Acta Math., 203 (2009), 149–233
DOI: 10.1007/s11511-009-0041-z

Quantum decay rates in chaotic scattering

by

Stéphane Nonnenmacher

Institut de Physique Théorique
CEA-Saclay, France

Maciej Zworski

University of California
Berkeley, CA, U.S.A.

1. Statement of results

In this article we prove that for a large class of operators, including Schrödinger operators,

$$P(h) = -h^2\Delta + V(x), \quad V \in \mathcal{C}_c^\infty(X), \ X = \mathbb{R}^2, \tag{1.1}$$

with hyperbolic classical flows, the smallness of dimension of the trapped set implies that there is a gap between the resonances and the real axis. In other words, the quantum decay rates are bounded from below if the classical repeller is sufficiently *filamentary*. The higher-dimensional statement is given in terms of the *topological pressure* and is presented in Theorem 3. Under the same assumptions, we also prove a useful resolvent estimate:

$$\|\chi(P(h)-E)^{-1}\chi\|_{L^2\to L^2} \leqslant C\frac{\log(1/h)}{h}, \tag{1.2}$$

for any compactly supported bounded function χ—see Theorem 5 and a remark following it for an example of applications.

We refer to §3.2 for the general assumptions on $P(h)$, keeping in mind that they apply to $P(h)$ of the form (1.1). The resonances of $P(h)$ are defined as poles of the meromorphic continuation of the resolvent:

$$R(z,h) \stackrel{\text{def}}{=} (P(h)-z)^{-1}: L^2(X) \longrightarrow L^2(X), \quad \operatorname{Im} z > 0,$$

through the continuous spectrum $[0,\infty)$. More precisely,

$$R(z,h): L^2_c(X) \longrightarrow L^2_{\text{loc}}(X), \quad z \in \mathbb{C}\setminus(-\infty,0],$$

is a meromorphic family of operators (here L^2_c and L^2_{loc} denote functions which are compactly supported and in L^2, and functions which are locally in L^2). The poles are called *resonances* and their set is denoted by $\operatorname{Res}(P(h))$—see [3] and [48] for introduction and references. Resonances are counted according to their multiplicities (which is generically one [21]).

In the case of (1.1) the classical flow is given by Newton's equations:

$$\begin{gathered}\Phi^t(x,\xi) \stackrel{\text{def}}{=} (x(t),\xi(t)),\\ x'(t)=\xi(t), \quad \xi'(t)=-dV(x(t)), \quad x(0)=x, \quad \xi(0)=\xi.\end{gathered} \tag{1.3}$$

This flow preserves the classical Hamiltonian

$$p(x,\xi) \stackrel{\text{def}}{=} |\xi|^2+V(x), \quad (x,\xi)\in T^*X, \ X=\mathbb{R}^2,$$

and the energy layers of p are denoted as follows:

$$\mathcal{E}_E \stackrel{\text{def}}{=} \{\varrho\in T^*X : p(\varrho)=E\} \quad \text{and} \quad \mathcal{E}_E^\delta \stackrel{\text{def}}{=} \bigcup_{|E'-E|\leqslant\delta} \mathcal{E}_{E'}, \ \delta>0. \tag{1.4}$$

The incoming and outgoing sets at energy E are defined as

$$\Gamma_E^\pm \stackrel{\text{def}}{=} \{\varrho\in T^*X : p(\varrho)=E \text{ and } \Phi^t(\varrho)\not\to\infty \text{ as } t\to\mp\infty\}\subset\mathcal{E}_E. \tag{1.5}$$

The *trapped set* at energy E,

$$K_E \stackrel{\text{def}}{=} \Gamma_E^+\cap\Gamma_E^-, \tag{1.6}$$

is a compact, locally maximal invariant set, contained inside $T^*_{B(0,R_0)}X$, for some R_0. That is clear for (1.1) but also follows from the general assumptions of §3.2.

We assume that the flow Φ^t is *hyperbolic* on K_E.

The definition of hyperbolicity is recalled in (3.11)—see §3.2 below. We recall that it is a structurally stable property, so that the flow is then also hyperbolic on $K_{E'}$, for E' near E. Classes of potentials satisfying this assumption at a range of non-zero energies are given in [26], [38, Appendix C] and [47]; see also Figure 1. The dimension of the trapped set appears in the fractal upper bounds on the number of resonances. We recall the following result [42] (see Sjöstrand [38] for the first result of this type).

Theorem 1. *Let $P(h)$ be given by* (1.1) *and suppose that the flow Φ^t is hyperbolic on K_E. Then, in the semiclassical limit,*

$$|\mathrm{Res}(P(h))\cap D(E,Ch)| = \mathcal{O}(h^{-d_H}), \tag{1.7}$$

where

$$2d_H+1 = \textit{Hausdorff dimension of } K_E. \tag{1.8}$$

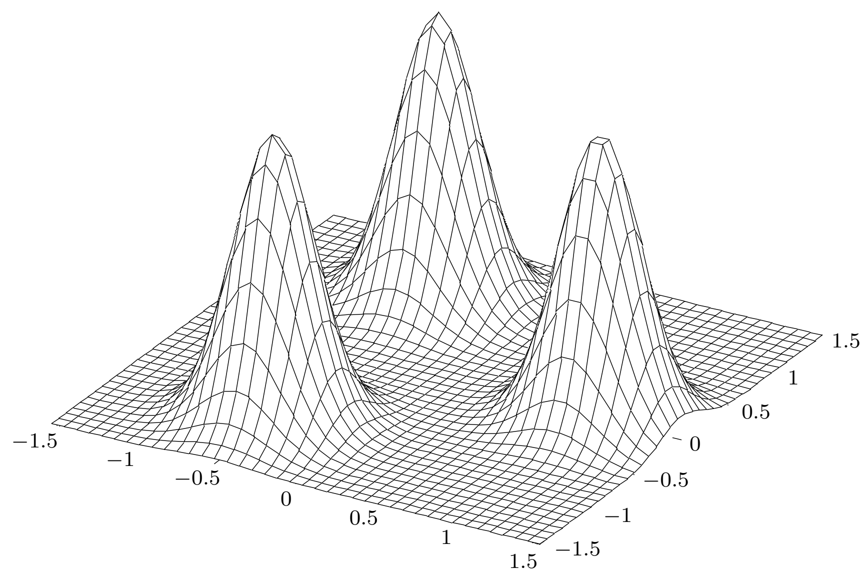

Figure 1. A three-bump potential exhibiting a hyperbolic trapped set for a range of energies. When the curve $\{x:V(x)=E(x)\}$ is made of three approximate circles of radii a and centers at equilateral distance R, the partial dimension d_H in (1.8) is approximately $\log 2/\log(R/a)$ when $R\gg a$.

We note that using [33, Theorem 4.1], and in dimension $n=2$, we strengthened the formulation of the result in [42] by replacing upper Minkowski (or box) dimension by the Hausdorff dimension. We refer to [42, Theorem 3] for the slightly more cumbersome general case.

In this article we address a different question which has been present in the physics literature at least since the seminal paper by Gaspard and Rice [14]. In the same setting of scattering by several convex obstacles, it has also been considered around the same time by Ikawa [18] (see also the careful analysis by Burq [5] and a recent paper by Petkov and Stoyanov [34]).

Question. What properties of the flow Φ^t, or of K_E alone, imply the existence of a *gap* $\gamma>0$ such that, for $h>0$ sufficiently small,

$$z\in \operatorname{Res}(P(h)),\ \operatorname{Re} z\sim E \quad\Longrightarrow\quad \operatorname{Im} z<-\gamma h\,?$$

In other words, what dynamical conditions guarantee a lower bound on the quantum decay rate?

Numerical investigations in different settings of semiclassical three-bump potentials [22], [23], three-disk scattering [14], [24], [46], Cantor-like Julia sets for $z\mapsto z^2+c$, with

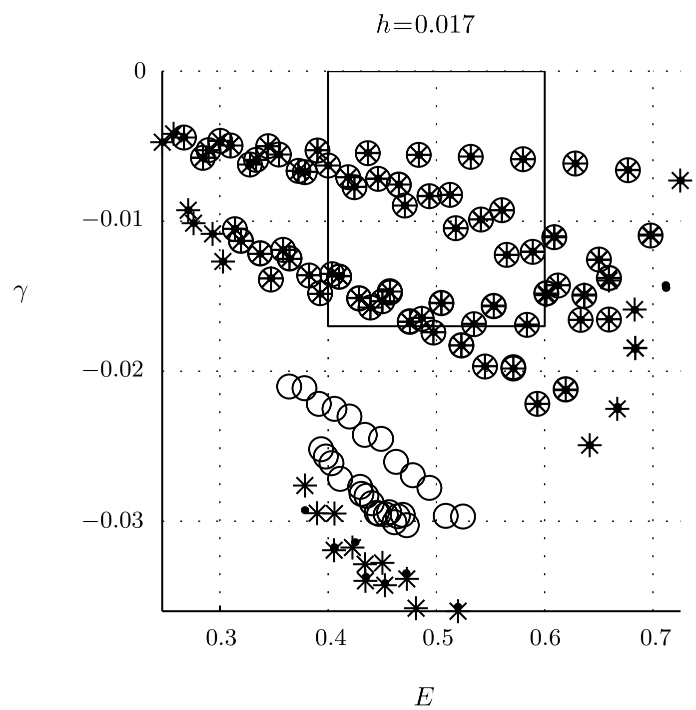

Figure 2. A sample of numerical results of [22]: the plot shows resonances for the potential of Figure 1 (h=0.017). For the energies inside the box, the fractal dimension is approximately $d_H \simeq 0.288 < 0.5$ (see [22, Table 2]), and resonances are separated from the real axis in agreement with Theorem 2.

$c<-2$ [43], and quantum maps [30], [31], [36], all indicate that a trapped set K_E of low dimension (a "filamentary" fractal set) guarantees the existence of a *resonance gap* $\gamma>0$.

Some of these works also confirm the fractal Weyl law of Theorem 1, which, unlike Theorem 2 below, was first conjectured in the mathematical works on counting resonances.

Here we provide the following result.

THEOREM 2. *Suppose that the assumptions of Theorem* 1 *hold and that the dimension* d_H *defined in* (1.8) *satisfies*

$$d_H < \tfrac{1}{2}. \tag{1.9}$$

Then there exist $\delta, \gamma>0$ *and* $h_{\delta,\gamma}>0$ *such that*

$$0 < h < h_{\delta,\gamma} \quad \Longrightarrow \quad \mathrm{Res}(P(h)) \cap ([E-\delta, E+\delta]-i[0, h\gamma]) = \varnothing. \tag{1.10}$$

The statement of the theorem can be made more general and more precise using a more sophisticated dynamical object, namely the *topological pressure* of the flow on K_E, associated with the (negative infinitesimal) unstable Jacobian

$$\varphi^+(\varrho) = -\frac{d}{dt}\log\det(d\Phi^t|_{E^+_\varrho})\Big|_{t=0},$$

namely

$$\mathcal{P}_E(s) = \text{pressure of the flow } \Phi^t \text{ on } K_E, \text{ with respect to the function } s\varphi^+.$$

We will give two equivalent definitions of the pressure below, the simplest to formulate (but not to use) given in (3.19).

The main result of this paper is the following.

THEOREM 3. *Suppose that X is a smooth manifold of the form* (3.1), *that the operator $P(h)$ defined on it satisfies the general assumptions of* §3.2 (*in particular it can be of the form* (1.1) *with $X=\mathbb{R}^n$*), *and that the flow Φ^t is hyperbolic on the trapped set K_E. Suppose that the topological pressure of the flow on K_E satisfies*

$$\mathcal{P}_E\big(\tfrac{1}{2}\big) < 0.$$

Then there exists $\delta>0$ such that for any γ satisfying

$$0 < \gamma < \min_{|E-E'|\leqslant\delta}\big(-\mathcal{P}_{E'}\big(\tfrac{1}{2}\big)\big), \tag{1.11}$$

there exits $h_{\delta,\gamma}>0$ such that

$$0 < h < h_{\delta,\gamma} \quad\Longrightarrow\quad \operatorname{Res}(P(h))\cap\big(\big[E-\tfrac{1}{4}\delta, E+\tfrac{1}{4}\delta\big]-i[0,h\gamma]\big)=\varnothing. \tag{1.12}$$

For $n=2$, the condition $d_H<\frac{1}{2}$ is equivalent to $\mathcal{P}_E\big(\frac{1}{2}\big)<0$, which shows that Theorem 2 follows from Theorem 3. The connection between the sign of $\mathcal{P}_E\big(\frac{1}{2}\big)$ and a resonance gap also holds in dimension $n\geqslant3$; however, for $n\geqslant3$ there is generally no simple link between the sign of $\mathcal{P}_E\big(\frac{1}{2}\big)$ and the value of d_H (except when the flow is "conformal" in the unstable and stable directions, respectively [33]).

The optimality of Theorem 3 is not clear. Except in some very special cases (for instance when K_E consists of one hyperbolic orbit) we do not expect the estimate on the width of the resonance free region in terms of the pressure to be optimal. In fact, in the analogous case of scattering on convex co-compact hyperbolic surfaces, the results of Naud (see [28] and references given there) show that the resonance free strip is wider at high energies than the strip predicted by the pressure. That relies on delicate zeta function analysis following the work of Dolgopyat: at zero energy there exists a Patterson–Sullivan resonance with the imaginary part (width) given by the pressure, but all other

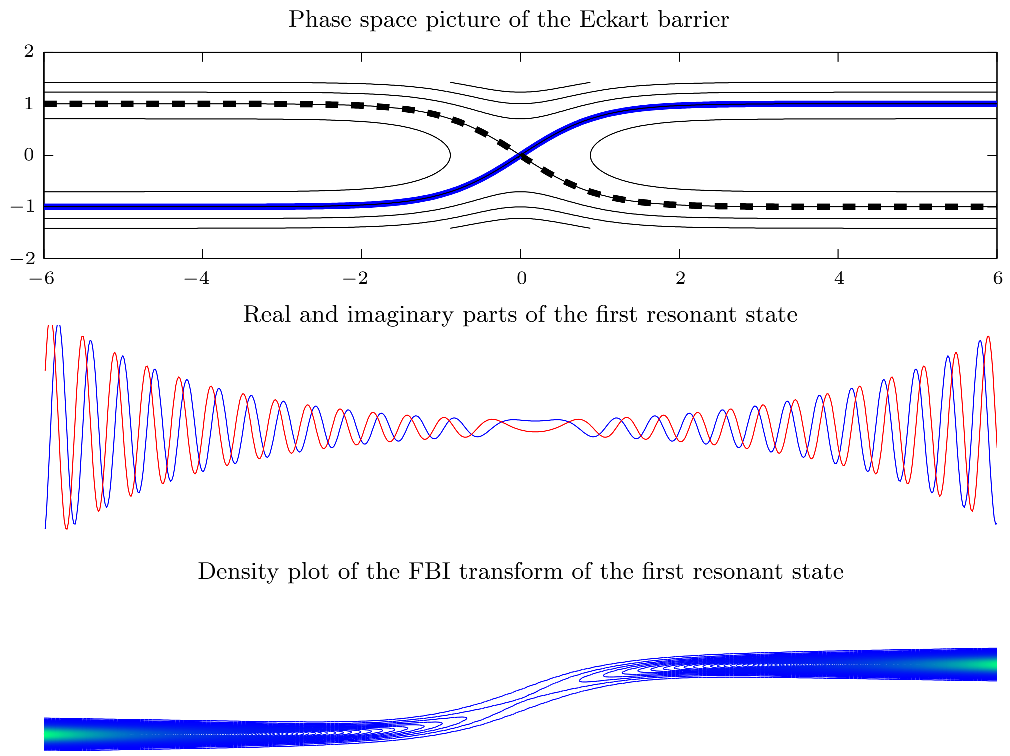

Figure 3. The top figure shows the phase portrait for the Hamiltonian $p(x,\xi)=\xi^2+\cosh^{-2}(x)$, with $\Gamma_1^{\pm}$ highlighted. The middle plot shows the resonant state corresponding to the resonance closest to the real axis at $h=\frac{1}{16}$, and the bottom plot shows the squared modulus of its Fourier–Bros–Iagolnitzer (FBI) transform. The resonance states were computed by D. Bindel (`http://cims.nyu.edu/~dbindel/resonant1d`) and the FBI transform was provided by L. Demanet. The result of Theorem 4 is visible in the mass of the FBI transform concentrated on Γ_1^+, with the exponential growth in the outgoing direction.

resonances have more negative imaginary parts. A similar phenomenon occurs in the case of Euclidean obstacle scattering as has recently been shown by Petkov and Stoyanov [34].

The proof of Theorem 3 is based on the ideas developed in the recent work of Anantharaman and the first author [1], [2] on semiclassical defect measures for eigenfuctions of the Laplacian on manifolds with Anosov geodesic flows. Although we do not use semiclassical defect measures in the proof of Theorem 3, the following result provides a connection.

THEOREM 4. *Let $P(h)$ satisfy the general assumptions of* §3.2 (*no hyperbolicity assumption here*). *Consider a sequence of values $h_k\to 0$ and a corresponding sequence of resonant states* (*see* (3.22) *in* §3.2 *below*) *satisfying*

$$\|u(h_k)\|_{L^2(\pi(K_E)+B(0,\delta))}=1, \quad \operatorname{Re} z(h_k)=E+o(1), \ \operatorname{Im} z(h)\geqslant -Ch, \tag{1.13}$$

where K_E is the trapped set at energy E, as defined by (1.6), *and* $\delta>0$. *Suppose that a semiclassical defect measure $d\mu$ on T^*X is associated with the sequence* $(u(h_k))$:

$$\begin{gathered}\langle a^w(x,h_kD)\chi u(h_k),\chi u(h_k)\rangle\to\int_{T^*X}a(\varrho)\,d\mu(\varrho),\quad \textit{as } k\to\infty,\\ a\in\mathcal{C}_c^\infty(T^*X),\quad \chi\in\mathcal{C}_c^\infty(X),\quad \pi^*\chi|_{\operatorname{supp}a}\equiv1,\quad \pi\colon T^*X\to X.\end{gathered}\tag{1.14}$$

Then

$$\operatorname{supp}\mu\subset\Gamma_E^+\tag{1.15}$$

and there exists $\lambda\geqslant0$ *such that*

$$\lim_{k\to\infty}\frac{\operatorname{Im}z(h_k)}{h_k}=-\frac{\lambda}{2}\quad\textit{and}\quad\mathcal{L}_{H_p}\mu=\lambda\mu.\tag{1.16}$$

See Figure 3 for a numerical result illustrating the theorem. A similar analysis of the phase space distribution for the resonant eigenstates of quantized open chaotic maps (discrete-time models for scattering Hamiltonian flows) has been recently performed in [20] and [29]. Connecting this theorem with Theorems 2 and 3, we see that the semiclassical defect measures associated with sequences of resonant states have decay rates λ bounded from below by $2\gamma>0$, once the dimension of the trapped set is small enough ($n=2$) or, more generally, the pressure at $\frac12$ is negative.

Our last result is the precise version of the resolvent estimate (1.2).

THEOREM 5. *Suppose that $P(h)$ satisfies the general assumptions of* §3.2 (*in particular it can be of the form* (1.1) *with* $X=\mathbb{R}^n$) *and that the flow Φ^t is hyperbolic on the trapped set K_E. If* $\mathcal{P}_E\left(\frac12\right)<0$ *then for any* $\chi\in\mathcal{C}_c^\infty(X)$ *we have*

$$\|\chi(P(h)-E)^{-1}\chi\|_{L^2(X)\to L^2(X)}\leqslant C\frac{\log(1/h)}{h},\quad 0<h<h_0.\tag{1.17}$$

Notice that the upper bound $C\log(1/h)/h$ is the same as the one obtained in the case of one hyperbolic orbit by Christianson [8]. To see how results of this type imply dynamical estimates see [6] and [8]. In the context of Theorem 5, applications are presented in [7]. Referring to that paper for details and pointers to the literatures, we present one application.

Let $P=-h^2\Delta_g$ be the Laplace–Beltrami operator satisfying the assumptions below, for instance on a manifold which is Euclidean outside of a compact set with the standard metric there. The Schrödinger propagator, $e^{-it\Delta_g}$, is unitary on any Sobolev space so regularity is not improved in propagation. Remarkably, when $K=\varnothing$, that is, when the metric is non-trapping, the regularity improves when we integrate in time and cutoff in space:

$$\int_0^T\|\chi e^{-it\Delta_g}u\|_{H^{1/2}(X)}^2\,dt\leqslant C\|u\|_{L^2(X)}^2,\quad \chi\in\mathcal{C}_c^\infty(X),$$

and this much exploited effect is known as *local smoothing*. As was shown by Doi [12], any trapping (for instance a presence of closed geodesics or more generally $K\neq\varnothing$) will destroy local smoothing. Theorem 5 implies that under the assumptions that the geodesic flow is hyperbolic on the trapped set $K\subset S^*X$, and that the pressure is negative at $\frac{1}{2}$ (or, when $\dim X=2$, that the dimension of $K\subset S^*X$ is less than 2) local smoothing holds with $H^{1/2}$ replaced by $H^{1/2-\varepsilon}$ for any $\varepsilon>0$.

Notation. In the paper, C denotes a constant the value of which may change from line to line. The constants which matter and have to be balanced against each other will always have a subscript: C_1, C_2 and alike. The notation $u=\mathcal{O}_V(f)$ means that $\|u\|_V=\mathcal{O}(f)$, and the notation $T=\mathcal{O}_{V\to W}(f)$ means that $\|Tu\|_W=\mathcal{O}(f)\|u\|_V$.

2. Outline of the proof

It this section we present the main ideas, with the precise definitions and references to previous works given in the main body of the paper. The operator to keep in mind is $P=P(h)=-h^2\Delta_g+V$, where $V\in\mathcal{C}_c^\infty(X)$, $X=\mathbb{R}^n$, and the metric g is Euclidean outside a compact set. The corresponding classical Hamiltonian is given by $p=\xi^2+V(x)$. Weaker assumptions, which in particular do not force the compact support of the perturbation, are described in §3.2.

First we outline the proof of Theorem 3 in the simplified case where resonances are replaced by the *eigenvalues* of an operator modified by a *complex absorbing potential*:

$$P_W=P_W(h)\stackrel{\text{def}}{=}P-iW,$$

where $W\in\mathcal{C}^\infty(X;[0,1])$ satisfies the following conditions:

$$W\geqslant 0,\quad \operatorname{supp}W\subset X\setminus B(0,R_1)\quad\text{and}\quad W|_{X\setminus B(0,R_1+r_1)}=1,$$

for R_1 and r_1 sufficiently large. In particular, R_1 is such that $\pi(K_E)\subset B(0,R_1)$, where K_E is the trapped set given by (1.6). The non-self-adjoint operator P_W has a discrete spectrum in $\operatorname{Im} z>-1/C$ and the analogue of Theorem 3 reads as follows.

THEOREM 3′. *Under the assumptions of Theorem* 3, *for*

$$0<\gamma<\min_{|E-E'|\leqslant\delta}\bigl(-\mathcal{P}_{E'}\bigl(\tfrac{1}{2}\bigr)\bigr),\tag{2.1}$$

there exits $h_0=h_0(\gamma,\delta)$ *such that for* $0<h<h_0$,

$$\operatorname{Spec}(P_W(h))\cap\bigl(\bigl[E-\tfrac{1}{4}\delta,E+\tfrac{1}{4}\delta\bigr]-i[0,h\gamma]\bigr)=\varnothing.\tag{2.2}$$

This means that the spectrum of $P_W(h)$ near E is separated from the real axis by $h\gamma$, where γ is given in terms of the pressure $\mathcal{P}_E\big(\frac{1}{2}\big)$ associated with half the (negative infinitesimal) unstable Jacobian.

This spectral gap is equivalent to the fact that the *decay rate* of any eigenstate is bounded from below:

$$P_W u = zu,\ z \in D\left(E, \frac{1}{C}\right) \text{ and } u \in L^2 \quad \Longrightarrow \quad \|e^{-itP_W/h}u\| \leqslant e^{-\gamma t}\|u\|.$$

This is the physical meaning of the gap between the spectrum (or resonances) and the real axis—a lower bound for the quantum decay rate—and the departing point for the proof. To show (2.2) we will show that for functions u which are microlocally concentrated near the energy layer $\mathcal{E}_E = p^{-1}(E)$ (that is, $u = \chi^w(x, hD)u + \mathcal{O}(h^\infty)$ for a χ supported near $\mathcal{E}_E$), we have

$$\begin{gathered} \|e^{-itP_W/h}u\| \leqslant Ch^{-n/2}e^{-\lambda t}\|u\|, \\ 0 < \lambda < \min_{|E-E'|\leqslant\delta}\left(-\mathcal{P}_{E'}\left(\frac{1}{2}\right)\right), \quad 0 \leqslant t \leqslant \widetilde{M}\log\frac{1}{h}, \end{gathered} \tag{2.3}$$

for any $\widetilde{M}$. Taking $\widetilde{M} \gg n/2\lambda$ and applying the estimate to an eigenstate u gives (2.2).

To prove (2.3) we decompose the propagator using an open cover $\{W_a\}_{a\in A}$ of the neighbourhood $\mathcal{E}_E^\delta$ of the energy surface. That cover is adapted to the definition of the pressure (see §5.2 and §5.3) and it leads to a microlocal partition of a neighbourhood of the energy surface:

$$\sum_{a\in A}\Pi_a = \chi^w(x, hD) + \mathcal{O}(h^\infty), \quad \chi \equiv 1 \text{ on } \mathcal{E}_E^{\delta/8}, \quad \operatorname{ess\,supp}\Pi_a \Subset W_a.$$

The definition of the pressure in §5.2 also involves a time $t_0 > 1$, independent of h, but depending on the classical cover. Taking

$$N \leqslant M\log\frac{1}{h}, \quad N \in \mathbb{N}, \quad M > 0 \text{ fixed but arbitrarily large,} \tag{2.4}$$

the propagator at time $t = Nt_0$ acting on functions u microlocalized inside $\mathcal{E}_E^{\delta/8}$ can be written as

$$e^{-iNt_0P_W/h}u = \sum_{\alpha\in A^N} U_{\alpha_N} \dots U_{\alpha_1}u + \mathcal{O}(h^\infty)\|u\|, \quad U_a \stackrel{\text{def}}{=} e^{-it_0P_W/h}\Pi_a. \tag{2.5}$$

The sequences $\alpha = (\alpha_1, \dots, \alpha_N)$ which are classically forbidden, that is, for which the corresponding sequences of neighbourhoods are *not* successively connected by classical propagation in time t_0, lead to negligible terms. So do the sequences for which the propagation crosses the region where $W=1$: the operator $e^{-it_0P_W/h}$ is negligible there, due to damping (or "absorption") by W.

As a result, the only terms relevant in the sum on the right-hand side of (2.5) come from $\alpha\in A_1^N\cap\mathcal{A}_N$, where A_1 indexes the element of the partition intersecting the trapped set K_E, and $\mathcal{A}_N$ are the classically allowed sequences—see (6.29). We then need the crucial *hyperbolic dispersion estimate* proved in §7 after much preliminary work in §4.3 and §5.1: for $N\leqslant M\log(1/h)$, $M>0$ arbitrary, we have for any sequence $\alpha\in A_1^N\cap\mathcal{A}_N$,

$$\|U_{\alpha_N}\dots U_{\alpha_1}\|\leqslant h^{-n/2}(1+\varepsilon_0)^N\prod_{j=1}^N\Big(\inf_{\varrho\in W_{\alpha_j}\cap K_E^\delta}\det(d\Phi^{t_0}(\varrho)|_{E_\varrho^{+0}})\Big)^{-1/2}. \tag{2.6}$$

The expression in parenthesis is the coarse-grained unstable Jacobian defined in (5.22), and $\varepsilon_0>0$ is a parameter depending on the cover $\{W_a\}_{a\in A}$, which can be taken arbitrarily small—see (5.24). From the definition of the pressure in §5.2, summing (2.6) over $\alpha\in A_1^N\cap\mathcal{A}_N$ leads to (2.3), with $\widetilde{M}=Mt_0$.

In §9 we show how to use (2.3) to obtain a resolvent estimate for P_W: at an energy E for which the flow is hyperbolic on K_E and $\mathcal{P}_E\big(\frac12\big)<0$, we have

$$\|(P_W-E)^{-1}\|_{L^2(X)\to L^2(X)}\leqslant C\frac{\log(1/h)}{h},\quad 0<h<h_0. \tag{2.7}$$

To prove Theorem 3, that is the gap between *resonances* and the real axis, we use the complex scaled operator P_θ: its eigenvalues near the real axis are resonances of P. If V is a decaying real-analytic potential extending to a conic neighbourhood of $\mathbb{R}^n$ (for instance a sum of three Gaussian bumps, as shown in Figure 1), then we can take $P_\theta=-h^2e^{-2i\theta}\Delta+V(e^{i\theta}x)$, though in this paper we will always use exterior complex scaling reviewed in §3.4, with $\theta\simeq M_1(1/h)\log(1/h)$, where M_1 is chosen depending on M in (2.4).

To use the same strategy of estimating $e^{-itP_\theta/h}$, we need to further modify the operator by conjugation with microlocal exponential weights. That procedure is described in §6. The methods developed there are also used in the proof of Theorem 4 and in showing how the estimate (2.7) implies Theorem 5.

Since we concentrate on the more complicated, and scientifically relevant, case of resonances, the additional needed facts about the study of P_W and its propagator are presented in the appendix.

3. Preliminaries and assumptions

In this section we recall basic concepts of semiclassical analysis, state the general assumptions on operators to which the theorems above apply, define hyperbolicity and topological pressure. We also define resonances using *complex scaling* which is the standard tool in the study of their distribution. Finally, we will review some results about semiclassical Lagrangian states and Fourier integral operators.

3.1. Semiclassical analysis

Let X be a $\mathcal{C}^\infty$ manifold which agrees with $\mathbb{R}^n$ outside a compact set, or more generally

$$X = X_0 \sqcup (\mathbb{R}^n \setminus B(0,R_0)) \sqcup \ldots \sqcup (\mathbb{R}^n \setminus B(0,R_0)), \quad X_0 \Subset X. \tag{3.1}$$

The class of symbols associated with the weight m is defined as

$$S_\delta^{m,k}(T^*X) = \{a \in \mathcal{C}^\infty(T^*X \times (0,1]) : |\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \leqslant C_\alpha h^{-k-\delta(|\alpha|+|\beta|)} \langle \xi \rangle^{m-|\beta|}\}.$$

Most of the time we will use the class with $\delta=0$ in which case we drop the subscript. When $m=k=0$, we simply write $S(T^*X)$ or S for the class of symbols. The reason for demanding the decay in ξ under differentiation is to have invariance under changes of variables.

We denote by $\Psi_{h,\delta}^{m,k}(X)$ or $\Psi_h^{m,k}(X)$ the corresponding class of pseudodifferential operators. We have surjective quantization and symbol maps:

$$\mathrm{Op}\colon S^{m,k}(T^*X) \longrightarrow \Psi_h^{m,k}(X) \quad \text{and} \quad \sigma_h\colon \Psi_h^{m,k}(X) \longrightarrow S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X).$$

Multiplication of symbols corresponds to composition of operators, to leading order:

$$\sigma_h(A \circ B) = \sigma_h(A)\sigma_h(B),$$

and

$$\sigma_h \circ \mathrm{Op}\colon S^{m,k}(T^*X) \longrightarrow S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X)$$

is the natural projection map. A finer filtration can be obtained by combining semiclassical calculus with the standard calculus (or the yet more general framework of the Weyl calculus)—see for instance [41, §3].

The class of operators and the quantization map are defined locally using the definition on $\mathbb{R}^n$:

$$\mathrm{Op}(a)u(x') = a^w(x,hD)u(x') = \frac{1}{(2\pi h)^n} \iint_{\mathbb{R}^{2n}} a\Big(\frac{x'+x}{2},\xi\Big) e^{i\langle x'-x,\xi\rangle/h} u(x)\,dx\,d\xi, \tag{3.2}$$

and we refer to [11, Chapter 7] for a detailed discussion of semiclassical quantization (see also [40, Appendix]), and to [13, Appendix D.2] for the semiclassical calculus on manifolds.

The semiclassical Sobolev spaces $H_h^s(X)$ are defined by choosing a globally elliptic, self-adjoint operator $A\in\Psi_h^{1,0}(X)$ (that is an operator satisfying $\sigma(A)\geqslant\langle\xi\rangle/C$ everywhere), and putting

$$\|u\|_{H_h^s}=\|A^s u\|_{L^2(X)}.$$

When $X=\mathbb{R}^n$,

$$\|u\|_{H_h^s}^2\sim\int_{\mathbb{R}^n}\langle\xi\rangle^{2s}|\mathcal{F}_h u(\xi)|^2\,d\xi,\quad \mathcal{F}_h u(\xi)\stackrel{\text{def}}{=}\frac{1}{(2\pi h)^{n/2}}\int_{\mathbb{R}^n}u(x)e^{-i\langle x,\xi\rangle/h}\,dx.$$

Unless otherwise stated, all norms in this paper, $\|\cdot\|$, are L^2 norms.

For $a\in S(T^*X)$ we follow [41] and say that the *essential support* is equal to a given compact set $K\Subset T^*X$,

$$\operatorname{ess\,supp}_h a=K\Subset T^*X,$$

if and only if for all $\chi\in S(T^*X)$,

$$\operatorname{supp}\chi\subset\complement K\quad\Longrightarrow\quad\chi a\in h^\infty\mathcal{S}(T^*X).$$

Here $\mathcal{S}$ denotes the Schwartz class which makes sense since X is Euclidean outside a compact set. In this article we are only concerned with a purely semiclassical theory and deal only with *compact* subsets of T^*X.

For $A\in\Psi_h(X)$, $A=\operatorname{Op}(a)$, we put

$$\operatorname{WF}_h(A)=\operatorname{ess\,supp}_h a,$$

noting that the definition does not depend on the choice of Op.

We introduce the following condition:

$$\begin{gathered}\text{for all } u\in\mathcal{C}^\infty((0,1]_h;\mathcal{D}'(X)) \text{ there exist } P \text{ and } h_0 \text{ such that}\\ \|\langle x\rangle^{-P}u\|_{L^2(X)}\leqslant h^{-P}\quad\text{for } h<h_0.\end{gathered}\tag{3.3}$$

We call families $u=u(h)$ satisfying (3.3) *h-tempered.* What we need is that for $u(h)$, h-tempered, $\chi^w(x,hD)u(h)\in h^\infty\mathcal{S}(X)$ for $\chi\in h^\infty\mathcal{S}(T^*X)$. That is, applying an operator in the residual class produces a negligible contribution.

For such h-tempered families we define the semiclassical L^2-wave front set as

$$\begin{aligned}\operatorname{WF}_h(u)=\complement\{(x,\xi):\ &\text{there exists } a\in S(T^*X) \text{ such that}\\ &a(x,\xi)=1 \text{ and } \|a^w(x,hD)u\|_{L^2}=\mathcal{O}(h^\infty)\}.\end{aligned}\tag{3.4}$$

The last condition in the definition can be equivalently replaced by

$$a^w(x,hD)u \in h^\infty \mathcal{C}^\infty((0,1]_h;\mathcal{C}^\infty(X)),$$

since we may always take $a\in\mathcal{S}(T^*X)$.

Equipped with the notion of semiclassical wave front set, it is useful and natural to consider the operators and their properties *microlocally.* For that we consider the class of *tempered* operators, $T=T(h):\mathcal{S}(X)\to\mathcal{S}'(X)$, defined by the condition

$$\text{there exist } P \text{ and } h_0 \text{ such that } \|\langle x\rangle^{-P}Tu\|_{H_h^{-P}(X)} \leqslant h^{-P}\|\langle x\rangle^{P}u\|_{H_h^{P}(X)} \text{ for } 0<h<h_0.$$

For open sets $V\subset\overline{V}\Subset T^*X$ and $U\subset\overline{U}\Subset T^*X$, the operators *defined microlocally* near $V\times U$ are given by the following equivalence classes of tempered operators:

$$\begin{gathered} T\sim T' \text{ if and only if there exist open sets} \\ \widetilde{U},\widetilde{V}\Subset T^*X,\ \overline{U}\Subset\widetilde{U} \text{ and } \overline{V}\Subset\widetilde{V} \text{ such that} \\ A(T-T')B=\mathcal{O}_{\mathcal{S}'\to\mathcal{S}}(h^\infty) \\ \text{for any } A,B\in\Psi_h(X) \text{ with } \mathrm{WF}_h(A)\subset\widetilde{V} \text{ and } \mathrm{WF}_h(B)\subset\widetilde{U}. \end{gathered} \tag{3.5}$$

For two such operators T and T', we say that $T=T'$ *microlocally* near $V\times U$. If we assumed that, say $A=a^w(x,hD)$, where $a\in\mathcal{C}_c^\infty(T^*X)$, then $\mathcal{O}_{\mathcal{S}'\to\mathcal{S}}(h^\infty)$ could be replaced by $\mathcal{O}_{L^2\to L^2}(h^\infty)$ in the condition. We should stress that "microlocally" is always meant in this semi-classical sense in our paper.

The operators in $\Psi_h(X)$ are bounded on L^2 uniformly in h. For future reference, we also recall the sharp Gårding inequality (see for instance [11, Theorem 7.12]):

$$a\in S(T^*X),\ a\geqslant 0 \quad\Longrightarrow\quad \langle a^w(x,hD)u,u\rangle \geqslant -Ch\|u\|_{L^2}^2,\ u\in L^2(X), \tag{3.6}$$

and Beals's characterization of pseudodifferential operators on X (see [11, Chapter 8] and [42, Lemma 3.5] for the S_δ case):

$$A\in\Psi_{h,\delta}(X) \quad\Longleftrightarrow\quad \begin{cases} \|\operatorname{ad}_{W_N}\ldots\operatorname{ad}_{W_1}A\|_{L^2\to L^2}=\mathcal{O}(h^{(1-\delta)N}), \\ \text{for all } W_j\in\operatorname{Diff}^1(X),\ j=1,\ldots,N, \\ \qquad W_j=\langle a,hD_x\rangle+\langle b,x\rangle,\ a,b\in\mathbb{R}^n, \text{ outside } X_0. \end{cases} \tag{3.7}$$

Here $\operatorname{ad}_B C=[B,C]$.

3.2. Assumptions on $P(h)$

We now state the general assumptions on the operator $P=P(h)$, stressing that the simplest case to keep in mind is

$$P=-h^2\Delta+V(x), \quad V\in\mathcal{C}_c^\infty(\mathbb{R}^n).$$

In general we consider

$$P(h)\in\Psi_h^{2,0}(X), \quad P(h)=P(h)^*,$$

and an energy level $E>0$, for which

$$\begin{gathered}P(h)=p^w(x,hD)+hp_1^w(x,hD;h), \quad p_1\in S^{2,0}(T^*X),\\ |\xi|\geqslant C \implies p(x,\xi)\geqslant\frac{\langle\xi\rangle^2}{C}, \quad p=E\implies dp\neq 0,\end{gathered}\tag{3.8}$$

and there exists R_0 such that $P(h)u(x)=Q(h)u(x)$ for all $u\in\mathcal{C}^\infty(X\setminus B(0,R_0))$.

Here the operator near infinity takes on each "infinite branch" $\mathbb{R}^n\setminus B(0,R_0)$ of X the following form:

$$Q(h)=\sum_{|\alpha|\leqslant 2}a_\alpha(x;h)(hD_x)^\alpha,$$

where $a_\alpha(x;h)=a_\alpha(x)$ is independent of h for $|\alpha|=2$, $a_\alpha(x;h)\in C_b^\infty(\mathbb{R}^n)$ is uniformly bounded with respect to h (here $C_b^\infty(\mathbb{R}^n)$ denotes the space of C^∞ functions with bounded derivatives of all orders), and

$$\begin{gathered}\sum_{|\alpha|=2}a_\alpha(x)\xi^\alpha\geqslant\frac{|\xi|^2}{c} \quad \text{for all } \xi\in\mathbb{R}^n \text{ and some constant } c>0,\\ \sum_{|\alpha|\leqslant 2}a_\alpha(x;h)\xi^\alpha\to\xi^2, \quad \text{as } |x|\to\infty, \text{ uniformly with respect to } h.\end{gathered}\tag{3.9}$$

We also need the following analyticity assumption in a neighbourhood of infinity: there exist $\theta_0\in[0,\pi)$ and $\varepsilon>0$ such that the coefficients $a_\alpha(x;h)$ of $Q(h)$ extend holomorphically in x to

$$\{r\omega:\omega\in\mathbb{C}^n,\operatorname{dist}(\omega,\mathbf{S}^n)<\varepsilon,r\in\mathbb{C},|r|>R_0 \text{ and } \arg r\in[-\varepsilon,\theta_0+\varepsilon)\},$$

with (3.9) valid also in this larger set of x's. Here for convenience we chose the same R_0 as the one appearing in (3.8), but that is clearly irrelevant.

We note that the analyticity assumption in a conic neighbourhood near infinity automatically strengthens (3.9) through an application of Cauchy inequalities:

$$\partial_x^\beta\bigg(\sum_{|\alpha|\leqslant 2}a_\alpha(x;h)\xi^\alpha-\xi^2\bigg)\leqslant|x|^{-|\beta|}f_{|\beta|}(|x|)\langle\xi\rangle^2, \quad \text{as } x\to\infty,\tag{3.10}$$

where for any $j\in\mathbb{N}$ the function $f_j(r)\searrow 0$ when $r\to\infty$.

3.3. Definitions of hyperbolicity and topological pressure

We use the notation

$$\Phi^t(\varrho)=e^{tH_p}(\varrho),\quad \varrho=(x,\xi)\in T^*X,$$

where H_p is the Hamilton vector field of p,

$$H_p \overset{\text{def}}{=} \sum_{j=1}^n\left(\frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j}-\frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}\right)=\{p,\cdot\},$$

in local coordinates in T^*X. The last expression is the Poisson bracket relative to the symplectic form $\omega=\sum_{j=1}^n d\xi_j\wedge dx_j$.

We assume that $p=p(x,\xi)$ and $E>0$ satisfy the assumptions (3.8) and (3.9) of §3.2, and study the flow Φ^t generated by p on $\mathcal{E}_E$. The incoming and outgoing sets, $\Gamma_E^\pm$, and the trapped set, K_E, are given by (1.5) and (1.6), respectively. The assumptions imply that K_E is compact.

We say that the flow Φ^t is *hyperbolic on* K_E if for any $\varrho\in K_E$ the tangent space to $\mathcal{E}_E$ at ϱ splits into flow, unstable and stable subspaces [19, Definition 17.4.1]:

(i) $T_\varrho(\mathcal{E}_E)=\mathbb{R}H_p(\varrho)\oplus E_\varrho^+\oplus E_\varrho^-$, $\dim E_\varrho^\pm=n-1$,

(ii) $d\Phi_\varrho^t(E_\varrho^\pm)=E^\pm_{\Phi^t(\varrho)}$ for all $t\in\mathbb{R}$, (3.11)

(iii) there exists $\lambda>0$ such that $\|d\Phi^t_\varrho(v)\|\leqslant Ce^{-\lambda|t|}\|v\|$ for all $v\in E_\varrho^\mp$, $\pm t\geqslant 0$.

K_E is a *locally maximal hyperbolic set* for the flow $\Phi^t|_{\mathcal{E}_E}$. The following properties are then satisfied:

(iv) $K_E\ni\varrho\mapsto E_\varrho^\pm\subset T_\varrho(\mathcal{E}_E)$ is Hölder continuous,

(v) any $\varrho\in K_E$ admits local (un)stable manifolds $W^\pm_{\text{loc}}(\varrho)$ tangent to $E^\pm_\varrho$,

(vi) there exists an "adapted" metric g_{ad} near K_E such that one can take $C=1$ in (iii). (3.12)

The adapted metric g_{ad} can be extended to the whole energy layer, so as to coincide with the standard Euclidean metric outside $T^*_{B(0,R_0)}X$. We call

$$E_\varrho^{+0}\overset{\text{def}}{=}E_\varrho^+\oplus\mathbb{R}H_p(\varrho)\quad\text{and}\quad E_\varrho^{-0}\overset{\text{def}}{=}E_\varrho^-\oplus\mathbb{R}H_p(\varrho) \tag{3.13}$$

the *weak unstable* and *weak stable* subspaces at the point ϱ, respectively. Similarly, we denote by $W^{+0}(\varrho)$ (resp. $W^{-0}(\varrho)$) the weak unstable (resp. stable) manifold. The family of all the (un)stable manifolds $W^\pm(\varrho)$ forms the (un)stable *lamination* on K_E, and one has

$$\Gamma_E^\pm=\bigcup_{\varrho\in K_E}W^\pm(\varrho).$$

If periodic orbits are dense in K_E, then the flow is said to be *axiom A* on K_E [4]. Such a hyperbolic set is *structurally stable* [19, Theorem 18.2.3], so that

$$\text{there exists } \delta>0 \text{ such that, for all } E'\in[E-\delta,E+\delta],\quad K_{E'} \text{ is a hyperbolic set for } \Phi^t|_{\mathcal{E}_{E'}}. \tag{3.14}$$

Besides, the total trapped set in the energy layer $\mathcal{E}_E^\delta$, that is

$$K_E^\delta \stackrel{\text{def}}{=} \bigcup_{|E'-E|\leqslant\delta} K_{E'}, \text{ is compact.} \tag{3.15}$$

Since the topological pressure plays a crucial role in the statement and proof of Theorem 3, we recall its definition in our context (see [19, Definition 20.2.1] or [33, Appendix A]).

Let d be the distance function associated with the adapted metric. We say that a set $\mathcal{S}\subset K_E$ is (ε,t)-*separated* if for $\varrho_1,\varrho_2\in\mathcal{S}$, $\varrho_1\neq\varrho_2$, we have $d(\Phi^{t'}(\varrho_1),\Phi^{t'}(\varrho_2))>\varepsilon$ for some $0\leqslant t'\leqslant t$. Obviously, such a set must be finite, but its cardinal may grow exponentially with t. The metric g_{ad} induces a volume form Ω on any n-dimensional subspace of $T(T^*\mathbb{R}^n)$. Using this volume form, we now define the unstable Jacobian on K_E. For any $\varrho\in K_E$, the determinant map

$$\bigwedge^n d\Phi^t(\varrho)|_{E_\varrho^{+0}}: \bigwedge^n E_\varrho^{+0} \longrightarrow \bigwedge^n E^{+0}_{\Phi^t(\varrho)}$$

can be identified with the real number

$$\det(d\Phi^t(\varrho)|_{E_\varrho^{+0}}) \stackrel{\text{def}}{=} \frac{\Omega_{\Phi^t(\varrho)}(d\Phi^t v_1\wedge d\Phi^t v_2\wedge\ldots\wedge d\Phi^t v_n)}{\Omega_\varrho(v_1\wedge v_2\wedge\ldots\wedge v_n)}, \tag{3.16}$$

where $(v_1,\ldots,v_n)$ can be any basis of E_ϱ^{+0}. This number defines the unstable Jacobian:

$$e^{\lambda_t^+(\varrho)} \stackrel{\text{def}}{=} \det(d\Phi^t(\varrho)|_{E_\varrho^{+0}}), \tag{3.17}$$

and its (negative) infinitesimal version

$$\varphi^+(\varrho) \stackrel{\text{def}}{=} -\frac{d\lambda_t^+(\varrho)}{dt}\bigg|_{t=0}.$$

From there we take

$$Z_t(\varepsilon,s) \stackrel{\text{def}}{=} \sup_{\mathcal{S}} \sum_{\varrho\in\mathcal{S}} e^{-s\lambda_t^+(\varrho)}, \tag{3.18}$$

where the supremum is taken over all (ε,t)-separated sets. The pressure is then defined as

$$\mathcal{P}_E(s) \stackrel{\text{def}}{=} \lim_{\varepsilon\to0} \limsup_{t\to\infty} \frac{1}{t} \log Z_t(\varepsilon,s). \tag{3.19}$$

This quantity is actually independent of the volume form Ω: after taking logarithms, a change in Ω produces a term $\mathcal{O}(1)/t$ which is irrelevant in the $t\to\infty$ limit.

From the identity $\lambda_t^+(\varrho)=-\int_0^t \varphi^+(\Phi^s(\varrho))\,ds$ we see that, in the ergodic theory terminology, $\mathcal{P}_E(s)$ is the topological pressure associated with the Hölder function ("potential") $s\varphi^+$. We remark that the standard definition of the unstable Jacobian consists in restricting $d\Phi^t(\varrho)$ to the *strong* unstable subspace E_ϱ^+ [4]; yet, including the (neutral) flow direction in the definition (3.17) of λ_t^+ (and hence of φ^+) does not alter the pressure, and is better suited for the applications in this article. In §5.2 we will give a different equivalent definition of the topological pressure, more adapted to our aims.

We end this section by stating a simple property of the topological pressure, which we will need further on. Although its proof its straightforward, we were unable to find it in the literature.

LEMMA 3.1. *For any $s\in\mathbb{R}$, the topological pressures $\mathcal{P}_E^\delta(s)$ and $\mathcal{P}_E(s)$ satisfy the relation*

$$\mathcal{P}_E(s)=\lim_{\delta\to 0}\mathcal{P}_E^\delta(s). \tag{3.20}$$

Proof. For any closed invariant set K, the pressure $\mathcal{P}_K(s)$ associated with the flow on K can be defined through the variational principle

$$\mathcal{P}_K(s)=\sup_{\mu\in\mathrm{Erg}(K)}\left(h_{KS}(\mu)-s\int_K \varphi^+\,d\mu\right),$$

where $\mathrm{Erg}(K)$ is the set of flow-invariant ergodic measures supported on K, and $h_{KS}(\mu)$ is the Kolmogorov–Sinai entropy of the measure [45, Corollary 9.10.1].

Take $K=K_E^\delta$. Because the flow leaves the foliation

$$K_E^\delta=\bigsqcup_{E'\in[E-\delta,E+\delta]}K_{E'}$$

invariant, any ergodic measure supported on K_E^δ is actually supported on a single $K_{E'}$. Hence, we deduce that

$$\mathcal{P}_E^\delta(s)=\sup_{E'\in[E-\delta,E+\delta]}\mathcal{P}_{E'}(s).$$

Now, from the structural stability of the flow on K_E, the function $E'\mapsto\mathcal{P}_{E'}(s)$ is continuous near E (this continuity is an obvious generalization of [4, Proposition 5.4]), from which we deduce (3.20). □

3.4. Definition of resonances through complex scaling

We briefly recall the complex scaling method—see [39] and references given there. Suppose that $P=P(h)$ satisfies the assumptions of §3.2. Here we can consider h as a fixed parameter which plays no role in the definition of resonances.

For any $\theta\in[0,\theta_0]$, let $\Gamma_\theta\subset\mathbb{C}^n$ be a totally real contour with the following properties:

$$\begin{gathered}\Gamma_\theta\cap B_{\mathbb{C}^n}(0,R_0)=B_{\mathbb{R}^n}(0,R_0),\\ \Gamma_\theta\cap\mathbb{C}^n\setminus B_{\mathbb{C}^n}(0,2R_0)=e^{i\theta}\mathbb{R}^n\cap\mathbb{C}^n\setminus B_{\mathbb{C}^n}(0,2R_0),\\ \Gamma_\theta=\{x+iF_\theta(x):x\in\mathbb{R}^n\},\quad \partial_x^\alpha F_\theta(x)=\mathcal{O}_\alpha(\theta).\end{gathered}\tag{3.21}$$

Notice that $F_\theta(x)=(\tan\theta)x$ for $|x|>2R_0$. By gluing $\Gamma_\theta\setminus B(0,R_0)$ to the compact piece X_0 in place of each infinite branch $\mathbb{R}^n\setminus B(0,R_0)$, we obtain a deformation of the manifold X, which we denote by X_θ.

The operator P then defines a dilated operator:

$$P_\theta\stackrel{\text{def}}{=}P^\sharp|_{X_\theta},\quad P_\theta u=P^\sharp(u^\sharp)|_{X_\theta},$$

where $P^\sharp$ is the holomorphic continuation of the operator P, and $u^\sharp$ is an almost analytic extension of $u\in\mathcal{C}_c^\infty(X_\theta)$.

For θ fixed and $E>0$, the scaled operator $P_\theta-E$ is uniformly elliptic in $\Psi_h^{2,0}(X_\theta)$, outside a compact set, hence the resolvent, $(P_\theta-z)^{-1}$, is meromorphic for $z\in D(E,1/C)$. We can also take θ to be h-dependent and the same statement holds for $z\in D(E,\theta/C)$. The spectrum of P_θ with $z\in D(E,\theta/C)$ is independent of θ and consists of *quantum resonances* of P. The latter are generally defined as the poles of the meromorphic continuation of

$$(P-z)^{-1}:\mathcal{C}_c^\infty(X)\longrightarrow\mathcal{C}^\infty(X)$$

from $D(E,\theta/C)\cap\{z:\operatorname{Im}z>0\}$ to $D(E,\theta/C)\cap\{z:\operatorname{Im}z<0\}$. The *resonant states* associated with a resonance z, $\operatorname{Re}z\sim E>0$, $|\operatorname{Im}z|<\theta/C$, are solutions to $(P-z)u=0$ satisfying

$$\begin{gathered}\text{there exists } U\in\mathcal{C}^\infty(\Omega_\theta),\text{ where } \Omega_\theta\stackrel{\text{def}}{=}\bigcup_{-\varepsilon<\theta'<\theta+\varepsilon}X_{\theta'},\text{ such that}\\ u=U|_X,\ u_{\theta'}=U|_{X_{\theta'}}\text{ and } (P_{\theta'}-z)u_{\theta'}=0\text{ for } 0<\theta'<\theta,\text{ and } u_\theta\in L^2(X_\theta).\end{gathered}\tag{3.22}$$

If the multiplicity of the pole is higher, there is a possibility of more complicated states but here, and in Theorem 4, we consider only resonant states satisfying $(P-z)u=0$. At any pole of the meromorphically continued resolvent, such states satisfying (3.22) always exist. We shall also call the state $u_\theta\in L^2(X_\theta)$ a resonant state.

If θ is small, as we shall always assume, we identify X with X_θ using the map

$$\begin{aligned} R: X_\theta &\longrightarrow X, \\ x &\longmapsto \operatorname{Re} x, \end{aligned} \tag{3.23}$$

and using this identification, consider P_θ as an operator on X, defined by $(R^{-1})^* P_\theta R^*$. We note that in the identification of $L^2(X)$ with $L^2(X_\theta)$ using $x \mapsto \operatorname{Re} x$,

$$C^{-1}\|u(h)\|_{L^2(X)} \leqslant \|u(h)\|_{L^2(X_\theta)} \leqslant C\|u(h)\|_{L^2(X)},$$

with C independent of θ if $0 \leqslant \theta \leqslant 1/C_1$.

For later use we conclude by describing the principal symbol of P_θ as an operator on $L^2(X)$ using the identification above:

$$p_\theta(x,\xi) = p(x+iF_\theta(x), (1+idF_\theta(x)^t)^{-1}\xi), \tag{3.24}$$

where the complex arguments are allowed due to the analyticity of $p(x,\xi)$ outside of a compact set—see §3.2. In this paper we will always take $\theta = \mathcal{O}(\log(1/h)h)$ so that $p_\theta(x,\xi) - p(x,\xi) = \mathcal{O}(\log(1/h)h)\langle\xi\rangle^2$. More precisely,

$$\begin{aligned} \operatorname{Re} p_\theta(x,\xi) &= p(x,\xi) + \mathcal{O}(\theta^2)\langle\xi\rangle^2, \\ \operatorname{Im} p_\theta(x,\xi) &= -d_\xi p(x,\xi)[dF_\theta(x)^t\xi] + d_x p(x,\xi)[F_\theta(x)] + \mathcal{O}(\theta^2)\langle\xi\rangle^2. \end{aligned} \tag{3.25}$$

In view of (3.9) and (3.10), we obtain the following estimate when $|x| \geqslant R_0$:

$$\operatorname{Im} p_\theta(x,\xi) = -2\langle dF_\theta(x)\xi, \xi\rangle + \mathcal{O}(\theta(f_0(|x|) + f_1(|x|)) + \theta^2)\langle\xi\rangle^2, \tag{3.26}$$

where $f_j(r) \to 0$ as $r \to \infty$. In particular, if R_0 is taken large enough,

$$(x,\xi) \in \mathcal{E}_E^\delta,\ |x| \geqslant 2R_0 \quad \Longrightarrow \quad \operatorname{Im} p_\theta(x,\xi) \leqslant -C\theta. \tag{3.27}$$

4. Semiclassical Fourier integral operators and their iteration

The crucial step in our argument is the analysis of compositions of a large number—of order $\log(1/h)$—of local Fourier integral operators. This section is devoted to general aspects of that procedure, which will then be applied in §7.

4.1. Definition of local Fourier integral operators

We will here review the *local* theory of these operators in the semiclassical setting. Let $\varkappa: T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ be a local diffeomorphism defined near $(0,0)$, and satisfying

$$\varkappa(0,0) = (0,0) \quad \text{and} \quad \varkappa^*\omega = \omega. \tag{4.1}$$

(Here ω is the standard symplectic form on $T^*\mathbb{R}^n$.) Let us also assume that the following projection from the graph of $\varkappa$,

$$T^*\mathbb{R}^n \times T^*\mathbb{R}^n \ni (x^1,\xi^1;x^0,\xi^0) \longmapsto (x^1,\xi^0) \in \mathbb{R}^n \times \mathbb{R}^n, \quad (x^1,\xi^1) = \varkappa(x^0,\xi^0), \tag{4.2}$$

is a diffeomorphism near the origin. It then follows that there exists a unique function $\psi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for (x^1,ξ^0) near $(0,0)$,

$$\varkappa(\psi'_\xi(x^1,\xi^0),\xi^0) = (x^1,\psi'_x(x^1,\xi^0)), \quad \det \psi''_{x\xi} \neq 0 \quad \text{and} \quad \psi(0,0) = 0.$$

The function ψ is said to *generate* the transformation $\varkappa$ near $(0,0)$. The existence of such a function ψ in a small neighbourhood of $(0,0)$ is equivalent to the following property: the $n\times n$ block $(\partial x^1/\partial x^0)$ in the tangent map $d\varkappa(0,0)$ is invertible.

A *local semiclassical quantization of* $\varkappa$ is an operator $T=T(h)$ acting as follows:

$$Tu(x^1) \stackrel{\text{def}}{=} \frac{1}{(2\pi h)^n} \iint_{\mathbb{R}^{2n}} e^{i(\psi(x^1,\xi^0)-\langle x^0,\xi^0\rangle)/h} \alpha(x^1,\xi^0;h) u(x^0)\, dx^0\, d\xi^0. \tag{4.3}$$

Here the amplitude α is of the form

$$\alpha(x,\xi;h) = \sum_{j=0}^{L-1} h^j \alpha_j(x,\xi) + h^L \widetilde{\alpha}_L(x,\xi;h) \quad \text{for all } L \in \mathbb{N},$$

with all the terms $\alpha_j, \widetilde{\alpha}_L \in S(1)$ supported in a fixed neighbourhood of $(0,0)$. Such an operator T is a local Fourier integral operator associated with $\varkappa$.

We list here several basic properties of T—see, e.g., [41, §3] and [13, Chapter 10]:

- We have $T^*T = A^w(x,hD)$, $A \in S(T^*\mathbb{R}^n)$,

$$A(\psi'_\xi(x^1,\xi^0),\xi^0) = \frac{|\alpha_0(x^1,\xi^0)|^2}{|\det \psi''_{x\xi}(x^1,\xi^0)|} + \mathcal{O}_{S(1)}(h). \tag{4.4}$$

In particular, T is bounded on L^2, uniformly with respect to h.

If $T^*T = I$ microlocally near $U \supset (0,0)$, then

$$|\alpha_0(x^1,\xi^0)| = |\det \psi''_{x\xi}(x^1,\xi^0)|^{1/2} \quad \text{for } (x^0,\xi^0) \text{ near } U. \tag{4.5}$$

- If $\alpha(0,0) \neq 0$, then T is microlocally invertible near $(0,0)$: there exists an operator S of the form (4.3) quantizing $\varkappa^{-1}$, such that $ST = I$ and $TS = I$ microlocally near $(0,0)$.
- For $b \in S(1)$,

$$Tb^w(x,hD) = c^w(x,hD)T + \mathcal{O}_{L^2\to L^2}(h), \quad \varkappa^* c \stackrel{\text{def}}{=} c \circ \varkappa = b.$$

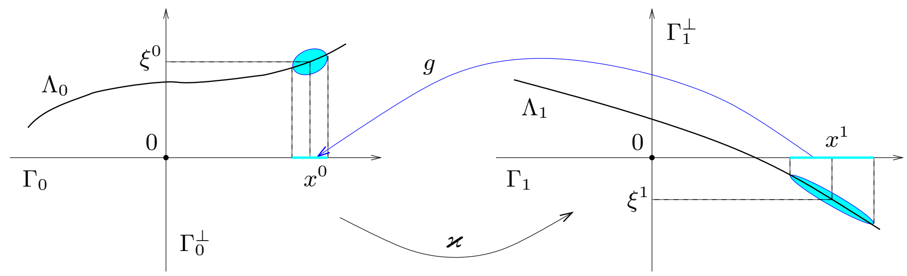

Figure 4. A schematic illustration of the objects appearing in Lemma 4.1. We labelled the x^j and ξ^j axes by Γ_j and $\Gamma_j^\perp$, respectively, in order to represent also the more general case of (4.15).

Moreover, if $\alpha(0,0)\neq 0$, then for any $b\in S(1)$ supported in a sufficiently small neighbourhood of $(0,0)$,

$$Tb^w(x,hD)=c^w(x,hD)T,\quad \varkappa^*c=b+\mathcal{O}_{S(1)}(h). \tag{4.6}$$

The converse is also true: if $\varkappa$ satisfies the projection properties (4.2) and T satisfies (4.6) for all $b\in S(1)$ with support near $(0,0)$, then T is equal to an operator of the form (4.3) microlocally near $(0,0)$. The relation (4.6) is a version of *Egorov's theorem* and we will frequently use it below.

- For $b\in S(1)$ we have $b^w(x,hD)T=\widetilde{T}+\mathcal{O}_{L^2\to L^2}(h^\infty)$, where $\widetilde{T}$ is of the form (4.3) with the same phase $\psi(x^1,\xi^0)$, but with a different symbol $\beta(x^1,\xi^0;h)\in S(1)$. Its principal symbol reads $\beta_0(x^1,\xi^0)=b(x^1,\psi'_x(x^1,\xi^0))\alpha_0(x^1,\xi^0)$, and the full symbol β is supported in $\operatorname{supp}\alpha$.

The proofs of these statements are similar to the proof of the next lemma, which is an application of the stationary phase method and a very special case of the composition formula for Fourier integral operators.

LEMMA 4.1. *We consider a Lagrangian $\Lambda_0=\{(x,\varphi_0'(x)):x\in\Omega_0\}$, $\varphi_0\in C_b^\infty(\Omega_0)$, contained in a small neighbourhood $V\subset T^*\mathbb{R}^n$, such that $\varkappa$ is generated by ψ near V. We assume that*

$$\varkappa(\Lambda_0)=\Lambda_1=\{(x,\varphi_1'(x)):x\in\Omega_1\},\quad \varphi_1\in C_b^\infty(\Omega_1). \tag{4.7}$$

Then, for any symbol $a\in\mathcal{C}_c^\infty(\Omega_0)$, the application of T to the Lagrangian state

$$a(x)e^{i\varphi_0(x)/h}$$

associated with Λ_0 satisfies

$$T(ae^{i\varphi_0/h})(x)=e^{i\varphi_1(x)/h}\biggl(\sum_{j=0}^{L-1}b_j(x)h^j+h^Lr_L(x,h)\biggr), \tag{4.8}$$

where the coefficients b_j *are described as follows. Consider the map*

$$\Omega_1 \ni x \longmapsto g(x) \stackrel{\text{def}}{=} \pi \circ \varkappa^{-1}(x, \varphi_1'(x)) \in \Omega_0, \tag{4.9}$$

where $\pi: T^*\mathbb{R}^n \to \mathbb{R}^n$ *is the standard projection along the fibers. Any point* $x^1 \in \Omega_1$ *is mapped by* g *to the unique point* x^0 *satisfying*

$$\varkappa(x^0, \varphi_0'(x^0)) = (x^1, \varphi_1'(x^1)).$$

The principal symbol b_0 *is then given by*

$$b_0(x^1) = e^{i\beta_0/h} \frac{\alpha_0(x^1, \xi^0)}{|\det \psi''_{x\xi}(x^1, \xi^0)|^{1/2}} |\det dg(x^1)|^{1/2} a \circ g(x^1), \quad \beta_0 \in \mathbb{R}, \ \xi^0 = \varphi_0' \circ g(x^1), \tag{4.10}$$

and it vanishes outside Ω_1*. Furthermore, we have, for any* $\ell \geqslant \mathbb{N}$,

$$\begin{aligned} \|b_j\|_{C^\ell(\Omega_1)} &\leqslant C_{\ell,j} \|a\|_{C^{\ell+2j}(\Omega_0)}, \qquad 0 \leqslant j \leqslant L-1, \\ \|r_L(\cdot, h)\|_{C^\ell(\Omega_1)} &\leqslant C_{\ell,L} \|a\|_{C^{\ell+2L+n}(\Omega_0)}. \end{aligned} \tag{4.11}$$

The constants $C_{\ell,j}$ *depend only on* $\varkappa$, α *and* $\sup_{\Omega_0} |\partial^\beta \varphi_0|$ *for* $0 < |\beta| \leqslant 2\ell + j$.

Proof. The stationary points of the phase in the integral defining $T(ae^{i\varphi_0/h})(x^1)$ are obtained by solving

$$d_{x^0,\xi^0}(\psi(x^1, \xi^0) - \langle x^0, \xi^0 \rangle + \varphi_0(x^0)) = 0 \quad \Longleftrightarrow \quad \begin{cases} \xi^0 = \varphi_0'(x^0), \\ x^0 = \psi_\xi'(x^1, \xi^0). \end{cases}$$

The assumption (4.7) implies, for $x^1 \in \Omega^1$, the existence of a unique solution $x^0 = g(x^1)$, $\xi^0 = \varphi_0' \circ g(x^1)$, and the non-degeneracy of the Hessian of the phase. One also checks that, after inserting the dependence $x^0(x^1)$, $\xi^0(x^1)$ in the critical phase, the derivative of the latter satisfies

$$d_{x^1}(\psi(x^1, \xi^0(x^1)) - \langle x^0(x^1), \xi^0(x^1) \rangle + \varphi_0(x^0(x^1))) = \varphi_1'(x^1).$$

This shows that the critical phase is equal to $\varphi_1(x^1)$, up to an additive constant.

The stationary phase theorem (see for instance [17, Theorem 7.7.6]) now shows that (4.8) holds with

$$b_0(x^1) = e^{i\beta_0/h} |\det(I - \psi''_{\xi\xi}(x^1, \xi^0) \circ \varphi_0''(x^0))|^{-1/2} \alpha_0(x^1, \xi^0) a(x^0), \tag{4.12}$$

$$b_j(x^1) = \sum_{j'=0}^{j} L_{j'}(x^1, D_{x,\xi})(\alpha_{j-j'}(x^1, \xi) a(x))|_{\xi=\xi^0, x=x^0}. \tag{4.13}$$

Each $L_j(x, D_{x,\xi})$ is a differential operator of order $2j$, with coefficients of the form

$$\frac{P_{j\gamma}(x^1)}{\det(I-\psi''_{\xi\xi}(x^1,\xi^0)\circ\varphi''_0(x^0))^{3j}},$$

where $P_{j\gamma}$ is a polynomial of degree $\leqslant 2j$ in the derivatives of ψ and φ_0, of order at most $2j+2$ (the right-hand side of (4.12) can also be written as $L_0(\alpha a)$). The remainder $r_L(x^1;h)$ is bounded by a constant (depending on M and n) times

$$\frac{\left(\sum_{|\alpha|\leqslant 2L}\sup_{x,\xi}|\partial^\alpha_{x,\xi}(\psi(x^1,\xi)-\langle x,\xi\rangle+\varphi_0(x))|\right)^{2L}\left(\sum_{|\alpha|\leqslant 2L+n}\sup_{x,\xi}|\partial^\alpha_{x,\xi}(\alpha(x^1,\xi)a(x))|\right)}{\inf_{x,\xi}|\det(I-\psi''_{\xi\xi}(x^1,\xi)\circ\varphi''_0(x))|^{3L}},$$

with similar estimates for the derivatives $\partial^\ell r_L(\cdot,h)$. The bounds (4.11) follow from the structure of the operators L_j, and the above estimate on the remainder.

It remains to identify the determinant appearing in (4.12) with the more invariant formulation in (4.10). The differential, $d\varkappa(x^0,\xi^0)$, is the map $(\delta x^0,\delta\xi^0)\mapsto(\delta x^1,\delta\xi^1)$, where

$$\begin{aligned}\delta x^0&=\psi''_{\xi x}\delta x^1+\psi''_{\xi\xi}\delta\xi^0,\\ \delta\xi^1&=\psi''_{x\xi}\delta\xi^0+\psi''_{xx}\delta x^1,\end{aligned}$$

and the ψ'' are evaluated at (x^1,ξ^0). By expressing δx^1 and $\delta\xi^1$ in terms of δx^0 and $\delta\xi^0$, we get

$$d\varkappa(x^0,\xi^0)=\begin{pmatrix}(\psi''_{\xi x})^{-1} & -(\psi''_{\xi x})^{-1}\psi''_{\xi\xi}\\ \psi''_{xx}(\psi''_{\xi x})^{-1} & \psi''_{x\xi}-\psi''_{xx}(\psi''_{\xi x})^{-1}\psi''_{\xi\xi}\end{pmatrix}. \tag{4.14}$$

The upper left block in this matrix is indeed invertible, as explained at the beginning of the section. From (4.14) we also see that the restriction of $d\varkappa$ to Λ_0 followed by the projection π is given by

$$\delta x^0\longmapsto\delta x^1=(\psi''_{\xi x})^{-1}(I-\psi''_{\xi\xi}\circ\varphi''_0)(\delta x^0).$$

Hence, noting that $g=\pi\circ\varkappa^{-1}\circ(\pi|_{\Lambda_1})^{-1}=(\pi\circ\varkappa\circ(\pi|_{\Lambda_0})^{-1})^{-1}$, we get

$$\det dg(x^1)=\frac{\det\psi''_{\xi x}(x^1,\xi^0)}{\det(I-\psi''_{\xi\xi}(x^1,\xi^0)\circ\varphi''_0(x^0))},$$

which completes the proof of (4.10). □

We want to generalize the above considerations by relaxing the structure of $\varkappa$: we only assume that $\varkappa$ is locally a canonical diffeomorphism such that $\varkappa(0,0)=(0,0)$. Without loss of generality, we can find linear Lagrangian subspaces, $\Gamma_j, \Gamma_j^\perp \subset T^*\mathbb{R}^n$, $j=0,1$, with the following properties:

- $\Gamma_j^\perp$ is transversal to Γ_j (that is, $\Gamma_j^\perp \cap \Gamma_j = \{0\}$);[1]
- if π_j (resp. $\pi_j^\perp$) is the projection $T^*\mathbb{R}^n \to \Gamma_j$ along $\Gamma_j^\perp$ (resp. the projection $T^*\mathbb{R}^n \to \Gamma_j^\perp$ along Γ_j), then, for some neighbourhood U of the origin, the map

$$\varkappa(U) \times U \ni (\varkappa(\varrho), \varrho) \longmapsto \pi_1(\varkappa(\varrho)) \times \pi_0^\perp(\varrho) \in \Gamma_1 \times \Gamma_0^\perp \tag{4.15}$$

is a local diffeomorphism from the graph of $\varkappa|_U$ to a neighbourhood of the origin in $\Gamma_1 \times \Gamma_0^\perp$. If we write the tangent map $d\varkappa(\varrho)$ as a matrix from $\Gamma_0 \oplus \Gamma_0^\perp$ to $\Gamma_1 \oplus \Gamma_1^\perp$, then the upper left block is invertible.

Let A_j, $j=0,1$, be linear symplectic transformations with the properties

$$A_j(\Gamma_j) = \{(x,0)\} \subset T^*\mathbb{R}^n \quad \text{and} \quad A_j(\Gamma_j^\perp) = \{(0,\xi)\} \subset T^*\mathbb{R}^n,$$

and let M_j be *metaplectic quantizations* of the A_j's (see [11, Appendix to Chapter 7] for a self-contained presentation in the semiclassical spirit). Then the rotated diffeomorphism

$$\widetilde{\varkappa} \stackrel{\text{def}}{=} A_1 \circ \varkappa \circ A_0^{-1} \tag{4.16}$$

has the properties of the map $\varkappa$ in Lemma 4.1. Let $\widetilde{T}$ be a quantization of $\widetilde{\varkappa}$ as in (4.3). Then

$$T \stackrel{\text{def}}{=} M_1^{-1} \circ \widetilde{T} \circ M_0 \tag{4.17}$$

is a quantization of $\varkappa$.

By transposing Lemma 4.1 to this framework, we may apply T to Lagrangian states supported on a Lagrangian Λ_0, $\varkappa(\Lambda_0)=\Lambda_1$, such that $\pi_j: \Lambda_j \to \Gamma_j$ is locally bijective, $j=0,1$. The action of $\varkappa^{-1}$ on Λ_1 can now be represented by the function

$$g = \pi_0 \circ \varkappa^{-1} \circ (\pi_1|_{\Lambda_1})^{-1}: \Gamma_1 \longrightarrow \Gamma_0. \tag{4.18}$$

Finally, performing phase-space translations, we may relax the condition

$$\varkappa(0,0) = (0,0).$$

---

[1] Here $\Gamma^\perp$ is *not* the symplectic annihilator of Γ—see for instance [17, §21.2].

4.2. The Schrödinger propagator as a Fourier integral operator

Using local coordinates on the manifold X, the above formalism applies to propagators acting on $L^2(X)$.

LEMMA 4.2. *Suppose that $P(h)$ satisfies the assumptions of* §3.2,

$$V_0 \Subset \mathcal{E}_E^\delta, \quad \chi \in S(1), \quad \chi|_{\mathcal{E}_E^\delta} \equiv 1 \quad \text{and} \quad V_1 \subset \Phi^t(V_0).$$

For a fixed time $t>0$, let

$$U_\chi(t) \stackrel{\text{def}}{=} \exp(-it\chi^w(x,hD)P(h)\chi^w(x,hD)/h), \tag{4.19}$$

be a modified unitary propagator of P, acting on $L^2(X)$.

Take some $\varrho_0 \in V_0 \cap \mathcal{E}_E$ and set $\varrho_1 = \Phi^t(\varrho_0) \in V_1$. Let $f_j : \pi(V_j) \to \mathbb{R}^n$, $j=0,1$, be local coordinates such that $f_0(\pi(\varrho_0)) = f_1(\pi(\varrho_1)) = 0 \in \mathbb{R}^n$. They induce on V_0 and V_1 the symplectic coordinates

$$F_j(x,\xi) \stackrel{\text{def}}{=} (f_j(x), (df_j(x)^t)^{-1}\xi - \xi^{(j)}), \quad j=0,1, \tag{4.20}$$

where $\xi^{(j)} \in \mathbb{R}^n$ is fixed by the condition $F_j(\varrho_j) = (0,0)$. Then the operator on $L^2(\mathbb{R}^n)$,

$$T^\sharp(t) \stackrel{\text{def}}{=} e^{-i\langle x,\xi^{(1)}\rangle/h}(f_1^{-1})^* U_\chi(t)(f_0)^* e^{i\langle x,\xi^{(0)}\rangle/h}, \tag{4.21}$$

is of the form (4.17) *for some choices of the A_j's, microlocally near* $(0,0)$.

Although complicated to write, the lemma simply states that the propagator is a Fourier integral operator in the sense of this section.

Sketch of the proof of Lemma 4.2. The first step is to prove that for $a \in S(1)$ with support in $\chi \equiv 1$ we have

$$U_\chi(t)^{-1}a^w(x,hD)U_\chi(t) = a_t^w(x,hD), \quad a_t = (\Phi^t)^*a + \mathcal{O}_{S(1)}(h). \tag{4.22}$$

This can be seen from differentiation with respect to t:

$$\partial_t a_t^w = \frac{i}{h}[\chi^w P \chi^w, a_t^w] = \frac{i}{h}[P, a_t^w] + \mathcal{O}(h^\infty), \quad a_0^w = a^w.$$

Since $(i/h)[P, a_t^w] = (H_p a_t)^w + \mathcal{O}(h)$ we conclude that $a_t^w = [(\Phi_t)^*a]^w + \mathcal{O}_{L^2 \to L^2}(h)$. An iteration of this argument shows (4.22) (see [13, Chapter 9] and the proof of Lemma 6.2 below). The converse to Egorov's theorem (see [41, Lemma 3.4] or [13, Theorem 10.7]) implies that (4.19) is a quantization of Φ^t, microlocally near $\varrho_0 \times \varrho_1$.

On the classical level, the symplectic coordinates F_0 and F_1 of (4.20) are such that the symplectic map

$$\varkappa \stackrel{\text{def}}{=} F_1 \circ \Phi^t \circ F_0^{-1} \quad \text{satisfies} \quad \varkappa(0,0) = (0,0).$$

Hence the operator $T^\sharp(t)$ is a quantization of $\varkappa$, and can be put in the form (4.17) for some choice of symplectic rotations A_j, microlocally near $(0,0)$. A possible choice of these rotations is given in Lemma 4.4 below. □

We will now describe a particular choice of coordinate chart in the neighbourhood U_ϱ of an arbitrary point $\varrho \in K_E$. Using the notation of the previous lemma, U_ϱ may be identified through a symplectic map F_ϱ with a neighbourhood of $(0,0) \in T^*\mathbb{R}^n$. This way, Lagrangian (resp. isotropic) subspaces in $T_\varrho(T^*X)$ are identified with Lagrangian (resp. isotropic) subspaces in $T_0(T^*\mathbb{R}^n)$.

We now recall that the weak stable and unstable subspaces $E_\varrho^{\pm 0}$ defined by (3.13) are Lagrangian. The proof of this well-known fact is simple: for any two vectors $v, w \in E_\varrho^+$, we have

$$\omega(v,w) = (\Phi^t)^* \omega(v,w) = \omega(\Phi_*^t v, \Phi_*^t w) \quad \text{for all } t \in \mathbb{R}.$$

By assumption, the vectors on the right-hand side converge to zero when $t \to -\infty$, which proves that the strong unstable subspaces are isotropic. The same method shows that $\omega(v, H_p) = 0$, so the weak unstable subspaces are Lagrangian. The same results apply to the stable subspaces. Besides, the isotropic subspace E_ϱ^- is transversal to the Lagrangian E_ϱ^{+0}, so the tangent space to the energy layer $\mathcal{E}_E$ at ϱ is decomposed into $T_\varrho \mathcal{E}_E = E_\varrho^{+0} \oplus E_\varrho^-$.

LEMMA 4.3. *Take any point $\varrho \in K_E$. As above, we may identify a neighbourhood $U_\varrho \subset T^*X$ of ϱ with a neighbourhood of $(0,0) \in T^*\mathbb{R}^n$. The tangent space $T_\varrho(T^*X)$ is then identified with $T_0(T^*\mathbb{R}^n) \equiv T^*\mathbb{R}^n$.*

The space $T^*\mathbb{R}^n$ can be equipped with a symplectic *basis $(e_1, ..., e_n; f_1, ..., f_n)$ such that $e_1 = H_p(\varrho)$, $E_\varrho^+ = \mathrm{span}\{e_2, ..., e_n\}$ and $E_\varrho^- = \mathrm{span}\{f_2, ..., f_n\}$. We also require that $\Omega_\varrho(e_1 \wedge ... \wedge e_n) = 1$, where Ω is the volume form on E_ϱ^{+0} induced by the adapted metric g_{ad} (see* (vi) *in* (3.12)*). The two Lagrangian subspaces*

$$\Gamma \stackrel{\text{def}}{=} E_\varrho^{+0} \quad \textit{and} \quad \Gamma^\perp \stackrel{\text{def}}{=} E_\varrho^- \oplus \mathbb{R} f_1$$

are transversal. Let us call $(\tilde{y}_1, ..., \tilde{y}_n, \tilde{\eta}_1, ..., \tilde{\eta}_n)$ the linear symplectic coordinates on $T^*\mathbb{R}^n$ dual to the basis $(e_1, ..., e_n; f_1, ..., f_n)$.

There exists a symplectic coordinate chart (y, η) near $\varrho \equiv (0,0)$ such that

$$\eta_1 = p - E, \quad \frac{\partial}{\partial y_j}(0,0) = e_j \ \textit{and} \ \frac{\partial}{\partial \eta_j}(0,0) = f_j, \ j = 1, ..., n. \tag{4.23}$$

Such a chart will be called adapted *to the dynamics. The point* (y,η) *is mapped to* $(\tilde{y},\tilde{\eta})$ *through a local symplectic diffeomorphism fixing the origin, and tangent to the identity at the origin.*

Proof. Once we select the Lagrangian $\Gamma=E_\varrho^{+0}$, with the isotropic E_ϱ^- plane transversal to Γ, it is always possible to complete E_ϱ^- into a Lagrangian $\Gamma^\perp$ transversal to Γ, by adjoining a certain subspace $\mathbb{R}v$ to E_ϱ^-. Since $\Gamma\oplus\Gamma^\perp$ spans the full space $T^*\mathbb{R}^n$, the vector v must be transversal to the energy hyperplane $T_\varrho\mathcal{E}_E$.

Since we took $e_1=H_p(\varrho)$, we can equip E_ϱ^+ with a basis $\{e_2,...,e_n\}$ satisfying

$$\Omega_\varrho(e_1\wedge e_2\wedge...\wedge e_n)=1.$$

There is a unique choice of vectors $\{f_1,...,f_n\}$ such that these vectors generate $\Gamma^\perp$ and satisfy $\omega(f_j,e_k)=\delta_{jk}$ for all $j,k=1,...,n$. The property $\omega(f_j,e_1)=0$ for $j>1$ implies that f_j is in the energy hyperplane, while $\omega(f_1,e_1)=dp(f_1)=1$ shows that $p((\tilde{y},\tilde{\eta}))=E+\tilde{\eta}_1+\mathcal{O}(\tilde{\eta}_1^2)$ when $\tilde{\eta}_1\to 0$.

From Darboux's theorem, there exists a (non-linear) symplectic chart $(y^\flat,\eta^\flat)$ near the origin such that $\eta_1^\flat=p-E$. There also exists a linear symplectic transformation A such that the coordinates $(y,\eta)=A(y^\flat,\eta^\flat)$ satisfy $\eta_1=\eta_1^\flat$ as well as the properties (4.23) on $T_0(T^*\mathbb{R}^n)$. The last statement concerning the mapping $(\tilde{y},\tilde{\eta})\mapsto(y,\eta)$ comes from the fact that the vectors $\partial/\partial\tilde{y}_j$ and $\partial/\partial\tilde{\eta}_j$ satisfy (4.23) as well. $\square$

LEMMA 4.4. *Suppose that* P *satisfies the assumptions of* §3.2 *and the hyperbolicity assumption* (3.11). *Fixing* $t>0$ *and using the notation of Lemmas* 4.2 *and* 4.3, *we consider the symplectic frames* $\Gamma_0\oplus\Gamma_0^\perp$ *and* $\Gamma_1\oplus\Gamma_1^\perp$, *constructed near* ϱ_0 *and* $\varrho_1=\Phi^t(\varrho_0)$, *respectively.*

Then, the graph of Φ^t *near* $\varrho_1\times\varrho_0$ *projects surjectively to* $\Gamma_1\times\Gamma_0^\perp$ (*see* (4.15)). *This implies that the operator* (4.21) *can be written in the form* (4.17), *where the metaplectic operators* M_j *quantize the coordinate changes* $F_j(x,\xi)\mapsto(\tilde{y}^j,\tilde{\eta}^j)$, *while* $\widetilde{T}(t)$ *quantizes* Φ^t *written in the coordinates* $(\tilde{y}^0,\tilde{\eta}^0)\mapsto(\tilde{y}^1,\tilde{\eta}^1)$.

The symplectic coordinate changes $(\tilde{y}^j,\tilde{\eta}^j)\mapsto(y^j,\eta^j)$ *can be quantized by Fourier integral operators* T_0 *and* T_1 *of the form* (4.3) *and microlocally unitary. If we set* $\mathcal{U}_j\stackrel{\text{def}}{=}T_j\circ M_j$, $j=0,1$, *the operator* (4.21) *can then be written as*

$$T^\sharp(t)=\mathcal{U}_1^*\circ T(t)\circ\mathcal{U}_0 \tag{4.24}$$

microlocally near $(0,0)$, *where* $T(t)$ *is a Fourier integral operator of the form* (4.3) *which quantizes the map* Φ^t, *when written in the adapted coordinates* $(y^0,\eta^0)\mapsto(y^1,\eta^1)$.

Proof. We may express the map Φ^t from V_0 to V_1 using the coordinate charts (y^0,η^0) on V_0 and (y^1,η^1) on V_1. The tangent map $d\Phi^t(\varrho_0)$ is then given by a matrix of the form

$$d\Phi^t(\varrho_0)\equiv\begin{pmatrix}1 & 0 & * & 0\\ 0 & A & * & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & * & {}^tA^{-1}\end{pmatrix}. \tag{4.25}$$

Since the full matrix is symplectic, the block

$$\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}$$

is necessarily invertible: this implies that the graph of Φ^t projects surjectively to $\Gamma_1\times\Gamma_0^{\perp}$ in some neighbourhood of $\varrho_1\times\varrho_0$. Equivalently, if we represent Φ^t near $\varrho_1\times\varrho_0$ as a map $\widetilde{\varkappa}$ in the "linear" coordinates $(\tilde y^0,\tilde\eta^0)$ and $(\tilde y^1,\tilde\eta^1)$, the graph of $\widetilde{\varkappa}$ projects surjectively to $(\tilde y^1,\tilde\eta^0)$, so that the operator $\widetilde{T}(t)=M_1\circ T^\sharp(t)\circ M_0^{-1}$ quantizing $\widetilde{\varkappa}$ can be put in the form (4.3) near the origin.

For each $j=0,1$, the tangency of the charts $(\tilde y^j,\tilde\eta^j)$ and (y^j,η^j) at the origin shows that the graph of the the coordinate change $(\tilde y^j,\tilde\eta^j)\mapsto(y^j,\eta^j)$ projects well on $(y^j,\tilde\eta^j)$, so this change can be quantized by an operator T_j of the form (4.3), microlocally unitary near the origin. The operator $T(t)=T_1\circ M_1\circ T^\sharp(t)\circ M_0^*\circ T_0^*$ quantizes Φ^t, when written in the coordinates $(y^0,\eta^0)\mapsto(y^1,\eta^1)$, and can also be written in the form (4.3) near the origin. □

4.3. Iteration of the propagators

Later we will compose operators of type $U(t_0)\Pi_a$, where Π_a is a microlocal cutoff to a small neighbourhood $W_a\subset\mathcal{E}_E^\delta$. In view of Lemma 4.2, the estimates on these compositions can be reduced to estimates on compositions of operators of type (4.3). The next proposition is similar to the results of [2, §3].

We take a sequence of symplectic maps $\{\varkappa_j\}_{j=1}^J$ defined in some open neighbourhood $V\subset T^*\mathbb{R}^n$ of the origin, which satisfy (4.2). Now the $\varkappa_j$'s do not necessarily leave the origin invariant, but we assume that $\varkappa_j(0,0)\subset V$ for all j. We then consider operators $\{T_j\}_{j=1}^J$ which quantize $\varkappa_j$ in the sense of (4.3) and are *microlocally unitary* near an open set $U\Subset V$ containing $(0,0)$. Let $\Omega\subset\mathbb{R}^n$ be an open set such that $U\Subset T^*\Omega$ and $\varkappa_j(U)\Subset T^*\Omega$ for all j.

For each j we take a smooth cutoff function $\chi_j\in\mathcal{C}_c^\infty(U;[0,1])$, and let

$$S_j\stackrel{\text{def}}{=}\chi_j^w(x,hD)\circ T_j. \tag{4.26}$$

We now consider a family of Lagrangian manifolds

$$\Lambda_k = \{(x, \varphi_k'(x)) : x \in \Omega\} \subset T^*\mathbb{R}^n, \quad k = 0, ..., N,$$

sufficiently close to the "position plane" $\{\xi=0\}$:

$$|\varphi_k'| < \varepsilon, \ |\partial^\alpha \varphi_k| \leqslant C_\alpha, \quad k = 0, ..., N, \ \alpha \in \mathbb{N}^n. \tag{4.27}$$

Furthermore, we assume that these manifolds are locally mapped to one another by the $\varkappa_j$'s: there exists a sequence of integers $j_k \in [1, J]$, $k=1, ..., N$, such that

$$\varkappa_{j_{k+1}}(\Lambda_k \cap U) \subset \Lambda_{k+1}, \quad k = 0, ..., N-1. \tag{4.28}$$

We want to propagate an initial Lagrangian state $a(x)e^{i\varphi_0(x)/h}$, $a \in C_c^\infty(\Omega)$, through the sequence of operators S_{j_k}, $k=1, ..., N$.

At each step, the action of $\varkappa_{j_k}^{-1}|_{\Lambda_k}$ can be projected on the position plane, to give a map g_k defined on $\pi\varkappa_{j_k}(U) \subset \Omega$:

$$g_k(x) = \pi \circ \varkappa_{j_k}^{-1}(x, \varphi_k'(x)). \tag{4.29}$$

For each $x = x^N \in \Omega$, we define iteratively $x^{k-1} = g_k(x^k)$, $k = N, ..., 1$: this procedure is possible as long as each x^k lies in the domain of definition of g_k. Let us state our crucial dynamical assumptions: we assume that for all such sequences $(x^N, ..., x^0)$, the Jacobian matrices, $\partial x^k / \partial x^l$, are uniformly bounded from above:

$$\left\| \frac{\partial x^k}{\partial x^l} \right\| = \left\| \frac{\partial (g^{k+1} \circ g^{k+2} \circ ... \circ g^l)}{\partial x^l}(x^l) \right\| \leqslant C_D, \quad 0 \leqslant k < l \leqslant N, \tag{4.30}$$

where C_D is independent of N. This assumption roughly means that the maps g_k are (weakly) contracting.

We will also use the notation

$$D_k \overset{\text{def}}{=} \sup_{x \in \Omega} |\det dg_k(x)|^{1/2}, \quad J_k \overset{\text{def}}{=} \prod_{k'=1}^{k} D_{k'}, \tag{4.31}$$

and assume that the D_k's are uniformly bounded: $1/C_D \leqslant D_k \leqslant C_D$.

We can now state the main propagation estimate of this section which describes an N-iteration of Lemma 4.1.

Proposition 4.1. *We use the above definitions and assumptions, and take N arbitrarily large, possibly varying with h.*

Take any $a\in C_c^\infty(\Omega)$ and consider the Lagrangian state $u=ae^{i\varphi_0/h}$ associated with the Lagrangian Λ_0.

Then we may write

$$(S_{j_N}\circ...\circ S_{j_1})(ae^{i\varphi_0/h})(x)=e^{i\varphi_N(x)/h}\left(\sum_{j=0}^{L-1}h^j a_j^N(x)+h^L R_L^N(x,h)\right), \tag{4.32}$$

where each $a_j^N\in C_c^\infty(\Omega)$ is independent of h, while $R_L^N\in C^\infty((0,1]_h,\mathcal{S}(\mathbb{R}^n))$.

If $x^N\in\Omega$ defines a sequence (see (4.29)) $x^{k-1}=g_k(x^k)$, $k=N,...,1$, then

$$|a_0^N(x^N)|=\left(\prod_{k=1}^N \chi_{j_k}(x^k,\varphi_k'(x^k))|\det dg_k(x^k)|^{1/2}\right)|a(x^0)|, \tag{4.33}$$

otherwise $a_j^N(x^N)=0$, $j=0,...,L-1$. Also, we have the bounds

$$\|a_j^N\|_{C^\ell(\Omega)}\leqslant C_{j,\ell}J_N(N+1)^{\ell+3j}\|a\|_{C^{\ell+2j}(\Omega)}, \qquad j=0,...,L-1,\ \ \ell\in\mathbb{N}, \tag{4.34}$$

$$\|R_L^N\|_{L^2(\mathbb{R}^n)}\leqslant C_L\|a\|_{C^{2L+n}(\Omega)}(1+C_0h)^N\sum_{k=1}^N J_k k^{3L+n}. \tag{4.35}$$

The constants $C_{j,\ell}$, C_0 and C_L depend on the constants in (4.27) and on the operators $\{S_j\}_{j=1}^J$.

A crucial point in the above proposition is the explicit dependence on N.

Proof. The proof of the proposition proceeds by iterating the results of Lemma 4.1, keeping track of the bounds on the symbols and remainders.

For each j, the operator $S_j=\chi_j^wT_j$ can also be written in the form (4.3), up to an error $\mathcal{O}_{L^2\to L^2}(h^\infty)$, with the symbol $\alpha^j(x^1,\xi^0;h)$ replaced by $\beta^j(x^1,\xi^0;h)$ of compact support, and principal symbol $\beta_0^j(x^1,\xi^0)=\chi_j(x^1,\psi_{jx}'(x^1,\xi^0))\alpha_0^j(x^1,\xi^0)$. From the unitarity of T_j, α_0^j satisfies (4.5) near U; as a result, when applying S_j to a Lagrangian state as in Lemma 4.1, the first ratio in (4.10) should be replaced by $\chi_j(x^1,\xi^1)$.

To abbreviate the formulas, we set

$$f_k(x)\stackrel{\text{def}}{=}e^{i\gamma_k(x)}\chi_{j_k}(x,\varphi_k'(x))|\det dg_k(x)|^{1/2}, \quad k=1,...,N,$$

where using unitarity (4.5),

$$e^{i\gamma_k(x)}=e^{i\beta_k/h}\frac{\alpha_0^{j_k}(x,\varphi_{k-1}'(g_k(x)))}{|\det\psi_{j_kx\xi}''(x,\varphi_{k-1}'(g_k(x)))|^{1/2}}.$$

Here β_k is a constant phase, as in (4.10). We will also use the short notation

$$a^N_{j,\ell} \stackrel{\text{def}}{=} \|a^N_j\|_{C^\ell(\Omega)}, \quad j=0,...,L-1, \ \ell\in\mathbb{N}.$$

We first analyze the principal symbol $a^N_0(x)$. The formula (4.10) and the definition of f_k give

$$a^N_0(x^N)=f_N(x^N)a^{N-1}_0(x^{N-1}), \tag{4.36}$$

which by iteration yields (4.33). From $\|f_k\|_{C^0}\leqslant D_k$ the recursive relation (4.36) also implies the bound $a^N_{0,0}\leqslant J_N\|a\|_{C^0}$.

To estimate higher C^ℓ norms we differentiate (4.36) with respect to x^N:

$$\frac{\partial a^N_0}{\partial x^N}=f_N(x^N)\frac{\partial x^{N-1}}{\partial x^N}\frac{\partial a^{N-1}_0}{\partial x^{N-1}}+\frac{\partial f_N}{\partial x^N}a^{N-1}_0(x^{N-1})$$

(to simplify the notation we omit the subscripts corresponding to the coordinates in $x^N=(x^N_1,...,x^N_n))$. Since we already control $a^{N-1}_{0,0}$, and the norms $\|f_N\|_{C^1}$ are bounded uniformly in N, the above expression can be schematically written as

$$\frac{\partial a^N_0}{\partial x^N}=f_N\frac{\partial x^{N-1}}{\partial x^N}\frac{\partial a^{N-1}_0}{\partial x^{N-1}}+\mathcal{O}(J_{N-1}\|a\|_{C^0}),$$

with an implied constant independent of N. Applying this equality iteratively to $\partial a^k_0/\partial x^k$ down to $k=0$, we obtain

$$\begin{aligned}\frac{\partial a^N_0}{\partial x^N}&=f_Nf_{N-1}...f_1\frac{\partial x^0}{\partial x^N}\frac{\partial a^0_0}{\partial x^0}\\&\quad+\mathcal{O}\biggl(J_{N-1}+f_N\frac{\partial x^{N-1}}{\partial x^N}J_{N-2}+f_Nf_{N-1}\frac{\partial x^{N-2}}{\partial x^N}J_{N-3}+...+f_Nf_{N-1}...f_2\frac{\partial x^1}{\partial x^N}\biggr)\|a\|_{C^0}.\end{aligned}$$

Notice that $a^0_0=a$. Using the uniform bounds for the Jacobian matrices $\partial x^k/\partial x^N$ and for the D_k, this expression leads to

$$a^N_{0,1}\leqslant CJ_N\|a\|_{C^1}+C\|a\|_{C^0}\sum_{k=1}^N\frac{J_N}{D_k}\leqslant C_{0,1}J_N(N+1)\|a\|_{C^1}.$$

The same procedure can be applied to higher derivatives of a^N_0: since $\|f_N\|_{C^\ell}$ is uniformly bounded, the chain rule shows that the ℓth derivatives of (4.36) can be written

$$\frac{\partial^\ell a^N_0}{(\partial x^N)^\ell}=f_N(x^N)\biggl(\frac{\partial x^{N-1}}{\partial x^N}\biggr)^\ell\frac{\partial^\ell a^{N-1}(x^{N-1})}{(\partial x^{N-1})^\ell}+\mathcal{O}(a^{N-1}_{0,\ell-1}).$$

Assume that we have proven the bounds (4.34) for the $a^k_{0,\ell-1}$, $k=0,...,N$. Iterating the above equality from $k=N-1$ down to $k=0$ yields the following estimate for $\partial^\ell a^N_0/(\partial x^N)^\ell$:

$$\frac{\partial^\ell a^N_0}{(\partial x^N)^\ell}=f_Nf_{N-1}...f_1\left(\frac{\partial x^0}{\partial x^N}\right)^\ell\frac{\partial^\ell a^0_0}{(\partial x^0)^\ell}+\mathcal{O}\biggl(J_{N-1}N^{\ell-1}+f_N\left(\frac{\partial x^{N-1}}{\partial x^N}\right)^\ell J_{N-2}(N-1)^{\ell-1}$$
$$+f_Nf_{N-1}\left(\frac{\partial x^{N-2}}{\partial x^N}\right)^\ell J_{N-3}(N-2)^{\ell-1}+...+f_Nf_{N-1}...f_2\left(\frac{\partial x^1}{\partial x^N}\right)^\ell\biggr)\|a\|_{C^{\ell-1}}. \tag{4.37}$$

Using the uniform bounds (4.30) for $\partial x^k/\partial x^N$ and D_k, we get

$$a^N_{0,\ell}\leqslant C_\ell J_N\|a\|_{C^\ell}+C\|a\|_{C^{\ell-1}}\sum_{k=1}^N\frac{J_N}{D_k}k^{\ell-1}\leqslant C_{0,\ell}J_N(N+1)^\ell\|a\|_{C^\ell}.$$

We can now deal with higher-order coefficients a^N_j by double induction on j and N. Above we have proved the bounds for $j=0$ and all N. Assume now that, for some $j\geqslant1$, we have proved the bounds (4.34) for $a^N_{j',\ell}$ for all $j'<j$, $\ell\geqslant0$ and all $N\geqslant1$. By induction on N we will prove the bounds for that j and all N.

Applying Lemma 4.1 term by term to

$$a^{N-1}\stackrel{\text{def}}{=}\sum_{j=0}^{L-1}h^ja^{N-1}_j+h^LR^{N-1}_L,$$

we see that each component a^N_j depends on the components $a^{N-1}_{j'}$, $0\leqslant j'\leqslant j$, and not on R^{N-1}_L. More precisely, from (4.13) we get

$$\begin{aligned}a^N_j(x^N)&=\sum_{j'=0}^j L^N_{j'}(\beta^{j_N}a^{N-1}_{j-j'})(x^N)\\&=f_N(x^N)a^{N-1}_j(x^{N-1})+\sum_{j'=1}^j\sum_{|\gamma|\leqslant2j'}\Gamma^N_{j'\gamma}(x^N)\partial^\gamma a^{N-1}_{j-j'}(x^{N-1}).\end{aligned}\tag{4.38}$$

As explained in the proof of Lemma 4.1, the functions $\Gamma^N_{j'\gamma}(x)$ can be expressed in terms of the map $\varkappa_{j_N}$ and the functions φ_{N-1} and β^{j_N}. From the assumptions on the latter, the norms $\|\Gamma^N_{j\gamma}\|_{C^\ell}$ are bounded uniformly with respect to N, so (4.38) implies the following upper bound:

$$a^N_{j,0}\leqslant D_Na^{N-1}_{j,0}+C\sum_{j'=1}^j a^{N-1}_{j-j',2j'} \tag{4.39}$$

$$\leqslant D_Na^{N-1}_{j,0}+CJ_{N-1}\sum_{j'=1}^j N^{2j'+3(j-j')}\|a\|_{C^{2j'}} \tag{4.40}$$

$$\leqslant D_Na^{N-1}_{j,0}+CjJ_{N-1}N^{3j-1}\|a\|_{C^{2j}}. \tag{4.41}$$

This inequality can be used in an induction with respect to N, starting from the trivial $a^0_{j,0}=0$. Assuming that $a^{N-1}_{j,0}\leqslant C_{j,0}J_{N-1}N^{3j}\|a\|_{C^{2j}}$ for some $C_{j,0}>0$, we obtain

$$a^N_{j,0}\leqslant C_{j,0}J_N\left(N^{3j}+\frac{Cj}{C_{j,0}D_N}N^{3j-1}\right)\|a\|_{C^{2j}}. \tag{4.42}$$

The constant $C_{j,0}$ can be chosen large enough, so that the brackets are smaller than $N^{3j}+3jN^{3j-1}\leqslant(N+1)^{3j}$, which proves the induction step for $a^N_{j,0}$.

Once we have proved the bounds for the sup-norms of the symbols a^N_j, we can estimate their derivatives by induction on ℓ, as we did above for the principal symbol a^N_0. Assume that we have proved the bounds (4.34) for all $a^N_{j,l}$, $N\geqslant1$, $0\leqslant l\leqslant\ell-1$. If we differentiate (4.38) ℓ times with respect to x^N, we get

$$\frac{\partial^\ell a^N_j}{(\partial x^N)^\ell}=f_N\left(\frac{\partial x^{N-1}}{\partial x^N}\right)^\ell\frac{\partial^\ell a^{N-1}_j}{(\partial x^{N-1})^\ell}+\mathcal{O}\left(a^{N-1}_{j,\ell-1}+\sum_{j'=1}^{j}a^{N-1}_{j-j',\ell+2j'}\right),$$

where the implied constant depends on the bounds on $\|\Gamma^N_{j\gamma}\|_{C^\ell}$. Taking into account what we already know on $a^{N-1}_{j,\ell-1}$ and $a^{N-1}_{j-j',\ell+2j'}$, this takes the form

$$\frac{\partial^\ell a^N_j}{(\partial x^N)^\ell}=f_N\left(\frac{\partial x^{N-1}}{\partial x^N}\right)^\ell\frac{\partial^\ell a^{N-1}_j}{(\partial x^{N-1})^\ell}+\mathcal{O}(J_{N-1}N^{\ell+3j-1}\|a\|_{C^{\ell+2j}}).$$

Applying iteratively this equality to $\partial^\ell a^k_j/(\partial x^k)^\ell$ down to $k=1$ (as in (4.37)) and using that $a^0_j(x)\equiv0$, $j>0$, we find

$$\begin{aligned}\frac{1}{\|a\|_{C^{\ell+2j}}}\frac{\partial^\ell a^N_j}{(\partial x^k)^\ell}=\mathcal{O}\biggl(&J_{N-1}N^{\ell+3j-1}+f_N\left(\frac{\partial x^{N-1}}{\partial x^N}\right)^\ell J_{N-2}(N-1)^{\ell+3j-1}\\&+f_Nf_{N-1}\left(\frac{\partial x^{N-2}}{\partial x^N}\right)^\ell J_{N-3}(N-2)^{\ell+3j-1}\\&+\ldots+f_Nf_{N-1}\ldots f_1\left(\frac{\partial x^1}{\partial x^N}\right)^\ell\biggr).\end{aligned} \tag{4.43}$$

From the uniform bound (4.30) and $\|f_k\|_{C^0}\leqslant D_k$, this gives

$$a^N_{1,1}\leqslant CJ_N\sum_{k=1}^N\frac{k^{\ell+3j-1}}{D_k}\leqslant C_{j,\ell}J_NN^{\ell+3j}\quad\text{for a certain } C_{1,\ell}>0.$$

This proves the induction step $\ell-1\to\ell$, so that we now have proved the bounds for $a^N_{j,\ell}$ for all $N\geqslant1$ and $\ell\geqslant0$. This achieves to show the induction step on j, and (4.34).

To estimate the remainder $R^N_L(x,h)$ we define $r^N_{k+1}(x,h)$ by

$$\begin{aligned}&S_{j_{k+1}}(e^{i\varphi_k/h}(a^k_0+ha^k_1+\ldots+h^{L-1}a^k_{L-1}))\\&\qquad=e^{i\varphi_k/h}(a^{k+1}_0+ha^{k+1}_1+\ldots+h^{L-1}a^{k+1}_{L-1}+h^Lr^{k+1}_L(\cdot,h)).\end{aligned}$$

Due to the cutoff function χ_j^w, the remainder will be

$$\mathcal{O}\left(\left(\frac{h}{h+d(\,\cdot\,,\pi\operatorname{supp}\chi_j)}\right)^{\infty}\right)$$

outside $\pi\operatorname{supp}\chi_j$, so it is essentially supported inside Ω. On the other hand, from Lemma 4.1 and the estimates (4.34), we get

$$\begin{aligned}\|r_L^{k+1}(\,\cdot\,,h)\|_{C^\ell(\mathbb{R}^n)} &\leqslant C_{L,\ell}\sum_{j=0}^{L-1}\|a_j^k\|_{C^{\ell+n+2(L-j)}} \leqslant C_{L,\ell}\sum_{j=0}^{L-1}J_k(k+1)^{j+\ell+n+2L}\|a\|_{C^{\ell+n+2L}}\\ &\leqslant C_{L,\ell}J_k(k+1)^{3L+\ell+n}\|a\|_{C^{\ell+n+2L}}.\end{aligned}$$

In particular,

$$\|r_L^{k+1}(\,\cdot\,,h)\|_{L^2(\mathbb{R}^n)} \leqslant C_L J_k(k+1)^{3L+n}\|a\|_{C^{2L+n}}. \tag{4.44}$$

The remainder $R_N^L(x,h)$ can now be written as

$$R_L^N = r_N^L + e^{-i\varphi_N/h}\sum_{k=1}^{N-1}(S_{j_N}\circ\ldots\circ S_{j_{k+1}})(r_L^k e^{i\varphi_k/h}).$$

Since we assumed that the T_j's are microlocally unitary on the support of the χ_j's, and that $0\leqslant\chi_j\leqslant 1$, we have, from the sharp Gårding inequality,

$$\|S_j\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leqslant 1+C_0h.$$

The above formula for R_N^L and (4.44) give the estimate (4.35). $\square$

Remark 4.1. We can also obtain slightly weaker pointwise estimates on R_L^N in place of the L^2 estimates of (4.32). In fact, since the χ_j's are compactly supported, we have

$$h^{n/2+\ell}\|R_L^N\|_{C^\ell(\mathbb{R}^n)} \leqslant C_\ell\|R_L^N\|_{H_h^\ell} \leqslant C'_\ell\|R_L^N\|_{L^2(\mathbb{R}^n)},$$

and hence

$$\|R_L^N(\,\cdot\,,h)\|_{C^\ell(\mathbb{R}^n)} \leqslant C_{L\ell}h^{-n/2-\ell}\|a\|_{C^{2L+n}}(1+C_0h)^N\sum_{k=1}^{N}J_k k^{3L+n}.$$

5. Classical dynamics

In this section we analyze the evolution of a family of Lagrangian leaves through the classical flow. We will check that these Lagrangians (which remain in the vicinity of the trapped set) stay "under control" uniformly with respect to time. Eventually, this uniform control, which implies that the conditions (4.27) hold, will allow us to apply Proposition 5.1 in §7.

5.1. Evolution of Lagrangian leaves

5.1.1. Poincaré sections and Poincaré maps

We describe the construction of Poincaré sections and maps associated with the flow Φ^t on $\mathcal{E}_E$ in the vicinity of K_E. This construction will be used in the next section.

Take $\varrho_0 \in K_E$. We use an adapted coordinate chart (y^0, η^0) centered at $\varrho_0 \equiv (0,0)$ to parametrize the neighbourhood of ϱ_0 in T^*X, with properties as described in Lemma 4.3. To keep in mind that

$$E^+_{\varrho_0} = \operatorname{span}\left\{ \frac{\partial}{\partial y_j}(0) : j = 2, ..., n \right\},$$

(and similarly for $E^-_{\varrho_0}$), we keep the "time" and "energy" coordinates y^0_1 and η^0_1, but rename the transversal coordinates as

$$u^0_j \stackrel{\text{def}}{=} y^0_{j+1} \text{ and } s^0_j \stackrel{\text{def}}{=} \eta^0_{j+1}, \quad j = 1, ..., n-1.$$

For any small $\varepsilon > 0$ and using the Euclidean disk $D_\varepsilon = \{u \in \mathbb{R}^{n-1} : |u| < \varepsilon\}$, we define a neighbourhood of ϱ_0 as the polydisk

$$U_0(\varepsilon) \equiv \{(y^0, \eta^0) : |y^0_1| < \varepsilon, |\eta^0_1| < \delta, u^0 \in D_\varepsilon \text{ and } s^0 \in D_\varepsilon\}. \tag{5.1}$$

Here $\delta > 0$ corresponds to an energy interval where the dynamics remains uniformly hyperbolic, as mentioned in (3.14). The intersection $U_0(\varepsilon) \cap \mathcal{E}_E$ is obtained by imposing the condition $\eta^0_1 = 0$, and a *Poincaré section* $\Sigma_0 = \Sigma_0(\varepsilon)$ transversal to the flow is obtained by imposing both $\eta^0_1 = 0$ and $y^0_1 = 0$. The chart (u^0, s^0) on Σ_0 is symplectic with respect to the induced symplectic structure on Σ_0.

Let us assume that the point $\Phi^1(\varrho_0)$ belongs to a polydisk $U_1(\varepsilon)$ constructed similarly around a certain point $\varrho_1 \in K_E$, using an adapted chart (y^1, η^1). As a result, the Poincaré section $\Sigma_1 \equiv \{(y^1, \eta^1) : y^1_1 = \eta^1_1 = 0\}$ will intersect the trajectory $(\Phi^s(\varrho_0))_{|s-1| \leqslant \varepsilon}$ at a single point, which we call ϱ'_0. The *Poincaré map* $\varkappa$ is defined, for $\varrho \in \Sigma_0(\varepsilon)$ near ϱ_0, by taking the intersection of the trajectory $(\Phi^t(\varrho))_{|t-1| \leqslant \varepsilon}$ with the section Σ_1 (this intersection consists of at most one point). This map is automatically *symplectic*. In general, the strong (un)stable spaces $E^\pm_{\varrho'_0}$ are not exactly tangent to Σ_1, but close to it: they form "angles" $\mathcal{O}(\varepsilon)$ with the intersections,

$$\widetilde{E}^\pm_{\varrho'_0} \stackrel{\text{def}}{=} E^{\pm 0}_{\varrho'_0} \cap T_{\varrho'_0} \Sigma_1.$$

Furthermore, since the (un)stable subspaces $E^\pm_\varrho$ are *Hölder continuous* with respect to $\varrho \in K_E$, with some Hölder exponent $\gamma > 0$, and $d(\varrho'_0, \varrho_1) \leqslant \varepsilon$, the subspaces $E^\pm_{\varrho'_0}$ form "angles" $\mathcal{O}(\varepsilon^\gamma)$ with $E^\pm_{\varrho_1}$. The tangent map $d\varkappa(\varrho_0)$ maps $E^\pm_{\varrho_0}$ to $\widetilde{E}^\pm_{\varrho'_0}$. Hence, using the coordinate frames $\{(u^0, s^0)\}$ on Σ_0 (centered at ϱ_0) and $\{(u^1, s^1)\}$ on Σ_1 (centered at ϱ_1), the symplectic matrix representing $d\varkappa(\varrho_0)$ can be written in the form

$$d\varkappa(\varrho_0) \equiv \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} + \varepsilon^\gamma \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}, \tag{5.2}$$

where the second matrix on the right has uniformly bounded entries. From the assumptions (3.11) on hyperbolicity, for ε small enough there exists

$$\nu = e^{-\lambda} + \mathcal{O}(\varepsilon^\gamma) < 1 \tag{5.3}$$

such that the matrix A satisfies

$$\|A^{-1}\| \leqslant \nu \quad \text{and} \quad \|{}^tA^{-1}\| \leqslant \nu, \tag{5.4}$$

where $\|{}^tA^{-1}\|$ is computed using the norms on $T_{\varrho_0}\Sigma_0$ and $T_{\varrho_1}\Sigma_1$ induced by the adapted metric g_{ad} (see §3.3). By extension, in the neighbourhood $V \subset \Sigma_0$ where it is defined, $\varkappa$ takes the following form in the coordinates $(u^0, s^0) \mapsto (u^1, s^1)$:

$$\varkappa(u^0, s^0) = (u^1, s^1)(\varrho_0') + (Au^0 + \alpha(u^0, s^0), {}^tA^{-1}s^0 + \beta(u^0, s^0)), \quad (u^0, s^0) \in V, \tag{5.5}$$

and the smooth functions α and β satisfy

$$\alpha(0,0) = \beta(0,0) = 0, \quad \|\alpha\|_{C^1(V)} \leqslant C\varepsilon^\gamma \quad \text{and} \quad \|\beta\|_{C^1(V)} \leqslant C\varepsilon^\gamma. \tag{5.6}$$

5.1.2. Evolving Lagrangian leaves

Given $\varepsilon > 0$, one can choose a finite set of points $\{\varrho_j \in K_E\}_{j \in I}$, adapted charts (y^j, η^j) centered on ϱ_j, such that the polydisks

$$U_j(\varepsilon) \equiv \{(y^j, \eta^j) : |y_1^j| < \varepsilon, |\eta_1^j| < \delta, u^j \in D_\varepsilon \text{ and } s^j \in D_\varepsilon\}$$

form an open cover of the trapped set K_E^δ in the energy layer $\mathcal{E}_E^\delta$:

$$K_E^\delta \subset \bigcup_{j \in I} U_j(\varepsilon). \tag{5.7}$$

For some index $j_0 \in I$, let $\Lambda = \Lambda^0_{\mathrm{loc}} \subset U_{j_0}(\varepsilon) \cap \mathcal{E}_E$ be a connected isoenergetic Lagrangian leaf.(2) For any $t > 0$ we call $\Lambda^t = \Phi^t(\Lambda)$.

(2) Here and below, a *leaf* is a contractible submanifold with piecewise smooth boundary.

We consider a point $\varrho_0 \in \Lambda$, and assume that there exists an integer $N>0$ such that, for each integer time $0 \leqslant k \leqslant N$, the point $\varrho_k = \Phi^k(\varrho_0)$ belongs to the set $U_{j_k}(\varepsilon)$ for some $j_k \in I$. We then call Λ^k_{loc} the connected part of $\big(\bigcup_{|s|<\varepsilon} \Phi^s \Lambda^k\big) \cap U_{j_k}(\varepsilon)$ containing ϱ_k.

We may use the symplectic coordinate chart (y^{j_k}, η^{j_k}) to represent Λ^k_{loc}. Being contained in a single energy shell $\mathcal{E}_E$, the Lagrangian leaf Λ^k_{loc} is foliated by flow trajectories (bicharacteristics). It can be put into the form

$$\Lambda = \bigcup_{|s|<\varepsilon} \Phi^s(S^k), \tag{5.8}$$

where $S^k = \Lambda^k_{\mathrm{loc}} \cap \Sigma_{j_k}$ is an $(n-1)$-dimensional Lagrangian leaf in the symplectic section

$$\Sigma_{j_k}(\varepsilon) = U_{j_k}(\varepsilon) \cap \{(y^{j_k}, \eta^{j_k}) : y_1^{j_k} = \eta_1^{j_k} = 0\}$$

(see Figure 5 for a representation of the above objects).

We will be interested in Lagrangian leaves which are "transversal enough" to the stable subspace $E^-_{\varrho_k}$, and can therefore be represented by graphs of smooth functions in the adapted charts:

$$\Lambda^k_{\mathrm{loc}} \equiv \{(y^{j_k}, \eta^{j_k}) : \eta^{j_k} = F^k(y^{j_k})\}. \tag{5.9}$$

The intersection $S^k = \Lambda^k_{\mathrm{loc}} \cap \Sigma_{j_k}$ is then also given by a graph:

$$S^k \equiv \{(u^{j_k}, s^{j_k}) : s^{j_k} = f^k(u^{j_k}) \text{ and } u^{j_k} \in D_\varepsilon\},$$

and (5.8) implies that $F^k(y^{j_k}) = (0, f^k(u^{j_k}))$, so that (5.9) takes the form

$$\Lambda^k_{\mathrm{loc}} \equiv \{(y_1^{j_k}, u^{j_k}; 0, f^k(u^{j_k})) : |y_1^{j_k}| < \varepsilon \text{ and } u^{j_k} \in D_\varepsilon\}. \tag{5.10}$$

Convention. In the rest of this section the norm $\|\cdot\|$ applying to an object living on $\Sigma_{j_k} \equiv D_\varepsilon \times D_\varepsilon$ corresponds to the Euclidean norm on $T_{\varrho_{j_k}} \Sigma_{j_k}$ relative to the adapted metric $g_{\mathrm{ad}}(\varrho_{j_k})$. The same convention applies to the norm $\|\cdot\|$ of a linear operator sending an object on Σ_{j_k} to an object on $\Sigma_{j_{k+1}}$ (or vice versa).

The following result (similar to the inclination lemma of [19, Proposition 6.2.23]) shows that, if ε has been chosen small enough and Λ is "transversal enough" to the stable manifolds (that is, in some "unstable cone"), then the local Lagrangian leaves Λ^k_{loc} remain in the same unstable cone, uniformly with respect to $k = 0, ..., N$.

Proposition 5.1. *Fix some* $\gamma_1>0$. *Then there exists* $\varepsilon_{\gamma_1}>0$ *such that, provided the diameter* $\varepsilon\in(0,\varepsilon_{\gamma_1})$, *the following holds*:

Suppose the Lagrangian $\Lambda=\Lambda^0_{\mathrm{loc}}\subset\mathcal{E}_E\cap U_{j_0}(\varepsilon)$ *is the graph of a smooth function* f^0 *in the adapted frame* (y^{j_0},η^{j_0}), *and is contained in the* unstable γ_1-cone:

$$\Lambda^0_{\mathrm{loc}}\equiv\{(y_1^{j_0},u^{j_0};0,f^0(u^{j_0})):|y_1^{j_0}|<\varepsilon \text{ and } u^{j_0}\in D_\varepsilon\},\quad \text{with } \sup_{u^{j_0}}\|df^0(u^{j_0})\|\leqslant\gamma_1.$$

(i) *Then, for any* $0\leqslant k\leqslant N$, *the connected component* $\Lambda^k_{\mathrm{loc}}\subset U_{j_k}(\varepsilon)$ *containing* ϱ_k *is also a graph in the frame* (y^{j_k},η^{j_k}), *and is also contained in the unstable* γ_1-*cone*:

$$\Lambda^k_{\mathrm{loc}}\equiv\{(y_1^{j_k},u^{j_k};0,f^k(u^{j_k})):|y_1^{j_k}|<\varepsilon \text{ and } u^{j_k}\in D_\varepsilon\},\quad \text{with } \sup_{u^{j_k}\in D_\varepsilon}\|df^k(u^{j_k})\|\leqslant\gamma_1.$$

(ii) *For any integer* $\ell\geqslant2$, *there exists* $\gamma_\ell>0$ *such that, if* f^0 *is in the unstable* γ_1-*cone and satisfies* $\|f^0\|_{C^\ell}\leqslant\gamma_\ell$, *then*

$$\|f^k\|_{C^\ell(D_\varepsilon)}\leqslant\gamma_\ell\quad \text{for all } k=0,...,N. \tag{5.11}$$

(iii) *From the above properties, near* ϱ_0 *the map* $\Phi^N|_\Lambda$ *can be projected on the planes* $\{(y^{j_0},\eta^{j_0}):\eta^{j_0}=0\}$ *and* $\{(y^{j_N},\eta^{j_N}):\eta^{j_N}=0\}$, *inducing a map* $y^{j_0}\mapsto y^{j_N}$.

In the case where the sets $U_{j_k}(\varepsilon)$ *contain a trajectory in* K^δ_E *(so these sets may be centered on* $\varrho_{j_k}=\Phi^k(\varrho_{j_0})$), *the projected map* $y^{j_0}\mapsto y^{j_N}$ *satisfies the following estimate on its domain of definition*:

$$\det\left(\frac{\partial y^{j_N}}{\partial y^{j_0}}\right)=(1+\mathcal{O}(\varepsilon))e^{\lambda^+_N(\varrho_{j_0})}.$$

Here λ^+_N *is the unstable Jacobian given in* (3.17). *The crucial point is that the implied constant is independent of* N.

Proof. We follow the proof of the stable/unstable manifold theorem for hyperbolic flows [19, Theorems 6.2.8 and 17.4.3].

For each $k=0,...,N$, the Poincaré section Σ_{j_k} does generally not contain ϱ_k, but it contains a unique iterate $\varrho'_k=\Phi^s\varrho_k$ for some $s\in(-\varepsilon,\varepsilon)$. The Poincaré map $\varkappa_k$ from $V_k\subset\Sigma_{j_k}(\varepsilon)$ to $\Sigma_{j_{k+1}}(\varepsilon)$ will satisfy $\varkappa_k(\varrho'_k)=\varrho'_{k+1}$.

Since $d(\varrho_{j_k},\varrho'_k)\leqslant\varepsilon$ and $d(\varrho_{j_{k+1}},\varrho'_{k+1})\leqslant\varepsilon$, there exists $C>1$ such that the extended Poincaré map from $\Sigma_{j_k}(\varepsilon)$ to $\Sigma_{j_{k+1}}(C\varepsilon)$ sends ϱ_{j_k} to a point $\varrho'_{j_k}\in\Sigma_{j_{k+1}}(C\varepsilon)$. We are thus in the situation of §5.1.1, with ϱ_0, ϱ'_0 and ϱ_1 being replaced by ϱ_{j_k}, ϱ'_{j_k} and $\varrho_{j_{k+1}}$, respectively (see Figure 5). In the charts $(u^{j_k},s^{j_k})\mapsto(u^{j_{k+1}},s^{j_{k+1}})$, the map $\varkappa_k$ takes the form

$$\varkappa_k(u^{j_k},s^{j_k})=(u^{j_{k+1}},s^{j_{k+1}})(\varrho'_{j_k})+(A_ku^{j_k}+\tilde{\alpha}_k(u^{j_k},s^{j_k}),{}^tA_k^{-1}s^{j_k}+\tilde{\beta}_k(u^{j_k},s^{j_k})), \tag{5.12}$$

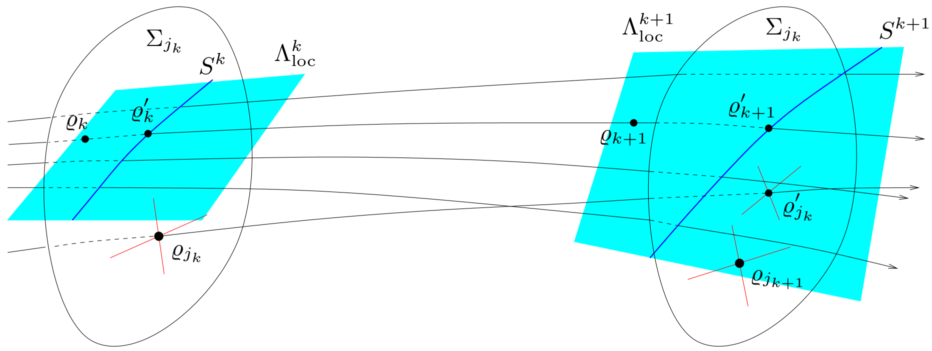

Figure 5. Illustration of the objects appearing in the proof of Proposition 5.1. The local Lagrangians $\Lambda^k_{\rm loc}$ and $\Lambda^{k+1}_{\rm loc}$ appear in light grey (light blue in the online version), and are foliated by bicharacteristics. The axes around ϱ_{j_k} and $\varrho_{j_{k+1}}$ represent the stable and unstable subspaces $E^{\pm}_{\varrho}$ on those points. The axes around $\varrho'_{j_k}=\varkappa_k(\varrho_{j_k})$ are the projected subspaces $\widetilde{E}^{\pm}_{\varrho'}$.

where $\|A_k^{-1}\|, \|{}^tA_k^{-1}\|\leqslant\nu$ and the smooth functions $\tilde{\alpha}_k$ and $\tilde{\beta}_k$ satisfy (5.6). It is convenient to shift the origin of the coordinates (u^{j_k}, s^{j_k}) (resp. $(u^{j_{k+1}}, s^{j_{k+1}})$) such as to center them at ϱ'_k (resp. at ϱ'_{k+1}). We call the shifted coordinates (u^k, s^k) (resp. (u^{k+1}, s^{k+1})). In these shifted coordinates, we get

$$\varkappa_k(u^k, s^k)=(A_k u^k+\alpha_k(u^k, s^k), {}^tA_k^{-1}s^k+\beta_k(u^k, s^k)), \quad (u^k, s^k)\in V_k. \tag{5.13}$$

The shifted functions α_k and β_k still satisfy (5.6), where $V=V_k$ corresponds to the neighbourhood of ϱ'_k where $\varkappa_k$ is defined.

After fixing the coordinate charts, we can study the behaviour of the intersections $S^k=\Lambda^k_{\rm loc}\cap\Sigma_k$ when k grows. We are exactly in the framework of [19, Theorem 6.2.8], and we will use the same method to control the S^k.

We first show that, if ε is chosen small enough, the unstable γ_1-cone in S^k is sent by $\varkappa_k$ inside the γ_1-cone in S^{k+1}. Let us assume that

$$S^k=\{(u^k, f^k(u^k))\}, \qquad \sup_{u^k\in D_\varepsilon}\|df^k\|\leqslant\gamma_1.$$

The projection of $\varkappa_k|_{S^k}$ on the horizontal subspace reads

$$u^k\longmapsto u^{k+1}=\pi\varkappa_k(u^k, f^k(u^k))=A_k u^k+\alpha_k(u^k, f^k(u^k)), \tag{5.14}$$

so by differentiation we get that it is uniformly expanding from some neighbourhood $D'_\varepsilon\subset D_\varepsilon$ to D_ε:

$$\frac{\partial u^{k+1}}{\partial u^k}=A_k+\frac{\partial\alpha_k}{\partial u^k}+\frac{\partial\alpha_k}{\partial s^k}\frac{\partial f^k}{\partial u^k}=A_k+\mathcal{O}(\varepsilon^\gamma(1+\gamma_1)). \tag{5.15}$$

The property $\|A_k^{-1}\|\leqslant\nu<1$ shows that, for $\varepsilon^\gamma(1+\gamma_1)$ small enough, this map is uniformly expanding. Hence, this map is *invertible*, and its inverse,

$$u^{k+1}\longmapsto u^k \stackrel{\text{def}}{=} \tilde{g}_{k+1}(u^{k+1}), \tag{5.16}$$

is uniformly contracting:

$$\|d\tilde{g}_{k+1}(u^{k+1})\| = \left\|\frac{\partial u^k}{\partial u^{k+1}}\right\| \leqslant \nu_1, \quad u^{k+1}\in D_\varepsilon, \tag{5.17}$$

with $\nu_1=\nu+C_\alpha(\varepsilon^\gamma(1+\gamma_1))<1$. As a result, since $\tilde{g}_{k+1}(0)=0$, we have

$$\|u^k\| = \|\tilde{g}_{k+1}(u^{k+1})\| \leqslant \nu_1\|u^{k+1}\|, \quad u^{k+1}\in D_\varepsilon.$$

We also see that the intersection $S^{k+1}=\varkappa_k(S^k)$ can be represented as the graph

$$S^{k+1} = \{(u^{k+1}, f^{k+1}(u^{k+1})) : u^{k+1}\in D_\varepsilon\}$$

in the coordinates centered at ϱ'_{k+1}, with the explicit expression

$$f^{k+1}(u^{k+1}) = {}^tA_k^{-1} f^k(u^k) + \beta_k(u^k, f^k(u^k)), \quad u^{k+1}\in D_\varepsilon, \quad u^k = \tilde{g}_{k+1}(u^{k+1}). \tag{5.18}$$

Differentiating this expression with respect to u^{k+1} leads to

$$\frac{\partial f^{k+1}}{\partial u^{k+1}} = \left(\frac{\partial u^k}{\partial u^{k+1}}\right)\left[({}^tA_k^{-1}+\partial_s\beta^k(u^k, f^k(u^k)))\frac{\partial f^k}{\partial u^k}(u^k)+\partial_u\beta^k(u^k, f^k(u^k))\right].$$

Since for ε small enough we have uniformly

$$\|{}^tA_k^{-1}+\partial_s\beta^k(u^k, f^k(u^k))\| \leqslant \nu_2, \quad \nu_2 = \nu+C_\beta\varepsilon^\gamma < 1,$$

the above Jacobian is bounded from above by

$$\left\|\frac{\partial f^{k+1}}{\partial u^{k+1}}\right\| \leqslant \nu_1(\nu_2\gamma_1+C\varepsilon^\gamma).$$

If $\varepsilon>0$ is small enough, the above right-hand side is smaller than $\nu_2\gamma_1$. We have thus proved that the γ_1-unstable cones in Σ_k are invariant through $\varkappa_k$, which proves the statement (i) of the proposition.

Let us now study the higher derivatives of the functions f^k, obtained by further differentiating (5.18). We use the norms

$$\|f\|_{C^\ell(V)} = \max_{\substack{\alpha\in\mathbb{N}^{n-1}\\ |\alpha|\leqslant\ell}} \sup_{u\in V} \|\partial^\alpha f(u)\|,$$

and will proceed by induction on the degree ℓ of differentiation. Let us assume that for some $\ell \geqslant 2$, there exists $\gamma_{\ell-1}$ such that all functions f^k, $0 \leqslant k \leqslant N$, satisfy $\|f^k\|_{C^{\ell-1}} \leqslant \gamma_{\ell-1}$. Above we have proved this property for $\ell=2$. By differentiating (5.18) ℓ times, we get

$$\frac{\partial^\ell f^{k+1}}{(\partial u^{k+1})^\ell} = \left(\frac{\partial u^k}{\partial u^{k+1}}\right)^\ell ({}^t A_k^{-1} + \partial_s \beta^k) \frac{\partial^\ell f^k}{(\partial u^k)^\ell} + P_{\ell,k}(\partial f^k, ..., \partial^{\ell-1} f^k),$$

which implies that

$$\left\| \frac{\partial^\ell f^{k+1}}{(\partial u^{k+1})^\ell} \right\| \leqslant \nu_1^\ell \nu_2 \left\| \frac{\partial^\ell f^k}{(\partial u^k)^\ell} \right\| + \|P_{\ell,k}(\partial f^k, ..., \partial^{\ell-1} f^k)\|.$$

Here $P_{\ell,k}$ is a polynomial of degree ℓ, with coefficients uniformly bounded with respect to k, and $u^k \in D_\varepsilon$. Using the assumption $\|f^k\|_{C^{\ell-1}} \leqslant \gamma_{\ell-1}$, there exists $C_{\ell-1}(\gamma_{\ell-1}) > 0$ such that the following inequality holds:

$$\left\| \frac{\partial^\ell f^{k+1}}{(\partial u^{k+1})^\ell} \right\| \leqslant \nu_1^\ell \nu_2 \left\| \frac{\partial^\ell f^k}{(\partial u^k)^\ell} \right\| + C_{\ell-1}(\gamma_{\ell-1}).$$

If we now choose $\gamma_\ell > 0$ such that

$$\gamma_\ell > \max\left\{ \frac{C_{\ell-1}(\gamma_{\ell-1})}{\nu_2(1-\nu_1^\ell)}, \gamma_{\ell-1}, \|f^0\|_{C^\ell} \right\},$$

we check that the condition $\|f^k\|_{C^\ell} \leqslant \gamma_\ell$ implies that $\|\partial^\ell f^{k+1}\|_{C^0} \leqslant \nu_2 \gamma_\ell$. Hence, all functions f^k, $0 \leqslant k \leqslant N$, satisfy $\|f^k\|_{C^\ell} \leqslant \gamma_\ell$, which proves statement (ii).

The important point in (iii) is the uniformity of the estimate with respect to N. To prove such a uniform estimate, one needs to analyze the trajectory $\{\varrho'_k\}_{k=0}^N$ with respect to the "reference trajectory" $\{\varrho_{j_k}\}_{k=0}^N$.(3) It is useful to replace the coordinates (u^{j_k}, η^{j_k}) on Σ^{j_k} by coordinates $(\tilde{u}^k, \tilde{s}^k)$ with the following properties. We define the local (un)stable manifolds on the Poincaré sections:

$$W_k^\pm \stackrel{\text{def}}{=} W_{\text{loc}}^{0\pm}(\varrho_{j_k}) \cap \Sigma^{j_k}.$$

The new coordinates $(\tilde{u}^k, \tilde{s}^k)$ satisfy

$$W_k^+ \equiv \{(\tilde{u}^k, 0) : \tilde{u}^k \in D_\varepsilon\}, \quad W_k^- \equiv \{(0, \tilde{s}^k) : \tilde{s}^k \in D_\varepsilon\}$$

and

$$(\tilde{u}^k, \tilde{s}^k) = (u^{j_k}, s^{j_k}) + \mathcal{O}(\|(u^{j_k}, s^{j_k})\|^2) \quad \text{near the origin,}$$

(3) As suggested in the statement of the proposition, we now assume that $\varrho_{j_k} = \Phi^k(\varrho_{j_0})$ for all $k = 0, ..., N$.

and they need not be symplectic. In these coordinates, the Poincaré map $\varkappa_k:\Sigma^{j_k}\to\Sigma^{j_{k+1}}$ has a more precise form than in (5.13): we can still write it as

$$\varkappa_k(\tilde{u}^k,\tilde{s}^k)=(A_k\tilde{u}^k+\alpha_k(\tilde{u}^k,\tilde{s}^k),{}^tA_k^{-1}\tilde{s}^k+\beta_k(\tilde{u}^k,\tilde{s}^k)),$$

but the smooth functions α_k and β_k satisfy more constraints than before:

$$\alpha_k(0,\tilde{s}^k)=\beta_k(\tilde{u}^k,0)\equiv 0 \quad\text{and}\quad d\tilde{\alpha}_k(0,0)=d\tilde{\beta}_k(0,0)=0.$$

This shows that, near the origin, $\alpha_k(\tilde{u}^k,\tilde{s}^k)=\mathcal{O}(\|\tilde{u}^k\|+\|\tilde{s}^k\|)\|\tilde{u}^k\|$ and similarly for β_k. Using these coordinates, we can show that most of the points along the trajectory $\{\varrho'_k\}_{k=0}^N$ are very close to the reference points $\{\varrho_{j_k}\}_{k=0}^N$. If we let $(\tilde{u}^k,\tilde{s}^k)$ be the coordinates of $\varrho'_k\in S^k$, we have

$$\begin{aligned}\tilde{u}^{k+1}&=A_k\tilde{u}^k+\alpha_k(\tilde{u}^k,\tilde{s}^k)=A_k\tilde{u}^k+\mathcal{O}(\varepsilon\|\tilde{u}^k\|),\\ \tilde{s}^{k+1}&={}^tA_k^{-1}\tilde{s}^k+\beta_k(\tilde{u}^k,\tilde{s}^k)={}^tA_k^{-1}\tilde{s}^k+\mathcal{O}(\varepsilon\|\tilde{s}^k\|).\end{aligned}$$

Taking into account the fact that $\|\tilde{u}^N\|\leqslant C\varepsilon$ and $\|\tilde{s}^0\|\leqslant C\varepsilon$, for ε small enough, there exists $\nu_3=\nu+\mathcal{O}(\varepsilon)<1$ such that

$$\|\tilde{u}^k\|\leqslant C\varepsilon\nu_3^{N-k} \text{ and } \|\tilde{s}^k\|\leqslant C\varepsilon\nu_3^k, \quad k=0,...,N.$$

These estimates prove that, if N is large, the points ϱ'_k for $k\gg1$, $N-k\gg1$ are close to ϱ_{j_k}. The tangent of the map $\tilde{u}^k\mapsto\tilde{u}^{k+1}$ induced by projecting $\varkappa_k|_{S^k}$ on the planes $\{(\tilde{u},\tilde{s}):\tilde{s}=0\}$ is given as in (5.15) by

$$\frac{\partial\tilde{u}^{k+1}}{\partial\tilde{u}^k}=A_k+\frac{\partial\alpha_k}{\partial\tilde{u}^k}+\frac{\partial\alpha_k}{\partial\tilde{s}^k}\frac{\partial f^k}{\partial\tilde{u}^k}=A_k+\mathcal{O}(\|\tilde{u}^k\|+\|\tilde{s}^k\|).$$

To obtain the last equality we used the fact that $\|df^k\|$ is uniformly bounded, as shown above. The tangent of the map obtained by projecting $\varkappa_{N-1}\circ...\circ\varkappa_0|_{S^0}$ on the planes $\{(\tilde{u},\tilde{s}):\tilde{s}=0\}$ then reads

$$\begin{aligned}\frac{\partial\tilde{u}^N}{\partial\tilde{u}^0}&=\prod_{k=0}^{N-1}(A_k+\mathcal{O}(\|\tilde{u}^k\|+\|\tilde{s}^k\|))=\prod_{k=0}^{N-1}(A_k+\mathcal{O}(\varepsilon(\nu_3^{N-k}+\nu_3^k)))\\ &=\biggl(\prod_{k=0}^{N-1}A_k\biggr)\prod_{k=0}^{N-1}(I+\mathcal{O}(\varepsilon(\nu_3^{N-k}+\nu_3^k))).\end{aligned}$$

The determinant of the last factor is of order $1+\mathcal{O}(\varepsilon)$, so we deduce

$$\det\biggl(\frac{\partial\tilde{u}^N}{\partial\tilde{u}^0}\biggr)=(1+\mathcal{O}(\varepsilon))\det\biggl(\prod_{k=0}^{N-1}A_k\biggr). \tag{5.19}$$

We then recall that the change of variables $(\tilde{u}^k,\tilde{s}^k)\mapsto(u^{j_k},s^{j_k})$ is close to the identity

$$\left(\frac{\partial(\tilde{u}^k,\tilde{s}^k)}{\partial(u^{j_k},s^{j_k})}\right)=I+\mathcal{O}(\varepsilon).$$

As a result, estimate (5.19) applies as well to the Jacobian of the map $\varkappa_{N-1}\circ...\circ\varkappa_0|_{S^0}$, projected in the planes $\{(u^{j_0},s^{j_0}):s^{j_0}=0\}$ and $\{(u^{j_N},s^{j_N}):s^{j_N}=0\}$, which we denote by $\det(\partial u^{j_N}/\partial u^{j_0})$.

We now consider the map $y^{j_0}\mapsto y^{j_N}$ induced by projecting $\Phi^N|_{\Lambda^0}$ on the planes $\{(y^{j_0},\eta^{j_0}):\eta^{j_0}=0\}$ and $\{(y^{j_N},\eta^{j_N}):\eta^{j_N}=0\}$. From the structure of the adapted coordinates, the tangent to this map has the form

$$\left(\frac{\partial y^{j_N}}{\partial y^{j_0}}\right)=\begin{pmatrix}1 & * \\ 0 & \partial u^{j_N}/\partial u^{j_0}\end{pmatrix},$$

so the estimate (5.19) also applies to $\det(\partial y^{j_N}/\partial y^{j_0})$.

Finally, we remark that if we take $\Lambda=W^{+0}_{\varrho_{j_0}}$, then the tangent map at $\varrho_0=\varrho_{j_0}$ is given by

$$\left(\frac{\partial u^{j_N}}{\partial u^{j_0}}\right)(0)=\prod_{k=0}^{N-1}A_k.$$

Hence in this case we find

$$\det\left(\prod_{k=0}^{N-1}A_k\right)=\det\left(\frac{\partial y^{j_N}}{\partial y^{j_0}}\right)=\det\left(d\Phi^N|_{E^{+0}_{\varrho_{j_0}}}\right)=e^{\lambda^+_N(\varrho_{j_0})}.$$

For the second equality we have used (3.16) and the fact that, for each k, the adapted coordinates satisfy $\Omega(\partial/\partial y_1^{j_k}\wedge...\wedge\partial/\partial y_n^{j_k})=1$ at the origin (see Lemma 4.3). □

Remark 5.1. Due to structural stability, the results of Proposition 5.1 apply to Lagrangian leaves $\Lambda\in\mathcal{E}_{E'}$ transversal to the stable lamination, for any energy

$$E'\in(E-\delta,E+\delta),$$

with the difference that the evolved local Lagrangians are of the form

$$\Lambda^k_{\mathrm{loc}}\equiv\{(y_1^{j_k},u^{j_k};E'-E,f^k(u^{j_k})):|y_1^{j_k}|\leqslant\varepsilon,|u^{j_k}|\leqslant\varepsilon\},\quad\text{with }\|df^k(u^{j_k})\|\leqslant\gamma_1.\qquad(5.20)$$

The Poincaré sections used in the proof are taken as $U_j(\varepsilon)\cap\{(y^j,\eta^j):y_1^j=0,\eta_1^j=E'-E\}$. All constants can be taken to be independent of $E'\in(E-\delta,E+\delta)$.

Remark 5.2. Each $f^k:(D_\varepsilon)_u\to\mathbb{R}^{n-1}_s$ representing the Lagrangian Λ^k_{loc} of (5.20) can be written as $f^k(u)=\phi'_k(u)$ for some function $\phi_k:(D_\varepsilon)_u\to\mathbb{R}$. Therefore, the function

$$\varphi_k(y_1,u)\stackrel{\mathrm{def}}{=}\phi_k(u)+(E'-E)y_1,\quad u\in D_\varepsilon,\ |y_1|\leqslant\varepsilon,$$

generates Λ^k_{loc} in the symplectic coordinates (y^{j_k},η^{j_k}).

5.2. An alternative definition of the topological pressure

To connect the resonance spectrum with the topological pressure (3.19) of the flow, we use an alternative definition of the pressure [35, §0.2 II], which will provide us with a convenient open cover of K_E^δ.

Taking $\delta>0$ small enough to satisfy (3.14), consider a finite cover $\mathcal{V}=\{V_b\}_{b\in B}$ of K_E^δ, made of sets of small diameters contained in the energy layer $\mathcal{E}_E^\delta$ and relatively open in that layer. For any integer $T>0$, the refined cover $\mathcal{V}^{(T)}$ is made of the sets

$$V_\beta \stackrel{\text{def}}{=} \bigcap_{k=0}^{T-1} \Phi^{-k}(V_{b_k}), \quad \beta = b_0 b_2 \dots b_{T-1} \in B^T. \tag{5.21}$$

The T-strings β such that $V_\beta \cap K_E^\delta \neq \varnothing$ make up a subset $\mathcal{B}'_T \subset B^T$. Below it is convenient to coarse-grain the unstable Jacobian (3.17) on subsets $W\subset\mathcal{E}_E^\delta$:

$$S_T(W) \stackrel{\text{def}}{=} -\inf_{\varrho\in W\cap K_E^\delta} \lambda_T^+(\varrho) \quad \text{for } W\subset\mathcal{E}_E^\delta \text{ such that } W\cap K_E^\delta\neq\varnothing. \tag{5.22}$$

We define the following quantity, similar to (3.18):

$$Z_T(\mathcal{V},s) \stackrel{\text{def}}{=} \inf\biggl\{ \sum_{\beta\in\mathcal{B}_T} e^{sS_T(V_\beta)} : \mathcal{B}_T\subset\mathcal{B}'_T \text{ and } K_E^\delta \subset \bigcup_{\beta\in\mathcal{B}_T} V_\beta \biggr\}.$$

The topological pressure of the flow on K_E^δ can then be obtained as follows:

$$\mathcal{P}_E^\delta(s) = \lim_{\operatorname{diam}\mathcal{V}\to 0} \lim_{T\to\infty} \frac{1}{T} \log Z_T(\mathcal{V},s).$$

Here the covers $\mathcal{V}$ are as above: they cover K_E^δ in the energy strip $\mathcal{E}_E^\delta$, and are relatively open. Finally, the pressure $\mathcal{P}_E(s)$ can be obtained through the limit (3.20).

From now on, we will restrict ourselves to the parameter $s=\frac{1}{2}$. Let us fix some small $\varepsilon_0>0$. From the above limits, there exists a cover $\mathcal{V}_0$ of K_E^δ in $\mathcal{E}_E^\delta$ (of arbitrarily small diameter $\varepsilon>0$) and an integer $t_0>0$ depending on $\mathcal{V}_0$, such that

$$\left| \frac{1}{t_0} \log Z_{t_0}\left(\mathcal{V}, \frac{1}{2}\right) - \mathcal{P}_E^\delta\left(\frac{1}{2}\right) \right| \leqslant \varepsilon_0. \tag{5.23}$$

As a consequence, there exists a subset $\mathcal{B}_{t_0}\subset\mathcal{B}'_{t_0}$, such that $\{V_\beta:\beta\in\mathcal{B}_{t_0}\}$ is an open cover of K_E^δ in $\mathcal{E}_E^\delta$, which satisfies

$$\sum_{\beta\in\mathcal{B}_{t_0}} e^{S_{t_0}(V_\beta)/2} \leqslant e^{t_0(\mathcal{P}_E^\delta(1/2)+\varepsilon_0)}.$$

We rename the family $\{V_\beta:\beta\in\mathcal{B}_{t_0}\}$ as $\{W_a:a\in A_1\}$, so the above bound reads

$$\sum_{a\in A_1} e^{S_{t_0}(W_a)/2} \leqslant e^{t_0(\mathcal{P}_E^\delta(1/2)+\varepsilon_0)}. \tag{5.24}$$

Each set W_a contains at least one point $\varrho_a\in K_E^\delta$, which we may set as reference point: following Lemma 4.3, we can represent W_a by an adapted chart (y^a,η^a) centered at ϱ_a. Similarly, we can also equip any $V_b\in\mathcal{V}_0$ with adapted charts (y^b,η^b) centered at some point $\varrho_b\in V_b\cap K_E^\delta$.

Each point $\varrho\in W_a=V_\beta$ evolves such that $\Phi^k(\varrho)\in V_{b_k}$ for all $k=0,...,t_0-1$. Therefore, as long as ε has been chosen small enough, we are in a position to apply Proposition 5.1 and Remarks 5.1 and 5.2 to isoenergetic γ_1-unstable Lagrangian leaves in W_a.

PROPOSITION 5.2. *Take any energy $E'\in[E-\delta,E+\delta]$ and any index $a\in A_1$. Assume that $\Lambda\subset\mathcal{E}_{E'}\cap W_a$ is a Lagrangian leaf generated in the chart (y^a,η^a) by a function φ defined on a subset $D_a\subset D_\varepsilon$, and is contained in the unstable γ_1-cone*:

$$\Lambda\simeq\{(y_1^a,u^a;E'-E,\varphi'(u^a)):u^a\in D_a\},\quad \textit{with } \|\varphi''\|_{C^0(D_a)}\leqslant\gamma_1.$$

Then, for any index $a'\in A_1$, the Lagrangian leaf $\Phi^{t_0}(\Lambda)\cap W_{a'}$ is also in the unstable γ_1-cone in the chart $(y^{a'},\eta^{a'})$.

Besides, the map $y^a\mapsto y^{a'}$ obtained by projecting $\Phi^{t_0}|_\Lambda$ on the planes $\{(y^a,\eta^a):\eta^a=0\}$ and $\{(y^{a'},\eta^{a'}):\eta^{a'}=0\}$ satisfies the following estimate on its domain of definition:

$$\det\left(\frac{\partial y^{a'}}{\partial y^a}\right)=(1+\mathcal{O}(\varepsilon^\gamma))e^{\lambda_{t_0}^+(\varrho_a)}.$$

Here $\lambda_{t_0}^+(\varrho_a)$ is the unstable Jacobian (3.17) of the reference point $\varrho_a\in W_a\cap K_E^\delta$, and $\gamma>0$ is the Hölder exponent of the unstable lamination. The implied constant is uniform with respect to t_0.

Proof. From Proposition 5.1, we know that for any $\varrho\in\Lambda$ and any $k=0,...,t_0-1$, the connected component Λ_{loc}^k of $\Phi^k(\Lambda)\cap V_{b_k}$ containing $\Phi^k(\varrho)$ lies in the unstable γ_1-cone with respect to the chart (y^{b_k},η^{b_k}). On the other hand, since Λ is a connected leaf inside W_a, at each step $k=0,...,t_0-1$ its image $\Phi^k(\Lambda)$ is fully contained in V_{b_k} and is *connected*, so that Λ_{loc}^k is actually equal to $\Phi^k(\Lambda)$ for all $k=0,...,t_0-1$. Finally, we apply one iteration of Proposition 5.1 to the leaf $\Lambda'=\Phi^{t_0-1}(\Lambda)\subset V_{b_{t_0-1}}\cap\mathcal{E}_{E'}$, and deduce that any intersection $\Phi(\Lambda')\cap W_{a'}=\Phi^{t_0}(\Lambda)\cap W_{a'}$ is also in the γ_1-unstable cone.

We now prove the statement concerning the Jacobian of the induced map. It is a direct consequence of part (iii) in Proposition 5.1, after replacing the time N by t_0. Let ϱ_a be the reference point in $W_a\cap K_E^\delta$, on which the coordinates (y^a,η^a) are centered. If V_b is a set containing $\Phi^{t_0}(\varrho_a)$, we may enlarge it into a set of diameter $C\varepsilon$, such that $\Phi^{t_0}(W_a)\subset V_b$ and $W_{a'}\subset V_b$. On V_b we may use adapted coordinates (y^b,η^b) centered on the point $\varrho_b\stackrel{\text{def}}{=}\Phi^{t_0}(\varrho_a)$, and represent $\Phi^{t_0}|_\Lambda$ by a map $y^a\mapsto y^b$. In this setting, Proposition 5.1 (iii) shows that the associated Jacobian satisfies

$$\det\left(\frac{\partial y^b}{\partial y^a}\right)=(1+\mathcal{O}(\varepsilon))e^{\lambda^+_{t_0}(\varrho_a)}.$$

There remains to compare the coordinates (y^b,η^b) with the coordinates $(y^{a'},\eta^{a'})$ centered on $\varrho_{a'}\in W_{a'}$. Since the (un)stable subspaces at ϱ_b and $\varrho_{a'}$ form angles $\mathcal{O}(\varepsilon^\gamma)$ and $d(\varrho_b,\varrho_{a'})=\mathcal{O}(\varepsilon)$, the representation of $\Phi^{t_0}|_\Lambda$ through $y^a\mapsto y^{a'}$ satisfies

$$\det\left(\frac{\partial y^{a'}}{\partial y^a}\right)=(1+\mathcal{O}(\varepsilon^\gamma))\det\left(\frac{\partial y^b}{\partial y^a}\right)=(1+\mathcal{O}(\varepsilon^\gamma))e^{\lambda^+_{t_0}(\varrho_a)}. \tag{5.25}$$

□

Notice that, even though t_0 (depending on the cover $\mathcal{V}_0$ in an unknown way) can be very large, applying Φ^{t_0} onto a near-unstable isoenergetic leaf $\Lambda\subset W_a$ does not fold it.

5.3. Completing the cover

We need to complete the family $\{W_a\}_{a\in A_1}$ in order to cover the full energy strip $\mathcal{E}_E^\delta$. Far from the interaction region (which we define using the radius R_0 of §3), we take the unbounded set

$$W_0=\mathcal{E}_E^\delta\cap\{\varrho:|x(\varrho)|>3R_0\}.$$

We complete the cover with a finite family of relatively open sets

$$\{W_a\subset\mathcal{E}_E^\delta\}_{a\in A_2},$$

with the following properties. These sets should have sufficiently small diameters, and for some uniform $d_1>0$ they should satisfy

$$d(W_a,\Gamma_E^{+\delta})+d(W_a,\Gamma_E^{-\delta})>d_1,\quad\text{where }\Gamma_E^{\pm\delta}\stackrel{\text{def}}{=}\bigcup_{|E'-E|<\delta}\Gamma_{E'}^{\pm},$$

and Γ_E^δ are the incoming/outgoing sets given in (1.5). Finally, the full family should cover $\mathcal{E}_E^\delta$:

$$\mathcal{E}_E^\delta=\bigcup_{a\in A}W_a,\quad\text{where } A=\{0\}\cup A_1\cup A_2.$$

Lemma 5.1. *Such a cover exists. Consequently, there exists* $N_0 \in \mathbb{N}$ *such that for any index* $a \in A_2$ *we have either*

$$\Phi^t(W_a) \cap \{\varrho : |x(\varrho)| < 3R_0\} = \varnothing \quad \textit{for any } t \geqslant N_0 t_0,$$

or

$$\Phi^{-t}(W_a) \cap \{\varrho : |x(\varrho)| < 3R_0\} = \varnothing \quad \textit{for any } t \geqslant N_0 t_0.$$

Proof. The complement of $\bigcup_{a \in A_1} W_a$ in $\mathcal{E}_E^\delta \cap T^*_{B(0,3R_0)} X$ is at a certain distance $D>0$ from K_E^δ. On the other hand, from the *uniform transversality* of stable and unstable manifolds on K_E^δ, there exists $d_1>0$ such that

$$\text{for all } \varrho \in \mathcal{E}_E^\delta \cap T^*_{B(0,3R_0)} X, \quad d(\varrho, \Gamma_E^{+\delta}) + d(\varrho, \Gamma_E^{-\delta}) \leqslant 4d_1 \implies d(\varrho, K_E^\delta) \leqslant D. \tag{5.26}$$

We first cover the set

$$S_- = \{\varrho \in \mathcal{E}_E^\delta \cap T^*_{B(0,3R_0)} X : d(\varrho, \Gamma_E^{-\delta}) > 2d_1\}$$

by small open sets $\{W_a : a \in A_2^-\}$ at distance $\geqslant d_1$ from $\Gamma_E^{-\delta}$. There exists $T_->0$ such that at any time $t \geqslant T_-$, the iterate $\Phi^t(W_a)$ has escaped outside $T^*_{B(0,3R_0)} X$ for any $a \in A_2^-$.

We then cover the set

$$S_+ = \{\varrho \in \mathcal{E}_E^\delta \cap T^*_{B(0,3R_0)} X : d(\varrho, \Gamma_E^{-\delta}) \leqslant 2d_1 \text{ and } d(\varrho, \Gamma_E^{+\delta}) > 2d_1\}$$

by small open sets $\{W_a : a \in A_2^+\}$ at distance $\geqslant d_1$ from $\Gamma_E^{+\delta}$. Now, there exists $T_+>0$ such that all these sets have escaped outside $T^*_{B(0,3R_0)} X$ for times $t \leqslant -T_+$. From (5.26), points $\varrho \in \mathcal{E}_E^\delta \cap T^*_{B(0,3R_0)} X$ which are neither in S_- nor in S_+ are at distance $\leqslant D$ from K_E^δ, and therefore already belong to some W_a, $a \in A_1$. Finally, we take $A_2 \stackrel{\text{def}}{=} A_2^- \cup A_2^+$ and $N_0 \in \mathbb{N}$ such that $N_0 t_0 \geqslant \max\{T_-, T_+\}$. □

6. Quantum dynamics

As reviewed in §3.4, resonances are the eigenvalues of the complex scaled operator P_θ. To prove the lower bound on the size of the imaginary part of a resonance $z(h)$, with a resonant state $u_\theta(h) \in L^2(X_\theta)$, $\|u_\theta\|=1$, we want to estimate

$$e^{-t|\operatorname{Im} z(h)|/h} = \|e^{-itP_\theta/h} u_\theta(h)\|, \quad t \gg 1,$$

where the exponential of $-itP_\theta/h$ is considered purely formally. In principle that could be done by estimating $\|e^{-itP_\theta/h} \chi^w(x, hD)\|$, where χ^w provides a localization to the energy surface. However, the imaginary part of P_θ can be positive of size $\sim\theta\sim Mh\log(1/h)$ and that poses problems for such estimates.

Hence the first step is to modify the operator P_θ without changing its spectrum. To make the notation simpler, we normalize the operator so that we work near energy 0. In the case of (1.1) that means considering

$$P(h)=-h^2\Delta+V(x)-E, \quad p(x,\xi)=|\xi|^2+V(x)-E.$$

Accordingly, the energy strips and trapped sets will be denoted by $\mathcal{E}^\delta$ and K^δ.

6.1. Modification of the scaled operator

To modify the operator P_θ we follow the presentation of [42, §§4.1, 4.2, 7.3] which is based on many earlier works cited there.

Thus, instead of P_θ we consider the operator $P_{\theta,\varepsilon}$ obtained by conjugation with an exponential weight:

$$P_{\theta,\varepsilon} \stackrel{\text{def}}{=} e^{-\varepsilon G^w/h} P_\theta e^{\varepsilon G^w/h}, \quad \varepsilon=M_2\theta, \quad \theta=M_1 h\log\frac{1}{h}. \tag{6.1}$$

This section is devoted to the construction of an appropriate weight $G^w=G^w(x,hD)$. The large constant M_1 will be of crucial importance for error estimates in our argument and will be chosen large enough to control propagation up to time $M\log(1/h)$, roughly $M_1\gg M$. The constant M_2 will also be given below.

We start with the construction of the weight $G(x,\xi)$.

Lemma 6.1. *Suppose that p satisfies the general assumptions* (3.8) (*with the energy $E>0$ now in the interval* $(-\delta,\delta)$). *Then, for any open neighbourhood V of K^δ such that $V\Subset T^*_{B(0,R_0)}X$, and any $\delta_0\in\left(0,\frac{1}{2}\right)$, there exists $G\in\mathcal{C}^\infty_c(T^*X)$ such that*

$$\begin{aligned} &H_pG(\varrho)\geqslant 0 && \text{if } \varrho\in T^*_{B(0,3R_0)}X,\\ &H_pG(\varrho)\geqslant 1 && \text{if } \varrho\in T^*_{B(0,3R_0)}X\cap(\mathcal{E}^\delta\setminus V),\\ &H_pG(\varrho)\geqslant -\delta_0 && \text{for all } \varrho\in T^*X.\end{aligned} \tag{6.2}$$

Proof. The construction of the function G is based on the following result of [15, Appendix]: for any open neighbourhoods U and V of K^δ, $\overline{U}\subset V$, there exists $G_0\in\mathcal{C}^\infty(T^*X)$ such that

$$G_0|_U\equiv 0, \quad H_pG_0\geqslant 0, \quad H_pG_0|_{\mathcal{E}^{2\delta}}\leqslant C \quad \text{and} \quad H_pG_0|_{\mathcal{E}^\delta\setminus V}\geqslant 1.$$

Such a G_0 is an escape function, and is necessarily of unbounded support. We need to truncate G_0 into a compactly supported function, without making H_pG_0 too negative. For $T>0$ and $\alpha\in(0,1)$ to be fixed later, let $\chi\in\mathcal{C}^\infty(\mathbb{R})$ satisfy

$$\chi(t)=\begin{cases}0, & \text{if } |t|>T,\\ t, & \text{if } |t|<\alpha T,\end{cases}\qquad |\chi(t)|\leqslant 2\alpha T \text{ and } \chi'(t)\geqslant -2\alpha,\quad t\in\mathbb{R}$$

(we obtain χ by regularizing a piecewise linear function with these properties). Let $\psi\in\mathcal{C}_c^\infty(\mathbb{R};[0,1])$ be equal to 1 for $|t|\leqslant 1$ and 0 for $|t|\geqslant 2$. For $R>0$ to be fixed later, we define

$$G(\varrho)\stackrel{\text{def}}{=}\chi(G_0(\varrho))\psi\left(\frac{p(\varrho)}{\delta}\right)\psi\left(\frac{|x(\varrho)|}{R}\right),$$

which vanishes on U, outside $\mathcal{E}^{2\delta}$ and for $|x|>2R$. We then compute

$$H_pG=\chi'(G_0)H_pG_0\psi\left(\frac{p}{\delta}\right)\psi\left(\frac{|x|}{R}\right)+\frac{1}{R}\chi(G_0)\psi\left(\frac{p}{\delta}\right)\psi'\left(\frac{|x|}{R}\right)H_p(|x|).$$

This is bounded from below by 0 for $\{\varrho:|x(\varrho)|<R \text{ and } |G_0(\varrho)|\leqslant\alpha T\}$, and by 1, if in addition $\varrho\in\mathcal{E}^\delta\setminus V$. For any $\varrho\in T^*X$ we have

$$H_pG(\varrho)\geqslant -C_0\alpha\left(1+\frac{T}{R}\right),$$

for some $C_0>0$, since (3.10) shows that $|H_p(|x|)|\leqslant C_1$ on $\mathcal{E}^{2\delta}$. Choosing $R>3R_0$ and $T=T(\alpha,R_0)$ large enough so that $|G_0(\varrho)|\leqslant\alpha T$ for $\varrho\in\mathcal{E}^{2\delta}\cap T^*_{B(0,3R_0)}X$, we have now guaranteed the first two conditions in (6.2). To obtain the last condition we need

$$C_0\alpha\left(1+\frac{T(\alpha,R_0)}{R}\right)<\delta_0,$$

and this follows from choosing α small enough and then R large enough. □

Using the identification (3.23), we consider G given in Lemma 6.1 as a function on T^*X_θ, and define $P_{\theta,\varepsilon}$ by (6.1). We note that $e^{\pm\varepsilon G^w(x,hD)/h}$ is a pseudodifferential operator with the symbol in the class $S_\delta^{-\infty,C_0}$ for any $\delta>0$ and some C_0, and that the operator

$$P_{\theta,\varepsilon}\stackrel{\text{def}}{=}e^{-\varepsilon G^w/h}P_\theta e^{\varepsilon G^w/h}=e^{-(\varepsilon/h)\operatorname{ad}_{G^w}}P_\theta\sim\sum_{k=0}^\infty\frac{\varepsilon^k}{k!}\left(-\frac{1}{h}\operatorname{ad}_{G^w}\right)^k(P_\theta)$$

has its symbol in the class $S_0^{2,0}$. This expansion shows that

$$\begin{aligned}P_{\theta,\varepsilon}(h)&=P_\theta(h)-i\varepsilon\{p_\theta,G\}^w(x,hD)+\varepsilon^2e_0^w(x,hD)\\&=p_\theta^w(x,hD)-i\varepsilon\{p_\theta,G\}^w(x,hD)+\varepsilon^2e_1^w(x,hD)+he_2^w(x,hD),\quad e_j\in S,\end{aligned}$$

where p_θ is the principal symbol of P_θ given by (3.24). In particular, denoting by $\mathcal{O}(\alpha)$ the quantization of a symbol in αS, we have

$$\begin{aligned} \operatorname{Re} P_{\theta,\varepsilon} &\stackrel{\text{def}}{=} (P_{\theta,\varepsilon}+P_{\theta,\varepsilon}^*)/2 = (\operatorname{Re} p_\theta)^w(x,hD)+\varepsilon\{\operatorname{Im} p_\theta, G\}^w(x,hD)+\mathcal{O}(h+\varepsilon^2) \\ &= \operatorname{Re} p_\theta^w(x,hD)+\mathcal{O}(h+\theta\varepsilon+\varepsilon^2), \\ \operatorname{Im} P_{\theta,\varepsilon} &\stackrel{\text{def}}{=} (P_{\theta,\varepsilon}-P_{\theta,\varepsilon}^*)/2i = \operatorname{Im} p_\theta^w(x,hD)-\varepsilon\{\operatorname{Re} p_\theta, G\}^w(x,hD)+\mathcal{O}(h+\varepsilon^2) \\ &= \operatorname{Im} p_\theta^w(x,hD)-\varepsilon(H_pG)^w(x,hD)+\mathcal{O}(h+\varepsilon^2). \end{aligned} \tag{6.3}$$

We can use our knowledge of p_θ, see (3.24)–(3.26), and the fact that the set V used to define G is contained in $T^*_{B(0,R_0)}X$, to deduce that, for any $\varrho\in\mathcal{E}^\delta$,

$$\operatorname{Im} p_\theta(\varrho)-\varepsilon H_pG(\varrho) \leqslant \begin{cases} 0, & \text{if } \varrho\in V, \\ C\theta-\varepsilon=-(M_2-C)\theta, & \text{if } \varrho\notin V \text{ and } |x(\varrho)|\leqslant 2R_0, \\ -C\theta+\varepsilon\delta_0=-\theta(C-\delta_0M_2), & \text{if } |x(\varrho)|>2R_0. \end{cases} \tag{6.4}$$

We now choose M_2 in (6.1) such that $C<M_2<C/\delta_0$, so that

$$\operatorname{Im} p_\theta(\varrho)-\varepsilon H_pG(\varrho)\leqslant 0 \quad \text{for any } \varrho\in\mathcal{E}^\delta. \tag{6.5}$$

The sharp Gårding inequality (3.6) and (6.5) give, in the sense of operators,

$$\operatorname{Im} \chi^w(x,hD)P_{\theta,\varepsilon}(h)\chi^w(x,hD)\leqslant Ch, \quad \operatorname{supp}\chi\subset\mathcal{E}^{\delta/2}, \tag{6.6}$$

where $\chi\in S(1)$ is real-valued. Achieving this approximate negativity was the main reason for introducing the weight G. Indeed, we notice that, before conjugating by this weight, we only had $\operatorname{Im}\chi^wP_\theta\chi^w\leqslant Ch\log(1/h)$.

6.2. The evolution operator

We take the energy width $\delta>0$ as in §5, and construct the weight G accordingly, as explained in the previous section. Let the function $\chi_\delta\in S(T^*X)$ satisfy

$$\operatorname{supp}\chi_\delta\subset\mathcal{E}^{\delta/2} \quad\text{and}\quad \chi_\delta|_{\mathcal{E}^{\delta/3}}\equiv 1. \tag{6.7}$$

In this section we will compare the two energy-localized operators

$$\widetilde{P}_0\stackrel{\text{def}}{=}\chi_\delta^w(x,hD)P(h)\chi_\delta^w(x,hD) \quad\text{and}\quad \widetilde{P}\stackrel{\text{def}}{=}\chi_\delta^w(x,hD)P_{\theta,\varepsilon}(h)\chi_\delta^w(x,hD). \tag{6.8}$$

$\widetilde{P}_0$ is obviously bounded and hermitian on $L^2(X)$, and $\widetilde{P}$ is bounded on $L^2(X_\theta)\simeq L^2(X)$ (using the map $x\mapsto\operatorname{Re}x$). We may thus define a unitary group and a non-unitary group as follows ($t\in\mathbb{R}$):

$$U_0(t)\stackrel{\text{def}}{=}e^{-it\widetilde{P}_0/h} \quad\text{and}\quad U(t)\stackrel{\text{def}}{=}e^{-it\widetilde{P}/h}, \quad\text{respectively.} \tag{6.9}$$

The need for the cutoff function χ_δ^w comes from the non-dissipative contributions of $\operatorname{Im} P_\theta$, which are compensated by the weight G only close to the energy surface. In view of the bound (6.6), we have

$$\|U(t)\|_{L^2\to L^2} \leqslant e^{Ct}, \quad t\geqslant 0. \tag{6.10}$$

We make the following observation based on §3.4 and the boundedness of $e^{\pm\varepsilon G^w/h}$ on L^2:

$$\operatorname{Res}(P(h))\cap D_{\delta,\theta/C} = \operatorname{Spec}(P_\theta(h))\cap D_{\delta,\theta/C} = \operatorname{Spec}(P_{\theta,\varepsilon}(h))\cap D_{\delta,\theta/C},$$
$$D_{\delta,\theta/C} \stackrel{\text{def}}{=} \{z : |\operatorname{Re} z|\leqslant\delta \text{ and } \operatorname{Im} z > -\theta/C\}.$$

Hence, from now on, by a normalized resonant state of $z(h)\in\operatorname{Res}(P(h))\cap D_{\delta,\theta/C}$ we mean that

$$u(h)\in L^2(X_\theta), \quad \|u(h)\|=1 \quad \text{and} \quad P_{\theta,\varepsilon}u(h)=z(h)u(h). \tag{6.11}$$

PROPOSITION 6.1. *Let us put* $\delta_1=\frac{1}{4}\delta$, $C>0$, *and let* $u(h)$ *be given by* (6.11) *with* $|\operatorname{Re} z(h)|<\delta_1$ *and* $\operatorname{Im} z(h)>-Ch$. *Then for any fixed* $M>0$ *and any* $0\leqslant t\leqslant M\log(1/h)$, *we have*

$$U(t)u(h)=e^{-itz(h)/h}u(h)+\mathcal{O}_{L^2}(h^\infty), \tag{6.12}$$

where $U(t)$ *is the modified propagator given by* (6.9). *More precisely, the* L^2 *norm of the error in* (6.12) *is bounded by* h^L *for any* L *and* $0<h<h_0=h_0(L,M)$.

Proof. Let $v(t)\stackrel{\text{def}}{=}U(t)u-e^{-itz/h}u$, so that

$$ih\partial_t v(t)=\widetilde{P}U(t)u-ze^{-itz/h}u=\widetilde{P}v(t)+e(t), \quad e(t)\stackrel{\text{def}}{=}e^{-itz/h}(\widetilde{P}-z)u.$$

Since $(P_{\theta,\varepsilon}-z)u=0$, we know that $\operatorname{WF}_h(u(h))$ lies in $\mathcal{E}^{\delta/3}$, so that $\chi_\delta^w u=u+\mathcal{O}_{L^2}(h^\infty)$. Hence, $\|e(t)\|=\mathcal{O}(h^\infty)$ and, using (6.6),

$$\partial_t\|v(t)\|^2=2\operatorname{Re}\langle\partial_t v(t),v(t)\rangle=\frac{2}{h}\langle\operatorname{Im}\widetilde{P}v(t),v(t)\rangle+2\operatorname{Im}\langle e(t),v(t)\rangle\leqslant C\|v(t)\|^2+\|e(t)\|^2,$$
$$v(0)=0.$$

The Gronwall inequality implies that

$$\|v(t)\|^2\leqslant e^{Ct}\int_0^t\|e(s)\|^2\,ds,$$

and the lemma follows from the logarithmic bound on t. □

The following lemma compares the two propagators in (6.9).

LEMMA 6.2. *For any fixed $t>0$, the operator*

$$V(t) \stackrel{\text{def}}{=} U_0(t)^{-1}U(t) \tag{6.13}$$

is a pseudodifferential operator with symbol $v(t)\in S_\gamma(T^*X)$ for any $\gamma\in\left(0,\frac{1}{2}\right)$.

Proof. To prove both statements, we simply differentiate $V(s)$ with respect to s:

$$\partial_s V(s) = \frac{1}{h} a(s)^w(x,hD)V(s), \qquad V(0)=I,$$
$$a(s)^w(x,hD) \stackrel{\text{def}}{=} \frac{1}{i} U_0(s)^{-1}(\widetilde{P}-\widetilde{P}_0)U_0(s).$$

Using Egorov's theorem, we obtain the following general bounds on the symbol $a(s)$, uniform for $s\in[0,t]$:

$$-Ch\log\frac{1}{h} \leqslant \operatorname{Re} a(s) \leqslant Ch \quad \text{and} \quad |\partial^\alpha a(s)| \leqslant C_\alpha h \log\frac{1}{h} \text{ for all } \alpha\in\mathbb{N}^{2n}.$$

To show that $V(t)$ is the quantization of a symbol $v(t)\in S_\gamma$ we use the Beals's characterization of pseudodifferential operators recalled in (3.7). We proceed by induction: suppose we know that

$$V_{N-1}(t) \stackrel{\text{def}}{=} \operatorname{ad}_{W_{N-1}} \dots \operatorname{ad}_{W_1} V(t) = \mathcal{O}_{L^2\to L^2}(h^{(1-\gamma)(N-1)}),\ N\geqslant 1, \quad V_0(t)=V(t),$$

where the W_j's are as in (3.7). We now consider the differential equation satisfied by

$$V_N(t) \stackrel{\text{def}}{=} \operatorname{ad}_{W_N} V_{N-1}(t).$$

Using the derivation property $\operatorname{ad}_W(AB)=(\operatorname{ad}_W A)B+A(\operatorname{ad}_W B)$ we see that

$$\partial_t V_N(t) = \operatorname{ad}_{W_N} \dots \operatorname{ad}_{W_1}\left(\left(\frac{a}{h}\right)^w(x,h)V(t)\right) = \left(\frac{a}{h}\right)^w(x,hD)V_N(t)+E_N(t),$$
$$E_N(t) = \mathcal{O}_{L^2\to L^2}(h^{N(1-\gamma)}),$$

where we used the induction hypothesis and the fact that

$$\operatorname{ad}_{W_{j_1}} \dots \operatorname{ad}_{W_{j_k}}\left(\frac{a}{h}\right)^w(x,hD) = \mathcal{O}_{L^2\to L^2}\left(h^k\log\frac{1}{h}\right) = \mathcal{O}_{L^2\to L^2}(h^{k(1-\gamma)}).$$

Since $V_N(0)=0$, Duhamel's formula shows that

$$V_N(t) = \int_0^t V(t-s)E_N(s)\,ds = \mathcal{O}_{L^2\to L^2}(h^{N(1-\gamma)}),$$

concluding the inductive step and the proof. □

The following lemma shows that the propagators $U(t)$ and $U_0(t)$ act very similarly on wavepackets localized close to the trapped set.

LEMMA 6.3. *Let* $U_G \Subset \widetilde{U}_G \Subset \mathcal{E}^{2\delta} \cap T^*_{B(0,R_0/2)}X$, *with* U_G *and* $\widetilde{U}_G$ *being open sets such that* U_G *is a neighbourhood of* K^δ, *while the weight* G *constructed in Lemma* 6.1 *vanishes identically on* $\widetilde{U}_G$.

Take $\delta_1 = \frac{1}{4}\delta$ *as in Proposition* 6.1 *and fix some* $t>0$. *Assume that the open set* V *is such that* $\Phi^s(V) \Subset U_G \cap \mathcal{E}^{\delta_1}$ *for all times* $s \in [0,t]$. *Take any* $\Pi \in C_c^\infty(V)$. *The propagators* $U(t)$ *and* $U_0(t)$ *then satisfy*

$$(U(t)-U_0(t))\Pi^w(x,hD) = \mathcal{O}_{L^2\to L^2}(h^\infty).$$

Proof. The proof is very similar to that of the previous lemma. The norm is equal to $\|(V(t)-1)\Pi^w\|_{L^2\to L^2}$. Differentiating this operator with respect to t, we find, for all $s\in[0,t]$,

$$\partial_s V(s)\Pi^w = \frac{1}{ih} U_0(s)^{-1}(\widetilde{P}-\widetilde{P}_0)U_0(s)V(s)\Pi^w.$$

From the dynamical assumption and using Egorov's theorem, we easily deduce that

$$U_0(s)^{-1}(\widetilde{P}-\widetilde{P}_0)U_0(s) = 0,$$

microlocally near V, uniformly for all $s\in[0,t]$. Since Π is supported inside V, we obtain $\partial_s V(s)\Pi^w = \mathcal{O}_{L^2\to L^2}(h^\infty)$. □

Using Lemma 6.2, we also prove a basic semiclassical propagation estimate for $U(t)$.

PROPOSITION 6.2. *Take* δ_1 *as in Proposition* 6.1 *and fix* $t>0$ *and* $\gamma\in\left[0,\frac{1}{2}\right)$.

(i) *Take* $\psi_0, \psi_1 \in S_\gamma(1)$ *such that* $\psi_1\circ\Phi^t$ *takes the value* 1 *near* $\operatorname{supp}\psi_0$: *precisely, assume that*

$$d(\operatorname{supp}\psi_0, \complement\{\varrho : \psi_1\circ\Phi^t(\varrho)=1\}) \geqslant \frac{h^\gamma}{C}, \quad \operatorname{supp}\psi_1 \subset \mathcal{E}^{\delta_1}, \tag{6.14}$$

where $d(\cdot,\cdot)$ *is a Riemannian distance on* T^*X *which coincides with the standard Euclidean distance outside* $T^*_{B(0,R_0)}X$. *Then*

$$\psi_1^w(x,hD)U(t)\psi_0^w(x,hD) = U(t)\psi_0^w(x,hD) + \mathcal{O}_{L^2\to L^2}(h^\infty). \tag{6.15}$$

(ii) *If* $\psi_0,\psi_1\in S_\gamma(1)$ *are such that* $\psi_0=1$ *near* $\operatorname{supp}\psi_1\circ\Phi^t$, *then*

$$\psi_1^w(x,hD)U(t)\psi_0^w(x,hD) = \psi_1^w(x,hD)U(t) + \mathcal{O}_{L^2\to L^2}(h^\infty). \tag{6.16}$$

Before proving the proposition, we remark that if instead $\psi_0, \psi_2 \in S(1)$ satisfy

$$d(\operatorname{supp}\psi_0, \operatorname{supp}\psi_2\circ\Phi^t) \geqslant \frac{1}{C}, \quad \operatorname{supp}\psi_j \subset \mathcal{E}^{\delta_1}, \tag{6.17}$$

then

$$\psi_2^w(x,hD)U(t)\psi_0^w(x,hD) = \mathcal{O}_{L^2\to L^2}(h^\infty). \tag{6.18}$$

Indeed, we can apply (6.15) with $\psi_1 = 1-\psi_2$.

Proof. We use Lemma 6.2 to write

$$\psi_1^w(x,hD)U(t)\psi_0^w(x,hD) = U_0(t)(U_0(t)^{-1}\psi_1^w(x,hD)U_0(t))V(t)\psi_0^w(x,hD). \tag{6.19}$$

Pseudodifferential calculus on $\Psi_{h,\gamma}$ (see for instance [11, Chapter 7] or [13, Chapter 4]) shows that the wavefront set of the operator $V(t)\psi_0^w(x,hD)$ is a subset of $\operatorname{supp}\psi_0$, while Egorov's theorem and the condition (6.14) implies that $U_0(t)^{-1}\psi_1^w(x,hD)U_0(t) = I$ microlocally in an h^γ-neighbourhood of $\operatorname{supp}\psi_0$. This operator can thus be omitted in (6.19), up to an error $\mathcal{O}(h^\infty)$, which proves the first statement.

The proof of the second statement goes similarly: $\psi_0^w(x,hD) = 1$ microlocally near the wavefront set of $U_0(t)^{-1}\psi_1^w(x,hD)U_0(t)$. □

We can use this proposition to show that the "deep complex scaling" region acts as an absorbing potential, that is, strongly damps the propagating wavepackets.

LEMMA 6.4. *Take δ_1 as in Proposition 6.1, R_0 as in (3.21) and fix some time $t_1 \geqslant 0$. Then, for any symbol $\psi \in S(T^*X)$ satisfying*

$$\operatorname{supp}(\psi\circ\Phi^{-t}) \subset \mathcal{E}^{4\delta_1/5} \cap \left\{\varrho : |x(\varrho)| > \tfrac{5}{2}R_0\right\} \quad \textit{for all } t\in[0,t_1], \tag{6.20}$$

we have

$$\|U(t_1)\psi^w(x,hD)\|_{L^2\to L^2} \leqslant e^{-\theta/hC_0}\|\psi^w(x,hD)\|_{L^2\to L^2} + \mathcal{O}_{L^2\to L^2}(h^\infty), \tag{6.21}$$

where $C_0 > 0$ is independent of the choice of ψ.

Proof. For any symbol $\psi_0 \in S(1)$ supported inside $\mathcal{E}^{\delta_1} \cap \{\varrho : x(\varrho) > 2R_0\}$, the estimates (6.4) imply that

$$\operatorname{Im}\langle \widetilde{P}\psi_0^w(x,hD)u, \psi_0^w(x,hD)u\rangle \leqslant -\frac{\theta}{C_1}\|\psi_0^w(x,hD)u\|^2 + \mathcal{O}(h^\infty)\|u\|^2 \tag{6.22}$$

for some $C_1 > 0$. From the hypothesis (6.20) on ψ, and assuming R_0 large enough, there exists a symbol $\psi_1 \in S(1)$ such that

$$\operatorname{supp}\psi_1 \subset \mathcal{E}^{\delta_1} \cap \{\varrho : x(\varrho) > 2R_0\} \quad \text{and} \quad d(\operatorname{supp}\psi, \complement\{\varrho : \psi_1\circ\Phi^t(\varrho) = 1\}) > \frac{1}{C}, \quad t\in[0,t_1].$$

Proposition 6.2 (i) then shows that

$$\psi_1^w(x,hD)U(t)\psi^w(x,hD) = U(t)\psi^w(x,hD) + \mathcal{O}_{L^2\to L^2}(h^\infty), \quad \text{uniformly for } t\in[0,t_1].$$

Combining this with (6.22) we obtain, uniformly for $t\in[0,t_1]$,

$$\begin{aligned}\partial_t\|U(t)\psi^w u\|^2 &= \frac{2}{h}\operatorname{Im}\langle \widetilde{P}\psi_1^w U(t)\psi^w u, \psi_1^w U(t)\psi^w u\rangle + \mathcal{O}(h^\infty)\|u\|^2\\ &\leqslant -\frac{2\theta}{C_1 h}\|U(t)\psi^w u\|^2 + \mathcal{O}(h^\infty)\|u\|^2,\end{aligned}$$

from which the lemma follows by Gronwall's inequality, with $1/C_0 = 2t_1/C_1$. □

6.3. Microlocal partition

We consider $\delta_1 = \frac{1}{4}\delta$ as in Proposition 6.1, and take a smooth partition of unity adapted to $\{W_a \cap \mathcal{E}^{\delta_1}\}_{a\in A}$, which by quantization produces a family $\{\Pi_a \in \Psi_h\}_{a\in A}$ such that

$$\mathrm{WF}_h(\Pi_a) \subset W_a \cap \mathcal{E}^{3\delta_1/4}, \quad \Pi_a = \Pi_a^* \quad \text{and} \quad \sum_{a\in A}\Pi_a = I \text{ microlocally near } \mathcal{E}^{\delta_1/2}.$$

The difference

$$\Pi_\infty \stackrel{\text{def}}{=} I - \sum_{a\in A}\Pi_a$$

is also a pseudodifferential operator in Ψ_h, and

$$\mathrm{WF}_h(\Pi_\infty) \cap \mathcal{E}^{\delta_1/2} = \varnothing.$$

Using this microlocal partition of unity, we decompose the modified propagator (6.9) at time t_0:

$$U(t_0) = \sum_{a\in A\cup\{\infty\}} U_a, \quad U_a \stackrel{\text{def}}{=} U(t_0)\Pi_a. \tag{6.23}$$

We then decompose the Nth power of the propagator as follows:

$$U(Nt_0) = \sum_{\alpha\in A^N} U_{\alpha_N} \dots U_{\alpha_1} + R_N. \tag{6.24}$$

The remainder R_N is the sum over all sequences α containing at least one index $\alpha_j = \infty$. The following lemma shows that the remainder R_N is irrelevant when applied to states microlocalized near $\mathcal{E}$.

Lemma 6.5. *Suppose that* $\chi\in\mathcal{C}_c^\infty(T^*\mathbb{R})$ *is supported inside* $\mathcal{E}^{\delta_1/5}$ *and that we consider logarithmic times in the semiclassical limit*:

$$N \leqslant M\log\frac{1}{h}, \quad M>0 \text{ fixed.} \tag{6.25}$$

Then the remainder term in (6.24) *satisfies*

$$\|R_N\chi^w(x,hD)\|_{L^2\to L^2}=\mathcal{O}(h^\infty),$$

with the implied constants depending only on M.

Proof. Let $\alpha\in A^N$ be a sequence containing at least one index $\alpha_j=\infty$. Call j_m the smallest integer such that $a_j=\infty$, so the corresponding term in R_N reads

$$U_{\alpha_N}\dots U_{\alpha_{j_m+1}}U(t_0)\Pi_\infty U_{\alpha_{j_m-1}}\dots U_{\alpha_1}, \quad \text{with } \alpha_1,\dots,\alpha_{j_m-1}\in A.$$

The lemma will be proved once we show that

$$\Pi_\infty U_{\alpha_{j_m-1}}\dots U_{\alpha_1}\chi^w(x,hD)=\mathcal{O}_{L^2\to L^2}(h^\infty), \tag{6.26}$$

with implied constants uniform with respect to the sequence α. Indeed, the remaining factor on the left is bounded as

$$\|U_{\alpha_N}\dots U_{\alpha_{j_m+1}}U(t_0)\|\leqslant Ce^{CN}\leqslant Ch^{-CM},$$

and the full number of sequences is $(|A|+1)^N=\mathcal{O}(h^{-C'M})$.

The estimate (6.26) is obvious if $j_m=0$, because $\mathrm{WF}_h(\Pi_\infty)$ and $\mathrm{WF}_h(\chi^w)$ are at a positive distance from each other. To treat the cases $j_m>0$, we will define a family of N nested symbols which cutoff in energy in various ranges between $\frac14\delta_1$ and $\frac12\delta_1$. Because $N\sim\log(1/h)$, we must use symbols in some class $S_{\delta'}(1)$, $\delta'\in\left(0,\frac12\right)$. We first define a sequence of functions $\tilde\chi_j\in\mathcal{C}_c^\infty(\mathbb{R},[0,1])$, $j=1,\dots,N$, as follows:

$$\tilde\chi_1(t)=\begin{cases}1, & \text{if } |t|\leqslant\frac14\delta_1,\\ 0, & \text{if } |t|\geqslant\frac14\delta_1+\frac12h^{\delta'},\end{cases}\qquad \tilde\chi_{j+1}(t)=\begin{cases}1, & \text{if } |t|\leqslant\frac14\delta_1,\\ \tilde\chi_j(|t|-h^{\delta'}), & \text{if } |t|\geqslant\frac14\delta_1,\end{cases}\qquad j\geqslant1.$$

The function $\tilde\chi_N$ vanishes for $|t|\geqslant\frac14\delta_1+Nh^{\delta'}$, and we will take h small enough so that $\frac14\delta_1+Nh^{\delta'}<\frac12\delta_1$. From there, the energy cutoff functions $\chi_j\in S_{\delta'}(1)$ are defined by

$$\chi_j(x,\xi)\stackrel{\text{def}}{=}\tilde\chi_j(p(x,\xi)), \quad j=1,\dots,N.$$

From the support properties of χ, the first cutoff function satisfies

$$\chi_1^w(x,hD)\chi^w(x,hD)=\chi^w(x,hD)+\mathcal{O}_{L^2\to L^2}(h^\infty). \tag{6.27}$$

For any $j=1,...,j_m-1$, we have $\chi_j=\chi_j\circ\Phi^{t_0}$, and the nesting between χ_j and χ_{j+1} allows us to apply the propagation results of Proposition 6.2 (i):

$$\chi_{j+1}^w(x,hD)U_{\alpha_j}\chi_j^w(x,hD)=U_{\alpha_j}\chi_j^w(x,hD)+\mathcal{O}(h^\infty),\quad j=1,...,j_m-1. \tag{6.28}$$

Therefore, inserting χ_{j+1}^w after each U_{α_j} leaves the operator (6.26) almost unchanged. Finally, the cutoff function χ_{j_m} is supported in the energy shell $\{\varrho:|p(\varrho)|\leqslant\frac{1}{4}\delta_1+Nh^{\delta'}\}$ which, for h small enough, is at finite distance from $\mathrm{WF}_h(\Pi_\infty)$, so that

$$\Pi_\infty\chi_{j_m}^w(x,hD)=\mathcal{O}_{L^2\to L^2}(h^\infty).$$

Combining this expression with (6.27) and (6.28) proves (6.26) and the lemma. □

The set A^N of N-sequences can be split between several subsets. Using the time N_0 characterized in Lemma 5.1, we define the set $\mathcal{A}_N\subset A^N$ as follows:

$$\alpha=\alpha_1\,...\,\alpha_N\in\mathcal{A}_N\quad\Longleftrightarrow\quad\begin{cases}\Phi^{t_0}(W_{\alpha_j})\cap W_{\alpha_{j+1}}\neq\varnothing & \text{for } j=1,...,N-1, \text{ and}\\ \alpha_j\in A_1 & \text{for } N_0<j<N-N_0.\end{cases} \tag{6.29}$$

The sequences in $\mathcal{A}_N$ spend most of the time in the vicinity of the trapped set.

The next lemma shows that we can discard all sequences except for those in $\mathcal{A}_N$.

LEMMA 6.6. *Suppose that* (6.25) *holds. Then there exists* $C_1>0$ *such that, for* h *small enough*,

$$\sum_{\alpha\in A^N\setminus\mathcal{A}_N}\|U_{\alpha_N}\,...\,U_{\alpha_1}\|\leqslant C_1|A|^Ne^{C_1Nt_0}e^{-\theta/C_1h}.$$

If $N\leqslant M\log(1/h)$, $\theta=M_1h\log(1/h)$ *and* $M_1\gg Mt_0$, *this implies that*

$$\sum_{\alpha\in A^N\setminus\mathcal{A}_N}\|U_{\alpha_N}\,...\,U_{\alpha_1}\|\leqslant h^{M_1/C_2},\quad 0<h<h_0(M,M_1,|A|). \tag{6.30}$$

Proof. Take $\alpha\in A^N\setminus\mathcal{A}_N$. If the first condition on the right in (6.29) is violated, then the property $\mathrm{WF}_h(\Pi_a)\Subset W_a\cap\mathcal{E}^{3\delta_1/4}$ for $a\in A$ and (6.18) imply that $\|U_\alpha\|=\mathcal{O}(h^\infty)$.

Assume that for some j, $N_0<j<N-N_0$, we have $\alpha_j\notin A_1$. We have three possibilities. First, assume that $\alpha_j=0$. In this case, the factor $U_{\alpha_j}=U(t_0)\Pi_0$ can be decomposed as

$$U(t_0-1)U(1)\Pi_0.$$

If R_0 has been chosen large enough, the set $W_0=\mathcal{E}^\delta\cap\{\varrho:x(\varrho)>3R_0\}$ satisfies the property

$$\Phi^t(W_0)\subset\{\varrho:|x(\varrho)|>\tfrac{8}{3}R_0\},\quad t\in[0,1].$$

Using the fact that $WF_h(\Pi_0)\subset W_0\cap\mathcal{E}^{3\delta_1/4}$ and applying Lemma 6.4 for $t_1=1$, we find that

$$\|U(t_0-1)U(1)\Pi_0\|\leqslant e^{C(t_0-1)}C_0e^{-\theta/hC_0}+\mathcal{O}(h^\infty). \tag{6.31}$$

Second, assume that $\alpha_j\in A_2^-$, using the same notation as in the proof of Lemma 5.1. In this case,

$$\Phi^t(W_{\alpha_j})\subset W_0 \quad \text{for any } t\geqslant N_0t_0. \tag{6.32}$$

Applying Proposition 6.2 (i) N_0 times, one realizes that the operator

$$\Pi_{\alpha_{j+N_0}}U_{\alpha_{j+N_0-1}}\dots U_{\alpha_{j+1}}U_{\alpha_j}$$

is negligible unless $\mathrm{WF}_h(\Pi_{\alpha_{j+N_0}})$ intersects W_0. This is the case if $\alpha_{j+N_0}=0$, or $\alpha_{j+N_0}\in A_2$ and $W_{\alpha_{j+N_0}}\cap W_0\neq\varnothing$. In both cases, we have (as long as R_0 has been taken large enough)

$$\Phi^t(W_{\alpha_{j+N_0}})\subset\big\{\varrho:|x(\varrho)|>\tfrac{8}{3}R_0\big\},\quad t\in[0,1],$$

and the estimate (6.31) applies to $\|U(t_0)\Pi_{\alpha_{j+N_0}}\|$.

Third, if $j\in A_2^+$, we have $\Phi^t(W_{\alpha_j})\in W_0$ for $t<-N_0t_0$. Again, iterating Proposition 6.2 (i) N_0 times shows that the operator

$$\Pi_{\alpha_j}U_{\alpha_{j-1}}\dots U_{\alpha_{j-N_0+1}}U(t_0)\Pi_{\alpha_{j-N_0}}$$

will be negligible unless $W_{\alpha_{j-N_0}}$ intersects W_0. This yields

$$\|U(t_0)\Pi_{\alpha_{j-N_0}}\|\leqslant e^{C(t_0-1)}C_0e^{-\theta/hC_0}+\mathcal{O}(h^\infty).$$

For these three cases, we find, using (6.10),

$$\|U_{\alpha_N}\dots U_{\alpha_1}\|\leqslant e^{C(N-N_0)t_0}e^{-\theta/hC_0}.$$

This estimate concerns an individual element $\alpha\in A^N\setminus\mathcal{A}_N$. Summing over all such elements produces a factor $|A|^N$, which proves the first estimate. The second estimate follows from the assumptions on N and θ. □

The following proposition, which is at the center of the method, controls the terms $\alpha\in\mathcal{A}_N$ in (6.24). The proof is more subtle than for the above lemmas, and uses the whole machinery of §4.3 and §5. In particular, a crucial use is made of the hyperbolicity of the classical dynamics on K^δ. For this reason, we call the following bound a *hyperbolic dispersion estimate.*

PROPOSITION 6.3. *Assume that* $N \leqslant M \log(1/h)$ *for some* $M > 0$. *Then, if the diameter* $\varepsilon > 0$ *of the cover* $\mathcal{V}_0$ *has been chosen small enough, for any* $\alpha \in \mathcal{A}_N \cap A_1^N$ *we have the bound*

$$\|U_{\alpha_N} \dots U_{\alpha_1}\| \leqslant h^{-n/2}(1+\varepsilon_0)^N \prod_{j=1}^{N} e^{S_{t_0}(W_{\alpha_j})/2}, \tag{6.33}$$

where the coarse-grained Jacobian $S_{t_0}(\cdot)$ *is defined in* (5.22) *and* ε_0 *is the parameter appearing in* (5.23).

Before proving this proposition in §7, we show how it implies Theorem 3.

6.4. End of the proof of Theorem 3

Suppose that $\|u(h)\|=1$ is an eigenfuction of $P_{\theta,\varepsilon}(h)$, with the same conditions as in Proposition 6.1: $P_{\theta,\varepsilon}(h)u(h)=z(h)u(h)$, $|\operatorname{Re} z(h)-E| \leqslant \delta_1$ and $\operatorname{Im} z(h) > -Ch$. Then, taking $t=Nt_0$, $N \leqslant M \log(1/h)$, in Proposition 6.1, we get

$$e^{Nt_0 \operatorname{Im} z(h)/h} = \|U(Nt_0)u(h)\| + \mathcal{O}(h^\infty).$$

Using the decomposition (6.24) and Lemmas 6.5 and 6.6, the state $U(Nt_0)u(h)$ can be decomposed as

$$U(Nt_0)u(h) = \sum_{\alpha \in \mathcal{A}_N} U_{\alpha_N} \dots U_{\alpha_1} u(h) + \mathcal{O}_{L^2}(h^{M_3}),$$

where M_3 can be as large as we like, if we take $\theta = M_1 h \log(1/h)$ with M_1 large, depending on Mt_0.

The norm of the right-hand side can be estimated by applying (6.33) to the factors $U_{\alpha_{N-N_0-1}} \dots U_{N_0+1}$. This leads to

$$e^{Nt_0 \operatorname{Im} z(h)/h} \leqslant Ch^{-n/2}(1+\varepsilon_0)^N \sum_{\alpha \in \mathcal{A}_N} \prod_{j=N_0+1}^{N-N_0-1} e^{S_{t_0}(W_{\alpha_j})/2} + \mathcal{O}(h^{M_3}). \tag{6.34}$$

The sum over $\mathcal{A}_N$ can be factorized:

$$\sum_{\alpha \in \mathcal{A}_N} \prod_{j=N_0+1}^{N-N_0-1} e^{S_{t_0}(W_{\alpha_j})/2} \leqslant |A|^{2N_0+1} \biggl(\sum_{a \in A_1} e^{S_{t_0}(W_a)/2} \biggr)^{N-2N_0-1}.$$

Combining this bound with (5.24), we finally obtain

$$e^{Nt_0 \operatorname{Im} z(h)/h} \leqslant C' h^{-n/2}(1+\varepsilon_0)^N e^{Nt_0(\mathcal{P}_E^\delta(1/2)+\varepsilon_0)} + \mathcal{O}(h^{M_3}). \tag{6.35}$$

Taking the logarithm and dividing by Nt_0, we get

$$\frac{\operatorname{Im} z(h)}{h} \leqslant \mathcal{P}_E^\delta\left(\frac{1}{2}\right)+3\varepsilon_0+n\frac{\log(1/h)}{2Nt_0}+\frac{\log C'}{Nt_0}.$$

We can take $N=M\log(1/h)$ with M arbitrarily large (and consequently with M_1, in the definition of θ, large), so that, for any h sufficiently small (say, $h<h(\delta,\varepsilon_0)$)

$$\frac{\operatorname{Im} z(h)}{h} \leqslant \mathcal{P}_E^\delta\left(\frac{1}{2}\right)+4\varepsilon_0.$$

In §5.2 we could take $\varepsilon_0>0$ as small as we wished. This proves Theorem 3.

7. Proof of the hyperbolic dispersion estimate

To prove the estimate in Proposition 6.3, we adapt the strategy of [1] and [2] to the present setting. We decompose an arbitrary state microlocalized inside W_{α_1} into a combination of Lagrangian states associated with "horizontal" Lagrangian leaves (namely, Lagrangian leaves situated in some unstable cone). By linearity, the evolution of the full initial state can be estimated by first evolving each of these Lagrangian states. Proposition 5.1 shows that, being in an unstable cone, the Lagrangians spread uniformly along the unstable direction, at a rate governed by the unstable Jacobian. Proposition 4.1 shows that this spreading implies a uniform exponential decay of the norm of the evolved Lagrangian state and, by linearity, a uniform decay of the full evolved state.

7.1. Decomposing localized states into a Lagrangian foliation

In this section we consider states $w\in L^2(\mathbb{R}^n)$ with wavefront sets contained in an open neighbourhood W of the origin, $\mathrm{WF}_h(w)\subset W\stackrel{\text{def}}{=}B(\varepsilon)_y\times B(\varepsilon)_\eta$. Here $B(\varepsilon)$ is the open ball of radius ε in $\mathbb{R}^n$. We will decompose such a state w into a linear combination of "local momentum states" $\{e_\eta\}_{\eta\in B(2\varepsilon)}$, associated with horizontal Lagrangian leaves $\{\Lambda_\eta\}_{\eta\in B(2\varepsilon)}$. Each Lagrangian leaf Λ_η is defined by

$$\Lambda_\eta \stackrel{\text{def}}{=} \{(y,\eta)\in T^*\mathbb{R}^n : y\in B(2\varepsilon)\},\quad \eta\in B(2\varepsilon).$$

This family of Lagrangian foliates $B(\varepsilon)\times B(\varepsilon)$:

$$W\Subset \bigcup_{\eta\in B(2\varepsilon)}\Lambda_\eta,\quad \Lambda_\eta\cap\Lambda_{\eta'}=\varnothing \text{ if } \eta\neq\eta'.$$

The associated Lagrangian states e_η are defined as follows. We start from the "full" momentum states $\widetilde{E}_\eta \in \mathcal{C}^\infty(\mathbb{R}^n)$:

$$\widetilde{E}_\eta(y) = e^{i\langle \eta, y\rangle/h}, \quad y \in \mathbb{R}^n, \eta \in \mathbb{R}^n,$$

and we smoothly truncate these states in a fixed ball:

$$e_\eta(y) \stackrel{\text{def}}{=} \widetilde{E}_\eta(y)\chi_\varepsilon(y), \quad \chi_\varepsilon \in \mathcal{C}_c^\infty(B(2\varepsilon)), \; \chi_\varepsilon|_{B(3\varepsilon/2)} = 1. \tag{7.1}$$

Notice that all states e_η satisfy

$$\|e_\eta\|_{L^2} = \|\chi_\varepsilon\|_{L^2} \leqslant C\varepsilon. \tag{7.2}$$

The h-Fourier decomposition of an arbitrary state $w \in L^2(\mathbb{R}^n_y)$ reads

$$w = \int_{\mathbb{R}^n} \frac{1}{(2\pi h)^{n/2}} (\mathcal{F}_h w)(\eta) \widetilde{E}_\eta \, d\eta.$$

With the assumption $\mathrm{WF}_h(w) \subset B(\varepsilon)_y \times B(\varepsilon)_\eta$, one deduces that

$$w = \int_{B(2\varepsilon)} \frac{1}{(2\pi h)^{n/2}} (\mathcal{F}_h w)(\eta) e_\eta \, d\eta + \mathcal{O}(h^\infty)\|w\|. \tag{7.3}$$

This is the decomposition into horizontal Lagrangian states we were aiming at. If we apply a semiclassically tempered operator T to this state (see §3.1), we obtain

$$Tw = \int_{B(2\varepsilon)} \frac{1}{(2\pi h)^{n/2}} (\mathcal{F}_h w)(\eta)(Te_\eta) \, d\eta + \mathcal{O}(h^\infty)\|w\|.$$

This gives the following bound for the norm of Tw:

$$\begin{aligned} \|Tw\|_{L^2} &\leqslant Ch^{-n/2} \int_{B(2\varepsilon)} |(\mathcal{F}_h w)(\eta)| \, \|Te_\eta\| \, d\eta + \mathcal{O}(h^\infty)\|w\| \\ &\leqslant Ch^{-n/2} \max_{\eta \in B(2\varepsilon)} \|Te_\eta\| \, \|w\| + \mathcal{O}(h^\infty)\|w\|. \end{aligned} \tag{7.4}$$

7.1.1. Decomposition of the initial state into near-unstable Lagrangian states

By using semiclassical Fourier integral operators, see for instance [13, Chapter 10], we can transplant the construction of the previous paragraph to any local coordinate representation. Here we will decompose states microlocalized in the sets W_a, $a \in A_1$. The horizontal Lagrangians are constructed with respect to the coordinate chart (y^a, η^a) centered at some point $\varrho_a \in W_a \cap K^\delta$, as described in Lemma 4.3. In order to cover the set W_a, we use the following family $(\Lambda_{\eta,a})$:

$$\Lambda_{\eta,a} \equiv \{(y^a, \eta^a) : y^a \in B(2\varepsilon) \text{ and } \eta^a = \eta\}, \quad \eta \in B(\delta, 2\varepsilon), \tag{7.5}$$

where $B(\delta,\varepsilon)\stackrel{\text{def}}{=}\{\eta=(\eta_1,s)\in\mathbb{R}^n:|\eta_1|<\delta \text{ and } |s|<\varepsilon\}$. Notice that these Lagrangians are isoenergetic ($\Lambda_{\eta,a}\subset\mathcal{E}_{\eta_1}$) and they belong to arbitrarily thin unstable cones in W_a, in particular to the cones used in Proposition 5.1 and Remark 5.1.

Using the Fourier integral operator $\mathcal{U}_a$ associated with the coordinate change $(x,\xi)\mapsto(y^a,\eta^a)$ (see Lemma 4.4), each state (7.1) can be brought to a Lagrangian state:

$$e_{\eta,a}=\mathcal{U}_a^*e_\eta \quad \text{associated with the Lagrangian leaf } \Lambda_{\eta,a}\subset\mathcal{E}_{\eta_1},$$

with norms bounded as in (7.2).

7.2. Evolving the Lagrangian states through $U_{\alpha_N}\ldots U_{\alpha_1}$

We now consider an arbitrary sequence $\alpha\in\mathcal{A}_N\cap A_1^N$. For any normalized $u\in L^2(X)$, the state

$$w\stackrel{\text{def}}{=}\Pi_{\alpha_1}u \quad \text{satisfies} \quad \mathrm{WF}_h(w)\subset W_{\alpha_1}\cap\mathcal{E}^{3\delta_1/4},$$

and can thus be decomposed into the Lagrangian states (e_{η,α_1}) associated with the leaves Λ_{η,α_1}, as in (7.3). In order to prove the estimate (6.33), we will first study the individual states

$$U_{\alpha_N}\ldots U_{\alpha_2}U(t_0)e_{\eta,\alpha_1}, \quad \eta\in B\big(\tfrac{3}{4}\delta_1,2\varepsilon\big). \tag{7.6}$$

We recall that each set W_a, $a\in A_1$, has the property

$$\Phi^k(W_a)\subset V_{b_k},\ k=0,\ldots,t_0-1, \quad \text{for some sequence } b_0,\ldots,b_{t_0-1}. \tag{7.7}$$

Therefore, to the sequence $\alpha=\alpha_1\ldots\alpha_N\in A_1^N$ corresponds a sequence $\beta=\beta_0\ldots\beta_{Nt_0-1}$ of neighbourhoods V_{β_k} visited at the times $k=0,\ldots,Nt_0-1$. For later convenience, we also consider a set V'_{Nt_0} (of diameter $C\varepsilon$), which contains $\Phi^{t_0}(W_{\alpha_N})$.

From now on, we fix some $\eta\in B\big(\frac{3}{4}\delta_1,2\varepsilon\big)$ and compute the state (7.6), making use of various properties proved in the preceding sections.

7.2.1. Evolution of the near-unstable Lagrangians Λ_{η,α_1}

The results of §4.2 and Lemma 4.1 show that it is relevant to study the evolution of the Lagrangian $\Lambda^0_{\mathrm{loc}}\stackrel{\text{def}}{=}\Lambda_{\eta,\alpha_1}\cap W_{\alpha_1}$ through the following operations: one evolves Λ^0_{loc} through Φ^{t_0}, then restricts the result to W_{α_2}, then evolves it through Φ^{t_0}, restricts to W_{α_3}, and so on. It is also useful to consider the intermediate steps, that is, for $k=mt_0+m'$, $0\leqslant m<N$, $0\leqslant m'<t_0$, we take

$$\begin{aligned}\Lambda^{mt_0}_{\mathrm{loc}} &\stackrel{\text{def}}{=}\Phi^{t_0}(\Lambda^{(m-1)t_0}_{\mathrm{loc}})\cap W_{\alpha_m}, && m=1,\ldots,N-1,\\ \Lambda^{mt_0+m'}_{\mathrm{loc}} &\stackrel{\text{def}}{=}\Phi^{m'}(\Lambda^{mt_0}_{\mathrm{loc}}), && m'=1,\ldots,t_0-1.\end{aligned}$$

Fix $\gamma_1=\frac{1}{2}$. By construction, Λ^0_{loc} is contained in the unstable γ_1-cone in the coordinates $(y^{\alpha_1},\eta^{\alpha_1})$. We can thus apply Proposition 5.1 (i) and Proposition 5.2 to this sequence of Lagrangian leaves: each Λ^k_{loc} is contained in the unstable γ_1-cone (when expressed in the coordinates $(y^{\beta_k},\eta^{\beta_k})$ on the set V_{β_k}). Furthermore, part (ii) of the proposition shows that the higher derivatives of the functions φ_k generating Λ^k_{loc} also remain uniformly bounded with respect to k. The sequence of Lagrangians is thus totally "under control", and the implied constants are independent of the choice of $\eta\in B\big(\frac{3}{4}\delta_1,2\varepsilon\big)$ parametrizing the initial state e_{η,α_1}.

7.2.2. Analysis of the operator $U_{\alpha_N}\dots U_{\alpha_1}$

We now show that all the propagators $U(1)$ in (7.6) may be replaced by the unitary propagators $U_0(1)$, up to a negligible error. For each $a\in A_1$ we recall that the set W_a satisfies (7.7). All the sets $V_b\in\mathcal{V}_0$ were chosen so that $\Phi^t(V_b)$ remains close to K^δ in the interval $t\in[0,1]$. As a result, one can apply Lemma 6.3 to the differences

$$(U(1)-U_0(1))\widetilde{\Pi}^w_b,$$

where $\widetilde{\Pi}_b\in\Psi_h$ satisfies

$$\widetilde{\Pi}_b=I \text{ near } V_b \quad\text{and}\quad \Phi^t(\mathrm{WF}_h(\widetilde{\Pi}_b))\Subset U_G \text{ for all } t\in[0,1].$$

Each factor $U_a=U(t_0)\Pi_a$ can then be decomposed as

$$\begin{aligned} U(t_0)\Pi_a &= U(1)\widetilde{\Pi}_{b_{t_0-1}}\dots U(1)\widetilde{\Pi}_{b_1}U(1)\Pi_a+\mathcal{O}(h^\infty)\\ &= U_0(1)\widetilde{\Pi}_{b_{t_0-1}}\dots U_0(1)\widetilde{\Pi}_{b_1}U_0(1)\Pi_a+\mathcal{O}(h^\infty). \end{aligned} \tag{7.8}$$

The first equality uses the propagation properties of Proposition 6.2 (i) and (7.7). The second one is obtained by applying Lemma 6.3 to all factors $U(1)\widetilde{\Pi}_{b_k}$. The operator (7.6) can thus be expanded as

$$U_{\alpha_N}\dots U_{\alpha_1}=S_{\beta_{Nt_0},\beta_{Nt_0-1}}\dots S_{\beta_1,\beta_0}\Pi_{\alpha_1}+\mathcal{O}(h^\infty), \tag{7.9}$$

where we called

$$\begin{aligned} S_{\beta_{k+1},\beta_k} &\stackrel{\text{def}}{=} \widetilde{\Pi}_{\beta_{k+1}}U_0(1), && k=0,\dots,Nt_0-1,\ t_0\nmid k+1,\\ S_{\beta_{k+1},\beta_k} &\stackrel{\text{def}}{=} \Pi_{\alpha_{m+1}}U_0(1), && k+1=mt_0,\ m=1,\dots,N-1,\\ S_{\beta_{Nt_0},\beta_{Nt_0-1}} &\stackrel{\text{def}}{=} \widetilde{\Pi}'_{Nt_0}U_0(1). \end{aligned}$$

The operator $\widetilde{\Pi}'_{Nt_0}\in\Psi_h$ on the last line has a compactly supported symbol, and is equal to the identity, microlocally near the set V'_{Nt_0}, so that $\widetilde{\Pi}'_{Nt_0}U(t_0)\Pi_{\alpha_N}=U(t_0)\Pi_{\alpha_N}+\mathcal{O}(h^\infty)$.

From Lemmas 4.2 and 4.4, each of the propagators S_{β_{k+1},β_k} can be put in the form

$$S_{\beta_{k+1},\beta_k}=\mathcal{U}^*_{\beta_{k+1}}T_{\beta_{k+1},\beta_k}\mathcal{U}_{\beta_k}+\mathcal{O}(h^\infty), \tag{7.10}$$

where $\mathcal{U}_{\beta_k}$ is the Fourier integral operator quantizing the local change of coordinates $(x,\xi)\mapsto(y^{\beta_k},\eta^{\beta_k})$ (see Lemma 4.4), while T_{β_{k+1},β_k} is an operator of the form (4.26), which quantizes the map $\varkappa_{\beta_k,\beta_{k-1}}$ obtained by expressing Φ^1 in the coordinates $(y^{b_k},\eta^{b_k})\mapsto(y^{b_{k+1}},\eta^{b_{k+1}})$.

Inserting (7.10) in (7.9), we obtain

$$U_{\alpha_N}\ldots U_{\alpha_2}U(t_0)e_{\eta,\alpha_1}=\mathcal{U}^*_{\beta_{Nt_0}}T_{\beta_{Nt_0},\beta_0}e_\eta+\mathcal{O}_{L^2}(h^\infty),$$

where we took for short

$$T_{\beta_{Nt_0},\beta_0}\stackrel{\text{def}}{=}T_{\beta_{Nt_0},\beta_{Nt_0-1}}\ldots T_{\beta_1,\beta_0}.$$

Here we used the fact that $\mathcal{U}^*_{\beta_k}\mathcal{U}_{\beta_k}=I$ microlocally near the wavefront set of $\widetilde{\Pi}^{(\prime)}_{\beta_k}$, Π_{α_m} or Π'_{Nt_0}.

7.2.3. Applying the semiclassical evolution estimate

The state $T_{\beta_{Nt_0},\beta_0}e_\eta$ has the same form as the left-hand side in (4.32). Since the Lagrangians

$$\Lambda^k_{\text{loc}}\equiv\{(y^{\beta_k},\eta^{\beta_k}=\varphi'_k(y^{\beta_k}))\}$$

remain under control uniformly for $1\leqslant k\leqslant N$, we can apply Proposition 4.1 to obtain a precise description of that state: for any integer $L>0$, we may write

$$T_{\beta_{Nt_0},\beta_0}e_\eta(y)=a^{Nt_0}(y)e^{i\varphi_{Nt_0}(y)/h}+h^LR^{Nt_0}_L(y),\quad y\in\mathbb{R}^n.$$

The symbol a^{Nt_0} admits an expansion,

$$a^{Nt_0}(y)=\sum_{j=0}^{L-1}h^ja_j^{Nt_0}(y),$$

which we now analyze. Starting from some $y\in B(C\varepsilon)$, let us assume that there exists no sequence of coordinates

$$\{y^k\}_{k=0}^{Nt_0},\quad y=y^{Nt_0},\quad y^{k-1}=g_k(y^k),$$

where g_k is the projection of the map $\varkappa_{\beta_k,\beta_{k-1}}^{-1}|_{\Lambda_{\rm loc}^k}$ on the axes $\{(y^{\beta_k},\eta^{\beta_k}):\eta^{\beta_k}=0\}$ and $\{(y^{\beta_{k-1}},\eta^{\beta_{k-1}}):\eta^{\beta_{k-1}}=0\}$. In this case, Proposition 4.1 shows that $a^{Nt_0}(y)=0$.

On the other hand, if such a sequence exists, the principal symbol $a_0^{Nt_0}(y)$ satisfies a formula of the type (4.33). The functions χ_{j_k} now correspond to the symbols of the operators $\widetilde{\Pi}_{\beta_k}$, Π_{α_m} or Π'_{Nt_0}, which are uniformly bounded from above by $1+\mathcal{O}(h)$.

The main factor in (4.33) is the product of determinants $|\det dg_k(y^k)|^{1/2}$, which corresponds to the uniform expansion of the Lagrangians along the horizontal direction. To estimate this product, we follow §4.3 and group these determinants by packets of length t_0. According to Proposition 5.2, for any t_0-packet we have

$$\prod_{k=mt_0+1}^{(m+1)t_0}|\det dg_k(y^k)|^{1/2}=\det\biggl(\frac{\partial y^{(m+1)t_0}}{\partial y^{mt_0}}\biggr)^{-1/2}=(1+\mathcal{O}(\varepsilon^\gamma))e^{-\lambda_{t_0}^+(\varrho_{\alpha_m})/2}$$

for $m=0,...,N-1$. Here we have used the coordinate frames $(y^{\beta_{mt_0}},\eta^{\beta_{mt_0}})$ to label points in W_{α_m} instead of the coordinates $(y^{\alpha_m},\eta^{\alpha_m})$ centered at $\varrho_{\alpha_m}\in W_{\alpha_m}$; this change does not modify the estimate of the corollary, as is clear from (5.25). The product of determinants is thus governed by the unstable Jacobian along the trajectory. Because the points $\varrho_{\alpha_m}\in W_{\alpha_m}\cap K^\delta$ are somewhat arbitrary, we prefer to use the coarse-grained Jacobian (5.22) to bound the above right-hand side. Taking the product over all t_0-packets, we thus obtain, for some $C>0$ independent of N,

$$|a_0^{Nt_0}(y)|\leqslant\prod_{m=0}^{N-1}(1+Ch)^{t_0}(1+C\varepsilon^\gamma)e^{S_{t_0}(W_{\alpha_m})/2},\quad y\in\operatorname{supp}a_0^{Nt_0}\subset B(C\varepsilon).\tag{7.11}$$

The proof of Proposition 5.1 (see (5.17)) also shows that the determinants $\det dg_k(y)$ satisfy

$$\sup_{y\in\mathrm{Dom}(g_k)}|\det dg_k(y)|\leqslant\det(A_k)^{-1}+C\varepsilon^\gamma,\quad k=1,...Nt_0.$$

We will assume that ε is small enough, so that the right-hand side is bounded from above by $\nu_3<1$. This implies that the Jacobians J_k of (4.31) decay exponentially when $k\to\infty$. Henceforth, the higher-order symbols $a_j^{Nt_0}$, bounded as in (4.34), are smaller than the principal symbol, so that the upper bound (7.11) also holds if we replace $a_0^{Nt_0}$ by the full symbol a^{Nt_0}. This decay of J_k also shows that the remainder $R_L^{Nt_0}$, estimated in (4.35), is uniformly bounded in L^2. As a result, the bound (7.11) implies the bound

$$\|T_{\beta_{Nt_0},\beta_0}e_\eta\|\leqslant C\varepsilon(1+C\varepsilon^\gamma)^N\prod_{m=0}^{N-1}e^{S_{t_0}(W_{\alpha_m})/2},\quad \eta\in B\bigl(\tfrac{3}{4}\delta_1,2\varepsilon\bigr).\tag{7.12}$$

To end the proof of Proposition 6.3, it remains to apply the decomposition (7.3) to the $w=\Pi_{\alpha_1}u$, with $u\in L^2(X)$ of norm unity, and the bound (7.4) that follows is

$$\|U_{\alpha_N}...U_{\alpha_1}u\|\leqslant C\varepsilon h^{-n/2}(1+C\varepsilon^\gamma)^N\prod_{m=0}^{N-1}e^{S_{t_0}(W_{\alpha_m})/2}+\mathcal{O}(h^\infty).$$

Notice that the main term on the right-hand side is larger than h^{M_3} for some $M_3>0$. This bound thus proves Proposition 6.3 if, given $\varepsilon_0>0$, we choose the diameter ε of the partition $\mathcal{V}_0$ small enough.

8. Microlocal properties of the resonant eigenstates

In this section we will use the results of §6.1 and §6.2 to prove Theorem 4. We will turn back to the notation of §3.2, that is, the operator to keep in mind is

$$P(h)=-h^2\Delta+V(x), \quad \text{with symbol} \quad p(x,\xi)=\xi^2+V(x).$$

We also recall that

$$P_{\theta,\varepsilon}=e^{-\varepsilon G^w/h}P_\theta e^{\varepsilon G^w/h}, \quad \varepsilon=M_2\theta, \quad \theta=M_1 h\log\frac{1}{h} \quad \text{and} \quad G^w=G^w(x,hD), \tag{8.1}$$

where G is given by Lemma 6.1, and $M_1>0$ can be arbitrarily large. In this section we will choose the set V in Lemma 6.1 to be

$$V=T^*_{B(0,3R_0/4)}X, \quad \text{and assume that} \quad G(x,\xi)=0 \text{ for } x\in B\big(0,\tfrac{1}{2}R_0\big). \tag{8.2}$$

We now consider a resonant state u in the sense of (3.22), in particular $u|_{B(0,R_0)}=u_\theta|_{B(0,R_0)}$. If u satisfies (3.22) for some choice of $R_0>0$ (which implies a choice of deformation X_θ, see §3.4), then it has the same property with any larger R_0 (and associated X_θ). The state

$$u_{\theta,\varepsilon}\stackrel{\text{def}}{=}e^{-\varepsilon G^w/h}u_\theta$$

is in $L^2(X_\theta)$ and satisfies

$$(P_{\theta,\varepsilon}-z)u_{\theta,\varepsilon}=0.$$

Furthermore, the support properties of G imply that

$$\|u_{\theta,\varepsilon}-u\|_{L^2(B(0,R_0/2))}=\mathcal{O}(h^\infty)\|u_{\theta,\varepsilon}\|_{L^2(X_\theta)}. \tag{8.3}$$

The following lemma provides control on the behaviour of $u_{\theta,\varepsilon}$ near infinity.

Lemma 8.1. *Let $P_{\theta,\varepsilon}$ be the operator given by* (8.1) *for some choice of $R_0\gg1$ and $M_1\gg1$. Suppose that*

$$(P_{\theta,\varepsilon}-z)u_{\theta,\varepsilon}=0, \quad \operatorname{Im} z\geqslant -Ch, \quad \operatorname{Re} z=E+o(1) \quad \textit{and} \quad \|u_{\theta,\varepsilon}\|_{L^2(X_\theta)}=1. \tag{8.4}$$

Then, there exist $R_1>4R_0$ and $C_0>1$, independent of M_1, such that

$$\|u_{\theta,\varepsilon}\|_{L^2(X_\theta\setminus B_{\mathbb{C}^n}(0,R_1))}=\mathcal{O}(h^{M_1/C_0}), \quad 0<h<h_0(M_1). \tag{8.5}$$

Proof. We will use the properties of the "deep complex scaling" region, explained in Lemma 6.4. The first step is localization in energy. Take $\psi\in\mathcal{C}_c^\infty((-2,2),[0,1])$ such that $\psi|_{[-1,1]}=1$ and define

$$\psi_0(\varrho)=\psi\biggl(\frac{4(p(\varrho)-E)}{\delta_1}\biggr)\quad\text{and}\quad\psi_1(\varrho)=\psi\biggl(\frac{8(p(\varrho)-E)}{\delta_1}\biggr). \tag{8.6}$$

Fix some time $t_1>0$ and consider spatial cutoff functions $\chi_0,\chi_1\in C^\infty(X,[0,1])$ localized near infinity:

$$\chi_j(x)=\begin{cases}0, & \text{if } x\in B(0,\widetilde{R}_j),\\ 1, & \text{if } x\in X\setminus B(0,\widetilde{R}_j+1),\end{cases}\qquad j=0,1,$$

where the radii $\widetilde{R}_1>\widetilde{R}_0+2\gg1$ are sufficiently large so that the following conditions are satisfied:

$$\operatorname{supp}((\chi_0\psi_0)\circ\Phi^{-t})\subset\mathcal{E}_E^{4\delta_1/5}\cap\bigl\{\varrho:|x(\varrho)|>\tfrac{5}{2}R_0\bigr\}\quad\text{for all } t\in[0,t_1], \tag{8.7}$$

$$(\chi_0\psi_0)(\varrho)=1\quad\text{near } \operatorname{supp}((\chi_1\psi_1)\circ\Phi^{t_1}). \tag{8.8}$$

We will now estimate the norm of the state

$$v\stackrel{\text{def}}{=}\chi_1\psi_1^w(x,hD)U(t_1)\chi_0\psi_0^w(x,hD)u_{\theta,\varepsilon}. \tag{8.9}$$

Using the condition (8.8), we apply Proposition 6.2 (ii) to the operator $\chi_1\psi_1^wU(t_1)\chi_0\psi_0^w$, and obtain

$$\begin{aligned}v&=\chi_1\psi_1^w(x,hD)U(t_1)u_{\theta,\varepsilon}+\mathcal{O}_{L^2}(h^\infty)\\&=e^{-it_1z/h}\chi_1\psi_1^w(x,hD)u_{\theta,\varepsilon}+\mathcal{O}_{L^2}(h^\infty)\\&=e^{-it_1z/h}\chi_1u_{\theta,\varepsilon}+\mathcal{O}_{L^2}(h^\infty).\end{aligned}$$

In the second equality we have applied Proposition 6.1. For the third one we used the microlocalization of $u_{\theta,\varepsilon}$ on $\mathcal{E}_E$:

$$\psi_1^w(x,hD)u_{\theta,\varepsilon}=u_{\theta,\varepsilon}+\mathcal{O}_{H_h^k}(h^\infty)\quad\text{for all } k. \tag{8.10}$$

On the other hand, the condition (8.7) allows us to apply Lemma 6.4:

$$\begin{aligned}\|U(t_1)\chi_0\psi_0^w(x,hD)u_{\theta,\varepsilon}\|&\leqslant e^{-\theta/hC_0}\|\chi_0\psi_0^w(x,hD)u_{\theta,\varepsilon}\|+\mathcal{O}(h^\infty)\\&\leqslant h^{M_1/C_0}\|\chi_0u_{\theta,\varepsilon}\|+\mathcal{O}(h^\infty).\end{aligned}$$

Here we have taken $\theta=M_1h\log(1/h)$ and used again the microlocalization of $u_{\theta,\varepsilon}$ near $\mathcal{E}_E$.

Using that $\operatorname{Im} z\geqslant -Ch$ and combining the above estimates, we find that

$$\|\chi_1u_{\theta,\varepsilon}\|\leqslant e^{Ct_1}h^{M_1/C_0}+\mathcal{O}(h^\infty).$$

This proves the proposition, once we take $R_1\geqslant\max\{\widetilde{R}_1+1,4R_0\}$. □

Remark. The statement of the lemma can be refined using exponential weights to give a stronger statement about $u_{\theta,\varepsilon}$ (including the case of $\varepsilon=0$):

$$\|e^{\theta|x|/C_2}u_{\theta,\varepsilon}\|_{L^2(X_\theta\setminus B_{\mathbb{C}^n}(0,R_1))}=\mathcal{O}(1),$$

see [37] for a similar argument.

LEMMA 8.2. *Let $K=K_E$ be the trapped set* (1.6) *for $p(x,\xi)$ at energy E. Suppose that $u_{\theta,\varepsilon}$ is as in Lemma* 8.1 *and G and ε have the properties in* (8.1) *and* (8.2). *Then for any $\delta>0$ there exists $C(\delta)>0$ such that*

$$\|u\|_{\theta,\varepsilon}\leqslant C(\delta)\|u\|_{L^2(\pi(K)+B_X(0,\delta))},\quad 0<h\leqslant h_0(\delta). \tag{8.11}$$

As a consequence, for any resonant state $u=u(h)$ with $\operatorname{Re}z-E=o(1)$ *and* $\operatorname{Im}z\geqslant -Ch$, *we have*

$$\begin{gathered}\text{for all } R>0 \text{ there exist } C(\delta,R) \text{ and } h_0(\delta,R) \text{ such that}\\ \|u\|_{L^2(B(0,R))}\leqslant C(\delta,R)\|u\|_{L^2(\pi(K)+B_X(0,\delta))}\quad \text{for } h\leqslant h_0(\delta,R).\end{gathered} \tag{8.12}$$

This means that a normalization in any small neighbourhood of $\pi(K)$ leads to an h-independent normalization in any compact set. This property allows us to define a global measure μ in Theorem 4.

Proof. Lemma 8.1 shows that, to establish (8.11), it is enough to prove that

$$\|u_{\theta,\varepsilon}\|_{B_{X_\theta}(0,R_1)}\leqslant C\|u\|_{L^2(\pi(K)+B_X(0,\delta))},\quad 0<h\leqslant h_0(\delta). \tag{8.13}$$

For $\delta_0>0$ small and R_1 as in Lemma 8.1, we consider the compact set

$$S\stackrel{\text{def}}{=}\overline{\mathcal{E}_E^{\delta_0}\cap T^*_{B(0,R_1)}X}.$$

If $\varrho\in S\cap\Gamma_0^{+\delta_0}$, there exist $T_\varrho\geqslant 0$ and a neighbourhood $U_\varrho\subset\mathcal{E}_E^{2\delta_0}$ of ϱ such that

$$\Phi^{-T_\varrho}(U_\varrho)\subset T^*(\pi(K)+B(0,\delta)), \tag{8.14}$$

provided δ_0 is small enough depending on δ (so that $K_E^{2\delta_0}\Subset T^*(\pi(K_E)+B(0,\delta))$).

On the other hand, if $\varrho\notin S\cap\Gamma_0^{+\delta_0}$, there exist $T_\varrho>0$ and a neighbourhood $U_\varrho\subset\mathcal{E}_E^{2\delta_0}$ of ϱ such that

$$\Phi^{-T_\varrho}(U_\varrho)\subset T^*(X\setminus B(0,2R_1)). \tag{8.15}$$

Since the set S is compact, we can cover it with the union of two families of sets

$$\{U_{\varrho_j}:j\in J_1\}\quad\text{and}\quad\{U_{\varrho_j}:j\in J_2\}$$

of the preceding two types, where J_1 and J_2 are (disjoint) finite index sets. We can also choose open sets $U'_{\varrho_j} \Subset U_{\varrho_j}$ such that $\bigcup_{j\in J_1\cup J_2} U'_{\varrho_j}$ still covers S. We note that these covers have different properties than the cover $\{W_a\}_{a\in A}$ constructed in §5.2.

We now construct a "quantum cover" adapted to the above classical cover:

$$A_j \in \Psi_h(X_\theta),\ \mathrm{WF}_h(A_j) \Subset U_{\varrho_j} \text{ and } A_j = I \text{ microlocally near } U'_{\varrho_j}, \quad j \in J_1 \cup J_2.$$

In view of the localization of $u_{\theta,\varepsilon}$ to the energy shell (see (8.10)), we have

$$\|u_{\theta,\varepsilon}\|_{L^2(B(0,R_1))} \leqslant C \sum_{j\in J_1\cup J_2} \|A_j u_{\theta,\varepsilon}\|.$$

Hence (8.13) will follow from the bounds

$$\|A_j u_{\theta,\varepsilon}\| \leqslant C\|u_{\theta,\varepsilon}\|_{L^2(\pi(K)+B(0,\delta))} + \mathcal{O}(h^\infty), \qquad j \in J_1, \tag{8.16}$$

$$\|A_j u_{\theta,\varepsilon}\| \leqslant C h^{M_1/C_0}, \qquad j \in J_2. \tag{8.17}$$

With $U(t)$ defined by (6.9), Proposition 6.1 and the condition $|\mathrm{Im}\, z| \leqslant Ch$ imply that for any bounded $t>0$,

$$\|A_j u_{\theta,\varepsilon}\| \leqslant e^{Ct}\|A_j U(t) u_{\theta,\varepsilon}\| + \mathcal{O}(h^\infty), \quad j \in J_1 \cup J_2.$$

Considering operators $tA_j \in \Psi_h(X_\theta)$ with the properties

$$\mathrm{WF}_h(\tilde{A}_j) \subset \Phi^{-T_{\varrho_j}}(U_{\varrho_j}) \quad \text{and} \quad \tilde{A}_j = I \text{ microlocally near } \Phi^{-T_{\varrho_j}}(\mathrm{WF}_h(A_j)),$$

we may apply Proposition 6.2 (ii):

$$\begin{aligned}\|A_j u_{\theta,\varepsilon}\| &\leqslant e^{CT_{\varrho_j}}\|A_j U(T_{\varrho_j}) u_{\theta,\varepsilon}\| + \mathcal{O}(h^\infty)\\ &\leqslant \|A_j U(T_{\varrho_j}) \tilde{A}_j u_{\theta,\varepsilon}\| + \mathcal{O}(h^\infty)\\ &\leqslant C' e^{C' T_{\varrho_j}} \|\tilde{A}_j u_{\theta,\varepsilon}\| + \mathcal{O}(h^\infty).\end{aligned}$$

In the last line we used the bound (6.10) and that $\|A_j\| \leqslant C'$. Notice that the times T_{ϱ_j} are uniformly bounded, depending on δ and R_1. From (8.14) we obtain the first estimate in (8.16). Lemma 8.1 and (8.15) provide the second estimate. This completes the proof of (8.11).

To see how (8.12) follows from (8.11), we choose $R_0 > 2R$ in the construction of $P_{\theta,\varepsilon}$ (see (8.1) and (8.2) above). From the support properties of the weight G, we have the following relationship between $u_{\theta,\varepsilon}$ and the corresponding resonant state u:

$$\|u - u_{\theta,\varepsilon}\|_{H_h^k(B(0,R_0/2))} = \mathcal{O}(h^\infty)\|u_{\theta,\varepsilon}\|_{L^2(X_\theta)} \quad \text{for all } k.$$

Then

$$\|u\|_{L^2(B(0,R))} \leqslant \|u_{\theta,\varepsilon}\|_{L^2(X_\theta)}(1 + \mathcal{O}(h^\infty)) \leqslant C(\delta, R)\|u\|_{L^2(\pi(K)+B(0,\delta))}. \qquad \square$$

The next proposition is a refined version of (1.15) appearing in Theorem 4.

PROPOSITION 8.1. *Suppose that u satisfies the assumptions of Theorem 4 and that $a\in\mathcal{C}_c^\infty$ is supported in $T^*X\setminus\Gamma_E^+$. Then, for any $\chi\in\mathcal{C}_c^\infty(X)$, we have*

$$\|a^w(x,hD)\chi u\|\leqslant C_M h^M \quad \text{for any } M>0, \tag{8.18}$$

that is, $u\equiv 0$ microlocally in $T^*X\setminus\Gamma_E^+$. The constant C_M in (8.18) depends on E, a and χ.

Proof. We choose R_0 such that $\operatorname{supp}\chi\subset B\big(0,\tfrac12 R_0\big)$ in the construction of $P_{\theta,\varepsilon}$ described in the beginning of this section. Then, by Lemma 8.2, the normalization in Theorem 4 is, up to uniform constants, equivalent to the normalization $\|u_{\theta,\varepsilon}\|_{L^2(X_\theta)}=1$. From (8.3) we see that

$$\|a^w(x,hD)\chi u\|=\|a^w(x,hD)\chi u_{\theta,\varepsilon}\|+\mathcal{O}(h^\infty).$$

The condition on the support of a shows that for $\delta_0>0$ small enough, $\operatorname{supp} a\cap\Gamma_E^{+\delta_0}=\varnothing$. Using an energy cutoff function ψ_0 of the form (8.6) supported inside $\mathcal{E}_E^{\delta_0}$, there exists a time $T>0$ such that

$$\Phi^{-T}\operatorname{supp}(a\psi_0)\Subset T^*(X\setminus B(0,2R_1))\cap\mathcal{E}_E^{\delta_0},$$

where R_1 is given by Lemma 8.1. Taking into account the microlocalization of type (8.10), we get

$$\|a^w(x,hD)\chi u\|=\|a^w(x,hD)\psi_0^w(x,h_D)\chi u_{\theta,\varepsilon}\|+\mathcal{O}(h^\infty).$$

We can now proceed as in the previous lemma:

$$\|a^w\psi_0^w\chi u_{\theta,\varepsilon}\|\leqslant C\|a^w\psi_0^w\chi U(T)u_{\theta,\varepsilon}\|\leqslant C\|a^w\psi_0^w\chi U(T)\psi_1^w\chi_1\chi u_{\theta,\varepsilon}\|,$$

where $\psi_1\in\mathcal{C}^\infty(T^*X,[0,1])$ satisfies $\psi_1|_{\mathcal{E}_E^{2\delta_0}}=1$, while $\chi_1\in\mathcal{C}^\infty(X)$ vanishes on $B(0,R_1)$ and takes the value 1 for $|x|\geqslant 2R_1$. The second line above is then due to Proposition 6.2 (ii). Lemma 8.1 shows that $\|\chi_1 u_{\theta,\varepsilon}\|=\mathcal{O}(h^{M_1/C_0})$, so we finally get

$$\|a^w(x,hD)\chi u\|=\mathcal{O}(h^{M_1/C_0}),$$

where M_1 can be taken arbitrarily large. □

Proof of Theorem 4. Inclusion (1.15) follows directly from Proposition 8.1, which shows that only points in Γ_E^+ can be in the support of the limit measure.

The proof of (1.16) follows the standard approach (see [16] and for a textbook presentation [13, Chapter 5]). Suppose that χ and u are as in (1.14). From Lemma 8.2 we know that $\|\chi u(h)\| \leqslant C_\chi$ with the constant C_χ independent of h. Hence, there exists a sequence $\{h_k\}_{k\in\mathbb{N}} \searrow 0$ for which (1.14) holds for any $A = a^w(x, hD)$ with $a \in \mathcal{C}_c^\infty(T^*X)$ and $\operatorname{supp} a \Subset (\pi^*\chi)^{-1}(1)$. From this support property we get

$$A[P,\chi] = \mathcal{O}_{L^2\to L^2}(h^\infty),$$

so that

$$\begin{aligned}\mathcal{O}(h^\infty) &= \operatorname{Im}\langle (P-z)\chi u, A\chi u\rangle = \operatorname{Im}\langle P\chi u, A\chi u\rangle - \operatorname{Im} z\langle A\chi u, \chi u\rangle \\ &= \tfrac{1}{2}h\langle (H_p a)^w(x,hD)\chi u, \chi u\rangle - \operatorname{Im} z\langle A\chi u, \chi u\rangle + \mathcal{O}(h^2)\|\chi u\|^2.\end{aligned}$$

For the sequence $\{h_k\}_{k\in\mathbb{N}}$ appearing in (1.14) we obtain

$$\frac{1}{2}\int_{T^*X} H_p a\, d\mu - \frac{\operatorname{Im} z(h_k)}{h_k}\int_{T^*X} a\, d\mu = o(1), \quad \text{as } k\to\infty.$$

Hence there exists $\lambda \geqslant 0$ such that $\operatorname{Im} z(h_k)/h_k \to -\frac{1}{2}\lambda$ and

$$\int_{T^*X} H_p a\, d\mu + \lambda\int_{T^*X} a\, d\mu = 0,$$

which is the same as (1.16). □

9. Resolvent estimates

In this section we will prove Theorem 5 and consequently we assume that the hypothesis of that theorem holds throughout this section. In particular $E>0$ is an energy level at which the pressure, $\mathcal{P}_E\left(\frac{1}{2}\right)$, is negative. We first need a result which is a simpler version of the estimates on the propagator $U(t)$ described in §6.4.

PROPOSITION 9.1. *Suppose that* $W \in \mathcal{C}^\infty(X;[0,1])$, $W \geqslant 0$, *satisfies the conditions*

$$\operatorname{supp} W \subset X\setminus B(0,R_1) \quad \textit{and} \quad W|_{X\setminus B(0,R_1+r_1)} = 1$$

for R_1 *and* r_1 *sufficiently large. Assume that* $\mathcal{P}_E\left(\frac{1}{2}\right) < 0$ *and choose* $\lambda \in \left(0, \left|\mathcal{P}_E\left(\frac{1}{2}\right)\right|\right)$. *Then there exists* $\delta_0 > 0$ *such that, for any* $\psi \in S(1)$ *supported inside* $\mathcal{E}_E^{\delta_0}$ *and any* $M > 0$,

$$\|e^{-it(P(h)-iW)/h}\psi^w(x,hD)\|_{L^2\to L^2} \leqslant Ch^{-n/2}e^{-\lambda t} + \mathcal{O}_M(h^\infty), \quad 0 \leqslant t \leqslant M\log\frac{1}{h}. \quad (9.1)$$

The proof of this proposition is very similar to the proof in the case of the complex-scaled operator $P_{\theta,\varepsilon}$ treated in §6. In fact the case of the absorbing potential is easier to deal with than complex scaling, and in particular we do not need the weights G. The modifications needed to apply §6 directly are given in the appendix.

Before proving (1.17), we will establish a resolvent estimate for the operator with the absorbing potential.

PROPOSITION 9.2. *Let $P=P(h)$, the energy $E>0$ and the absorbing potential W be as in Proposition 9.1. Then, for any $\varepsilon>0$,*

$$\|(P(h)-iW-E)^{-1}\|_{L^2(X)\to L^2(X)} \leqslant \frac{n(1+\varepsilon)}{2\left|\mathcal{P}_E\left(\frac{1}{2}\right)\right|}\frac{\log(1/h)}{h}, \quad 0<h<h_0(\varepsilon). \tag{9.2}$$

Proof. We will use Proposition 9.1 and h-dependent complex interpolation similar to that in [44].

If we put

$$U_1(t) \stackrel{\text{def}}{=} e^{-it(P(h)-iW)/h}\psi^w(x,hD),$$

where ψ is as in (9.1), then the following estimates hold for any $M>0$ and $0<h<h_M$:

$$\|U_1(t)\| \leqslant \begin{cases} 1+\mathcal{O}(h), & 0\leqslant t\leqslant T_E, \qquad T_E(h)\stackrel{\text{def}}{=} n\log(1/h)/2\lambda, \\ C_0 h^{-n/2}e^{-\lambda t}, & T_E\leqslant t\leqslant T_M, \qquad T_M(h)\stackrel{\text{def}}{=} M\log(1/h), \\ h^{M/C_0}, & t\geqslant T_M, \end{cases} \tag{9.3}$$

where C_0 is independent of M. The notation $T_E(h)$ comes from the analogy with the *Ehrenfest time* (the time the system needs to delocalize a Gaussian wavepacket).

The first estimate in (9.3) follows from the subunitarity of $e^{-it(P(h)-iW)/h}$ and the bound $\|\psi^w\|_{L^2\to L^2}\leqslant 1+\mathcal{O}(h)$. The second estimate follows from Proposition 9.1 by absorbing the remainder $\mathcal{O}_M(h^\infty)$ in the leading term by taking $h<h_M$ small enough. The last estimate follows by writing

$$U_1(t) = e^{-i(t-T_M)(P(h)-iW)/h}U_1(T_M),$$

and using subunitarity for the first factor and the previous estimate for $\|U_1(T_M)\|$.

The estimates (9.3) and ellipticity away from the energy surface give the following lemma.

LEMMA 9.1. *In the notation of Proposition* 9.1 *and* (9.3) *we have*, *for any* $N>0$,

$$\|(P(h)-iW-z)^{-1}\|_{L^2\to L^2}\leqslant\frac{C_1+T_E(h)}{h}+\frac{h^N}{\operatorname{Im} z},\quad \operatorname{Im} z>0,\ |z-E|<\delta,\ 0<h<h_N. \tag{9.4}$$

Proof. We first prove the same estimate for the energy-localized operator

$$(P(h)-iW-z)^{-1}\psi^w(x,hD)=\frac{1}{h}\int_0^\infty U_1(t)e^{itz/h}\,dt,\quad \operatorname{Im} z>0.$$

From (9.3) we obtain

$$\begin{aligned}\|(P(h)-iW-z)^{-1}\psi^w(x,hD)\| &\leqslant \frac{1}{h}\biggl(\int_0^{T_E}+\int_{T_E}^{T_M}+\int_{T_M}^\infty\biggr)\|U_1(t)\|e^{-\operatorname{Im} zt/h}dt\\ &\leqslant \frac{T_E(h)}{h}+\frac{C}{h\lambda+\operatorname{Im} z}+\frac{h^{M/C_0}}{\operatorname{Im} z}.\end{aligned}$$

This is the estimate on the right-hand side of (9.4), once we take M large enough and h small enough.

To solve $(P-iW-z)u=(1-\psi^w)f$, $f\in L^2(X)$, we follow the following standard procedure—see for instance [13, Proof of Theorem 6.4] for a simple example. There exists $\psi_1\in \mathcal{C}_c^\infty(T^*X;[0,1])$ supported near the energy surface $\mathcal{E}_E$, such that the pseudodifferential operator $(P-iW-z-i\psi_1^w)^{-1}$ is uniformly bounded in L^2 for z as in the lemma and $h\in(0,h_0)$, while $\psi_1^w(1-\psi^w)=\mathcal{O}_{L^2\to L^2}(h^\infty)$. It follows that

$$(P-iW-z)(P-iW-z-i\psi_1^w)^{-1}(1-\psi^w)f=(1-\psi^w)f+Rf,$$

where $R=\mathcal{O}_{L^2\to L^2}(h^\infty)$. If we put

$$L\stackrel{\text{def}}{=}(P-iW-z-i\psi_1^w)^{-1}(1-\psi^w)+(P-iW-z)^{-1}\psi^w,$$

then

$$(P-iW-z)L=I+R,\quad \|L\|\leqslant\frac{C/\lambda+T_E(h)}{h}+\frac{h^N}{\operatorname{Im} z},\quad \|R\|=\mathcal{O}(h^\infty)$$

and $(P-iW-z)^{-1}=L(I+R)^{-1}$ satisfies the estimate (9.4). □

To estimate the norm of the resolvent on the energy axis, $\|(P-iW-E)^{-1}\|$, we need the following parametric version of the maximum principle.

LEMMA 9.2. *Suppose that* $\zeta\mapsto F(\zeta)$ *is holomorphic in a neighbourhood of* $[-1,1]+i[-c_-,c_+]$, *for some fixed* $c_\pm>0$, *and that*

$$\begin{aligned}\log|F(\zeta)|&\leqslant M, &&\zeta\in[-1,1]+i[-c_-,c_+],\\ |F(\zeta)|&\leqslant\alpha+\frac{\gamma}{\operatorname{Im}\zeta}, &&\zeta\in[-1,1]+i(0,c_+],\end{aligned}$$

where $M,\alpha\gg1$, *while* $\gamma\ll1$. *Then*, *for* ε *satisfying* $\gamma M^{3/2}/\alpha\ll\varepsilon^{5/2}\ll1$, *we have*

$$|F(0)|\leqslant(1+\varepsilon)\alpha.$$

Proof. Let $g(\zeta)=e^{-3M\zeta^2+ia\zeta}$, with $a\in\mathbb{R}$ to be chosen later. Then $g(0)=1$ and

$$|g(\zeta)|\leqslant \exp(-3M(\operatorname{Re}\zeta)^2+3M(\operatorname{Im}\zeta)^2+|a||\operatorname{Im}\zeta|).$$

Let $1\gg\delta_-\gg\delta_+>0$. The following bounds hold on the boundary of $[-1,1]+i[-\delta_-,\delta_+]$:

$$\log|F(\zeta)g(\zeta)|\leqslant\begin{cases} -2M+3M\delta_-^2+|a|\delta_-, & \operatorname{Re}\zeta=\pm1,\ -\delta_-\leqslant\operatorname{Im}\zeta\leqslant\delta_+,\\ M+3M\delta_-^2+a\delta_-, & |\operatorname{Re}\zeta|\leqslant1,\ \operatorname{Im}\zeta=-\delta_-,\\ \log(\alpha+\gamma/\delta_+)+3M\delta_+^2-a\delta_+, & |\operatorname{Re}\zeta|\leqslant1,\ \operatorname{Im}\zeta=\delta_+.\end{cases}$$

Following the standard "three-line" argument, we select

$$a=\frac{1}{\delta_++\delta_-}\Big(-M+\log\Big(\alpha+\frac{\gamma}{\delta_+}\Big)\Big)\simeq\frac{1}{\delta_-}\Big(-M+\log\Big(\alpha+\frac{\gamma}{\delta_+}\Big)\Big),$$

so that the bounds for $\operatorname{Im}\zeta=\pm\delta_\pm$ and $|\operatorname{Re}z|\leqslant1$ are the same:

$$\begin{aligned}\log|F(\zeta)g(\zeta)|&\leqslant M\frac{\delta_+}{\delta_++\delta_-}+\log\Big(\alpha+\frac{\gamma}{\delta_+}\Big)\frac{\delta_-}{\delta_++\delta_-}+3M\delta_-^2\\ &\lesssim M\delta_+\delta_-^{-1}+\log\Big(\alpha+\frac{\gamma}{\delta_+}\Big)+3M\delta_-^2.\end{aligned}$$

To ensure that the above right-hand side is smaller than $\log(\alpha(1+\varepsilon))$, we need the following conditions to be satisfied:

$$\delta_+\delta_-^{-1}M\ll\varepsilon,\quad M\delta_-^2\ll\varepsilon\quad\text{and}\quad\frac{\gamma}{\alpha\delta_+}\ll\varepsilon.$$

These conditions can be arranged if $\varepsilon^{5/2}$ is large enough compared with $\gamma M^{3/2}/\alpha$, which is the condition in the statement of the lemma. One easily checks that the bound $\log|F(\zeta)g(\zeta)|\leqslant\log(\alpha(1+\varepsilon))$ then also holds for $|\operatorname{Re}\zeta|=1$ and $\operatorname{Im}\zeta=\pm\delta_\pm$, and therefore for $\zeta=0$ by the maximum principle. □

We now complete the proof of Proposition 9.2. To apply Lemma 9.2 we need the estimate of Lemma 9.1, but also an estimate of $\|(P-iW-z)^{-1}\|$ for $|\operatorname{Re}z-E|\leqslant\delta$ and $|\operatorname{Im}z|\leqslant\lambda h$ (where we recall that $(P-iW-z)^{-1}$ has no poles in that strip for h small enough). We can cite [10, Lemma 6.1](4) and obtain

$$\|(P-iW-z)^{-1}\|\leqslant C_\varepsilon e^{C_\varepsilon h^{-n-\varepsilon}},\quad \operatorname{Im}z>-\lambda h.$$

Lemma 9.2 applied to the data

$$F(\zeta)=\langle(P-iW-E-h\zeta)^{-1}f,g\rangle,\quad f,g\in L^2(X),\quad \|f\|=\|g\|=1,$$
$$M=Ch^{-n-1},\quad \alpha=\frac{C_1+T_E(h)}{h},\quad \gamma=h^N,$$

proves the corollary (observe that the condition $\gamma M^{3/2}/\alpha\ll1$ is satisfied for h small enough). □

(4) Strictly speaking the quoted lemma is stated for P with bounded symbols. However, since the symbol of $P-iW-z$ is bounded away from zero outside a compact set, exactly the same argument applies.

To pass from the estimate (9.2) to an estimate on $\chi(P-z)^{-1}\chi$, $\chi\in\mathcal{C}_c^\infty(X)$, we first recall (see for instance [44]) that if $\operatorname{supp}\chi\subset B(0,R_0)$, where R_0 is as in §3.4, then

$$\chi(P-z)^{-1}\chi=\chi(P_\theta-z)^{-1}\chi.$$

Also, if $\operatorname{supp}\pi^*\chi\cap\operatorname{supp}G=\varnothing$, then

$$\begin{aligned}\chi(P_\theta-z)^{-1}\chi&=\chi e^{\varepsilon G^w/h}(P_{\theta,\varepsilon}-z)^{-1}e^{-\varepsilon G^w/h}\chi\\&=\chi(P_{\theta,\varepsilon}-z)^{-1}\chi+\mathcal{O}_{L^2\to L^2}(h^\infty)\|(P_{\theta,\varepsilon}-z)^{-1}\|.\end{aligned}$$

Hence,

$$\|\chi(P_\theta-z)^{-1}\chi\|=(1+\mathcal{O}(h^\infty))\|(P_{\theta,\varepsilon}-z)^{-1}\|. \tag{9.5}$$

For future use, we now consider an auxiliary simpler scattering situation, namely an operator $P^\sharp=P^\sharp(h)$ satisfying the assumptions of §3.2 and for which the associated classical flow is *non-trapping* at energy E, that is, $K_E=\varnothing$. From a result of Martinez [25], we have

$$\chi(P^\sharp-z)^{-1}\chi=\mathcal{O}\left(\frac{1}{h}\right),\quad z\in D(E,Ch),$$

see [27, Proposition 3.1].([5]) Below we will need the following estimate for the resolvent of $P^\sharp_{\theta,\varepsilon}$.

LEMMA 9.3. *Suppose that $P^\sharp=P^\sharp(h)$ is an operator satisfying the assumptions of* §3.2 *and that the flow of $p^\sharp$ is* non-trapping *at energy E, that is, $K_E=\varnothing$. Then, in the notation of* §6.1,

$$(P^\sharp_{\theta,\varepsilon}-z)^{-1}=\mathcal{O}_{L^2\to L^2}\left(\frac{1}{h}\right),\quad z\in D(E,Ch). \tag{9.6}$$

Proof. Since $P^\sharp_{\theta,\varepsilon}-z$ is a Fredholm operator on $L^2(X)$ (as elsewhere we identify X_θ with X), the estimate will follow if we find $Q(z)$ such that, for $z\in D(E,Ch)$,

$$(P^\sharp_{\theta,\varepsilon}-z)Q(z)=I+A(z),\quad Q(z)=\mathcal{O}_{L^2\to L^2}\left(\frac{1}{h}\right)\quad\text{and}\quad A(z)=\mathcal{O}_{L^2\to L^2}(h). \tag{9.7}$$

We will solve this problem in two steps, away and near the energy layer $\mathcal{E}_E$. Consider the two nested energy cutoff functions

$$\psi_0(x,\xi)=\psi\left(\frac{p(x,\xi)-E}{\delta}\right)\quad\text{and}\quad\psi_1(x,\xi)=\psi\left(\frac{8(p(x,\xi)-E)}{\delta}\right), \tag{9.8}$$

([5]) The statement of that proposition should be corrected to include a cutoff function χ, or, without a cutoff function, a factor $\log(1/h)$ on the right-hand side of [27, (3.2)]. Lemma 9.3 gives a correct global version without the logarithmic loss.

where $\psi \in \mathcal{C}_c^\infty((-2,2),[0,1])$ and $\psi|_{[-1,1]} \equiv 1$. Since $P_{\theta,\varepsilon}^\sharp$ is elliptic on $\operatorname{supp}(1-\psi_1)$ (that is, away from $\mathcal{E}_E$), standard symbolic calculus (as in the proof of Lemma 9.1) provides an operator $Q_0(z)$ such that

$$(P_{\theta,\varepsilon}^\sharp - z)Q_0(z) = I - \psi_1^w(x,hD) + A_0(z), \quad Q_0(z) = \mathcal{O}_{L^2\to L^2}(1), \quad A_0(z) = \mathcal{O}_{L^2\to L^2}(h).$$

We now treat the problem near the energy layer. We want to produce an operator $Q_1(z)$ such that

$$(P_{\theta,\varepsilon}^\sharp - z)Q_1(z) = \psi_1^w(x,hD) + A_1(z), \quad Q_1(z) = \mathcal{O}_{L^2\to L^2}\left(\frac{1}{h}\right), \quad A_1(z) = \mathcal{O}_{L^2\to L^2}(h).$$

To this aim we use the tools developed in §6.2 and consider the energy-localized propagator

$$U^\sharp(t) \stackrel{\text{def}}{=} e^{-it\widetilde{P}_{\theta,\varepsilon}^\sharp/h}, \quad \widetilde{P}_{\theta,\varepsilon}^\sharp \stackrel{\text{def}}{=} \psi_0^w(x,hD)P_{\theta,\varepsilon}^\sharp \psi_0^w(x,hD),$$

which satisfies $\|U^\sharp(t)\| \leqslant e^{Ct}$ for any $t>0$. The non-trapping assumption at energy E implies that

$$\text{there exists } T>0 \text{ such that, for all } \varrho \in \mathcal{E}_E^{\delta/4} \cap T^*_{B(0,3R_0)}X,\ |\pi(\Phi^t(\varrho)| > 3R_0 \text{ for } t>T. \tag{9.9}$$

We claim that we can take

$$Q_1(z) \stackrel{\text{def}}{=} \frac{i}{h}\int_0^T U^\sharp(t)\psi_1^w(x,hD)e^{itz/h}\,dt. \tag{9.10}$$

Indeed,

$$\begin{aligned}(P_{\theta,\varepsilon}^\sharp - z)Q_1(z) &= \psi_1^w(x,hD) + A_1(z),\\ A_1(z) &\stackrel{\text{def}}{=} -U^\sharp(T)\psi_1^w(x,hD) + \frac{i}{h}\int_0^T (P_{\theta,\varepsilon}^\sharp - \widetilde{P}_{\theta,\varepsilon}^\sharp)U^\sharp(t)\psi_1^w(x,hD)e^{itz/h}\,dt.\end{aligned}$$

The escape property (9.9) shows that there exists a time $0<T_{\min}<T$ such that points in $\mathcal{E}_E^{\delta/4} \cap T^*_{B(0,3R_0)}X$ will have escaped outside $B\big(0,\frac{5}{2}R_0\big)$ after $T-T_{\min}$, while points in $\mathcal{E}_E^{\delta/4} \cap T^*(X \setminus B(0,3R_0))$ cannot penetrate inside $B\big(0,\frac{5}{2}R_0\big)$ before the time $T_{\min}$. In both cases, Lemma 6.4 provides the following estimate:

$$\|U^\sharp(T)\psi_1^w(x,hD)\| = \mathcal{O}(h^{M_1/C_0})$$

for some $C_0 = C_0(T-T_{\min})$. On the other hand, M_1 can be chosen arbitrarily large, in particular we assume that $M_1/C_0 > 1$.

To analyze the second term in the definition of A_1, we use the energy cutoff function

$$\psi_{1/2}(\varrho) \stackrel{\text{def}}{=} \psi\bigg(\frac{4(p(\varrho)-E)}{\delta}\bigg),$$

which is nested between ψ_1 and ψ_0, and write

$$P^\sharp_{\theta,\varepsilon} - \widetilde{P}^\sharp_{\theta,\varepsilon} = P^\sharp_{\theta,\varepsilon}(1-\psi_0^w) + (1-\psi_0^w)P^\sharp_{\theta,\varepsilon}\psi_0^w(1-\psi_{1/2}^w) + \mathcal{O}_{L^2\to L^2}(h^\infty).$$

From the support properties of the ψ_j and using (6.18), we get

$$(1-\psi_0^w)U^\sharp(t)\psi_1^w,\, (1-\psi_{1/2}^w)U^\sharp(t)\psi_1^w = \mathcal{O}_{L^2\to L^2}(h^\infty).$$

These estimates show that $A_1(z)=\mathcal{O}_{L^2\to L^2}(h)$.

As a result, the operators $Q(z)=Q_0(z)+Q_1(z)$ and $A(z)=A_0(z)+A_1(z)$ satisfy (9.7), completing the proof. □

Proof of Theorem 5. We now return to our original operator $P(h)$ with the properties described in §3.3. As is seen from (9.5), it is sufficient to prove the bound

$$(P_{\theta,\varepsilon}-E)^{-1} = \mathcal{O}_{L^2\to L^2}\bigg(\frac{\log(1/h)}{h}\bigg).$$

As in Lemma 9.3 above, we will construct an approximate inverse

$$(P_{\theta,\varepsilon}-E)Q = I + A, \quad Q = \mathcal{O}\bigg(\frac{\log(1/h)}{h}\bigg) \quad \text{and} \quad A = \mathcal{O}(h).$$

We consider the cutoff functions (9.8). Once again, the operator can be easily inverted away from the energy shell. We then need to solve

$$(P_{\theta,\varepsilon}-E)Q_1 = \psi_1^w(x,hD) + A_1, \quad Q_1 = \mathcal{O}\bigg(\frac{\log(1/h)}{h}\bigg) \quad \text{and} \quad A_1 = \mathcal{O}(h). \tag{9.11}$$

We will now use our knowledge of the absorbing-potential resolvent, see Propositions 9.1 and 9.2: we will use the fact that the operators $P_{\theta,\varepsilon}$ and $P-iW$ are very similar near the trapped set.

Assume that $1 \ll R_4 < R_3 < R_2 < R_1 < \tfrac{1}{2}R_0$, where the radius R_1 is used to define the absorbing potential W, while R_0 is used in the complex deformation of X (see §3.4), and the weight G is supposed to vanish on $\pi^{-1}B\big(0, \tfrac{1}{2}R_0\big)$. Consider the spatial cutoff functions $\chi_j \in \mathcal{C}_c^\infty(X,[0,1])$, $j=1,2$, satisfying

$$\operatorname{supp}\chi_j \Subset B(0,R_j) \;\text{ and }\; \chi_j|_{B(0,R_{j+1})} \equiv 1, \quad j=1,2,3.$$

To solve (9.11), we first put

$$Q_2 \stackrel{\text{def}}{=} \chi_1(P-iW-E)^{-1}\chi_2\psi_1^w.$$

We can then compute

$$(P_{\theta,\varepsilon}-E)Q_2 = \chi_2\psi_1^w + [P,\chi_1](P-iW-E)^{-1}\chi_2\psi_1^w + \mathcal{O}_{L^2\to L^2}(h^\infty), \tag{9.12}$$

where the error term is due to the weight G, which vanishes near the supports of χ_j:

$$\chi_j e^{\varepsilon G/h} = \chi_j + \mathcal{O}_{L^2\to H_h^k}(h^\infty) \quad \text{for all } k.$$

On the other hand, Proposition 9.2 implies that

$$Q_2 = \mathcal{O}_{L^2\to L^2}\left(\frac{\log(1/h)}{h}\right) \quad \text{and} \quad [P,\chi_1](P-iW-E)^{-1}\chi_2\psi_1^w = \mathcal{O}_{L^2\to L^2}\left(\log\frac{1}{h}\right).$$

To treat the operator on the right, we observe that the differential operator $[P,\chi_1]$ vanishes outside $B(0,R_2)$, while χ_1 vanishes outside $B(0,R_1)$. We are thus in position to apply Lemma A.2. For any $v\in L^2$, $\|v\|=1$, set $f\stackrel{\text{def}}{=}\chi_2\psi_1^w v$. The support of f is contained inside $B(0,R_2)$, and its wavefront set lies inside $\mathcal{E}_E^\delta$. As a consequence, the state $u\stackrel{\text{def}}{=}(P-iW-E)^{-1}f$ also satisfies $\mathrm{WF}_h(u)\subset\mathcal{E}_E^\delta$, and the wavefront set of the state $[P,\chi_1]u$ is contained inside $\mathrm{WF}_h(u)\cap T^*(X\setminus B(0,R_2))$. According to Lemma 9.3,

$$\Phi^t(\mathrm{WF}_h([P,\chi_1]u))\cap T^*_{B(0,3R_0)}X = \varnothing \quad \text{for any } t\geqslant T\big(R_2,R_0,\tfrac{1}{2}E\big). \tag{9.13}$$

Using $T=T\big(R_2,R_0,\frac{1}{2}E\big)$, we put

$$Q_3 \stackrel{\text{def}}{=} -\frac{i}{h}\int_0^T U(t)e^{itE/h}[P,\chi_1](P-iW-E)^{-1}\chi_2\psi_1^w\,dt = \mathcal{O}_{L^2\to L^2}\left(\frac{\log(1/h)}{h}\right).$$

Like in the proof of Lemma 9.3, the outgoing property (9.13) implies that

$$(P_{\theta,\varepsilon}-E)Q_3 = -[P,\chi_1](P-iW-E)^{-1}\chi_2\psi_1^w + \mathcal{O}_{L^2\to L^2}(h^{M_1/C_0}).$$

Hence, assuming that $M_1\gg 1$, we have

$$(P_{\theta,\varepsilon}-E)(Q_2+Q_3) = \chi_2\psi_1^w + \mathcal{O}_{L^2\to L^2}(h).$$

It remains to find an approximate solution with the right-hand side given by

$$(1-\chi_2)\psi_1^w + \mathcal{O}_{L^2\to L^2}(h).$$

Since we chose R_3 large enough to contain $\pi(K_E)$, we can choose some $1 \ll R_4 < R_3$, and construct an operator $P^\sharp$ which is non-trapping in the sense of Lemma 9.3, and satisfies

$$P^\sharp|_{X\setminus B(0,R_4)} = P|_{X\setminus B(0,R_4)}.$$

From the discussion leading to (9.5), it follows that

$$P^\sharp_{\theta,\varepsilon}|_{X\setminus B(0,R_4)} = P_{\theta,\varepsilon}|_{X\setminus B(0,R_4)} + \mathcal{O}_{L^2\to H^k_h}(h^\infty).$$

Using the cutoff function χ_3, we put

$$Q_4 \stackrel{\text{def}}{=} (1-\chi_3)(P^\sharp_{\theta,\varepsilon}-E)^{-1}(1-\chi_2)\psi_1^w$$

and then check that

$$\begin{aligned}(P_{\theta,\varepsilon}-E)Q_4 &= (1-\chi_2)\psi_1^w - A_4 + \mathcal{O}_{L^2\to L^2}(h^\infty), \qquad Q_4 = \mathcal{O}_{L^2\to L^2}(1/h),\\ A_4 &\stackrel{\text{def}}{=} [P,\chi_3](P^\sharp_{\theta,\varepsilon}-E)^{-1}(1-\chi_2)\psi_1^w, \qquad A_4 = \mathcal{O}_{L^2\to L^2}(1).\end{aligned}$$

We have $A_4 = \widetilde{\chi}_2 A_4$, where $\widetilde{\chi}_2$ has the same properties as χ_2 (in particular, $\widetilde{\chi}_2|_{\operatorname{supp}\chi_3} \equiv 1$). For any $v \in L^2$, the state $A_4 v$ will be supported inside $B(0,R_3)$, and its wavefront set will be contained in $\mathcal{E}_E^\delta$. One can thus adapt the construction of Q_2+Q_3 when replacing $\chi_2\psi_1^w$ by A_4, to obtain an approximate inverse Q_5 with the properties

$$(P_{\theta,\varepsilon}-E)Q_5 = A_4 + \mathcal{O}_{L^2\to L^2}(h), \quad Q_5 = \mathcal{O}_{L^2\to L^2}\left(\frac{\log(1/h)}{h}\right).$$

We conclude that $Q_1 \stackrel{\text{def}}{=} Q_2+Q_3+Q_4+Q_5$ satisfies (9.11), which proves the theorem. □

Appendix

In this appendix we explain how the methods of §6 apply to the case in which the deformed operator $P_{\theta,\varepsilon}$ is replaced by the operator with the absorbing-potential operator, $P-iW$, where W is described in Proposition 9.1. The arguments are easier in the case of $P-iW$ and the only complication comes with the following replacement of Lemma 6.2.

LEMMA A.1. *Let W satisfy the conditions given in Proposition* 9.1. *Then*, *for any fixed $t>0$, the operator*

$$V(t) \stackrel{\text{def}}{=} e^{itP/h}e^{-it(P-iW)/h} \tag{A.1}$$

satisfies

$$V(t) = v(t)^w(x,hD) + \mathcal{O}_{L^2\to L^2}(h^\infty), \quad v(t) \in S_{1/2}(T^*X). \tag{A.2}$$

Proof. We start as in the proof of Lemma 6.2: differentiating $V(s)$ with respect to s gives

$$\partial_s V(s) = \frac{1}{h} a(s)^w(x,hD)V(s), \quad V(0)=I,\ a(s)^w(x,hD) \stackrel{\text{def}}{=} -e^{isP/h} W e^{-isP/h},$$

with $a \in S$. Let

$$A(t) \stackrel{\text{def}}{=} \int_0^t a(s)\,ds \quad \text{and} \quad v_0(t) \stackrel{\text{def}}{=} e^{A(t)/h}.$$

We claim that the function $v_0 \in S_{1/2}$. If fact, by Egorov's theorem,

$$A = A_0 + \mathcal{O}(h), \quad A_0(t)(x,\xi) = -\int_0^t W(\pi(\Phi^s(x,\xi)))\,ds \leqslant 0,$$

hence we only need to check the claim for $e^{A_0(t)/h}$. The non-negativity and the C^2-boundedness of $-A_0$ imply the standard estimate $|\partial^\alpha_{(x,\xi)} A_0| \leqslant C|A_0|^{1/2}$, $|\alpha|=1$, from which we see that for any $\beta \in \mathbb{N}^n$,

$$\begin{aligned} \partial^\beta e^{A_0(t)/h} &= \bigg(\sum_{\sum_{l=1}^k \beta_l = \beta} h^{-k} \prod_{l=1}^k \partial^{\beta_l} A_0 \bigg) e^{A_0(t)/h} \\ &= \sum_{\sum_{l=1}^k \beta_l = \beta} \mathcal{O}(h^{-k}) \prod_{l=1}^k (|A_0(t)|^{\delta_{1,|\beta_l|}/2} e^{A_0(t)/kh}) \\ &\leqslant C_\beta \sum_{\sum_{l=1}^k \beta_l = \beta} \prod_{l=1}^k h^{-1+\delta_{1,|\beta_l|}/2} \leqslant C'_\beta \prod_{l=1}^k h^{-|\beta_l|/2} = \mathcal{O}(h^{-|\beta|/2}), \end{aligned}$$

that is, $v_0(t) \in S_{1/2}$. It follows that

$$\partial_s v_0(s)^w(x,hD) = \frac{1}{h}(a(s)v_0(s))^w(x,hD) = \frac{1}{h} a(s)^w(x,hD) v_0(s)^w(x,hD) - r(s)^w(x,hD),$$

where the symbolic calculus shows that $r(s) \in h^{1/2} S_{1/2}$. By Duhamel's formula,

$$E(t) \stackrel{\text{def}}{=} V(t) - v_0(t)^w(x,hD) = \int_0^t V(t-s) r(s)^w(x,hD)\,ds = \mathcal{O}_{L^2\to L^2}(h^{1/2})$$

and

$$\begin{aligned} V(t) &= v_0(t)^w(x,hD) + \int_0^t (v_0(t-s) \# r(s))^w(x,hD)\,ds \\ &\quad + \int_0^t \int_0^s V(t-s-s') E(s-s') r(s')^w(x,hD)\,ds'\,ds \\ &= v_0(t)^w(x,hD) + \int_0^t (v_0(t-s) \# r(s))^w(x,hD)\,ds + \mathcal{O}_{L^2\to L^2}(h). \end{aligned}$$

The iteration of this argument gives the full expansion of a symbol $v(t) \in S_{1/2}$, the quantization of which is equal to $V(t)$ modulo an error $\mathcal{O}_{L^2\to L^2}(h^\infty)$. □

Using Lemma A.1, we obtain the analogues of all the results of §6.2, for $t\geqslant 0$, with $U(t)$ replaced by $e^{-it(P-iW)/h}$, and errors given by $\mathcal{O}(h^\infty)$ instead of $\mathcal{O}(h^{M_1/C_0})$. The proof of the modified Proposition 6.3 is then the same, and Proposition 9.1 follows from the argument presented in §6.4. For instance, here is a version of the propagation results of Proposition 6.2 (see also Proposition 8.1).

PROPOSITION A.1. *Fix $T>0$. Then, for any $v=v(h)\in L^2$, $\|v\|=\mathcal{O}(h^{-M})$ (in particular, v is h-tempered in the sense of* (3.3)*),*

$$\mathrm{WF}_h(e^{-it(P-iW)/h}v)\subset \Phi^t(\mathrm{WF}_h(v)),$$

where WF_h is defined by (3.4)*.*

Proof. In the notation of Lemma A.1, we write

$$e^{-it(P-iW)/h}v=e^{-itP/h}V(t)v$$

and observe that the symbolic calculus on $S_{1/2}$ and (A.2) give $\mathrm{WF}_h(V(t)v)\subset\mathrm{WF}_h(v)$. Indeed, if $a(x,hD)^w v=\mathcal{O}_{L^2}(h^\infty)$ and $a(x,\xi)\equiv 1$ in a neighbourhood of (x_0,ξ_0) (that is, $(x_0,\xi_0)\notin\mathrm{WF}_h(v)$), then for any symbol b with $\operatorname{supp} b\Subset\{(x,\xi):a(x,\xi)=1\}$,

$$b^w(x,hD)V(t)=b^w v(t)^w a^w+\mathcal{O}_{L^2\to L^2}(h^\infty).$$

Hence, $b^w(x,hD)V(t)v=\mathcal{O}_{L^2}(h^\infty)$ and $(x_0,\xi_0)\notin\mathrm{WF}(V(t)v)$. It follows that all we need is the inclusion

$$\mathrm{WF}_h(e^{-itP/h}V(t)v)\subset\Phi^t(\mathrm{WF}_h(V(t)v)),$$

and that follows from the h-temperedness of $V(t)v$ and Egorov's theorem. □

In §9 we also need the following propagation result.

LEMMA A.2. *Let P satisfy the general assumptions of* §3.2 *and let W be as in Proposition* 9.1*, in particular $W|_{B(0,R_1)}\equiv 0$. Suppose that, for some radii $1\ll R_2<R_1$,*

$$(P-iW-z)u=f,\quad \operatorname{Im} z=\mathcal{O}(h),\ \|u\|=\mathcal{O}(h^{-M}),\ \|f\|=\mathcal{O}(1),\ \operatorname{supp} f\Subset B(0,R_2).$$

Then,

$$\begin{gathered}\text{for all } \varepsilon>0 \text{ there exists } T=T(R_2,R_0,\varepsilon)>0 \text{ such that}\\ \text{for all } (x,\xi)\in\mathrm{WF}_h(u)\setminus T^*_{B(0,R_2)}X \text{ with } p(x,\xi)\geqslant\varepsilon,\\ |\pi(\Phi^t(x,\xi))|>3R_0\quad\text{for all } t>T.\end{gathered}\tag{A.3}$$

Here $\pi: T^*X\to X$ is the natural projection. In other words, $u|_{X\setminus B(0,R_2)}$ is outgoing.

Proof. The principal symbol satisfies $\operatorname{Im}(p-iW-\operatorname{Re} z)\leqslant 0$, hence we have backward propagation:

$$\mathrm{WF}_h(u)\subset \Phi^t(\mathrm{WF}_h(u))\cup \bigcup_{0\leqslant s\leqslant t} \Phi^s(\mathrm{WF}_h(f)) \quad \text{for all } t\geqslant 0. \tag{A.4}$$

Indeed, we check that

$$(ih\partial_t-(P-iW))(U(t)u-e^{-itz/h}u)=e^{-itz/h}f,$$

and thus, by Duhamel's formula,

$$e^{-itz/h}u=U(t)u+\frac{i}{h}\int_0^t e^{-i(t-s)(P-iW)/h}e^{-isz/h}f\,ds,$$

from which (A.4) follows by applying Proposition A.1.

From the ellipticity of $P-iW-z$ in $X\setminus B(0,R_1+r_1)$ we have

$$\|u\|_{L^2(X\setminus B(0,R_1+r_1))}=\mathcal{O}(h^\infty).$$

Together with (A.4), this implies that

$$\mathrm{WF}_h(u)\subset\Gamma_+\cup\bigcup_{s\geqslant 0}\Phi^s(\mathrm{WF}_h(f)), \quad \Gamma_+\stackrel{\text{def}}{=}\{(x,\xi): e^{tH_p}(x,\xi)\not\to\infty \text{ as } t\to-\infty\}.$$

The assumptions on P in §3.2 (essentially the fact that it is close to the Euclidean Laplacian near infinity) show that for $x(t)\stackrel{\text{def}}{=}\pi(\Phi^t(x_0,\xi_0))$, $p(x_0,\xi_0)\geqslant\varepsilon$,

$$\frac{d}{dt}|x(t)|^2\bigg|_{t=0}\geqslant 0,\ |x_0|>R \quad\Longrightarrow\quad \frac{d}{dt}|x(t)|^2>0,\ t\geqslant 0, \tag{A.5}$$

if R is large enough. Indeed,

$$\begin{aligned}\frac{d^2}{dt^2}|x(t)|^2&=2\frac{d}{dt}\langle x(t),x'(t)\rangle=2\frac{d}{dt}\langle x(t),p'_\xi(x(t),\xi(t))\rangle\\&=2|p'_\xi|^2+2\langle x(t),p''_{\xi x}[p'_\xi]-p''_{\xi\xi}[p'_x]\rangle\geqslant 4\xi^2-o(1)\langle\xi\rangle^2,\end{aligned}$$

where we used (3.10) to obtain

$$p''_{\xi x}=o(\langle\xi\rangle|x|^{-1}) \quad\text{and}\quad p'_x=o(\langle\xi\rangle^2|x|^{-1}),$$

(here $o(1)\to 0$ as $x\to\infty$). Hence, $t\mapsto|x(t)|^2$ is strictly convex and that proves (A.5).

Now observe that, for any point $\varrho\in\mathrm{WF}_h(u)\setminus T^*_{B(0,R_2)}X$, we have $\varrho\in\Gamma_+\setminus T^*_{B(0,R_2)}X$, or $\varrho\in\Phi^s(\mathrm{WF}_h(f))$ for some $s>0$. In both cases, there exists $1\ll\widetilde{R}_2<R_2$ and $t>0$ such that $\Phi^{-t}(\varrho)\in T^*_{B(0,\widetilde{R}_2)}$. Thus, the trajectory $(\Phi^s(\varrho))_{s\in[-t,0]}$ has necessarily crossed the sphere $\{x:|x|=R_2\}$ for some t_0, coming from inside. From the above discussion, the trajectory is then strictly outgoing ($d|x(s)|/ds>0$) for $s>t_0$. In particular, there exists a time $T=T(R_2,R_0,\varepsilon)$ (uniform for all such ϱ) such that $\Phi^s(\varrho)$ will be outside $B(0,3R_0)$ for $s\geqslant T$. □

Acknowledgements. We would like to thank the National Science Foundation for partial support. The first author is grateful to UC Berkeley for its hospitality in April 2006; he was partially supported by the Agence Nationale de la Recherche, under the grant ANR-05-JCJC-0107-01. We would also like to thank Nicolas Burq for a helpful discussion of resolvent estimates.

Note added in proof. The authors have obtained a direct proof of resolvent estimates valid in a complex h-sized neighbourhood of the real axis [32]. That replaces parts of the argument in §9. On the other hand, K. Datchev [9] has extended the validity of the resolvent estimate on the real axis to operators on asymptotically Euclidean manifolds, eliminating the need for analyticity of the symbol near infinity.

References

[1] Anantharaman, N., Entropy and the localization of eigenfunctions. *Ann. of Math.*, 168 (2008), 435–475.

[2] Anantharaman, N. & Nonnenmacher, S., Half-delocalization of eigenfunctions for the Laplacian on an Anosov manifold. *Ann. Inst. Fourier* (*Grenoble*), 57 (2007), 2465–2523.

[3] Bindel, D. & Zworski, M., Theory and computation of resonances in 1D scattering. `http://www.cs.cornell.edu/~bindel/cims/resonant1d`.

[4] Bowen, R. & Ruelle, D., The ergodic theory of Axiom A flows. *Invent. Math.*, 29 (1975), 181–202.

[5] Burq, N., Contrôle de l'équation des plaques en présence d'obstacles strictement convexes. *Mém. Soc. Math. France*, 55 (1993).

[6] Burq, N. & Zworski, M., Geometric control in the presence of a black box. *J. Amer. Math. Soc.*, 17 (2004), 443–471.

[7] Christianson, H., Cutoff resolvent estimates and the semilinear Schrödinger equation. *Proc. Amer. Math. Soc.*, 136:10 (2008), 3513–3520.

[8] — Dispersive estimates for manifolds with one trapped orbit. *Comm. Partial Differential Equations*, 33 (2008), 1147–1174.

[9] Datchev, K., Local smoothing for scattering manifolds with hyperbolic trapped sets. *Comm. Math. Phys.*, 286 (2009), 837–850.

[10] Dencker, N., Sjöstrand, J. & Zworski, M., Pseudospectra of semiclassical (pseudo-) differential operators. *Comm. Pure Appl. Math.*, 57 (2004), 384–415.

[11] Dimassi, M. & Sjöstrand, J., *Spectral Asymptotics in the Semi-Classical Limit.* London Mathematical Society Lecture Note Series, 268. Cambridge University Press, Cambridge, 1999.

[12] Doi, S., Smoothing effects of Schrödinger evolution groups on Riemannian manifolds. *Duke Math. J.*, 82 (1996), 679–706.

[13] Evans, L. C. & Zworski, M., *Lectures on Semiclassical Analysis.* `http://math.berkeley.edu/~zworski/semiclassical.pdf`.

[14] Gaspard, P. & Rice, S. A., Semiclassical quantization of the scattering from a classically chaotic repellor. *J. Chem. Phys.*, 90:4 (1989), 2242–2254.

[15] Gérard, C. & Sjöstrand, J., Semiclassical resonances generated by a closed trajectory of hyperbolic type. *Comm. Math. Phys.*, 108 (1987), 391–421.

[16] Gérard, P., Mesures semi-classiques et ondes de Bloch, in *Séminaire sur les Équations aux Dérivées Partielles* (1990–1991), Exp. No. XVI. École Polytech., Palaiseau, 1991.

[17] Hörmander, L., *The Analysis of Linear Partial Differential Operators.* I, II. Grundlehren der Mathematischen Wissenschaften, 256, 257. Springer, Berlin–Heidelberg, 1983.

[18] Ikawa, M., Decay of solutions of the wave equation in the exterior of several convex bodies. *Ann. Inst. Fourier* (*Grenoble*), 38:2 (1988), 113–146.

[19] Katok, A. & Hasselblatt, B., *Introduction to the Modern Theory of Dynamical Systems.* Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.

[20] Keating, J. P., Novaes, M., Prado, S. D. & Sieber, M., Semiclassical structure of chaotic resonance eigenfunctions. *Phys. Rev. Lett.*, 97:15 (2006), 150406.

[21] Klopp, F. & Zworski, M., Generic simplicity of resonances. *Helv. Phys. Acta*, 68 (1995), 531–538.

[22] Lin, K. K., Numerical study of quantum resonances in chaotic scattering. *J. Comput. Phys.*, 176 (2002), 295–329.

[23] Lin, K. K. & Zworski, M., Quantum resonances in chaotic scattering. *Chem. Phys. Lett.*, 355 (2002), 201–205.

[24] Lu, W. T., Sridhar, S. & Zworski, M., Fractal Weyl laws for chaotic open systems. *Phys. Rev. Lett.*, 91:15 (2003), 154101.

[25] Martinez, A., Resonance free domains for non globally analytic potentials. *Ann. Henri Poincaré*, 3 (2002), 739–756.

[26] Morita, T., Periodic orbits of a dynamical system in a compound central field and a perturbed billiards system. *Ergodic Theory Dynam. Systems*, 14 (1994), 599–619.

[27] Nakamura, S., Stefanov, P. & Zworski, M., Resonance expansions of propagators in the presence of potential barriers. *J. Funct. Anal.*, 205 (2003), 180–205.

[28] Naud, F., Classical and quantum lifetimes on some non-compact Riemann surfaces. *J. Phys. A*, 38:49 (2005), 10721–10729.

[29] Nonnenmacher, S. & Rubin, M., Resonant eigenstates for a quantized chaotic system. *Nonlinearity*, 20:6 (2007), 1387–1420.

[30] Nonnenmacher, S. & Zworski, M., Fractal Weyl laws in discrete models of chaotic scattering. *J. Phys. A*, 38:49 (2005), 10683–10702.

[31] — Distribution of resonances for open quantum maps. *Comm. Math. Phys.*, 269 (2007), 311–365.

[32] — Semiclassical resolvent estimates in chaotic scattering. *Appl. Math. Res. Express*, 2009 (2009), 1–13.

[33] Pesin, Y. B. & Sadovskaya, V., Multifractal analysis of conformal Axiom A flows. *Comm. Math. Phys.*, 216 (2001), 277–312.

[34] Petkov, V. & Stoyanov, L., Analytic continuation of the resolvent of the Laplacian and the dynamical zeta function. *C. R. Math. Acad. Sci. Paris*, 345 (2007), 567–572.

[35] Ruelle, D., *Thermodynamic Formalism.* Encyclopedia of Mathematics and its Applications, 5. Addison-Wesley, Reading, MA, 1978.

[36] Schomerus, H. & Tworzydło, J., Quantum-to-classical crossover of quasibound states in open quantum systems. *Phys. Rev. Lett.*, 93:15 (2004), 154102.

[37] Shubin, M. A. & Sjöstrand, J., Appendix to Weak Bloch property and weight estimates for elliptic operators, in *Séminaire sur les Équations aux Dérivées Partielles* (1989–1990), Exp. No. V. École Polytech., Palaiseau, 1990.

[38] Sjöstrand, J., Geometric bounds on the density of resonances for semiclassical problems. *Duke Math. J.*, 60 (1990), 1–57.
[39] — A trace formula and review of some estimates for resonances, in *Microlocal Analysis and Spectral Theory* (Lucca, 1996), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 490, pp. 377–437. Kluwer Acad. Publ., Dordrecht, 1997.
[40] — Eigenvalue distribution for non-self-adjoint operators with small multiplicative random perturbations. Preprint, 2008. arXiv:0802.3584 [math.SP].
[41] Sjöstrand, J. & Zworski, M., Quantum monodromy and semi-classical trace formulae. *J. Math. Pures Appl.*, 81 (2002), 1–33.
[42] — Fractal upper bounds on the density of semiclassical resonances. *Duke Math. J.*, 137 (2007), 381–459.
[43] Strain, J. & Zworski, M., Growth of the zeta function for a quadratic map and the dimension of the Julia set. *Nonlinearity*, 17:5 (2004), 1607–1622.
[44] Tang, S. H. & Zworski, M., From quasimodes to reasonances. *Math. Res. Lett.*, 5 (1998), 261–272.
[45] Walters, P., *An Introduction to Ergodic Theory.* Graduate Texts in Mathematics, 79. Springer, New York, 1982.
[46] Wirzba, A., Quantum mechanics and semiclassics of hyperbolic n-disk scattering systems. *Phys. Rep.*, 309 (1999).
[47] Wojtkowski, M. P., Design of hyperbolic billiards. *Comm. Math. Phys.*, 273 (2007), 283–304.
[48] Zworski, M., Resonances in physics and geometry. *Notices Amer. Math. Soc.*, 46 (1999), 319–328.

Stéphane Nonnenmacher
Institut de Physique Théorique
CNRS URA 2306
CEA-Saclay
FR-91191 Gif-sur-Yvette
France
snonnenmacher@cea.fr

Maciej Zworski
University of California, Berkeley
Department of Mathematics
Berkeley, CA 94720
U.S.A.
zworski@math.berkeley.edu

Received August 13, 2007