Acta Math., 209 (2012), 83–131
DOI: 10.1007/s11511-012-0082-6

Algebraic geometry of topological spaces I

by

Guillermo Cortiñas

Universidad de Buenos Aires
Buenos Aires, Argentina

Andreas Thom

Universität Leipzig
Leipzig, Germany

Contents

Cortiñas' research was partly supported by grants PICT 2006-00836, UBACyT-X057 and MTM2007-64704. Thom's research was partly supported by the DFG (GK *Gruppen und Geometrie* Göttingen).

1. Introduction

In his foundational paper [39], Jean-Pierre Serre asked whether all finitely generated projective modules over the polynomial ring $k[t_1,...,t_n]$ over a field k are free. This question, which became known as Serre's conjecture, remained open for about twenty years. An affirmative answer was given independently by Daniel Quillen [35] and Andrei Suslin [41]. Richard G. Swan observed in [45] that the Quillen–Suslin theorem implies that all finitely generated projective modules over the Laurent polynomial ring $k[t_1,t_1^{-1},...,t_n,t_n^{-1}]$ are free. This was later generalized by Joseph Gubeladze [19], [20], who proved, among other things, that if M is an abelian, cancellative, torsion-free, semi-normal monoid, then every finitely generated projective module over $k[M]$ is free. Quillen–Suslin's theorem and Swan's theorem are the special cases $M=\mathbb{N}_0^n$ and $M=\mathbb{Z}^n$ of Gubeladze's result. On the other hand, it is classical that if X is a contractible compact Hausdorff space, then all finitely generated projective modules over the algebra $C(X)$ of complex-valued continuous functions on X—which by another theorem of Swan, are the same thing as locally trivial complex vector bundles on X—are free. In this paper we prove the following result (see Theorem 6.3).

THEOREM 1.1. *Let X be a contractible compact space and M be a countable, cancellative, torsion-free, semi-normal, abelian monoid. Then every finitely generated projective module over $C(X)[M]$ is free.*

Moreover we show (Theorem 6.10) that bundles of finitely generated free $\mathbb{C}[M]$-modules over a not necessarily contractible, compact Hausdorff space X which are direct summands of trivial bundles, are locally trivial. The case $M=\mathbb{N}_0^n$ of Theorem 1.1 gives a parameterized version of Quillen–Suslin's theorem. The case $M=\mathbb{Z}^n$ is connected with a conjecture of Jonathan Rosenberg [38] which predicts that the negative algebraic K-theory groups of $C(X)$ are homotopy invariant for compact Hausdorff spaces X. Indeed, if R is any ring, then the negative algebraic K-theory group $K_{-n}(R)$ is defined as a certain canonical direct summand of $K_0(R[\mathbb{Z}^n])$; the theorem above thus implies that $K_{-n}(C(X))=0$ if X is contractible. Using this and excision, we derive the following result (see Theorem 6.5).

THEOREM 1.2. *Let $\mathfrak{Comp}$ be the category of compact Hausdorff spaces and let $n>0$. Then the functor*

$$\mathfrak{Comp} \longrightarrow \mathfrak{Ab},$$
$$X \longmapsto K_{-n}(C(X)),$$

is homotopy invariant.

A partial result in the direction of Theorem 1.2 was obtained by Eric Friedlander and Mark E. Walker in [15]. They proved that $K_{-n}(C(\Delta^p))=0$ for $p\geqslant 0$, $n>0$. In §7.2 we give a second proof of Theorem 1.2 which uses the Friedlander–Walker result. Elaborating on their techniques, and combining them with our own methods, we obtain the following general criterion for homotopy invariance (see Theorem 7.6).

THEOREM 1.3. *Let F be a functor on the category $\mathfrak{Comm}/\mathbb{C}$ of commutative $\mathbb{C}$-algebras with values in the category $\mathfrak{Ab}$ of abelian groups. Assume that the following three conditions are satisfied*:

(i) *F is split-exact on C^*-algebras*;

(ii) *F vanishes on coordinate rings of smooth affine varieties*;

(iii) *F commutes with filtering colimits.*

Then the functor

$$\mathfrak{Comp} \longrightarrow \mathfrak{Ab},$$
$$X \longmapsto F(C(X)),$$

is homotopy invariant and $F(C(X))=0$ for X contractible.

Observe that K_{-n} satisfies all the hypothesis of the theorem above ($n>0$). This gives a third proof of Theorem 1.2. We also use Theorem 1.3 to prove the following vanishing theorem for homology theories (see Theorem 7.7). In this paper a homology theory on a category $\mathfrak{C}$ of algebras is simply a functor $E:\mathfrak{C}\to\mathfrak{Spt}$ to the category of spectra which preserves finite products up to homotopy.

THEOREM 1.4. *Let $E:\mathfrak{Comm}/\mathbb{C}\to\mathfrak{Spt}$ be a homology theory of commutative $\mathbb{C}$-algebras and let $n_0\in\mathbb{Z}$. Assume that the following three conditions are satisfied*:

(i) *E satisfies excision on commutative C^*-algebras*;

(ii) *E_n commutes with filtering colimits for $n\geqslant n_0$*;

(iii) *$E_n(\mathcal{O}(V))=0$ for each smooth affine algebraic variety V for $n\geqslant n_0$.*

Then $E_n(A)=0$ for every commutative C^*-algebra A for $n\geqslant n_0$.

Recall that a ring R is called K-*regular* if

$$\operatorname{coker}(K_n(R)\to K_n(R[t_1,...,t_p]))=0,\quad p\geqslant 1,\ n\in\mathbb{Z}.$$

As an application of Theorem 1.4 to the homology theory

$$F^p(A)=\operatorname{hocofiber}(K(A\otimes_{\mathbb{C}}\mathcal{O}(V))\to K(A\otimes_{\mathbb{C}}\mathcal{O}(V)[t_1,...,t_p])),$$

where V is a smooth algebraic variety, we obtain the following result (Theorem 8.1).

THEOREM 1.5. *Let V be a smooth affine algebraic variety over $\mathbb{C}$, $R=\mathcal{O}(V)$ and A be a commutative C^*-algebra. Then $A\otimes_{\mathbb{C}}R$ is K-regular.*

The case $R=\mathbb{C}$ of the previous result was discovered by Jonathan Rosenberg, see Remark 8.2.

We also give an application of Theorem 1.4 which concerns the algebraic Hochschild and cyclic homology of $C(X)$. We use the theorem in combination with the celebrated results of Gerhard Hochschild, Bertram Kostant and Alex Rosenberg [24] and of Daniel Quillen and Jean-Louis Loday [30] on the Hochschild and cyclic homology of smooth affine algebraic varieties, and the spectral sequence of Christian Kassel and Arne Sletsjøe [27], to prove the following result (see Theorem 8.6 for a full statement of our result and for the appropriate definitions).

THEOREM 1.6. *Let $k\subset\mathbb{C}$ be a subfield. Write $\mathrm{HH}_*(\cdot/k)$, $\mathrm{HC}_*(\cdot/k)$, $\Omega^*_{\cdot/k}$, d and $H^*_{\mathrm{dR}}(\cdot/k)$ for algebraic Hochschild and cyclic homology, algebraic Kähler differential forms, exterior differentiation, and algebraic de Rham cohomology, all taken relative to the field k. Let X be a compact Hausdorff space. Then, for $n\in\mathbb{Z}$,*

$$\begin{aligned}\mathrm{HH}_n(C(X)/k)&=\Omega^n_{C(X)/k},\\ \mathrm{HC}_n(C(X)/k)&=\frac{\Omega^n_{C(X)/k}}{d\Omega^{n-1}_{C(X)/k}}\oplus\bigoplus_{2\leqslant 2p\leqslant n}H^{n-2p}_{\mathrm{dR}}(C(X)/k).\end{aligned}$$

We also apply Theorem 1.4 to the K-theoretic isomorphism conjecture of Farrell–Jones and to the Beĭlinson–Soulé conjecture. The K-theoretic isomorphism conjecture for the group Γ with coefficients in a ring R asserts that a certain assembly map

$$\mathcal{A}^\Gamma(R)\colon\mathbb{H}^\Gamma(E_{\mathcal{VC}}(\Gamma),K(R))\longrightarrow K(R[\Gamma])$$

is an equivalence. Applying Theorem 1.4 to the cofiber of the assembly map, we obtain that if $\mathcal{A}^\Gamma(\mathcal{O}(V))$ is an equivalence for each smooth affine algebraic variety V over $\mathbb{C}$, then

$\mathcal{A}^\Gamma(A)$ is an equivalence for any commutative C^*-algebra A. The (rational) Beĭlinson–Soulé conjecture concerns the decomposition of the rational K-theory of a commutative ring into the sum of eigenspaces of the Adams operations

$$K_n(R)\otimes\mathbb{Q}=\bigoplus_{i\geqslant 0}K_n^{(i)}(R).$$

The conjecture asserts that if R is regular noetherian, then

$$K_n^{(i)}(R)=0 \quad \text{for } n\geqslant \max\{1,2i\}.$$

It is well known that the validity of the conjecture for $R=\mathbb{C}$ would imply that it also holds for $R=\mathcal{O}(V)$ whenever V is a smooth algebraic variety over $\mathbb{C}$. We use Theorem 1.4 to show that the validity of the conjecture for $\mathbb{C}$ would further imply that it holds for every commutative C^*-algebra.

Next we give an idea of the proofs of our main results, Theorems 1.1 and 1.3.

The basic idea of the proof of Theorem 1.1 goes back to Rosenberg's article [36] and ultimately to the usual proof of the fact that locally trivial bundles over a contractible compact Hausdorff space are trivial. It consists of translating the question of the freedom of projective modules into a lifting problem:

$$\begin{array}{ccccc} & & & & \mathrm{GL}(\mathbb{C}[M]) \\ & & & \nearrow & \big\downarrow \pi \\ X & \xrightarrow[e]{} & P_n(\mathbb{C}[M]) & \xleftarrow[\iota]{} & \dfrac{\mathrm{GL}(\mathbb{C}[M])}{\mathrm{GL}_{[1,n]}(\mathbb{C}[M])\times\mathrm{GL}_{[n+1,\infty)}(\mathbb{C}[M])}. \end{array} \tag{1.1}$$

Here we think of a projective module of constant rank n over $C(X)$ as a map e to the set of all rank-n idempotent matrices, which by Gubeladze's theorem is the same as the set $P_n(\mathbb{C}[M])$ of those matrices which are conjugate to the diagonal matrix $1_n\oplus 0_\infty$. Thus $g\mapsto g(1_n\oplus 0_\infty)g^{-1}$ defines a surjective map $\mathrm{GL}(\mathbb{C}[M])\to P_n(\mathbb{C}[M])$ which identifies the latter set with the quotient of $\mathrm{GL}(\mathbb{C}[M])$ by the stabilizer of $1_n\oplus 0_\infty$, which is precisely the subgroup $\mathrm{GL}_{[1,n]}(\mathbb{C}[M])\times\mathrm{GL}_{[n+1,\infty)}(\mathbb{C}[M])$. For this setup to make sense we need to equip each set involved in (1.1) with a topology in such a way that all maps in the diagram are continuous. Moreover for the lifting problem to have a solution, it will suffice to show that ι is a homeomorphism and that π is a *compact fibration*, i.e. that it restricts to a fibration over each compact subset of the base.

In §4 we show that any countable-dimensional $\mathbb{R}$-algebra R is equipped with a canonical compactly generated topology which makes it into a topological algebra. A subset

$F \subset R$ is closed in this topology if and only if $F \cap B \subset B$ is closed for every compact semi-algebraic subset B of every finite-dimensional subspace of R. In particular this applies to $M_\infty(R)$. The subset $P_n(R) \subset M_\infty(R)$ carries the induced topology, and the map e of (1.1) is continuous for this topology. The group $\mathrm{GL}(R)$ also carries a topology, generated by the compact semi-algebraic subsets $\mathrm{GL}_n(R)^B$. Here $B \subset M_n(R)$ is any compact semi-algebraic subset as before, and $\mathrm{GL}_n(R)^B$ consists of those $n \times n$ invertible matrices g for which both g and g^{-1} belong to B. The subgroup $\mathrm{GL}_{[1,n]}(R) \times \mathrm{GL}_{[n+1,\infty)}(R) \subset \mathrm{GL}(R)$ turns out to be closed, and we show in §4.2—with the aid of Gregory Brumfiel's theorem on quotients of semi-algebraic sets (see Theorem 3.4)—that for a topological group G of this kind, the quotient G/H by a closed subgroup H is again compactly generated by the images of the compact semi-algebraic subsets defining the topology of G, and these images are again compact, semi-algebraic subsets. Moreover the restriction of the projection $\pi: G \to G/H$ over each compact semi-algebraic subset $S \subset G$ is semi-algebraic.

We also show (Theorem 4.19) that π is a compact fibration. This boils down to showing that if $S \subset G$ is compact semi-algebraic, and $T = f(S)$, then we can find an open covering of T such that π has a section over each open set in the covering. Next we observe that if U is any space, then the group $\mathrm{map}(U, G)$ acts on the set $\mathrm{map}(U, G/H)$, and a map $U \to G/H$ lifts to $U \to G$ if and only if its class in the quotient $F(U) = \mathrm{map}(U, G/H)/\mathrm{map}(U, G)$ is the class of the trivial element: the constant map $u \mapsto H$ $(u \in U)$. For example the class of the composition of π with the inclusion $S \subset G$ is the trivial element of $F(S)$. Hence if $p = \pi|_S: S \to T$, then $F(p)$ sends the inclusion $T \subset G/H$ to the trivial element of $F(S)$.

In §2 we introduce a notion of (weak) split-exactness for contravariant functors of topological spaces with values in pointed sets; for example the functor F introduced above is split-exact (Lemma 2.2). The key technical tool for proving that π is a fibration is the following result (see Theorem 3.14); its proof uses the good topological properties of semi-algebraic sets and maps, especially Hardt's triviality theorem (Theorem 3.10).

THEOREM 1.7. *Let T be a compact semi-algebraic subset of $\mathbb{R}^k$. Let S be a semi-algebraic set and let $f: S \to T$ be a proper continuous semi-algebraic surjection. Then, there exists a semi-algebraic triangulation of T such that for every weakly split-exact contravariant functor F from the category $\mathfrak{Pol}$ of compact polyhedra to the category $\mathfrak{Set}_-$ of pointed sets, and every simplex Δ^n in the triangulation, we have*

$$\ker(F(\Delta^n) \to F(f^{-1}(\Delta^n))) = *.$$

Here ker is the kernel in the category of pointed sets, i.e. the fiber over the base point. In our situation Theorem 1.7 applies to show that there is a triangulation of $T \subset G/H$ such that the projection π has a section over each simplex in the triangulation. A standard argument now shows that T has an open covering (by open stars of a subdivision of the previous triangulation) such that π has section over each open set in the covering. Thus in diagram (1.1) we have that π is a compact fibration and that e is continuous. The map

$$\iota: \frac{\mathrm{GL}(R)}{\mathrm{GL}_{[1,n]}(R)\times \mathrm{GL}_{[n+1,\infty)}(R)} \longrightarrow P_n(R)$$

is continuous for every countable-dimensional $\mathbb{R}$-algebra R (see §5.1). We show in Proposition 5.5 that it is a homeomorphism whenever the map

$$K_0(\ell^\infty(R)) \longrightarrow \prod_{n\geqslant 1} K_0(R) \tag{1.2}$$

is injective. Here $\ell^\infty(R)$ is the set of all sequences $\mathbb{N}\to R$ whose image is contained in one of the compact semi-algebraic subsets $B\subset R$ which define the topology of R; it is isomorphic to $\ell^\infty(\mathbb{R})\otimes R$ (Lemma 5.4). The algebraic compactness theorem (Theorem 5.9) says that if R is a countable-dimensional $\mathbb{C}$-algebra such that

$$K_0(\mathcal{O}(V)) \xrightarrow{\sim} K_0(\mathcal{O}(V)\otimes_{\mathbb{C}} R) \quad \text{for all smooth affine } V, \tag{1.3}$$

then (1.2) is injective. A theorem of Swan (see Theorem 6.2) implies that $R=\mathbb{C}[M]$ satisfies (1.3). Thus the map ι of diagram (1.1) is a homeomorphism. This concludes the sketch of the proof of Theorem 1.1.

The proof of the algebraic compactness theorem uses the following result (see Theorem 5.7).

Theorem 1.8. *Let F and G be functors from commutative $\mathbb{C}$-algebras to sets. Assume that both F and G preserve filtering colimits. Let $\tau: F\to G$ be a natural transformation. Assume that $\tau(\mathcal{O}(V))$ is injective (resp. surjective) for each smooth affine algebraic variety V over $\mathbb{C}$. Then $\tau(\ell^\infty(\mathbb{C}))$ is injective (resp. surjective).*

The proof of Theorem 1.8 uses a technique which we call *algebraic approximation*, which we now explain. Any commutative $\mathbb{C}$-algebra is the colimit of its subalgebras of finite type, which form a filtered system. If the algebra contains no nilpotent elements, then each of its subalgebras of finite type is of the form $\mathcal{O}(Y)$ for an *affine variety* Y, by which we mean a reduced affine scheme of finite type over $\mathbb{C}$. If $\mathbb{C}[f_1, ..., f_n]\subset\ell^\infty(\mathbb{C})$ is the subalgebra generated by $f_1, ..., f_n$, and $\mathbb{C}[f_1, ..., f_n]\cong\mathcal{O}(Y)$, then Y is isomorphic to a closed subvariety of $\mathbb{C}^n$, and $f=(f_1, ..., f_n)$ defines a map from $\mathbb{N}$ to a precompact subset of the space Y_{an} of closed points of Y equipped with the topology inherited by the euclidean topology on $\mathbb{C}^n$. The space Y_{an} is equipped with the structure of a (possibly singular) analytic variety, whence the subscript. Summing up, we have

$$\ell^\infty(\mathbb{C}) = \operatorname*{colim}_{\mathbb{N}\to Y_{\mathrm{an}}} \mathcal{O}(Y), \tag{1.4}$$

where the colimit runs over all affine varieties Y and all maps with precompact image. The proof of Theorem 1.8 consists of showing that in (1.4) we can restrict to maps $\mathbb{N}\to V$ with V smooth. This uses Hironaka's desingularization [23] to lift a map $f:\mathbb{N}\to V$ with V affine and singular, to a map $f':\mathbb{N}\to\widetilde{V}$ with $\widetilde{V}$ smooth and possibly non-affine, and Jouanoulou's device [25] to further lift f' to a map $f'':\mathbb{N}\to W$ with W smooth and affine.

The idea of algebraic approximation appears in the work of Rosenberg [36], [37], [38], and later in the article of Friedlander and Walker [15]. One source of inspiration is the work of Suslin [42]. In [42], Suslin studies an inclusion of algebraically closed fields $L\subset K$ and analyzes K successfully in terms of its finitely generated L-subalgebras.

Next we sketch the proof of Theorem 1.3. The first step is to reduce to the polyhedral case. For this we use Theorem 1.9 below, proved in Theorem 7.2. Its proof uses another algebraic approximation argument, together with a result of Allan Calder and Jerrold Siegel, which says that the right Kan extension to $\mathfrak{Comp}$ of a homotopy invariant functor defined on $\mathfrak{Pol}$ is homotopy invariant on $\mathfrak{Comp}$.

THEOREM 1.9. *Let $F:\mathfrak{Comm}\to\mathfrak{Ab}$ be a functor. Assume that F satisfies each of the following conditions*:

(i) *F commutes with filtered colimits*;

(ii) *F is split-exact on C^*-algebras*;

(iii) *the functor $\mathfrak{Pol}\to\mathfrak{Ab}$, $D\mapsto F(C(D))$, is homotopy invariant.*

Then the functor

$$\begin{aligned}\mathfrak{Comp} &\longrightarrow \mathfrak{Ab},\\ X &\longmapsto F(C(X)),\end{aligned}$$

is homotopy invariant.

Next, Proposition 2.3 says that we can restrict to showing that F vanishes on contractible polyhedra. Since any contractible polyhedron is a retract of its cone, which is a starlike polyhedron, we further reduce to showing that F vanishes on starlike polyhedra. Using excision, we may restrict once more, to proving that $F(\Delta^p)=0$ for all p. For this we follow the strategy used by Friedlander–Walker in [15]. To start, we use algebraic approximation again. We write

$$C(\Delta^p)=\operatorname*{colim}_{\Delta^p\to Y_{\mathrm{an}}}\mathcal{O}(Y), \tag{1.5}$$

where the colimit runs over all continuous maps from Δ^p to affine algebraic varieties, equipped with the euclidean topology. Since F is assumed to vanish on $\mathcal{O}(V)$ for smooth affine V, it would suffice to show that any map $\Delta^p\to Y_{\mathrm{an}}$ factors as $\Delta^p\to V_{\mathrm{an}}\to Y_{\mathrm{an}}$ with

V smooth and affine. Actually using excision again we may restrict to showing this for each simplex in a sufficiently fine triangulation of Δ^p. As in the proof of Theorem 1.1, this is done using Hironaka's desingularization, Jouanoulou's device and Theorem 1.7.

The rest of this paper is organized as follows. In §2 we give the appropriate definitions and first properties of split-exactness. We also recall some facts about algebraic K-theory and cyclic homology, such as the key results of Andrei Suslin and Mariusz Wodzicki on excision for algebraic K-theory and algebraic cyclic homology. In §3, we recall some facts from real algebraic geometry, and prove Theorem 1.7 (Theorem 3.14). Large semi-algebraic groups and their associated compactly generated topological groups are the subject of §4. The main result of this section is the fibration theorem (Theorem 4.19) which says that the quotient map of such a group by a closed subgroup is a compact fibration. Then, §5 is devoted to algebraic compactness, that is, to the problem of giving conditions on a countable-dimensional algebra R so that the map

$$\iota\colon \frac{\mathrm{GL}(R)}{\mathrm{GL}_{[1,n]}(R)\times\mathrm{GL}_{[n+1,\infty)}(R)}\longrightarrow P_n(R)$$

be a homeomorphism. The connection between this problem and the algebra $\ell^\infty(R)$ of bounded sequences is established by Proposition 5.5. Theorem 1.8 is proved in Theorem 5.7. Theorem 5.9 establishes that the map (1.2) is injective whenever (1.3) holds. §6 contains the proofs of Theorems 1.1 and 1.2 (Theorems 6.3 and 6.5). We also show (Theorem 6.10) that if M is a monoid as in Theorem 1.1 then any bundle of finitely generated free $\mathbb{C}[M]$ modules over a compact Hausdorff space which is a direct summand of a trivial bundle is locally trivial. In §7 we deal with homotopy invariance. Theorems 1.9, 1.3 and 1.4 (Theorems 7.2, 7.6 and 7.7) are proved in this section, where also a second proof of Rosenberg's conjecture, using a result of Friedlander and Walker, is given (see §7.2). Finally, §8 is devoted to applications of the homotopy invariance and vanishing homology theorems, including Theorems 1.5 and 1.6 (Theorems 8.1 and 8.6) and also to the applications to the conjectures of Farrell–Jones (Theorems 8.7 and 8.10) and of Beĭlinson–Soulé (Theorem 8.14).

2. Split-exactness, homology theories and excision

2.1. Set-valued split-exact functors on the category of compact Hausdorff spaces

In this section we consider contravariant functors from the category of compact Hausdorff topological spaces to the category $\mathfrak{Set}_-$ of pointed sets. Recall that if T is a pointed set and $f\colon S \to T$ is a map, then

$$\ker f = \{s \in S : f(s) = *\}.$$

We say that a functor $F\colon \mathfrak{Comp} \to \mathfrak{Set}_{-}$ is *split-exact* if for each push-out square

$$\begin{array}{ccc} X_{12} & \xrightarrow{\iota_1} & X_1 \\ {\scriptstyle\iota_2}\downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array} \tag{2.1}$$

of topological spaces with ι_1 or ι_2 split-injective, the map

$$F(X) \longrightarrow F(X_1) \times_{F(X_{12})} F(X_2)$$

is a surjection with trivial kernel. We say that F is *weakly split-exact* if the map above has trivial kernel. In case the functor takes values in abelian groups, the notion of split-exactness above is equivalent to the usual one. For more details on split-exact functors taking values in the category $\mathfrak{Ab}$ of abelian groups, see §2.4.

In the next lemma and elsewhere, if X and Y are topological spaces, we write

$$\operatorname{map}(X, Y) = \{f : X \to Y \text{ continuous}\}$$

for the set of continuous maps from X to Y.

Lemma 2.1. *Let (Y, y) be a pointed topological space. The contravariant functor*

$$X \longmapsto \operatorname{map}(X, Y),$$

from compact Hausdorff topological spaces to pointed sets, is split-exact.

Proof. Note that $\operatorname{map}(X, Y)$ is naturally pointed by the constant map taking the value $y \in Y$. Let (2.1) be a push-out of compact Hausdorff topological spaces and assume that ι_1 is a split-injection. It is sufficient to show that the diagram

$$\begin{array}{ccc} \operatorname{map}(X_{12}, Y) & \longleftarrow & \operatorname{map}(X_1, Y) \\ \uparrow & & \uparrow \\ \operatorname{map}(X_2, Y) & \longleftarrow & \operatorname{map}(X, Y) \end{array}$$

is a pull-back. But this is immediate from the universal property of a push-out. □

Lemma 2.2. *Let $H\subset G$ be an inclusion of topological groups. Then the pointed set* $\mathrm{map}(X,G/H)$ *carries a natural left action of the group* $\mathrm{map}(X,G)$, *and the functor*

$$X\longmapsto \frac{\mathrm{map}(X,G/H)}{\mathrm{map}(X,G)}$$

is split-exact.

Proof. We need to show that the map

$$\frac{\mathrm{map}(X,G/H)}{\mathrm{map}(X,G)}\longrightarrow \frac{\mathrm{map}(X_1,G/H)}{\mathrm{map}(X_1,G)}\times_{\mathrm{map}(X_{12},G/H)/\mathrm{map}(X_{12},G)}\frac{\mathrm{map}(X_2,G/H)}{\mathrm{map}(X_2,G)} \tag{2.2}$$

is a surjection with trivial kernel. Let $f\colon X\to G/H$ be a map such that its pull-backs $f_i\colon X_i\to G/H$ admit continuous lifts $\hat{f}_i\colon X_i\to G$. Although the pull-backs of $\hat{f}_1$ and $\hat{f}_2$ to X_{12} might not agree, we can fix this problem. Let σ be a continuous splitting of the inclusion $X_{12}\hookrightarrow X_1$. Define the map

$$\begin{aligned}\gamma\colon X_1 &\longrightarrow H,\\ x&\longmapsto (\hat{f}_1|_{X_{12}}(\sigma(x)))^{-1}(\hat{f}_2|_{X_{12}}(\sigma(x))).\end{aligned}$$

Note that $\hat{f}_1\gamma$ is still a lift of f_1 and agrees with $\hat{f}_2$ on X_{12}; hence they define a map $\hat{f}\colon X\to G$ which lifts f. This proves that (2.2) has trivial kernel. Let now $f_1\colon X_1\to G/H$ and $f_2\colon X_2\to G/H$ be such that there exists a function $\theta\colon X_{12}\to G$ with $\theta(x)f_1(x)=f_2(x)$ for all $x\in X_{12}$. Using the splitting σ of the inclusion $X_{12}\hookrightarrow X_1$ again, we can extend θ to X_1 to obtain $f_1'(x)=\theta(\sigma(x))f_1(x)$ for $x\in X_1$. Note that f_1' is just another representative of the class of f_1. Since f_1' and f_2 agree on X_{12}, we conclude that there exists a continuous map $f\colon X\to G/H$ which pulls back to f_1' on X_1 and to f_2 on X_2. This proves that (2.2) is surjective. □

Proposition 2.3. *Let $\mathfrak{C}$ be either the category $\mathfrak{Comp}$ of compact Hausdorff spaces or the full subcategory $\mathfrak{Pol}$ of compact polyhedra. Let $F\colon\mathfrak{C}\to\mathfrak{Ab}$ be a split-exact functor. Assume that $F(X)=0$ for contractible $X\in\mathfrak{C}$. Then F is homotopy invariant.*

Proof. We have to prove that if $X\in\mathfrak{C}$ and $1_X\times 0\colon X\to X\times[0,1]$ is the inclusion, then $F(1_X\times 0)\colon F(X\times[0,1])\to F(X)$ is a bijection. Since it is obviously a split-surjection it remains to show that this map is injective. Consider the push-out diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X\times 0} & X\times[0,1]\\ \downarrow & & \downarrow\\ * & \longrightarrow & cX.\end{array}$$

By split-exactness, the map $F(cX)\to P:=F(*)\times_{F(X)}F(X\times[0,1])$ is onto. Since we are also assuming that F vanishes on contractible spaces, we further have $F(*)=F(cX)=0$, whence $P=\ker(F(1_X\times 0))=0$. □

2.2. Algebraic *K*-theory

In the previous subsection we considered contravariant functors on spaces; now we turn our attention to the dual picture of covariant functors from categories of algebras to pointed sets or abelian groups. The most important example for us is algebraic K-theory. Before we go on, we want to quickly recall some definitions and results. Let R be a unital ring. The abelian group $K_0(R)$ is defined to be the Grothendieck group of the monoid of isomorphism classes of finitely generated projective R-modules with direct sum as addition. We define

$$K_n(R) = \pi_n(B\,\mathrm{GL}(R)^+), *) \quad \text{for all } n \geqslant 1,$$

where $X \mapsto X^+$ denotes Quillen's plus-construction [34]. Bass' nil K-groups of a ring are defined as

$$NK_n(R) = \mathrm{coker}(K_n(R) \to K_n(R[t])). \tag{2.3}$$

The so-called fundamental theorem gives an isomorphism

$$K_n(R[t,t^{-1}]) = K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R) \quad \text{for all } n \geqslant 1, \tag{2.4}$$

which holds for all unital rings R. One can use this to define K-groups and nil groups in negative degrees. Indeed, if one puts

$$K_{n-1}(R) = \mathrm{coker}(K_n(R[t]) \oplus K_n(R[t^{-1}]) \to K_n(R[t,t^{-1}])),$$

negative K-groups can be defined inductively. There is a functorial spectrum $K(R)$, such that

$$K_n(R) = \pi_n K(R), \quad n \in \mathbb{Z}. \tag{2.5}$$

This spectrum can be constructed in several equivalent ways (see e.g. [17], [32], [33, §5], [49, §6] and [50]). Functors from the category of algebras to spectra and their properties will be studied in more detail in the next subsection.

A ring R is called K_n-*regular* if the map $K_n(R) \to K_n(R[t_1, ..., t_m])$ is an isomorphism for all m; it is called K-*regular* if it is K_n-regular for all n. It is well known that if R is a regular noetherian ring then R is K-regular and $K_n R = 0$ for $n < 0$. In particular this applies when R is the coordinate ring of a smooth affine algebraic variety over a field. We think of the Laurent polynomial ring $R[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$ as the group ring $R[\mathbb{Z}^n]$ and use the fact that if the natural map $K_0(R) \to K_0(R[\mathbb{Z}^n])$ is an isomorphism for all $n \in \mathbb{N}$, then all negative algebraic K-groups and all (iterated) nil K-groups in negative degrees vanish. This can be proved with an easy induction argument.

Remark 2.4. Iterating the nil-group construction, one obtains the following formula for the K-theory of the polynomial ring in m-variables

$$K_n(R[t_1, ..., t_m]) = \bigoplus_{p=0}^{m} N^p K_n(R) \otimes \bigwedge^p \mathbb{Z}^m. \tag{2.6}$$

Here $\bigwedge^p$ is the exterior power and $N^p K_n(R)$ denotes the iterated nil group defined using the analogue of formula (2.3). Thus a ring R is K_n-regular if and only if $N^p K_n(R)=0$ for all $p \geqslant 0$. In [1] Hyman Bass raised the question of whether the condition that $NK_n(R)=0$ is already sufficient for K_n-regularity. This question was settled in the negative in [9, Theorem 4.1], where an example of a commutative algebra R of finite type over $\mathbb{Q}$ was given such that $NK_0(R)=0$ but $N^2K_0(R)\neq 0$. On the other hand, it was proved [8, Corollary 6.7] (see also [21]) that if R is of finite type over a large field such as $\mathbb{R}$ or $\mathbb{C}$, then $NK_n(R)=0$ does imply that R is K_n-regular. This is already sufficient for our purposes, since the rings this paper is concerned with are algebras over $\mathbb{R}$. For completeness let us remark further that if R is any ring such that $NK_n(R)=0$ for *all* n then R is K-regular, i.e. K_n-regular for all $n\in\mathbb{Z}$. As observed by Jim Davis in [11, Corollary 3] this follows from Frank Quinn's theorem that the Farrell–Jones conjecture is valid for the group $\mathbb{Z}^n$ (see also [8, Theorem 4.2]).

2.3. Homology theories and excision

We consider functors and homology theories of associative, not necessarily unital algebras over a fixed field k of characteristic zero. In what follows, $\mathfrak{C}$ will denote either the category $\mathfrak{Ass}/k$ of associative k-algebras or the full subcategory $\mathfrak{Comm}/k$ of commutative algebras. A *homology theory* on $\mathfrak{C}$ is a functor $E: \mathfrak{C} \to \mathfrak{Spt}$ to the category of spectra which preserves finite products up to homotopy. That is, $E\left(\prod_{i\in I} A_i\right) \to \prod_{i\in I} E(A_i)$ is a weak equivalence for finite I. If $A\in\mathfrak{C}$ and $n\in\mathbb{Z}$, we write $E_n(A)=\pi_n E(A)$ for the nth stable homotopy group. Let E be a homology theory and let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{2.7}$$

be an exact sequence (or extension) in $\mathfrak{C}$. We say that E *satisfies excision* for (2.7), if $E(A)\to E(B)\to E(C)$ is a homotopy fibration. The algebra A is E*-excisive* if E satisfies excision on any extension (2.7) with kernel A. If $\mathfrak{A}\subset\mathfrak{C}$ is a subcategory, and E satisfies excision for every sequence (2.7) in $\mathfrak{A}$, then we say that E satisfies excision on $\mathfrak{A}$.

Remark 2.5. If we have a functor E which is only defined on the subcategory $\mathfrak{C}_1\subset\mathfrak{C}$ of unital algebras and unital homomorphisms, and which preserves finite products up to homotopy, then we can extend it to all of $\mathfrak{C}$ by setting

$$E(A)=\operatorname{hofiber}(E(A_k^+)\to E(k)).$$

Here A_k^+ denotes the unitalization of A as a k-algebra. The restriction of the new functor E to unital algebras is not the same as the old one, but it is homotopy equivalent to it. Indeed, for A unital, we have $A_k^+\cong A\oplus k$ as k-algebras. Since E preserves finite products, this implies the claim. In this article, whenever we encounter a homology theory defined only on unital algebras, we shall implicitly consider it extended to non-unital algebras by the procedure just explained. Similarly, if $F\colon\mathfrak{C}_1\to\mathfrak{Ab}$ is a functor to abelian groups which preserves finite products, it extends to all of $\mathfrak{C}$ by

$$F(A)=\ker(F(A_k^+)\to F(k)).$$

In particular this applies when $F=E_n$ is the homology functor associated with a homology theory as above.

The main examples of homology theories we are interested in are K-theory, Hochschild homology and the various variants of cyclic homology. A milestone in understanding excision in K-theory is the following result of Suslin and Wodzicki [43], [44].

Theorem 2.6. (Suslin–Wodzicki) *A $\mathbb{Q}$-algebra R is K-excisive if and only if for the $\mathbb{Q}$-algebra unitalization $R_{\mathbb{Q}}^+=R\oplus\mathbb{Q}$ we have*

$$\operatorname{Tor}_n^{R_{\mathbb{Q}}^+}(\mathbb{Q},R)=0 \quad \textit{for all } n\geqslant 0.$$

For example it was shown in [44, Theorem C] that any ring satisfying a certain "triple factorization property" is K-excisive; since any C^*-algebra has this property, ([44, Proposition 10.2]) we have the following result.

Theorem 2.7. (Suslin–Wodzicki) *C^*-algebras are K-excisive.*

Excision for Hochschild and cyclic homology of k-algebras, denoted respectively by $\operatorname{HH}(\cdot/k)$ and $\operatorname{HC}(\cdot/k)$, has been studied in detail by Wodzicki in [52]; as a particular case of his results, we cite the following theorem.

Theorem 2.8. (Wodzicki) *The following are equivalent for a k-algebra A:*

(1) *A is $\operatorname{HH}(\cdot/k)$-excisive;*
(2) *A is $\operatorname{HC}(\cdot/k)$-excisive;*
(3) $\operatorname{Tor}_*^{A_k^+}(k,A)=0$.

Note that it follows from (2.6) and (2.8) that a k-algebra A is K-excisive if and only if it is $\operatorname{HH}(\cdot/\mathbb{Q})$-excisive.

Remark 2.9. Wodzicki has proved (see [53, Theorems 1 and 4]) that if A is a C^*-algebra then A satisfies the conditions of Theorem 2.8 for any subfield $k\subset\mathbb{C}$.

2.4. Milnor squares and excision

We now record some facts about Milnor squares of k-algebras and excision.

Definition 2.10. A square of k-algebras

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array} \tag{2.8}$$

is said to be a *Milnor square* if it is a pull-back square and either f or g is surjective. It is said to be *split* if either f or g has a section.

Let F be a functor from $\mathfrak{C}$ to abelian groups and let

$$0 \longrightarrow A \longrightarrow B \rightleftarrows C \longrightarrow 0 \tag{2.9}$$

be a split-extension in $\mathfrak{C}$. We say that F is *split-exact* on (2.9) if

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

is (split-)exact. If $\mathfrak{A}\subset\mathfrak{C}$ is a subcategory and F is split-exact on all split-exact sequences contained in $\mathfrak{A}$, then we say that F is split-exact on $\mathfrak{A}$.

LEMMA 2.11. *Let $E{:}\,\mathfrak{C}\to\mathfrak{Spt}$ be a homology theory and (2.8) be a Milnor square. Assume that* $\ker(f)$ *is E-excisive. Then E maps (2.8) to a homotopy cartesian square.*

LEMMA 2.12. *Let $F{:}\,\mathfrak{C}\to\mathfrak{Ab}$ be a functor, $\mathfrak{A}\subset\mathfrak{C}$ be a subcategory closed under kernels and (2.8) be a split Milnor square in $\mathfrak{A}$. Assume that F is split-exact on $\mathfrak{A}$. Then the sequence*

$$0 \longrightarrow F(A) \longrightarrow F(B)\oplus F(C) \longrightarrow F(D) \longrightarrow 0$$

is split-exact.

3. Real algebraic geometry and split-exact functors

In this section, we recall several results from real algebraic geometry and prove a theorem on the behavior of weakly split-exact functors with respect to proper semi-algebraic surjections (see Theorem 3.14). Recall that a *semi-algebraic set* is a priori a subset of $\mathbb{R}^n$ which is described as the solution set of a finite number of polynomial equalities and inequalities. A map between semi-algebraic sets is *semi-algebraic* if its graph is a semi-algebraic set. For general background on semi-algebraic sets, consult [2].

3.1. General results about semi-algebraic sets

Let us start with recalling the following two propositions.

PROPOSITION 3.1. (See [2, Proposition 3.1]) *The closure of a semi-algebraic set is semi-algebraic.*

PROPOSITION 3.2. (See [2, Proposition 2.83]) *Let S and T be semi-algebraic sets, $S' \subset S$ and $T' \subset T$ be semi-algebraic subsets and $f: S \to T$ be a semi-algebraic map. Then $f(S')$ and $f^{-1}(T')$ are semi-algebraic.*

Note that a semi-algebraic map does not need to be continuous. Moreover, within the class of continuous maps, there are surjective maps $f: S \to T$ for which the quotient topology induced by S does not agree with the topology on T. An easy example is the projection map from $\{(0,0)\} \cup \{(t, t^{-1}) : t > 0\}$ to $[0, \infty)$. This motivates the following definition.

Definition 3.3. Let S and T be semi-algebraic sets. A continuous semi-algebraic surjection $f: S \to T$ is said to be *topological*, if for every semi-algebraic map $g: T \to Q$ the composition $g \circ f$ is continuous if and only if g is continuous.

Brumfiel proved the following result, which says that (under certain conditions) semi-algebraic equivalence relations lead to good quotients.

THEOREM 3.4. (See [5, Theorem 1.4]) *Let S be a semi-algebraic set and let $R \subset S \times S$ be a closed semi-algebraic equivalence relation. If $\pi_1: R \to S$ is proper, then there exists a semi-algebraic set T and a topological semi-algebraic surjection $f: S \to T$ such that*

$$R = \{(s_1, s_2) \in S \times S : f(s_1) = f(s_2)\}.$$

Remark 3.5. Note that the properness assumption in the previous theorem is automatically fulfilled if S is compact. This is the case we are interested in.

COROLLARY 3.6. *Let S, S' and T be compact semi-algebraic sets and $f: T \to S$ and $f': T \to S'$ be continuous semi-algebraic maps. Then, the topological push-out $S \cup_T S'$ carries a canonical semi-algebraic structure such that the natural maps $\sigma: S \to S \cup_T S'$ and $\sigma': S' \to S \cup_T S'$ are semi-algebraic.*

For semi-algebraic sets, there is an intrinsic notion of connectedness, which is given by the following definition.

Definition 3.7. A semi-algebraic set $S \subset \mathbb{R}^k$ is said to be *semi-algebraically connected* if it is not a non-trivial union of semi-algebraic subsets which are both open and closed in S.

One of the first results on connectedness of semi-algebraic sets is the following theorem.

THEOREM 3.8. (See [2, Theorem 5.20]) *Every semi-algebraic set S is the disjoint union of a finite number of semi-algebraically connected semi-algebraic sets which are both open and closed in S.*

Next we come to aspects of semi-algebraic sets and continuous semi-algebraic maps which differ drastically from the expected results for general continuous maps. In fact, there is a far-reaching generalization of Ehresmann's theorem about local triviality of submersions. Let us consider the following definition.

Definition 3.9. Let S and T be two semi-algebraic sets and $f: S \to T$ be a continuous semi-algebraic function. We say that f is a *semi-algebraically trivial fibration* if there exist a semi-algebraic set F and a semi-algebraic homeomorphism $\theta: T \times F \to S$ such that $f \circ \theta$ is the projection onto T.

A seminal theorem is Hardt's triviality result, which says that away from a subset of T of smaller dimension, every map $f: S \to T$ looks like a semi-algebraically trivial fibration.

THEOREM 3.10. ([2] or [22, §4]) *Let S and T be two semi-algebraic sets and $f: S \to T$ be a continuous semi-algebraic function. Then there exists a closed semi-algebraic subset $V \subset T$ with $\dim V < \dim T$, such that f is a semi-algebraically trivial fibration over every semi-algebraic connected component of $T \setminus V$.*

We shall also need the following result about semi-algebraic triangulations.

THEOREM 3.11. ([2, Theorem 5.41]) *Let $S \subset \mathbb{R}^k$ be a compact semi-algebraic set, and let $S_1, ..., S_q$ be semi-algebraic subsets. There exists a simplicial complex K and a semi-algebraic homeomorphism $h: |K| \to S$ such that each S_j is the union of images of open simplices of K.*

Remark 3.12. In the preceding theorem, the case where the subsets S_j are closed is of special interest. Indeed, if the subsets S_j are closed, the theorem implies that the triangulation of S induces triangulations of S_j for each $j \in \{1, ..., q\}$.

The following proposition is an application of Theorems 3.10 and 3.11.

PROPOSITION 3.13. *Let $T \subset \mathbb{R}^m$ be a compact semi-algebraic subset, S be a semi-algebraic set and $f: S \to T$ be a continuous semi-algebraic map. Then there exist a semi-algebraic triangulation of T and a finite sequence of closed subcomplexes*

$$\varnothing = V_{r+1} \subset V_r \subset V_{r-1} \subset ... \subset V_1 \subset V_0 = T$$

such that the following conditions are satisfied:

(i) *for each* $k\in\{0,...,r\}$ *we have* $\dim V_{k+1}<\dim V_k$ *and the map*

$$f|_{f^{-1}(V_k\setminus V_{k+1})}: f^{-1}(V_k\setminus V_{k+1}) \longrightarrow V_k\setminus V_{k+1}$$

is a semi-algebraically trivial fibration over every semi-algebraic connected component;

(ii) *each simplex in the triangulation lies in some* V_k *and has at most one face of codimension* 1 *which intersects* V_{k+1}.

Proof. We set $n=\dim T$. By Theorem 3.10, there exists a closed semi-algebraic subset $V_1\subset T$, with $\dim V_1<n$, such that f is a semi-algebraic trivial fibration over every semi-algebraic connected component of $T\setminus V_1$. Consider now $f|_{f^{-1}(V_1)}: f^{-1}(V_1)\to V_1$ and proceed as before to find $V_2\subset V_1$. By induction, we find a chain

$$\varnothing\subset V_r\subset V_{r-1}\subset ...\subset V_1\subset V_0=T$$

such that $V_k\subset V_{k-1}$ is a closed semi-algebraic subset and

$$f|_{f^{-1}(V_{k-1}\setminus V_k)}: f^{-1}(V_{k-1}\setminus V_k) \longrightarrow V_{k-1}\setminus V_k$$

is a semi-algebraically trivial fibration over every semi-algebraic connected component, for all $k\in\{1,...,r\}$. Using Theorem 3.11, we may now choose a semi-algebraic triangulation of T such that the subsets V_k are subcomplexes. Taking a barycentric subdivision, each simplex lies in V_k for some $k\in\{0,...,r\}$ and has at most one face of codimension 1 which intersects the set V_{k+1}. □

3.2. The theorem on split-exact functors and proper maps

The following is our main technical result. It is the key to the proofs of Theorems 4.19 and 7.6.

THEOREM 3.14. *Let* T *be a compact semi-algebraic subset of* $\mathbb{R}^k$. *Let* S *be a semi-algebraic set and let* $f: S\to T$ *be a proper continuous semi-algebraic surjection. Then, there exists a semi-algebraic triangulation of* T *such that for every weakly split-exact contravariant functor* $F: \mathfrak{Pol}\to\mathfrak{Set}_-$ *and every simplex* Δ^n *in the triangulation, we have*

$$\ker(F(\Delta^n)\to F(f^{-1}(\Delta^n)))=*.$$

Proof. Choose a triangulation of T and a sequence of subcomplexes $V_k\subset T$ as in Proposition 3.13. We shall show that $\ker(F(\Delta^n)\to F(f^{-1}(\Delta^n)))=*$ for each simplex in the chosen triangulation. The proof is by induction on the dimension of the simplex. The statement is clear for zero-dimensional simplices, since f is surjective. Let Δ^n be an n-dimensional simplex in the triangulation. By assumption, f is a semi-algebraically trivial fibration over $\Delta^n\setminus\Delta^{n-1}$ for some face $\Delta^{n-1}\subset\Delta^n$. Hence, there exists a semi-algebraic set K and a semi-algebraic homeomorphism

$$\theta\colon (\Delta^n\setminus\Delta^{n-1})\times K \longrightarrow f^{-1}(\Delta^n\setminus\Delta^{n-1})$$

over $\Delta^n\setminus\Delta^{n-1}$. Consider the inclusion $f^{-1}(\Delta^{n-1})\subset f^{-1}(\Delta^n)$. Since $f^{-1}(\Delta^{n-1})$ is an absolute neighborhood retract, there exists a compact neighborhood N of $f^{-1}(\Delta^{n-1})$ in $f^{-1}(\Delta^n)$ which retracts onto $f^{-1}(\Delta^{n-1})$. We claim that the set $f(N)$ contains some standard neighborhood A of Δ^{n-1}. Indeed, assume that $f(N)$ does not contain standard neighborhoods. Then there exists a sequence in the complement of $f(N)$ converging to Δ^{n-1}. Lifting this sequence, one can choose a convergent sequence in the complement of N converging to $f^{-1}(\Delta^{n-1})$. This contradicts the fact that N is a neighborhood, and hence there exists a standard compact neighborhood A of Δ^{n-1} in Δ^n such that $f^{-1}(A)\subset N$. Since $(\Delta^n\setminus\Delta^{n-1})\times K\cong f^{-1}(\Delta^n\setminus\Delta^{n-1})$, any retraction of Δ^n onto A yields a retraction of $f^{-1}(\Delta^n)$ onto $f^{-1}(A)$. We have that $f^{-1}(A)\subset N$, and thus we can conclude that $f^{-1}(\Delta^{n-1})$ is a retract of $f^{-1}(\Delta^n)$.

By Corollary 3.6, the topological push-out

$$\begin{array}{ccc} f^{-1}(\Delta^{n-1}) & \longrightarrow & f^{-1}(\Delta^n) \\ \downarrow & & \vdots \\ \Delta^{n-1} & \cdots\cdots\rightarrow & Z \end{array}$$

carries a semi-algebraic structure. Moreover, by weak split-exactness, we have

$$\ker(F(Z)\to F(\Delta^{n-1})\times_{F(f^{-1}(\Delta^{n-1}))}F(f^{-1}(\Delta^n)))=*. \tag{3.1}$$

Note that $f^{-1}(\Delta^n\setminus\Delta^{n-1})\subset Z$ by the definition of Z. We claim that the natural map $\sigma\colon Z\to\Delta^n$ is a semi-algebraic split-surjection. Indeed, identify

$$f^{-1}(\Delta^n\setminus\Delta^{n-1})=(\Delta^n\setminus\Delta^{n-1})\times K,$$

pick $k\in K$, and consider

$$G_k=\overline{\{(d,(d,k))\in\Delta^n\times Z : d\in\Delta^n\setminus\Delta^{n-1}\}}.$$

Then G_k is semi-algebraic by Proposition 3.1, and therefore defines a continuous semi-algebraic map $\varrho_k\colon\Delta^n\to Z$ which splits σ. Thus $F(\sigma)\colon F(\Delta^n)\to F(Z)$ is injective, whence

$$\ker(F(\Delta^n)\to F(\Delta^{n-1})\times_{F(f^{-1}(\Delta^{n-1}))}F(f^{-1}(\Delta^n)))=*, \tag{3.2}$$

by (3.1). But the kernel of $F(\Delta^{n-1})\to F(f^{-1}(\Delta^{n-1}))$ is trivial by induction, so the same must be true of $F(\Delta^n)\to F(f^{-1}(\Delta^n))$, by (3.2). □

4. Large semi-algebraic groups and the compact fibration theorem

4.1. Large semi-algebraic structures

Recall that a partially ordered set (poset) Λ is *filtered* if for any $\lambda, \gamma \in \Lambda$ there exists μ such that $\mu \geqslant \lambda$ and $\mu \geqslant \gamma$. We shall say that a filtered poset Λ is *archimedian* if there exists a monotone map $\phi: \mathbb{N} \to \Lambda$, from the ordered set of natural numbers, which is cofinal, i.e. it is such that for every $\lambda \in \Lambda$ there exists an $n \in \mathbb{N}$ such that $\phi(n) \geqslant \lambda$. If X is a set, we write $P(X)$ for the partially ordered set of all subsets of X, ordered by inclusion.

A *large semi-algebraic structure* on a set X consists of the following:

(i) an archimedian filtered partially ordered set Λ;

(ii) a monotone map $X: \Lambda \to P(X)$, $\lambda \mapsto X_\lambda$, such that $X = \bigcup_\lambda X_\lambda$;

(iii) a compact semi-algebraic structure on each X_λ such that if $\lambda \leqslant \mu$ then the inclusion $X_\lambda \subset X_\mu$ is semi-algebraic and continuous.

We think of a large semi-algebraic structure on X as an exhaustive filtration $\{X_\lambda\}_\lambda$ by compact semi-algebraic sets. We say that a structure $\{X_\gamma : \gamma \in \Gamma\}$ is *finer* than a structure $\{X_\lambda : \lambda \in \Lambda\}$ if for every $\gamma \in \Gamma$ there exists $\lambda \in \Lambda$ such that $X_\gamma \subset X_\lambda$ and the inclusion is continuous and semi-algebraic. Two structures are *equivalent* if each of them is finer than the other. A *large semi-algebraic set* is a set X together with an equivalence class of semi-algebraic structures on X. If X is a large semi-algebraic set, then any large semi-algebraic structure $\{X_\lambda\}_\lambda$ in the equivalence class defining X is called a *defining structure* for X. If $X = (X, \Lambda)$ and $Y = (Y, \Gamma)$ are large semi-algebraic sets, then a set map $f: X \to Y$ is called a *morphism* if for every $\lambda \in \Lambda$ there exists $\gamma \in \Gamma$ such that $f(X_\lambda) \subset Y_\gamma$ and such that the induced map $f: X_\lambda \to Y_\gamma$ is semi-algebraic and continuous. We write $\mathcal{V}_\infty$ for the category of large semi-algebraic sets.

Remark 4.1. If $f: X \to Y$ is a morphism of large semi-algebraic sets, then we may choose structures $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ such that f strictly preserves filtrations, i.e. $f(X_n) \subset Y_n$ for all n. However, if $X = Y$, then there may not exist a structure $\{X_\lambda\}_\lambda$ such that $f(X_\lambda) \subset X_\lambda$.

Remark 4.2. Consider the category $\mathcal{V}_{s,b}$ of compact semi-algebraic sets with *continuous* semi-algebraic mappings and its ind-category ind-$\mathcal{V}_{s,b}$. The objects are functors

$$T: (X_T, \leqslant) \longrightarrow \mathcal{V}_{s,b},$$

where $(X_T, \leqslant)$ is a filtered partially ordered set. We set

$$\hom(T, S) = \lim_{d \in X_T} \operatorname*{colim}_{e \in X_S} \hom_{\mathcal{V}_{s,b}}(T(d), S(e)).$$

The category of large semi-algebraic sets is equivalent to the subcategory of those ind-objects whose structure maps are injective and whose index posets are archimedean. In particular, a filtering colimit of an archimedian system of injective homomorphisms of large semi-algebraic sets is again a large semi-algebraic set.

If X and Y are large semi-algebraic sets with structures $\{X_\lambda:\lambda\in\Lambda\}$ and $\{Y_\gamma:\gamma\in\Gamma\}$ then the cartesian product $X\times Y$ is again a large semi-algebraic set, with structure $\{X_\lambda\times Y_\gamma:(\lambda,\gamma)\in\Lambda\times\Gamma\}$. A *large semi-algebraic group* is a group object G in $\mathcal{V}_\infty$. Thus G is a group which is a large semi-algebraic set and each of the maps defining the multiplication, unit and inverse are homomorphisms in $\mathcal{V}_\infty$. We shall additionally assume that G admits a structure $\{G_\lambda\}_\lambda$ such that $G_\lambda^{-1}\subset G_\lambda$. This hypothesis, although not strictly necessary, is satisfied by all the examples we shall consider, and makes proofs technically simpler. We shall also need the notions of large semi-algebraic vector space and of large semi-algebraic ring, which are defined similarly.

Example 4.3. Any semi-algebraic set S can be considered as a large semi-algebraic set, with the structure defined by its compact semi-algebraic subsets, which is equivalent to the structure defined by any exhaustive filtration of S by compact semi-algebraic subsets. In particular this applies to any finite-dimensional real vector space V; moreover the vector space operations are semi-algebraic and continuous, so that V is a (large) semi-algebraic vector space. Any linear map between finite-dimensional vector spaces is semi-algebraic and continuous, whence it is a homomorphism of semi-algebraic vector spaces. Moreover, the same is true of any multilinear map $f\colon V_1\times\ldots\times V_n\to V_{n+1}$ between finite-dimensional vector spaces.

Definition 4.4. Let V be a real vector space of countable dimension. The *fine* large semi-algebraic structure $\mathcal{F}(V)$ is that given by all the compact semi-algebraic subsets of all the finite-dimensional subspaces of V.

Remark 4.5. The fine large semi-algebraic structure is reminiscent of the *fine locally convex topology* which makes every complex algebra of countable dimension into a locally convex algebra. For details, see [4, §II.2, Exercise 5].

LEMMA 4.6. *Let $n\geqslant 1$, $V_1,\ldots,V_{n+1}$ be countable-dimensional $\mathbb{R}$-vector spaces and $f\colon V_1\times\ldots\times V_n\to V_{n+1}$ be a multilinear map. Equip each V_i with the fine large semi-algebraic structure and $V_1\times\ldots\times V_n$ with the product large semi-algebraic structure. Then f is a morphism of large semi-algebraic sets.*

Proof. In view of the definition of the fine large semi-algebraic structure, the general case is immediate from the finite-dimensional case. □

PROPOSITION 4.7. *Let A be a countable-dimensional $\mathbb{R}$-algebra, equipped with the fine large semi-algebraic structure.*

(i) *A is a large semi-algebraic ring.*

(ii) *Assume that A is unital. Then the group $\mathrm{GL}_n(A)$ together with the structure*

$$\mathcal{F}(\mathrm{GL}_n(A)) = \{\{g\in \mathrm{GL}_n(A) : g, g^{-1}\in F\} : F\in \mathcal{F}(M_n(A))\}$$

is a large semi-algebraic group, and if $A\to B$ is an algebra homomorphism, then the induced group homomorphism $\mathrm{GL}_n(A)\to \mathrm{GL}_n(B)$ is a homomorphism of large semi-algebraic sets.

Proof. Part (i) is immediate from Lemma 4.6. If $F\in\mathcal{F}(M_n(A))$, write

$$\mathrm{GL}_n(A)^F = \{g\in \mathrm{GL}_n(A) : g, g^{-1}\in F\}. \tag{4.1}$$

We will show that $\mathrm{GL}_n(A)^F$ is a compact semi-algebraic set.

Write $m, \pi_i: M_n(A)\times M_n(A)\to M_n(A)$ for the multiplication and projection maps, $i=1,2$, and $\tau: M_n(A)\times M_n(A)\to M_n(A)\times M_n(A)$ for the permutation of factors. If A is unital, then

$$\mathrm{GL}_n(A)^F = \pi_1((m|_{F\times F})^{-1}(1)\cap\tau(m|_{F\times F})^{-1}(1)),$$

which is compact semi-algebraic, by Proposition 3.2. □

We will also study the large semi-algebraic group $\mathrm{GL}_S(A)$ for a subset $S\subset\mathbb{Z}$. This is understood to be the group of matrices g indexed by $\mathbb{Z}$, where $g_{i,j}=\delta_{i,j}$ if $i\notin S$ or $j\notin S$.

COROLLARY 4.8. *If A is unital and $S\subset\mathbb{Z}$, then $\mathrm{GL}_S(A)$ carries a natural large semi-algebraic group structure, namely that of the colimit $\mathrm{GL}_S(A)=\bigcup_T \mathrm{GL}_T(A)$, where T runs among the finite subsets of S.*

Remark 4.9. If A is any not necessarily unital ring, and $a,b\in M_n(A)$, then

$$a\star b = a+b+ab$$

is an associative operation, with the zero matrix as neutral element; the group $\mathrm{GL}_n(A)$ is defined as the set of all matrices which are invertible under $\star$. If A happens to be unital, the resulting group is isomorphic to that of invertible matrices via $g\mapsto g+1$. If A is any countable-dimensional $\mathbb{R}$-algebra, then part (ii) of Proposition 4.7 still holds if we replace g^{-1} by the inverse of g under the operation $\star$ in the definition of $\mathrm{GL}_n(A)^F$. Corollary 4.8 also remains valid in the non-unital case, and the proof is the same.

4.2. Construction of quotients of large semi-algebraic groups

LEMMA 4.10. *Let $X \subset Y$ be an inclusion of large semi-algebraic sets. The following are equivalent*:

(i) *there exists a defining structure $\{Y_\lambda\}_\lambda$ of Y such that $\{Y_\lambda \cap X\}_\lambda$ is a defining structure for X*;

(ii) *for each defining structure $\{Y_\lambda\}_\lambda$ of Y, $\{Y_\lambda \cap X\}_\lambda$ is a defining structure for X.*

Definition 4.11. Let $X \subset Y$ be an inclusion of large semi-algebraic sets. We say that X is *compatible* with Y if the equivalent conditions of Lemma 4.10 are satisfied.

PROPOSITION 4.12. *Let $H \subset G$ be an inclusion of large semi-algebraic groups, $\{G_\lambda\}_\lambda$ be a defining structure for G and $\pi: G \to G/H$ be the projection. Assume that the inclusion is compatible in the sense of Definition 4.11. Then $(G/H)_\lambda = \pi(G_\lambda)$ is a large semi-algebraic structure, and the resulting large semi-algebraic set G/H is the categorical quotient in $\mathcal{V}_\infty$.*

Proof. The map $G_\lambda \to (G/H)_\lambda$ is the set-theoretical quotient modulo the relation $R_\lambda = \{(g_1, g_2) : g_1^{-1} g_2 \in H\} \subset G_\lambda \times G_\lambda$. Let μ be such that the product map m sends $G_\lambda \times G_\lambda$ into G_μ; write $\mathrm{inv}: G \to G$ for the map $\mathrm{inv}(g) = g^{-1}$. Then

$$R_\lambda = (m \circ (\mathrm{inv}, \mathrm{id}))^{-1}(H \cap G_\mu).$$

Because $H \subset G$ is compatible, $H \cap G_\mu \subset G_\mu$ is closed and semi-algebraic, whence the same is true of R_λ. By Theorem 3.4, $(G/H)_\lambda$ is semi-algebraic and $G_\lambda \to (G/H)_\lambda$ is semi-algebraic and continuous. It follows that the $(G/H)_\lambda$ define a large semi-algebraic structure on G/H and that the projection is a morphism in $\mathcal{V}_\infty$. The universal property of the quotient is straightforward. □

Example 4.13. Let R be a unital, countable-dimensional $\mathbb{R}$-algebra. The set

$$P_n(R) = \{g(1_n \oplus 0_\infty) g^{-1} : g \in \mathrm{GL}(R)\}$$

of all finite idempotent matrices which are conjugate to the $n \times n$ identity matrix can be written as a quotient of a compatible inclusion of large semi-algebraic groups. We have

$$P_n(R) = \frac{\mathrm{GL}(R)_{[1,\infty)}}{\mathrm{GL}_{[1,n]}(R) \times \mathrm{GL}_{[n+1,\infty)}(R)}.$$

On the other hand, since $P_n(R) \subset M_\infty(R)$, it also carries another large semi-algebraic structure, induced by the fine structure on $M_\infty(R)$. Since $g \mapsto g(1_n \oplus 0_\infty) g^{-1}$ is semi-algebraic, the universal property of the quotient (Proposition 4.12) implies that the quotient structure is finer than the subspace structure. Similarly, we may write the set of those idempotent matrices which are stably conjugate to $1_n \oplus 0_\infty$ as

$$P_n^\infty(R) = \{g(1_\infty \oplus 1_n \oplus 0_\infty)g^{-1} : g \in \mathrm{GL}_\mathbb{Z}(R)\} = \frac{\mathrm{GL}_{(-\infty,\infty)}(R)}{\mathrm{GL}_{(-\infty,n]}(R) \times \mathrm{GL}_{[n+1,\infty)}(R)}.$$

Again this carries two large semi-algebraic structures: the quotient structure, and that coming from the inclusion $P_n^\infty(R) \subset M_2(M_\infty(R)^+)$ into the 2×2 matrices of the unitalization of $M_\infty(R)$.

4.3. Large semi-algebraic sets as compactly generated spaces

A topological space X is said to be *compactly generated* if it carries the inductive topology with respect to its compact subsets, i.e. a map $f: X \to Y$ is continuous if and only if its restriction to any compact subset of X is continuous. In other words, a subset $U \subset X$ is open (resp. closed) if and only if $U \cap K$ is open (resp. closed) in K for every compact $K \subset X$. Observe that any filtering colimit of compact spaces is compactly generated. In particular, if X is a large semi-algebraic set, with defining structure $\{X_\lambda\}_\lambda$, then $X = \bigcup_\lambda X_\lambda$ equipped with the colimit topology is compactly generated, and this topology depends only on the equivalence class of the structure $\{X_\lambda\}_\lambda$. In what follows, whenever we regard a large semi-algebraic set as a topological space, we will implicitly assume it equipped with the compactly generated topology just defined. Note further that any morphism of large semi-algebraic sets is continuous for the compactly generated topology. Lemma 4.14 characterizes those inclusions of large semi-algebraic sets which are closed subspaces, and Lemma 4.15 concerns quotients of large semi-algebraic groups with the compactly generated topology. Both lemmas are straightforward.

LEMMA 4.14. *An inclusion $X \subset Y$ of large semi-algebraic sets is compatible if and only if X is a closed subspace of Y with respect to the compactly generated topologies.*

LEMMA 4.15. *Let $H \subset G$ be a compatible inclusion of large semi-algebraic groups. View G and H as topological groups equipped with the compactly generated topologies. Then the compactly generated topology associated with the quotient large semi-algebraic set G/H is the quotient topology.*

We shall be concerned with large semi-algebraic groups which are Hausdorff for the compactly generated topology. The main examples are countable-dimensional $\mathbb{R}$-vector spaces and groups such as $\mathrm{GL}_S(A)$, for some subset $S \subset \mathbb{Z}$ and some countable-dimensional unital $\mathbb{R}$-algebra A.

LEMMA 4.16. *Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$, V be a countable-dimensional $\mathbb{F}$-vector space, A be a countable-dimensional $\mathbb{F}$-algebra, $S\subset\mathbb{Z}$, X be a compact Hausdorff topological space and $C(X)=\operatorname{map}(X,\mathbb{F})$. Equip V, A and $\mathrm{GL}_S(A)$ with the compactly generated topologies. Then the natural homomorphisms*

$$C(X)\otimes_{\mathbb{F}} V \longrightarrow \operatorname{map}(X,V) \quad \textit{and} \quad \mathrm{GL}_S(\operatorname{map}(X,A)) \longrightarrow \operatorname{map}(X,\mathrm{GL}_S(A))$$

are bijective.

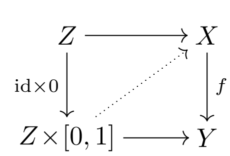
Proof. It is clear that both homomorphisms are injective. The image of the first one consists of those continuous maps whose image is contained in a finitely generated subspace of V. But since X is compact and V has the inductive topology of all closed balls in a finitely generated subspace, every continuous map is of that form. Next note that

$$\mathrm{GL}_S(A) = \bigcup_{S',F} \mathrm{GL}_{S'}(A)^F,$$

where the union runs among the finite subsets of S and the compact semi-algebraic sets of the form $F=M_{S'}(B)$, with B being a compact semi-algebraic subset of some finitely generated subspace of A. Hence any continuous map $f\colon X\to\mathrm{GL}_S(A)$ sends X into some $M_{S'}(B)$, and thus each of the entries $f(x)_{i,j}$, $i,j\in S'$, is a continuous function. Hence $f(x)$ comes from an element of $\mathrm{GL}_S(\operatorname{map}(X,A))$. □

4.4. The compact fibration theorem for quotients of large semi-algebraic groups

Recall that a continuous map $f\colon X\to Y$ of topological spaces is said to have the *homotopy lifting property* (HLP) with respect to a space Z if for any solid arrow diagram



of continuous maps, the continuous dotted arrow exists and makes both triangles commute. The map f is a (Hurewicz) fibration if it has the HLP with respect to any space Z, and is a Serre fibration if it has the HLP with respect to all disks D^n, $n\geqslant 0$.

Definition 4.17. Let X and Y be topological spaces and $f\colon X\to Y$ be a continuous map. We say that f is a *compact fibration* if for every compact subspace K, the map $f^{-1}(K)\to K$ is a fibration.

Remark 4.18. Note that every compact fibration is a Serre fibration. Also, since every map $p: E\to B$, with compact B, which is a locally trivial bundle is a fibration, any map $f: X\to Y$, such that the restriction $f^{-1}(K)\to K$ to any compact subspace $K\subset Y$ is a locally trivial bundle, is a compact fibration. The notion of compact fibration comes up naturally in the study of homogenous spaces of infinite-dimensional topological groups.

THEOREM 4.19. *Let $H\subset G$ be a compatible inclusion of large semi-algebraic groups. Then the quotient map $\pi: G\to G/H$ is a compact fibration.*

Proof. Choose defining structures $\{H_p\}_p$ and $\{G_p\}_p$ indexed over $\mathbb{N}$ and such that $H_p=G_p\cap H$; let $\{(G/H)_p\}_p$ be as in Proposition 4.12. As any compact subspace $K\subset G/H$ is contained in some $(G/H)_p$, it suffices to show that the projection

$$\pi_p = \pi|_{\pi^{-1}((G/H)_p)}: \pi^{-1}((G/H)_p) \longrightarrow (G/H)_p$$

is a locally trivial bundle. By a well-known argument (see e.g. [47, Theorem 4.13]), if the quotient map of a group by a closed subgroup admits local sections, then it is a locally trivial bundle; the same argument applies in our case to show that if π_p admits local sections then it is a locally trivial bundle. Consider the functor

$$\begin{aligned} F: \mathfrak{Comp} &\longrightarrow \mathfrak{Set}_-, \\ X &\longmapsto \frac{\operatorname{map}(X, G/H)}{\operatorname{map}(X, G)}. \end{aligned}$$

By Lemma 2.2, F is split-exact. By Theorem 3.14 applied to F and to the proper semi-algebraic surjection $G_p\to(G/H)_p$, there is a triangulation of $(G/H)_p$ such that

$$\ker(F(\Delta^n)\to F(\pi_p^{-1}(\Delta^n))) = * \tag{4.2}$$

for each simplex Δ^n in the triangulation. The diagram

$$\begin{array}{ccccc} \pi^{-1}(\Delta^n)\cap G_p & \longrightarrow & G_p & \longrightarrow & G \\ \downarrow & & \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \pi} \\ \Delta^n & \longrightarrow & (G/H)_p & \longrightarrow & G/H \end{array}$$

shows that the class of the inclusion $\Delta^n\subset G/H$ is an element of that kernel, and therefore it can be lifted to a continuous map $\Delta^n\to G$, by (4.2). Thus π_p admits a continuous section over every simplex in the triangulation. Therefore, using split-exactness of F, it admits a continuous section over each of the open stars $\operatorname{st}^{\circ}(x)$ of the vertices of the barycentric subdivision. As the open stars of vertices form an open covering of $(G/H)_p$, we conclude that π_p admits local sections. This finishes the proof. $\square$

5. Algebraic compactness, bounded sequences and algebraic approximation

5.1. Algebraic compactness

Let R be a countable-dimensional unital $\mathbb{R}$-algebra, equipped with the fine large semi-algebraic structure. Consider the large semi-algebraic sets

$$M_\infty(R) \supset P_n(R) = \frac{\mathrm{GL}(R)_{[1,\infty)}}{\mathrm{GL}_{[1,n]}(R)\times \mathrm{GL}_{[n+1,\infty)}(R)}, \tag{5.1}$$

$$M_2(M_\infty(R)^+) \supset P_n^\infty(R) = \frac{\mathrm{GL}_{(-\infty,\infty)}(R)}{\mathrm{GL}_{(-\infty,n]}(R)\times \mathrm{GL}_{[n+1,\infty)}(R)} \tag{5.2}$$

introduced in Example 4.13. Recall that each of $P_n(R)$ and $P_n(R)^\infty$ carries two large semi-algebra structures: the homogenous ones, coming from the quotients, and those induced by the inclusions above. As the homogeneous structures are finer than the induced ones, the same is true of the corresponding compactly generated topologies; they agree if and only if the corresponding large semi-algebraic structures are equivalent, or, in other terms, if every subset which is compact in the homogeneous topology is also compact in the induced one. This motivates the following definition.

Definition 5.1. Let R be a countable-dimensional unital $\mathbb{R}$-algebra equipped with the fine large semi-algebraic structure. We say that R has the *algebraic compactness property* if for every $n\geqslant 1$ the homogeneous and the induced large semi-algebraic structure of $P_n^\infty(R)$ agree.

We show in Proposition 5.3 below that if R satisfies algebraic compactness, then the two topologies in (5.1) also agree. For this we need some properties of compactly generated topological groups. All topological groups under consideration are assumed to be Hausdorff.

LEMMA 5.2. *Let H and H' be closed subgroups of a Hausdorff compactly generated group G. Then the quotient topology on $H/(H\cap H')$ is the subspace topology inherited from the quotient topology on G/H'.*

Proof. First of all, it is clear that the canonical inclusion map $\iota: H/(H\cap H')\to G/H'$ is continuous. Indeed, let $\pi: G\to G/H'$ be the projection; identify $H\to H/(H\cap H')$ with the restriction of π. A subset $A\subset G/H'$ is closed if and only if $\pi^{-1}(A)\cap K$ is closed for every compact $K\subset G$. Hence, $\pi^{-1}(A)\cap H\cap K=\pi^{-1}(A\cap\pi(H))\cap K$ is closed and the claim follows since compact subsets of H are also compact in G. Let now $A\subset H/(H\cap H')$ be closed, i.e. $\pi^{-1}(A)\cap K'$ is closed for every compact $K'\subset H$. For compact $K\subset G$, the set $K'=K\cap H$ is compact in H and we get that $\pi^{-1}(A)\cap K$ is closed in H and hence in G. This finishes the proof. □

Proposition 5.3. *Let R be a unital, countable-dimensional $\mathbb{R}$-algebra, equipped with the fine large semi-algebraic structure. Assume that R has the algebraic compactness property. Then the homogeneous and induced large semi-algebraic structures of $P_n(R)$ agree.*

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathrm{GL}_{(-\infty,n]}(R)\times\mathrm{GL}_{[n+1,\infty)}(R) & \longrightarrow & \mathrm{GL}_{(-\infty,\infty)}(R) & \longrightarrow & P_n^{\infty}(R) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{GL}_{[1,n]}(R)\times\mathrm{GL}_{[n+1,\infty)}(R) & \longrightarrow & \mathrm{GL}_{[1,\infty)}(R) & \longrightarrow & P_n(R). \end{array}$$

Now apply Lemma 5.2. □

5.2. Bounded sequences, algebraic compactness and K_0-triviality

Let X be a large semi-algebraic set and let $\{X_\lambda\}_\lambda$ be a defining structure. The space of *bounded sequences* in X is

$$\ell^\infty(X)=\ell^\infty(\mathbb{N},X)=\{z:\mathbb{N}\to X \text{ such that there exists } \lambda \text{ with } z(\mathbb{N})\subset X_\lambda\}.$$

Note that with our definition, the objects $\ell^\infty(\mathbb{R})$ and $\ell^\infty(\mathbb{C})$ coincide with the well-known spaces of bounded sequences.

Lemma 5.4. *Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$ and V be a countable-dimensional $\mathbb{F}$-vector space equipped with the fine large semi-algebraic structure. Then the natural map*

$$\ell^\infty(\mathbb{F})\otimes_{\mathbb{F}}V\longrightarrow\ell^\infty(V)$$

is an isomorphism.

Proof. Choose a basis $\{v_q\}_q$ of V. Every element of $\ell^\infty(\mathbb{F})\otimes_{\mathbb{F}}V$ can be written uniquely as a finite sum $\sum_q\lambda_q\otimes v_q$; this gets mapped to the sequence $\left\{\sum_q\lambda_q(n)v_q\right\}_n$, which vanishes if and only if all the λ_q are zero. This proves the injectivity statement. Let $z\in\ell^\infty(V)$; by definition, there is a finite-dimensional subspace $W\subset V$ and a bounded closed semi-algebraic subset $S\subset W$ such that $z(\mathbb{N})\subset S$. We may assume that S is a closed ball centered at zero, and that W is the smallest subspace containing $z(\mathbb{N})$. Hence there exist $i_1<...<i_p\in\mathbb{N}$ such that $B=\{z_{i_1},...,z_{i_p}\}$ is a basis of W. The map $W\to\mathbb{R}^p$, $w\mapsto[w]_B$, which sends a vector w to the p-tuple of its coordinates with respect to B, is linear and therefore bounded. In particular there exists $C>0$ such that $\|[w]_B\|_\infty<C$ for all $w\in S$. Thus we may write $z=\sum_{j=1}^p\lambda_jz_{i_j}$ with $\lambda_j\in\ell^\infty(\mathbb{F})$. This proves the surjectivity assertion of the lemma. □

PROPOSITION 5.5. *Let R be a unital, countable-dimensional $\mathbb{R}$-algebra equipped with the fine large semi-algebraic structure. Then the following are equivalent*:

(i) *R has the algebraic compactness property*;

(ii) *for every n, the map*

$$P_n^\infty(\ell^\infty(R)) \longrightarrow \ell^\infty(P_n^\infty(R)) \tag{5.3}$$

is surjective;

(iii) *the map*

$$K_0(\ell^\infty(R)) \longrightarrow \prod_{r\geqslant 1} K_0(R)$$

is injective.

Proof. Choose countable indexed structures $\{G_n\}_n$ on $G=\mathrm{GL}_\mathbb{Z}(R)$ and $\{X_n\}_n$ on $X=M_2(M_\infty(R)^+)$. Let $\pi: G\to G/H=P_n^\infty(R)$ be the projection. We know already (see Example 4.13) that the induced structure on $P_n^\infty(R)$ is coarser than the homogeneous one, i.e. each $(G/H)_r=\pi(G_r)$ is contained in some X_m. Assertion (i) is therefore equivalent to saying that each $P_n^\infty(R)\cap X_m$ is contained in some $\pi(G_r)$. Negating this, we obtain a bounded sequence $e=\{e_r\}_r$ of idempotent matrices, i.e. $e\in\ell^\infty(P_n^\infty(R))$ with respect to the induced large semi-algebraic structure on $P_n^\infty(R)$, each e_r is equivalent to $1_\infty\oplus(1_n\oplus 0_\infty)$ in $M_2(M_\infty(R)^+)$, but there is no sequence $\{g_r\}_r$ of invertible matrices in $\mathrm{GL}_2(M_\infty(R)^+)$ such that $g_re_rg_r^{-1}=1_\infty\oplus(1_n\oplus 0_\infty)$ and both $\{g_r\}_r$ and $\{g_r^{-1}\}_r$ are bounded. In other words, e is not in $P_n^\infty(\ell^\infty(R))$. We have shown that (i) is equivalent to (ii). Next note that every element $x\in K_0(\ell^\infty(R))$ can be written as a difference $x=[e]-[1_\infty\oplus 0_\infty]$ with $e\in M_2(M_\infty(R)^+)$ idempotent and $e\equiv 1_\infty\oplus 0_\infty$ modulo the ideal $M_2(M_\infty(R))$. The idempotent e is determined, up to conjugation, by $\mathrm{GL}_2(M_\infty(R)^+)$. The element x goes to zero in $\prod_{r\geqslant 1} K_0(R)$ if and only if each e_r is conjugate to $1_\infty\oplus 0_\infty$. Hence condition (iii) is satisfied if (ii) is. The converse follows easily. Indeed, for any sequence $\{e_r\}_r$ as above, we see that the image of the classes $[e]-[1_\infty\oplus 0_\infty]$ and $[1_\infty\oplus(1_n\oplus 0_\infty)]-[1_\infty\oplus 0_\infty]$ in $\prod_{p\geqslant 1} K_0(R)$ coincide. Hence, by injectivity of the comparison map, e is conjugate to $1_\infty\oplus(1_n\oplus 0_\infty)$, and we get a sequence of invertible elements $\{g_r\}_r$ in $\mathrm{GL}_2(M_\infty(R)^+)$ such that g_r conjugates e_r to $1_\infty\oplus(1_n\oplus 0_\infty)$ and the sequences $\{g_r\}_r$ and $\{g_r^{-1}\}_r$ are bounded. This completes the proof. □

Example 5.6. Both $\mathbb{R}$ and $\mathbb{C}$ have the algebraic compactness property since the third condition is well known to be satisfied. Indeed, $\ell^\infty(\mathbb{R})$ and $\ell^\infty(\mathbb{C})$ are (real) C^*-algebras, and one can easily compute that

$$K_0(\ell^\infty(\mathbb{C})) = \ell^\infty(\mathbb{Z}) \subset \prod_{n\geqslant 1}\mathbb{Z} = \prod_{n\geqslant 1} K_0(\mathbb{C}).$$

The same computation applies to $\mathbb{R}$ in place of $\mathbb{C}$.

5.3. Algebraic approximation and bounded sequences

THEOREM 5.7. *Let F and G be functors from commutative $\mathbb{C}$-algebras to sets. Assume that both F and G preserve filtering colimits. Let $\tau: F \to G$ be a natural transformation. Assume that $\tau(\mathcal{O}(V))$ is injective (resp. surjective) for each smooth affine algebraic variety V over $\mathbb{C}$. Then $\tau(\ell^\infty(\mathbb{C}))$ is injective (resp. surjective).*

Proof. Let $\mathcal{F} \subset \ell^\infty(\mathbb{C})$ be a finite subset. Put $A_\mathcal{F} = \mathbb{C}\langle\mathcal{F}\rangle \subset \ell^\infty(\mathbb{C})$ for the unital subalgebra generated by $\mathcal{F}$. Because $A_\mathcal{F}$ is reduced, it corresponds to an affine algebraic variety $V_\mathcal{F}$, and the inclusion $A_\mathcal{F} \subset \ell^\infty(\mathbb{C})$ is dual to a map $\iota_\mathcal{F}: \mathbb{N} \to (V_\mathcal{F})_{\mathrm{an}}$, with precompact image, to the analytic variety associated with $V_\mathcal{F}$. Thus we may write $\ell^\infty(\mathbb{C})$ as the filtering colimit

$$\ell^\infty(\mathbb{C}) = \operatorname*{colim}_{\mathbb{N} \to V_{\mathrm{an}}} \mathcal{O}(V). \tag{5.4}$$

Here the colimit is taken over all maps $\iota: \mathbb{N} \to V_{\mathrm{an}}$ whose codomain is the associated analytic variety of the closed points of some affine algebraic variety V, and which have precompact image in the euclidean topology. We claim that every such map factors through a map $V'_{\mathrm{an}} \to V_{\mathrm{an}}$, with V' smooth and affine. Note that the claim implies that we may write (5.4) as a colimit of smooth algebras; the theorem is immediate from this. Recall that Hironaka's desingularization (see [23]) provides a proper surjective homomorphism of algebraic varieties $\pi: \widetilde{V} \to V$ from a smooth quasi-projective variety. Thus the induced map $\pi_{\mathrm{an}}: \widetilde{V}_{\mathrm{an}} \to V_{\mathrm{an}}$ between the associated analytic varieties is proper and surjective for the usual euclidean topologies. It follows from this that we can lift ι along π_{an}. Next, Jouanoulou's device (see [25]) provides a smooth affine vector bundle torsor $\sigma: V' \to \widetilde{V}$; the associated map σ_{an} is also a bundle torsor, and in particular a fibration and weak equivalence. Because $\widetilde{V}_{\mathrm{an}}$ is a CW-complex, σ_{an} admits a continuous section. Thus ι_F finally factors through the smooth affine variety $V'_\mathcal{F}$. □

Remark 5.8. The proof above does not work in the real case, since a desingularization $\widetilde{V} \to V$ of real algebraic varieties need not induce a surjective map between the corresponding real analytic (or semi-algebraic) varieties. For example, consider

$$R = \frac{\mathbb{R}[x,y]}{\langle x^2 + y^2 - x^3 \rangle}.$$

The homomorphism

$$\begin{aligned} f: R &\longrightarrow \mathbb{R}[t], \\ p(x,y) &\longmapsto p(t^2+1, t(t^2+1)), \end{aligned}$$

is injective and $\mathbb{R}[t]$ is integral over R. Thus, the induced scheme homomorphism

$$f_{\#}\colon \mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec}\mathbb{R}[t] \longrightarrow V = \operatorname{Spec} R.$$

is a desingularization; it is finite (whence proper) and surjective, and an isomorphism outside of the point zero, represented by the maximal ideal $\mathfrak{M}=\langle x,y\rangle \in V$. But note that the preimage of $\mathfrak{M}$ consists just of the maximal ideal $\langle t^2+1\rangle$, which has residue field $\mathbb{C}$; this means that the preimage of zero has no real points. Therefore the restriction of $f_{\#}$ to real points is not surjective.

5.4. The algebraic compactness theorem

THEOREM 5.9. *Let R be a countable-dimensional unital $\mathbb{C}$-algebra. Assume that the map $K_0(\mathcal{O}(V)) \to K_0(\mathcal{O}(V)\otimes R)$ is an isomorphism for every affine smooth algebraic variety V over $\mathbb{C}$. Then R has the algebraic compactness property.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} K_0(\ell^\infty(R)) & \longrightarrow & \prod_{p\geqslant 1} K_0(R) \\ \uparrow & & \uparrow \\ K_0(\ell^\infty(\mathbb{C})) & \longrightarrow & \prod_{p\geqslant 1} K_0(\mathbb{C}). \end{array}$$

The bottom row is a monomorphism by Example 5.6. Our hypothesis on R together with Theorem 5.7 applied to the natural transformation $K_0(\cdot)\to K_0(\cdot\otimes_{\mathbb{C}} R)$ imply that both columns are isomorphisms. It follows that the top row is injective, which by Proposition 5.5 says that R satisfies algebraic compactness. □

6. Applications: projective modules, lower *K*-theory and bundle theory

6.1. Parametrized Gubeladze's theorem and Rosenberg's conjecture

All monoids considered are commutative, cancellative and torsion-free. If M is cancellative then it embeds into its total quotient group $G(M)$. A cancellative monoid M is said to be *semi-normal* if for every element x of the total quotient group $G(M)$ for which $2x$ and $3x$ are contained in the monoid M, it follows that x is contained in the monoid M.

The following is a particular case of a theorem of Gubeladze, which in turn generalized the celebrated theorem of Quillen [35] and Suslin [41] which settled Serre's conjecture: every finitely generated projective module over a polynomial ring over a field is free.

Theorem 6.1. (See [19] and [20]) *Let D be a principal ideal domain and M be a commutative, cancellative, torsion-free, semi-normal monoid. Then every finitely generated projective module over the monoid algebra $D[M]$ is free.*

We shall also need the following generalization of Gubeladze's theorem, due to Swan. Recall that if $R\to S$ is a homomorphism of unital rings and M is an S-module, then we say that M is *extended* from R if there exists an R-module N such that $M\cong S\otimes_R N$ as S-modules.

Theorem 6.2. (See [46]) *Let $R=\mathcal{O}(V)$ be the coordinate ring of a smooth affine algebraic variety over a field, and let $d=\dim V$. Also let M be a torsion-free, semi-normal, cancellative monoid. Then all finitely generated projective $R[M]$-modules of rank $n>d$ are extended from R.*

In the next theorem and elsewhere below, we shall consider only the complex case; thus, in what follows, $C(X)$ shall always mean $\mathrm{map}(X,\mathbb{C})$.

Theorem 6.3. *Let X be a contractible compact space and M be an abelian, countable, torsion-free, semi-normal, cancellative monoid. Then every finitely generated projective module over $C(X)[M]$ is free.*

Proof. The assertion of the theorem is equivalent to the assertion that every idempotent matrix with coefficients in $C(X)[M]$ is conjugate to a diagonal matrix with only zeroes and ones in the diagonal. By Lemma 4.16, an idempotent matrix with coefficients in $C(X)[M]$ is the same as a continuous map from X to the space $\mathrm{Idem}_\infty(\mathbb{C}[M])$ of all idempotent matrices in $M_\infty(\mathbb{C}[M])$, equipped with the induced topology. Now observe that, since the trace map $M_\infty(\mathbb{C}[M])\to\mathbb{C}[M]$ is continuous, so is the rank map $\mathrm{Idem}_\infty(\mathbb{C}[M])\to\mathbb{N}_0$. Hence, by Theorem 6.1, the space $\mathrm{Idem}_\infty(\mathbb{C}[M])$ is the topological coproduct

$$\mathrm{Idem}_\infty(\mathbb{C}[M])=\coprod_n P_n(\mathbb{C}[M]),$$

and thus any continuous map $e\colon X\to\mathrm{Idem}_\infty(\mathbb{C}[M])$ factors through a map

$$e\colon X\to P_n(\mathbb{C}[M]).$$

By Theorems 6.2 and 5.9, the induced topology of

$$P_n(\mathbb{C}[M])=\frac{\mathrm{GL}(\mathbb{C}[M])}{\mathrm{GL}_{[1,n]}(\mathbb{C}[M])\times\mathrm{GL}_{[n+1,\infty)}(\mathbb{C}[M])}$$

coincides with the quotient topology. By Theorem 4.19, e lifts to a continuous map $g\colon X\to\mathrm{GL}(\mathbb{C}[M])$. By Lemma 4.16, $g\in\mathrm{GL}(C(X)[M])$ and conjugates e to $1_n\oplus 0_\infty$. This concludes the proof. □

THEOREM 6.4. *The functor*

$$\begin{aligned} \mathfrak{Comp} &\longrightarrow \mathfrak{Ab}, \\ X &\longmapsto K_0(C(X)[M]), \end{aligned}$$

is homotopy invariant.

Proof. This is immediate from Theorem 6.3 and Proposition 2.3. □

THEOREM 6.5. (Rosenberg's conjecture) *The functor*

$$\begin{aligned} \mathfrak{Comp} &\longrightarrow \mathfrak{Ab}, \\ X &\longmapsto K_{-n}(C(X)), \end{aligned}$$

is homotopy invariant for $n>0$.

Proof. Because of the isomorphism (2.4), $K_{-n}(C(X))$ is naturally a direct summand of $K_0(C(X)[\mathbb{Z}^n])$, whence it is homotopy invariant by Theorem 6.4. □

Remark 6.6. Let X be a compact topological space, S^1 be the circle and $j\geqslant 0$. By (2.4), Theorem 6.5 and excision, we have

$$K_{-j}(C(X\times S^1)) = K_{-j}(C(X))\oplus K_{-j-1}(C(X)) = K_{-j}(C(X)[t,t^{-1}]).$$

Thus, the effect on negative K-theory of the cartesian product of the maximal ideal spectrum $X=\operatorname{Max}(C(X))$ with S^1 is the same as that of taking the product of the prime ideal spectrum $\operatorname{Spec}(C(X))$ with the algebraic circle $\operatorname{Spec}(\mathbb{C}[t,t^{-1}])$. More generally, for the C^*-algebra tensor product $\otimes_{\min}$ and any commutative C^*-algebra A, we have

$$K_{-j}(A\otimes_{\min} C(S^1)) = K_{-j}(A\otimes_{\mathbb{C}}\mathbb{C}[t,t^{-1}]), \quad j>0.$$

Remark 6.7. Theorem 6.5 was stated by Jonathan Rosenberg in [36, Theorem 2.4] and again in [37, Theorem 2.3] for the real case. Later, in [38], Rosenberg acknowledges that the proof was faulty, but conjectures the statement to be true. Indeed, a mistake was pointed out by Walker (see [15, p. 799, line 8] or [38, p. 26, line 12]). In their work on semi-topological K-theory, Friedlander and Walker prove [15, Theorem 5.1] that the negative algebraic K-theory of the ring $C(\Delta^n)$ of complex-valued continuous functions on the simplex vanishes for all n. We show in §7.2 how another proof of Rosenberg's conjecture can be obtained using the Friedlander–Walker result.

Remark 6.8. The proof of Theorem 6.5 does not need the detour of the proof of our main results in the case $n=1$. Indeed, the ring of germs of continuous functions at a point in X is a Hensel local ring with residue field $\mathbb{C}$, and Vladimir Drinfeld proves that K_{-1} vanishes for Hensel local rings with residue field $\mathbb{C}$, see [12, Theorem 3.7]. This solves the problem locally and reduces the remaining complications to bundle theory. (This was observed by the second author in discussions with Charles Weibel at Institut Henri Poincaré, Paris, in 2004.) No direct approach like this is known for K_{-2} or in lower dimensions.

Already in [36], Rosenberg computed the values of negative algebraic K-theory on commutative unital C^*-algebras, assuming the homotopy invariance result.

COROLLARY 6.9. (Rosenberg, [38]) *Let X be a compact topological space. Let* **bu** *denote the connective K-theory spectrum. Then*,

$$K_{-i}(C(X)) = \mathrm{bu}^i(X) = [\Sigma^i X, \mathbf{bu}], \quad i \geqslant 0.$$

The fact that connective K-theory shows up in this context was further explored and clarified in the thesis of the second author [48], which was also partially built on the validity of Theorem 6.5.

6.2. Application to bundle theory: local triviality

Let R be a countable-dimensional $\mathbb{R}$-algebra. Any finitely generated R-module M is a countable-dimensional vector space, and thus it can be regarded as a compactly generated topological space. We consider not necessarily locally trivial bundles of finitely generated free R-modules over compact spaces, such that each fiber is equipped with the compactly generated topology just recalled. We call such a gadget a *quasi-bundle of finitely generated free R-modules.*

THEOREM 6.10. *Let X be a compact space and M be a countable, torsion-free, semi-normal, cancellative monoid. Let $E \to X$ be a quasi-bundle of finitely generated free $\mathbb{C}[M]$-modules. Assume that there exist $n \geqslant 1$, another quasi-bundle E' and a quasi-bundle isomorphism $E \oplus E' \cong X \times \mathbb{C}[M]^n$. Then E is locally trivial.*

Proof. Put $R=\mathbb{C}[M]$. The isomorphism $E \oplus E' \cong X \times R^n$ gives a continuous function $e: X \to P_n(R)$; E is locally trivial if e is locally conjugate to an idempotent of the form $1_r \oplus 0_\infty$, i.e. if it can be lifted locally along the projection $\mathrm{GL}(R) \to P_n(R)$ to a continuous map $X \to \mathrm{GL}(R)$. Our hypothesis on M together with Theorems 6.3, 5.9, 6.2 and 4.19 imply that such local liftings exist. □

7. Homotopy invariance

7.1. From compact polyhedra to compact spaces: a result of Calder–Siegel

Consider the category $\mathfrak{Comp}$ of compact Hausdorff topological spaces with continuous maps and its full subcategory $\mathfrak{Pol}\subset\mathfrak{Comp}$ formed by those spaces which are compact polyhedra. In this subsection we show that for a functor which commutes with filtering colimits and is split-exact on C^*-algebras, homotopy invariance on $\mathfrak{Pol}$ implies homotopy invariance on $\mathfrak{Comp}$. For this, we shall need a particular case of a result of Calder and Siegel [6], [7] that we recall below. We point out that the Calder–Siegel results have been further generalized by Armin Frei in [14]. For each object $X\in\mathfrak{Comp}$ we consider the comma category $(X\downarrow\mathfrak{Pol})$, whose objects are morphisms $f: X\to\operatorname{cod}(f)$, where the codomain $\operatorname{cod}(f)$ is a compact polyhedron. Morphisms are commutative diagrams as usual. Let $G:\mathfrak{Pol}\to\mathfrak{Ab}$ be a (contravariant) functor to the category of abelian groups. Its right Kan extension $G^{\mathfrak{Pol}}:\mathfrak{Comp}\to\mathfrak{Ab}$ is defined by

$$G^{\mathfrak{Pol}}(X)=\operatorname*{colim}_{f\in(X\downarrow\mathfrak{Pol})} G(\operatorname{cod}(f))\quad\text{for all } X\in\mathfrak{Comp}.$$

Note that $\mathfrak{Pol}$ has finite products so that $(X\downarrow\mathfrak{Pol})$ is a filtered category. The result of Calder–Siegel (see Corollary 2.7 and Theorem 2.8 in [7]) gives that homotopy invariance properties of G give rise to homotopy invariance properties of $G^{\mathfrak{Pol}}$. More precisely, we have the following result.

THEOREM 7.1. (Calder-Siegel) *If* $G:\mathfrak{Pol}\to\mathfrak{Ab}$ *is a* (*contravariant*) *homotopy invariant functor*, *then the functor* $G^{\mathfrak{Pol}}:\mathfrak{Comp}\to\mathfrak{Ab}$ *is homotopy invariant.*

We want to apply the theorem when G is of the form $D\mapsto E(C(D))$, the functor E commutes with (algebraic) filtering colimits and is split-exact on C^*-algebras. For this we have to compare E with the right Kan extension of G; we need some preliminaries. Let $X\in\mathfrak{Comp}$ and $D\subset\mathbb{C}$ be the unit disk. Since X is compact, for each $f\in C(X)$ there is an $n\in\mathbb{N}$ such that $f/n\in C(X,D)$. Thus any finitely generated subalgebra $A\subset C(X)$ is generated by a finite subset $F\subset C(X,D)$. Let $\mathcal{F}$ be the set of all finite subsets of $C(X,D)$. As $C(X)$ is the colimit of its finitely generated subalgebras $\mathbb{C}\langle F\rangle$, we have

$$\operatorname*{colim}_{F\in\mathcal{F}}\mathbb{C}\langle F\rangle=C(X).$$

For $F\in\mathcal{F}$, write $Y_F\subset\mathbb{C}^F$ for the Zariski closure of the image of the map

$$\begin{aligned}\alpha_F: X&\longrightarrow\mathbb{C}^F,\\ x&\longmapsto\{f(x)\}_{f\in F}.\end{aligned}$$

The image of α_F is contained in the compact semi-algebraic set

$$P_F = D \cap Y_F$$

In particular, P_F is a compact polyhedron. Note that α_F induces an isomorphism between the ring $\mathcal{O}(Y_F)$ of regular polynomial functions and the subalgebra $\mathbb{C}\langle F\rangle \subset C(X)$ generated by F. Hence the inclusion $P_F \subset Y_F$ induces a homomorphism β_F which makes the diagram

$$\begin{array}{ccc} \mathbb{C}\langle F\rangle & \longrightarrow & C(X) \\ & {}_{\beta_F}\searrow \quad \nearrow & \\ & C(P_F) & \end{array}$$

commute. Taking colimits, we obtain

$$\begin{array}{ccc} C(X) & = & C(X) \\ & {}_{\beta}\searrow \quad \nearrow_{\pi} & \\ & \operatorname*{colim}_{F\in\mathcal{F}} C(P_F). & \end{array} \tag{7.1}$$

Thus the map π is a split surjection.

THEOREM 7.2. *Let E: $\mathfrak{Comm} \to \mathfrak{Ab}$ be a functor. Assume that E satisfies each of the following conditions*:

(1) *E commutes with filtered colimits*;

(2) $\mathfrak{Pol} \to \mathfrak{Ab}$, $D \mapsto E(C(D))$, *is homotopy invariant.*

Then the functor

$$\mathfrak{Comp} \longrightarrow \mathfrak{Ab}, \quad X \longmapsto E(C(X)),$$

is homotopy invariant on the category of compact topological spaces.

Proof. By (7.1) and the first hypothesis, the map $E(\beta)$ is a right inverse of the map

$$E(\pi): E\Big(\operatorname*{colim}_{F\in\mathcal{F}} C(P_F)\Big) = \operatorname*{colim}_{F\in\mathcal{F}} E(C(P_F)) \longrightarrow E(C(X)).$$

On the other hand, we have a commutative diagram

$$\begin{array}{ccc} \operatorname*{colim}_{F\in\mathcal{F}} E(C(P_F)) & \xrightarrow{\theta} & \operatorname*{colim}_{f\in(X\downarrow\mathfrak{Pol})} E(C(\operatorname{cod}(f))) \\ & {}_{E(\pi)}\searrow & \downarrow{\scriptstyle \pi'} \\ & & E(C(X)). \end{array} \tag{7.2}$$

Hence the map π' in the diagram above is split by the composite $\theta E(\beta)$, and therefore $E(C(\cdot))$ is naturally a direct summand of the functor

$$\begin{aligned} G^{\mathfrak{Pol}}\colon \mathfrak{Comp} &\longrightarrow \mathfrak{Ab}, \\ X &\longmapsto \operatorname*{colim}_{f\in(X\downarrow\mathfrak{Pol})} E(C(\operatorname{cod}(f))). \end{aligned} \tag{7.3}$$

But (7.3) is the right Kan extension of the functor $G\colon\mathfrak{Pol}\to\mathfrak{Ab}$, $K\mapsto E(C(K))$, and thus it is homotopy invariant by the second hypothesis and Calder–Siegel's theorem. It follows that $E(C(\cdot))$ is homotopy invariant, as we had to prove. □

Remark 7.3. If in Theorem 7.2 the functor E is split-exact, then $A\mapsto E(A)$ is homotopy invariant on the category of commutative C^*-algebras. Indeed, since every commutative unital C^*-algebra is of the form $C(X)$ for some compact space X, it follows that F is homotopy invariant on unital commutative C^*-algebras. Using this and split-exactness, we get that it is also homotopy invariant on all commutative C^*-algebras.

Remark 7.4. In general, one cannot expect that the homomorphism π' in (7.2) be an isomorphism. For the injectivity one would need the following implication: if D is a compact polyhedron, and $f\colon X\to D$ and $s\colon D\to\mathbb{C}$ are continuous maps such that $0=s\circ f\colon X\to\mathbb{C}$, then there exist a compact polyhedron D' and continuous maps $g\colon X\to D'$ and $h\colon D'\to D$ such that $h\circ g=f\colon X\to D$ and $0=h\circ s\colon D'\to\mathbb{C}$. But this is too strong if X is a pathological space. To give a concrete example: let X be a Cantor set inside $[0,1]$, f be the natural inclusion and s be the distance function to the Cantor set, and suppose that g and h as above exist. If $0=h\circ s\colon D'\to\mathbb{C}$, then the image of D' in $[0,1]$ has to be contained in X. But the image has only finitely many connected components, since D' has this property. Hence, since X is totally disconnected, the image of D' in $[0,1]$ cannot be all of X. This is a contradiction.

7.2. Second proof of Rosenberg's conjecture

A second proof of Rosenberg's conjecture (Theorem 6.5) can be obtained by combining Theorem 7.2 with the following theorem, which is due to Friedlander and Walker.

Theorem 7.5. ([15, Theorem 5.1]) *If $n>0$ and $q\geqslant 0$, then*

$$K_{-n}(C(\Delta^q))=0.$$

Second proof of Rosenberg's conjecture (*Theorem* 6.5). By Proposition 2.3 and Theorem 7.2 it suffices to show that $K_n(C(D))=0$ for contractible $D\in\mathfrak{Pol}$. If D is contractible, then the identity $1_D\colon D\to D$ factors over the cone cD. Hence, it is sufficient to show that $K_n(C(cD))=0$. The cone cD is a star-like simplicial complex and for any subcomplexes $A, B \subset D$ with $A \cup B = D$, we get a Milnor square

$$\begin{array}{ccc} C(cD) & \longrightarrow & C(cA) \\ \downarrow & & \downarrow \\ C(cB) & \longrightarrow & C(c(A\cap B)). \end{array}$$

Since cA is contractible, it retracts onto $c(A\cap B)$, and therefore the square above is split. Using excision, we obtain the split-exact sequence of abelian groups

$$0 \longrightarrow K_n(C(cD)) \longrightarrow K_n(C(cA)) \oplus K_n(C(cB)) \longrightarrow K_n(C(c(A\cap B))) \longrightarrow 0.$$

Decomposing cD like this, we see that the result of Theorem 7.5 is sufficient for the vanishing of $K_n(C(cD))$. □

7.3. The homotopy invariance theorem

The aim of this subsection is to prove the following result.

THEOREM 7.6. *Let F be a functor on the category of commutative $\mathbb{C}$-algebras with values in abelian groups. Assume that the following three conditions are satisfied*:

(i) *F is split-exact on C^*-algebras*;

(ii) *F vanishes on coordinate rings of smooth affine varieties*;

(iii) *F commutes with filtering colimits.*

Then the functor

$$\mathfrak{Comp} \longrightarrow \mathfrak{Ab},$$
$$X \longmapsto F(C(X)),$$

is homotopy invariant on the category of compact Hausdorff topological spaces and

$$F(C(X)) = 0$$

for contractible X.

Proof. Note that, since a point is a smooth algebraic variety, our hypotheses imply that $F(\mathbb{C})=0$. Thus, if F is homotopy invariant and X is contractible, then we have $F(X)=F(\mathbb{C})=0$. Let us prove then that $X \mapsto F(C(X))$ is homotopy invariant on the category of compact Hausdorff topological spaces. Proceeding as in the proof of Theorem 7.5, we see that it is sufficient to show that $F(C(\Delta^n))=0$ for all $n \geqslant 0$. Any finitely generated subalgebra of $C(\Delta^n)$ is reduced, and hence corresponds to an algebraic variety over $\mathbb{C}$. Since F commutes with filtered colimits, we obtain

$$F(C(\Delta^n)) = \operatorname*{colim}_{\Delta^n \to Y_{\mathrm{an}}} F(\mathcal{O}(Y)),$$

where the colimit runs over all continuous maps from Δ^n to the analytic variety Y_{an} equipped with the usual euclidean topology. For ease of notation, we will from now on just write Y for both the algebraic variety and the analytic variety associated with it. Let $\iota\colon \Delta^n \to Y$ be a continuous map. As in the proof of the algebraic approximation theorem (Theorem 5.7), we consider Hironaka's desingularization $\pi\colon \widetilde{Y} \to Y$ and Jouanoulou's affine bundle torsor $\sigma\colon Y' \to \widetilde{Y}$. Let $T \subset Y$ be a compact semi-algebraic subset such that $\iota(\Delta^n) \subset T$. Since π is a proper morphism, $\widetilde{T} = \pi^{-1}(T)$ is compact and semi-algebraic. By definition of vector bundle torsor ([51]), there is a Zariski cover of $\widetilde{Y}$ such that the pull-back of σ over each open subscheme $U \subset \widetilde{Y}$ of the covering is isomorphic, as a scheme over U, to an algebraic trivial vector bundle. Thus σ is a locally trivial fibration for the euclidean topologies, and the trivialization maps are (semi-)algebraic. Hence, as $\widetilde{T}$, being compact, is locally compact, we may find a finite covering $\{\widetilde{T}_i\}_i$ of $\widetilde{T}$ by closed semi-algebraic subsets such that σ is a trivial fibration over each $\widetilde{T}_i$, and compact semi-algebraic subsets $S_i \subset Y'$ such that $\sigma(S_i) = \widetilde{T}_i$. Put $S = \bigcup_i S_i$. Then S is compact semi-algebraic, and $f = (\pi \circ \sigma)|_S\colon S \to T$ is a continuous semi-algebraic surjection. By Theorem 3.14, there exists a semi-algebraic triangulation of T such that $\ker(F(\Delta^m) \to F(f^{-1}(\Delta^m))) = 0$ for each simplex Δ^m in the triangulation. Consider the diagram

$$\begin{array}{ccccc}
F(C(f^{-1}(\Delta^m))) & \longleftarrow & F(C(S)) & \longleftarrow & F(\mathcal{O}(Y')) \\
\uparrow & & \uparrow & & \uparrow \\
F(C(\Delta^m)) & \longleftarrow & F(C(T)) & \longleftarrow & F(\mathcal{O}(Y)).
\end{array}$$

If $\alpha \in F(\mathcal{O}(Y))$, then its image in $F(C(f^{-1}(\Delta^m)))$ vanishes since $f^{-1}(\Delta^m) \to \Delta^m$ factors through the smooth affine variety Y', and $F(\mathcal{O}(Y')) = 0$. Hence, by Theorem 3.14, we have $\alpha|_{\Delta^m} = 0$ for each simplex in the triangulation. Coming back to the map $\iota\colon \Delta^n \to Y$, we have the diagram

$$\begin{array}{ccc}
\Delta^n & \xrightarrow{\iota} & Y \\
& \searrow^{\jmath} & \uparrow \\
\Delta^m & \xrightarrow[\theta]{} & T.
\end{array}$$

Here $\theta\colon \Delta^m \to T$ is the inclusion of a simplex in the triangulation and $\jmath$ is the corestriction of ι. We need to conclude that $\iota^*(\alpha) = 0$, knowing only that $\theta^*(\alpha) = 0$ for each simplex in a triangulation of T. This is done using split-exactness and barycentric subdivisions. Indeed, we perform the barycentric subdivision of Δ^n sufficiently many times so that each n-dimensional simplex is mapped to the closed star $\mathrm{st}(x)$ of some vertex x in the triangulation of T. Since Δ^n is star-like, the reduction argument of the proof of Theorem 5.7 shows that it is enough to show the vanishing of $\iota^*(\alpha)$ for the (top-dimensional) simplices in this subdivision of Δ^n. If $(\Delta')^n$ is one of these top-dimensional simplices, and $\Delta^m \subset \mathrm{st}(x)$, we can complete the diagram above to a diagram

$$\begin{array}{ccccc} (\Delta')^n & \longrightarrow & \Delta^n & \xrightarrow{\ \iota\ } & Y \\ & \searrow & & \searrow^{\jmath} & \uparrow \\ \Delta^m & \longrightarrow & \mathrm{st}(x) & \longrightarrow & T. \end{array}$$

Hence, it suffices to show that the pull-back of α to $\mathrm{st}(x)$ vanishes. But since $\mathrm{st}(x)$ is star-like, then, by the same reduction argument as before, the vanishing of the pull-back of α to each of the top simplices $\Delta^m \subset \mathrm{st}(x)$ is sufficient to conclude that $\alpha|_{\mathrm{st}(x)}=0$. This finishes the proof. □

As an application, we obtain the following proof.

Third proof of Rosenberg's conjecture (*Theorem* 6.5). If $n<0$ then K_n is split-exact and vanishes on coordinate rings of smooth affine algebraic varieties. By Theorem 7.6, this implies that $X \mapsto K_n(C(X))$ is homotopy invariant. □

7.4. A vanishing theorem for homology theories

THEOREM 7.7. *Let $E: \mathfrak{Comm}/\mathbb{C} \to \mathfrak{Spt}$ be a homology theory of commutative $\mathbb{C}$-algebras and let $n_0 \in \mathbb{Z}$. Assume that*

(i) *E is excisive on commutative C^*-algebras*;

(ii) *E_n commutes with algebraic filtering colimits for $n \geqslant n_0$*;

(iii) *$E_n(\mathcal{O}(V))=0$ for each smooth affine algebraic variety V for $n \geqslant n_0$.*

Then $E_n(A)=0$ for every commutative C^*-algebra A and every $n \geqslant n_0$.

Proof. Let $n \geqslant n_0$. We have to show that we have $E_n(A)=0$ for every commutative C^*-algebra A. Because, by (i), each E_n is split-exact on commutative C^*-algebras, it suffices to show that $E_n(A)=0$ for unital A, i.e. for $A=C(X)$, $X \in \mathfrak{Comp}$. Since, by (ii), E_n preserves filtering colimits, the proof of Theorem 7.2 shows that $E_n(C(X))$ is a direct summand of

$$\operatorname*{colim}_{f \in (X \downarrow \mathfrak{Pol})} E(C(\mathrm{cod}(f))).$$

Hence it suffices to show that $E_n(C(D))=0$ for every compact polyhedron D. By (iii) and excision, this is true if $\dim D=0$. Let $m\geqslant 1$ and assume that the assertion of the theorem holds for compact polyhedra of dimension less than m. By Theorem 7.6, $D\mapsto E_n(C(D))$ is homotopy invariant; in particular $E_n(C(\Delta^m))=0$. If $\dim D=m$ and D is not a simplex, write $D=\Delta^m\cup D'$ as the union of an m-simplex and a subcomplex D' which has fewer m-dimensional simplices. Put $L=\Delta^m\cap D'$; then $\dim L<m$, and we have the exact sequence

$$E_{n+1}(C(L)) \longrightarrow E_n(C(D)) \longrightarrow E_n(C(\Delta^m))\oplus E(C(D')) \longrightarrow E_n(C(L)).$$

We have seen above that $E_n(C(\Delta^m))=0$; moreover $E_n(C(L))=E_{n+1}(C(L))=0$ because $\dim L<m$, and $E_n(C(D'))=0$ because D' has fewer m-dimensional simplices than D. This concludes the proof. □

8. Applications of the homotopy invariance and vanishing homology theorems

8.1. *K*-regularity for commutative C^*-algebras

THEOREM 8.1. *Let V be a smooth affine algebraic variety over $\mathbb{C}$, $R=\mathcal{O}(V)$ and A be a commutative C^*-algebra. Then $A\otimes_{\mathbb{C}}R$ is K-regular.*

Proof. For each fixed $p\geqslant 1$ and $i\in\mathbb{Z}$, write

$$F^p(A)=\operatorname{hocofiber}(K(A\otimes R)\to K(A[t_1,...,t_p]\otimes R))$$

for the homotopy cofiber. It suffices to prove that the homology theory

$$F^p\colon \mathfrak{Comm}/\mathbb{C}\longrightarrow \mathfrak{Spt}$$

satisfies the hypotheses of Theorem 7.7. By [52, Corollary 9.7], $A[t_1,...,t_p]\otimes_{\mathbb{C}}R$ is K-excisive for every C^*-algebra A and every $p\geqslant 1$. It follows that the homology theory $F^p\colon\mathfrak{Ass}/\mathbb{C}\to\mathfrak{Spt}$ is excisive on C^*-algebras. In particular, its restriction to $\mathfrak{Comm}/\mathbb{C}$ is excisive on commutative C^*-algebras. Moreover, if W is any smooth affine algebraic variety, then $R\otimes_{\mathbb{C}}\mathcal{O}(W)=\mathcal{O}(V\times W)$ is regular noetherian, and therefore K-regular. Finally, F^p_* preserves filtering colimits, because both K_* and $(\cdot)\otimes\mathbb{Z}[t_1,...,t_p]\otimes R$ do. □

Remark 8.2. The case $R=\mathbb{C}$ of the previous theorem was discovered by Jonathan Rosenberg. Unfortunately, the two proofs he has given, in [37, Theorem 3.1] and [38, p. 866] turned out to be problematic. A version of Theorem 8.1 for $A=C(D)$, $D\in\mathfrak{Pol}$, was given by Friedlander and Walker in [15, Theorem 5.3]. Furthermore, Rosenberg acknowledges in [38, p. 24] that Walker also found a proof of this in the general case, but that he did not publish it. Anyhow, as Rosenberg observed in [37, p. 91], in this situation, the polyhedral case implies the general case by a short reduction argument (this also follows from Theorem 7.2 above). Hence, an essentially complete argument for the proof of Theorem 8.1 existed already in the literature, although it was scattered in various sources.

The following result compares Quillen's algebraic K-theory with Weibel's homotopy algebraic K-theory, KH, introduced in [51].

Corollary 8.3. *If A is a commutative C^*-algebra, then the map $K_*(A) \to \mathrm{KH}_*(A)$ is an isomorphism.*

Proof. Weibel proved in [51, Proposition 1.5] that if A is a unital K-regular ring, then $K_*(A) \xrightarrow{\sim} \mathrm{KH}_*(A)$. Using excision, it follows that this is true for all commutative C^*-algebras. Now apply Theorem 8.1. □

8.2. Hochschild and cyclic homology of commutative C^*-algebras

In the following paragraph we recall some basic facts about Hochschild and cyclic homology that we shall need; the standard reference for these topics is Loday's book [29].

Let k be a field of characteristic zero. Recall that a *mixed complex* of k-vector spaces is a graded vector space $\{M_n\}_{n\geqslant 0}$ together with maps

$$b\colon M_* \longrightarrow M_{*-1} \quad \text{and} \quad B\colon M_* \to M_{*+1}$$

satisfying $b^2=B^2=bB+Bb=0$. One can associate various chain complexes with a mixed complex M, giving rise to the Hochschild, cyclic, negative cyclic and periodic cyclic homologies of M, respectively denoted by HH_*, HC_*, HN_* and HP_*. For example, $\mathrm{HH}_*(M)=H_*(M,b)$. A map of mixed complexes is a homogeneous map which commutes with both b and B. It is called a *quasi-isomorphism* if it induces an isomorphism at the level of Hochschild homology; this automatically implies that it also induces an isomorphism for HC and all the other homologies mentioned above. For a k-algebra A there is defined a mixed complex $(C(A/k), b, B)$, with

$$C_n(A/k) = \begin{cases} \tilde{A}_k \otimes_k A^{\otimes_k n}, & \text{if } n>0,\\ A, & \text{if } n=0. \end{cases}$$

We write $\mathrm{HH}_*(A/k)$, $\mathrm{HC}_*(A/k)$, etc. for $\mathrm{HH}_*(C(A/k))$, $\mathrm{HC}_*(C(A/k))$, etc. If furthermore A is unital and $\bar{A}=A/k$, then there is also a mixed complex $\overline{C}(A/k)$ with

$$\overline{C}_n(A/k) = A \otimes_k \bar{A}^{\otimes_k n},$$

and the natural surjection $C(A/k)\to\overline{C}(A/k)$ is a quasi-isomorphism. Note also that

$$\ker(\overline{C}(\tilde{A}_k/k)\to\overline{C}(k/k))=C(A/k). \tag{8.1}$$

If A is commutative and unital, we have a third mixed complex $(\Omega_{A/k},0,d)$ given in degree n by $\Omega^n_{A/k}$, the module of n-Kähler differential forms, where d is the exterior derivation of forms. A natural map of mixed complexes $\mu\colon\overline{C}(A/k)\to\Omega_{A/k}$ is defined by

$$\mu(a_0\otimes_k\bar{a}_1\otimes_k\ldots\otimes_k\bar{a}_n)=\frac{1}{n!}a_0\,da_1\wedge\ldots\wedge da_n. \tag{8.2}$$

It was shown by Loday and Quillen in [30] (using a classical result of Hochschild–Kostant–Rosenberg [24]) that μ is a quasi-isomorphism if A is a *smooth* k-algebra, i.e. $A=\mathcal{O}(V)$ for some smooth affine algebraic variety over k. It follows from this (see [29]) that for $Z\Omega^n_{A/k}=\ker(d\colon\Omega^n_{A/k}\to\Omega^{n+1}_{A/k})$ and $H^*_{\mathrm{dR}}(A/k)=H^*(\Omega_{A/k},d)$, we have, for $n\in\mathbb{Z}$,

$$\begin{aligned}\mathrm{HH}_n(A/k)&=\Omega^n_{A/k},\\ \mathrm{HC}_n(A/k)&=\frac{\Omega^n_{A/k}}{d\Omega^{n-1}_{A/k}}\oplus\bigoplus_{0\leqslant 2i<n}H^{n-2i}_{\mathrm{dR}}(A/k),\\ \mathrm{HN}_n(A/k)&=Z\Omega^n_{A/k}\oplus\prod_{p>0}H^{n+2p}_{\mathrm{dR}}(A/k),\\ \mathrm{HP}_n(A/k)&=\prod_{p\in\mathbb{Z}}H^{2p-n}_{\mathrm{dR}}(A/k).\end{aligned} \tag{8.3}$$

For arbitrary commutative unital A, there is a decomposition

$$C_n(A/k)=\bigoplus_{p=0}^{n}C^{(p)}(A/k)$$

such that b maps $C^{(p)}$ to itself, while $B(C^{(p)})\subset C^{(p+1)}$ (see [29]). One defines

$$\mathrm{HH}^{(p)}_n(A/k)=H_nC^{(p)}(A/k).$$

We have

$$\mathrm{HH}^{(p)}_q(A/k)=\begin{cases}0, & \text{for } q<p,\\ \Omega^p_{A/k}, & \text{for } q=p,\end{cases}$$

but in general, for $q>p$, $\mathrm{HH}^{(p)}_q(A/k)\neq 0$. The map (8.2) is still a quasi-isomorphism if A is smooth over a field $F\supset k$; this follows from the Loday–Quillen result using the base change spectral sequence of Kassel–Sletsjøe, which we recall below.

LEMMA 8.4. (Kassel–Sletsjøe, [27, Special cases 4.3a]) *Let $k \subseteq F$ be fields of characteristic zero. For each $p \geqslant 1$ there exists a bounded second-quadrant homological spectral sequence, for $0 \leqslant i < p$ and $j \geqslant 0$,*

$$ {}_pE^1_{-i,i+j} = \Omega^i_{F/k} \otimes_F \mathrm{HH}^{(p-i)}_{p-i+j}(R/F) \Longrightarrow \mathrm{HH}^{(p)}_{p+j}(R/k). $$

COROLLARY 8.5. *If A is a smooth F-algebra, then (8.2) is a quasi-isomorphism.*

THEOREM 8.6. *Let X be a compact topological space, $A = C(X)$ and $k \subset \mathbb{C}$ be a subfield. Then the map (8.2) is a quasi-isomorphism, and we have the identities (8.3).*

Proof. Extend $C^{(p)}(\cdot/k)$ (and $\mathrm{HH}^{(p)}_n(\cdot/k)$) to non-unital algebras by

$$ C^{(p)}(A/k) = \ker(C^{(p)}(\tilde{A}_k/k) \to C^{(p)}(k/k)). $$

Let $E^{(p)}(A/k)$ be the spectrum associated with $C^{(p)}(A/k)$ by the Dold–Kan correspondence. Regard $E^{(p)}$ as a homology theory of $\mathbb{C}$-algebras. Then $E^{(p)}$ is excisive on C^*-algebras, by Remark 2.9 and naturality. Furthermore, $E^{(p)}_n(A/k) = \mathrm{HH}^{(p)}_n(A/k) = 0$ whenever $n > p$ and A is smooth over $\mathbb{C}$, by Corollary 8.5. It is also clear that $\mathrm{HH}^{(p)}_*(\cdot/k)$ preserves filtering colimits, since $\mathrm{HH}_*(\cdot/k)$ does. Thus, we may apply Theorem 7.7 to conclude the proof. □

8.3. The Farrell–Jones isomorphism conjecture

Let A be a ring, Γ be a group and $q \leqslant 0$. Put

$$ \mathrm{Wh}^A_q(\Gamma) = \mathrm{coker}(K_q(A) \to K_q(A[\Gamma])) $$

for the cokernel of the map induced by the natural inclusion $A \subset A[\Gamma]$. Recall [31, p. 708, Conjecture 1] that the Farrell–Jones conjecture with coefficients for a torsion-free group implies that if Γ is torsion-free and A is a noetherian regular unital ring, then

$$ \mathrm{Wh}^A_q(\Gamma) = 0, \quad q \leqslant 0. \tag{8.4} $$

Note that the conjecture in particular implies that $K_q(A[\Gamma]) = 0$ for $q < 0$ if A is noetherian regular, for in this case we have $K_q(A) = 0$ for $q < 0$.

THEOREM 8.7. *Let Γ be a torsion-free group which satisfies (8.4) for every commutative smooth $\mathbb{C}$-algebra A. Also let $q \leqslant 0$. Then the functor $A \mapsto \mathrm{Wh}^A_q(\Gamma)$ is homotopy invariant on commutative C^*-algebras.*

Proof. It follows from Theorem 7.6 applied to $A \mapsto \mathrm{Wh}^A_q(\Gamma)$. □

Corollary 8.8. *Let* Γ *be as above and* X *be a contractible compact space. Then* $K_0(C(X)[\Gamma])=\mathbb{Z}$ *and* $K_q(C(X)[\Gamma])=0$ *for* $q<0$.

Corollary 8.9. *Let* Γ *be as above. Then, the functor*

$$X \longmapsto K_q(C(X)[\Gamma])$$

is homotopy invariant on the category of compact topological spaces for $q\leqslant 0$.

Proof. This follows directly from Proposition 2.3 and the preceding corollary. □

The general case of the Farrell–Jones conjecture predicts that for any group Γ and any unital ring R, the assembly map

$$\mathcal{A}^\Gamma(R): \mathbb{H}^\Gamma(E_{\mathcal{VC}}(\Gamma), K(R)) \longrightarrow K(R[\Gamma]) \tag{8.5}$$

is an equivalence. Here $\mathbb{H}^\Gamma(\cdot\,, K(R))$ is the equivariant homology theory associated with the spectrum $K(R)$ and $E_{\mathcal{VC}}(\Gamma)$ is the classifying space with respect to the class of virtually cyclic subgroups (see [31] for definitions of these objects).

We also get the following result.

Theorem 8.10. *Let* Γ *be a group such that the map* (8.5) *is an equivalence for every smooth commutative* $\mathbb{C}$*-algebra* A*. Then* (8.5) *is an equivalence for every* C^**-algebra* A.

Proof. It follows from Theorem 7.7 applied to $E(R)=\mathrm{hocofiber}(\mathcal{A}^\Gamma(R))$. □

Remark 8.11. We have

$$K_q(C(X)[\Gamma]) = \pi_q^S(\mathrm{map}_-(X_-, KD(\mathbb{C}[\Gamma]))), \quad q\leqslant 0,$$

where $KD(\mathbb{C}[\Gamma])$ denotes the diffeotopy K-theory spectrum, see [10, Definition 4.1.3]. This follows from the study of a suitable coassembly map and is not carried out in detail here. The homotopy groups of $KD(\mathbb{C}[\Gamma])$ can be computed from the equivariant connective K-homology of $\underline{E}\Gamma$ using the Farrell–Jones assembly map.

8.4. Adams operations and the decomposition of rational *K*-theory

The rational K-theory of a unital commutative ring A carries a natural decomposition

$$K_n(A)\otimes\mathbb{Q} = \bigoplus_{i\geqslant 0} K_n(A)^{(i)}.$$

Here

$$K_n(A)^{(i)} = \bigcap_{k\neq 0} \{x\in K_n(A) : \psi^k(x) = k^i x\},$$

where ψ^k is the Adams operation. For example, $K_0^{(0)}(A)=H^0(\operatorname{Spec} A,\mathbb{Q})$ is the rank component, and $K_n^{(0)}(A)=0$ for $n>0$ ([28, Corollaire 6.8]). A conjecture of Alexander Beĭlinson and Christophe Soulé (see [3] and [40]) asserts that

$$K_n^{(i)}(A)=0 \quad \text{for } n\geqslant \max\{1,2i\}. \tag{8.6}$$

The conjecture as stated was proved wrong for non-regular A (see [16] and [13, Remark 7.5.6]), but no regular counterexamples have been found. Moreover, the original statement has been formulated in terms of motivic cohomology (with rational, torsion and integral coefficients) and generalized to regular noetherian schemes [26, §4.3.4]. For example, if $X=\operatorname{Spec} R$ is smooth then $K_n^{(i)}(R)=H^{2i-n}(X,\mathbb{Q}(i))$ is the motivic cohomology of X with coefficients in the twisted sheaf $\mathbb{Q}(i)$.

We shall need the well-known fact that the validity of (8.6) for $\mathbb{C}$ implies its validity for all smooth $\mathbb{C}$-algebras; this is Proposition 8.13 below. In turn this uses the also well-known fact that rational K-theory sends field inclusions to monomorphisms. We include proofs of both facts for the sake of completeness.

LEMMA 8.12. *Let $F\subset E$ be fields. Then $K_*(E)\otimes\mathbb{Q}\to K_*(F)\otimes\mathbb{Q}$ is injective.*

Proof. Since K-theory commutes with filtering colimits, we may assume that E/F is a finitely generated field extension, which we may write as a finite extension of a finitely generated purely transcendental extension. If E/F is purely transcendental, then, by induction, we are reduced to the case $E=F(t)$, which follows from [18, Theorem 1.3]. If $d=\dim_F E$ is finite, then the transfer map $K_*(E)\to K_*(F)$ [34, p. 111] splits $K_*(F)\to K_*(E)$ up to d-torsion. □

PROPOSITION 8.13. *If (8.6) holds for $\mathbb{C}$, then it holds for all smooth $\mathbb{C}$-algebras.*

Proof. The Gysin sequence argument at the beginning of [26, §4.3.4] shows that if (8.6) is an isomorphism for all finitely generated field extensions of $\mathbb{C}$, then it is an isomorphism for all smooth R. If $E\supset\mathbb{C}$ is a finitely generated field extension, then we may write $E=F[\alpha]$ for some purely transcendental field extension $F\cong\mathbb{C}(t_1,...,t_n)\supset\mathbb{C}$ and some algebraic element α. From this and the fact that $\mathbb{C}$ is algebraically closed and of infinite transcendence degree over $\mathbb{Q}$, we see that E is isomorphic to a subfield of $\mathbb{C}$. Now apply Lemma 8.12. □

THEOREM 8.14. *Assume that (8.6) holds for the field $\mathbb{C}$. Then it also holds for all commutative C^*-algebras.*

Proof. By Proposition 8.13, our current hypotheses imply that (8.6) is true for smooth A. In particular, we have that the homology theory $K^{(i)}$ vanishes on smooth

A for $n \geqslant n_0 = \max\{2i, 1\}$. Because K-theory satisfies excision for C^*-algebras and commutes with algebraic filtering colimits, the same is true of $K^{(i)}$. Hence, we may apply Theorem 7.7, concluding the proof. □

Acknowledgements. Part of the research for this article was carried out during a visit of the first author to Universität Göttingen. He is indebted to this institution for their hospitality. He also wishes to thank Charles A. Weibel for a useful email discussion on Beĭlinson–Soulé's conjecture. A previous version of this article contained a technical mistake in the proof of Theorem 7.2; we are thankful to Emanuel Rodríguez Cirone for bringing this to our attention.

References

[1] Bass, H., Some problems in "classical" algebraic K-theory, in *Algebraic K-Theory, II: "Classical" Algebraic K-Theory and Connections with Arithmetic* (Seattle, WA, 1972), Lecture Notes in Math., 342, pp. 3–73. Springer, Berlin–Heidelberg, 1973.

[2] Basu, S., Pollack, R. & Roy, M. F., *Algorithms in Real Algebraic Geometry.* Algorithms and Computation in Mathematics, 10. Springer, Berlin–Heidelberg, 2003.

[3] Beĭlinson, A. A., Higher regulators and values of L-functions, in *Current Problems in Mathematics*, Vol. 24, Itogi Nauki i Tekhniki, pp. 181–238. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984 (Russian).

[4] Bourbaki, N., *Éléments de mathématique. Espaces vectoriels topologiques.* Chapitres 1 à 5. Masson, Paris, 1981.

[5] Brumfiel, G. W., Quotient spaces for semialgebraic equivalence relations. *Math. Z.*, 195 (1987), 69–78.

[6] Calder, A. & Siegel, J., Homotopy and Kan extensions, in *Categorical Topology* (Mannheim, 1975), Lecture Notes in Math., 540, pp. 152–163. Springer, Berlin–Heidelberg, 1976.

[7] — Kan extensions of homotopy functors. *J. Pure Appl. Algebra*, 12 (1978), 253–269.

[8] Cortiñas, G., Haesemeyer, C., Walker, M. E. & Weibel, C., Bass' *NK* groups and cdh-fibrant Hochschild homology. *Invent. Math.*, 181 (2010), 421–448.

[9] — A negative answer to a question of Bass. *Proc. Amer. Math. Soc.*, 139 (2011), 1187–1200.

[10] Cortiñas, G. & Thom, A., Comparison between algebraic and topological K-theory of locally convex algebras. *Adv. Math.*, 218 (2008), 266–307.

[11] Davis, J. F., Some remarks on Nil groups in algebraic K-theory. Preprint, 2008. arXiv:0803.1641 [math.KT].

[12] Drinfeld, V., Infinite-dimensional vector bundles in algebraic geometry: an introduction, in *The Unity of Mathematics*, Progr. Math., 244, pp. 263–304. Birkhäuser, Boston, MA, 2006.

[13] Feĭgin, B. L. & Tsygan, B. L., Additive K-theory, in *K-Theory, Arithmetic and Geometry* (Moscow, 1984–1986), Lecture Notes in Math., 1289, pp. 67–209. Springer, Berlin–Heidelberg, 1987.

[14] Frei, A., Kan extensions along full functors: Kan and Čech extensions of homotopy invariant functors. *J. Pure Appl. Algebra*, 17 (1980), 285–292.

[15] Friedlander, E. M. & Walker, M. E., Comparing K-theories for complex varieties. *Amer. J. Math.*, 123 (2001), 779–810.

[16] Geller, S. C. & Weibel, C. A., Hodge decompositions of Loday symbols in K-theory and cyclic homology. *K-Theory*, 8 (1994), 587–632.
[17] Gersten, S. M., On the spectrum of algebraic K-theory. *Bull. Amer. Math. Soc.*, 78 (1972), 216–219.
[18] — Some exact sequences in the higher K-theory of rings, in *Algebraic K-theory*, I: *Higher K-theories* (Seattle, WA, 1972), Lecture Notes in Math., 341, pp. 211–243. Springer, Berlin–Heidelberg, 1973.
[19] Gubeladze, J., The Anderson conjecture and projective modules over monoid algebras. *Soobshch. Akad. Nauk Gruzin. SSR*, 125 (1987), 289–291 (Russian).
[20] — The Anderson conjecture and a maximal class of monoids over which projective modules are free. *Mat. Sb.*, 135 (177) (1988), 169–185, 271 (Russian); English translation in *Math. USSR–Sb.*, 63 (1989), 165–180.
[21] — On Bass' question for finitely generated algebras over large fields. *Bull. Lond. Math. Soc.*, 41 (2009), 36–40.
[22] Hardt, R. M., Semi-algebraic local-triviality in semi-algebraic mappings. *Amer. J. Math.*, 102 (1980), 291–302.
[23] Hironaka, H., Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math.*, 79 (1964), 109–203, 205–326.
[24] Hochschild, G., Kostant, B. & Rosenberg, A., Differential forms on regular affine algebras. *Trans. Amer. Math. Soc.*, 102 (1962), 383–408.
[25] Jouanolou, J. P., Une suite exacte de Mayer–Vietoris en K-théorie algébrique, in *Algebraic K-theory*, I: *Higher K-theories* (Seattle, WA, 1972), Lecture Notes in Math., 341, pp. 293–316. Springer, Berlin–Heidelberg, 1973.
[26] Kahn, B., Algebraic K-theory, algebraic cycles and arithmetic geometry, in *Handbook of K-Theory*. Vol. 1, pp. 351–428. Springer, Berlin–Heidelberg, 2005.
[27] Kassel, C. & Sletsjøe, A. B., Base change, transitivity and Künneth formulas for the Quillen decomposition of Hochschild homology. *Math. Scand.*, 70 (1992), 186–192.
[28] Kratzer, C., λ-structure en K-théorie algébrique. *Comment. Math. Helv.*, 55 (1980), 233–254.
[29] Loday, J.-L., *Cyclic Homology*. Grundlehren der Mathematischen Wissenschaften, 301. Springer, Berlin–Heidelberg, 1998.
[30] Loday, J.-L. & Quillen, D., Cyclic homology and the Lie algebra homology of matrices. *Comment. Math. Helv.*, 59 (1984), 569–591.
[31] Lück, W. & Reich, H., The Baum–Connes and the Farrell–Jones conjectures in K- and L-theory, in *Handbook of K-theory*. Vol. 2, pp. 703–842. Springer, Berlin–Heidelberg, 2005.
[32] Pedersen, E. K. & Weibel, C. A., A nonconnective delooping of algebraic K-theory, in *Algebraic and Geometric Topology* (New Brunswick, NJ, 1983), Lecture Notes in Math., 1126, pp. 166–181. Springer, Berlin–Heidelberg, 1985.
[33] — K-theory homology of spaces, in *Algebraic Topology* (Arcata, CA, 1986), Lecture Notes in Math., 1370, pp. 346–361. Springer, Berlin–Heidelberg, 1989.
[34] Quillen, D., Higher algebraic K-theory. I, in *Algebraic K-Theory*, I: *Higher K-Theories* (Seattle, WA, 1972), Lecture Notes in Math., 341, pp. 85–147. Springer, Berlin–Heidelberg, 1973.
[35] — Projective modules over polynomial rings. *Invent. Math.*, 36 (1976), 167–171.
[36] Rosenberg, J., K and KK: topology and operator algebras, in *Operator Theory*: *Operator Algebras and Applications*(Durham, NH, 1988), Proc. Sympos. Pure Math., 51, Part 1, pp. 445–480. Amer. Math. Soc., Providence, RI, 1990.
[37] — The algebraic K-theory of operator algebras. *K-Theory*, 12 (1997), 75–99.

[38] — Comparison between algebraic and topological K-theory for Banach algebras and C^*-algebras, in *Handbook of K-Theory*. Vol. 2, pp. 843–874. Springer, Berlin–Heidelberg, 2005.
[39] Serre, J.-P., Faisceaux algébriques cohérents. *Ann. of Math.*, 61 (1955), 197–278.
[40] Soulé, C., Opérations en K-théorie algébrique. *Canad. J. Math.*, 37 (1985), 488–550.
[41] Suslin, A. A., Projective modules over polynomial rings are free. *Dokl. Akad. Nauk SSSR*, 229 (1976), 1063–1066 (Russian); English translation in *Soviet Math. Dokl.*, 17 (1976), 1160–1164.
[42] — On the K-theory of algebraically closed fields. *Invent. Math.*, 73 (1983), 241–245.
[43] Suslin, A. A. & Wodzicki, M., Excision in algebraic K-theory and Karoubi's conjecture. *Proc. Nat. Acad. Sci. USA*, 87:24 (1990), 9582–9584.
[44] — Excision in algebraic K-theory. *Ann. of Math.*, 136 (1992), 51–122.
[45] Swan, R. G., Projective modules over Laurent polynomial rings. *Trans. Amer. Math. Soc.*, 237 (1978), 111–120.
[46] — Gubeladze's proof of Anderson's conjecture, in *Azumaya Algebras*, *Actions*, *and Modules* (Bloomington, IN, 1990), Contemp. Math., 124, pp. 215–250. Amer. Math. Soc., Providence, RI, 1992.
[47] Switzer, R. M., *Algebraic Topology—Homotopy and Homology*. Classics in Mathematics. Springer, Berlin–Heidelberg, 2002.
[48] Thom, A. B., *Connective E-Theory and Bivariant Homology*. Ph.D. Thesis, Universität Münster, Münster, 2003.
[49] Thomason, R. W. & Trobaugh, T., Higher algebraic K-theory of schemes and of derived categories, in *The Grothendieck Festschrift*, Vol. III, Progr. Math., 88, pp. 247–435. Birkhäuser, Boston, MA, 1990.
[50] Wagoner, J. B., Delooping classifying spaces in algebraic K-theory. *Topology*, 11 (1972), 349–370.
[51] Weibel, C. A., Homotopy algebraic K-theory, in *Algebraic K-Theory and Algebraic Number Theory* (Honolulu, HI, 1987), Contemp. Math., 83, pp. 461–488. Amer. Math. Soc., Providence, RI, 1989.
[52] Wodzicki, M., Excision in cyclic homology and in rational algebraic K-theory. *Ann. of Math.*, 129 (1989), 591–639.
[53] — Homological properties of rings of functional-analytic type. *Proc. Nat. Acad. Sci. USA*, 87:13 (1990), 4910–4911.

Guillermo Cortiñas
Departamento de Matemática
Universidad de Buenos Aires
Ciudad Universitaria Pab 1
C1428EGA Buenos Aires
Argentina
gcorti@dm.uba.ar

Andreas Thom
Mathematisches Institut
Universität Leipzig
PF 100920
DE-04009 Leipzig
Germany
thom@math.uni-leipzig.de

Received March 18, 2010
Received in revised form August 19, 2010