downarrow & & & & u \downarrow & & u \downarrow \\ B_0 & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & B_i & \xrightarrow{\sim} & \cdots \xrightarrow{\sim} B_r \end{array}$$

with each map  $A_i \rightarrow B_i$  in  $u_* S'_p \mathcal{P}_0^{n+1}$ . Then, for each  $r$ ,  $w_r F'_*(S'_p \mathcal{P}_0^{n+1}, S'_p \mathcal{P}_0^n)$  is the nerve  $u_* w_r S'_p$  of the category  $u_* w_r S'_p$ , and the map in question is the inclusion of  $u_* w_0 S'_p$  in  $u_* w_* S'_p$ . Thus, to show that it is a weak equivalence, it suffices to show that each iterated degeneracy  $u_* w_0 S'_p \rightarrow u_* w_r S'_p$  is a weak equivalence. The map  $u_* w_0 S'_p \rightarrow u_* w_r S'_p$  is the nerve of the functor that sends  $A$  to the sequence of identity maps  $A = \dots = A$ . This is left adjoint to the functor  $uw_r S'_p \rightarrow uw_0 S'_p$  that sends a sequence  $A_0 \rightarrow \dots \rightarrow A_r$  to its zeroth object  $A_0$ . It follows that  $u_* w_0 S'_p \rightarrow u_* w_r S'_p$  is a homotopy equivalence.  $\square$

At this point we could state Theorem 3.10, although it would appear somewhat cryptic. To explain the form it takes, consider the further simplification of replacing the upper triangular diagrams of  $S'_p$  with a sequence of composable maps (at the cost of losing the simplicial structure in  $p$ ). For a Waldhausen category  $\mathcal{C}$ , let  $F_{p-1}\mathcal{C} = F_{p-1}(\mathcal{C}, \mathcal{C})$  be the Waldhausen category whose objects are the sequences of  $p-1$  composable w-cofibrations in  $\mathcal{C}$ . Then the functor  $wS_p\mathcal{C} \rightarrow wF_{p-1}\mathcal{C}$  that sends  $(A_{i,j})$  to  $A_{0,1} \rightarrow \dots \rightarrow A_{0,p}$  is an equivalence of categories and therefore induces a weak equivalence of nerves. When  $\mathcal{C}$  is a saturated Waldhausen category that admits FFWC, we have the Waldhausen category  $F'_{p-1}\mathcal{C} = F'_{p-1}(\mathcal{C}, \mathcal{C})$  and the analogous functor  $wS'_p\mathcal{C} \rightarrow wF'_{p-1}\mathcal{C}$  induces a weak equivalence of nerves. This argument does not apply to the map

$$u_* S'_p \mathcal{P}_0^{n+1} \longrightarrow u_* F'_{p-1} \mathcal{P}_0^{n+1},$$

however. The reason is that when  $A_1 \rightarrow A_2$  and  $B_1 \rightarrow B_2$  are weak w-cofibrations in  $\mathcal{P}_0^{n+1}$  (maps that are injective on  $\pi_{n+1}$ ), and

$$\begin{array}{ccc} A_1 & \longrightarrow & A_2 \\ u \downarrow & & \downarrow u \\ B_1 & \longrightarrow & B_2 \end{array}$$

is a commutative diagram with the vertical maps in  $\mathcal{U}_{n+1}$ , then the induced map on homotopy pushouts (with notation as in (2.1))

$$M(A_1 \rightarrow A_2) \longrightarrow M(B_1 \rightarrow B_2)$$

is generally not a map in  $\mathcal{U}_{n+1}$ . The key is to look instead at the “weak fibration” analogue: if we assume instead that  $A_1 \rightarrow A_2$  and  $B_1 \rightarrow B_2$  are maps in  $\mathcal{P}_0^{n+1}$  that induce epimorphisms on  $\pi_0$ , then for a commutative diagram as above, the induced map of homotopy pullbacks

$$N(A_1 \rightarrow A_2) \longrightarrow N(B_1 \rightarrow B_2)$$

is a map in  $\mathcal{U}_{n+1}$  by the five lemma. This suggests the following definition.

*Definition 3.7.* Let  $F_q^f \mathcal{P}_0^{n+1}$  be the category where an object is a sequence of  $q$  composable maps  $A_0 \rightarrow \dots \rightarrow A_q$  in  $\mathcal{P}_0^{n+1}$  that induce epimorphisms on  $\pi_0$ , and a morphism is a commutative diagram. Let  $uF_q^f \mathcal{P}_0^{n+1}$  denote the subcategory containing all objects of  $F_q^f \mathcal{P}_0^{n+1}$ , but consisting of only those morphisms  $A \rightarrow B$  such that  $A_i \rightarrow B_i$  is in  $\mathcal{U}_{n+1}$  for all  $i$ .

For any object  $A = (A_{i,j})$  in  $S'_p \mathcal{P}_0^{n+1}$ , the maps  $A_{i,p} \rightarrow A_{j,p}$  induce epimorphisms on  $\pi_0$  for all  $0 \leq i < j \leq p-1$ , and so we get a functor  $\phi: S'_p \mathcal{P}_0^{n+1} \rightarrow F_{p-1}^f \mathcal{P}_0^{n+1}$  that sends  $A$  to  $A_{0,p} \rightarrow A_{1,p} \rightarrow \dots \rightarrow A_{p-1,p}$ . Moreover, since the squares in  $A$  are homotopy (co)cartesian, we can recover  $A_{i,j}$  from  $\phi A$ , up to weak equivalence, as the homotopy fiber of the map  $A_{i,p} \rightarrow A_{j,p}$  for all  $0 \leq i < j \leq p-1$ . Likewise, for any object  $(A_i)$  in  $F_{p-1}^f \mathcal{P}_0^{n+1}$ , the homotopy pullback of each map  $A_i \rightarrow A_j$  is an object of  $\mathcal{P}_0^{n+1}$ , and the mapping path-space construction analogous to the construction in the proof of Theorem 2.9 constructs a functor  $\Phi: F_{p-1}^f \mathcal{P}_0^{n+1} \rightarrow S'_p \mathcal{P}_0^{n+1}$  and natural weak equivalences relating  $\phi \circ \Phi$  and  $\Phi \circ \phi$  to the identity functors in  $F_{p-1}^f \mathcal{P}_0^{n+1}$  and  $S'_p \mathcal{P}_0^{n+1}$ . As mentioned above, the five lemma implies that a morphism  $A \rightarrow B$  in  $S'_p \mathcal{P}_0^{n+1}$  is in  $uS'_p \mathcal{P}_0^{n+1}$  if and only if  $\phi A \rightarrow \phi B$  is in  $uF_{p-1}^f \mathcal{P}_0^{n+1}$ ; thus,  $\phi$  and  $\Phi$  restrict to functors  $\phi: uS'_p \mathcal{P}_0^{n+1} \rightarrow uF_{p-1}^f \mathcal{P}_0^{n+1}$  and  $\Phi: uF_{p-1}^f \mathcal{P}_0^{n+1} \rightarrow uS'_p \mathcal{P}_0^{n+1}$ . As a consequence, we obtain the following proposition.

**PROPOSITION 3.8.** *The functor  $\phi: uS'_p \mathcal{P}_0^{n+1} \rightarrow uF_{p-1}^f \mathcal{P}_0^{n+1}$  induces a weak equivalence of nerves for each  $p$ .*

For any object  $A$  of  $\mathcal{P}_0^{n+1}$ , the  $n$ -connected cover  $A\langle n \rangle$  is an Eilenberg–MacLane  $R$ -module  $K(\pi_{n+1}A, n+1)$ , and the map  $A\langle n \rangle \rightarrow A$  is in  $\mathcal{U}_{n+1}$ . This suggests that the nerve  $u_* F_{p-1}^f \mathcal{P}_0^{n+1}$  is weakly equivalent to the nerve of the full subcategory of sequences of maps of  $K(\pi, n+1)$ 's (see Lemma 4.2 below for a precise statement). This further suggests that we can understand  $u_* F_{p-1}^f \mathcal{P}_0^{n+1}$ , and therefore  $u_* S'_p \mathcal{P}_0^{n+1}$ , in terms of  $\pi_{n+1}$ . This leads to the following definition.

*Definition 3.9.* Let  $Z = \pi_0 R$ . Let  $M_p Z$  be the category whose objects are sequences of  $p-1$  composable maps of finitely generated  $Z$ -modules  $X_0 \rightarrow \dots \rightarrow X_{p-1}$  and whose morphisms are commutative diagrams. Let  $uM_p Z$  be the subcategory of  $M_p Z$  consisting of all objects but only those maps  $X \rightarrow Y$  that are isomorphisms  $X_i \rightarrow Y_i$  for all  $0 \leq i \leq p-1$ .

We understand  $uM_0 Z$  to be the trivial category consisting of a single object (the empty sequence of maps) with only the identity map. We then make  $uM_\bullet Z$  into a simplicial category as follows: for  $0 \leq i \leq p-1$ , the face map  $\partial_i: M_p Z \rightarrow M_{p-1} Z$  is obtained by dropping  $X_i$  (and composing), and the degeneracy map  $s_i: M_{p-1} Z \rightarrow M_p Z$  is obtained by repeating  $X_i$  (with the identity map). The face map  $\partial_p: M_p Z \rightarrow M_{p-1} Z$  sends  $X_0 \rightarrow \dots \rightarrow X_{p-1}$  to  $K_0 \rightarrow \dots \rightarrow K_{p-2}$ , where  $K_i$  is the kernel of the composite map  $X_i \rightarrow X_{p-1}$ . The last degeneracy  $s_{p-1}: M_{p-1} Z \rightarrow M_p Z$  inserts 0 as the last object in the sequence. With this definition, the functors  $\psi_p: uS'_p \mathcal{P}_0^{n+1} \rightarrow uM_p Z$  that take  $A = (A_{i,j})$  in  $S'_p \mathcal{P}_0^{n+1}$  to

$$\pi_{n+1}(A_{0,p}) \longrightarrow \pi_{n+1}(A_{1,p}) \longrightarrow \dots \longrightarrow \pi_{n+1}(A_{p-1,p})$$

in  $M_p Z$  assemble to a simplicial functor  $\psi: uS'_\bullet \mathcal{P}_0^{n+1} \rightarrow uM_\bullet Z$ . Note also that for each  $p$ , the functor  $\psi_p$  factors as the composite  $\pi_{n+1} \circ \phi$ . We will exploit this factorization to prove the following theorem in the next section.

**THEOREM 3.10.** *The simplicial functor  $\psi: uS'_\bullet \mathcal{P}_0^{n+1} \rightarrow uM_\bullet Z$  induces a weak equivalence of nerves.*

Finally, to reduce Theorem 1.2 to Theorem 3.10, we need to show that the nerve  $uM_\bullet Z$  is weakly contractible. For this, consider the functors  $c_p: uM_p Z \rightarrow uM_{p+1} Z$  that take the sequence  $X_0 \rightarrow \dots \rightarrow X_{p-1}$  to the sequence  $0 \rightarrow X_0 \rightarrow \dots \rightarrow X_{p-1}$ . These functors satisfy the following formulas:

$$\partial_i c_p = \begin{cases} \text{id}, & \text{if } i=0, \\ c_{p-1} \partial_{i-1}, & \text{if } i>0, \end{cases} \quad s_i c_p = \begin{cases} c_{p+1} c_p, & \text{if } i=0, \\ c_{p+1} s_{i-1}, & \text{if } i>0. \end{cases}$$

It follows that the functors  $c_\bullet$  specify a simplicial contraction of  $uM_\bullet Z$ , i.e., a retraction  $C(uM_\bullet Z) \rightarrow uM_\bullet Z$  of the inclusion  $uM_\bullet Z \rightarrow C(uM_\bullet Z)$ , where  $C(uM_\bullet Z)$  is the join of a point with  $uM_\bullet Z$  (cf. [5, Definition 5.1 and Proposition 6.2]).

#### 4. The proof of Theorems 1.3 and 3.10

In this section we prove Theorem 1.3 from the introduction and Theorem 3.10 from the previous section, completing the proof of Theorem 1.2. The heart of the argument is Lemma 4.1 below that compares the nerves of topologized categories with the nerves of the discrete categories obtained from forgetting the topology. This allows us to convert nerves constructed out of sets of maps to nerves constructed out of the corresponding spaces of maps. This facilitates the comparison of the categories of  $R$ -modules with the corresponding algebraic categories in Theorems 1.3 and 3.10 because, when the domain is nice (homotopy equivalent to a cell  $R$ -module), the space of maps between objects of  $\mathcal{P}_0^0$  (in the case of Theorem 1.3) and the space of maps from a  $K(\pi, n+1)$   $R$ -module to an object in  $\mathcal{P}_0^{n+1}$  (in the case of Theorem 3.10) are homotopy discrete with components the appropriate corresponding sets of maps of  $\pi_0 R$ -modules.

Because the hypotheses for Lemma 4.1 are complicated, we state it only for the categories  $F'_q \mathcal{P}$  (and  $wF'_q \mathcal{P}$ ), but the proof uses few of the particulars of these categories and works very generally for tensored topological categories. For the statement, let  $B$  denote a fixed object of  $F'_q \mathcal{P}$ , and consider the over-category  $F'_q \mathcal{P}/B$ : an object is a map  $\alpha: A \rightarrow B$  in  $F'_q \mathcal{P}$ , and a morphism is a commutative triangle. A morphism  $\alpha \rightarrow \alpha'$  between objects in  $F'_q \mathcal{P}/B$  is a weak equivalence if the underlying map  $A \rightarrow A'$  in  $F'_q \mathcal{P}$  is a weak equivalence; let  $w(F'_q \mathcal{P}/B)$  denote the subcategory of weak equivalences. Let  $\mathcal{D}$  be a full subcategory of  $w(F'_q \mathcal{P}/B)$  such that

- (i) if  $\alpha_0: A \rightarrow B$  is an object of  $\mathcal{D}$  and  $\alpha_1: A \rightarrow B$  is homotopic to  $\alpha_0$  in  $F'_q \mathcal{P}$ , then  $\alpha_1$  is also an object of  $\mathcal{D}$ ;
- (ii) if  $\alpha: A \rightarrow B$  is an object of  $\mathcal{D}$ , then  $A \wedge I_+ \rightarrow B$  is an object of  $\mathcal{D}$ .

In (ii) we have in mind the map obtained by composing the projection  $A \wedge I_+ \rightarrow A$  with  $\alpha$ , but, by (i), any map that restricts at some point on  $I$  to a map homotopic to  $\alpha: A \rightarrow B$  is an object of  $\mathcal{D}$ . Note that since  $I^n$  is homeomorphic to  $\Delta[n]$ , when  $A \rightarrow B$  is an object of  $\mathcal{D}$  we can regard  $A \wedge \Delta[n]_+ \rightarrow B$  as an object of  $\mathcal{D}$ .

We regard  $\mathcal{D}$  as a discrete category, but there is a natural topology on both the objects and the morphisms of  $\mathcal{D}$ ; let  $\mathcal{D}_\bullet^{\text{sing}}$  denote the simplicial category obtained by taking the total singular complex of the objects and morphisms. More concretely, an object of  $\mathcal{D}_n^{\text{sing}}$  is a map  $\alpha: A \wedge \Delta[n]_+ \rightarrow B$  in  $F'_q \mathcal{P}$  that at each point of  $\Delta[n]$  restricts to an object of  $\mathcal{D}$  (or, equivalently, that is itself an object of  $\mathcal{D}$ ). A map  $f: \alpha \rightarrow \alpha'$  in  $\mathcal{D}_n^{\text{sing}}$  is a map  $f: A \wedge \Delta[n]_+ \rightarrow A'$  in  $wF'_q \mathcal{P}$  such that the composite

$$A \wedge \Delta[n]_+ \xrightarrow{\text{id}_A \wedge \Delta} A \wedge \Delta[n]_+ \wedge \Delta[n]_+ \xrightarrow{f \wedge \text{id}_{\Delta[n]}} A' \wedge \Delta[n]_+ \xrightarrow{\alpha'} B$$

is  $\alpha$ . The category  $\mathcal{D}_0^{\text{sing}}$  is  $\mathcal{D}$ , and so we get a simplicial functor  $\mathcal{D} \rightarrow \mathcal{D}_\bullet^{\text{sing}}$ , regarding  $\mathcal{D}$  as a constant simplicial category.

LEMMA 4.1. *With hypotheses and notation above, the inclusion  $\mathcal{D} \rightarrow \mathcal{D}_\bullet^{\text{sing}}$  induces a weak equivalence on nerves.*

*Proof.* It suffices to show that the iterated degeneracy functor  $s_0^n: \mathcal{D} \rightarrow \mathcal{D}_n^{\text{sing}}$  induces a weak equivalence on nerves. This functor has a left adjoint  $U$  that takes an object  $\alpha: A \wedge \Delta[n]_+ \rightarrow B$  in  $\mathcal{D}_n^{\text{sing}}$  and regards it as an object of  $\mathcal{D}$ , and takes a map  $f: \alpha \rightarrow \alpha'$  in  $\mathcal{D}_n^{\text{sing}}$  to the composite

$$A \wedge \Delta[n]_+ \xrightarrow{\text{id}_A \wedge \Delta} A \wedge \Delta[n]_+ \wedge \Delta[n]_+ \xrightarrow{f \wedge \text{id}_{\Delta[n]}} A' \wedge \Delta[n]_+,$$

a map in  $\mathcal{D}$ . The unit  $\text{Id} \rightarrow s_0^n \circ U$  in  $\mathcal{D}_n^{\text{sing}}$  is induced by the identity map in  $\mathcal{D}$ : the identity on  $A \wedge \Delta[n]_+$  in  $\mathcal{D}$  specifies a map from  $\alpha: A \wedge \Delta[n]_+ \rightarrow B$  to

$$s_0^n U \alpha: (A \wedge \Delta[n]_+) \wedge \Delta[n]_+ \longrightarrow B$$

in  $\mathcal{D}_n^{\text{sing}}$ . The counit  $U \circ s_0^n \rightarrow \text{Id}$  in  $\mathcal{D}$  is induced by the projection  $(-) \wedge \Delta[n]_+ \rightarrow (-)$ .  $\square$

We now move on to the proof of Theorem 1.3. Recall that  $\mathcal{E}$  is the exact category of finitely generated left  $\pi_0 R$ -modules. Since the forgetful functor from  $wS_p \mathcal{P}_0^0$  to  $wF_{p-1} \mathcal{P}_0^0$  and the forgetful functor from  $wS_p \mathcal{E}$  to  $wF_{p-1} \mathcal{E}$  are equivalences of categories, it suffices to show that the functor  $\pi_0: wF_{p-1} \mathcal{P}_0^0 \rightarrow wF_{p-1} \mathcal{E}$  induces a weak equivalence on nerves. Let  $\mathcal{C}_0^0$  be the full subcategory of  $\mathcal{P}_0^0$  of objects homotopy equivalent to cell  $R$ -modules; then  $\mathcal{C}_0^0$  is a Waldhausen subcategory of  $\mathcal{P}_0^0$  and  $F_{p-1} \mathcal{C}_0^0$  is a Waldhausen subcategory of  $F_{p-1} \mathcal{P}_0^0$ . The following lemma allows us to work with  $F_{p-1} \mathcal{C}_0^0$  in place of  $F_{p-1} \mathcal{P}_0^0$ .

LEMMA 4.2. *The inclusion  $wF_{p-1} \mathcal{C}_0^0 \rightarrow wF_{p-1} \mathcal{P}_0^0$  induces a weak equivalence of nerves.*

*Proof.* Applying Quillen's Theorem A [7], it suffices to show that for every object  $B$  in  $wF_{p-1} \mathcal{P}_0^0$ , the comma category  $\mathcal{D} = wF_{p-1} \mathcal{C}_0^0 \downarrow B$  is weakly contractible. Since the functor in question is the inclusion of a full subcategory, the comma category  $\mathcal{D}$  is the full subcategory of  $w(F_{p-1} \mathcal{P}_0^0 / B)$  of maps  $A \rightarrow B$ , where  $A$  is an object in  $wF_{p-1} \mathcal{C}_0^0$ , and so is therefore also a full subcategory of  $w(F'_{p-1} \mathcal{P} / B)$ . Applying Lemma 4.1, it suffices to show that the nerve of  $\mathcal{D}_\bullet^{\text{sing}}$  is weakly contractible. Consider the functor  $\pi_0$  from  $\mathcal{D}_\bullet^{\text{sing}}$  to the over-category  $wF_{p-1} \mathcal{E} / \pi_0 B$ . Because the constituent  $R$ -module  $A_i$  of an object  $A$  of  $F_{p-1} \mathcal{C}_0^0$  is homotopy equivalent to a cell  $R$ -module, the space of  $R$ -module maps  $A_i \rightarrow B_i$  has the correct homotopy type and so is homotopy discrete with components  $\text{Hom}_Z(\pi_0 A_i, \pi_0 B_i)$ . Moreover, since the constituent maps of  $A$  are Hurewicz cofibrations, the space of maps  $A \rightarrow B$  in  $wF_{p-1} \mathcal{P}_0^0$  is likewise homotopy discrete and with

components  $wF_{p-1}\mathcal{E}(\pi_0 A, \pi_0 B)$ ; the natural topology on the set of objects of  $wF_{p-1}\mathcal{C}_0^0 \downarrow B$  is homeomorphic to the disjoint union over the objects  $A$  of  $F_{p-1}\mathcal{C}_0^0$  of these spaces. The natural topology on the set of all morphisms in  $wF_{p-1}\mathcal{C}_0^0 \downarrow B$  decomposes into a disjoint union over pairs of objects  $A, A' \in F_{p-1}\mathcal{C}_0^0$  of the spaces  $wF_{p-1}\mathcal{P}(A, A') \times wF_{p-1}\mathcal{P}(A', B)$  with their natural topology. Again this is homotopy discrete with components the disjoint union of  $wF_{p-1}\mathcal{E}(\pi_0 A, \pi_0 A') \times wF_{p-1}\mathcal{E}(\pi_0 A', \pi_0 B)$ . We conclude that the category  $\mathcal{D}_\bullet^{\text{sing}}$  has a homotopy discrete simplicial set of objects, has a homotopy discrete simplicial set of morphisms, has category of components equivalent to  $wF_{p-1}\mathcal{E}/\pi_0 B$ , and the induced map on nerves is a weak equivalence. Since  $wF_{p-1}\mathcal{E}/\pi_0 B$  has a final object (namely,  $\pi_0 B$ ), it has a contractible nerve.  $\square$

We can also apply Lemma 4.1 with  $\mathcal{D}=wF_{p-1}\mathcal{C}_0^0$  (with  $B$  the final object). The simplicial category  $\mathcal{D}_\bullet^{\text{sing}}$  then has a homotopy discrete simplicial set of objects, has a homotopy discrete simplicial set of maps, has category of components equivalent to  $wF_{p-1}\mathcal{E}$ , and has nerve weakly equivalent to the nerve of  $wF_{p-1}\mathcal{E}$ . Since the composite functor  $wF_{p-1}\mathcal{C}_0^0 \rightarrow wF_{p-1}\mathcal{E}$  is  $\pi_0$ , it follows that  $\pi_0: wF_{p-1}\mathcal{P}_0^0 \rightarrow wF_{p-1}\mathcal{E}$  induces a weak equivalence of nerves, and this completes the proof of Theorem 1.3.

The proof of Theorem 3.10 is only slightly more complicated. By Proposition 3.8 in the previous section, it suffices to show that the functor  $\pi_{n+1}: uF_{p-1}^f\mathcal{P}_0^{n+1} \rightarrow uM_p Z$  induces a weak equivalence on nerves. Now let  $C_{n+1}^{n+1}F_{p-1}^f$  be the full subcategory of  $uF_{p-1}^f\mathcal{P}_0^{n+1}$  consisting of those sequences  $A_0 \rightarrow \dots \rightarrow A_{p-1}$  where each map  $A_i \rightarrow A_{i+1}$  is a Hurewicz cofibration, each  $A_i$  is homotopy equivalent to a cell  $R$ -module, and  $\pi_q A_i = 0$  for  $q \neq n+1$ . In other words, the objects of  $C_{n+1}^{n+1}F_{p-1}^f$  are the sequences of Hurewicz cofibrations between homotopy cell  $K(\pi, n+1)$   $R$ -modules. In particular, any map between objects of  $C_{n+1}^{n+1}F_{p-1}^f$  is a weak equivalence, and so we can also consider  $C_{n+1}^{n+1}F_{p-1}^f$  as a full subcategory of  $wF_{p-1}\mathcal{P}$ . Then, as above, we have the following lemma.

**LEMMA 4.3.** *The inclusion  $C_{n+1}^{n+1}F_{p-1}^f \rightarrow uF_{p-1}^f\mathcal{P}_0^{n+1}$  induces a weak equivalence of nerves.*

The proof is essentially the same combination of Lemma 4.1 and Quillen's Theorem A as in the proof of Lemma 4.2 above, using  $M_p Z$  in place of  $F_{p-1}\mathcal{E}$ : for an object  $B$  in  $uF_{p-1}^f\mathcal{P}_0^{n+1}$ , the space of  $R$ -module maps from a homotopy cell  $K(\pi, n+1)$   $R$ -module into the  $R$ -module  $B_i$  (which has zero homotopy groups above degree  $n+1$ ) is homotopy discrete with components  $\text{Hom}_Z(\pi, \pi_{n+1}B_i)$ , and for any object  $A$  of  $C_{n+1}^{n+1}F_{p-1}^f$  the space of maps  $A \rightarrow B$  in  $uF_{p-1}^f\mathcal{P}_0^{n+1}$  is homotopy discrete with components  $uM_p Z(\pi_{n+1}A, \pi_{n+1}B)$ . As in the proof of Lemma 4.2, it follows that the functor  $\pi_{n+1}$  induces a weak equivalence of nerves from  $C_{n+1}^{n+1}F_{p-1}^f \downarrow B$  to  $uM_p Z/\pi_{n+1}B$ , which is contractible.

Applying Lemma 4.1 again with  $\mathcal{D}=C_{n+1}^{n+1}F_{p-1}^f$  and  $B$  the final object, we see that the functor  $\pi_{n+1}$  from  $C_{n+1}^{n+1}F_{p-1}^f$  to  $uM_pZ$  induces a weak equivalence of nerves. It follows that the functor  $\pi_{n+1}$  from  $uF_{p-1}^f\mathcal{P}_0^{n+1}$  to  $uM_pZ$  induces a weak equivalence of nerves, and this completes the proof of Theorem 3.10.

### References

- [1] AUSONI, C., Topological Hochschild homology of connective complex  $K$ -theory. *Amer. J. Math.*, 127:6 (2005), 1261–1313.
- [2] AUSONI, C. & ROGNES, J., Algebraic  $K$ -theory of topological  $K$ -theory. *Acta Math.*, 188 (2002), 1–39.
- [3] BAAS, N. A., DUNDAS, B. I. & ROGNES, J., Two-vector bundles and forms of elliptic cohomology, in *Topology, Geometry and Quantum Field Theory*, London Math. Soc. Lecture Note Ser., 308, pp. 18–45. Cambridge Univ. Press, Cambridge, 2004.
- [4] ELMENDORF, A. D., KRIZ, I., MANDELL, M. A. & MAY, J. P., *Rings, Modules, and Algebras in Stable Homotopy Theory*. Mathematical Surveys and Monographs, 47. Amer. Math. Soc., Providence, RI, 1997.
- [5] MAY, J. P., *Simplicial Objects in Algebraic Topology*. Van Nostrand Mathematical Studies, 11. Van Nostrand, Princeton, NJ, 1967.
- [6] MCCLURE, J. E. & STAFFELDT, R. E., The chromatic convergence theorem and a tower in algebraic  $K$ -theory. *Proc. Amer. Math. Soc.*, 118:3 (1993), 1005–1012.
- [7] QUILLEN, D., Higher algebraic  $K$ -theory, I, in *Algebraic  $K$ -theory, I: Higher  $K$ -Theories* (Battelle Memorial Inst., Seattle, WA, 1972), Lecture Notes in Math., 341, pp. 85–147. Springer, Berlin–Heidelberg, 1973.
- [8] ROGNES, J., Galois extensions of structured ring spectra. Stably dualizable groups. *Mem. Amer. Math. Soc.*, 192:898 (2008).
- [9] THOMASON, R. W. & TROBAUGH, T., Higher algebraic  $K$ -theory of schemes and of derived categories, in *The Grothendieck Festschrift*, Vol. III, Progr. Math., 88, pp. 247–435. Birkhäuser, Boston, MA, 1990.
- [10] WALDHAUSEN, F., Algebraic  $K$ -theory of spaces, a manifold approach, in *Current Trends in Algebraic Topology*, Part 1 (London, Ont., 1981), CMS Conf. Proc., 2, pp. 141–184. Amer. Math. Soc., Providence, RI, 1982.
- [11] ——, Algebraic  $K$ -theory of spaces, in *Algebraic and Geometric Topology* (New Brunswick, NJ, 1983), Lecture Notes in Math., 1126, pp. 318–419. Springer, Berlin–Heidelberg, 1985.

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