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Astala's conjecture on distortion of Hausdorff measures under quasiconformal maps in the plane

by

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1. Introduction

An orientation-preserving homeomorphism $\phi\colon\Omega\to\Omega'$ between planar domains $\Omega,\Omega'\subset\mathbb{C}$ is called *K-quasiconformal* if it belongs to the Sobolev space $W^{1,2}_{\mathrm{loc}}(\Omega)$ and satisfies the *distortion inequality*

$$\max_\alpha|\partial_\alpha\phi|\leqslant K\min_\alpha|\partial_\alpha\phi|\quad \text{a.e. in } \Omega. \tag{1.1}$$

Infinitesimally, quasiconformal mappings carry circles to ellipses with eccentricity at most K. Finer properties of quasiconformal mappings can be identified by studying their mapping properties with respect to the Hausdorff measure, the primary focus of this paper. It has been known since the work of Ahlfors [1] that quasiconformal mappings preserve sets of Lebesgue measure zero. It is also well known that they preserve sets of Hausdorff dimension zero, since K-quasiconformal mappings are Hölder continuous with exponent $1/K$; see [14]. However, they need not preserve Hausdorff dimension bigger than zero. Gehring and Reich [10] identified as a conjecture the precise bounds for the area distortion under quasiconformal mappings, a conjecture verified by the the groundbreaking work of Astala [2]. As a consequence of area distortion, Astala obtained the theorem below, which proved the case $n=2$ of a conjecture of Iwaniec and Martin in $\mathbb{R}^n$ [11].

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Astala's Hausdorff dimension distortion theorem 1.1. *For any compact set E with Hausdorff dimension $0<t<2$ and any K-quasiconformal mapping ϕ we have*

$$\frac{1}{K}\left(\frac{1}{t}-\frac{1}{2}\right) \leqslant \frac{1}{\dim(\phi E)}-\frac{1}{2} \leqslant K\left(\frac{1}{t}-\frac{1}{2}\right). \tag{1.2}$$

Finally, these bounds are optimal, in that equality may occur in either estimate.

The question that we study concerns refinement of the left-hand endpoint above. Can it be improved to the level of Hausdorff measures $\mathcal{H}^t$? Indeed, this is the case. The next theorem, the main result of this paper, answers in the affirmative Astala's Question 4.4 in [2].

Main theorem 1.2. *If ϕ is a planar K-quasiconformal mapping, $0\leqslant t\leqslant 2$ and*

$$t' = \frac{2Kt}{2+(K-1)t},$$

then we have the following implication for all compact sets $E\subset\mathbb{C}$:

$$\mathcal{H}^t(E)=0 \quad \textit{implies} \quad \mathcal{H}^{t'}(\phi E)=0. \tag{1.3}$$

Since the inverse of a K-quasiconformal mapping is also a K-quasiconformal mapping, the following refinement of the right-hand endpoint in (1.2) follows: for a compact set F, we have that $\mathcal{H}^{t'}(F)>0$ implies $\mathcal{H}^t(\phi F)>0$.

The above theorem is sharp in two senses. Firstly, the hypothesis $\mathcal{H}^t(E)=0$ cannot be weakened to $\mathcal{H}^t(E)<\infty$ while keeping the same conclusion (i.e. the statement "$\mathcal{H}^t(E)<\infty$ implies $\mathcal{H}^{t'}(\phi E)=0$" under the same conditions as in Theorem 1.2, which has a weaker hypothesis than Theorem 1.2 and hence is a stronger statement than Theorem 1.2, is false). Secondly, if we keep the hypothesis $\mathcal{H}^t(E)=0$, the conclusion $\mathcal{H}^{t'}(\phi E)=0$ cannot be strengthened, to Hausdorff measure zero with respect to a gauge. For any gauge function h satisfying

$$\lim_{s\to 0}\frac{s^{t'}}{h(s)}=0,$$

there exists a compact set E and a K-quasiconformal mapping ϕ with $\mathcal{H}^t(E)=0$ but $\mathcal{H}^h(\phi E)=\infty$. See Theorem 1.7 (a) in [25] for the relevant examples.

Some instances of this theorem are known, and have connections to significant further properties of quasiconformal maps. Note that the above classical result of Ahlfors asserts that the theorem is true when $t=2$, while the theorem is obviously true when $t=0$, since ϕ is a homeomorphism. In fact, for the Lebesgue measure, there is the following precise quantitative bound due to [2] for a properly normalized K-quasiconformal mapping ϕ:

$$|\phi E| \leqslant C|E|^{1/K}.$$

This bound leads to the sharp Sobolev regularity estimate $\phi \in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ for every

$$p < \frac{2K}{K-1}.$$

A positive answer was also given for the special case $t'=1$ (hence $t=2/(K+1)$) in [3]. This special case is important due to its applications towards removability of sets for bounded K-quasiregular mappings, i. e. a quasiconformal analogue of the celebrated Painlevé's problem. We refer the reader to [23] and [3] for details. The same paper [3] contains other related results, as does Prause [19].

Let us give an overview of the proof and the paper. The highest levels of the argument follow a familiar line of reasoning. Matters are reduced to the case of small dilatation in Lemma 2.1. Thus, we take a compact set E with t-Hausdorff measure equal to zero and a K-quasiconformal map ϕ. To provide the conclusion that the t'-Hausdorff measure of ϕE is zero, we should exhibit a covering of ϕE by (quasi)disks that is arbitrarily small in $\mathcal{H}^{t'}$-measure. To do this we should begin with a corresponding covering of E that is small in the $\mathcal{H}^t$-measure. The first novelty is that we show that this can be done with certain dyadic cubes (denoted $P \in \mathcal{P}$ below) that admit one key additional feature, that they obey a t-packing condition described in Proposition 2.2.

Associated with $\mathcal{P}$ is a measure $w_{t,\mathcal{P}}$, defined in (2.6), which exhibits "t-dimensional" behaviour, reflective of the t-packing condition. The second novelty is that the Beurling operator, and more generally a standard Calderón–Zygmund operator, is bounded on $L^2(w_{t,\mathcal{P}})$; see Proposition 2.3. This fact does *not* follow from standard weighted theory of singular integrals, but this new class of measures have enough additional combinatorial structure that a proof of this fact is not difficult to supply.

The mapping ϕ is then factored into $\phi = \phi_1 \circ h$, where ϕ_1 is the "conformal inside" part and h is the "conformal outside" part. The conformal inside part admits a relevant estimate that can be found in [3], and is recalled below. The relevant estimate on the conformal outside part is new, and uses in an essential way the two novelties just mentioned. See the proof of Lemma 5.2. It uses Astala's approach for distortion of area [2]. The conformal inside/outside order of the factorization $\phi = \phi_1 \circ h$ appears also in [19].

The principal lemmas are in §2. The new lemma on approximating Hausdorff content, with control of a packing condition, namely Proposition 2.2, is given in §3. In §4 the proof of the weighted estimate for the Beurling operator, Proposition 2.3, is given. These two propositions are combined in §5.

As usual, in a string of inequalities, the letter C might denote different constants from one inequality to the next.

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2. Principal propositions

We state the principal propositions of the paper, with the first being a restatement of the main theorem for a specific class of quasiconformal mappings, namely those of small dilatation.

LEMMA 2.1. *Let* $0<t<2$. *Then there is a small constant* $0<\varkappa_0<1$ ($\varkappa_0=\varkappa_0(t)$ *is a decreasing function of* t) *so that the following holds. Let* $g:\mathbb{C}\to\mathbb{C}$ *be a* K*-quasiconformal map with*

$$\frac{K-1}{K+1}\leqslant \varkappa_0.$$

Then we have the following implication for all compact subsets $E\subset\mathbb{C}$:

$$\mathcal{H}^t(E)=0 \quad \textit{implies} \quad \mathcal{H}^{t'}(gE)=0, \tag{2.1}$$

where

$$t'=\frac{2Kt}{2+(K-1)t}.$$

Proof of Theorem 1.2. We use the usual factorization of a K-quasiconformal mapping into those with small dilatation. For a fixed K-quasiconformal mapping g, we can write

$$g=g_\lambda\circ\ldots\circ g_2\circ g_1, \tag{2.2}$$

so that each g_i is K_i-quasiconformal, $K=K_1\ldots K_\lambda$, and

$$K_i\leqslant\frac{1+\varkappa_0}{1-\varkappa_0}$$

for all $i=1,2,\ldots,\lambda$, with $\varkappa_0=\varkappa_0(t')$. (See [1] or [12].) It follows that the dilatation of each g_i satisfies

$$\frac{K_i-1}{K_i+1}\leqslant\varkappa_0(t'),$$

that is Lemma 2.1 applies to each g_i individually.

Indeed, let us set

$$\tau(t,K)=\frac{2Kt}{2+(K-1)t},$$

and inductively define $\tau_1=\tau(t,K_1)$, and $\tau_{i+1}=\tau(\tau_i,K_{i+1})$. Let $E\subset\mathbb{C}$ be a compact subset of the plane with $\mathcal{H}^t(E)=0$. It follows from an inductive application of Lemma 2.1 (since $\varkappa_0(t')\leqslant\varkappa_0(\tau_i)$ for all $i=1,2,...,\lambda$) that we have

$$\mathcal{H}^{\tau_j}(g_j\circ...\circ g_1(E))=0,\quad 1\leqslant j\leqslant\lambda.$$

And it is easily checked that

$$\tau_\lambda=\frac{2Kt}{2+(K-1)t},$$

which is the dimension t' in Theorem 1.2. □

We state our proposition on the approximation of Hausdorff content with the t-packing condition. Let $\mathcal{P}$ be a finite collection of disjoint dyadic cubes in the plane. Let $0<t<2$. We denote the *t-Carleson packing norm* of $\mathcal{P}$ as follows:

$$\|\mathcal{P}\|_{t\text{-pack}}=\sup_Q\Bigg(\ell(Q)^{-t}\sum_{\substack{P\in\mathcal{P}\\ P\subset Q}}\ell(P)^t\Bigg)^{1/t}, \tag{2.3}$$

where the supremum is taken over all dyadic cubes Q. In this formula and throughout this paper, $\ell(Q)$ denotes the side-length of the cube Q. And we say that $\mathcal{P}$ satisfies the *t-Carleson packing condition* if $\|\mathcal{P}\|_{t\text{-pack}}<\infty$.

Recall that for a set E, $0\leqslant s\leqslant 2$ and $0<\delta\leqslant\infty$, one defines

$$\mathcal{H}^s_\delta(E)=\inf\Bigg\{\sum_{i=1}^\infty \operatorname{diam}(B_i)^s : E\subset\bigcup_{i=1}^\infty B_i \text{ and } \operatorname{diam}(B_i)\leqslant\delta\Bigg\}, \tag{2.4}$$

where $B_i\subset\mathbb{C}$ is a set and $\operatorname{diam}(B_i)$ denotes its diameter. Then one defines the *Hausdorff s-measure* of E to be

$$\mathcal{H}^s(E)=\lim_{\delta\to 0}\mathcal{H}^s_\delta(E)=\sup_{\delta>0}\mathcal{H}^s_\delta(E). \tag{2.5}$$

The quantity $\mathcal{H}^s_\infty(E)$ is usually referred to as the *Hausdorff content* of E.

It is well known that in the definition of Hausdorff measure, if instead of covering with balls or arbitrary sets, one covers with dyadic cubes, one obtains an equivalent measure. We will take the dyadic cubes to be closed unless otherwise stated, i.e. of the form $[2^{-k}m_1,2^{-k}(m_1+1)]\times[2^{-k}m_2,2^{-k}(m_2+1)]$, with k a non-negative integer, and m_1 and m_2 integers. Recall also that $\mathcal{H}^t(E)=0$ if and only if $\mathcal{H}^t_\infty(E)=0$. For these and related facts, see e.g. [13] or [8].

Only the case $m=2$ of the following proposition is used below. As usual, for $a>0$, we denote by aQ the cube concentric to the cube Q, but such that $\ell(aQ)=a\ell(Q)$.

PROPOSITION 2.2. *Let $m\geqslant 0$ be an integer. Then there is a positive constant C such that, for any compact $E\subset(0,1)^2\subset\mathbb{C}$, $0<t<2$ and $\varepsilon>0$, there is a finite collection of closed dyadic cubes $\mathcal{P}=\{P_i\}_{i=1}^N$ such that*

(a) $2^m P_i\cap 2^m P_j=\varnothing$ *for* $i\neq j$;

(b) $E\subset\bigcup_{i=1}^N 3\cdot 2^m P_i$;

(c) $\|\mathcal{P}\|_{t\text{-}pack}\leqslant 1$;

(d) $\sum_{i=1}^N \ell(P_i)^t\leqslant C(\mathcal{H}^t_\infty(E)+\varepsilon)$.

Given $0<t\leqslant 2$ and a collection $\mathcal{P}$ of pairwise disjoint dyadic cubes, we define the measure $w_{t,\mathcal{P}}$ associated with $\mathcal{P}$ by

$$w_{t,\mathcal{P}}(x)=\sum_j \ell(P_j)^{t-2}\,\chi_{P_j}(x), \tag{2.6}$$

where χ_{P_j} denotes the characteristic function of P_j and $\ell(P_j)$ denotes the side-length of P_j. Define also

$$\overline{P}=\bigcup_{i=1}^N P_i. \tag{2.7}$$

The measure $w_{t,\mathcal{P}}$ behaves as a t-dimensional measure, namely if Q is an arbitrary cube (dyadic or not) with sides parallel to the coordinate axes, then

$$w_{t,\mathcal{P}}(Q)\leqslant 16\|\mathcal{P}\|^t_{t\text{-pack}}\ell(Q)^t. \tag{2.8}$$

We will be concerned with a quasiconformal map f that is conformal outside of $\overline{P}$, and we will need an estimate on the diameters of $f(P_i)$. The map f will have an explicit expression as a Neumann series involving the Beurling operator, which we recall here. Let

$$(\mathcal{S}f)(z)=-\frac{1}{\pi}\text{p.v.}\int_{\mathbb{C}}\frac{f(\tau)}{(z-\tau)^2}\,dA(\tau), \tag{2.9}$$

be the *Beurling transform*. This is an example of a standard singular integral bounded on $L^2(\mathbb{C})$ (see [21].) The second proposition gives a weighted norm inequality with respect to the weight $w_{t,\mathcal{P}}$ for the compression of $\mathcal{S}$ to the set $\overline{P}$, i.e. the operator $\chi_{\overline{P}}\mathcal{S}\chi_{\overline{P}}$, assuming that $\mathcal{P}$ satisfies a Carleson t-packing condition.

PROPOSITION 2.3. *Let $0<t<2$ and let $\mathcal{P}=\{P_i\}_{i=1}^N$ be a collection of open dyadic cubes with pairwise disjoint triples, i.e. $3P_i\cap 3P_j=\varnothing$ for $i\neq j$. Assume further that $\|\mathcal{P}\|_{t\text{-pack}}\leqslant 1$. Then there exists an absolute positive constant $C=C(t)$ such that*

$$\|\mathcal{S}(\chi_{\overline{P}}f)\|_{L^2(w_{t,\mathcal{P}})}\leqslant C\|f\|_{L^2(w_{t,\mathcal{P}})} \tag{2.10}$$

for all $f\in L^2(\mathbb{C})$. Moreover, $C(t)$ is an increasing function of t.

The proof of this proposition is presented in §4, and follows from elementary bounds on the Beurling operator, and combinatorial properties of the measure $w_{t,\mathcal{P}}$. This estimate is new, and does not follow from standard weighted theory. The theory of A_2 weights is built around the assumption that the weights are positive a.e., while the weights $w_{t,\mathcal{P}}$ are zero on a large set, and do not admit extensions to A_2 weights uniformly in the A_2 characteristic. (Cf. Wolff's theorem in [9, p. 439].)

3. The proof of Proposition 2.2

Given $\varepsilon>0$, by the definition of dyadic Hausdorff content at dimension t, there exists a (possibly infinite) collection $\{Q_n\}_n$ of closed dyadic cubes such that $E\subseteq\bigcup_n Q_n$, and

$$\sum_n \ell(Q_n)^t \leqslant \mathcal{H}^t_\infty(E)+\varepsilon. \tag{3.1}$$

By compactness of E, after relabeling indices, there is a finite number N for which

$$E\subseteq \bigcup_{n=1}^N (3Q_n)^\circ,$$

where A° denotes the interior of the set A. Since each cube of the form $3Q_n$ is the union of 9 dyadic cubes of the same size as Q_n, we can write, after relabeling, $E\subseteq\bigcup_{n=1}^{N'} Q_n$, where Q_n are closed dyadic cubes (possibly with overlapping or repeated cubes).

By selecting the maximal cubes among the Q_n, and eliminating those Q_n not intersecting E, we may now assume, after a relabeling, that

$$\sum_{n=1}^N \ell(Q_n)^t \leqslant 9(\mathcal{H}^t_\infty(E)+\varepsilon), \tag{3.2}$$

and that the cubes Q_n are dyadic, intersect E, and have pairwise disjoint interiors.

Let $\min\{\ell(Q_n)\}=2^{-M}$, and call a finite collection of cubes $\mathcal{R}$ *admissible*, denoted by $\mathcal{R}\in\mathrm{Adms}$, if

(1) $\mathcal{R}$ is a finite collection of dyadic cubes that intersect E, thus $\mathcal{R}=\{R_i\}_{i=1}^H$ for a finite H and $R_i\cap E\neq\varnothing$ for all i;

(2) $2^{-M}\leqslant\ell(R_i)\leqslant 1$;

(3) $E\subseteq\bigcup_{i=1}^H R_i$;

(4) they have pairwise disjoint interiors.

We have just seen that Adms is non-empty. The minimum

$$\min\biggl\{\sum_{R_i\in\mathcal{R}} \ell(R_i)^t : \mathcal{R}\in\mathrm{Adms}\biggr\},$$

is achieved, as there are only finitely many admissible collections of cubes. Denote an admissible collection that achieves the minimum as $\mathcal{T}=\{T_i\}_{i=1}^{M'}$. By (3.2), we have

$$\sum_{i=1}^{M'} \ell(T_i)^t \leqslant \sum_{j=1}^{N} \ell(Q_j)^t \leqslant 9(\mathcal{H}_\infty^t(E)+\varepsilon). \tag{3.3}$$

Any minimizer also satisfies a local property: for any dyadic cube Q such that $2^{-M}\leqslant\ell(Q)\leqslant 2^0$, it is true that

$$\sum_{T_i\subset Q} \ell(T_i)^t \leqslant \ell(Q)^t. \tag{3.4}$$

Indeed, if Q intersects E, and this inequality did not hold, the cube Q would have been selected instead of the cubes T_i with $T_i\subset Q$, contradicting the property of achieving the minimum. If the cube Q does not intersect E, then the inequality is trivial.

As an immediate consequence, we get that for any dyadic cube Q, irrespective of its size,

$$\sum_{T_i\subset Q} \ell(T_i)^t \leqslant \ell(Q)^t. \tag{3.5}$$

In other words, the cubes T_i satisfy (c) in the statement of Proposition 2.2.

Thus, $\mathcal{T}$ satisfies conditions (c) and (d) of the conclusion. To accommodate (a) and (b) as well, fix an integer $m\in\mathbb{N}\setminus\{0\}$, and fix a cube $T_i\in\mathcal{T}$. Subdivide T_i into its 2^{2m+2} dyadic descendants of side-length $2^{-m-1}\ell(T_i)$. Let $\widehat{T}_i$ be the dyadic descendant of T_i of side-length $2^{-m-1}\ell(T_i)$ whose upper right corner is the center of T_i. It is now easy to check that the cubes $\widehat{T}_i$ satisfy (d) in the statement of Proposition 2.2 (with a larger constant C than the constant obtained for the cubes T_i), as well as (c), (b) and (a). Since $t<2$, notice that C, which depends on m, can be taken independent of t.

4. Weighted norm inequalities for the Beurling transform

We prove the following estimate on the Beurling operator acting on $L^p(w_{t,\mathcal{P}})$ spaces. Note that the same proof applies to any standard Calderón–Zygmund singular integral, so we exhibit a whole new class of weights with respect to which singular integrals are bounded and yet do not admit extension to A_p weights with uniformly bounded A_p characteristic. For more on non-doubling measures see, e.g., [9, p. 439], [16], [17], [20], [23], [24] and the references therein.

LEMMA 4.1. *Under the assumptions of Proposition* 2.3, *for any* $1<p<\infty$, *and for two subsets* $F,G\subset\overline{P}$, *we have the estimate*

$$\int_G |\mathcal{S}\chi_F(x)|w_{t,\mathcal{P}}\,dx \leqslant C_{p,t}|F|_{w_{t,\mathcal{P}}}^{1/p}|G|_{w_{t,\mathcal{P}}}^{1-1/p}. \tag{4.1}$$

Here $C_{p,t}$ *is a constant that only depends on* p *and* t. *For fixed* p, $C_{p,t}$ *is an increasing function of* t.

Here and throughout,

$$|A|_{w_{t,\mathcal{P}}} = w_{t,\mathcal{P}}(A) = \int_A w_{t,\mathcal{P}}\,dx.$$

This is the restricted weak-type estimate for $\mathcal{S}$ as a bounded operator from the Lorentz space $L^{p,1}(w_{t,\mathcal{P}})$ to $L^{p,\infty}(w_{t,\mathcal{P}})$. A standard interpolation then proves Proposition 2.3 (see, e.g., Theorem 3.15 in [22, p. 197]).

To prove this, we split $\mathcal{S}$ into a local and non-local part, $\mathcal{S}=\mathcal{S}_{\text{local}}+\mathcal{S}_{\text{non}}$, where writing the kernel of $\mathcal{S}$ as $K(x,y)$, we define the kernel of $\mathcal{S}_{\text{local}}$ to be

$$K_{\text{local}}(x,y) = K(x,y)\sum_{P\in\mathcal{P}} \chi_P(x)\chi_P(y).$$

On each $P\in\mathcal{P}$, $w_{t,\mathcal{P}}$ is a constant multiple of Lebesgue measure, hence we can estimate the local part directly, using the $L^p(dx)$-bound for $\mathcal{S}$:

$$\begin{aligned}\|\mathcal{S}_{\text{local}}f\|_{L^p(w_{t,\mathcal{P}})}^p &= \sum_{P\in\mathcal{P}} \|\chi_P S_{\text{local}}(\chi_P f)\|_{L^p(w_{t,\mathcal{P}})}^p \\ &\leqslant C_p \sum_{P\in\mathcal{P}} \|\chi_P f\|_{L^p(w_{t,\mathcal{P}})}^p \leqslant C_p\|f\|_{L^p(w_{t,\mathcal{P}})}^p.\end{aligned}$$

On the non-local part, we abandon cancellation, and only use the homogeneity of the Beurling kernel. It is also convenient to pass to a combinatorial analog of the non-local operator. To this end, let us say that a collection of (not necessarily dyadic) cubes $\mathcal{Q}$ is a *grid* if and only if for all $Q,Q'\in\mathcal{Q}$ we have $Q\cap Q'\in\{\varnothing,Q,Q'\}$. One can construct a collection of cubes $\widetilde{\mathcal{Q}}$ so that these conditions hold:

(1) $\widetilde{\mathcal{Q}}$ is a union of at most 9 grids;

(2) for each dyadic cube P there is a cube $Q\in\widetilde{\mathcal{Q}}$ with $P\subset Q$ and $|Q|\leqslant C|P|$;

(3) for each pair of dyadic cubes P,P' with $3P\cap 3P'=\varnothing$, there is a cube $Q\in\widetilde{\mathcal{Q}}$ with $P,P'\subset Q$ and $|Q|\leqslant C\operatorname{dist}(P,P')^2$.

Here C is an absolute constant.

Proof. We recall a standard notion used e.g. in [15, §5]. Define a *shifted dyadic mesh* in two dimensions to be

$$\widetilde{\mathcal{Q}} = \left\{2^j(k+(0,1)^2+(-1)^i\alpha) : i\in\{0,1\}, j\in\mathbb{Z}, k\in\mathbb{Z}^2 \text{ and } \alpha\in\left\{0,\tfrac{1}{3},\tfrac{2}{3}\right\}^2\right\}.$$

Observe that for each cube $Q\subset\mathbb{R}^2$, there is a $Q'\in\widetilde{\mathcal{Q}}$ with $Q\subset\frac{9}{10}Q'$ and $\ell(Q')\leqslant 9\ell(Q)$. This is easier to check in one dimension. □

Then, it follows that for all functions f supported on $\overline{P}$, and for a point $x\in P$ with $P\in\mathcal{P}$,

$$|\mathcal{S}_{\mathrm{non}}f(x)| \leqslant \sum_{\substack{P'\in\mathcal{P}\\ P'\neq P}} \int_{P'} |K(x,y)f(y)|\,dy \leqslant C \sum_{\substack{P'\in\mathcal{P}\\ P'\neq P}} \int_{P'} |f(y)| \frac{dy}{\operatorname{dist}(P,P')^2} \leqslant C\,\mathrm{S}_{\widetilde{\mathcal{Q}}}\,|f|(x),$$

where we define for any collection of cubes $\mathcal{Q}$,

$$\mathrm{S}_{\mathcal{Q}}\,f(x) = \sum_{\substack{Q\in\mathcal{Q}\\ Q \text{ non-local}}} \frac{\chi_Q(x)}{\ell(Q)^2}\int_Q f(y)\,dy. \tag{4.2}$$

Here we say that an arbitrary cube Q (dyadic or not) with sides parallel to the coordinate axes is *non-local* if there exist $P_1, P_2\in\mathcal{P}$ such that $P_i\cap Q\neq\varnothing$ for $i=1,2$. It follows (since $3P_1\cap 3P_2=\varnothing$) that if Q is non-local, then $\ell(P)\leqslant\ell(Q)$ if $P\in\mathcal{P}$ and $P\cap Q\neq\varnothing$.

Given the collection of cubes $\mathcal{P}$ (which is fixed throughout this section), and given a grid $\mathcal{Q}$, there is a unique subcollection of cubes

$$\mathcal{Q}' = \{\text{non-local cubes of the underlying grid } \mathcal{Q}\}.$$

Thus, for the proof of Lemma 4.1, it suffices to consider only collections of cubes $\mathcal{Q}'$ (and we restrict our attention to such collections of cubes for the rest of this section) and to prove the following lemma.

LEMMA 4.2. *Under the assumptions of Lemma* 4.1, *for the collection of non-local cubes $\mathcal{Q}'$ associated with any grid $\mathcal{Q}$, we have the inequality*

$$\int_G [\mathrm{S}_{\mathcal{Q}'}\,\chi_F] w_{t,\mathcal{P}}\,dx \leqslant C_{p,t}|F|^{1/p}_{w_{t,\mathcal{P}}}|G|^{1-1/p}_{w_{t,\mathcal{P}}}, \quad 1<p<\infty. \tag{4.3}$$

For fixed p, $C_{p,t}$ is an increasing function of t.

We turn to the proof. There are two points to observe. Consider the $w_{t,\mathcal{P}}$-maximal function defined by

$$\mathrm{M}_t\,g = \sup_{Q\in\mathcal{Q}'} \frac{\chi_Q}{|Q|_{w_{t,\mathcal{P}}}} \int_Q g(y) w_{t,\mathcal{P}}(y)\,dy \tag{4.4}$$

This operator maps $L^1(w_{t,\mathcal{P}})$ to $L^{1,\infty}(w_{t,\mathcal{P}})$, that is,

$$\lambda|\{x : \mathrm{M}_t\, g(x) > \lambda\}|_{w_{t,\mathcal{P}}} \leqslant \|g\|_{L^1(w_{t,\mathcal{P}})}, \quad 0 < \lambda < \infty. \tag{4.5}$$

Indeed, this is a maximal inequality true for all weights, and follows immediately from the usual covering lemma proof, which is quite simple in this context, as $\mathcal{Q}$ is a grid.

For $F, G \subset \overline{P}$, if $8|F|_{w_{t,\mathcal{P}}} \leqslant |G|_{w_{t,\mathcal{P}}}$, we take $F' = F$. Otherwise we define

$$F' = F \cap \left\{ x : \mathrm{M}_t\, \chi_G(x) \leqslant \frac{2w_{t,\mathcal{P}}(G)}{w_{t,\mathcal{P}}(F)} \right\}. \tag{4.6}$$

By the weak-$L^1(w_{t,\mathcal{P}})$ inequality for M_t, we see that $|F'|_{w_{t,\mathcal{P}}} \geqslant \frac{1}{2}|F|_{w_{t,\mathcal{P}}}$. (In the argument of [15, §3], F' is a major subset of F.) We show that

$$\int_G [\mathrm{S}_{\mathcal{Q}'}\, \chi_{F'}] w_{t,\mathcal{P}}\, dx \leqslant C_t \min\{w_{t,\mathcal{P}}(F)\,,\, w_{t,\mathcal{P}}(G)\}. \tag{4.7}$$

Upon iteration of inequality (4.7), we see that we actually have inequality (4.7) with $F' = F$ on the left-hand side and C_t replaced by

$$C_t \log\left(2 + \frac{|F|_{w_{t,\mathcal{P}}}}{|G|_{w_{t,\mathcal{P}}}}\right).$$

Indeed, with $F = F_0$ and $F' = F_1$ we now apply (4.7) with F_0 replaced by $F_0 \setminus F_1$, and F_2 by the corresponding major subset of $F_0 \setminus F_1$. We continue the iteration until

$$8|F_n|_{w_{t,\mathcal{P}}} \leqslant |G|_{w_{t,\mathcal{P}}},$$

which occurs with

$$n \lesssim \log\left(2 + \frac{|F|_{w_{t,\mathcal{P}}}}{|G|_{w_{t,\mathcal{P}}}}\right).$$

From this inequality we immediately obtain (4.3):

$$\begin{aligned} \int_G [\mathrm{S}_{\mathcal{Q}'}\, \chi_F] w_{t,\mathcal{P}}\, dx &\leqslant C_t \log\left(2 + \frac{|F|_{w_{t,\mathcal{P}}}}{|G|_{w_{t,\mathcal{P}}}}\right) \min\{w_{t,\mathcal{P}}(F), w_{t,\mathcal{P}}(G)\} \\ &\leqslant C_{p,t} |F|_{w_{t,\mathcal{P}}}^{1/p} |G|_{w_{t,\mathcal{P}}}^{1-1/p} \end{aligned} \tag{4.8}$$

for $1 < p < \infty$, which reduces the proof of Lemma 4.2 to showing (4.7).

We now turn to the proof of (4.7):

$$\begin{aligned}
\int_G [\mathrm{S}_{\mathcal{Q}'}\,\chi_{F'}] w_{t,\mathcal{P}}\,dx &= \sum_{Q\in\mathcal{Q}'} \frac{|F'\cap Q|}{\ell(Q)^2} |G\cap Q|_{w_{t,\mathcal{P}}} \\
&= \sum_{Q\in\mathcal{Q}'} \frac{|F'\cap Q|}{\ell(Q)^{2-t}} \frac{|G\cap Q|_{w_{t,\mathcal{P}}}}{\ell(Q)^t} \\
&\leqslant \underbrace{\min\biggl\{16\|\mathcal{P}\|_{t\text{-pack}}^t, 32\frac{w_{t,\mathcal{P}}(G)}{w_{t,\mathcal{P}}(F)}\biggr\}}_{=:A} \sum_{Q\in\mathcal{Q}'} \frac{|F'\cap Q|}{\ell(Q)^{2-t}} \qquad (4.9)\\
&= A \sum_{Q\in\mathcal{Q}'} \sum_{P:P\cap Q\neq\varnothing} \frac{|F'\cap P\cap Q|}{\ell(Q)^{2-t}} \\
&\leqslant A \sum_{P\in\mathcal{P}} \sum_{\substack{Q\in\mathcal{Q}'\\ Q\cap P\neq\varnothing}} |F'\cap P| \frac{1}{\ell(Q)^{2-t}} \\
&\leqslant AC_t \sum_{P\in\mathcal{P}} \frac{|F'\cap P|}{\ell(P)^{2-t}} \qquad (4.10)\\
&= AC_t |F'|_{w_{t,\mathcal{P}}} \\
&\leqslant C_t' \min\{w_{t,\mathcal{P}}(F'), w_{t,\mathcal{P}}(G)\} \\
&\leqslant C_t' \min\{w_{t,\mathcal{P}}(F), w_{t,\mathcal{P}}(G)\}.
\end{aligned}$$

In passing to (4.9), we have used the packing condition (see (2.8)) and the definition of F' in (4.6), to wit if $|Q\cap F'|\neq 0$, then necessarily

$$\frac{|G\cap Q|_{w_{t,\mathcal{P}}}}{\ell(Q)^t} \leqslant 16 \frac{|G\cap Q|_{w_{t,\mathcal{P}}}}{|Q|_{w_{t,\mathcal{P}}}} \leqslant 32 \frac{|G|_{w_{t,\mathcal{P}}}}{|F|_{w_{t,\mathcal{P}}}}.$$

In passing to (4.10), we have used that for any fixed scale $2^{-\ell}$, there are at most 4 cubes $Q\in\mathcal{Q}'$ such that $Q\cap P\neq\varnothing$ and $\ell(Q)=2^{-\ell}$, and also that any such Q satisfies $\ell(Q)\geqslant\ell(P)$, since Q is non-local. Note that C_t and C_t' are increasing functions of t.

5. The proof of Lemma 2.1

We use a familiar scheme, which we recall here. We have already seen how to approximate the t-Hausdorff content of a set E by a finite union of cubes. We can therefore assume that E is in fact a finite union of cubes, and we approximate the Hausdorff content of the image of E. Applying Stoilow factorization methods, a normalized version of the mapping ϕ is written as $\phi=\phi_1\circ h$, where both $h,\phi_1:\mathbb{C}\to\mathbb{C}$ are principal K-quasiconformal mappings, such that h is conformal in the complement of the set E and ϕ_1 is conformal on the set $F=h(E)$. One then studies the mapping properties of the two functions ϕ_1 and h separately, referred to the 'conformal inside' and the 'conformal outside' parts, respectively. Recall that a principal K-quasiconformal mapping is a K-quasiconformal mapping that is conformal outside $\bar{\mathbb{D}}$ and is normalized by $\phi(z)-z=O(1/|z|)$ as $|z|\to\infty$.

The conformal inside part has already been addressed, in [3], and we recall the relevant result in Theorem 5.3 below. The conformal outside part is new, and the point we turn to now.

The following lemma is often used in the theory of extrapolation of A_p weights, and we use it in a similar way to the way it is used in that theory.

LEMMA 5.1. *Let* $f,g\geqslant 0$ *be measurable functions. Then*, *if* $0<p<1$,

$$\int fg \geqslant \|f\|_p \|g\|_{p'}, \tag{5.1}$$

where $1/p+1/p'=1$ (*and hence* $p'<0$),

$$\|f\|_p = \left(\int |f|^p\right)^{1/p} \quad \textit{and} \quad \|g\|_{p'} = \left(\int |g|^{p'}\right)^{1/p'} = \frac{1}{\left(\int \dfrac{1}{|g|^{-p'}}\right)^{1/(-p')}}.$$

As a consequence,

$$\|f\|_p = \inf_{g:\|g\|_{p'}=1} \int fg. \tag{5.2}$$

Proof. The inequality (5.1) follows easily from the usual Hölder's inequality (i.e. with $p>1$.) The case of equality in (5.2) is attained by taking $g=f^{p-1}/\|f\|_p^{p-1}$. □

We will use the following notation. For a finite collection of pairwise disjoint dyadic cubes $\mathcal{P}=\{P_j\}_{j=1}^N$, let

$$\beta_j = \frac{(\ell(P_j)^2)^{t/2-1}}{\left(\sum_{k=1}^N (\ell(P_k)^2)^{t/2}\right)^{(t/2-1)\cdot 2/t}}. \tag{5.3}$$

(Compare with g in the proof of Lemma 5.1.) Also, let $E=\overline{P}=\bigcup_{j=1}^n P_j$ and let

$$\widetilde{w}_{t,\mathcal{P}}(x) = \sum_{j=1}^n \beta_j \chi_{P_j}(x), \tag{5.4}$$

which is a constant multiple of $w_{t,\mathcal{P}}$, as defined in (2.6).

The conformal outside lemma states that the quasiconformal image of $\mathcal{P}$ has controlled distortion, in the ℓ^t-quasinorm.

Lemma 5.2. *Let* $0<t<2$. *There is a positive constant* ε_0 (*which is a decreasing function of* t) *so that the following holds.*

Let $\mathcal{P}=\{P_j\}_{j=1}^N$ *be a finite collection of dyadic cubes which satisfy the* t*-Carleson packing condition* $\|\mathcal{P}\|_{t\text{-pack}}\leqslant C$. *Assume further that the cubes* P_j *are pairwise disjoint.*

Let $E=\overline{P}=\bigcup_{j=1}^n P_j$ *and let* $f:\mathbb{C}\to\mathbb{C}$ *be a principal* K*-quasiconformal mapping which is conformal outside the compact set* E, *with*

$$\frac{K-1}{K+1}<\varepsilon_0.$$

Then, there is a constant $C(K,t)$ *which depends only on* K *and* t (*which, for fixed* K, *is an increasing function of* t) *such that*

$$\sum_{j=1}^N \operatorname{diam}(f(P_j))^t \leqslant C(K,t)\sum_{j=1}^N \ell(P_j)^t. \tag{5.5}$$

Prause [19] proved results somewhat in the spirit of Lemma 5.2 above, but for different Hausdorff measures, which give a weaker conclusion than the statement (1.3). Our lemma, and in particular the hypothesis on t-packing, is informed by the counterexample of Bishop [7].

Proof. By Lemma 5.1, with β_j as in (5.3) and $\widetilde{w}_{t,\mathcal{P}}(x)$ as in (5.4), by quasi-symmetry we get

$$\begin{aligned}\biggl(\sum_{j=1}^N \operatorname{diam}(f(P_j))^t\biggr)^{2/t} &= \inf_{\substack{\alpha_j>0\\ 1=\|\{\alpha_j\}\|_{\ell(t/2)'}}} \sum_{j=1}^N \operatorname{diam}(f(P_j))^2\alpha_j\\ &\leqslant \sum_{j=1}^N \operatorname{diam}(f(P_j))^2\beta_j\\ &\leqslant C(K)\int_E J(z,f)\widetilde{w}_{t,\mathcal{P}}(z)\,dA(z).\end{aligned} \tag{5.6}$$

Here $J(z,f)$ denotes the Jacobian (determinant) of f at z.

We follow Astala's approach for his area distortion theorem [2, p. 50] (see also [4]), equipped with the new results of this paper. The central role of the Beurling operator is indicated by the identity

$$f_z = 1+\mathcal{S}(f_{\bar z}). \tag{5.7}$$

Using the trivial inequality $|2\operatorname{Re} a|\leqslant 2|a|\leqslant|a|^2+1$, and that $J(z,f)=|f_z|^2-|f_{\bar z}|^2$ (see e.g. (9) in [1, p. 6], or [4]), we can estimate

$$\begin{aligned}\int_E J(z,f)\widetilde{w}_{t,\mathcal{P}}(z)\,dA(z) &= \int_E (|f_z|^2-|f_{\bar z}|^2)\widetilde{w}_{t,\mathcal{P}}(z)\,dA(z)\\ &= \int_E (1+2\operatorname{Re}\mathcal{S}(f_{\bar z})+|\mathcal{S}(f_{\bar z})|^2-|f_{\bar z}|^2)\widetilde{w}_{t,\mathcal{P}}(z)\,dA(z)\\ &\leqslant 2\int_E (1+|\mathcal{S}(f_{\bar z})|^2)\widetilde{w}_{t,\mathcal{P}}(z)\,dA(z)\\ &= 2\biggl(\underbrace{\int_E \widetilde{w}_{t,\mathcal{P}}(z)\,dA(z)}_{=:I_1}+\underbrace{\int_E |\mathcal{S}(f_{\bar z})|^2\widetilde{w}_{t,\mathcal{P}}(z)\,dA(z)}_{=:I_2}\biggr).\end{aligned}\tag{5.8}$$

Notice that $I_1=\sum_{j=1}^N \ell(P_j)^2\beta_j$. We shall bound the other term by a multiple of I_1. Indeed, with respect to I_2, since $\widetilde{w}_{t,\mathcal{P}}$ and $w_{t,\mathcal{P}}$ only differ by a multiplicative constant the Beurling operator has the same operator norm on $L^2(\widetilde{w}_{t,\mathcal{P}})$ and $L^2(w_{t,\mathcal{P}})$. And so, by Proposition 2.3,

$$I_2=\int_E |\mathcal{S}(f_{\bar z})|^2\widetilde{w}_{t,\mathcal{P}}(z)\,dA(z)\leqslant C(t)\underbrace{\int_E |f_{\bar z}|^2\widetilde{w}_{t,\mathcal{P}}(z)\,dA(z)}_{=:I_3}.\tag{5.9}$$

Turning to I_3, the Beurling operator is again decisive. Recall the representation of $f_{\bar z}$ as a power series in the Beltrami coefficient μ. Namely,

$$f_{\bar z}=\mu f_z=\mu+\mu\mathcal{S}(\mu)+\mu\mathcal{S}(\mu\mathcal{S}(\mu))+\dots.\tag{5.10}$$

This is obtained upon multiplying (5.7) by μ, writing $f_{\bar z}=(\operatorname{Id}-\mu\mathcal{S})^{-1}(\mu)$ and using the standard Neumann series

$$(\operatorname{Id}-\mu\mathcal{S})^{-1}=\operatorname{Id}+\mu\mathcal{S}+\mu\mathcal{S}\mu\mathcal{S}+\mu\mathcal{S}\mu\mathcal{S}\mu\mathcal{S}+\dots.\tag{5.11}$$

As we shall see, this series converges in $L^2(w_{t,\mathcal{P}})$ for small (depending on t) $\|\mu\|_\infty$ by Proposition 2.3.

Observe the two inequalities

$$\biggl(\int_E |\mu|^2\widetilde{w}_{t,\mathcal{P}}(z)\,dA(z)\biggr)^{1/2}\leqslant\|\mu\|_\infty\biggl(\int_E \chi_E\widetilde{w}_{t,\mathcal{P}}(z)\,dA(z)\biggr)^{1/2}=\|\mu\|_\infty I_1^{1/2},\tag{5.12}$$

$$\biggl(\int_E |\mu\mathcal{S}(g)|^2\widetilde{w}_{t,\mathcal{P}}(z)\,dA(z)\biggr)^{1/2}\leqslant\|\mu\|_\infty\|\mathcal{S}\|_{L^2(\widetilde{w}_{t,\mathcal{P}})}\biggl(\int_E |g|^2\widetilde{w}_{t,\mathcal{P}}(z)\,dA(z)\biggr)^{1/2}.\tag{5.13}$$

The second inequality is applied to the sequence of functions

$$g=\mu, \quad g=\mu\mathcal{S}(\mu), \quad g=\mu\mathcal{S}(\mu\mathcal{S}(\mu))$$

and so on. Using the triangle inequality in (5.10) in the $L^2(\widetilde{w}_{t,\mathcal{P}})$ norm gives

$$I_3^{1/2} \leqslant \|\mu\|_\infty \biggl(\sum_{n=1}^{\infty} (\|\mu\|_\infty \|\mathcal{S}\|_{L^2(\widetilde{w}_{t,\mathcal{P}})})^n \biggr) I_1^{1/2}. \tag{5.14}$$

The middle term on the right is bounded if we demand that

$$\|\mu\|_\infty < \varepsilon_0 = (2\|\mathcal{S}\|_{L^2(\widetilde{w}_{t,\mathcal{P}})\to L^2(\widetilde{w}_{t,\mathcal{P}})})^{-1} < 1. \tag{5.15}$$

This is the ε_0 required in the statement of Lemma 5.2 (and hence ε_0 is a decreasing function of t). It follows that

$$I_3 \leqslant I_1. \tag{5.16}$$

From (5.6), (5.8), (5.9) and (5.16), it follows that

$$\biggl(\sum_{j=1}^{N} \operatorname{diam}(f(P_j))^t \biggr)^{2/t} \leqslant C(K) \int_E J(z,f)\,\widetilde{w}_{t,\mathcal{P}}(z)\,dA(z) \leqslant C'(K,t) I_1. \tag{5.17}$$

It remains to bound I_1 by the right-hand side of (5.5).

But it follows by construction (recall the parenthetical comment right after (5.3)) that

$$I_1 = \sum_{j=1}^{N} \ell(P_j)^2 \beta_j = \|\{\ell(P_j)^2\}_{j=1}^N\|_{\ell^{t/2}} = \biggl(\sum_{j=1}^{N} \ell(P_j)^t \biggr)^{2/t}. \tag{5.18}$$

This completes the proof. □

Recall that it is known how to deal with the quasiconformal map which is "conformal inside". Namely, we recall the following result.

THEOREM 5.3. *Let $\phi:\mathbb{C}\to\mathbb{C}$ be a principal K-quasiconformal mapping which is conformal outside $\mathbb{D}$. Let $\{S_j\}_{j=1}^N$ be a finite family of pairwise disjoint quasi-disks in $\mathbb{D}$, such that $S_j=f(D_j)$ for a single K-quasiconformal map f and for disks* (*or cubes*) *D_j, and assume that ϕ is conformal in $\Omega=\bigcup_j S_j$. Then for any $t\in(0,2]$ and*

$$t' = \frac{2Kt}{2+(K-1)t},$$

we have

$$\biggl(\sum_{j=1}^{N} \operatorname{diam}(\phi(S_j))^{t'} \biggr)^{1/t'} \leqslant C(K) \biggl(\sum_{j=1}^{N} \operatorname{diam}(S_j)^t \biggr)^{1/tK}. \tag{5.19}$$

Theorem 5.3 can be found in [3, (2.6)] stated for disks D_j, but the proof works for K-quasi-disks (more precisely, we use it for "K-quasi-squares", i.e. the image under a single K-quasiconformal map—where K will be typically close to 1—of squares.) It should be emphasized that for a general quasiconformal mapping ϕ we have $J(z,\phi)\in L^p_{\mathrm{loc}}$ only for $p<K/(K-1)$. The improved integrability $p=K/(K-1)$ under the extra assumption that $\phi|_\Omega$ is conformal was shown in [6, Lemma 5.2]. This phenomenon is crucial for the proof of Theorem 5.3, since we are studying Hausdorff measures rather than dimension. Note that Theorem 5.3 is also implicit in [2] (see Corollary 2.3 and the variational principle on p. 48).

At this point we prove Astala's conjecture for the case of small dilatation, Lemma 2.1.

Proof of Lemma 2.1. We first give the argument that allows us to reduce to the usual normalizations. It is a standard argument, but we give it for convenience.

Let τ be a Möbius transformation fixing ∞. The dilatation K of g, let us call it K_g, is the same as that of $g\circ\tau$, i.e. $K_g=K_{g\circ\tau}$. Also,

$$\mathcal{H}^t(E)=0 \quad \text{if and only if} \quad \mathcal{H}^t(\tau(E))=0.$$

Consequently, without loss of generality, we may assume that

$$E\subset\Big(\frac{1}{32},\frac{1}{16}\Big)^2\subset\frac{1}{8}\mathbb{D}.$$

Let μ_g be the Beltrami coefficient for g. Let φ be the (unique) principal homeomorphic solution to the Beltrami equation

$$\bar\partial\varphi=(\chi_{\mathbb{D}}\mu_g)\partial\varphi.$$

Then, by Stoilow's factorization, we have that $g=\psi\circ\varphi$, where $K_g=K_\psi=K_\varphi$, both ψ and φ are K-quasiconformal maps, φ is principal and ψ is conformal in $\varphi(\mathbb{D})$.

Since ψ is conformal in a neighbourhood of $\varphi(E)$, by Koebe's distortion theorem (see e.g. [18]),

$$0<c_\psi\leqslant\inf_{\varphi(E)}|\psi'(z)|\leqslant\sup_{\varphi(E)}|\psi'(z)|\leqslant C_\psi<\infty,$$

and hence ψ is bi-Lipschitz in $\varphi(E)$. Therefore,

$$\mathcal{H}^{t'}(S)=0 \quad \text{if and only if} \quad \mathcal{H}^{t'}(\psi(S))=0,$$

for $S\subset\varphi([\frac{1}{32},\frac{1}{16}]^2)$. Consequently, without loss of generality, we may further assume that g is a principal mapping.

Consider $\varepsilon>0$ and use Proposition 2.2, with $m=2$, to obtain a collection of cubes $\mathcal{P}=\{P_i\}$ satisfying the conclusions of Proposition 2.2 with respect to the compact set E. Write

$$\Omega=\Big(\bigcup_i P_i\Big)^{\circ}.$$

Following [2], decompose $g=\phi\circ f$, where ϕ and f are principal K-quasiconformal mappings, f is conformal outside $\bar{\Omega}$, and ϕ is conformal in $f(\Omega)\cup(\mathbb{C}\setminus\mathbb{D})$. Recall that Lemma 5.2 only applies to quasiconformal mappings with dilatation (by which we mean $\|\mu\|_\infty$) at most ε_0. If we assume that the dilatation of g is at most ε_0, then the dilatation of f satisfies the same bound, so that Lemma 5.2 applies to it.

Then, by quasi-symmetry, Theorem 5.3 and Lemma 5.2,

$$\begin{aligned}
\mathcal{H}^{t'}_\infty(gE) &\leqslant \mathcal{H}^{t'}_\infty\Big(g\Big(\bigcup_i 12P_i\Big)\Big) \\
&\leqslant \sum_i \operatorname{diam}(g(12P_i))^{t'} \\
&\leqslant C(K)\sum_i \operatorname{diam}(g(P_i))^{t'} \\
&\leqslant C(K)\Big(\sum_i \operatorname{diam}(f(P_i))^{t}\Big)^{t'/tK} \\
&\leqslant C(K,t)\Big(\sum_i \ell(P_i)^{t}\Big)^{t'/tK} \\
&\leqslant C(K,t)(\mathcal{H}^t_\infty(E)+\varepsilon)^{t'/tK} \\
&\leqslant C(K,t)\varepsilon^{t'/tK}.
\end{aligned}$$

The parameter $\varepsilon>0$ was arbitrary, so the proof of Lemma 2.1 is complete. □

Remark 5.20. The proof of Lemma 2.1 actually gives the following quantitative estimate for Hausdorff content. Let $0<t<2$ and

$$t'=\frac{2Kt}{2+(K-1)t}.$$

Assume that f is a principal K-quasiconformal mapping with

$$\frac{K-1}{K+1}\leqslant \varkappa_0(t),$$

and let $E\subset\big(\frac{1}{32},\frac{1}{16}\big)^2$ be compact. Then

$$\mathcal{H}^{t'}_\infty(fE)\leqslant C(\varkappa_0(t))\mathcal{H}^t_\infty(E)^{t'/tK}. \tag{5.21}$$

We claim that this can be rewritten in the following invariant form. Assume now that f is a K-quasiconformal mapping with

$$\frac{K-1}{K+1} \leqslant \varkappa_0(t),$$

and let E be a compact set contained in a ball B. Let t and t' be as in (5.21). Then

$$\frac{\mathcal{H}^{t'}_\infty(fE)}{[\operatorname{diam} fB]^{t'}} \leqslant C(\varkappa_0(t)) \left(\frac{\mathcal{H}^t_\infty(E)}{[\operatorname{diam} B]^t} \right)^{t'/tK}. \tag{5.22}$$

Indeed this follows using the method of Corollary 10 in [5] (see also [4]). Finally, for arbitrary $K>1$, iteration of (5.22) (with $\varkappa_0(t')$ instead of $\varkappa_0(t)$) shows that (5.22) holds with $C(\varkappa_0(t))$ replaced by $C(K,t)$.

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