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Forcing axioms and the continuum hypothesis

by

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1. Introduction

One way to formulate the Baire category theorem is that no compact space can be covered by countably many nowhere dense sets. Soon after Cohen's discovery of forcing, it was realized that it was natural to consider strengthenings of this statement in which one replaces *countably many* with $\aleph_1$*-many*. Even taking the compact space to be the unit interval, this already implies the failure of the continuum hypothesis and therefore is a statement not provable in ZFC. Additionally, there are ZFC examples of compact spaces which can be covered by $\aleph_1$-many nowhere dense sets. For instance, if K is the one-point compactification of an uncountable discrete set, then K^ω can be covered by $\aleph_1$-many nowhere dense sets. Hence some restriction must be placed on the class of compact spaces in order to obtain even a consistent statement.

Still, there are natural classes of compact spaces for which the corresponding statement about Baire category—commonly known as a *forcing axiom*—is consistent. The first and best known example is *Martin's Axiom for* $\aleph_1$*-dense sets* ($\mathrm{MA}_{\aleph_1}$) whose consistency was isolated from the solution of Souslin's problem [21]. This is the forcing axiom for compact spaces which do not contain uncountable families of pairwise disjoint

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open sets. For broader classes of spaces, it is much more natural to formulate the class and state the corresponding forcing axiom in terms of the equivalent language of forcing notions.

Foreman, Magidor and Shelah have isolated the broadest class of forcings for which a forcing axiom is relatively consistent—those forcings which preserve stationary subsets of ω_1 [9]. The corresponding forcing axiom is known as *Martin's maximum* (MM) and has a vast wealth of consequences which are still being developed (many are in fact consequences of the weaker *proper forcing axiom* (PFA)—see [18] for a recent survey).

Many consequences of MM (and in fact $\mathrm{MA}_{\aleph_1}$ itself) are examples of Π_2-sentences concerning the structure $H(\aleph_2)=(H(\aleph_2),\in,\omega_1,\mathrm{NS}_{\omega_1})$. Woodin has produced a forcing extension of $L(\mathbb{R})$, under an appropriate large cardinal assumption, which is provably optimal in terms of the Π_2-sentences which its $H(\aleph_2)$ satisfies [22]. Not surprisingly, the theory of the $H(\aleph_2)$ of this model largely coincides with the consequences of MM which concern $H(\aleph_2)$.

What will concern us in the present paper is the extent to which there is a corresponding strongest forcing axiom which is consistent with the continuum hypothesis (CH). More specifically, Woodin has posed the following problem.

Problem 1.1. ([22]) Are there two Π_2-sentences ψ_1 and ψ_2 in the language of the structure $(H(\aleph_2),\in,\omega_1,\mathrm{NS}_{\omega_1})$ such that ψ_1 and ψ_2 are each individually Ω-consistent with CH but such that $\psi_1\wedge\psi_2$ Ω-implies $\neg$CH?

For the present discussion, it is sufficient to know that "Ω-consistent" means something weaker than "provably forceable from large cardinals" and "Ω-implies" means something weaker than just "implies."

Even though CH implies that $[0,1]$ can be covered by $\aleph_1$-many nowhere dense sets, some forcing axioms are in fact compatible with CH. Early on in the development of iterated forcing, Jensen established that Souslin's hypothesis was consistent with CH (see [4]). Shelah then developed a general framework for establishing consistency results with CH by iterated forcing [20]. The result was a largely successful but ad-hoc method which Shelah and others used to prove that many consequences of MM are consistent with CH (see [2], [7], [8], [14], [17] and [20]). Moreover, with a few exceptions, it was known that starting from a ground model with a supercompact cardinal, these consequences of MM could all be made to hold in a single forcing extension which satisfies CH.

The purpose of the present paper is to prove the following theorem, which shows that Problem 1.1 has a positive answer if it is consistent that there is an inaccessible limit of measurable cardinals (usually this question is discussed in the context of much stronger large cardinal hypotheses).

THEOREM 1.2. *There exist sentences* ψ_1 *and* ψ_2 *which are* Π_2 *over the structure* $(H(\omega_2), \in, \omega_1)$ *such that the following conditions hold*:

- ψ_2 *can be forced by a proper forcing not adding* ω*-sequences of ordinals*;
- *if there exists a strongly inaccessible limit of measurable cardinals, then* ψ_1 *can be forced by a proper forcing which does not add* ω*-sequences of ordinals*;
- *the conjunction of* ψ_1 *and* ψ_2 *implies that* $2^{\aleph_0} = 2^{\aleph_1}$.

Note that neither of ψ_1 and ψ_2 requires the use of the non-stationary ideal on ω_1 as a predicate. The first conclusion follows from Theorem 3.3 and Lemmas 3.5 and 3.6. The second conclusion follows from Theorem 3.10 and Lemmas 4.2 and 4.3. The third conclusion of Theorem 1.2 is proved in Proposition 2.5.

The relative consistency of these sentences with CH is obtained by adapting Eisworth and Roitman's preservation theorems for not adding reals [8] (which are closely based on Shelah's framework noted above) in two different—and necessarily incompatible—ways. Traditionally, the two ingredients in any preservation theorem of this sort are *completeness* and some form of $(<\omega_1)$-*properness*. For the preservation theorem for one of our sentences (which is essentially proved in [8]), the *completeness* condition is weakened while maintaining the other requirement. In the other preservation theorem the *completeness* condition is strengthened slightly from the condition in [8], but $(<\omega_1)$-*properness* is replaced by the weaker combination of *properness and* $(<\omega_1)$-*semiproperness*.

The paper is organized as follows. In §2 we formulate the two Π_2-sentences and outline the tasks which must be completed to prove the main theorem. §3 contains a discussion of the preservation theorems which will be needed for the main result, including the proof of a new preservation theorem for not adding reals. §4 is devoted to the analysis of the single step forcings associated with one of the Π_2-sentences. §5 contains some concluding remarks.

The reader is assumed to have familiarity with proper forcing and with countable support iterated forcing constructions. While we aim to keep the present paper relatively self-contained, readers will benefit from familiarizing themselves with the arguments of [3], [6] and [8]. We will also deal with revised countable support and will use [15] as a reference. The notation is mostly standard for set theory and we will generally follow the conventions of [12] and [13]. We will now take the time to fix some notational conventions which are not entirely standard. If A is a set of ordinals, $\mathrm{ot}(A)$ will denote the order-type of A. If θ is a regular cardinal, then $H(\theta)$ will denote the collection of all sets of hereditary cardinality less than θ. Unless explicitly stated otherwise, θ will always denote an uncountable regular cardinal. If X is an uncountable set, we will let $[X]^{\aleph_0}$ denote the collection of all countable subsets of X. If X has cardinality ω_1, then an ω_1-*club* in $[X]^{\aleph_0}$ is a cofinal subset which is closed under taking countable unions and is well ordered in type ω_1 by containment. At certain points we will need to code hereditarily countable sets as elements of 2^ω. If $r \in 2^\omega$ and A is in $H(\aleph_1)$, then we say that r *codes* A if $(\mathrm{tc}(A), \in, A)$ is isomorphic to (ω, R_1, R_2), where $R_1 \subseteq \omega^2$ and $R_2 \subseteq \omega$ are defined by

$$(i,j) \in R_1 \iff r(2^{i+1}(2j+1)) = 1,$$
$$i \in R_2 \iff r(2i+1) = 1.$$

(Here tc denotes the transitive closure operation.) While not every r in 2^ω codes an element of $H(\aleph_1)$, every element of $H(\aleph_1)$ has a code in 2^ω. Also, if f is a finite-to-one function from a set of ordinals of order-type ω into $2^{<\omega}$, then we will say that f *codes* $A \in H(\aleph_1)$ if, for some cofinite subset X of the domain of f, $\bigcup f[X]$ is a single infinite length sequence which codes A in the sense above. Finally, if r and s are elements of $2^{\leqslant\omega}$, we will let $\Delta(r,s)$ denote the least i such that $r(i) \neq s(i)$ (if no such i exists, we define $\Delta(r,s) = \min\{|r|, |s|\}$).

2. Two Π_2-sentences

In this section we will present the two Π_2-sentences which are used to resolve Problem 1.1 and will prove that their conjunction implies $2^{\aleph_0} = 2^{\aleph_1}$. This will be done by appealing to the following theorem of Devlin and Shelah.

THEOREM 2.1. ([5]) *The equality* $2^{\aleph_0} = 2^{\aleph_1}$ *is equivalent to the following statement: There is an* $F: H(\aleph_1) \to 2$ *such that, for every* $g: \omega_1 \to 2$, *there is an* $X \in H(\aleph_2)$ *such that, whenever* M *is a countable elementary submodel of* $(H(\aleph_2), \in, X)$,

$$F(\overline{X}) = g(\delta),$$

where $\overline{X}$ *and* δ *are the images of* X *and* ω_1, *respectively, under the transitive collapse of* M.

Let us also note the following equivalent formulation of Jensen's principle $\diamondsuit$.

PROPOSITION 2.2. $\diamondsuit$ *holds if and only if there is a sequence* $\langle X_\alpha : \alpha < \omega_1 \rangle$ *of elements of* $H(\aleph_1)$ *such that, whenever* $Y \in H(\aleph_2)$, *there is a countable elementary submodel* M *of* $(H(\aleph_2), \in, Y)$ *such that* $X_\delta = \overline{Y}$, *where* $\overline{Y}$ *and* δ *are the images of* X *and* ω_1, *respectively, under the transitive collapse of* M.

The first of our Π_2-sentences is essentially the same as one used by Caicedo and Veličković in [3] in order to prove that the bounded proper forcing axiom implies that there is a well ordering of $H(\aleph_2)$ which is Δ_1-definable from a parameter in $H(\aleph_2)$. (This coding has its roots in work of Gitik [11].) We will now take some time to recall the definitions associated with this coding. Given $x\subseteq\omega$, let $\sim_x$ be the equivalence relation on $\omega\setminus x$ defined by letting $m\sim_x n$ if and only if $[m,n]\cap x=\varnothing$. Given two further subsets y and z of ω, let $\{I_k\}_{k<t}$ (for some $t\leqslant\omega$) be the increasing enumeration of the set of $\sim_x$-equivalence classes intersecting both y and z, and let the oscillation of x, y and z be the function $o(x,y,z)\colon t\to 2$ defined by

$$o(x,y,z)=0 \quad \text{if and only if} \quad \min(I_k\cap y)\leqslant\min(I_k\cap z).$$

Let $\vec{C}=\langle C_\delta:\delta\in\mathrm{Lim}(\omega_1)\rangle$ be a *ladder system* on ω_1 (so that each C_δ is a cofinal subset of δ of order-type ω), and let $\alpha<\beta<\gamma$ be limit ordinals greater than ω_1. Let $N\subseteq M$ be countable subsets of γ with $\{\omega_1,\alpha,\beta\}\subseteq N$ such that, for all $\xi\in\{\omega_1,\alpha,\beta,\gamma\}$,

$$\sup(N\cap\xi)<\sup(M\cap\xi)$$

and $\sup(M\cap\xi)$ is a limit ordinal. We are going to specify a way of decoding a finite binary sequence from $\vec{C}$, N, M, α and β. This decoding will be a very minor variation of the one defined in [3].

Let $\overline{M}$ be the transitive collapse of M, and let $\pi\colon M\to\overline{M}$ be the corresponding collapsing function. Let $\omega_1^{\overline{N}}$ and $\omega_1^{\overline{M}}$ denote the respective order-types of $N\cap\omega_1$ and $M\cap\omega_1$. Let $\alpha_M=\pi(\alpha)$, $\beta_M=\pi(\beta)$ and $\gamma_M=\mathrm{ot}(M)$. The *height of N in M with respect to $\vec{C}$* is defined as $n(N,M)=|\omega_1^{\overline{N}}\cap C_{\omega_1^{\overline{M}}}|$. Set

$$\begin{aligned} x&=\{|\pi(\xi)\cap C_{\alpha_M}|:\xi\in\alpha\cap N\},\\ y&=\{|\pi(\xi)\cap C_{\beta_M}|:\xi\in\beta\cap N\},\\ z&=\{|\pi(\xi)\cap C_{\gamma_M}|:\xi\in N\}.\end{aligned}$$

If the length of $o(x,y,z)$ is at least $n(N,M)$, then we define

$$s(N,M)=s^{\vec{C}}_{\alpha,\beta}(N,M)=o(x,y,z).$$

Otherwise we leave $s(N,M)$ undefined. If s is a finite-length binary sequence, we define $\bar{s}$ to be the sequence of the same length l with its digits reversed: $\bar{s}(i)=s(l-i)$.

If $\alpha<\beta<\gamma$ are ordinals of cofinality ω_1 in the interval (ω_1,ω_2), and f is a function from ω_1 to 2^ω, then we say that (α,β,γ) *codes f (relative to $\vec{C}$)* if there is an ω_1-club $\langle N_\xi:\xi<\omega_1\rangle$ in $[\gamma]^{\aleph_0}$ such that

- $\{\omega_1,\alpha,\beta\}\subseteq N_0$;
- for all $\nu<\omega_1$ and all $\xi\in\{\omega_1,\alpha,\beta,\gamma\}$, $\sup(N_\nu\cap\xi)$ is a limit ordinal;
- for all $\nu_0<\nu_1<\omega_1$ and all $\xi\in\{\omega_1,\alpha,\beta,\gamma\}$, $\sup(N_{\nu_0}\cap\xi)<\sup(N_{\nu_1}\cap\xi)$;

- for every limit $\nu<\omega_1$, there is a $\nu_0<\nu$ such that if $\nu_0<\xi<\nu$, then

$$\Delta(\bar{s}(N_\xi, N_\nu), f(N_\nu\cap\omega_1)) \geqslant n(N_\xi, N_\nu),$$

where the functions s and n are computed using the parameters $\vec{C}$, α, β and γ.

It is not difficult to show that if (α,β,γ) codes both f and g with respect to some $\vec{C}$, then there is a closed unbounded set of δ such that $f(\delta)=g(\delta)$.

Let us pause for a moment to note that the assertion "*for some* $\vec{C}$, *every* f *is coded by some triple* (α,β,γ)" implies that $2^{\aleph_0}=2^{\aleph_1}$. To see this, define F as follows:

- $F(\overline{\mathcal{N}},\bar{\alpha},\bar{\beta})=1$ whenever there exist an ω_1-club $\mathcal{N}$ in $[\gamma]^{\aleph_0}$, ordinals $\alpha<\beta<\gamma<\omega_2$ as above, and a countable elementary submodel M of $H(\aleph_2)$ containing $\{\mathcal{N},\alpha,\beta\}$, such that $s(N_\xi,N_\nu)(0)=1$ for a cobounded set of $\xi<\nu=M\cap\omega_1$, and $(\overline{M},\in,\overline{\mathcal{N}},\bar{\alpha},\bar{\beta})$ is the collapse of $(M,\in,\mathcal{N},\alpha,\beta)$;
- $F(X)=0$ for all other X in $H(\aleph_1)$.

Now let $g:\omega_1\to 2$ be given and define $f:\omega_1\to 2^\omega$ by letting $f(\delta)$ be the real which takes the constant value $g(\delta)$. If $\mathcal{N}$ witnesses that (α,β,γ) codes f, and M is a countable elementary submodel of $H(\aleph_2)$ containing $\{\mathcal{N},\alpha,\beta\}$, then $F(\overline{\mathcal{N}},\bar{\alpha},\bar{\beta})=g(\delta)$. By Theorem 2.1, this implies $2^{\aleph_0}=2^{\aleph_1}$.

Definition 2.3. ψ_1 is the assertion that for every $A:\omega_1\to 2$ and for every ladder system $\vec{C}$, there is a triple (α,β,γ) and a function $f:\omega_1\to 2^\omega$ such that (α,β,γ) codes f relative to $\vec{C}$, and for each $\delta<\omega_1$, $f(\delta)$ is a code for $A\restriction\delta$.

We will prove in §4 that the conjunction of ψ_1 and CH can be forced over any model in which there is an inaccessible limit of measurable cardinals.

Now we will turn to the task of defining a Π_2-sentence ψ_2 which, together with ψ_1, provides a solution to Problem 1.1. Suppose for a moment that $\langle N_\xi:\xi<\omega_1\rangle$ witnesses that (α,β,γ) codes $A:\omega_1\to 2$ relative to $\vec{C}$. If X_i, $i<\omega$, is an infinite increasing sequence in $\{N_\xi:\xi<\omega_1\}$, define the *height* of $\{X_i\}_{i<\omega}$ to be $\delta=\omega_1\cap\bigcup_{i<\omega}X_i$. Observe that, together with α, β and $\vec{C}$, $\{X_i\}_{i<\omega}$ uniquely determines $A\restriction\delta$. Moreover, $A\restriction\delta$ can be recovered from just the isomorphism type of the structure

$$(N,\in,\omega_1,\alpha,\beta;X_i:i<\omega),$$

where $N=\bigcup_{i<\omega}X_i$. We will refer to a structure arising in this way as a ψ_1-structure and say that this structure *codes* $A\restriction\delta$.

Definition 2.4. ψ_2 is the assertion that for every ladder system $\vec{C}$, every triple $\alpha<\beta<\gamma$ of ordinals strictly between ω_1 and ω_2, and every ω_1-club $\mathcal{N}$ in $[\gamma]^{\aleph_0}$, there is a function $f\colon\omega_1\to 2^{<\omega}$ such that for every limit $\delta<\omega_1$, $f\restriction C_\delta$ codes (in the sense discussed at the end of the introduction) the transitive collapse of a structure

$$(N,\in,\omega_1,\alpha,\beta;X_i:i<\omega),$$

where $\{X_i\}_{i<\omega}$ is an increasing sequence in $\mathcal{N}$ of height greater than δ and $N=\bigcup_{i<\omega}X_i$.

In §3, we will prove that ψ_2 is relatively consistent with CH. We now have the following proposition.

PROPOSITION 2.5. *$\psi_1\wedge\psi_2$ implies $2^{\aleph_0}=2^{\aleph_1}$. In fact, $2^{\aleph_0}=2^{\aleph_1}$ follows from the existence of a ladder system $\vec{C}$ for which the conjunction of ψ_1 and ψ_2, both relative to $\vec{C}$, holds.*

Proof. Fix a ladder system $\vec{C}$ and suppose that ψ_1 and ψ_2 are true. If $t\colon\delta\to 2^{<\omega}$ for some countable limit ordinal δ, and if $t\restriction C_\delta$ codes a ψ_1-structure which in turn codes $g\restriction\delta^*$ for some $\delta^*>\delta$ and $g\colon\omega_1\to 2$, then define $F(t)=g(\delta)$. Now, if (α,β,γ) codes $g\colon\omega_1\to 2$ relative to $\vec{C}$ as witnessed by $\mathcal{N}$, and $f\colon\omega_1\to 2^{<\omega}$ witnesses the corresponding instance of ψ_2, then $F(f\restriction\delta)=g(\delta)$ for every limit ordinal δ. By Theorem 2.1, $2^{\aleph_0}=2^{\aleph_1}$. □

We will finish this section by showing that both ψ_1 and ψ_2 imply that $\diamondsuit$ fails. Let us say that an ω_1-club of $[\gamma]^\omega$ (for some $\gamma<\omega_2$ of uncountable cofinality) is *typical* in case for all $\nu_0<\nu_1<\omega_1$, $N_{\nu_0}\cap\omega_1$ and $\sup(N_{\nu_0})$ are limit ordinals, $N_{\nu_0}\cap\omega_1<N_{\nu_1}\cap\omega_1$ and $\sup(N_{\nu_0})<\sup(N_{\nu_1})$. The following fact shows that our methods do not extend to show non-existence of a Π_2-maximal model for $\diamondsuit$.

FACT 2.6. *$\diamondsuit$ implies the failure of ψ_1. In fact, $\diamondsuit$ implies that there is a ladder system $\vec{C}$ with the property that for every ordinal γ in ω_2 of uncountable cofinality and every typical ω_1-club $\langle N_\nu:\nu<\omega_1\rangle$ of $[\gamma]^\omega$, there are stationary many $\nu<\omega_1$ such that, for unboundedly many $\xi<\nu$, one has $|C_{N_\nu\cap\omega_1}\cap N_\xi|>|C_{\mathrm{ot}(N_\nu)}\cap\sup(\pi[N_\xi])|$, where π is the collapsing function of N_ν.*

Proof. It is easy to fix a natural notion of coding in such a way that for every $\gamma<\omega_2$ and every ω_1-club $\langle N_\nu:\nu<\omega_1\rangle$ of $[\gamma]^\omega$ there is a set $X\subseteq\omega_1$ and there is a closed unbounded set of $\delta<\omega_1$ such that $X\cap\delta$ codes a directed system $\mathcal{S}=\langle\delta_\nu,i_{\nu,\nu'}:\nu\leqslant\nu'<\delta\rangle$, where, for all $\nu\leqslant\nu'<\delta$, $\delta_\nu=\mathrm{ot}(N_\nu)$ and $i_{\nu,\nu'}=\pi_{N_{\nu'}}\circ\pi_{N_\nu}^{-1}$ (where π_{N_ν} denotes the collapsing function of N_ν). Let us fix such a notion of coding. Let $\vec{X}=\{X_\nu\}_{\nu<\omega_1}$ be a $\diamondsuit$-sequence. We recursively define from $\vec{X}$ a ladder system $\vec{C}=\langle C_\delta:\delta\in\mathrm{Lim}(\omega_1)\rangle$ in the following way.

Let $\delta\in\mathrm{Lim}(\omega_1)$ and suppose that X_δ codes a directed system $\mathcal{S}=\langle\delta_\nu,i_{\nu,\nu'}:\nu\leqslant\nu'<\delta\rangle$ with well-founded direct limit, where the δ_ν's are countable limit ordinals, and each $i_{\nu,\nu'}$ is an order-preserving map from δ_ν to $\delta_{\nu'}$, $i_{\nu,\nu'}\neq\mathrm{id}$. Suppose that, for all $\nu<\delta$,

$\mathrm{crit}(i_{\nu,\nu+1})$ is a limit ordinal and $\nu\leqslant\mathrm{crit}(i_{\nu,\nu+1})<\mathrm{crit}(i_{\nu+1,\nu+2})$, where $\mathrm{crit}(i_{\nu,\nu'})$ is the least ordinal moved by $i_{\nu,\nu'}$, and that $\sup(\mathrm{range}(i_{\nu,\nu+1}))<\delta_{\nu+1}$. Let η_δ be the direct limit of $\mathcal{S}$ and let $i_{\nu,\delta}\colon\delta_\nu\to\eta_\delta$ be the corresponding limit map for each $\nu<\delta$. We identity η_δ with an ordinal. Suppose that $\delta>\eta_{\delta'}$ for all limit ordinals $\delta'<\delta$. Then we pick C_δ and C_{η_δ} in such a way that for unboundedly many ν below δ, $|C_\delta\cap\mathrm{crit}(i_{\nu,\delta})|$ is bigger than $|C_{\eta_\delta}\cap\sup(\mathrm{range}(i_{\nu,\delta}))|$. Now, using the fact that $\vec{X}$ is a $\diamondsuit$-sequence, it is not difficult to check that $\vec{C}$ is a ladder system as required. $\square$

It is also easy to see that $\diamondsuit$—and in fact Ostaszewski's principle $\clubsuit$—implies the failure of ψ_2. To see this, let $\langle C_\delta:\delta\in\mathrm{Lim}(\omega_1)\rangle$ be a $\clubsuit$-sequence. Suppose that $f\colon\omega_1\to 2^{<\omega}$ is such that for all limit $\delta<\omega_1$ there is a cofinite set $X\subseteq C_\delta$ such that $\bigcup f[X]$ is a member of 2^ω. Then there is some $n<\omega$ such that $S=\{\nu\in\omega_1:|f(\nu)|=n\}$ is unbounded in ω_1. But if δ is such that $C_\delta\subseteq S$, then $\bigcup f[C_\delta]$ is finite, which is a contradiction.

3. Iteration theorems

In this section we will review and adapt Eisworth and Roitman's general framework for verifying that an iteration of forcings does not add new reals. We will need two preservation results, one of which is essentially established in [8] (and was known to Eisworth), and one of which is an adaptation of the result in [8] to iterations of totally proper α-semiproper forcings. In the course of the section, we will also establish that ψ_2 is relatively consistent with CH.

Before we begin, we will review some of the definitions which we will need in this section. A *forcing* $\mathbb{Q}$ is a partial order with a greatest element $1_{\mathbb{Q}}$. A cardinal θ is *sufficiently large* for a forcing $\mathbb{Q}$ if $\mathcal{P}(\mathcal{P}(\mathbb{Q}))$ is an element of $H(\theta)$. We will say that M is a *suitable model* for $\mathbb{Q}$ if $\mathbb{Q}$ is in M and M is a countable elementary submodel of $H(\theta)$ for some θ which is sufficiently large for $\mathbb{Q}$. If M is a suitable model for $\mathbb{Q}$ and q is in $\mathbb{Q}$, then we will say that q is *$(M,\mathbb{Q})$-generic* if whenever $r\leqslant q$ and $D\in M$ is a dense subset of $\mathbb{Q}$, r is compatible with an element of $D\cap M$. If, moreover, $\{p\in\mathbb{Q}\cap M:q\leqslant p\}$ is an $(M,\mathbb{Q})$-generic filter, then we say that q is *totally $(M,\mathbb{Q})$-generic*. $\mathbb{Q}$ is (*totally*) *proper* if whenever M is a suitable model for $\mathbb{Q}$ and q is in $\mathbb{Q}\cap M$, q has a (totally) $(M,\mathbb{Q})$-generic extension. It is easily verified that a forcing is totally proper if and only if it is proper and does not add any new reals.

Remark 3.1. It is important to note that if $\mathbb{Q}$ is totally proper and M is suitable for $\mathbb{Q}$, it need not be the case that every $(M,\mathbb{Q})$-generic condition is totally $(M,\mathbb{Q})$-generic. It is true that every $(M,\mathbb{Q})$-generic condition can be *extended* to a totally $(M,\mathbb{Q})$-generic condition. This distinction is very important in the discussion of when an iteration of forcings adds new reals.

A *suitable tower* (in $H(\theta)$) for $\mathbb{Q}$ is a set $\mathcal{N}=\{N_\xi:\xi<\eta\}$ (for some ordinal η) such that for some θ which is sufficiently large for $\mathbb{Q}$:

- each N_ξ is a countable elementary submodel of $H(\theta)$ having $\mathbb{Q}$ as a member;
- if $\nu<\eta$ is a limit ordinal, then $N_\nu=\bigcup_{\xi<\nu} N_\xi$;
- if $\nu<\eta$ is a successor ordinal, then $\{N_\xi:\xi<\nu\}$ is in N_ν.

Since a tower is naturally ordered by $\in$, we notationally identify it with the corresponding sequence. A condition q is $(\mathcal{N},\mathbb{Q})$-generic if it is $(N,\mathbb{Q})$-generic for each N in $\mathcal{N}$. A partial order $\mathbb{Q}$ is *η-proper* if whenever $\mathcal{N}=\langle N_\xi:\xi<\eta\rangle$ is a suitable tower for $\mathbb{Q}$ and q is in N_0, then q has an $(\mathcal{N},\mathbb{Q})$-generic extension. If a forcing is η-proper for every $\eta<\omega_1$, we will say that it is $(<\omega_1)$*-proper*.

Now we will return to our discussion of iterated totally proper forcing.

Definition 3.2. Suppose that η is a countable ordinal and $\mathbb{P}*\dot{\mathbb{Q}}$ is a two-step forcing iteration such that $\mathbb{P}$ is η-proper. The iteration $\mathbb{P}*\dot{\mathbb{Q}}$ is *η-complete* if, whenever

(1) $\langle N_\xi:\xi<1+\eta\rangle$ is a suitable tower of models for $\mathbb{P}*\dot{\mathbb{Q}}$;
(2) $G\subseteq\mathbb{P}\cap N_0$ is $(N_0,\mathbb{P})$-generic;
(3) $(p,\dot{q})$ is in $\mathbb{P}*\dot{\mathbb{Q}}\cap N_0$ with p in G;

there is a $G^*\subseteq\mathbb{P}*\dot{\mathbb{Q}}\cap N_0$ extending G with $(p,\dot{q})\in G^*$ such that, whenever r is a lower bound for G which is $(\langle N_\xi:\xi<1+\eta\rangle,\mathbb{P})$-generic, r forces that G^*/G has a lower bound in $\dot{\mathbb{Q}}$.

Notice that, if $\eta<\zeta$ and $\mathbb{P}*\dot{\mathbb{Q}}$ is η-complete, then $\mathbb{P}*\dot{\mathbb{Q}}$ is ζ-complete. By routine adaptations to the proof of Theorem 4 of [8], we obtain the following iteration theorem.

THEOREM 3.3. *Let η and γ be ordinals, with $\eta<\omega_1$, and let*

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma\rangle$$

be a countable support iteration with countable support limit $\mathbb{P}_\gamma$. Suppose that, for all $\alpha<\gamma$,

- $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ *is* $(<\omega_1)$*-proper;*
- *the iteration $\mathbb{P}_\alpha*\dot{\mathbb{Q}}_\alpha$ is η-complete.*

Then $\mathbb{P}_\gamma$ is totally proper.

We will now argue that this theorem is sufficient to prove that the conjunction of ψ_2 and CH can be forced over any model of ZFC. First we will recall a general fact which we will use repeatedly below.

LEMMA 3.4. *Suppose that* $X \subseteq H(\aleph_1)$ *and that* $\mathbb{Q} \subseteq X^{<\omega_1}$ *is a partial order, ordered by extension, with the following properties*:

- $\mathbb{Q}$ *is closed under initial segments*;
- *for every* $\alpha < \omega_1$, $\{q \in \mathbb{Q} : |q| \geqslant \alpha\}$ *is dense*;
- *if* q *is in* $\mathbb{Q}$ *with* $|q| = \alpha$, $p: \alpha \to X$ *and*

$$\{\xi < \alpha : q(\xi) \neq p(\xi)\}$$

is finite, then p *is in* $\mathbb{Q}$.

Then if

- M *is a suitable model for* $\mathbb{Q}$;
- q *is in* $\mathbb{Q} \cap M$;
- $C \subseteq (M \cap \omega_1) \setminus |q|$ *is cofinal in* $M \cap \omega_1$ *with order-type* ω;
- f *is a function from* C *into* $X \cap M$;

then there is a $q': M \cap \omega_1 \to X$ *extending* q *such that* $q'(\xi) = f(\xi)$ *for all* $\xi \in C$ *and*

$$\{q' \restriction \xi : \xi \in M \cap \omega_1\}$$

is an M*-generic filter for* $\mathbb{Q}$.

Proof. It is sufficient to prove that if $\mathbb{Q}$, M, q, C and f are as in the statement of the lemma and $D \subseteq \mathbb{Q}$ is dense and in M, then there is a $q' \leqslant q$ in $M \cap D$ such that $q'(\xi) = f(\xi)$ for all ξ in $|q'| \cap C$. By the elementarity of M, there is a countable elementary $N \prec H(\aleph_1)$ in M such that q is in N, $D \cap N$ is dense in $\mathbb{Q} \cap N$, and $\{p \in \mathbb{Q} \cap N : \xi \leqslant |p|\}$ is dense in N for every $\xi \in N \cap \omega_1$. Let $\nu = N \cap \omega_1$ and let $C' = C \cap \nu$. Since ν is a limit ordinal and C' is finite, there is a $q_0 \leqslant q$ in N such that $(f \restriction C') \subseteq q_0$. By the density of $D \cap N$ in $\mathbb{Q} \cap N$, there is a $q' \leqslant q_0$ in $D \cap N$. Since $|q'| \cap C = |q_0| \cap C$, we are done. □

By performing a preliminary forcing if necessary, we may assume that our ground model satisfies $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Suppose that $\vec{C}$, α, β, γ and $\mathcal{N}$ represent an instance of ψ_2, i.e., $\vec{C}$ is a ladder system on ω_1, $\omega_1 < \alpha < \beta < \gamma < \omega_2$ and $\mathcal{N}$ is an ω_1-club contained in $[\gamma]^{\aleph_0}$. Define $\mathbb{Q} = \mathbb{Q}_{\vec{C},\alpha,\beta,\mathcal{N}}$ to be the collection of all q such that the domain of q is η for some countable limit ordinal η, q maps into $2^{<\omega}$, and q satisfies the conclusion of ψ_2 for $\delta \leqslant \eta$. Note that $\mathbb{Q}$ has cardinality $2^{\aleph_0} = \aleph_1$.

LEMMA 3.5. $\mathbb{Q}$ *is totally* α*-proper for each* $\alpha < \omega_1$.

Proof. First observe that $\mathbb{Q}$ satisfies the hypothesis of Lemma 3.4. (We leave it to the reader to verify that if $\xi < \omega_1$, then $\{q \in \mathbb{Q} : |q| \geqslant \xi\}$ is dense in $\mathbb{Q}$.) We will prove by induction on α the following statement: *If* $\mathcal{M} = \langle M_\xi : \xi \leqslant \alpha \rangle$ *is a suitable tower for* $\mathbb{Q}$,

$q_0 \in M_0 \cap \mathbb{Q}$ and f_0 is a finite partial function from $\omega_1^{M_\alpha} \setminus |q_0|$ to $2^{<\omega}$, then there is a totally $(\mathcal{M}, \mathbb{Q})$-generic $\bar{q} \leqslant q_0$ with $f_0 \subset \bar{q}$.

If $\alpha = 0$, this is vacuously true. If $\alpha = \beta + 1$, then by our inductive assumption there is a $q' \leqslant q_0$ such that q' is totally $(M_\xi, \mathbb{Q})$-generic for all $\xi < \beta$ and such that $\bar{q}(\xi) = f_0(\xi)$ whenever $\xi \in \text{dom}(f_0) \cap \text{dom}(\bar{q})$. By elementarity, such a q' can be moreover found in M_β. Define $\delta = M_\beta \cap \omega_1$ and let $f : C_\delta \to 2^{<\omega}$ be such that for some $\{X_i\}_{i<\omega} \subseteq \mathcal{N}$ of height greater than δ, f codes the ψ_1-structure corresponding to $\{X_i\}_{i<\omega}$. By modifying f if necessary, we may assume that $q' \cup f \cup f_0$ is a function. By Lemma 3.4, there is a $\bar{q} : \delta \to 2^{<\omega}$ such that $\bar{q}$ extends q', $\{\bar{q} \restriction \xi : \xi \in N_\beta \cap \omega_1\}$ is an $(\langle N_\xi : \xi \leqslant \beta \rangle, \mathbb{Q})$-generic filter, and $\bar{q} \cup f \cup f_0$ is a function. Notice that this implies that $\bar{q}$ is in $\mathbb{Q}$ and is therefore as desired.

If α is a limit ordinal, let α_n, $n < \omega$, be an increasing sequence of ordinals converging to α with $\alpha_0 = 0$. Define $\delta = M_\alpha \cap \omega_1$ and as above let $f : C_\delta \to 2^{<\omega}$ be such that for some $\{X_i\}_{i<\omega} \subseteq \mathcal{N}$ of height greater than δ, f codes the ψ_1-structure corresponding to $\{X_i\}_{i<\omega}$. Let q_0 be a given element of $M_0 \cap \mathbb{Q}$ and let f_0 be a given finite partial function from $\omega_1 \setminus |q_0|$. By modifying f if necessary, we may assume that $f \cup f_0$ is a function. Construct a $\leqslant$-descending sequence q_n, $n < \omega$, such that

- q_{n+1} is totally $(\langle M_\xi : \xi \leqslant \alpha_n \rangle, \mathbb{Q})$-generic;
- q_{n+1} is in M_{α_n+1} and has domain $M_{\alpha_n} \cap \omega_1$;
- q_{n+1} extends $f_0 \cup f \cap M_{\alpha_n}$.

Given q_n, q_{n+1} can be found in $H(\theta)$ by applying our induction hypothesis to q_n and to $(f \cup f_0) \cap M_{\alpha_n}$. Such a q_{n+1} moreover exists in M_{α_n+1} by elementarity, completing the inductive construction. It now follows that $\bar{q} = \bigcup_{n<\omega} q_n$ is a totally $(\langle M_\xi : \xi \leqslant \alpha \rangle, \mathbb{Q})$-generic extension of q_0 as desired. □

Under CH, length-ω_2 countable support iterations of proper forcings which are forced to have cardinality at most $\aleph_1$ are $\aleph_2$-c.c. and hence preserve cardinals (see for instance [1, Theorem 2.10]). Standard book-keeping arguments then reduce our task to verifying that an iteration of forcings of the form $\mathbb{Q}_{\vec{C}, \alpha, \beta, \mathcal{N}}$ is ω-complete.

LEMMA 3.6. *Suppose that*

- *$\mathbb{P}$ is a totally proper forcing;*
- *for each $\delta \in \text{Lim}(\omega_1)$, $\dot{C}_\delta$ is a $\mathbb{P}$-name for a cofinal subset of δ of order-type ω;*
- *$\vec{C}$ is a $\mathbb{P}$-name for the ladder system on ω_1 induced by the names $\dot{C}_\delta$, $\delta \in \text{Lim}(\omega_1)$;*
- *$\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ are $\mathbb{P}$-names for an increasing sequence of ordinals between ω_1 and ω_2;*
- *$\dot{\mathcal{N}}$ is a $\mathbb{P}$-name for an ω_1-club contained in $[\dot{\gamma}]^{\aleph_0}$;*
- *$\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for the partial order $\mathbb{Q}_{\dot{\vec{C}}, \dot{\alpha}, \dot{\beta}, \dot{\mathcal{N}}}$.*

Then $\mathbb{P} * \dot{\mathbb{Q}}$ is an ω-complete iteration.

Proof. Let $\langle N_k : k<\omega\rangle$ be a tower of models with $\mathbb{P}*\dot{\mathbb{Q}}$ in N_0, $G\subseteq\mathbb{P}\cap N_0$ be an $(N_0,\mathbb{P})$-generic filter and $(p,\dot q)$ be in $\mathbb{P}*\dot{\mathbb{Q}}\cap N_0$ such that p is in G. Notice that some condition in G decides $\dot q$, $\dot\alpha$, $\dot\beta$ and $\dot\gamma$ to be some q, α, β and γ, respectively. Let r be a real which codes the transitive collapse of

$$\Big(\bigcup_{k<\omega} N_k\cap\gamma, \in, \alpha, \beta; N_k\cap\gamma : k<\omega\Big).$$

The key point is that, if $\bar p$ is $(\langle N_k : k<\omega\rangle,\mathbb{P})$-generic, then $\bar p$ forces that $N_k\cap\gamma$ is in $\dot{\mathcal{N}}$ for all $k<\omega$.

Set $\delta=N_0\cap\omega_1$. Notice that there is a ladder $\widehat{C}_\delta$ on δ such that, whenever C' is a ladder on δ which is in N_1, $C'\setminus\widehat{C}_\delta$ is finite and $\widehat{C}_\delta$ consists only of ordinals not in the domain of $\dot q$ as decided by G. In particular, if $\bar p$ is $(N_1,\mathbb{P})$-generic, then $\bar p$ forces that $\dot C_\delta$ is contained in $\widehat{C}_\delta$ except for a finite set. Let f_δ be a bijection between $\widehat{C}_\delta$ and $\{r\restriction n : n<\omega\}$. Lemma 3.4 now allows us to build a $G^*\subseteq\mathbb{P}*\dot{\mathbb{Q}}$ such that $(p,\dot q)$ is in G^* and if

$$g=\bigcup\{s\in H(\omega)^{<\delta} : \text{there exists } p\in G \text{ such that } (p,\check s)\in G^*\},$$

then f_δ is a restriction of g. It follows that, whenever r is a lower bound for G which is $(\langle N_k : k<\omega\rangle,\mathbb{P})$-generic, then r forces that G^*/G has a lower bound. □

Putting together Theorem 3.3 with Lemmas 3.5 and 3.6, we have (in ZFC) that there exists a partial order forcing ψ_2 + CH. Corresponding results for ψ_1 are proved in §4.

Unlike ψ_2, it is generally not possible to force an instance of ψ_1 with an ω-proper forcing. Fortunately, assuming the existence of three measurable cardinals, there is a forcing to force an instance of ψ_1 which is $(<\omega_1)$-semiproper. In the remainder of this section, we formulate and prove a version of [8, Theorem 4] which applies to iterations of totally proper $(<\omega_1)$-semiproper iterands. This seems to provide the first example of a forcing which is proper and $(<\omega_1)$-semiproper, but not $(<\omega_1)$-proper.

In order to state this definition, we will borrow the following pieces of notation from [8] (originating in [20]): Given a set N and a forcing notion $\mathbb{P}\in N$, $\mathrm{Gen}(N,\mathbb{P})$ denotes the set of all $(N,\mathbb{P})$-generic filters $G\subseteq N\cap\mathbb{P}$. Furthermore, if $p\in N\cap\mathbb{P}$,

$$\mathrm{Gen}(N,\mathbb{P},p)=\{G\in\mathrm{Gen}(N,\mathbb{P}) : p\in G\}$$

and $\mathrm{Gen}^+(N,\mathbb{P},p)$ denotes the set of all $G\in\mathrm{Gen}(N,\mathbb{P},p)$ such that G has a lower bound in $\mathbb{P}$.

Given a partial order $\mathbb{P}$, a regular cardinal θ which is sufficiently large for $\mathbb{P}$, and a countable $N\prec H(\theta)$ with $\mathbb{P}\in N$, we say that a condition $p\in\mathbb{P}$ is $(\mathbb{P},N)$-*semigeneric* if

$p \Vdash \tau \in \check{\omega}_1 \cap \check{N}$ for all $\mathbb{P}$-names τ in N for countable ordinals. Given a countable ordinal η, $\mathbb{P}$ is said to be η-*semiproper* if for every suitable tower $\langle N_\xi : \xi < \eta \rangle$ with $\mathbb{P} \in N_0$, and every $p \in \mathbb{P} \cap N_0$, there is a condition $q \leqslant p$ in $\mathbb{P}$ which is $(\langle N_\xi : \xi < \eta \rangle, \mathbb{P})$-semigeneric, i.e., which is $(N_\xi, \mathbb{P})$-semigeneric for all $\xi < \eta$.

Given a countable elementary substructure N of $H(\theta)$ with $\mathbb{P} \in N$, and given $G \in \mathrm{Gen}(N, \mathbb{P})$, we let $N[G]$ denote the set of G-interpretations of $\mathbb{P}$-names which are in N (see [8, §3] for details).

In the following definition, we have replaced the condition that r be

$$(\langle N_\xi : \xi < 1+\eta \rangle, \mathbb{P})\text{-generic}$$

from Definition 3.2 with the condition that it be merely

$$(\langle N_\xi : \xi < 1+\eta \rangle, \mathbb{P})\text{-semigeneric.}$$

We call the corresponding notion η-semicompleteness, and note that it is a stronger condition than η-completeness.

Definition 3.7. Suppose that η is a countable ordinal and $\mathbb{P} * \dot{\mathbb{Q}}$ is a two-step forcing iteration such that $\mathbb{P}$ is η-semiproper. The iteration $\mathbb{P} * \dot{\mathbb{Q}}$ is η-*semicomplete* if, whenever

(1) $\langle N_\xi : \xi < 1+\eta \rangle$ is a suitable tower of models for $\mathbb{P} * \dot{\mathbb{Q}}$;
(2) $G \subseteq \mathbb{P} \cap N_0$ is $(N_0, \mathbb{P})$-generic;
(3) $(p, \dot{q})$ is in $\mathbb{P} * \dot{\mathbb{Q}} \cap N_0$ with p in G;

there is a $G^* \subseteq \mathbb{P} * \dot{\mathbb{Q}} \cap N_0$ extending G with $(p, \dot{q}) \in G^*$ such that, whenever r is a lower bound for G which is $(\langle N_\xi : \xi < 1+\eta \rangle, \mathbb{P})$-semigeneric, r forces that G^*/G has a lower bound in $\dot{\mathbb{Q}}$.

Even though we will be working exclusively with iterations of proper forcings in this paper, we will use the terminology of *revised countable support* iterations in order to prove the analogue of Theorem 3.3 for η-semicomplete iterations. By *revised countable support* (RCS) we mean either the original presentation of RCS due to Shelah [20], or the later reformulation due to Miyamoto [15]. Theorem 3.8 and Fact 3.9 below are proved in [15] but are already implicit in [20]. In [14] it is claimed, erroneously, that these facts apply to the presentation of RCS due to Donder and Fuchs [10]. Under the Donder–Fuchs presentation of RCS, an RCS iteration of proper forcings is identical to the corresponding countable support iteration, for which Theorem 3.8 fails. For the Shelah and Miyamoto versions, an RCS limit of proper forcings and the corresponding countable support limit are merely isomorphic on a dense set. It follows, in the end, that Theorem 3.10 is true when one uses countable support in place of revised countable support, though again our proof of this fact requires RCS. A similar situation holds in [14].

To facilitate the statements below, we let "$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ has RCS limit $\mathbb{P}_\gamma$" include the case that $\gamma = \beta + 1$ and $\mathbb{P}_\gamma = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$ (and similarly for countable support).

THEOREM 3.8. ([15, Corollary 4.12]) *Let γ be an ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ be an* RCS *iteration with* RCS *limit $\mathbb{P}_\gamma$. Fix $\beta < \gamma$ and $p \in \mathbb{P}_\beta$. Suppose that τ is a $\mathbb{P}_\beta$-name for a condition in $\mathbb{P}_\gamma$ for which p forces that $\tau \restriction \beta \in G_\beta$. Then there is a condition p' in $\mathbb{P}_\gamma$ such that $p' \restriction \beta = p$ and p forces that $p' \restriction [\beta, \gamma) = \tau \restriction [\beta, \gamma)$.*

The following fact is extracted from [15, pp. 7–10].

FACT 3.9. *Let γ be a limit ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ be an* RCS *iteration with* RCS *limit $\mathbb{P}_\gamma$. Then, for each $p \in \mathbb{P}_\gamma$, there exists a sequence of $\mathbb{P}_\gamma$-names τ_i, $i \in \omega$, for elements of $\gamma + 1$ such that*

- *for any condition q, any $i \in \omega$ and any $\alpha \leqslant \gamma$, if $q \Vdash \tau_i = \check{\alpha}$, then $(q \restriction \alpha)^\frown 1_{\mathbb{P}_\gamma / \mathbb{P}_\alpha}$ forces $\tau_i = \check{\alpha}$;*
- *for all $i \in \omega$, $p \Vdash \tau_i < \check{\gamma}$;*
- *the empty condition in $\mathbb{P}_\gamma$ forces that for every*

$$\beta \geqslant \sup\{\tau_i : i \in \omega\},$$

$p(\beta) = 1_{\dot{Q}_\beta}$.

The following is our extension of Theorem 3.3 to η-semicomplete iterations. We will introduce two more useful facts before we start the proof.

THEOREM 3.10. *Let η and γ be ordinals, with $\eta < \omega_1$, and let*

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$$

be an RCS *iteration with* RCS *limit $\mathbb{P}_\gamma$. Suppose that, for all $\alpha < \gamma$,*

- *$\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ is $(<\omega_1)$-semiproper;*
- *the iteration $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ is η-semicomplete;*
- *$\Vdash_{\alpha+1} |\mathbb{P}_\alpha| \leqslant \aleph_1$.*

Then $\mathbb{P}_\gamma$ is totally proper.

A proof of the following fact appears in [14].

FACT 3.11. *Let η be a countable ordinal, γ be an ordinal and*

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$$

be an RCS *iteration with* RCS *limit $\mathbb{P}_\gamma$. Suppose that, for all $\alpha < \gamma$,*

- *$\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ is η-semiproper;*
- *$\Vdash_{\alpha+1} |\mathbb{P}_\alpha| \leqslant \aleph_1$.*

Then $\mathbb{P}_\gamma$ *is* η*-semiproper.*

The proof of Theorem 3.10 uses the following lemma, a simplified (and ostensibly weaker) version of [14, Lemma 4.10] which is used in the course of proving Fact 3.11 above.

LEMMA 3.12. *Let* γ *be an ordinal and* η *be a countable ordinal. Suppose that* $\mathbb{P}_\gamma$ *is the* RCS *limit of an* RCS *iteration* $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ *such that, for each* $\alpha < \gamma$,

- $1_{\mathbb{P}_\alpha}$ *forces* $\dot{\mathbb{Q}}_\alpha$ *to be* η*-semiproper;*
- $1_{\mathbb{P}_{\alpha+1}}$ *forces* $\mathbb{P}_\alpha$ *to have cardinality* $\aleph_1$.

Let θ *be sufficiently large for* $\mathbb{P}_\gamma$. *Fix* $\alpha \leqslant \beta \leqslant \gamma$, *and fix a suitable tower* $\langle N_\xi : \xi < \eta \rangle$ *for* $\mathbb{P}_\gamma$ *with* $\alpha, \beta \in N_0$. *Let* $s \in \mathbb{P}_\gamma$ *and* $r \in \mathbb{P}_\alpha$ *be such that*

- r *is* $(N_\xi, \mathbb{P}_\alpha)$*-semigeneric for all* $\xi < \eta$;
- $s \restriction \alpha \geqslant r$;
- r *forces that* $s \restriction [\alpha, \gamma) = t \restriction [\alpha, \gamma)$ *for some* $t \in \mathbb{P}_\gamma \cap N_0$.

Then there exists $r^\dagger \in \mathbb{P}_\beta$ *such that*

- $r^\dagger$ *is* $(N_\xi, \mathbb{P}_\beta)$*-semigeneric for all* $\xi < \eta$;
- $r^\dagger \leqslant s \restriction \beta$;
- $r^\dagger \restriction \alpha = r$.

To prove Theorem 3.10, let θ be a regular cardinal which is sufficiently large for $\mathbb{P}_\gamma$. Let $N \prec H(\theta)$ be countable with $\mathbb{P}$ and η in N, and let $p \in \mathbb{P}_\gamma \cap N$ be an arbitrary condition. We must produce a totally $(N, \mathbb{P}_\gamma)$-generic condition $q \leqslant p$. For each $\alpha \in N \cap (\gamma+1)$, let α^* denote the order-type of $N \cap \alpha$. Fix a suitable tower $\overline{N} = \langle N_\xi : \xi \leqslant \eta\gamma^* \rangle$ with $N_0 = N$. The following claim is a variation of [8, Claim 6.2]. In order to facilitate the statement of the claim, we let $N_{\eta\gamma^*+1}$ stand for $H(\theta)$.

CLAIM 3.13. *Given* $\alpha < \beta$ *in* $N_0 \cap (\gamma+1)$, $p \in \mathbb{P}_\beta$ *and*

$$G \in \mathrm{Gen}^+(N_0, \mathbb{P}_\alpha, p \restriction \alpha) \cap N_{\eta\alpha^*+1},$$

there is a $G^\dagger \in \mathrm{Gen}(N_0, \mathbb{P}_\beta, p) \cap N_{\eta\beta^*+1}$ *such that, whenever* $r \in \mathbb{P}_\alpha$ *is a lower bound for* G *that is* $(N_\xi, \mathbb{P}_\alpha)$*-semigeneric for all* $\xi \in (\eta\alpha^*, \eta\gamma^*]$, *there is an* $r^\dagger \in \mathbb{P}_\beta$ *such that*

(1) $r^\dagger$ *is a lower bound for* $G^\dagger$;
(2) $r^\dagger \restriction \alpha = r$;
(3) $r^\dagger$ *is* $(N_\xi, \mathbb{P}_\beta)$*-semigeneric for every* $\xi \in (\eta\beta^*, \eta\gamma^*]$.

Theorem 3.10 follows from taking $\alpha = 0$ and $\beta = \gamma$ in Claim 3.13. Inducting primarily on γ, we assume that the claim holds for all $\gamma' < \gamma$ in place of γ, for this fixed sequence of N_ξ's. This will be useful in the limit case below.

Remark 3.14. Item (1) above implies that $\{q\restriction\alpha : q\in G^\dagger\}=G$, since otherwise these two generic filters could not have the same lower bound r.

Since $\mathbb{P}_0$ is the trivial forcing, the case $\alpha=0$ and $\beta=1$ follows from the assumption that $\dot{\mathbb{Q}}_0$ is totally proper and $(<\omega_1)$-semiproper.

Now consider the case where $\beta=\beta_0+1$. We are given a

$$G\in \mathrm{Gen}^+(N_0,\mathbb{P},p\restriction\alpha)\cap N_{\eta\alpha^*+1},$$

and, applying the induction hypothesis, we may fix a

$$G_0^\dagger \in \mathrm{Gen}(N_0,\mathbb{P}_{\beta_0},p\restriction\beta_0)\cap N_{\eta\beta_0^*+1}$$

satisfying the claim with β_0 in the role of β. Since $\mathbb{P}_\alpha$ is $(<\omega_1)$-semiproper, the conclusion of the claim implies that $G_0^\dagger\in\mathrm{Gen}^+(N_0,\mathbb{P}_{\beta_0},p\restriction\beta_0)$. We apply the definition of "$\dot{\mathbb{Q}}_{\beta_0}$ is η-semicomplete for $\mathbb{P}_{\beta_0}$" in $N_{\eta\beta^*+1}$ with $\{N_0\}\cup\{N_\xi:\eta\beta_0^*+1\leqslant\xi\leqslant\eta\beta^*\}$, $G_0^\dagger$ and $p(\beta_0)$ in place of $\langle N_\xi:\xi<1+\eta\rangle$, $\overline{G}$ and $\dot{q}$, respectively, there. This gives us a

$$G^\dagger \in \mathrm{Gen}(N_0,\mathbb{P}_\beta,p)\cap N_{\eta\beta^*+1}$$

extending $G_0^\dagger$ such that, whenever $r_0^\dagger$ is a lower bound for $G_0^\dagger$ which is

$$(\{N_0\}\cup\{N_\xi:\eta\beta_0^*+1\leqslant\xi\leqslant\eta\beta^*\},\mathbb{P}_{\beta_0})\text{-semigeneric},$$

$r_0^\dagger$ forces that $G^\dagger/G_0^\dagger$ has a lower bound in $\mathbb{Q}_{\beta_0}$.

Now, whenever $r\in\mathbb{P}_\alpha$ is a lower bound for G that is $(N_\xi,\mathbb{P}_\alpha)$-semigeneric for all $\xi\in(\eta\alpha^*,\eta\gamma^*]$, there is, by the choice of $G_0^\dagger$, a condition $r_0^\dagger\in\mathbb{P}_{\beta_0}$ such that

- $r_0^\dagger$ is a lower bound for $G_0^\dagger$;
- $r_0^\dagger\restriction\alpha=r$;
- $r_0^\dagger$ is $(N_\xi,\mathbb{P}_\alpha)$-semigeneric for every $\xi\in(\eta\beta_0^*,\eta\gamma^*]$.

By Theorem 3.8, there is a condition $s\in\mathbb{P}_\beta\cap N_{\eta\beta^*+1}$ such that $s\restriction\beta_0$ is $1_{\mathbb{P}_{\beta_0}}$ and $1_{\mathbb{P}_{\beta_0}}$ forces that $s(\beta_0)$ is a lower bound for $G^\dagger/G_0^\dagger$ if such a lower bound exists. By Lemma 3.12, then there is an $r^\dagger$ as desired, with $r^\dagger\restriction\beta_0=r_0^\dagger$ and $s\geqslant r^\dagger$. This takes care of the case where β is a successor ordinal.

Finally, suppose that β is a limit ordinal. Fix a strictly increasing sequence

$$\langle\alpha_n:n\in\omega\rangle\in N_{\eta\beta^*+1}$$

which is cofinal in $N_0\cap\beta$, with $\alpha_0=\alpha$, and let $\langle D_n:n\in\omega\rangle\in N_{\eta\beta^*+1}$ be a listing of the dense open subsets of $\mathbb{P}_\beta$ in N_0.

SUBCLAIM 3.15. *There exist sequences* $\langle p_n : n \in \omega\rangle$ *and* $\langle G_n : n \in \omega\rangle$ *in* $N_{\eta\beta^*+1}$ *such that* $p_0 = p$, $G_0 = G$ *and, for all* $n \in \omega$,

- $p_{n+1} \in N_0 \cap D_n$;
- $p_{n+1} \leqslant p_n$;
- $p_{n+1} \restriction \alpha_n \in G_n$;
- $G_n \in \mathrm{Gen}(N_0, \mathbb{P}_{\alpha_n}, p_n \restriction \alpha_n) \cap N_{\eta\alpha_n^*+1}$;
- *whenever* $r \in \mathbb{P}_{\alpha_n}$ *is a lower bound for* G_n *that is* $(N_\xi, \mathbb{P}_{\alpha_n})$*-semigeneric for all* $\xi \in (\eta\alpha_n^*, \eta\gamma^*]$, *there is an* $r^+ \in \mathbb{P}_{\alpha_{n+1}}$ *such that*
  - r^+ *is a lower bound for* G_{n+1};
  - $r^+ \restriction \alpha_n = r$;
  - r^+ *is* $(N_\xi, \mathbb{P}_{\alpha_{n+1}})$*-semigeneric whenever*

$$\xi \in (\eta\alpha_{n+1}^*, \eta\gamma^*].$$

Given $n \in \omega$, $r \in \mathbb{P}_{\alpha_n}$ and $\delta \in (\alpha_n, \gamma] \cap N_0$, let $A(r, n, \delta^*)$ denote the statement that r is a lower bound for G_n and r is $(N_\xi, \mathbb{P}_{\alpha_n})$-semigeneric for all $\xi \in (\eta\alpha_n^*, \eta\delta^*]$ (this is just for notational convenience, and we will use it only when the G_n in question has already been established). Then the last item of the subclaim says that for all $r \in \mathbb{P}_{\alpha_n}$ satisfying $A(r, n, \gamma^*)$, there exists an $r^+ \in \mathbb{P}_{\alpha_{n+1}}$ such that

- $r^+ \restriction \alpha_n = r$;
- r^+ satisfies $A(r^+, n+1, \gamma^*)$ (again, for G_{n+1} as chosen).

To verify the subclaim, suppose that p_n and G_n are given. We will verify that p_{n+1} and G_{n+1} exist as described in the subclaim. First note that $E = \{t \restriction \alpha_n : t \in D_n \text{ and } t \leqslant p_n\}$ is dense in $\mathbb{P}_{\alpha_n}$ below $p_n \restriction \alpha_n$, and that $E \in N_0$. Since $p_n \restriction \alpha_n \in G_n$, there exists a $t \in E \cap G_n$. Applying the definition of E inside N_0, we get a $p_{n+1} \in N_0 \cap D_n$ with $p_{n+1} \leqslant p_n$ and $p_{n+1} \restriction \alpha_n \in G_n$, as desired.

Applying the induction hypothesis inside of $N_{\eta\beta^*+1}$, with α_n, α_{n+1} and β in place of α, β and γ, respectively, we can find a filter

$$G_{n+1} \in \mathrm{Gen}(N_0, \mathbb{P}_{\alpha_{n+1}}, p_{n+1} \restriction \alpha_{n+1})$$

so that for any condition $r \in \mathbb{P}_{\alpha_n}$ satisfying $A(r, n, \beta^*)$ there is an $r' \in \mathbb{P}_{\alpha_{n+1}}$ satisfying $A(r', n+1, \beta^*)$ such that $r' \restriction \alpha_n = r$. We need to see that for this G_{n+1}, for any condition $r \in \mathbb{P}_{\alpha_n}$ satisfying $A(r, n, \gamma^*)$ there is an $r^+ \in \mathbb{P}_{\alpha_{n+1}}$ satisfying $A(r', n+1, \gamma^*)$ such that $r^+ \restriction \alpha_n = r$.

Fix such an r. Since r satisfies $A(r, n, \gamma^*)$, it satisfies $A(r, n, \beta^*)$. Fix $r' \in \mathbb{P}_{\alpha_{n+1}}$ such that $r' \restriction \alpha_n = r$ and r' satisfies $A(r, n+1, \beta^*)$. In order to apply Lemma 3.12, we want to see that there is an $r''\in\mathbb{P}_{\alpha_{n+1}}$ satisfying $A(r'',n+1,\beta^*)$ such that $r''\restriction\alpha_n=r$ and such that

$$r''\Vdash_{\mathbb{P}_{\alpha_n}} r''/G_{\alpha_n}\in N_{\eta\beta^*+1}[G_{\alpha_n}],$$

that is, that r forces (in $\mathbb{P}_{\alpha_n}$) that there is a $\mathbb{P}_{\alpha_n}$-name t in $N_{\eta\beta^*+1}$ such that

$$r''/G_{\alpha_n}=t_{G_{\alpha_n}}.$$

If we force with $\mathbb{P}_{\alpha_n}$ below $r\restriction\alpha_n$, in $V[G_{\alpha_n}]$, $r'/G_{\alpha_n}\in\mathbb{P}_{\alpha_{n+1}}/G_{\alpha_n}$ satisfies the following condition:

(**) is a lower bound for $\{s/G_{\alpha_n}:s\in G_{n+1}\}$ and is semigeneric for

$$(N_\xi[G_{\alpha_n}],\mathbb{P}_{\alpha_{n+1}}/G_{\alpha_n})$$

for all $\xi\in(\eta\alpha^*_{n+1},\eta\beta^*]$.

So there exists a condition satisfying (**) in $N_{\eta\beta^*+1}[G_{\alpha_n}]$.

Let τ be a $(\mathbb{P}_{\alpha_n}\restriction r)$-name for an element of $\mathbb{P}_{\alpha_{n+1}}/G_n$ in $N_{\eta\beta^*+1}[G_{\alpha_n}]$ satisfying (**). Viewing $\mathbb{P}_{\alpha_{n+1}}$ as $\mathbb{P}_{\alpha_n}*\dot{Q}_{\alpha_n,\alpha_{n+1}}$ (so that $\dot{Q}_{\alpha_n,\alpha_{n+1}}$ is a $\mathbb{P}_{\alpha_n}$-name for the rest of the iteration $\mathbb{P}_{\alpha_{n+1}}$), let $r''=(r,\tau)$.

Now apply Lemma 3.12. We have that

- r is $(N_\xi,\mathbb{P}_{\alpha_n})$-semigeneric for all $\xi\in(\eta\beta^*,\eta\gamma^*]$;
- $r''\restriction\alpha_n=r$;
- r forces that there is a $t\in\mathbb{P}_{\alpha_{n+1}}\cap N_{\eta\beta^*+1}$ such that $r''\restriction[\alpha_n,\alpha_{n+1})=t\restriction[\alpha_n,\alpha_{n+1})$.

Then by the lemma, there exists an r^+ which is $(N_\xi,\mathbb{P}_{\alpha_{n+1}})$-semigeneric for all $\xi\in(\eta\beta^*,\eta\gamma^*]$ such that $r^+\leqslant r''$ and $r^+\restriction\alpha_n=r$. This verifies the subclaim.

Let $G^\dagger=\{t\in N_0\cap\mathbb{P}_\beta:\text{there exists } n \text{ such that } t\geqslant p_n\}$. Then

$$G^\dagger\in\mathrm{Gen}(N_0,\mathbb{P}_\beta,p)\cap N_{\eta\beta^*+1}.$$

SUBCLAIM 3.16. *$G^\dagger$ has a lower bound.*

In order to see this, let r be a lower bound for G that is $(N_\xi,\mathbb{P}_\alpha)$-semigeneric for all $\xi\in(\eta\alpha^*,\eta\gamma^*]$. The properties of the sequence $\langle G_n:n\in\omega\rangle$ allow us to build a sequence $\langle r_n:n\in\omega\rangle$ satisfying the following properties:

- $r_0=r$;
- r_n is a lower bound for G_n in $\mathbb{P}_{\alpha^*}$;
- r_n is $(N_\xi,\mathbb{P}_{\alpha_n})$-semigeneric for all $\xi\in(\eta\alpha^*_n,\eta\gamma^*]$;
- $r_{n+1}\restriction\alpha_n=r_n$.

Finally let $r^+ = \bigcup_{n\in\omega} r_n \in \mathbb{P}_\beta$. Let us check that r^+ is a lower bound for $G^\dagger$. First note that, by the argument presented in Remark 3.14, $\{q\restriction\alpha_n : q\in G_m\} = G_n$, whenever $n\leqslant m$. When $m\leqslant n$, one has $p_m \geqslant p_n$, so $p_m\restriction\mathbb{P}_{\alpha_n} \geqslant p_n\restriction\mathbb{P}_{\alpha_n}$. Since for each $n\in\omega$ we have $p_n\restriction\alpha_n\in G_n$, we get that for each such n, $\{p_m\restriction\alpha_n : m\in\omega\}\subseteq G_n$.

For each $n\in\omega$, let τ_i^n be the names as in Fact 3.9 corresponding to p_n. Since the p_m's collectively meet all dense subsets of $\mathbb{P}_\beta$ in N_0, a value for each τ_i^n is decided by some p_m, and since p_n and p_m are compatible this value is decided to be some value in $N_0\cap\beta$. Since for each $m\in\omega$, $r\restriction\mathbb{P}_{\alpha_m}$ is a lower bound for G_m, we have that $r\restriction\alpha_m\leqslant p_n\restriction\alpha_m$ for all $m\in\omega$, and thus $r\leqslant p$. It follows that r is a lower bound for $G^\dagger$. This proves the subclaim, and thereby the limit case of Claim 3.13 and thereby Theorem 3.10.

4. The single step forcing for ψ_1

In this section we examine the single step forcings associated with ψ_1. Before proceeding, we will recall some terminology from [16]. Let X be an uncountable set and let θ be a regular cardinal with $\mathcal{P}([X]^{\aleph_0})$ in $H(\theta)$. $[X]^{\aleph_0}$ is topologized by declaring sets of the form

$$[a, M] = \{N\in[X]^{\aleph_0} : a\subseteq N\subseteq M\}$$

to be *open* whenever M is in $[X]^{\aleph_0}$ and a is a finite subset of M. If M is a countable elementary submodel of $H(\theta)$ with X in M, then $\Sigma\subseteq[X]^{\aleph_0}$ is *M-stationary* if $M\cap E\cap\Sigma$ is non-empty whenever $E\subseteq[X]^{\aleph_0}$ is a club in M. If Σ is a function whose domain is a club of countable elementary submodels of $H(\theta)$, then we say that Σ is an *open stationary set mapping* if $\Sigma(M)$ is open and M-stationary whenever M is in the domain of Σ. If $\mathcal{N} = \langle N_\xi : \xi<\omega_1\rangle$ is a continuous $\subseteq$-chain, where $\langle N_\xi : \xi\leqslant\nu\rangle$ is in $N_{\nu+1}$ for each ν, then we say that $\mathcal{N}$ is a *reflecting sequence* for Σ if, whenever $\nu<\omega_1$ is a limit ordinal, there is a $\nu_0<\nu$ such that

$$N_\xi\cap X\in\Sigma(N_\nu)$$

whenever $\nu_0<\xi<\nu$. If $\mathcal{N} = \langle N_\xi : \xi\leqslant\delta\rangle$ is a sequence of countable successor length which has the above properties for all limit $\nu\leqslant\delta$, then we will say that $\mathcal{N}$ is a *partial reflecting sequence* for Σ. In [16] it is shown that PFA implies that all open stationary set mappings admit reflecting sequences and that the forcing $\mathbb{P}_\Sigma$ of all countable partial reflecting sequences for an open stationary set mapping Σ is always totally proper.

Except for trivial cases, $\mathbb{P}_\Sigma$ is not ω-proper. Moreover it will be $(<\omega_1)$-semiproper only under rather special circumstances. The following lemma gives a useful sufficient condition for when we can build generic conditions in $\mathbb{P}_\Sigma$ for a given suitable tower of models.

LEMMA 4.1. *Let Σ be an open stationary set mapping whose domain consists of elements of $H(\theta)$ and let λ be sufficiently large for $\mathbb{P}_\Sigma$. Suppose that $\mathcal{M}=\langle M_\delta:\delta\leqslant\alpha\rangle$ is a tower of countable elementary submodels of $H(\lambda)$ which is suitable for $\mathbb{P}_\Sigma$ and such that $\langle M_\delta\cap H(\theta):\delta\leqslant\alpha\rangle$ is a partial reflecting sequence for Σ. Then every condition in M_0 can be extended to a totally $(\mathcal{M},\mathbb{P}_\Sigma)$-generic condition.*

Proof. This follows from the properness of $\mathbb{P}_\Sigma$ when $\alpha=0$, and by the induction hypothesis, elementarity and the total properness of $\mathbb{P}_\Sigma$ when α is a successor ordinal. When α is a limit ordinal, choose an increasing sequence $\langle\beta_i:i<\omega\rangle$ converging to α, such that for all δ in the interval $[\beta_0,\alpha)$ one has $M_\delta\cap X\in\Sigma(M_\alpha\cap H(\theta))$. Note that any condition in $\mathbb{P}_\Sigma$ which is $(M_\delta,\mathbb{P}_\Sigma)$-generic for all $\delta<\alpha$ will be $(M_\alpha,\mathbb{P}_\Sigma)$-generic. The difficulty in what follows is in ensuring that a final segment of the generic sequence we build falls inside of $\Sigma(M_\alpha\cap H(\theta))$. We will ensure that this happens for all members of the sequence containing M_{β_0}. We have that for each δ in the interval $[\beta_0,\alpha)$ there is a finite set $a_\delta\subset M_\delta\cap X$ such that $[a_\delta,M_\delta\cap X]\subset\Sigma(M_\alpha\cap H(\theta))$.

By elementarity and the induction hypothesis, we may assume first that s_0 is a condition in M_{β_0+1} which is $(M_\delta,\mathbb{P}_\Sigma)$-generic for all $\delta\leqslant\beta_0$, and which extends any given condition $s\in M_0$. We may assume that the last member of s_0 is $M_{\beta_0}\cap X$, and we have then that a tail of s_0 is contained in $\Sigma(M_\alpha\cap H(\theta))$. Suppose now that $i\in\omega$, that s_i is a condition in M_{β_i+1} which is $(M_\delta,\mathbb{P}_\Sigma)$-generic for all $\delta\leqslant\beta_i$, which extends s_0, whose last member is $M_{\beta_i}\cap X$, and is such that every member of s_i containing $M_{\beta_0}\cap X$ is in $\Sigma(M_\alpha\cap H(\theta))$. We show how to choose s_{i+1} satisfying these conditions for $i+1$. If $a_{\beta_{i+1}}\subseteq M_{\beta_i+1}$, then we let s_i' be a condition in M_{β_i+1} extending s_i by one set which contains $a_{\beta_{i+1}}$, and, applying the induction hypothesis and elementarity, we let s_{i+1} be a condition in $M_{\beta_{i+1}+1}$ as desired, extending s_i'.

If $a_{\beta_{i+1}}$ is not in M_{β_i+1}, we need to work harder to extend s_i while staying inside $\Sigma(M_\alpha\cap H(\theta))$. In this case, let $a(i,0)=a_{\beta_{i+1}}$ and let $\gamma(i,0)$ be the largest δ in (β_i,β_{i+1}) such that $a(i,0)$ is not contained in M_δ. Let $a(i,1)$ be a finite subset of $M_{\gamma(i,0)}\cap X$ such that

$$[a(i,1),M_{\gamma(i,0)}\cap X]\subseteq\Sigma(M_\alpha\cap H(\theta)).$$

Continue in this way, letting $\gamma(i,j+1)$ be the largest δ in $[\beta_i,\gamma(i,j))$ such that $\delta=\beta_i$ or $a(i,j+1)$ is not contained in M_δ, and, if $\gamma(i,j+1)>\beta_i$, letting $a(i,j+2)$ be a finite subset of $M_{\gamma(i,j+1)}\cap X$ such that

$$[a(i,j+2),M_{\gamma(i,j+1)}\cap X]\subseteq\Sigma(M_\alpha\cap H(\theta)).$$

As the $\gamma(i,j)$'s are decreasing, this sequence must stop at a point where $a(i,j)\subseteq M_{\beta_i+1}$ and $\gamma(i,j)=\beta_i$. Let k be this j. As $\langle a(i,j):j\leqslant k\rangle$ is in $M_{\beta_{i+1}+1}$, we can argue in $M_{\beta_{i+1}+1}$, as follows.

Let $t(i,k)$ be a condition in M_{β_i+1} extending s_i such that every member of $t(i,k)\setminus s_i$ contains $a(i,k)$. Applying the induction hypothesis and elementarity, let $s(i,k)$ be a condition in $M_{\gamma(i,k-1)+1}$ extending $t(i,k)$ which is $(M_\delta, \mathbb{P}_\Sigma)$-generic for every $\delta \leqslant \gamma(i,k-1)$, and whose last member is $M_{\gamma(i,k-1)} \cap X$. For each positive $j<k$, let $t(i,j)$ be a condition in $M_{\gamma(i,j)+1}$ extending $s(i,j+1)$ such that every member of $t(i,j)\setminus s(i,j+1)$ contains $a(i,j)$, and let $s(i,j)$ be a condition in $M_{\gamma(i,j-1)+1}$ extending $t(i,j)$ which is $(M_\delta, \mathbb{P}_\Sigma)$-generic for every $\delta \leqslant \gamma(i,j-1)$, and whose last element is $M_{\gamma(i,j-1)} \cap X$. Finally, let $t(i,0)$ be a condition in $M_{\gamma(i,0)+1}$ extending $s(i,1)$ such that every member of $t(i,0)\setminus s(i,1)$ contains $a(i,0)$, and let s_{i+1} be a condition in $M_{\beta_{i+1}+1}$ extending $t(i,0)$ which is $(M_\delta, \mathbb{P}_\Sigma)$-generic for all $\delta \leqslant \beta_{i+1}$ and whose last member is $M_{\beta_{i+1}} \cap X$.

Then every member of $s_{i+1}\setminus s_i$ is in $\Sigma(M_\alpha \cap H(\theta))$, as desired. Continuing in this way, the union of the s_i's will be the desired condition. □

Now we return to our discussion of ψ_1. Let $\vec{C}$ be a ladder system and let $\varkappa_i$, $i<3$, be an increasing sequence of cardinals greater than ω_2. For a fixed $A: \omega_1 \to 2$, we will define a totally proper forcing $\mathbb{Q}_{A,\vec{\varkappa},\vec{C}}$ which collapses $\varkappa_2$ to have cardinality ω_1 and adds a function $f: \omega_1 \to 2^\omega$ such that $f(\delta)$ is a code for $A \restriction \delta$ for each $\delta < \omega_1$, together with a witness $\mathcal{N}$ of the statement that $(\varkappa_0, \varkappa_1, \varkappa_2)$ codes f with respect to $\vec{C}$. In order to improve readability, we will suppress terms from subscripts which are either clear from the context or which do not influence the truth of a given statement.

The forcing $\mathbb{Q}_{A,\vec{\varkappa},\vec{C}}$ is the collection of all q such that

(1) q is a function from some countable successor ordinal $\delta+1$ into $[\varkappa_2]^{\aleph_0}$;

(2) q is continuous and strictly $\subseteq$-increasing;

(3) if $\nu \leqslant \delta$ is a limit ordinal, then there are a $\nu_0 < \nu$ and an $r \in 2^\omega$ such that r codes $A \restriction \nu$ and, for all $\nu_0 < \xi < \nu$,

$$\Delta(\bar{s}_{\vec{\varkappa}}(N_\xi, N_\nu), r) \geqslant n(N_\xi, N_\nu).$$

This forcing can be viewed as a two-step iteration in which we first add, by countable approximations, a function $f: \omega_1 \to 2^\omega$ with the property that $f(\delta)$ codes $A \restriction \delta$ for each δ. Then we force to add a reflecting sequence (using the partial order described above) for the set mapping Σ_f, where $\Sigma_f(N)$ is the set of all M in $[\varkappa_2]^{\aleph_0}$ such that $M \subseteq N$, $M \cap \varkappa$ is bounded in $N \cap \varkappa$ for $\varkappa$ in $\{\omega_1, \varkappa_0, \varkappa_1, \varkappa_2\}$ and

$$\Delta(\bar{s}_{\vec{\varkappa}}(M,N), f(N \cap \omega_1)) \geqslant n(M,N).$$

It is not difficult to verify that this is an open set mapping, and it will follow from arguments below that it is in fact an open stationary set mapping. Hence $\mathbb{Q}_{A,\vec{\varkappa},\vec{C}}$ can be regarded as a two-step iteration of a σ-closed forcing followed by a forcing of the form $\mathbb{P}_\Sigma$.

Our goal in this section is to prove the following two lemmas. It then follows from Theorem 3.10 and standard book-keeping and chain condition arguments (see, e.g., [13, Chapter VIII] and [20]) that if there is an inaccessible cardinal which is a limit of measurable cardinals, then there is a proper forcing extension with the same reals which satisfies ψ_1.

LEMMA 4.2. *If* $\varkappa_i$, $i<3$, *is an increasing sequence of measurable cardinals, then* $\mathbb{Q}_{A,\vec{\varkappa},\vec{C}}$ *is* $(<\omega_1)$*-semiproper.*

LEMMA 4.3. *If* $\mathbb{P}$ *is a totally proper forcing and* $\vec{\varkappa}$, $\vec{C}$ *and* $\dot{A}$ *are* $\mathbb{P}$*-names for objects as described above, then* $\mathbb{P}*\dot{\mathbb{Q}}_{\dot{A}}$ *is* 1*-semicomplete. In particular* $\dot{\mathbb{Q}}_{\dot{A}}$ *is totally proper.*

Remark 4.4. The reader may be puzzled as to why we have constructed $\mathbb{Q}_{A,\vec{\varkappa},\vec{C}}$ by first forcing to produce the function f, since there are certainly functions f in V such that $f(\delta)$ codes $A\restriction\delta$. The problem arises in proving Lemma 4.3—the argument below does not go through unless we force the function f as we are building the corresponding reflecting sequence.

Remark 4.5. It is interesting to note that it is much easier to obtain the consistency of $\psi_1[\vec{C}]$ with CH for *some* ladder system $\vec{C}$. Suppose that $\vec{C}$ is a ladder system on ω_1, $\mathbb{P}$ is a totally proper forcing, and $\dot{A}$ is a $\mathbb{P}$-name for an element of 2^{ω_1}. If M is a suitable model for $\mathbb{P}*\dot{\mathbb{Q}}_{\dot{A},\vec{C}}$, p is totally $(M,\mathbb{P})$-generic, and $\dot{q}$ is forced by p to be an element of $M[G]\cap\dot{\mathbb{Q}}_{\dot{A},\vec{C}}$, then there is an $\dot{r}$ such that $(p,\dot{r})$ is a totally $(M,\mathbb{P}*\dot{\mathbb{Q}}_{\dot{A},\vec{C}})$-generic extension of $(p,\dot{q})$ (those familiar with preservation theorems for not adding reals with proper forcing should notice that this almost never happens). This allows one to easily prove that if $\vec{C}$ is a fixed ladder system, then we can iterate forcings of the form $\dot{\mathbb{Q}}_{\dot{A},\vec{C}}$ without adding reals (and without the complex iteration machinery which we are about to employ). This shows that if we allow a fixed ladder system as a parameter, we can force $\psi_1[\vec{C}]\wedge$CH over any model of ZFC (recall that $2^{\aleph_0}=2^{\aleph_1}$ follows from the existence of a ladder system $\vec{C}$ such that both $\psi_1[\vec{C}]$ and $\psi_2[\vec{C}]$ hold). The difficulty arises when we want to quantify out the parameter $\vec{C}$ in order to obtain a Π_2-sentence. The final section of [17] contains an example of a pair $\psi_1'[\vec{C}]$ and ψ_2' of Π_2-sentences having these same properties except that $\forall\vec{C}\,\psi_1'[\vec{C}]$ implies $2^{\aleph_0}=2^{\aleph_1}$.

In [3], the proof of Lemma 5 actually yields the following lemma (stated in our notation).

LEMMA 4.6. *Suppose that* $\varkappa_i$, $i<3$, *is an increasing sequence of regular cardinals above* ω_1 *and* $\vec{C}^i$, $i<l$ *(for some* $l\in\omega$*), is a sequence of ladder systems on* ω_1*. If* M *is a countable elementary submodel of* $H(\theta)$ *for* θ *sufficiently large and* $E\subseteq[\varkappa_2]^{\aleph_0}$ *is a club*

in M, then there is an n so that for any σ in $2^{<\omega}$ there is an N in $E\cap M$ such that

$$o(x^i\backslash n, y^i\backslash n, z^i\backslash n)=\sigma \quad \textit{and} \quad n^i(N,M)\leqslant n$$

for all $i<l$, where x^i, y^i, z^i and n^i are computed from M and N as in the computation of $s_{\vec{\varkappa}}^{\vec{C}^i}(N,M)$.

We will now prove Lemmas 4.2 and 4.3.

Proof of Lemma 4.3. Let $\mathbb{P}$ be proper and force that

- $\dot{\varkappa}_i$, $i<3$, is an increasing sequence of regular cardinals above ω_1;
- $\langle \dot{C}_\xi : \xi\in\mathrm{Lim}(\omega_1)\rangle$ is a ladder system on ω_1;
- $\dot{A}$ is a function from ω_1 to 2.

Let $\dot{\mathbb{Q}}$ denote $\dot{\mathbb{Q}}_{\dot{A},\vec{\varkappa},\vec{C}}$, $N_0\in N_1$ be suitable models for $\mathbb{P}*\dot{\mathbb{Q}}$, G be an $(N_0,\mathbb{P})$-generic filter and q be an element of $\mathbb{Q}^{N_0[G]}$. Observe that there is a condition in G deciding $\dot{\varkappa}_i$ to be some $\varkappa_i$ for each $i<3$. Let $\bar{\varkappa}_i$ denote the image of $\varkappa_i$ under the transitive collapse of N_0. Furthermore, if $\delta=N_0\cap\omega_1$, then there is an $A:\delta\to 2$ such that for every $\alpha<\delta$, there is a condition in G forcing $\dot{A}\restriction\check{\alpha}=\check{A}\restriction\check{\alpha}$. Fix an r in 2^ω such that r codes $A\restriction\delta$.

Notice that, by CH, if p is $(N_1,\mathbb{P})$-semigeneric and a lower bound for G, then p forces that the value of $\dot{C}_\delta$ is some element of N_1, where $\delta=N_0\cap\omega_1$ (although it need not decide which this value is). A similar statement is true concerning $\dot{C}_{\bar{\varkappa}_i}$ for each $i<3$. Let C_δ^j, $j<\omega$, and $C_{\bar{\varkappa}_i}^j$, $j<\omega$, enumerate all cofinal subsets of δ and $\bar{\varkappa}_i$, respectively, of order-type ω which are elements of N_1. Let D_i, $i<\omega$, enumerate all dense open subsets of $\mathbb{Q}$ which are in $N_0[G]$.

We will now build a sequence q_i, $i<\omega$, of conditions in $\mathbb{Q}^{N_0[G]}$ such that

- $q_{i+1}\leqslant q_i$ and q_{i+1} is in $N_0[G]\cap D_i$;
- if ξ is in $\mathrm{dom}(q_{i+1})\backslash\mathrm{dom}(q_i)$, then $M=q_{i+1}(\xi)$ satisfies

$$\Delta(\bar{s}^j(M,N_0\cap\varkappa_2),r)\geqslant n^j(M,N_0\cap\varkappa_2)\quad\text{for all } j\leqslant i,$$

where s^j and n^j are computed using C_δ^j and $C_{\bar{\varkappa}_i}^j$, $i<3$.

If this can be done, then any condition $\bar{p}$ which is an $(N_1,\mathbb{P})$-semigeneric lower bound for G will force that there is some $i_0<\omega$ such that $\dot{C}_\delta=\breve{C}_\delta^{i_0}$, and therefore that q_i, $i<\omega$, will have a lower bound (namely the union of this sequence).

Suppose that we have constructed q_i and we wish to construct q_{i+1}. Following [16, Theorem 3.1] (or Lemma 4.1), it is sufficient to demonstrate that there is a countable elementary submodel M of $H((2^{\varkappa_2})^+)$ such that D_i and q_i are in M and

$$\Delta(\bar{s}^j(M\cap\varkappa_2,N_0\cap\varkappa_2),r)\geqslant n^j(M,N_0\cap\varkappa_2)$$

holds for all $j\leqslant i$. Let E be the collection of all sets of the form $M\cap\varkappa_2$ such that M is a countable elementary submodel of $H((2^{\varkappa_2})^+)$ and both q_i and D_i are in M. Let n be given as in Lemma 4.6 and let $\sigma=r\restriction(n+1)$. Find an M in E such that

$$o(x^j\setminus n, y^j\setminus n, z^j\setminus n)=\bar\sigma \quad \text{and} \quad n^j(M, N_0\cap\varkappa_2)\leqslant n$$

for all $j\leqslant i$. Then $\bar s^j(M,N_0\cap\varkappa_2)$ contains $r\restriction n$ as an initial part and therefore

$$\Delta(\bar s^j(M,N_0\cap\varkappa_2), r)\geqslant n^j(M,N_0\cap\varkappa_2).$$

This finishes the proof. □

We are now ready to turn to the proof of Lemma 4.2. Since $\mathbb{Q}_{A,\vec\varkappa,\vec C}$ decomposes as an iteration of a σ-closed partial order followed by a forcing of the form $\mathbb{P}_\Sigma$, it is sufficient to verify the $({<}\omega_1)$-semiproperness of the second factor. In fact we will show that if $\vec\varkappa$ consists of measurable cardinals, $f:\omega_1\to 2^\omega$ is any function and $\Sigma_{f,\vec\varkappa}$ is the open set mapping associated with f as above, then $\mathbb{P}_{\Sigma_{f,\vec\varkappa}}$ is $({<}\omega_1)$-semiproper.

For the rest of this section, let $\vec\varkappa=\langle\varkappa_0,\varkappa_1,\varkappa_2\rangle$ be a fixed increasing sequence of three measurable cardinals, and fix a normal ultrafilter U_i on each $\varkappa_i$. Let f be any fixed function from ω_1 to 2^ω and let $\vec C$ be a fixed ladder system on ω_1. We will denote $\mathbb{P}_{\Sigma_{f,\vec\varkappa}}$ by $\mathbb{P}$.

Let θ be sufficiently large for $\mathbb{P}$ and let $\lhd$ be a well ordering of $H(\theta)$. Given subsets M and I of $H(\theta)$, with $I\subseteq\varkappa_2\in M$, we use $\operatorname{cl}(M,I)$ to denote the set of values $g(\eta_0,...,\eta_{n-1})$, where g is a function in M with domain $\varkappa_2^{<\omega}$ and $\{\eta_0,...,\eta_{n-1}\}$ is a finite subset of I.

Still fixing θ, $\lhd$, $\vec\varkappa$ and $\vec U$, given $i\leqslant 2$ and an elementary submodel M of $(H(\theta),\in,\lhd)$ of cardinality less than $\varkappa_i$, we will say that $\{M_\xi\}_{\xi<\varkappa_i}$ is the *iteration of M relative to U_i* in case $\{M_\xi\}_{\xi<\varkappa_i}$ is the unique $\subseteq$-continuous sequence such that $M_0=M$ and, for all $\xi<\varkappa_i$, $M_{\xi+1}=\operatorname{cl}(M_\xi,\{\eta_\xi\})$, where $\eta^i_\xi=\min(\bigcap(U_i\cap M_\xi))$. We will also call $\{\eta^i_\xi\}_{\xi<\varkappa}$ the *critical sequence of M relative to U_i*. We will use the following well-known facts repeatedly in the proof of Lemma 4.10.

FACT 4.7. *For θ, $\lhd$ and $\vec\varkappa$ as above, if M is an elementary submodel of $(H(\theta),\in,\lhd)$ and $I\subseteq\varkappa_2\in M$, then $\operatorname{cl}(M,I)$ is an elementary submodel of $(H(\theta),\in,\lhd)$.*

FACT 4.8. *Let θ, $\lhd$, $\vec\varkappa$ and $\vec U$ be as above. Fix $i\leqslant 2$ and let M be an elementary submodel of $H(\theta)$ such that $U_i,\varkappa_2\in M$. If $\eta\in\bigcap(M\cap U_i)$, then $\operatorname{cl}(M,\{\eta\})\cap\varkappa_i$ is an end-extension of $M\cap\varkappa_i$.*

FACT 4.9. *Let θ, $\lhd$, $\vec\varkappa$ and $\vec U$ be as above. Fix $i\leqslant 2$ and let M be an elementary submodel of $H(\theta)$ such that $U_i,\varkappa_2\in M$. Let I be a subset of $\varkappa_i$ and let $\mu\in M$ be a regular cardinal greater than $\varkappa_i$. Then*

$$\sup(\operatorname{cl}(M,I)\cap\mu)=\sup(M\cap\mu).$$

Lemma 4.2 follows from combining Lemma 4.1 with Lemma 4.10, since whenever N and N^* are suitable models for a partial order $\mathbb{P}$ with $N\subseteq N^*$ and $N\cap\omega_1=N^*\cap\omega_1$, any $q\in\mathbb{P}$ which is $(N^*,\mathbb{P})$-generic is $(N,\mathbb{P})$-semigeneric.

LEMMA 4.10. *Let $\alpha<\omega_1$ be a limit ordinal and let $\langle N_\xi:\xi\leqslant\alpha\rangle$ be a suitable tower in $H(\theta)$ for $\mathbb{P}$ such that each N_ξ is a countable elementary submodel of $(H(\theta),\in,\lhd)$. Then there is a suitable tower $\langle N^*_\xi:\xi\leqslant\alpha\rangle$ in $H(\theta)$ such that, for each $\xi\leqslant\alpha$,*

- *N^*_ξ is a countable elementary submodel of $(H(\theta),\in,\lhd)$ of the form $\mathrm{cl}(N_\xi,I)$ for some $I\subseteq\varkappa_2$, with $N^*_\xi\cap\omega_1=N_\xi\cap\omega_1$;*
- *if $\xi\leqslant\alpha$ is a limit ordinal, then there is a $\xi_0<\xi$ such that*

$$\Delta(\bar{s}(N^*_\nu\cap\varkappa_2,N^*_\xi\cap\varkappa_2),f(N^*_\xi\cap\omega_1))\geqslant n(N^*_\nu\cap\varkappa_2,N^*_\xi\cap\varkappa_2),$$

whenever $\xi_0<\nu<\xi$, where s denotes $s^{\vec{C}}_{\varkappa_0,\varkappa_1}$.

Proof. We proceed by induction on α. We start by proving the lemma for $\alpha=\omega$, in which case we will prove the lemma with one additional conclusion, discussed below. Let $\{N^0_j\}_{j<\omega}$ and $\{\eta^0_i\}_{i<\omega}$ be the respective initial segments of length ω of the iteration of N_ω and the critical sequence of N_ω, both relative to U_0. Let $N^0=\bigcup_{j<\omega}N^0_j$. Let $\{N^1_j\}_{j<\omega}$ and $\{\eta^1_i\}_{i<\omega}$ be the respective initial segments of length ω of the iteration of N^0 and the critical sequence of N^0, both relative this time to U_1. Let $N^1=\bigcup_{j<\omega}N^1_j$. Finally, let $\{N^2_j\}_{j<\omega}$ and $\{\eta^2_i\}_{i<\omega}$ be the respective initial segments of length ω of the iteration of N^1 and the critical sequence of N^1, both relative to U_2.

Each model N^*_j will be of the form $\mathrm{cl}\big(N_j,\bigcup_{r<3}\{\eta^r_i:i\in I^r_j\}\big)$ for suitable finite subsets I^r_j of ω (for $r<3$). It will follow in particular from Fact 4.8 that $N_j\cap\omega_1=N^*_j\cap\omega_1$, so that $n(N_j,N_\omega)=n(N^*_j,N^*_\omega)$. Furthermore, we will choose the sets I^r_j so that $j\subseteq I^r_j\subseteq I^r_{j+1}$ for all $r<3$ and all $j<\omega$. This will ensure that each N^*_j is a member of N^*_{j+1}, and also that we already know at the beginning of the construction exactly which set $N^*_\omega=\bigcup_{j<\omega}N^*_j$ is going to be. Specifically, N^*_ω will be

$$\mathrm{cl}\left(N_\omega,\bigcup_{r<3}\{\eta^r_i:i<\omega\}\right)=\bigcup_{j<\omega}\mathrm{cl}\left(N_j,\bigcup_{r<3}\{\eta^r_i:i<\omega\}\right)=\bigcup_{j<\omega}N^2_j.$$

Let $\delta=N_\omega\cap\omega_1$. Let π be the collapsing function of N^*_ω, and let $C^0=\pi^{-1}[C_{\pi(\varkappa_0)}]$, $C^1=\pi^{-1}[C_{\pi(\varkappa_1)}]$ and $C^2=\pi^{-1}[C_{\pi(\varkappa_2)}]$.

For each $j<\omega$ and $r<3$, I^r_j will be of the form

$$j\cup\left(\bigcup_{j'<j}I^r_{j'}\right)\cup\{i^r_k:k<n\}$$

for a suitable increasing sequence $\{i^r_k\}_{k<n}$ of integers above $\bigcup_{j'<j} I^r_{j'}$ to be defined as follows. Let $j<\omega$ be given and suppose that $I^r_{j'}$ have been chosen for all $r<3$ and $j'<j$.

Set

$$M_0 = \mathrm{cl}\left(N_j, \bigcup_{r<3}\left\{\eta^r_i : i \in j \cup \bigcup_{j'<j} I^r_{j'}\right\}\right).$$

Let $n=n(N_j, N_\omega)$. If $n=0$, we can let $N^*_j=M_0$. Otherwise, let $\langle p_0, ..., p_{n-1}\rangle$ be $f(\delta)\restriction n$. By the choice of $\{\eta^r_i\}_{i<\omega}$ (for $r<3$) together with Fact 4.8, each of $\{\eta^0_i\}_{i<\omega}$, $\{\eta^1_i\}_{i<\omega}$ and $\{\eta^2_i\}_{i<\omega}$ is cofinal in $\varkappa_0\cap N^*_\omega$, $\varkappa_1\cap N^*_\omega$ and $\varkappa_2\cap N^*_\omega$, respectively. Choose integers i^0_k, i^1_k and i^2_k, $0\leqslant k\leqslant n-1$, and models M_t, $1\leqslant t\leqslant 2n$, satisfying the following conditions:

- $j\cup\bigcup_{j'<j} I^0_{j'}<i^0_0<...<i^0_{n-1}$;
- $j\cup\bigcup_{j'<j} I^1_{j'}<i^1_0<...<i^1_{n-1}$;
- $j\cup\bigcup_{j'<j} I^2_{j'}<i^2_0<...<i^2_{n-1}$;
- for all $k\in\{0, ..., n-1\}$,
  - $\sup(M_{2k}\cap\varkappa_0)<\eta^0_{i^0_k}$ and $C^0\cap\eta^0_{i^0_k}$ has size strictly bigger than both
  $$|C^1\cap\sup(M_{2k}\cap\varkappa_1)| \quad\text{and}\quad |C^2\cap\sup(M_{2k}\cap\varkappa_2)|;$$
  - $M_{2k+1}=\mathrm{cl}_j(M_{2k}\cup\{\eta^0_{i^0_k}\})$;
  - if $p_{n-1-k}=0$, then
  $$|C^0\cap\sup(M_{2k+1}\cap\varkappa_0)| < |C^1\cap\eta^1_{i^1_k}| < |C^2\cap\eta^2_{i^2_k}|;$$
  - if $p_{n-1-k}=1$, then
  $$|C^0\cap\sup(M_{2k+1}\cap\varkappa_0)| < |C^1\cap\eta^2_{i^2_k}| < |C^2\cap\eta^1_{i^1_k}|;$$
  - $M_{2k+2}=\mathrm{cl}(M_{2k+1}, \{\eta^1_{i^1_k}, \eta^2_{i^2_k}\})$.

Note the following consequences of these choices (and Facts 4.8 and 4.9, and the fact that each η^r_i is regular), for all $k\in\{0, ..., n-1\}$:

- $M_{2k+1}\cap[\sup(M_{2k}\cap\varkappa_0), \eta^0_{i^0_k})=\varnothing$;
- $M_{2k+2}\cap\varkappa_0=M_{2k+1}\cap\varkappa_0$;
- for all $\mu\in\{\eta^1_{i^1_{k'}}:k'<k\}\cup\{\varkappa_1\}$,
$$\sup(M_{2k}\cap\mu) = \sup(M_{2k+1}\cap\mu) < \eta^1_{i^1_k};$$
- $M_{2k+2}\cap[\sup(M_{2k+1}\cap\varkappa_1), \eta^1_{i^1_k})=\varnothing$;
- for all $\mu\in\{\eta^2_{i^2_{k'}}:k'<k\}\cup\{\varkappa_2\}$,
$$\sup(M_{2k}\cap\mu) = \sup(M_{2k+1}\cap\mu) < \eta^2_{i^2_k};$$
- $M_{2k+2}\cap[\sup(M_{2k+1}\cap\varkappa_2), \eta^2_{i^2_k})=\varnothing$.

Now it is not hard to check that the string $\langle p_{n-1}, ..., p_0\rangle$ is a terminal segment of $s(M_{2n}\cap\varkappa_2, N^*\cap\varkappa_2)$, which means that we can let $N^*_j = M_{2n}$.

We note one additional aspect of this construction: each η^r_i is in each member of $N_\omega\cap U_i$. From this and Fact 4.8 it follows that for each $j\in\omega$ and any countable elementary submodel P of $(H(\theta), \in, \lhd)$ such that $\vec{U}\in P\in N_\omega$,

$$\mathrm{cl}\Bigg(P, \bigcup_{r<3}\{\eta^r_i : i\in I^r_j\}\Bigg)\cap\omega_1 = P\cap\omega_1.$$

This completes the proof for $\alpha=\omega$.

Now we can prove the lemma for a general $\alpha<\omega_1$ by induction on α, α being a limit ordinal. Let $\{N_\nu\}_{\nu\leqslant\alpha}$ be a tower as in the hypothesis of the lemma for α, and assume that the lemma is true for all $\beta<\alpha$. Let $\{\alpha_j\}_{j<\omega}$ be any increasing sequence of limit ordinals with supremum α. Apply the case $\alpha=\omega$ to the sequence $\{N_{\alpha_j+1}\}_{j<\omega}$ to obtain the models $N^*_{\alpha_j+1}$ for $j<\omega$. Let $N^*_\alpha = \bigcup_{j<\omega} N^*_{\alpha_j+1}$. Applying the additional conclusion of the case $\alpha=\omega$, we have that there is a $\subseteq$-increasing sequence of finite sets $\langle E_j : j<\omega\rangle$ such that, for each $j<\omega$,

- $E_j\subseteq N^*_{\alpha_j+1}\cap\varkappa_2$;
- $N^*_{\alpha_j+1} = \mathrm{cl}(N_{\alpha_j+1}, E_j)$;
- for all countable elementary submodels P of $(H(\theta), \in, \lhd)$ in N_α,

$$\mathrm{cl}(P, E_j)\cap\omega_1 = P\cap\omega_1;$$

- for all Q such that $E_j\subseteq Q\subseteq N^*_{\alpha_j+1}$,

$$\Delta(\bar{s}(Q, N^*_\alpha\cap\varkappa_2), f(N^*_\alpha\cap\omega_1)) \geqslant n(N^*_{\alpha_j+1}\cap\varkappa_2, N^*_\alpha\cap\varkappa_2) = n(Q\cap\varkappa_2, N^*_\alpha\cap\varkappa_2).$$

We can now build the rest of the sequence of N^*_β's by working separately on each interval $[\alpha_j+2, \alpha_{j+1}]$ inside of $N^*_{\alpha_{j+1}+1}$ (this omits the construction for the first interval, which can be taken care of by setting $\alpha_{-1}=-2$). Fixing such a j, for each $\beta\in[\alpha_j+2, \alpha_{j+1}]$, let $N^0_\beta = \mathrm{cl}(N_\beta, E_{j+1})$. Now apply the induction hypothesis inside of $N^*_{\alpha_{j+1}+1}$ to the sequence $\langle N^0_\beta : \alpha_j+2\leqslant\beta\leqslant\alpha_{j+1}\rangle$ to obtain the desired sequence $\langle N^*_\beta : \alpha_j+2\leqslant\beta\leqslant\alpha_{j+1}\rangle$. $\square$

5. Concluding remarks

The Π_2-sentences which we employed to resolve Problem 1.1 are quite ad hoc in nature and it is natural to ask whether there are simpler examples. In particular, it is unclear whether there are Π_2-sentences which have already been studied in the literature which solve Problem 1.1. Also, while it is reasonable to expect that the consistency of ψ_1 (with or without CH) requires an inaccessible cardinal, it is unclear whether consistency of ψ_1 with CH requires, for instance, a measurable cardinal. This seems largely a technical question, but one whose solution (either positive or negative) would likely involve new ideas and give new insight how models of CH can (or cannot) be obtained by iterated forcing.

Until the present article, the study of preservation theorems for not adding reals largely centered on the degree to which ($<\omega_1$)-properness can be dispensed with in theorems like [20, Theorem VIII.4.5] (which is the precursor to [8] and Theorems 3.3 and 3.10 above). This question has now been resolved as well in the third author's paper [19]: some hypothesis beyond a *completeness assumption* is necessary.

The present article underscores that the notion of *completeness* is not as robust as one might hope. The results in this paper show that there is an important distinction between 1-semicomplete iterations and ω-complete iterations. In [8], the apparent added flexibility of 2-complete over 1-complete iterations was important to the argument. While this was largely dismissed as a technical detail at the time, it may now warrant further investigation.

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