Acta Math., 209 (2012), 179–196
DOI: 10.1007/s11511-012-0084-4

Strict comparison and $\mathcal{Z}$-absorption of nuclear C^*-algebras

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1. Introduction

X. Jiang and H. Su [5] constructed a unital separable simple infinite-dimensional nuclear C^*-algebra $\mathcal{Z}$, called the Jiang–Su algebra, whose K-theoretic invariant is isomorphic to that of the complex numbers. The Jiang–Su algebra has recently started to play a central role in Elliott's classification program for nuclear C^*-algebras. We say that a unital C^*-algebra is *$\mathcal{Z}$-absorbing* if $A \cong A \otimes \mathcal{Z}$. H. Lin, Z. Niu and W. Winter proved that certain $\mathcal{Z}$-absorbing C^*-algebras are classified by their ordered K-groups [13], [26]. Indeed, all classes of unital simple nuclear C^*-algebras for which Elliott's classification conjecture have been confirmed consist of $\mathcal{Z}$-absorbing algebras. One may view $\mathcal{Z}$ as being the stably finite analogue of the Cuntz algebra $\mathcal{O}_\infty$. W. Winter also showed that $\mathcal{Z}$ is the initial object in the category of strongly self-absorbing C^*-algebras [24].

In view of this, it is desirable to characterize $\mathcal{Z}$-absorbing C^*-algebras in various manners. In 2008, A. S. Toms and W. Winter conjectured that the properties of strict comparison, finite nuclear dimension, and $\mathcal{Z}$-absorption are equivalent for unital separable simple infinite-dimensional nuclear C^*-algebras (see [22] and [27], for example). M. Rørdam proved that $\mathcal{Z}$-absorption implies strict comparison for unital simple exact C^*-algebras [17]. W. Winter showed that any unital separable simple infinite-dimensional C^*-algebra with finite nuclear dimension is $\mathcal{Z}$-absorbing [25]. In the present paper we provide another partial answer to the conjecture above. Namely, it will be shown that strict comparison implies $\mathcal{Z}$-absorption under the assumption that the algebra has finitely many extremal traces.

The following is the main result of this paper.

The first author was supported in part by the Grant-in-Aid for Young Scientists (B) of JSPS. The second author was supported by the JSPS Research Fellowships for Young Scientists.

Theorem 1.1. *Let A be a unital separable simple infinite-dimensional nuclear C^*-algebra with finitely many extremal traces. Then the following are equivalent*:

(i) $A\otimes\mathcal{Z}\cong A$;

(ii) *A has strict comparison*;

(iii) *any completely positive map $A\to A$ can be excised in small central sequences*;

(iv) *A has property* (SI).

Here, we recall the definition of strict comparison. In this paper we denote by A_+ the positive cone of A and by $T(A)$ the set of tracial states on A. We define the dimension function d_τ associated with $\tau\in T(A)$ by $d_\tau(a)=\lim_{n\to\infty}\tau(a^{1/n})$ for $a\in M_k(A)_+$, where τ is regarded as an unnormalized trace on $M_k(A)$. We say that a separable nuclear C^*-algebra A has *strict comparison* if for $a,b\in M_k(A)_+$, with $d_\tau(a)<d_\tau(b)$ for any $\tau\in T(A)$, there exist $r_n\in M_k(A)$, $n\in\mathbb{N}$, such that $r_n^*br_n\to a$. The definition of excision in small central sequences is given in Definition 2.1, and the definition of property (SI) is given in Definition 4.1. As mentioned above, (i) $\Rightarrow$ (ii) was proved by M. Rørdam [17, Corollary 4.6] without assuming that A has finitely many extremal tracial states. The implication (iii) $\Rightarrow$ (iv) is immediate from the definitions and does not need the assumption of finitely many extremal traces. We use the full assumption on A for the implications (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i). It will also be shown that (i) implies (iii) and (iv) without the assumption of finitely many extremal tracial states in Theorem 4.2. In §5, using the same method, we shall show approximate divisibility of unital separable simple nuclear C^*-algebras with tracial rank zero.

The main technical device in this paper is excision of completely positive maps. In [1], C. A. Akemann, J. Anderson and G. K. Pedersen proved that any pure state on a C^*-algebra can be excised by positive norm-1 elements. By using their result, E. Kirchberg obtained a Stinespring dilation type theorem for unital nuclear completely positive maps from a unital purely infinite simple C^*-algebra to itself. This theorem is one of the technical cornerstones in the proof of Kirchberg's celebrated embedding theorem for exact C^*-algebras [7], [8]. In this article, by using the result of [1], we will establish a similar 'dilation' type result for completely positive maps in the setting of stably finite C^*-algebras. To this end, we have to work with central sequences and to take into account the values of traces on them (Definition 2.1).

The other ingredient in this paper is property (SI) (SI stands for 'small isometries'). The idea of property (SI) originates with A. Kishimoto (see [11, Lemma 3.6]). Using it, he proved that certain automorphisms of AT algebras (i.e., inductive limits of finite-dimensional C^*-algebras over $C(\mathbb{T})$) have the Rokhlin property. (See [14] and [19] for further developments.) In [15] and [20], property (SI) was used to show $\mathcal{Z}$-absorption of crossed products by strongly outer actions. The main theorem in the present paper implies that this property is not so restrictive but is shared by 'many' stably finite nuclear C^*-algebras.

We recall the notion of central sequence algebras of C^*-algebras. Let A be a separable C^*-algebra. Set

$$A^\infty = \ell^\infty(\mathbb{N}, A)/\{\{a_n\}_n \in \ell^\infty(\mathbb{N}, A) : \lim_{n\to\infty} \|a_n\| = 0\}.$$

We identify A with the C^*-subalgebra of A^∞ consisting of equivalence classes of constant sequences. We let

$$A_\infty = A^\infty \cap A'$$

and call it the *central sequence algebra* of A. A sequence $\{x_n\}_n \in \ell^\infty(\mathbb{N}, A)$ is called a *central sequence* if $\|[a, x_n]\| \to 0$ as $n \to \infty$ for all $a \in A$. A central sequence is a representative of an element in A_∞.

2. Excision in small central sequences

In this section, we prove Proposition 2.2, which plays an important role in §3.

Definition 2.1. Let A be a separable C^*-algebra with $T(A) \neq \varnothing$, and let $\varphi: A \to A$ be a completely positive map. We say that φ *can be excised in small central sequences* if for any central sequences $\{e_n\}_n$ and $\{f_n\}_n$ of positive contractions in A satisfying

$$\lim_{n\to\infty} \max_{\tau\in T(A)} \tau(e_n) = 0 \quad \text{and} \quad \lim_{m\to\infty} \liminf_{n\to\infty} \min_{\tau\in T(A)} \tau(f_n^m) > 0,$$

there exist $s_n \in A$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \|s_n^* a s_n - \varphi(a) e_n\| = 0 \text{ for any } a \in A \quad \text{and} \quad \lim_{n\to\infty} \|f_n s_n - s_n\| = 0.$$

The following proposition is our main tool for the proof of the implication (ii) $\Rightarrow$ (iii) in Theorem 1.1. This may be thought of as a stably finite analogue of Kirchberg's Stinespring type theorem [7] (see also [8, Proposition 1.4]).

PROPOSITION 2.2. *Let A be a unital separable simple infinite-dimensional C^*-algebra with $T(A) \neq \varnothing$. Suppose that A has strict comparison. Let ω be a state on A and let $c_i, d_i \in A$, $i = 1, 2, ..., N$. Then the completely positive map $\varphi: A \to A$ defined by*

$$\varphi(a) = \sum_{i,j=1}^N \omega(d_i^* a d_j) c_i^* c_j, \quad a \in A,$$

can be excised in small central sequences.

In order to prove this proposition, we need a couple of lemmas.

Lemma 2.3. *Let A be a separable C^*-algebra with $T(A)\neq\varnothing$. For any central sequence $\{f_n\}_n$ of positive contractions in A, there exists a central sequence $\{\tilde{f}_n\}_n$ of positive contractions in A such that $\{\tilde{f}_n f_n\}_n=\{\tilde{f}_n\}_n$ in A_∞ and*

$$\lim_{m\to\infty}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(\tilde{f}_n^m)=\lim_{m\to\infty}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^m).$$

Proof. We can find a natural number $N_m\in\mathbb{N}$ such that the inequality

$$\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^m)<\min_{\tau\in T(A)}\tau(f_l^m)+\frac{1}{m}$$

holds for every $l>N_m$. We may assume that $N_m<N_{m+1}$. Define a sequence $\{m_n\}_n$ of natural numbers such that $m_n=m$ when $N_m<n\leqslant N_{m+1}$. Note that $\{m_n\}_n$ is an increasing sequence such that $m_n\to\infty$ and

$$\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^{m_n})\geqslant\lim_{m\to\infty}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^m).$$

Let $\{\widetilde{m}_n\}_n$ be a sequence of natural numbers such that $\widetilde{m}_n\to\infty$, $\widetilde{m}_n\leqslant m_n^{1/2}$ and $\{f_n^{\widetilde{m}_n}\}_n$ is a central sequence. Let $\tilde{f}_n=f_n^{\widetilde{m}_n}$. It is easy to see that $\{f_n\tilde{f}_n\}_n=\{\tilde{f}_n\}_n$ in A_∞. Also,

$$\begin{aligned}\lim_{m\to\infty}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^m)&\geqslant\lim_{m\to\infty}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(\tilde{f}_n^m)\\&=\lim_{m\to\infty}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^{m\widetilde{m}_n})\\&\geqslant\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^{m_n})\\&\geqslant\lim_{m\to\infty}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^m).\end{aligned}$$

□

Lemma 2.4. *Let A be a unital separable simple C^*-algebra with $T(A)\neq\varnothing$ and let $a\in A$ be a non-zero positive element. Then there exists $\alpha>0$ such that*

$$\alpha\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n)\leqslant\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^{1/2}af_n^{1/2})$$

for any central sequence $\{f_n\}_n$ of positive contractions in A.

Proof. Since A is unital and simple, there exist $v_1,v_2,...,v_m\in A$ such that

$$\sum_{i=1}^m v_i^*av_i=1.$$

Set

$$\alpha=\left(\sum_{i=1}^m\|v_i\|^2\right)^{-1}>0.$$

Then we have

$$\begin{aligned}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n) &= \liminf_{n\to\infty}\min_{\tau\in T(A)}\sum_{i=1}^{m}\tau(v_i^* a v_i f_n)\\ &= \liminf_{n\to\infty}\min_{\tau\in T(A)}\sum_{i=1}^{m}\tau(v_i^* a^{1/2} f_n a^{1/2} v_i)\\ &= \liminf_{n\to\infty}\min_{\tau\in T(A)}\sum_{i=1}^{m}\tau(f_n^{1/2} a^{1/2} v_i v_i^* a^{1/2} f_n^{1/2})\\ &\leqslant \frac{1}{\alpha}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^{1/2} a f_n^{1/2}). \qquad \square\end{aligned}$$

LEMMA 2.5. *Let A be a unital separable simple C^*-algebra with $T(A)\neq\varnothing$. Suppose that A has strict comparison. Let $\{e_n\}_n$ and $\{f_n\}_n$ be as in Definition 2.1. Then for any norm-1 positive element $a\in A$, there exists a sequence $\{r_n\}_n$ in A such that*

$$\lim_{n\to\infty}\|r_n^* f_n^{1/2} a f_n^{1/2} r_n - e_n\| = 0 \quad \textit{and} \quad \limsup_{n\to\infty}\|r_n\| = \limsup_{n\to\infty}\|e_n\|^{1/2}.$$

Proof. By [20, Lemma 3.2 (i)], we may assume that $\lim_{n\to\infty}\max_{\tau\in T(A)} d_\tau(e_n)=0$. Set

$$c = \lim_{m\to\infty}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^m) > 0.$$

Take $\varepsilon>0$. It suffices to show that there exist $r_n\in A$, $n\in\mathbb{N}$, such that

$$\limsup_{n\to\infty}\|r_n^* f_n^{1/2} a f_n^{1/2} r_n - e_n\| \leqslant \varepsilon \quad \text{and} \quad \lim_{n\to\infty}\|r_n^* r_n - e_n\| = 0.$$

As $\|a\|=1$, using continuous functional calculus, we get non-zero positive contractions $a_0, a_1\in A$ such that $\|a_0-a\|\leqslant\varepsilon$ and $a_1\leqslant a_0^m$ for all $m\in\mathbb{N}$. Applying Lemma 2.4 to $a_1\in A_+\setminus\{0\}$, we obtain $\alpha>0$. Then for any $m\in\mathbb{N}$ it follows that

$$\begin{aligned}\alpha c &\leqslant \alpha\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^m)\\ &\leqslant \liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^{m/2} a_1 f_n^{m/2})\\ &\leqslant \liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^{m/2} a_0^m f_n^{m/2}).\end{aligned}$$

Put $b_n=f_n^{1/2} a_0 f_n^{1/2}$. Since $\{f_n\}_n$ is central, one has

$$\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(b_n^m) = \liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^{m/2} a_0^m f_n^{m/2}) \geqslant \alpha c$$

for any $m\in\mathbb{N}$. Then we have an increasing sequence $m_n\in\mathbb{N}$ of natural numbers such that $m_n\to\infty$ and $\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(b_n^{m_n})\geqslant\alpha c$.

For $\delta>0$, define a continuous function $g_\delta\in C([0,1])$ by $g_\delta(t)=\max\{0,\delta^{-1}(t-1+\delta)\}$. Let $\varepsilon_n>0$, $n\in\mathbb{N}$, be a decreasing sequence such that $\varepsilon_n\to 0$ and $(1-\varepsilon_n)^{m_n}\to 0$. Then we have

$$\begin{aligned}\liminf_{n\to\infty}\min_{\tau\in T(A)} d_\tau(g_{\varepsilon_n}(b_n)) &\geqslant \liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(g_{\varepsilon_n}(b_n)) \geqslant \liminf_{n\to\infty}\Big(\min_{\tau\in T(A)}\tau(b_n^{m_n})-(1-\varepsilon_n)^{m_n}\Big)\\ &=\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(b_n^{m_n})\geqslant \alpha c>0.\end{aligned}$$

Because A has strict comparison, we can find a sequence $\{q_n\}_n$ in A such that

$$\lim_{n\to\infty}\|q_n^* g_{\varepsilon_n}(b_n)q_n-e_n\|=0.$$

Note that $\{q_n\}_n$ is not necessarily bounded. We define $r_n=g_{\varepsilon_n}^{1/2}(b_n)q_n$ for $n\in\mathbb{N}$. Then it follows that

$$\|(1-b_n)r_n\|\leqslant\varepsilon_n\|r_n\|\to 0 \quad\text{and}\quad \|r_n^*b_nr_n-e_n\|\leqslant\|r_n^*(b_n-1)r_n\|+\|r_n^*r_n-e_n\|\to 0,$$

as $n\to\infty$. Consequently, we have

$$\limsup_{n\to\infty}\|r_n^*f_n^{1/2}af_n^{1/2}r_n-e_n\|\leqslant\limsup_{n\to\infty}\|r_n^*f_n^{1/2}a_0f_n^{1/2}r_n-e_n\|+\varepsilon=\varepsilon. \qquad \square$$

Now we are ready to prove Proposition 2.2.

Proof of Proposition 2.2. Let $\varphi: A\to A$ be as in the statement. Replacing c_i and d_i by $c_i/\|c_i\|$ and $\|c_i\|d_i$, we may assume that $\|c_i\|\leqslant 1$. Let F be a finite subset of the unit ball of A and let $\varepsilon>0$. It suffices to show that there exist $s_n\in A$, $n\in\mathbb{N}$, such that

$$\limsup_{n\to\infty}\|s_n^*xs_n-\varphi(x)e_n\|<\varepsilon \quad\text{and}\quad \lim_{n\to\infty}\|f_ns_n-s_n\|=0$$

for any $x\in F$. Set $G=\{d_i^*xd_j\in A: x\in F,\ i=1,2,...,N\}$ and $\delta=\varepsilon/N^2$.

Since A is unital simple infinite-dimensional, by Glimm's lemma, any state on A can be approximated by pure states in the weak*-topology. Hence we may assume that ω is a pure state on A. By [1, Proposition 2.2], there exists $a\in A_+$ such that $\|a\|=1$ and $\|a(\omega(x)-x)a\|<\delta$ for every $x\in G$. Let $\{e_n\}_n$ and $\{f_n\}_n$ be as in Definition 2.1. By Lemma 2.3, we obtain a central sequence $\{\tilde f_n\}_n$ of positive contractions in A satisfying $\{\tilde f_nf_n\}_n=\{\tilde f_n\}_n$ in A_∞ and

$$\lim_{m\to\infty}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(\tilde f_n^m)=\lim_{m\to\infty}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^m)>0.$$

Applying Lemma 2.5 to $\{e_n\}_n$, $\{\tilde f_n\}_n$ and a^2, we obtain $r_n\in A$, $n\in\mathbb{N}$, satisfying

$$\lim_{n\to\infty}\|r_n^*\tilde f_n^{1/2}a^2\tilde f_n^{1/2}r_n-e_n\|=0 \quad\text{and}\quad \limsup_{n\to\infty}\|r_n\|\leqslant 1.$$

We define

$$s_n = \sum_{i=1}^{N} d_i a \tilde{f}_n^{1/2} r_n c_i, \quad n \in \mathbb{N}.$$

Since $\{f_n\}_n$ is central and $\{r_n\}_n$ is bounded it follows that

$$\begin{aligned}\limsup_{n\to\infty} \|f_n s_n - s_n\| &\leqslant \limsup_{n\to\infty} \sum_{i=1}^{N} \|(1-f_n) d_i a \tilde{f}_n^{1/2}\| \, \|r_n\| \\ &= \limsup_{n\to\infty} \sum_{i=1}^{N} \|d_i a (1-f_n) \tilde{f}_n^{1/2}\| = 0.\end{aligned}$$

Also, for any $x \in F$, we have

$$\begin{aligned}\limsup_{n\to\infty} \|s_n^* x s_n - \varphi(x) e_n\| &= \limsup_{n\to\infty} \left\| \sum_{i,j=1}^{N} c_i^* (r_n^* \tilde{f}_n^{1/2} a d_i^* x d_j a \tilde{f}_n^{1/2} r_n - \omega(d_i^* x d_j) e_n) c_j \right\| \\ &\leqslant \limsup_{n\to\infty} \sum_{i,j=1}^{N} \|r_n^* \tilde{f}_n^{1/2} a d_i^* x d_j a \tilde{f}_n^{1/2} r_n - \omega(d_i^* x d_j) e_n\| \\ &= \limsup_{n\to\infty} \sum_{i,j=1}^{N} \|r_n^* \tilde{f}_n^{1/2} (a d_i^* x d_j a - \omega(d_i^* x d_j) a^2) \tilde{f}_n^{1/2} r_n\| \\ &\leqslant \limsup_{n\to\infty} \sum_{i,j=1}^{N} \|a (d_i^* x d_j - \omega(d_i^* x d_j)) a\| \\ &< \varepsilon.\end{aligned}$$

□

Remark 2.6. In the argument above, the assumption of strict comparison is used in the proof of Lemma 2.5. But it should be pointed out that we need much less than the full strength of strict comparison. Indeed, what we used in the proof of Lemma 2.5 is as follows: if $\{e_n\}_n$ and $\{f_n\}_n$ are sequences of positive contractions in A satisfying

$$\lim_{n\to\infty} \max_{\tau \in T(A)} d_\tau(e_n) = 0 \quad \text{and} \quad \liminf_{n\to\infty} \min_{\tau \in T(A)} d_\tau(f_n) > 0,$$

then there exists a sequence $\{r_n\}_n$ in A such that

$$\lim_{n\to\infty} \|r_n^* f_n r_n - e_n\| = 0.$$

3. Proof of (ii) ⇒ (iii) in Theorem 1.1

In this section, we give a proof of the implication (ii) ⇒ (iii) in Theorem 1.1, by using Proposition 2.2. We begin with the following well-known fact. This is a special case of [9, Proposition 4.2].

Lemma 3.1. *Let A be a unital separable simple infinite-dimensional nuclear C^*-algebra, and let ω be a pure state of A. Then any completely positive map $A \to A$ can be approximated in the pointwise norm topology by completely positive maps φ of the form*

$$\varphi(a) = \sum_{l=1}^{N} \sum_{i,j=1}^{N} \omega(d_i^* a d_j) c_{l,i}^* c_{l,j}, \quad a \in A,$$

where $c_{l,i}, d_i \in A$, $l, i = 1, 2, ..., N$.

Proof. Let $\varrho: A \to M_N$ and $\sigma: M_N \to A$ be completely positive maps. Because A is nuclear, any completely positive map is approximated by completely positive maps which factor through full matrix algebras. Thus it suffices to show that $\sigma \circ \varrho$ can be approximated in the pointwise norm topology by completely positive maps φ as in the lemma. Replacing ϱ and σ by $\varrho(1_A)^{-1/2} \varrho(\cdot) \varrho(1_A)^{-1/2}$ and $\sigma(\varrho(1_A)^{1/2} \cdot \varrho(1_A)^{1/2})$ with inverses taken in the respective hereditary subalgebra, we may assume that ϱ is unital.

We denote by $(\pi, \mathcal{H}, \xi)$ the Gelfand–Naimark–Segal (GNS) representation associated with ω. Since A is unital separable simple infinite-dimensional, $\pi(A)$ does not contain non-zero compact operators on $\mathcal{H}$. Applying Voiculescu's theorem (see, for example, [3, Theorem 1.7.8]) to the unital completely positive map $\varrho \circ \pi^{-1}: \pi(A) \to M_N$, we can find isometries $V_n: \mathbb{C}^N \to \mathcal{H}$, $n \in \mathbb{N}$, such that

$$\lim_{n \to \infty} \|\varrho(a) - V_n^* \pi(a) V_n\| = 0$$

for any $a \in A$. Let $\{e_1, e_2, ..., e_N\}$ be a basis for $\mathbb{C}^N$ and set $\xi_{i,n} = V_n e_i \in \mathcal{H}$. By Kadison's transitivity theorem, we obtain $d_{i,n} \in A$, $i = 1, 2, ..., N$, $n \in \mathbb{N}$, such that $\pi(d_{i,n})\xi = \xi_{i,n}$. Then we have

$$\omega(d_{i,n}^* a d_{j,n}) = (\pi(a)\xi_{j,n} \mid \xi_{i,n})_{\mathcal{H}} = (V_n^* \pi(a) V_n e_i \mid e_j)$$

for $i, j = 1, 2, ..., N$ and $a \in A$, where $(\cdot \mid \cdot)_{\mathcal{H}}$ and $(\cdot \mid \cdot)$ denote the inner products. This implies that

$$\lim_{n \to \infty} \|\varrho(a) - (\omega(d_{i,n}^* a d_{j,n}))_{i,j}\| = 0, \quad a \in A.$$

Let $e_{i,j}$ be the standard matrix units for M_N. Since $\sigma: M_N \to A$ is a completely positive map, the matrix $(\sigma(e_{i,j}))_{i,j} \in M_N(A)$ is positive (see, for example, [3, Proposition 1.5.12]). Hence there exist $c_{l,j} \in A$, $l, j = 1, 2, ..., N$, such that

$$\sigma(e_{i,j}) = \sum_{l=1}^{N} c_{l,i}^* c_{l,j}. \qquad \square$$

The proof of the following lemma relies on A. Kishimoto's technique used in the proof of the implication (2) ⇒ (1) in [10, Theorem 4.5]. For a state ω on A, we define the seminorm $\|\cdot\|_\omega$ by $\|a\|_\omega = \omega(a^* a)^{1/2}$ for $a \in A$.

LEMMA 3.2. *Let ω be a state on a unital separable C^*-algebra A and let $k\in\mathbb{N}$. Let $\{e_n\}_n$ be a central sequence of positive contractions in A and let $\{u_n\}_n$ be a central sequence of unitary operators in A. If*

$$\lim_{n\to\infty} \|\mathrm{Ad}\, u_n^i(e_n)e_n\|_\omega = 0$$

holds for every $i=1,2,...,k-1$, then there exists a central sequence $\{e_n'\}_n$ of positive contractions in A such that

$$e_n' \leqslant e_n, \quad \lim_{n\to\infty} \omega(e_n - e_n') = 0, \quad \text{and} \quad \lim_{n\to\infty} \|\mathrm{Ad}\, u_n^i(e_n')e_n'\| = 0$$

for every $i=1,2,...,k-1$.

Proof. For $m\in\mathbb{N}$, we let f_m denote the continuous function on $[0,\infty)$ defined by $f_m(t)=\min\{1,mt\}$. Define central sequences $\{g_n\}_n$ and $\{e_{m,n}'\}_n$ by

$$g_n = e_n^{1/2}\left(\sum_{i=1}^{k-1} \mathrm{Ad}\, u_n^i(e_n)\right)e_n^{1/2}$$

and

$$e_{m,n}' = e_n^{1/2}(1-f_m(g_n))e_n^{1/2}.$$

Note that $e_{m,n}' \leqslant e_n$ for any $m,n\in\mathbb{N}$. By the assumption of e_n and u_n, for any $j\in\mathbb{N}$ it follows that

$$\begin{aligned}
\omega(e_n^{1/2}g_n^j e_n^{1/2}) &\leqslant \|g_n\|^{j-1}\omega(e_n^{1/2}g_n e_n^{1/2}) \\
&\leqslant (k-1)^{j-1}\sum_{i=1}^{k-1} \omega(e_n \mathrm{Ad}\, u_n^i(e_n)e_n) \\
&\leqslant (k-1)^{j-1}\sum_{i=1}^{k-1} \|\mathrm{Ad}\, u_n^i(e_n)e_n\|_\omega \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Then we have

$$\omega(e_n - e_{m,n}') = \omega(e_n^{1/2} f_m(g_n) e_n^{1/2}) \to 0 \quad \text{as } n\to\infty$$

for any $m\in\mathbb{N}$. Furthermore, for $i=1,2,...,k-1$ we have

$$\begin{aligned}
\|\mathrm{Ad}\, u_n^i(e_{m,n}')e_{m,n}'\|^2 &\leqslant \|e_{m,n}'\mathrm{Ad}\, u_n^i(e_{m,n}')e_{m,n}'\| \\
&\leqslant \left\| e_{m,n}' \sum_{i=1}^{k-1} \mathrm{Ad}\, u_n^i(e_n)e_{m,n}' \right\| \\
&= \left\| e_n^{1/2}(1-f_m(g_n))e_n^{1/2} \sum_{i=1}^{k-1} \mathrm{Ad}\, u_n^i(e_n)e_n^{1/2}(1-f_m(g_n))e_n^{1/2} \right\| \\
&\leqslant \|(1-f_m(g_n))g_n\| \\
&< \frac{1}{m}.
\end{aligned}$$

Since A is separable and $\{e'_{m,n}\}_n$ is a central sequence, we can find an increasing sequence $\{m_n\}_n$ of natural numbers such that $m_n \to \infty$, $\omega(e_n - e'_{m_n,n}) \to 0$ and $\{e'_{m_n,n}\}_n$ is a central sequence. Therefore $e'_n = e'_{m_n,n}$, $n \in \mathbb{N}$, satisfy the desired conditions. $\square$

In the proof of the following lemma, we use [21, Lemma 2.1]. We remark that this lemma in [21] heavily depends on U. Haagerup's theorem [4, Theorem 3.1], which says that any nuclear C^*-algebra has a virtual diagonal in the sense of B. E. Johnson [6].

For the definition of order-zero maps, the reader should see [27, §1].

Lemma 3.3. *Let A be a unital separable simple infinite-dimensional nuclear C^*-algebra with finitely many extremal tracial states. For any $k \in \mathbb{N}$, there exist a completely positive contractive order zero map $\psi\colon M_k \to A_\infty$ and a central sequence $\{c_n\}_n$ of positive contractions in A such that*

$$\lim_{n\to\infty} \max_{\tau\in T(A)} \left| \tau(c_n^m) - \frac{1}{k} \right| = 0$$

for any $m \in \mathbb{N}$ and $\psi(e) = \{c_n\}_n$, where e is a minimal projection in M_k.

Proof. Let $\{\tau_1, \tau_2, ..., \tau_N\}$ be the set of extremal points of $T(A)$. Set

$$\tau = \frac{1}{N} \sum_{i=1}^{N} \tau_i$$

and let π be the GNS representation associated with $\tau \in T(A)$. Clearly τ_i and τ extend to tracial states on $\pi(A)''$. In what follows, we regard A as a subalgebra of $\pi(A)''$ and omit π. Since A is nuclear, A'' is isomorphic to the direct sum of N copies of the approximately finite-dimensional II_1-factor $\mathcal{R}$. In particular, $A'' \overline{\otimes} \mathcal{R} \cong A''$. Hence, we have a sequence of matrix units $E_{i,j,n} \in A''$ for M_k such that

$$\lim_{n\to\infty} \|[E_{i,j,n}, x]\|_\tau = 0$$

holds for any $x \in A''$. Define a unitary $U_n \in A''$ by

$$U_n = \sum_{i=1}^{k} E_{i,i+1,n},$$

where $i+1$ is understood modulo k. By [21, Lemma 2.1], we can find a central sequence $\{e_n\}_n$ of positive contractions in A and a central sequence $\{u_n\}_n$ of unitary operators in A such that

$$\lim_{n\to\infty} \|e_n - E_{1,1,n}\|_\tau = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n - U_n\|_\tau = 0.$$

Then we have

$$\lim_{n\to\infty} \|\mathrm{Ad}\, u_n^j(e_n)e_n\|_\tau = 0$$

for every $j=1,2,\ldots,k-1$. From Lemma 3.2, we may assume that $\{e_n\}_n$ and $\{u_n\}_n$ satisfy

$$\lim_{n\to\infty} \|\mathrm{Ad}\, u_n^j(e_n)e_n\| = 0.$$

It follows from [18, Proposition 2.4] that there exists a completely positive contractive order zero map $\psi: M_k \to A_\infty$ such that $\psi(e)=\{e_n\}_n$, where e is a minimal projection in M_k. Because $\tau_i \leqslant N\tau$ for any $i=1,2,\ldots,N$, one has

$$\left|\tau_i(e_n^m) - \frac{1}{k}\right| = |\tau_i(e_n^m - E_{1,1,n})| \leqslant \|e_n^m - E_{1,1,n}\|_{\tau_i}^{1/2} \leqslant N^{1/2}\|e_n^m - E_{1,1,n}\|_\tau^{1/2} \to 0$$

as $n\to\infty$ for any $m\in\mathbb{N}$. The proof is complete. □

LEMMA 3.4. *Let A be a unital separable simple infinite-dimensional nuclear C^*-algebra with $T(A)\neq\varnothing$. Suppose that the conclusion of Lemma 3.3 holds for A. Then for any central sequence $\{f_n\}_n$ of positive contractions in A and any $k\in\mathbb{N}$, there exist central sequences $\{f_{i,n}\}_n$, $i=1,2,\ldots,k$, of positive contractions in A such that $\{f_n f_{i,n}\}_n = \{f_{i,n}\}_n$ and $\{f_{i,n} f_{j,n}\}_n = 0$ for $i\neq j$, in A_∞, and*

$$\lim_{m\to\infty} \liminf_{n\to\infty} \min_{\tau\in T(A)} \tau(f_{i,n}^m) = \frac{1}{k} \lim_{m\to\infty} \liminf_{n\to\infty} \min_{\tau\in T(A)} \tau(f_n^m).$$

Proof. Set $c=\lim_{m\to\infty} \liminf_{n\to\infty} \min_{\tau\in T(A)} \tau(f_n^m)$. Take a finite subset $F\subset A$, $\varepsilon>0$ and $N\in\mathbb{N}$ arbitrarily. It suffices to show that there exist sequences $\{f_{i,n}\}_n$, $i=1,2,\ldots,k$, of positive contractions in A satisfying $\limsup_{n\to\infty} \|[f_{i,n}, a]\| < \varepsilon$ for each $a\in F$,

$$\limsup_{n\to\infty} \|f_n f_{i,n} - f_{i,n}\| < \varepsilon \quad \text{and} \quad \limsup_{n\to\infty} \|f_{i,n} f_{j,n}\| < \varepsilon$$

for $i\neq j$, and

$$\left|\liminf_{n\to\infty} \min_{\tau\in T(A)} \tau(f_{i,n}^m) - \frac{c}{k}\right| < \varepsilon$$

for any $m\leqslant N$. Let $l\in\mathbb{N}$ be such that $|(t-1)t^l|<\varepsilon$ for $t\in[0,1]$ and

$$\left|\liminf_{n\to\infty} \min_{\tau\in T(A)} \tau(f_n^{2lm}) - c\right| < \frac{\varepsilon}{2}$$

for any $m\leqslant N$. Because we assumed that the conclusion of Lemma 3.3 holds for A, we obtain positive contractions $e_i\in A$, $i=1,2,\ldots,k$, such that $\|[e_i,a]\|<\varepsilon$ for $a\in F$, $\|e_i e_j\|<\varepsilon$ for $i\neq j$, and $\max_{\tau\in T(A)} |\tau(e_i^m) - 1/k| < \varepsilon/4$ for any $m\leqslant N$.

Set $f_{i,n}=f_n^l e_i f_n^l$, $i=1,2,...,k$. Clearly it follows that $\limsup_{n\to\infty}\|[f_{i,n},a]\|<\varepsilon$ for $a\in F$ and $\|f_nf_{i,n}-f_{i,n}\|\leqslant\|f_nf_n^l-f_n^l\|<\varepsilon$ for $n\in\mathbb{N}$. For $i\neq j$ we have

$$\limsup_{n\to\infty}\|f_{i,n}f_{j,n}\|\leqslant\limsup_{n\to\infty}\|e_if_n^{2l}e_j\|<\varepsilon.$$

By [15, Lemma 4.6], we have

$$\limsup_{n\to\infty}\max_{\tau\in T(A)}\left|\tau(f_n^{lm}e_i^mf_n^{lm})-\frac{\tau(f_n^{2lm})}{k}\right|\leqslant 2\max_{\tau\in T(A)}\left|\tau(e_i^m)-\frac{1}{k}\right|<\frac{\varepsilon}{2}$$

for any $m\leqslant N$. Since $\|f_{i,n}^m-f_n^{lm}e_i^mf_n^{lm}\|\to 0$ as $n\to\infty$, we conclude that

$$\begin{aligned}\left|\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_{i,n}^m)-\frac{c}{k}\right|&=\lim_{N\to\infty}\left|\inf_{n>N}\min_{\tau\in T(A)}\tau(f_{i,n}^m)-\frac{c}{k}\right|\\&\leqslant\limsup_{N\to\infty}\left|\inf_{n>N}\min_{\tau\in T(A)}\tau(f_n^{lm}e_i^mf_n^{lm})-\frac{1}{k}\inf_{n>N}\min_{\tau\in T(A)}\tau(f_n^{2lm})\right|\\&\quad+\frac{1}{k}\lim_{N\to\infty}\left|\inf_{n>N}\min_{\tau\in T(A)}\tau(f_n^{2lm})-c\right|\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2k}\\&\leqslant\varepsilon\end{aligned}$$

for any $m\leqslant N$. □

We are now ready to prove the implication (ii) ⇒ (iii) in Theorem 1.1.

Proof of (ii) ⇒ (iii) *in Theorem* 1.1. Let φ be a completely positive map $A\to A$. We would like to show that φ can be excised in small central sequences. Let $\{e_n\}_n$ and $\{f_n\}_n$ be as in Definition 2.1. By Lemma 3.1, we may assume that there exist a pure state ω on A and $c_{l,i},d_i\in A$, $l,i=1,2,...,N$, such that

$$\varphi(a)=\sum_{l=1}^N\sum_{i,j=1}^N\omega(d_i^*ad_j)c_{l,i}^*c_{l,j},\quad a\in A.$$

Set

$$\varphi_l(a)=\sum_{i,j=1}^N\omega(d_i^*ad_j)c_{l,i}^*c_{l,j},\quad a\in A,$$

so that $\varphi=\varphi_1+\varphi_2+...+\varphi_N$.

Applying Lemma 3.4 to $\{f_n\}_n$, we have central sequences $\{f_{l,n}\}_n$, $l=1,2,...,N$, of positive contractions in A satisfying $\{f_{l,n}f_n\}_n=\{f_{l,n}\}_n$ and $\{f_{l,n}f_{l',n}\}_n=0$ for $l\neq l'$, in A_∞, and

$$\lim_{m\to\infty}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_{l,n}^m)>0.$$

Applying Proposition 2.2 to φ_l, $\{e_n\}_n$ and $\{f_{l,n}\}_n$, we obtain a sequence $\{s_{l,n}\}_n$ in A such that

$$\lim_{n\to\infty}\|s_{l,n}^*as_{l,n}-\varphi_l(a)e_n\|=0,\quad a\in A,\quad \text{and}\quad \lim_{n\to\infty}\|f_{l,n}s_{l,n}-s_{l,n}\|=0.$$

We define

$$s_n=\sum_{l=1}^N s_{l,n}$$

for $n\in\mathbb{N}$. Since $\limsup_{n\to\infty}\|s_{l,n}\|\leqslant\|\varphi_l(1)\|$, it follows that

$$\begin{aligned}\|f_ns_n-s_n\| &\leqslant \sum_{l=1}^N \|f_ns_{l,n}-s_{l,n}\|\\ &\leqslant \sum_{l=1}^N \|f_n\|\,\|s_{l,n}-f_{l,n}s_{l,n}\|+\|f_nf_{l,n}-f_{l,n}\|\,\|s_{l,n}\|\to 0\quad \text{as } n\to\infty.\end{aligned}$$

If $l\neq l'$, then

$$\lim_{n\to\infty}\|s_{l,n}^*as_{l',n}\|=\lim_{n\to\infty}\|s_{l,n}^*f_{l,n}af_{l',n}s_{l',n}\|=0$$

for any $a\in A$. Therefore, we conclude that

$$\lim_{n\to\infty}\|s_n^*as_n-\varphi(a)e_n\|=\lim_{n\to\infty}\left\|\sum_{l=1}^N s_{l,n}^*as_{l,n}-\varphi_l(a)e_n\right\|=0. \qquad \square$$

4. Proof of (iii) ⇒ (iv) ⇒ (i) in Theorem 1.1

In this section we prove the implications (iii) ⇒ (iv) ⇒ (i) in Theorem 1.1. First, let us recall the definition of property (SI) from [15].

Definition 4.1. ([15, Definition 4.1]) Let A be a separable C^*-algebra with $T(A)\neq\varnothing$. We say that A has *property* (SI) if for any central sequences $\{e_n\}_n$ and $\{f_n\}_n$ of positive contractions in A satisfying

$$\lim_{n\to\infty}\max_{\tau\in T(A)}\tau(e_n)=0\quad \text{and}\quad \lim_{m\to\infty}\liminf_{n\to\infty}\min_{\tau\in T(A)}\tau(f_n^m)>0,$$

there exists a central sequence $\{s_n\}_n$ in A such that

$$\lim_{n\to\infty}\|s_n^*s_n-e_n\|=0\quad \text{and}\quad \lim_{n\to\infty}\|f_ns_n-s_n\|=0.$$

Proof of (iii)$\Rightarrow$(iv) *in Theorem* 1.1. Let $\{e_n\}_n$ and $\{f_n\}_n$ be as in Definition 4.1. By the assumption in statement (iii), id_A can be excised in small central sequences. Thus we have $s_n \in A$, $n\in\mathbb{N}$, such that $\|s_n^* a s_n - a e_n\|\to 0$ for any $a\in A$ and $\|f_n s_n - s_n\|\to 0$. Since A is unital, we get $\|s_n^* s_n - e_n\|\to 0$. Also, for any $a\in A$, we obtain

$$\limsup_{n\to\infty} \|[s_n, a]\|^2 = \limsup_{n\to\infty} \|a^* s_n^* s_n a - a^* s_n^* a s_n - s_n^* a^* s_n a + s_n^* a^* a s_n\| = 0,$$

which means that $\{s_n\}_n$ is central. □

Proof of (iv)$\Rightarrow$(i) *in Theorem* 1.1. By Lemma 3.3, we get central sequences $\{c_{i,n}\}_n$ in A, $i=1,2,...,k$, such that $\{c_{i,n}c_{j,n}^*\}_n = \delta_{i,j}\{c_{1,n}^2\}_n$ in A_∞ and

$$\lim_{n\to\infty} \max_{\tau\in T(A)} \left| \tau(c_{1,n}^m) - \frac{1}{k} \right| = 0, \quad m\in\mathbb{N},$$

and $c_{1,n}$ is a positive contraction for all $n\in\mathbb{N}$. Let $\{e_n\}_n$ be a central sequence of positive contractions in A such that

$$\{e_n\}_n = \left\{ 1 - \sum_{i=1}^{k} c_{i,n}^* c_{i,n} \right\}_n \quad \text{in } A_\infty.$$

Then we have

$$\limsup_{n\to\infty} \max_{\tau\in T(A)} \tau(e_n) = \limsup_{n\to\infty} \max_{\tau\in T(A)} \tau\left(1 - \sum_{i=1}^{k} c_{i,n}^* c_{i,n}\right) = \limsup_{n\to\infty} \max_{\tau\in T(A)} (1 - k\tau(c_{1,n}^2)) = 0$$

and

$$\lim_{m\to\infty} \liminf_{n\to\infty} \min_{\tau\in T(A)} \tau(c_{1,n}^m) = \frac{1}{k} > 0.$$

Due to property (SI), we obtain a central sequence $\{s_n\}_n$ in A such that

$$\left\{ s_n^* s_n + \sum_{i=1}^{k} c_{i,n}^* c_{i,n} \right\}_n = 1 \text{ and } \{c_{1,n} s_n\}_n = \{s_n\}_n \quad \text{in } A_\infty,$$

which means that $\{\{c_{i,n}\}_n\}_{i=1}^k \cup \{\{s_n\}_n\} \subset A_\infty$ satisfies relation $\mathcal{R}_k$ defined in [20, §2]. It follows from [18, Proposition 5.1] (see also [20, Proposition 2.1]) that there exists a unital homomorphism from the prime dimension drop algebra $I(k, k+1)$ to A_∞. The Jiang–Su algebra $\mathcal{Z}$ is an inductive limit of such $I(k, k+1)$'s. By [23, Proposition 2.2], we can conclude that $A\otimes\mathcal{Z}\cong A$. □

In the same way as the proof above, we can show the following. Notice that we do not need the assumption of finitely many extremal traces for this theorem.

THEOREM 4.2. *Let A be a unital separable simple infinite-dimensional nuclear C^*-algebra with $T(A)\neq\varnothing$. Suppose that A is $\mathcal{Z}$-absorbing. Then any completely positive map $A\to A$ can be excised in small central sequences. Moreover, A has property* (SI).

Proof. By [17, Corollary 4.6], A has strict comparison. Since $\mathcal{Z}$ is a unital separable simple infinite-dimensional nuclear C^*-algebra with a unique trace, Lemma 3.3 is valid for $\mathcal{Z}$. Hence the conclusion of Lemma 3.3 also holds for $A\cong A\otimes\mathcal{Z}$. Then the proof of the implication (ii) $\Rightarrow$ (iii) in Theorem 1.1 (see §3) works for A, and whence any completely positive map $A\to A$ can be excised in small central sequences. By the proof of (iii) $\Rightarrow$ (iv) in Theorem 1.1, we can conclude that A has property (SI). □

5. C^*-algebras with tracial rank zero

In this section we prove that any unital separable simple nuclear infinite-dimensional C^*-algebra with tracial rank zero is approximately divisible (Theorem 5.4).

LEMMA 5.1. *Let A be a unital separable simple infinite-dimensional C^*-algebra with tracial rank zero and let $k\in\mathbb{N}$. There exists a sequence $\{\varphi_n\}_n$ of homomorphisms from M_k to A such that $\{\varphi_n(x)\}_n$ is a central sequence for any $x\in M_k$ and*

$$\lim_{n\to\infty}\max_{\tau\in T(A)}\tau(1-\varphi_n(1))=0.$$

Proof. Let C be a unital simple infinite-dimensional C^*-algebra with real rank zero. We first claim that for any $\varepsilon>0$ there exists a homomorphism $\varphi\colon M_k\to C$ such that one has $\tau(1-\varphi(1))<\varepsilon$ for all $\tau\in T(C)$. Choose $m\in\mathbb{N}$ such that $k/2^m$ is less than ε. By [28, Theorem 1.1 (i)], there exists a partition of unity $1=p_1+p_2+...+p_{2^m}+q$ consisting of projections in C such that p_1 is Murray–von Neumann equivalent to p_i for every $i=1,2,...,2^m$ and q is Murray–von Neumann equivalent to a subprojection of p_1. There is a unital homomorphism from M_{2^m} to $(1-q)C(1-q)$ and $\tau(q)<2^{-m}$ for any $\tau\in T(C)$. It follows that there exists a homomorphism $\varphi\colon M_k\to C$ such that

$$\tau(1-\varphi(1))\leqslant 2^{-m}(k-1)+\tau(q)<2^{-m}k<\varepsilon.$$

We now prove the statement. Since A has tracial rank zero, there exist a sequence of projections $e_n\in A$, a sequence of finite-dimensional subalgebras B_n of A, with $1_{B_n}=e_n$, and a sequence of unital completely positive maps $\pi_n\colon A\to B_n$ such that

- $\|[a,e_n]\|\to 0$ as $n\to\infty$ for any $a\in A$;
- $\|\pi_n(a)-e_nae_n\|\to 0$ as $n\to\infty$ for any $a\in A$;
- $\tau(1-e_n)<1/2n$ for all $\tau\in T(A)$.

Choose a family of mutually orthogonal minimal projections $p_{n,1}, p_{n,2}, ..., p_{n,k_n}$ of B_n such that $e_nAe_n \cap B'_n \cong \bigoplus_{i=1}^{k_n} p_{n,i}Ap_{n,i}$. As A has real rank zero, so does $p_{n,i}Ap_{n,i}$. It follows from the claim above that we can find a homomorphism $\varphi_{n,i}: M_k \to p_{n,i}Ap_{n,i}$ such that $\tau(p_{n,i} - \varphi_{n,i}(1))$ is arbitrarily small for all $\tau \in T(A)$. By taking a direct sum of the $\varphi_{n,i}$'s, we get a homomorphism $\varphi_n: M_k \to e_nAe_n \cap B'_n$ such that $\tau(e_n - \varphi_n(1)) < 1/2n$ for all $\tau \in T(A)$. It is easy to see that $\{\varphi_n(x)\}_n$ is a central sequence for any $x \in M_k$ and $\tau(1 - \varphi_n(1)) < 1/n$ for every $\tau \in T(A)$. The proof is complete. □

LEMMA 5.2. *Let A be a unital separable simple nuclear infinite-dimensional C^*-algebra with tracial rank zero. Then any completely positive map $A \to A$ can be excised in small central sequences.*

Proof. By [12, Theorem 3.7.2] and [16, Corollary 3.10], A has strict comparison. Then we can prove this lemma in the same way as the proof of the implication (ii) ⇒ (iii) in Theorem 1.1 (see §3), by using the lemma above instead of Lemma 3.3. □

LEMMA 5.3. *Let A be a unital separable simple nuclear infinite-dimensional C^*-algebra with tracial rank zero. Then A has property* (SI).

Proof. This follows from the lemma above and the proof of the implication (iii) ⇒ (iv) in Theorem 1.1 (see §4). □

THEOREM 5.4. *Let A be a unital separable simple nuclear infinite-dimensional C^*-algebra with tracial rank zero. Then A is approximately divisible. In particular, A is $\mathcal{Z}$-absorbing.*

Proof. In order to prove that A is approximately divisible, it suffices to construct a unital homomorphism from $M_2 \oplus M_3$ to A_∞ ([2, Proposition 2.7]). By Lemma 5.1, there exists a sequence $\{\varphi_n\}_n$ of homomorphisms from M_2 to A such that $\{\varphi_n(x)\}_n$ is a central sequence for any $x \in M_2$ and

$$\lim_{n\to\infty} \max_{\tau \in T(A)} \tau(1 - \varphi_n(1)) = 0.$$

By the lemma above, A has property (SI). It follows that there exists a central sequence $\{s_n\}_n$ such that

$$\lim_{n\to\infty} \|s_n^* s_n - (1 - \varphi_n(1))\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\varphi_n(e_{11}) s_n - s_n\| = 0,$$

where $e_{11} \in M_2$ is a rank-1 projection in M_2. Hence, there exists a unital homomorphism from $M_2 \oplus M_3$ to A_∞. Thus, A is approximately divisible. By [23, Theorem 2.3], a unital separable approximately divisible C^*-algebra is $\mathcal{Z}$-absorbing. The proof is complete. □

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Received November 15, 2011