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Correction to "Distinguished varieties"

by

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Lemma 4.6 is incorrect. The theorem it supports, Theorem 4.1, is correct as stated, and can be proved with a slight modification of the argument in the paper.

The error in the lemma is that there can be points of the distinguished variety

$$V=\{(z,w)\in\mathbb{D}^2:\det[\Psi(z)-wI]=0\},$$

where the dimension of the null space of $\Psi(z)-wI$ is discontinuous, and at these points one may not be able to choose $\hat{u}^1$ continuously.

If these singularities are disjoint from the set of nodes $\{\lambda_1,...,\lambda_N\}$, then the proof of Theorem 4.1 is not materially affected. However, to include the case that a node be a singular point, we must modify the argument, and replace Lemma 4.6 with a correct version, Lemma 4.16 below.

Lemma 4.16. *Every admissible kernel on a set* $\{\lambda_1,...,\lambda_N\}$ *can be extended to an admissible kernel* k *on a distinguished variety* V *that contains the points* $\lambda_1,...,\lambda_N$. V *can be represented as*

$$V=\{(z,w)\in\mathbb{D}^2:\det[\Psi(z)-wI]=0\}$$

for some matrix-valued inner function Ψ. *Moreover*, *the extension can be chosen in such a way that*

$$k(z,w)=s_{\bar{z}}\otimes\hat{u}^1(z,w),$$

where each vector $\hat{u}^1(z,w)$ *is in the null-space of* $\det[\Psi(z)-wI]$, *and so that*, *at each node* λ_j, *there are* q_j *sequences*

$$\{\alpha_m,\beta_{p,m}\}_{m=1}^{\infty},\quad 1\leqslant p\leqslant q_j,$$

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that converge to λ_j *and such that the vector* $\hat{u}^1(\lambda_j)=u_j^1$ *is the limit of vectors in the linear span of*

$$\{\hat{u}^1(\alpha_m,\beta_{p,m}) : 1 \leqslant p \leqslant q_j\}.$$

Proof. Everything in the proof of Lemma 4.6 is correct except for the assertion that $(\hat{u}^1, \hat{u}^2)$ can be chosen continuously. We wish to show that these vectors can be chosen so that the conclusion of Lemma 4.16 holds.

Fix some node, λ_1 say. By Gaussian elimination, after permutation of the coordinates, there are analytic functions f_{ij}, $1 \leqslant i \leqslant j \leqslant d_1$, on a neighborhood of λ_1 such that

$$\operatorname{Ker}(\Psi(z)-wI) = \operatorname{Ker}\begin{pmatrix} f_{11}(z,w) & f_{12}(z,w) & \dots & f_{1d_1}(z,w) \\ 0 & f_{22}(z,w) & \dots & f_{2d_1}(z,w) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_{d_1d_1}(z,w) \end{pmatrix},$$

and, moreover, the diagonal functions f_{11} through f_{rr} do not vanish at λ_1, and the functions f_{jj} do vanish for $r < j \leqslant d_1$.

Note that the zero sets of f_{jj} are (unions of) sheets of V near λ_1. Choose any sheet of the variety near λ_1. With the exception of a possible jump at λ_1, the dimension of $\operatorname{Ker}(\Psi(z)-wI)$ will be locally constant, t say. After a permutation of the last d_1-r coordinates if necessary, we may assume that the sheet corresponds to the vanishing of $f_{r+1,r+1}$ through $f_{r+t,r+t}$ (we are not assuming that they are coprime). Now choose $\hat{u}^1$ on this sheet so that its $(r+1)$-st through $(r+t)$-th coordinates agree with those of u_1^1, its $(r+t+1)$-st through d_1-th coordinates are zero, and it lies in

$$\operatorname{Ker}\begin{pmatrix} f_{11}(z,w) & f_{12}(z,w) & \dots & f_{1d_1}(z,w) \\ 0 & f_{22}(z,w) & \dots & f_{2d_1}(z,w) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_{d_1d_1}(z,w) \end{pmatrix}.$$

Repeat this for each sheet, and the sum of the chosen vectors will converge to u_1^1. $\square$

To prove Theorem 4.1, given Lemma 4.16, we will change the second and last paragraphs of Step 2.

We claim that at every node λ_j, the formula (4.10) uniquely defines the solution to the Pick problem on *some* sheet through λ_j. The union of all the irreducible components containing these particular sheets will give a distinguished variety containing every node on which the solution is unique.

Indeed, if the denominator in equation (4.10) vanished on every sheet of V that meets λ_j, then any linear combination of the q vectors

$$\{w_j \widehat{K}((\alpha_m, \beta_{p,m}), \lambda_j)\}_{j=1}^N$$

would be orthogonal to γ, and by taking a limit we obtain (4.11) again. The remainder of the proof proceeds as before, showing that (4.11) would contradict minimality of the problem.

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