Acta Math., 210 (2013), 95–171
DOI: 10.1007/s11511-013-0091-0

Invariant Peano curves of expanding Thurston maps

by

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The author was partially supported by an NSF post-doctoral fellowship, the Swiss National Science Foundation, and the Academy of Finland (projects SA-11842 and SA-118634).

1. Introduction

A *Thurston map* is a branched covering of the sphere $f\colon S^2\to S^2$ that is *post-critically finite*. A celebrated theorem of Thurston gives a *topological characterization* of rational maps among Thurston maps (see [DH3]). In this paper we consider such maps that are *expanding* (see §2 for precise definitions). In the case when f is a rational map this means that the Julia set of f is the whole sphere.

The following is the main theorem.

THEOREM 1.1. *Let f be an expanding Thurston map. Then for each sufficiently high iterate $F=f^n$ there is a* Peano curve $\gamma\colon S^1\to S^2$ (*onto*) *such that $F(\gamma(z))=\gamma(z^d)$ (for all $z\in S^1$). Here $d=\deg F$. This means that the following diagram commutes*:

$$\begin{array}{ccc} S^1 & \xrightarrow{z^d} & S^1 \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ S^2 & \xrightarrow[F]{} & S^2. \end{array}$$

Furthermore, we can approximate the Peano curve γ as follows. There is a homotopy $\Gamma\colon S^2\times[0,1]\to S^2$, *with $\Gamma(z,0)=z$, such that*

$$\Gamma(z,1)=\gamma(z) \quad \textit{for all } z\in S^1.$$

Here we view $S^1\subset S^2$ as the equator.

In fact Γ may be chosen to be a *pseudo-isotopy*, meaning that it is an isotopy on $[0,1)$.

The result may be paraphrased as follows. Via γ we can view the sphere S^2 as a parameterized circle S^1. Wrapping this parameterized circle (which is S^2) around itself d times yields the map F.

The existence of such a *semi-conjugacy* γ as above follows for many rational maps F of degree 2 by work of Tan, Rees and Shishikura (see [Ta], [Re2] and [Sh]); the relevant construction of *mating* is reviewed in §1.2. Milnor constructs such a Peano curve γ (i.e., a semi-conjugacy) for one specific example F (see [Mi1]) in this setting. Kameyama gives a sufficient criterion for the existence of γ (in [Ka, Theorem 3.5]).

Note that the result is purely topological, i.e., does not depend on F being (equivalent to) a rational map or not.

We also prove the following converse statement to Theorem 1.1.

THEOREM 1.2. *Let $f\colon S^2\to S^2$ be a Thurston map such that for some iterate $F=f^n$ there exists a Peano curve $\gamma\colon S^1\to S^2$ (onto) satisfying $F(\gamma(z))=\gamma(z^d)$ for all $z\in S^1$. Then f is expanding.*

According to *Sullivan's dictionary* there is a close correspondence between the dynamics of rational maps and of Kleinian groups [Su]. Cannon and Thurston construct (in [CT]) an invariant Peano curve $\gamma\colon S^1\to S^2$ for the fundamental group of a (hyperbolic) 3-manifold M^3 that *fibers over the circle*. Theorem 1.1 may be viewed as the corresponding result in the case of rational maps. Thus it provides another entry in Sullivan's dictionary.

1.1. Group invariant Peano curves

We review the Cannon–Thurston construction from [CT]. The purpose is to put Theorem 1.1 into perspective.

Let Σ be a compact hyperbolic 2-manifold, and $\varphi\colon\Sigma\to\Sigma$ be a *pseudo-Anosov* homeomorphism. Consider the equivalence relation on the product $\Sigma\times[0,1]$ given by

$$(x,0)\sim(\varphi(x),1).$$

Then the 3-manifold $M^3:=\Sigma\times[0,1]/\sim$ is called a *manifold that fibers over the circle*. Thurston has proved that M^3 admits a *hyperbolic* metric, see [Ot].

The fundamental groups $\pi_1(\Sigma)$ and $\pi_1(M^3)$ are *Gromov hyperbolic*, see [Gr] as well as [GH]. Thus they have *boundaries at infinity*, which in this case are $\partial_\infty\pi_1(\Sigma)=S^1$ and $\partial_\infty\pi_1(M^3)=S^2$.

This is seen by noting that $\pi_1(\Sigma)$ and hyperbolic 2-space $\mathbb{H}^2$, as well as $\pi_1(M^3)$ and hyperbolic 3-space $\mathbb{H}^3$, are *quasi-isometric*. The boundary at infinity of $\mathbb{H}^2$ is S^1, the boundary at infinity of $\mathbb{H}^3$ is S^2, the boundary of the disk, resp. the unit ball, in the Poincaré model of hyperbolic space.

The inclusion $\Sigma \to \Sigma \times \{0\} \to M^3$ induces an inclusion of the fundamental groups $\imath\colon \pi_1(\Sigma) \to \pi_1(M^3)$, which is a group homomorphism. In fact $\imath(\pi_1(\Sigma))$ is a normal subgroup of $\pi_1(M^3)$. The map $\imath$ extends to the boundaries at infinity $S^1 = \partial_\infty \pi_1(\Sigma)$ and $S^2 = \partial_\infty \pi_1(M^3)$ to a continuous map $\sigma\colon S^1 \to S^2$.

It is well known (and not very hard to show), that a non-trivial normal subgroup $N \lhd G$ of a Gromov hyperbolic group G has the same boundary at infinity as G. Thus $\partial_\infty \imath(\pi_1(\Sigma)) = \partial_\infty(\pi_1(M^3)) = S^2$. It follows that the map σ is *onto*, i.e., a Peano curve.

Each element $g \in \pi_1(\Sigma)$ acts (by left-multiplication) on $\pi_1(\Sigma)$; this action extends to $S^1 = \partial_\infty \pi_1(\Sigma)$. Similarly each element $g \in \pi_1(M^3)$ acts on $\pi_1(M^3)$ and this action extends to $S^2 = \partial_\infty \pi_1(M^3)$. The map σ is *invariant* with respect to this group action, meaning that for every $g \in \pi_1(\Sigma)$ one has $\imath(g)(\sigma(t)) = \sigma(g(t))$ for all $t \in S^1$. Thus the following diagram commutes:

$$\begin{array}{ccc} S^1 & \xrightarrow{\ g\ } & S^1 \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ S^2 & \xrightarrow[\imath(g)]{} & S^2. \end{array}$$

The invariant Peano curve γ from Theorem 1.1 is the object corresponding to the group invariant Peano curve σ according to Sullivan's dictionary.

The Cannon–Thurston construction has been extended by Minsky in [Min] and McMullen in [Mc2] to (some) cases where Σ is not compact.

In [Th1] Thurston asked whether (in a sense) all hyperbolic 3-manifolds arise as manifolds that fiber over the circle. This has now become known as the *virtual fibering conjecture*. It stipulates that every hyperbolic 3-manifold has a finite cover which fibers over the circle. This would mean that we can understand every hyperbolic 3-manifold in terms of 2-manifolds. See [Ga] for more background on this conjecture, and [Ag] for recent progress.

Theorem 1.1 may be viewed as the solution of the problem corresponding to the virtual fibering conjecture according to Sullivan's dictionary.

1.2. Consequences of Theorem 1.1

To not further increase the size of the present paper, we will develop the implications of the main theorem in a follow-up paper [Me1]. They are outlined here briefly to put the result into perspective.

Using the invariant Peano curve $\gamma\colon S^1 \to S^2$ from Theorem 1.1, an equivalence relation on S^1 is defined by

$$s \sim t \quad \Longleftrightarrow \quad \gamma(s) = \gamma(t) \tag{1.1}$$

for all $s,t\in S^1$. Elementary topology yields that $S^1/\!\sim$ is homeomorphic to S^2 and that $z^d/\!\sim : S^1/\!\sim \to S^1/\!\sim$ is topologically conjugate to the map F.

THEOREM 1.3. *The following diagram commutes*:

$$\begin{array}{ccc} S^1/\!\sim & \xrightarrow{z^d/\sim} & S^1/\!\sim \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ S^2 & \xrightarrow[F]{} & S^2. \end{array}$$

Here the homeomorphism $h\colon S^1/\!\sim \to S^2$ *is given by* $h([s])=\gamma(s)$ *for all* $s\in S^1$.

The equivalence relation (1.1) may be constructed from *finite data*, more precisely from two finite families of finite sets of rational numbers.

The proper setting is as follows. For each $n\in\mathbb{N}$, two equivalence relations, $\overset{n,w}{\sim}$ and $\overset{n,b}{\sim}$, are defined. The equivalence relation $\sim$ defined in (1.1) is the *closure* of the union of all $\overset{n,w}{\sim}$ and $\overset{n,b}{\sim}$. Each $\overset{n,w}{\sim}$ is the *pull-back* of $\overset{n-1,w}{\sim}$ by z^d (similarly $\overset{n,b}{\sim}$ is the pull-back of $\overset{n-1,b}{\sim}$). Thus F can be recovered (up to topological conjugacy) from the equivalence relations $\overset{1,w}{\sim}$ and $\overset{1,b}{\sim}$.

This provides a way to *describe* expanding Thurston maps effectively.

The description above may be viewed as a two-sided version of the viewpoint introduced by Douady–Hubbard and Thurston ([DH1], [DH2], [Th2], [Th3], see also [Re2] and [Ke]), namely the combinatorial description of Julia sets in terms of *external rays*.

Recently (analogously defined) *random laminations* have been used to study the scaling limits of planar maps (see [Le] and [LP]).

The description of F above yields in addition that F arises as a *mating* of two polynomials. Mating of polynomials was introduced by Douady and Hubbard [Do] as a way to geometrically combine two polynomials to form a rational map. We recall the construction briefly.

Consider two monic polynomials p_1 and p_2 of the same degree with connected and locally connected Julia sets. Let K_1 and K_2 be their filled-in Julia sets. For $j=1,2$ let

$$\phi_j\colon \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}}\setminus K_j$$

be the Riemann maps, normalized by $\phi_j(\infty)=\infty$ and

$$\phi_j'(\infty) = \lim_{z\to\infty} \frac{z}{\phi_j(z)} > 0$$

(in fact then $\phi_j'(\infty)=1$). By *Carathéodory's theorem* ϕ_j extends continuously to

$$\sigma_j\colon S^1 = \partial\overline{\mathbb{D}} \longrightarrow \partial K_j.$$

The *topological mating* of K_1 and K_2 is obtained by identifying $\sigma_1(z)\in\partial K_1$ with $\sigma_2(\bar{z})\in\partial K_2$. More precisely, we consider the disjoint union of K_1 and K_2 and let $K_1 \amalg K_2$ be the quotient obtained from the equivalence relation generated by $\sigma_1(z)\sim\sigma_2(\bar{z})$ (for all $z\in S^1=\partial\mathbb{D}$). The map

$$p_1 \amalg p_2 \colon K_1 \amalg K_2 \longrightarrow K_1 \amalg K_2,$$

given by

$$(p_1 \amalg p_2)|_{K_j} = p_j \quad \text{for } j=1,2,$$

is well defined. If a map f is topologically conjugate to $p_1 \amalg p_2$, we say that f is obtained as a (topological) *mating*. If both K_1 and K_2 have empty interior, each of the maps σ_1 and σ_2 descends to a Peano curve $\gamma\colon S^1\to K_1 \amalg K_2$ which provides a semi-conjugacy of $z^d\colon S^1\to S^1$ to $p_1 \amalg p_2$ (here $d=\deg p_1=\deg p_2$).

In particular it is known (see [Ta], [Sh] and [Re2]) that the *mating* of two quadratic polynomials $p_1=z^2+c_1$ and $p_2=z^2+c_2$, where c_1 and c_2 are *Misiurewicz points* (i.e., the critical point 0 is strictly pre-periodic for p_j) not contained in conjugate limbs of the Mandelbrot set, results in a map that is topologically conjugate to a rational map F. The filled-in Julia sets of p_1 and p_2 have empty interior. The Julia set of F is the whole sphere, and hence F is expanding. Thus a Peano curve γ as in Theorem 1.1 exists for such a map F.

Recall that a *periodic critical point* (of a Thurston map f) is a critical point c such that $f^k(c)=c$ for some $k\geqslant 1$.

THEOREM 1.4. ([Me1]) *Let $f\colon S^2\to S^2$ be an expanding Thurston map without periodic critical points. Then every sufficiently high iterate $F=f^n$ is obtained as a* topological mating *of two polynomials.*

If at least one of the filled-in Julia sets K_1 and K_2 has non-empty interior, we can take a further quotient of $K_1 \amalg K_2$ by identifying the points of the closure of each bounded Fatou component. Technically we take the *closure* of the equivalence relation (on the disjoint union of K_1 and K_2) obtained from $\sigma_1(z)\sim\sigma_2(\bar{z})$ (for all $z\in S^1=\partial\mathbb{D}$) as well as $x\sim y$ if x and y are in the closure of the *same* bounded Fatou component of p_1 or p_2.

The maps p_1 and p_2 descend to the quotient map $p_1 \widehat{\amalg} p_2$.

THEOREM 1.5. ([Me1]) *Let $f\colon S^2\to S^2$ be an expanding Thurston map with (at least one) periodic critical point. Then every sufficiently high iterate $F=f^n$ is topologically conjugate to a map $p_1 \widehat{\amalg} p_2$ as above.*

The next theorem investigates the *measure-theoretic* mapping properties of γ.

THEOREM 1.6. ([Me1]) *The Peano curve γ maps* Lebesgue measure *of S^1 to the* measure of maximal entropy *(with respect to F) on S^2.*

The polynomials into which F *unmates*, i.e., the polynomials p_1 and p_2 from Theorems 1.4 and 1.5, can be found by a simple explicit combinatorial algorithm. This is explained in [Me2].

As another application of Theorem 1.1 one obtains *fractal tilings*. Namely divide the circle $S^1=\mathbb{R}/\mathbb{Z}$ into d intervals $[j/d,(j+1)/d]$, $j=0,...,d-1$. It follows from Theorem 1.1 that F maps each set $\gamma([j/d,(j+1)/d])$ to the whole sphere. The tiling lifts to the *orbifold covering*, which is either the Euclidean or the hyperbolic plane.

1.3. Outline

The construction of the invariant Peano curve, i.e., the proof of Theorem 1.1, forms the core of this work.

In §1.5 an example is introduced that serves to illustrate the construction throughout the paper. §2 gives precise definitions of expanding Thurston maps, as well as gathers facts from [BM] relevant here.

We will fix a Jordan curve $\mathcal{C}$ containing the set of all post-critical points (=post(F)). We construct *approximations* $\gamma^n\colon S^1\to S^2$, that will go through $F^{-n}(\mathcal{C})$. The limit

$$\gamma=\lim_{n\to\infty}\gamma^n$$

will be the desired Peano curve.

The construction of γ consists of two parts. In the first part (which is logically the second) we assume that we can deform $\mathcal{C}$ to $\gamma^1=F^{-1}(\mathcal{C})$ by a *pseudo-isotopy relative to* post(F). The approximations γ^n can then be constructed inductively by repeated lifts. This is done in §3. The correct *parametrization* of γ^n is done in §4.

The second part is the construction of the pseudo-isotopy H^0 relative to post(F), which deforms the Jordan curve $\mathcal{C}$ to the first approximation γ^1.

We *color* one component of $S^2\setminus\mathcal{C}$ white, and the other black. Preimages of these Jordan domains by F then form the *white* and *black* 1-*tiles*. At each vertex (of 1-tiles) we will declare which white and black 1-tiles are *connected*. These connections will be described by *complementary non-crossing partitions*.

Connections at all vertices will be defined in such a way that the *white tile graph* forms a *spanning tree*. The "outline" of this spanning tree forms the first approximation γ^1. The main work consists of making sure that γ^1 lies in the right homotopy class (that $\mathcal{C}$ can be deformed to γ^1 by a pseudo-isotopy relative to post(F)).

§5 assembles some standard topological lemmas needed in the following. In §6 the necessary background about connections and complementary non-crossing partitions is developed.

The desired pseudo-isotopy H^0 (equivalently the spanning tree of white 1-tiles) is constructed in §7. It is here that we (possibly) need to take an iterate $F=f^n$ (in order to be in the right homotopy class).

In §8 an alternative *combinatorial* way to construct the approximations γ^n is presented. An *n-tile* is the preimage of a component of $S^2\backslash\mathcal{C}$ by F^n. At each *n-vertex* of such an n-tile we define which n-tiles are connected. Following the "outline" of one connected component as before yields the approximation γ^n. These *connections of n-tiles* are constructed inductively in a purely combinatorial fashion.

Theorem 1.2 (existence of a Peano curve which semi-conjugates z^d to F implies expansion) is proved in §9.

The question arises whether it is necessary to take an iterate $F=f^n$ in Theorem 1.1. While we do not have a definite answer, we give an example in §10 which shows (in the opinion of the author) that the answer is likely yes. More precisely, for the considered example h, there exists no pseudo-isotopy H^0 as required (there is one for the second iterate h^2).

We finish with some open problems in §11.

1.4. Acknowledgments

The author wishes to thank Juan Rivera-Letelier for many fruitful discussions; Stanislav Smirnov, Mario Bonk and Kari Astala for their hospitality. Kevin Pilgrim and Tan Lei pointed out that Theorem 1.1 should have a converse, i.e., that Theorem 1.2 should hold.

1.5. An example

We illustrate the proof using the following map g. It is a *Lattès map* (see [La] and [Mi3]).

Map the square $\left[0,\frac{1}{2}\right]^2\subset\mathbb{C}$ to the upper half-plane by a Riemann map, normalized by mapping the vertices 0, $\frac{1}{2}$, $\frac{1}{2}+\frac{1}{2}i$ and $\frac{1}{2}i$ to 0, 1, ∞ and -1, respectively (here i denotes the imaginary unit). By Schwarz reflection this map can be extended to a meromorphic function $\wp\colon\mathbb{C}\to\widehat{\mathbb{C}}$. This is the *Weierstraß $\wp$-function* (up to a Möbius transformation), it is (doubly) periodic with respect to the lattice $L:=\mathbb{Z}^2$. Thus we may view $\wp$ as a (double) branched covering map of the sphere by the torus $\mathbb{T}^2:=\mathbb{C}/L$.

Color preimages of the upper half-plane by $\wp$ white, and preimages of the lower half-plane by $\wp$ black. The plane is then colored in a *checkerboard* fashion. Consider the map

$$\begin{aligned}\psi\colon\mathbb{C}&\longrightarrow\mathbb{C},\\ z&\longmapsto 2z.\end{aligned}$$

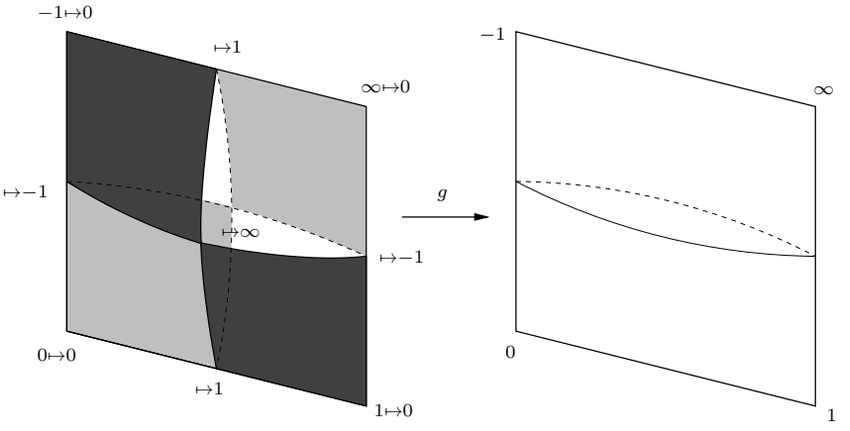

Figure 1. The Lattès map g.

We may view ψ as a self-map of the torus $\mathbb{T}^2$. One checks that there is a (unique and well defined) map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\psi} & \mathbb{C} \\ {\scriptstyle \wp}\downarrow & & \downarrow{\scriptstyle \wp} \\ \widehat{\mathbb{C}} & \xrightarrow[g]{} & \widehat{\mathbb{C}} \end{array}$$

commutes. The map g is *rational*, in fact

$$g = 4\frac{z(1-z^2)}{(z^2+1)^2}.$$

The Julia set of g is the whole sphere.

One may describe g as follows. Push the Euclidean metric of $\mathbb{C}$ to the (Riemann) sphere $\widehat{\mathbb{C}}$ by $\wp$. In this metric the sphere looks like a *pillow* (technically this is an *orbifold*, see for example [Mi2, Appendix E] and [Mc1, Appendix A]). Indeed, by construction, the upper and lower half-planes are then both isometric to the square $\left[0, \frac{1}{2}\right]^2$. Two such squares glued along their boundary form the sphere. We *color* one of these squares (say the upper half-plane) *white*, and the other square (the lower half-plane) *black*. The map g is now given as follows. Divide each of the two squares into four small squares (of side-length $\frac{1}{4}$). Color these eight small squares in a checkerboard fashion white and black. Map one such small white square to the big white square. This extends by reflection to the whole pillow, which yields the map g. There are obviously many different ways to color and map the small squares. The "right" way to do so (in order to obtain g) is indicated in Figure 1.

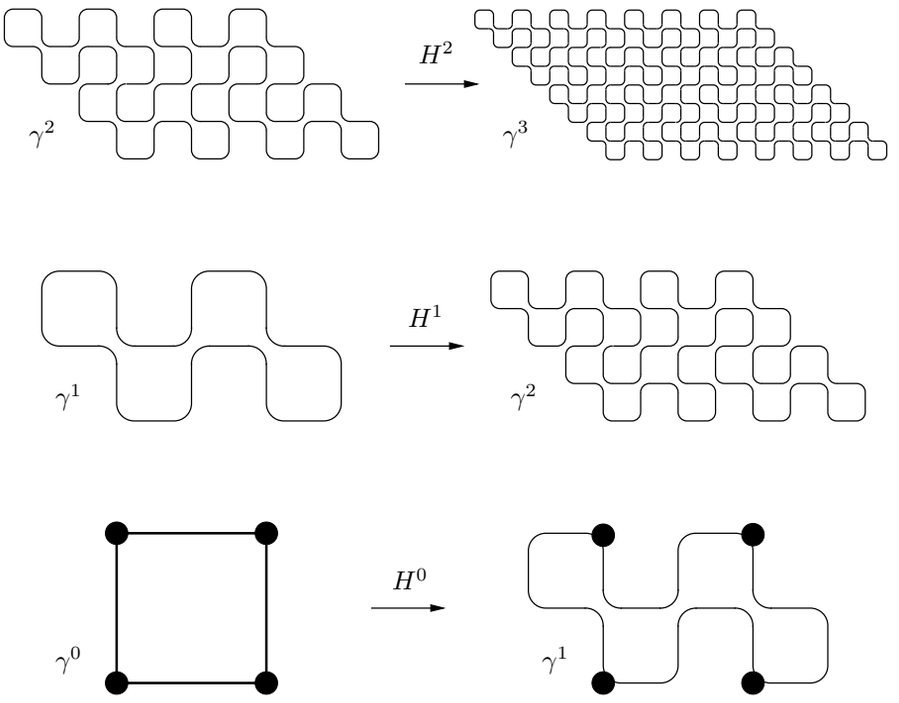

Figure 2. Construction of γ for the map g.

The six vertices of the small squares at which four small squares intersect are the *critical points* of g. They are mapped by g to $\{1, \infty, -1\}$; these points in turn are mapped to 0, which is a fixed point. The set $\{0, 1, \infty, -1\}$=post(g) is the set of all *post-critical points*.

The map $\wp$ is the *orbifold covering map*. The pictures explaining our construction will all be in the *orbifold covering*, i.e., in $\mathbb{C}$. For example, the Peano curve will be constructed by certain approximating curves. These are more easily visualized when lifted to $\mathbb{C}$.

1.6. The construction for the example

The construction is explained using the example g defined in the last section.

The 0-*th approximation* γ^0 of the Peano curve is the extended real line

$$\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \subset \widehat{\mathbb{C}}.$$

Note that $\widehat{\mathbb{R}}$ contains all post-critical points of g. In the "pillow" model, $\widehat{\mathbb{R}}$ is the common boundary of the two squares. The picture in the orbifold covering is shown in Figure 2

in the lower left. The (lifts of the) post-critical points are the dots at the vertices.

The upper and lower half-planes (the two squares from which the "pillow" was constructed) are called the 0-*tiles*. Their preimages by g (the small squares to the left in Figure 1) are called the 1-*tiles*. We *color* them white if they are preimages of the upper half-plane, and black otherwise. There are four white as well as four black 1-tiles. The white 1-tiles intersect at the *critical points*, of which there are six. At each critical point (1-*vertex*) we define a *connection*. This is an assignment of which 1-tiles are connected and which are disconnected at this 1-vertex. Connections are defined in such a way that the resulting *white tile graph* is a spanning tree. This means that it contains all white 1-tiles and no loops. In our example the white 1-tiles are connected at the three critical points labeled by "$\mapsto -1$" and "$\mapsto \infty$" in Figure 1, and disconnected at the others. The corresponding picture in the orbifold covering is shown in the lower right of Figure 2.

Following the boundary of this spanning tree gives the *first approximation* of the Peano curve γ^1 (again indicated in the lower right of Figure 2). To obtain the curve γ^1 on the pillow, one needs to "fold the two squares that are overlapping to the left and right on the back" (where they intersect in a critical point).

We will need the following additional assumption on the spanning tree. We have to be able to deform γ^0 to γ^1 by a *pseudo-isotopy* H^0 that keeps the post-critical points fixed. Recall that a pseudo-isotopy $H^0\colon S^2\times[0,1]\to S^2$ is a homotopy that ceases to be an isotopy only at $t=1$.

The pseudo-isotopy is *lifted* to (pseudo-isotopies) H^n by iterates g^n. The approximations of the Peano curve are constructed inductively. Namely γ^{n+1} is obtained as the deformation of γ^n by H^n. Each curve γ^n goes through $g^{-n}(\text{post})$. The limiting curve γ is the desired Peano curve.

1.7. Notation

The Riemann sphere is denoted by $\widehat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. We denote the 2-sphere by S^2, when it is not assumed to be equipped with a conformal structure. By $\operatorname{int} U$ we denote the *interior* of a set U. The cardinality of a (finite) set S is denoted by $\#S$. The circle S^1 will often be identified with $\mathbb{R}/\mathbb{Z}$ whenever convenient.

For two non-negative expressions A and B we write $A\lesssim B$ if there is a constant $C>0$ such that $A\leqslant CB$. We refer to C as $C(\lesssim)$. Similarly we write $A\asymp B$ if $A/C\leqslant B\leqslant CA$ for a constant $C\geqslant 1$.

- The *n-iterate* of a map f is denoted by f^n; $f^{-n}(A)$ denotes the preimage of a set A by the iterate f^n.
- *Upper indices* indicate the *order* of an object, meaning that U^n is the preimage

of some object U^0 by f^n or F^n.

- By crit=crit(f) and post=post(f) we denote the sets of *critical* and *post-critical points*, respectively (see the next section).
- The *degree* of F is denoted by d, the *number of post-critical points* by k.
- The *local degree* of F at $v\in S^2$ is denoted by $\deg_F(v)$ (see Definition 2.1 (1)).
- $\mathcal{C}$ is a Jordan curve containing all post-critical points.
- *Lower indices* w and b denote whether objects are colored *white* or *black*.
- X^0_w and X^0_b denote the white and black 0-*tiles*, respectively (§2).
- The *sets of all* n*-tiles*, n*-edges* and n*-vertices* are denoted by $\mathbf{X}^n$, $\mathbf{E}^n$ and $\mathbf{V}^n$, respectively (§2).
- The *expansion factor* of a fixed *visual metric* for F is denoted by Λ, see (2.3).
- γ^n is the n*-th approximation* of the invariant Peano curve (§3).
- H^0 is the *pseudo-isotopy* that deforms $\mathcal{C}$ to γ^1. H^n is the *lift* of H^0 by F^n; it is a pseudo-isotopy that deforms γ^n to γ^{n+1} (see Definition 3.2 and Lemma 3.4).
- $\alpha^n_j\subset\mathbb{R}/\mathbb{Z}$ is a point that is mapped by γ^n (and subsequently by γ) to an n-vertex (§4.2).
- $\pi_w\cup\pi_b$ is a *complementary non-crossing partition*. It describes which white and black 1-tiles are connected at some 1-vertex (§6.1).
- A lower index ε indicates a geometric realization of an object, where in a small neighborhood of each 1-vertex we change tiles to "geometrically represent the connection" (Definition 6.8).

2. Expanding Thurston maps as subdivisions

Definition 2.1. A *Thurston map* is an orientation-preserving, post-critically finite, branched covering of the sphere

$$f\colon S^2\to S^2.$$

To elaborate:

(1) f is a *branched cover* of the sphere S^2, meaning that locally we can write f as $z\mapsto z^q$ after orientation-preserving homeomorphic changes of coordinates in domain and range.

More precisely, for each point $v\in S^2$ there exist $q\in\mathbb{N}$, (open) neighborhoods V and W of v and $w=f(v)$, respectively, and orientation-preserving homeomorphisms $\varphi\colon V\to\mathbb{D}$ and $\psi\colon W\to\mathbb{D}$, with $\varphi(v)=0$ and $\psi(w)=0$, satisfying

$$\psi\circ f\circ\varphi^{-1}(z)=z^q$$

for all $z\in\mathbb{D}$. The integer $q=\deg_f(v)\geqslant 1$ is called the *local degree* of the map at v. A point c at which the local degree $\deg_f(c)\geqslant 2$ is called a *critical point*. The set of all critical points is denoted by $\mathrm{crit}=\mathrm{crit}(f)$. There are only finitely many critical points, since S^2 is compact. Note that no assumptions about the smoothness of f are made.

(2) The map f is *post-critically finite*, meaning that the set of *post-critical points*

$$\mathrm{post}=\mathrm{post}(f):=\bigcup_{n\geqslant 1}\{f^n(c):c\in\mathrm{crit}(f)\}$$

is finite. As usual f^n denotes the nth iterate. We are only interested in the case when $\#\,\mathrm{post}(f)\geqslant 3$.

(3) Consider a Jordan curve $\mathcal{C}\supset\mathrm{post}$. The Thurston map f is called *expanding* if

$$\mathrm{mesh}\, f^{-n}(\mathcal{C})\to 0 \quad \text{as } n\to\infty.$$

Here $\mathrm{mesh}\, f^{-n}(\mathcal{C})$ is the maximal diameter of a component of $S^2\setminus f^{-n}(\mathcal{C})$. It was shown in [BM, Lemma 6.1] that this definition is independent of the chosen curve $\mathcal{C}$. This notion of "expansion" agrees with the one by Haïssinsky–Pilgrim in [HP] (see [BM, Proposition 6.2]).

Fix a Jordan curve $\mathcal{C}\supset\mathrm{post}$. Here and in the following, we always assume that such a curve $\mathcal{C}$ is *oriented*. Let U_w and U_b be the two components of $S^2\setminus\mathcal{C}$, where $\mathcal{C}$ is positively oriented as boundary of U_w. The closures of U_w and U_b are denoted by X_w^0 and X_b^0, respectively. We *color* X_w^0 *white*, and X_b^0 *black*. We refer to X_w^0 (resp. X_b^0) as the white (resp. black) 0-*tile*.

The closure of one component of $f^{-n}(U_w)$ or of $f^{-n}(U_b)$ is called an n-*tile*. It was shown in [BM, Proposition 5.17] that for such an n-tile X the map

$$f^n\colon X\to X_{w,b}^0 \quad \text{is a homeomorphism.} \tag{2.1}$$

This means in particular that each n-tile is a closed Jordan domain. The set of all n-tiles is denoted by $\mathbf{X}^n$. The definition of "expansion" implies that n-tiles become arbitrarily small, this is the (only) reason we require expansion.

In [BM, Theorem 14.2] (see also [CFP3]) it was shown that if f is expanding, then for every sufficiently high iterate $F=f^n$ we can choose $\mathcal{C}$ to be *invariant* with respect to F. This means that $F(\mathcal{C})\subset\mathcal{C}$ ($\Leftrightarrow\mathcal{C}\subset F^{-1}(\mathcal{C})$). It implies that each n-tile is contained in exactly one $(n-1)$-tile. Furthermore, F may be represented as a *subdivision* (see [BM, Chapter 12] as well as the ongoing work of Cannon, Floyd and Parry [CFP1], [CFP2]). We will require $\mathcal{C}$ to be F-invariant only in §7. This is clearly a convenience in the proof, the author however feels that this assumption is not strictly necessary.

The set of all n-vertices is defined as

$$\mathbf{V}^n = f^{-n}(\text{post}). \tag{2.2}$$

Note that post$=\mathbf{V}^0\subset\mathbf{V}^1\subset\dots$. Each point $v\in\mathbf{V}^n$ is called an *n-vertex*.

The post-critical points (or 0-vertices) divide the curve $\mathcal{C}$ into $k=\#\,\text{post}(f)$ closed Jordan arcs called 0-*edges*. The closure of one component of $f^{-n}(\mathcal{C})\setminus\mathbf{V}^n$ is called an n-*edge*. For each n-edge E^n there is a 0-edge E^0 such that $f^n(E^n)=E^0$. Furthermore the map $f^n\colon E^n\to E^0$ is a homeomorphism ([BM, Proposition 5.17]). The set of all n-edges is denoted by $\mathbf{E}^n$, so that $f^{-n}(\mathcal{C})=\bigcup\mathbf{E}^n$. There are $\#\mathbf{E}^n=k\deg(f)^n$ n-edges.

Each n-edge will have an *orientation*, meaning that it has an *initial* and a *terminal* point. A 0-edge is *positively oriented* if its orientation agrees with the one of the Jordan curve $\mathcal{C}$. Similarly, an n-edge E^n is called *positively oriented* if f^n maps the initial (resp. terminal) point of E^n to the initial (resp. terminal) point of (the 0-edge) $f^n(E^n)$.

Each n-tile contains exactly $k=\#$ post n-edges and k n-vertices in its boundary.

The n-tiles, n-edges and n-vertices form a *cell complex* when viewed as 2-, 1- and 0-cells, respectively (see [BM, Chapter 5]).

The n-edges and n-vertices form a *graph* in the natural way. Note that this graph may have multiple edges, but no loops.

We *color* the n-tiles *white* if they are preimages of X_w^0, and *black* if they are preimages of X_b^0. Each n-edge is shared by two n-tiles of different color. Thus n-tiles are colored in a "checkerboard fashion". An oriented n-edge is positively oriented if and only if it is positively oriented as boundary of the white n-tile it is contained in (and negatively oriented as boundary of the black n-tile it is contained in). The set of white n-tiles is denoted by $\mathbf{X}_w^n$, and the set of black n-tiles by $\mathbf{X}_b^n$.

LEMMA 2.2. *The n-tiles of each color are connected, meaning that*

$$\bigcup\mathbf{X}_w^n \text{ and } \bigcup\mathbf{X}_b^n \text{ are connected sets.}$$

Proof. Note that $\bigcup\mathbf{X}_w^n$ (and $\bigcup\mathbf{X}_b^n$) is connected if and only if $\bigcup\mathbf{E}^n$ is connected.

If $\bigcup\mathbf{E}^n$ is not connected, one component of $S^2\setminus\bigcup\mathbf{E}^n$ is not simply connected. This contradicts the fact that each such component is the interior of an n-tile, and thus simply connected. □

In [BM, Chapter 8] *visual metrics* for an expanding Thurston map f were considered. If n-tiles have been defined (in terms of a Jordan curve $\mathcal{C}\supset\text{post}$), we define $m=m_{f,\mathcal{C}}$ by

$$m(x,y) := \max\{n\in\mathbb{N}\colon \text{there exist non-disjoint } n\text{-tiles } X\ni x \text{ and } Y\ni y\}$$

for all $x,y\in S^2$, $x\neq y$. We set $m(x,x)=\infty$. A metric ϱ on S^2 is called a *visual metric* for f if there is a constant $\lambda>1$ (called the *expansion factor* of ϱ) such that

$$\varrho(x,y)\asymp\lambda^{-m(x,y)} \tag{2.3}$$

for all $x,y\in S^2$ and a constant $C=C(\asymp)$ independent of x and y. Here it is understood that $\lambda^{-\infty}=0$.

Visual metrics always exist, see [BM, Theorem 15.1], as well as [HP]. In fact ϱ can be chosen such that f is an *expanding local similarity* with respect to ϱ. More precisely, for each $x\in S^2$ there exists a neighborhood $U_x\ni x$ such that

$$\frac{\varrho(f(x),f(y))}{\varrho(x,y)}=\lambda \tag{2.4}$$

for all $y\in U_x\setminus\{x\}$. We do however not need this stronger form.

We fix a curve $\mathcal{C}\supset\operatorname{post}(f)$ as well as an iterate $F=f^n$ for now, assuming that they have certain properties (more precisely, that there is a pseudo-isotopy H^0 as in the next section). In §7 they will be chosen properly. Note that the post-critical set of F equals the post-critical set of f, which is thus just denoted by "post". Throughout the construction, we set

$$d:=\deg F=(\deg f)^n \quad\text{and}\quad k:=\#\operatorname{post}.$$

From now on m-tiles, m-edges and m-vertices are understood to be with respect to $(F,\mathcal{C})$, meaning that they are mn-tiles, mn-edges and mn-vertices, respectively, with respect to $(f,\mathcal{C})$.

Clearly expansion of f implies expansion of F. A visual metric for f with expansion factor λ is a visual metric for F with expansion factor $\Lambda=\lambda^n$. Expression (2.4) continues to hold, where we have to replace λ by $\Lambda:=\lambda^n>1$.

LEMMA 2.3. *Let ϱ be a visual metric for F with expansion factor Λ. Then there are $\varepsilon_0>0$ and a constant $K\geqslant 1$ such that the following holds: For any $\varepsilon\in(0,\varepsilon_0)$ let $\mathcal{N}(\mathbf{V}^1,\varepsilon)$ be the ε-neighborhood of $\mathbf{V}^1$ (defined in terms of ϱ). Then there is a neighborhood V^1_ε of $\mathbf{V}^1$ such that*

$$\mathcal{N}\Big(\mathbf{V}^1,\frac{\varepsilon}{K}\Big)\subset V^1_\varepsilon\subset\mathcal{N}(\mathbf{V}^1,\varepsilon)$$

and, for all $n\in\mathbb{N}$, the set $V=V^{n+1}_{\Lambda^{-n}\varepsilon}:=F^{-n}(V^1_\varepsilon)$ satisfies

$$\mathcal{N}\Big(\mathbf{V}^{n+1},\Lambda^{-n}\frac{\varepsilon}{K}\Big)\subset V\subset\mathcal{N}(\mathbf{V}^{n+1},\Lambda^{-n}\varepsilon).$$

The proof of this lemma follows immediately from [BM, Lemmas 8.9 and 8.10].

3. The approximations γ^n

We begin the proof of Theorem 1.1. We assume (until the end of §7) that F ($=f^n$, the index "n" however will be "recycled") is an expanding Thurston map and that $\mathcal{C}\supset$post is a fixed Jordan curve. The n-tiles and n-edges are defined in terms of $(F,\mathcal{C})$; see the previous section. Furthermore we fix a visual metric ϱ for F with expansion factor $\Lambda>1$; see (2.3). Metrical properties and objects, such as the diameter and neighborhoods, will always be defined in terms of this metric.

The desired invariant Peano curve γ will be constructed as the limit of *approximations* γ^n. Here γ^0 is the Jordan curve $\mathcal{C}\supset$post. The first approximation γ^1 will be constructed in §7, more precisely a *pseudo-isotopy* H^0 (relative to post) that deforms γ^0 to γ^1 will be constructed.

In this section the approximations γ^n of the invariant Peano curve will be constructed by repeated *lifts* of H^0. These curves are however not yet parameterized, they are *Eulerian circuits*.

3.1. Pseudo-isotopies

Definition 3.1. (Pseudo-isotopies) A homotopy $H\colon S^2\times[0,1]\to S^2$ is called a *pseudo-isotopy* if it is an isotopy on $S^2\times[0,1)$. We always require that $H(x,0)=x$ on S^2. If $H(\cdot\,,t)$ is constant on a set $A\subset S^2$ it is a *pseudo-isotopy relative to* A (from now on we will use the abbreviation "rel." for "relative to"). Alternatively we then say that H is *supported* on $S^2\setminus A$. We interchangeably write $H_t(x)=H(x,t)$ to unclutter notation.

Remark. Given a pseudo-isotopy H_t as above, it follows that H_1 is *surjective* (in fact $S^2\setminus\{\text{point}\}$ has different homotopy type than S^2) and *closed* (since we are dealing with compact Hausdorff spaces). A pseudo-isotopy on a general space S is required to end in a surjective, closed map.

Our starting point is a pseudo-isotopy $H^0=H^0(x,t)$ as follows. This is the central object of the whole construction. In this and the following sections we show that such an H^0 is sufficient to construct the invariant Peano curve as desired. The construction of H^0 itself will be done in §7. In Lemma 7.2 an equivalent condition for the existence of H^0 will be given.

Definition 3.2. (Pseudo-isotopy H^0) We consider a pseudo-isotopy H^0 with the following properties:

(H^0 1) H^0 is a pseudo-isotopy rel. $\mathbf{V}^0=$post (the set of all post-critical points).

(H^0 2) The set of all 0-edges $\bigcup \mathbf{E}^0 = \mathcal{C}$ is deformed by H^0 to $\bigcup \mathbf{E}^1$, that is

$$H_1^0\Big(\bigcup \mathbf{E}^0\Big) = \bigcup \mathbf{E}^1.$$

To simplify the discussion we require that H^0 deforms the 0-edges to 1-edges "as nicely as possible" (see Lemma 3.3 below). The construction would still work however, without imposing the following two properties:

(H^0 3) Let $\varepsilon_0 > 0$ be the constant from Lemma 2.3, $0 < \varepsilon < \min\{\varepsilon_0, \frac{1}{2}\}$ and V_ε^1 be a neighborhood of $\mathbf{V}^1$ as in Lemma 2.3; we require that

$$H^0 \colon S^2 \times [1-\varepsilon, 1] \longrightarrow S^2 \quad \text{is supported on } V_\varepsilon^1.$$

So H^0 "freezes" on $S^2 \setminus V_\varepsilon^1$.

(H^0 4) Consider a 1-vertex v. Only finitely many points of $\mathcal{C} = \bigcup \mathbf{E}^0$ are deformed by H^0 to v. In other words, we require that

$$\Big\{ x \in \bigcup \mathbf{E}^0 : H_1^0(x) = v \Big\} \quad \text{is a finite set.}$$

One final assumption will be made on H^0. However the precise meaning will only be explained in §3.4.

(H^0 5) View $\gamma^0 = \mathcal{C}$ as a circuit of 0-edges. Let γ^1 be the Eulerian circuit obtained from H^0, see Definition 3.8 (iv). Then

$$F \colon \gamma^1 \longrightarrow \gamma^0,$$

is a d-fold cover; see Definition 3.10.

Consider $\{x_j\}_j := (H_1^0)^{-1}(\mathbf{V}^1) \cap \mathcal{C}$, the set of points on $\mathcal{C} = \bigcup \mathbf{E}^0$ that are mapped by H_1^0 to some 1-vertex (each x_j possibly to a different one). Note that $\{x_j\}_j$ is finite by (H^0 4) and $\{x_j\}_j \supset \text{post} = \mathbf{V}^0$ by (H^0 1). Thus the points $\{x_j\}_j$ divide $\mathcal{C}$ (and each 0-edge) into closed arcs A_j. Recall that $d = \deg F$ and $k = \#\,\text{post}$.

LEMMA 3.3. *There are kd arcs A_j as above. Furthermore*

$$E_j^1 := H_1^0(A_j) \text{ is a 1-edge} \quad \text{and} \quad H_1^0 \colon A_j \longrightarrow E_j^1 \text{ is a homeomorphism,}$$

for each j. On the other hand

$$\text{each 1-edge } E^1 \text{ is the image of one such } A_j \text{ by } H_1^0.$$

Proof. Consider one arc A_j as in the statement with endpoints x_j and x_{j+1}. Note that $\bigcup \mathbf{E}^1 \setminus \mathbf{V}^1$ is disconnected, each component is the interior of a 1-edge. Thus

$$H_1^0(\operatorname{int} A_j) \subset \operatorname{int} E_j^1$$

for some 1-edge E_j^1. Assume that $H_1^0 \colon A_j \to E_j^1$ is not a homeomorphism.

Assume first that $H_1^0(A_j) \neq E_j^1$. Then $H_1^0(x_j) = H_1^0(x_{j+1})$ and there are distinct points $x, y \in \operatorname{int} A_j$ mapped to the same point z by H_1^0. But $z \in S^2 \setminus V_\varepsilon^1$ for sufficiently small ε. Then

$$H_{1-\varepsilon}^0(x) = H_1^0(x) = H_1^0(y) = H_{1-\varepsilon}^0(y),$$

which is a contradiction ($H_{1-\varepsilon}^0$ is a homeomorphism). Thus $H_1^0(A_j) = E_j^1$. Exactly the same argument shows that $H_1^0 \colon A_j \to E_j^1$ is bijective, and hence a homeomorphism.

Using the previous argument again shows that distinct arcs A_i and A_j map to distinct 1-edges E_i^1 and E_j^1, respectively.

Finally, since $H_1^0(\bigcup \mathbf{E}^0) = \bigcup \mathbf{E}^1$ (by $(H_1^0\, 2)$), each 1-edge E^1 is the image of one such arc A_j by H_1^0.

Thus, there is exactly one A_j for each 1-edge, and so there are kd such arcs. □

3.2. Lifts of pseudo-isotopies

LEMMA 3.4. (Lift of pseudo-isotopy) *Let $H \colon S^2 \times [0,1] \to S^2$ be a pseudo-isotopy rel.* post$=\mathbf{V}^0$. *Then H can be* lifted uniquely *by F to a pseudo-isotopy $\widetilde{H}$ rel. $\mathbf{V}^1$. This means that $F(\widetilde{H}(x,t)) = H(F(x),t)$ for all $x \in S^2$ and all $t \in [0,1]$, i.e., the following diagram commutes*:

$$\begin{array}{ccc} S^2 & \xrightarrow{\widetilde{H}} & S^2 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ S^2 & \xrightarrow[H]{} & S^2. \end{array}$$

Furthermore, the following are true:

(1) *If H is a pseudo-isotopy rel. a set $S \subset S^2$, then the lift $\widetilde{H}$ is a pseudo-isotopy rel. $F^{-1}(S)$.*

(2) *Let H^n be the lift of H by an iterate F^n. Then*

$$\operatorname{diam} H^n := \max_{x \in S^2} \operatorname{diam}\{H^n(x,t) : t \in [0,1]\} \lesssim \Lambda^{-n}.$$

Here the diameter is measured with respect to the fixed visual metric with expansion factor $\Lambda > 1$. The constant $C(\lesssim)$ is independent of n.

The proof follows from the standard lifting of paths, see [BM, Proposition 11.1]. For property (2) see [BM, Lemma 11.3].

We now lift the pseudo-isotopy from the last subsection. Lifts retain the properties of H^0.

LEMMA 3.5. (Properties of H^n) *Let H^0 be a pseudo-isotopy as in the last subsection. Let H^n be the lift of H^0 by F^n (equivalently the lift of H^{n-1} by F). The lifts satisfy the following properties*:

(H^n 1) *H^n is a pseudo-isotopy rel. $\mathbf{V}^n$ (the set of all n-vertices).*

(H^n 2) *The set of all n-edges $\bigcup \mathbf{E}^n$ is deformed by H^n to $\bigcup \mathbf{E}^{n+1}$, that is*

$$H_1^n\left(\bigcup \mathbf{E}^n\right) = \bigcup \mathbf{E}^{n+1}.$$

(H^n 3) *Let V_ε^1 be the neighborhood of $\mathbf{V}^1$ as in (H^0 3), see also Lemma 2.3. The set $V = V_{\Lambda^{-n}\varepsilon}^{n+1} := F^{-n}(V_\varepsilon^1)$, which is a neighborhood of $\mathbf{V}^{n+1}$, is such that*

$$H^n \colon S^2 \times [1-\varepsilon, 1] \longrightarrow S^2 \quad \text{is supported on } V.$$

So H^n "freezes" on $S^2 \setminus V$.

(H^n 4) *Consider an $(n+1)$-vertex v. Only finitely many points of $\bigcup \mathbf{E}^n$ are deformed by H^n to v. In other words,*

$$\left\{x \in \bigcup \mathbf{E}^n : H_1^n(x) = v\right\} \quad \text{is a finite set.}$$

We list the final property here. Again it will be explained and proved only in §3.4.

(H^n 5) *Let γ^n and γ^{n+1} be the Eulerian circuits from Definition 3.8 (iv). Then*

$$F \colon \gamma^{n+1} \longrightarrow \gamma^n$$

is a d-fold cover in the sense of Definition 3.10.

Proof. (H^n 1) is clear from Lemma 3.4 (1).

(H^n 3) follows directly from Lemma 2.3 and Lemma 3.4 (1).

(H^n 2) Since H^n is the lift of H^0 by F^n, we have

$$F^n\left(H_1^n\left(\bigcup \mathbf{E}^n\right)\right) = H_1^0\left(F^n\left(\bigcup \mathbf{E}^n\right)\right) = H_1^0\left(\bigcup \mathbf{E}^0\right) = \bigcup \mathbf{E}^1.$$

Thus,

$$H_1^n\left(\bigcup \mathbf{E}^n\right) \subset \bigcup \mathbf{E}^{n+1}.$$

To prove equality in the last expression consider $\operatorname{int} E^1$, the interior of a 1-edge. Let $U^0 = \operatorname{int} A^0 = (H_1^0)^{-1}(\operatorname{int} E^1) \cap \bigcup \mathbf{E}^0$ be the set in $\bigcup \mathbf{E}^0$ that is deformed by H_1^0 to $\operatorname{int} E^1$. This is an arc that does not contain a post-critical point (see Lemma 3.3).

Consider $U_1^n, ..., U_{d^n}^n \subset \bigcup \mathbf{E}^n$, the preimages of U^0 by F^n; they are disjoint arcs. Each U_j^n is deformed by H_1^n to (the interior of) an $(n+1)$-edge (since $F^n(H_1^n(U_j^n)) = H_1^0(F^n(U_j^n)) = H_1^0(U^0) = \operatorname{int} E^1$).

We remind the reader of the following elementary fact about lifts. Let

$$\sigma\colon [0,1] \longrightarrow S^2 \setminus \operatorname{post}(F)$$

be a path, and let $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ be two lifts by F^n with distinct initial points. Then the endpoints of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are distinct. Indeed otherwise the lift of the reversed path $\sigma(1-t)$ would fail to be unique.

Therefore the U_j^n are deformed by H^n to (the interior of) d^n *distinct* $(n+1)$-edges. It follows that $\bigcup \mathbf{E}^n$ is deformed by H^n to kd^{n+1} $(n+1)$-edges, i.e., all of them.

$(H^n\, 4)$ Assume distinct points $\{x_j^n\}_{j\in\mathbb{N}} \subset \bigcup \mathbf{E}^n$ are deformed to some $(n+1)$-vertex v^{n+1} by H_1^n. Then the (infinitely many different) points $x_j^0 := F^n(x_j^n) \in \bigcup \mathbf{E}^0$ are deformed by H_1^0 to the 1-vertex $v^1 := F^n(v^{n+1})$, contradicting property $(H^0\, 4)$. □

From now on we assume that the pseudo-isotopies H^n are given as above.

Consider $\{x_j\}_j := (H_1^n)^{-1}(\mathbf{V}^{n+1}) \cap \bigcup \mathbf{E}^n$, the set of points on $\bigcup \mathbf{E}^n$ that are mapped by H_1^n to some $(n+1)$-vertex (each x_j possibly to a different one). Note that $\{x_j\}_j$ is finite by $(H^n\, 4)$ and $\{x_j\}_j \supset \mathbf{V}^n$ by $(H^n\, 1)$. Thus the points $\{x_j\}_j$ divide $\bigcup \mathbf{E}^n$ (and each n-edge) into closed arcs A_j.

LEMMA 3.6. *There are kd^{n+1} such arcs A_j as above. Furthermore*

$$E_j' := H_1^n(A_j) \text{ is an } (n+1)\text{-edge} \quad \text{and} \quad H_1^n\colon A_j \longrightarrow E_j' \text{ is a homeomorphism},$$

for each j. On the other hand

$$\text{each } (n+1)\text{-edge } E' \text{ is the image of one such } A_j \text{ by } H_1^n.$$

Proof. This follows exactly as in Lemma 3.3. □

3.3. Eulerian circuits γ^n

We construct γ^n, the nth approximation of the invariant Peano curve, from the pseudo-isotopies H^n. The curves γ^n however do not yet have the "right" parametrization. Thus γ^n will for now be an *Eulerian circuit* in $\bigcup \mathbf{E}^n$. However the *parametrization* of this Eulerian circuit will later still be denoted by $\gamma^n(t)$.

Definition 3.7. An *Eulerian circuit* is a closed edge path that traverses each edge exactly once.

Consider now the graph of n-edges $\bigcup \mathbf{E}^n$, containing kd^n n-edges. In this graph an Eulerian circuit is a finite sequence of oriented n-edges

$$\gamma^n = E_0, ..., E_{kd^n-1},$$

such that the following holds (indices are taken mod kd^n). Each n-edge appears exactly once, and the terminal point of E_j is the initial point of E_{j+1}. In particular, the terminal point of E_{kd^n-1} is the initial point of E_0. If v is the terminal point of E_j (the initial point of E_{j+1}), we say that E_{j+1} *succeeds* E_j in γ^n at v.

Cyclical permutations of indices are not considered to change γ^n, but orientation reversing does.

The approximations γ^n of the invariant Peano curve are defined as follows.

Definition 3.8. (Eulerian circuits γ^n) Recall that the Jordan curve $\mathcal{C}=\bigcup \mathbf{E}^0$ is positively oriented as boundary of the white 0-tile X_w^0. Let

$$\gamma^0 = S^1 \longrightarrow \mathcal{C}$$

be an orientation-preserving homeomorphism. We define inductively

$$\begin{aligned} \gamma^{n+1}\colon S^1 &\longrightarrow \bigcup \mathbf{E}^{n+1}, \\ t &\longmapsto H_1^n(\gamma^n(t)), \end{aligned}$$

for all $n \geqslant 0$. Let us note the following properties:

(i) The map is surjective by $(H^n\, 2)$.

(ii) The set $\mathbf{W}^n := (\gamma^n)^{-1}(\mathbf{V}^n) \subset S^1$ is finite by $(H^n\, 4)$.

(iii) For each n-edge E there is exactly one closed arc $[w_j, w_{j+1}] \subset \mathbb{R}/\mathbb{Z} = S^1$, formed by consecutive points $w_j, w_{j+1} \in \mathbf{W}^n$, such that

$$\gamma^n\colon [w_j, w_{j+1}] \longrightarrow E \quad \text{is a homeomorphism.}$$

This follows directly from Lemma 3.6.

(iv) The map γ^n induces an Eulerian circuit (still denoted by γ^n) on $\bigcup \mathbf{E}^n$ in the obvious way, namely the n-edges are given the orientation and ordering induced by γ^n.

We record how the Eulerian circuit γ^n is related to the Eulerian circuit γ^{n+1}. Consider an n-edge E, which is subdivided into arcs $A_0, ..., A_m$ as in Lemma 3.6. An orientation of E induces an orientation of the arcs A_j. As before we say that A_j succeeds A_i in E if the terminal point of A_i is the initial point of A_j.

LEMMA 3.9. *Let D' and E' be two $(n+1)$-edges. Let $A', B' \subset \bigcup \mathbf{E}^n$ be the two arcs that are mapped (homeomorphically) to D' and E', respectively, by H_1^n. Then E' succeeds D' in γ^{n+1} if and only if either*

(a) *A' and B' are contained in the same n-edge E, and B' succeeds A' in E (oriented by γ^n); or*

(b) *A' and B' are contained in different n-edges $E(A')$ and $E(B')$, the terminal point of A' is the terminal point of $E(A')$, the initial point of B' is the initial point of $E(B')$, and $E(B')$ succeeds $E(A')$ (in γ^n).*

Proof. This is again obvious from the construction. □

3.4. γ^{n+1} is a d-fold cover of γ^n

We are now ready to give the definition of properties $(H^0\,5)$ and $(H^n\,5)$.

Definition 3.10. (Cover of Eulerian circuits) Let γ^{n+1} and γ^n be the Eulerian circuits constructed in Definition 3.8 (iv). We call

$$F\colon \gamma^{n+1} \longrightarrow \gamma^n$$

a *d-fold cover* if F maps succeeding $(n+1)$-edges (in γ^{n+1}) to succeeding n-edges (in γ^n). An equivalent definition is as follows. Let

$$\gamma^n = E_0, \ldots, E_{d^n-1} \quad \text{and} \quad \gamma^{n+1} = E'_0, \ldots, E'_{d^{n+1}-1}$$

be two Eulerian circuits. Here each E_j is an (oriented) n-edge and each E'_j an (oriented) $(n+1)$-edge. Let m be the index such that $F(E'_0)=E_m$. Then γ^{n+1} is a d-fold cover of γ^n by F if

$$F(E'_j) = E_{m+j}$$

for all $j=0, \ldots, d^{n+1}-1$.

Convention. Indices of n-edges (and n-vertices) are taken mod kd^n here and in the following.

Property $(H^0\,5)$ is equivalent to the following (seemingly weaker) condition. Recall that each 0-edge $E_j \subset \mathcal{C}$ is *positively oriented* if its orientation agrees with the one induced by $\mathcal{C}$. Similarly each n-edge E^n is positively oriented if $F^n\colon E^n \to E_j$ preserves orientation. Recall furthermore that n-tiles are colored white or black if they are preimages by F^n of the 0-tile X_w^0 or X_b^0, respectively. Each n-edge E^n is contained in the boundary of exactly one white and one black n-tile. Then E^n is *positively oriented* if it is positively oriented as boundary arc of the white n-tile in $X^n \supset E^n$.

LEMMA 3.11. *Let γ^1 be an Eulerian circuit in $\bigcup \mathbf{E}^1$. Then the following conditions are equivalent:*

(H^0 5) *$F\colon \gamma^1 \to \gamma^0$ is a d-fold cover;*

(H^0 5′) *Each 1-edge in γ^1 is positively oriented.*

Proof. Let $p_0, \dots, p_{k-1} \subset \mathcal{C}$ be the post-critical points, labeled positively on $\mathcal{C}$. Consider an oriented 1-edge E^1 with initial point $v \in \mathbf{V}^1$ and terminal point $v' \in \mathbf{V}^1$. It is positively oriented if and only if $F(v')$ succeeds $F(v)$, i.e., if $F(v)=p_j$ and $F(v')=p_{j+1}$ for some j (indices are taken mod k).

Let γ^1 go through 1-vertices $v_0, \dots, v_{kd^n-1}$ in this order. Then $F\colon \gamma^1 \to \gamma^0$ is a d-fold cover if and only if $F(v_{j+1})$ succeeds $F(v_j)$ (for all j; indices are taken mod kd^n), which holds if and only if each edge in γ^1 is positively oriented. □

Remark. It is not very hard to show that if γ^1 is obtained as in Definition 3.8 (without assuming (H^0 5)), then either all 1-edges are positively oriented, or all 1-edges are negatively oriented in γ^1 (see [Me2, Lemma 6.7]). In the latter case our construction would result in a semi-conjugacy of F to z^{-d}. Indeed a Peano curve $\gamma\colon S^1 \to S^2$ that semi-conjugates $F=f^n$ to z^{-d} exists by a slight variation of the construction presented here. Namely in §7 the role of the white and black 1-tiles has to be reversed.

We now show how property (H^0 5) implies (H^n 5), i.e., finish the proof of Lemma 3.5.

LEMMA 3.12. *Let H^0 be a pseudo-isotopy as in Definition 3.1 and H^n be the lifts of H^0 by F^n. The Eulerian circuits γ^n are the ones from Definition 3.8. Then*

(H^n 5) *$F\colon \gamma^{n+1} \to \gamma^n$ is a d-fold cover.*

Proof. The reader is advised to consult Figure 3 for reference. Roughly speaking, by deforming $\bigcup \mathbf{E}^0$ via H^0 and $\bigcup \mathbf{E}^1$ via H^1, one can push the d-fold cover $F\colon \gamma^1 \to \gamma^0$ to a d-fold cover $F\colon \gamma^2 \to \gamma^1$. We give however a more pedestrian (combinatorial) proof.

The proof is by induction. Thus assume that $F\colon \gamma^n \to \gamma^{n-1}$ is a d-fold cover.

Assume that the $(n+1)$-edge E' succeeds the $(n+1)$-edge D' in γ^{n+1}. We need to show that the n-edge $E:=F(E')$ succeeds the n-edge $D:=F(D')$ in γ^n.

Let $A', B' \subset \bigcup \mathbf{E}^n$ be the two arcs that are mapped by H_1^n to D' and E', respectively, see Lemma 3.6. Let $A:=F(A'), B:=F(B') \subset \bigcup \mathbf{E}^{n-1}$ be their images. Since H^n is the lift of H^{n-1} by F (the diagram commutes), we have

$$H_1^{n-1}(A) = D \quad \text{and} \quad H_1^{n-1}(B) = E.$$

There are two cases to consider, by Lemma 3.9.

Case 1. A' and B' are contained in the same n-edge E^n, and B' succeeds A' (given the orientation of E^n by γ^n).

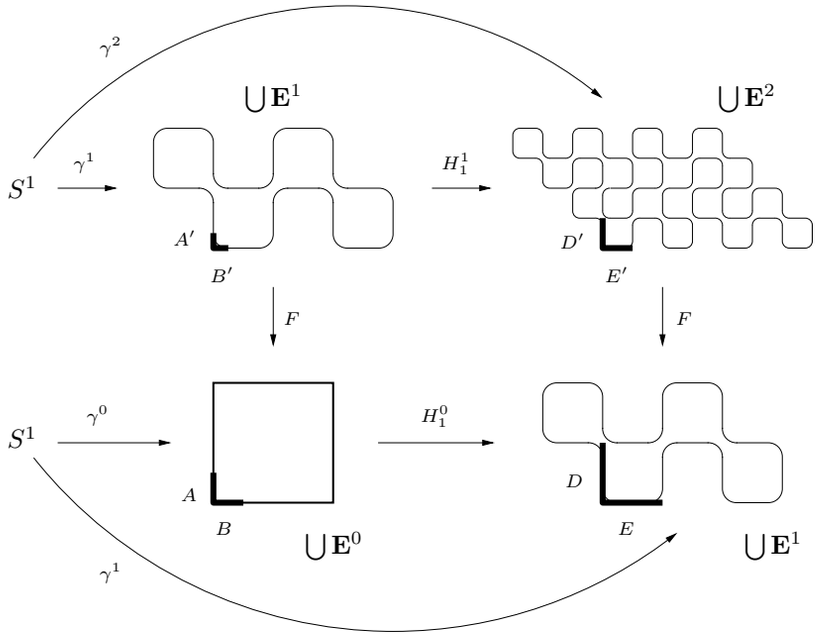

Figure 3. Commutative diagram for Lemma 3.12.

Note that since $F\colon \gamma^n \to \gamma^{n-1}$ is a d-fold cover, F maps n-edges oriented by γ^n to $(n-1)$-edges oriented by γ^{n-1}.

Therefore A and B are contained in the same $(n-1)$-edge $E^{n-1}=F(E^n)$, and B succeeds A (given the orientation of E^{n-1} by γ^{n-1}). Thus E succeeds D in γ^n.

Case 2. A' and B' are contained in different n-edges $E(A')$ and $E(B')$, such that A' and $E(A')$ have the same terminal points, B' and $E(B')$ have the same initial points, and $E(A')$ and $E(B')$ are succeeding in γ^n.

Thus the $(n-1)$-edge $F(E(B'))\supset B$ succeeds $F(E(A'))\supset A$ in γ^{n-1}, since

$$F\colon \gamma^n \longrightarrow \gamma^{n-1}$$

is a d-fold cover. Furthermore, the terminal point of A is the terminal point of $F(E(A'))$, which is the initial point of both B and $F(E(B'))$. Thus E succeeds D in γ^n by Lemma 3.9. □

By repeating the argument in Lemma 3.11 we obtain inductively the following consequence.

Corollary 3.13. *All n-edges in the Eulerian circuit γ^n are positively oriented (for each n).*

4. Construction of γ

In this section we complete the construction of γ, i.e., the proof of Theorem 1.1, under the assumption of the existence of a pseudo-isotopy H^0 as in Definition 3.2.

LEMMA 4.1. *To construct $\gamma\colon S^1\to S^2$ as in Theorem 1.1, it is enough to show that there is a Peano curve $\widetilde{\gamma}\colon S^1\to S^2$ such that the diagram*

$$\begin{array}{ccc} S^1 & \xrightarrow{\widetilde{\varphi}} & S^1 \\ {\scriptstyle\widetilde{\gamma}}\downarrow & & \downarrow{\scriptstyle\widetilde{\gamma}} \\ S^2 & \xrightarrow[F]{} & S^2 \end{array}$$

commutes, where $\widetilde{\varphi}(z)=e^{2\pi i\theta_0}z^d$.

(Here and in other powers of e, throughout the paper, i denotes the imaginary unit.)

Proof. Let $\mu:=e^{2\pi i\theta_0/(1-d)}$. This means that

$$e^{2\pi i\theta_0}\mu^d = e^{2\pi i\theta_0}\mu^{d-1}\mu = \mu.$$

Consider $\gamma(z):=\widetilde{\gamma}(\mu z)$. Then

$$F(\gamma(z)) = F(\widetilde{\gamma}(\mu z)) = \widetilde{\gamma}(e^{2\pi i\theta_0}\mu^d z^d) = \widetilde{\gamma}(\mu z^d) = \gamma(z^d). \qquad \square$$

In this section however we will drop the "$\sim$" from the notation. This means that we will write γ, γ^n and so on; when in fact we mean $\widetilde{\gamma}$, $\widetilde{\gamma}^n$, which become our desired objects by composing with a rotation as above.

4.1. The length of n-arcs

The circle S^1 will be divided into *n-arcs*, each of which will be mapped by γ^n to an n-edge. We first need to find the right "length" of such n-arcs. It will be convenient to parameterize those lengths by the corresponding n-edges. Thus $l(E)$ will be the length of the n-arc (in S^1) that is mapped by γ^n to the n-edge E. We require the following properties:

(l 1) $l(E)>0$ for every n-edge E.

(l 2) For all n,

$$\sum_{E\in\mathbf{E}^n} l(E) = 1.$$

(l 3) Given an $(n+1)$-edge E' let $E=F(E')\in\mathbf{E}^n$. Then

$$l(E) = d\,l(E').$$

(l 4) Let E be an n-edge. Then $H_1^n(E)$ is a chain $E'_1, ..., E'_N$ of $(n+1)$-edges. We require that

$$l(E) = \sum_{i=1}^{N} l(E'_i).$$

To this end, consider (all) 0-edges $E_0, ..., E_{k-1}$ ordered by the first approximation γ^0 (positively on $\mathcal{C}$). We say that an n-edge E^n is of *type* j if $F^n(E^n)=E_j$. Recall that H^0 deforms each 0-edge to several 1-edges. We define a matrix $M=(m_{ij})_{i,j}$, which keeps track of those deformations, by

$$m_{ij} \text{ is the number of 1-edges in } H_1^0(E_i) \text{ that are of type } j.$$

LEMMA 4.2. *Consider an n-edge E_i^n of type i. Let $\widetilde{m}_{ij}$ be the number of $(n+1)$-edges of type j in $H_1^n(E_i^n)$. Then*

$$\widetilde{m}_{ij} = m_{ij}.$$

Furthermore, let m_{ij}^n be the number of n-edges of type j contained in

$$H_1^{n-1} \circ H_1^{n-2} \circ ... \circ H_1^0(E_i).$$

Then

$$(m_{ij}^n)_{i,j} = M^n.$$

Proof. Let $E_1^{n+1}, ..., E_m^{n+1}$ be the $(n+1)$-edges in $H_1^n(E_i^n)$. Since H^n is the lift of H^0 by F^n, it follows that H^0 deforms (the 0-edge) $E_i=F^n(E_i^n)$ to the 1-edges

$$E_1^1 = F^n(E_1^{n+1}), \quad ..., \quad E_m^1 = F^n(E_m^{n+1}).$$

The first statement follows, since F^n preserves the type of edges.

The second statement follows immediately from the first. □

LEMMA 4.3. *The matrix M is* primitive, *i.e., $M^n>0$ for some n.*

Proof. Recall from §3.4 that $F\colon \gamma^{n+1} \to \gamma^n$ is a d-fold cover. Thus, by induction, $F^n\colon \gamma^n \to \gamma^0$ is a d^n-fold cover. Therefore, along γ^n the type of n-edges varies cyclically, in γ^n an n-edge of type j is succeeded by one of type $j+1$. This means that every chain of k n-edges in γ^n contains exactly one n-edge of each type.

Fix a 0-edge E_i connecting two post-critical points p and q. Consider

$$H_1^{n-1} \circ H_1^{n-2} \circ ... \circ H_1^0(E_i).$$

This is a chain of n-edges in γ^n that connects the points p and q. Since F is expanding (see Definition 2.1 (3)), the diameter of n-edges goes to 0 (uniformly) with n. Thus, by choosing n large enough, our chain contains at least k n-edges, and therefore at least one n-edge of each type.

With this choice of n the claim follows from Lemma 4.2. □

Note that there are d 1-edges of each type, and thus $\sum_i m_{ij}=d$. The Perron–Frobenius theorem (see, for example, [HJ, Theorems 8.2.11 and 8.1.21]) implies that d is a simple eigenvalue of M (in fact its spectral radius). Furthermore, there is a unique eigenvector $l=\{l_j\}_{j=0}^{k-1}$ to d, such that $l_j>0$ (for all $j=0,...,k-1$) and $\sum_{j=0}^{k-1} l_j=1$. We note that $l_j\subset\mathbb{Q}$ for all $j=0,...,k-1$. The *length* of (an n-arc in S^1 corresponding to) an n-edge E_j^n of type j is now defined as

$$l(E_j^n) := d^{-n} l_j. \tag{4.1}$$

LEMMA 4.4. *The* length *defined above satisfies properties* $(l\,1)$–$(l\,4)$.

Proof. $(l\,1)$ This follows immediately, since $l_j>0$ for all j.

There are d^n n-edges of each type. Thus

$$\sum_{E\in\mathbf{E}^n} l(E) = \sum_{j=0}^{k-1} l_j = 1,$$

which is property $(l\,2)$.

$(l\,3)$ This is again clear, since F maps $(n+1)$-edges to n-edges of the same type.

Property $(l\,4)$ follows from the fact that $Ml=dl$. Let E_i^n be an n-edge of type i, and $E_1^{n+1},...,E_N^{n+1}$ be the $(n+1)$-edges contained in $H_1^n(E_i^n)$. Then, by Lemma 4.2,

$$\sum_m l(E_m^{n+1}) = d^{-n-1}\sum_{j=0}^{k-1} m_{ij} l_j = d^{-n} l_i = l(E_i^n). \qquad \square$$

Note that the lengths depend on the particular pseudo-isotopy H^0 chosen, it is not a property of the edges alone.

4.2. Parameterizing γ^n

Fix a post-critical point p_0. Consider the Eulerian circuit $\gamma^0=\mathcal{C}=\bigcup\mathbf{E}^0$, that is

$$\gamma^0 = E_0,...,E_{k-1}, \quad E_j\in\mathbf{E}^0.$$

It is labeled in such a way that the initial point of E_0 is p_0. Recall that we want to parameterize γ so that $\varphi=e^{2\pi i\theta_0}z^d$ is semi-conjugate to F (see Lemma 4.1). We now define θ_0. If p_0 is a fixed point of F set $\theta_0:=0$. Otherwise let $E_0,...,E_{m^0-1}$ be the (unique) positively oriented chain in γ^0 from p_0 to $F(p_0)$. Then

$$\theta_0 := l(E_0)+...+l(E_{m^0-1}). \tag{4.2}$$

Label $\gamma^1 = E^1_0, ..., E^1_{kd-1}$ in such a way that E^1_0 is the initial 1-edge of the chain $H^0_1(E_0)$ in γ^1. In the same fashion label (the Eulerian circuit)

$$\gamma^n = E^n_0, ..., E^n_{kd^n-1}, \quad E^n_j \in \mathbf{E}^n,$$

so that E^n_0 is the initial n-edge in $H^{n-1}_1(E^{n-1}_0)$ (for each n). Thus the initial point of each E^n_0 is p_0. Note, however, that γ^n may go through p_0 several times.

It will be convenient to identify S^1 with $\mathbb{R}/\mathbb{Z}$. Divide the circle $\mathbb{R}/\mathbb{Z}$ into k arcs a_j as follows. Let

$$\alpha_0 := 0 \quad \text{and} \quad \alpha_j := l(E_0) + ... + l(E_{j-1}) \tag{4.3}$$

for $j = 1, ..., k-1$. Then $a_j := [\alpha_j, \alpha_{j+1}]$ (where indices are taken mod k).

Convention. When writing $[\alpha, \beta] \subset \mathbb{R}/\mathbb{Z}$ for an arc on the circle, we always mean the *positively oriented* arc from α to β. In particular $a_{k-1} = [\alpha_{k-1}, 0] = [\alpha_{k-1}, 1]$.

In the same fashion we divide the circle $\mathbb{R}/\mathbb{Z}$ into kd^n *n-arcs* a^n_j (for each n) by

$$\alpha^n_0 := 0 \quad \text{and} \quad \alpha^n_j := l(E^n_0) + ... + l(E^n_{j-1})$$

for $j = 1, ..., kd^n - 1$. Then $a^n_j := [\alpha^n_j, \alpha^n_{j+1}]$.

Convention. The (lower) indices of points α^n_j, n-arcs a^n_j and n-edges E^n_j are always taken mod kd^n. In particular, $\alpha^n_{kd^n} = \alpha^n_0$ and $a^n_{kd^n-1} = [\alpha^n_{kd^n-1}, 0] = [\alpha^n_{kd^n-1}, 1] \subset \mathbb{R}/\mathbb{Z}$.

We now define the approximations γ^n on each n-arc $a^n_j \subset \mathbb{R}/\mathbb{Z}$ by

$$\gamma^n : a^n_j \longrightarrow E^n_j \quad \text{is (any) orientation-preserving homeomorphism,}$$

as *parameterized curves*. Thus initial and terminal points are mapped onto each other by γ^n. Note that $\gamma^n(0) = p_0$ for all n.

In $\mathbb{R}/\mathbb{Z}$ the map $\varphi(z) = e^{2\pi i\theta_0} z^d$ is given by

$$\phi : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}, \quad \phi(t) = dt + \theta_0 \bmod 1.$$

LEMMA 4.5. *The parameterized curves γ^n satisfy the following properties*:

(1) *If $m \geqslant n$, then each point α^n_j is a point α^m_i. Furthermore,*

$$\gamma^m(\alpha^n_j) = \gamma^n(\alpha^n_j),$$

for all $j = 0, ..., kd^n - 1$. Note that $\{\alpha^n_j\}_{j=0}^{kd^n-1} = (\gamma^n)^{-1}(\mathbf{V}^n)$. So the n-th approximation determines the preimages (on the circle) of the n-vertices.

(2) *The map ϕ maps each point α_j^{n+1} to a point α_i^n. For any point $\alpha_j^{n+1}\in\mathbb{R}/\mathbb{Z}$,*

$$F(\gamma^{n+1}(\alpha_j^{n+1}))=\gamma^n(\phi(\alpha_j^{n+1})).$$

Thus we have the following commutative diagram:

$$\begin{array}{ccc}\{\alpha_j^{n+1}\}_{j=0}^{kd^{n+1}-1}\subset\mathbb{R}/\mathbb{Z} & \xrightarrow{\phi} & \{\alpha_j^n\}_{j=0}^{kd^n-1}\subset\mathbb{R}/\mathbb{Z}\\ \Big\downarrow{\scriptstyle\gamma^{n+1}} & & \Big\downarrow{\scriptstyle\gamma^n}\\ \mathbf{V}^{n+1}\subset S^2 & \xrightarrow[F]{} & \mathbf{V}^n\subset S^2.\end{array}$$

This will imply the desired semi-conjugacy.

(3) *The supremum norm is given in terms of the visual metric* (2.3). *Then*

$$\|\gamma^{n+1}-\gamma^n\|_\infty\lesssim\Lambda^{-n}$$

for all n. Here $C(\lesssim)$ does not depend on n.

Proof. (1) Consider E_0, the first 0-edge in γ^0. Then $H_1^0(E_0)$ is the chain $E_0^1,\dots,E_{m-1}^1$ of 1-edges in γ^1. Note that the terminal point of E_0 is the terminal point of E_{m-1}^1. By property $(l\,4)$,

$$\alpha_1=l(E_0)=l(E_0^1)+\dots+l(E_{m-1}^1)=\alpha_m^1.$$

Thus

$$\gamma^1(\alpha_1)=\gamma^1(\alpha_m^1)=\text{terminal point of } E_m^1=\text{terminal point of } E_0=\gamma^0(\alpha_1).$$

In the same fashion one shows that each α_j is a point α_i^1, and $\gamma^1(\alpha_j)=\gamma^0(\alpha_j)$ for all $j=0,\dots,k-1$. The general statement follows by induction (see Lemma 4.2).

(2) Recall from the definitions of θ_0 (4.2) and $\{\alpha_j\}_{j=0}^{k-1}$ (4.3) that $\alpha_{m^0}=\theta_0$. Then, by (1) and the definition of θ_0, we have

$$\gamma^n(\theta_0)=\gamma^0(\theta_0)=F(p_0).$$

Let $m^n=m^n(\theta_0)$ be the index such that $\alpha_{m^n}^n=\theta_0$.

Consider E_0^{n+1}, the initial $(n+1)$-edge in γ^{n+1}. It is clear that $F(E_0^{n+1})$ is an n-edge with initial point $F(p_0)$ (by Corollary 3.13). There may be several such n-edges in general however. We next show that $F(E_0^{n+1})$ is in fact the "right" n-edge, namely the image (by γ^n) of the n-arc (on $\mathbb{R}/\mathbb{Z}$) with initial point θ_0.

Claim 1. $F(E_0^{n+1})=\gamma^n(a_{m^n}^n)=E_{m^n}^n$.

This is clear for $n=0$, since there is only one 0-edge with initial point $F(p_0)$. To prove the claim by induction, we assume it is true for $n-1$.

Consider E_0^n; by assumption, $F(E_0^n)=\gamma^{n-1}(a^{n-1}_{m^{n-1}})=E^{n-1}_{m^{n-1}}$. Let $A^n\subset E_0^n$ be the (initial) n-arc that is deformed by H^n to E_0^{n+1}. Let $A^{n-1}:=F(A^n)\subset E^{n-1}_{m^{n-1}}$; it is an n-arc that is deformed by H^{n-1} to an n-edge E_j^n (since H^n is the lift of H^{n-1} by F):

$$\begin{array}{ccc} A^n\subset E_0^n & \xrightarrow{H_1^n} & E_0^{n+1} \\ \Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle F} \\ A^{n-1}\subset E^{n-1}_{m^{n-1}} & \xrightarrow[H_1^{n-1}]{} & E_j^n. \end{array}$$

The crucial property is that by construction $j=m^n$. This is seen as follows. By (l 4) the total length of the $(n-1)$-edges preceding $E^{n-1}_{m^{n-1}}$ (which is θ_0) is the same as the total length of all n-edges preceding E_j^n:

$$\theta_0=l(E_0^{n-1})+\ldots+l(E^{n-1}_{m^{n-1}-1})=l(E_0^n)+\ldots+l(E^n_{j-1}),$$

and thus $j=m^n$.

Hence, $F(E_0^{n+1})=E^n_{m^n}$, since the diagram above commutes. This proves Claim 1.

Claim 2. $F(E_j^{n+1})=E^n_{m^n+j}$ for $j=0,\ldots,kd^{n+1}-1$.

This follows from Claim 1, and the fact that $F\colon\gamma^{n+1}\to\gamma^n$ is a d-fold covering in the sense of Definition 3.10. The reader is reminded (for the last time) that the index m^n+j is taken mod kd^n.

Claim 3. The map ϕ maps points α_j^{n+1} to points α_i^n, in fact

$$\phi(\alpha_j^{n+1})=\alpha^n_{m^n+j}.$$

To prove this claim note first that

$$\phi(\alpha_0^{n+1})=\phi(0)=\theta_0=\alpha^n_{m^n}$$

by definition. In the following we write $\alpha\equiv\beta$ if α and β represent the same point on the circle $\mathbb{R}/\mathbb{Z}$, i.e., if $\alpha-\beta\in\mathbb{Z}$.

By the previous claim, $F(E_j^{n+1})=E^n_{m^n+j}$, and thus

$$l(E^n_{m^n+j})=dl(E_j^{n+1})$$

by Property ($l\,3$). Therefore

$$\begin{aligned}\alpha^n_{m^n+j} &\equiv \alpha^n_{m^n}+l(E^n_{m^n})+l(E^n_{m^n+1})+\ldots+l(E^n_{m^n+j-1})\\ &= \theta_0+d\,(l(E^{n+1}_0)+\ldots+l(E^{n+1}_{j-1}))=\theta_0+d\alpha^{n+1}_j\equiv\phi(\alpha^{n+1}_j)\end{aligned}$$

for $j=0,\ldots,kd^{n+1}-1$. Thus Claim 3 is proved.

It remains to show the semi-conjugacy. Note that, by construction, γ^n maps α^n_j to the initial point of E^n_j. Thus, by Claims 2 and 3,

$$\begin{aligned}F(\gamma^{n+1}(\alpha^{n+1}_j)) &= F(\text{initial point of } E^{n+1}_j) = \text{ initial point of } E^n_{m^n+j}\\ &= \gamma^n(\alpha^n_{m^n+j})=\gamma^n(\phi(\alpha^{n+1}_j)).\end{aligned}$$

This finishes the proof of property (2).

(3) The diameter of each n-edge E^n in the visual metric (2.3) is given by

$$\operatorname{diam} E^n \asymp \Lambda^{-n},$$

see [BM, Lemma 8.4].

Consider one n-arc $a^n_j=[\alpha^n_j,\alpha^n_{j+1}]$. Then $\gamma^n(a^n_j)=E^n_j$. The pseudo-isotopy H^n deforms E^n_j to a $(n+1)$-chain $E^{n+1}_i,\ldots,E^{n+1}_{i+m-1}$. The number m (of $(n+1)$-edges in this chain) is uniformly bounded by Lemma 4.2. By (the proof of) property (1), we have $\alpha^n_j=\alpha^{n+1}_i$ and $\alpha^n_{j+1}=\alpha^{n+1}_{i+m}$, and so

$$a^n_j = a^{n+1}_i\cup\ldots\cup a^{n+1}_{i+m-1},$$

where

$$\gamma^{n+1}(a^{n+1}_i)=E^{n+1}_i,\quad \ldots,\quad \gamma^{n+1}(a^{n+1}_{i+m-1})=E^{n+1}_{i+m-1}.$$

Furthermore, the $(n+1)$-chain $E^{n+1}_i,\ldots,E^{n+1}_{i+m-1}$ and the n-edge E^n_j intersect in (the endpoints of E^n_j) $\gamma^n(\alpha^n_j)=\gamma^{n+1}(\alpha^{n+1}_i)$ and $\gamma^n(\alpha^n_{j+1})=\gamma^{n+1}(\alpha^{n+1}_{i+m})$, again by property (1). Thus, on a^n_j,

$$\|\gamma^n-\gamma^{n+1}\|_\infty \leqslant \operatorname{diam} E^n_j+\operatorname{diam} E^{n+1}_i+\ldots+\operatorname{diam} E^{n+1}_{i+m-1}\lesssim\Lambda^{-n}+m\Lambda^{-n-1}\lesssim\Lambda^{-n},$$

as desired. □

4.3. Construction of the invariant Peano curve γ

We now come to the proof of the main result, assuming the existence of a pseudo-isotopy H^0 as in Definition 3.2.

Define

$$\gamma\colon \mathbb{R}/\mathbb{Z} \longrightarrow S^2,$$
$$t \longmapsto \lim_{n\to\infty} \gamma^n(t).$$

Since the sequence $\{\gamma^n\}_{n=1}^{\infty}$ converges uniformly by Lemma 4.5 (3), this is a parameterized curve.

Claim 1. γ is a Peano curve (onto).

This is clear since the curve γ contains by construction $\bigcup_{n=1}^{\infty} \mathbf{V}^n$ (all n-vertices). This set is dense in S^2.

Claim 2. $F(\gamma(t))=\gamma(\phi(t))$ for all $t\in\mathbb{R}/\mathbb{Z}$.

Note that, by properties (1) and (2) of Lemma 4.5, this is true for all $t=\alpha_j^n$. The claim follows, since the set of all such points α_j^n is dense in the circle $\mathbb{R}/\mathbb{Z}$.

Thus we "just" need to construct the pseudo-isotopy H^0 (with properties $(H^0\,1)$–$(H^0\,5)$) to finish the proof of Theorem 1.1.

4.4. γ is the end of a pseudo-isotopy

The homotopy $\Gamma\colon S^2\times[0,1]\to S^2$ from Theorem 1.1 is constructed as follows. Roughly speaking we concatenate the homotopies H^n. The precise definition is as follows. Break up the unit interval into intervals

$$I=[0,1]=\left[0,\tfrac{1}{2}\right]\cup\left[\tfrac{1}{2},\tfrac{3}{4}\right]\cup\ldots\cup\left[1-2^{-n},1-2^{-n-1}\right]\cup\ldots\cup\{1\}.$$

The nth interval in this union is denoted by $I^n=[1-2^{-n},1-2^{-n-1}]$. Let $s_n\colon I^n\to I$, $s_n(t)=2^{n+1}(t-(1-2^{-n}))$ for $n\in\mathbb{N}_0$. We define $\Gamma\colon S^2\times I\to S^2$ by $\Gamma(x,t)=H^0(x,s_0(t))$ for $t\in I^0$ and $\Gamma(x,t)=H^1(H_1^0(x),s_1(t))$ for $t\in I^1$. In general,

$$\Gamma(x,t) := H^n(H_1^{n-1}\circ\ldots\circ H_1^0(x),s_n(t))$$

for $t\in I^n$ (for some $n\in\mathbb{N}_0$) and all $x\in S^2$. Since the diameters of H^n tend to 0 exponentially (see Lemma 3.4 (2)), it follows that Γ extends to $t=1$ by $\Gamma(x,1):=\lim_{t\to 1}\Gamma(x,t)$ continuously. This is the desired homotopy.

It is possible to choose Γ to be a pseudo-isotopy. This can be done explicitly by slightly altering the above construction. We do not work out the details here. It is however a direct consequence of the general theory of decomposition spaces. Namely it follows from the fact that every *cell-like upper semi-continuous decomposition* of a 2-manifold is *shrinkable* [Da, Theorem 25.1].

5. Some topological lemmas

Here we collect some topological theorems and lemmas for future reference. We first note the following form of the Jordan–Schönflies theorem.

THEOREM 5.1. (Isotopic Schönflies theorem) *Let $\gamma, \sigma \subset \mathbb{D}$ be two Jordan arcs with common endpoints $p, q \in \overline{\mathbb{D}}$. Then there is an isotopy of $\overline{\mathbb{D}}$ rel. $\partial\mathbb{D} \cup \{p, q\}$ that deforms γ to σ.*

We give a quick outline of how this form can be obtained from the standard Schönflies theorem.

THEOREM 5.2. (Schönflies theorem [Mo, Theorem 10.4]) *Let $h\colon J \subset \mathbb{R}^2 \to \tilde{J} \subset \mathbb{R}^2$ be a homeomorphism, where J is a Jordan curve. Then h may be extended to a homeomorphism $h\colon \mathbb{R}^2 \to \mathbb{R}^2$.*

We remind the reader of the *Alexander trick*.

THEOREM 5.3. (Alexander [Mo, Theorem 11.1]) *Let $h\colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a homeomorphism such that $h|_{S^1} = \mathrm{id}_{S^1}$. Then the map $\phi\colon \overline{\mathbb{D}} \times [0,1]$ defined by*

$$\phi(x,t) := \begin{cases} th\left(\dfrac{x}{t}\right), & \text{if } 0 \leqslant |x| \leqslant t, \\ x, & \text{if } t \leqslant |x| \leqslant 1, \end{cases}$$

is an isotopy with $\phi(\cdot, 0) = \mathrm{id}_{\overline{\mathbb{D}}}$ and $\phi(\cdot, 1) = h$.

Outline of proof of Theorem 5.1. Consider first $p, q \in S^1 = \partial\overline{\mathbb{D}}$. Let $C_1, C_2 \subset S^1$ be the two arcs bounded by p and q. Let $h_j\colon \gamma \cup C_j \to \sigma \cup C_j$ be homeomorphisms which are constant on S^1, $j = 1, 2$. Using Theorem 5.2, they can be extended to a homeomorphism of $\overline{\mathbb{D}}$. Theorem 5.3 gives the desired isotopy.

If $p = 0$ and $q \in S^1$, extend γ and σ to arcs with common endpoints $\tilde{p}, q \in S^1$. The previous procedure yields the isotopy.

If $p \in \mathbb{D}$ and $q \in S^1$, we use the same construction as before. Then we post-compose with the isotopy that maps the rays between $\phi(p,t)$ and $\zeta \in S^1$ to the rays between p and $\zeta \in S^1$.

Finally let $p, q \in \mathbb{D}$. By the above, we may assume that $p = 0$. Extend γ and σ to curves $\widetilde{\gamma}$ and $\tilde{\sigma}$, respectively, with common endpoints $\tilde{p}$ and $\tilde{q}$. As above we obtain an isotopy $\phi(x,t)$ rel. $S^1 \cup \{p\}$ deforming $\widetilde{\gamma}$ to $\tilde{\sigma}$. We may assume that $\phi(q,1) = q$ (choose the homeomorphisms h_j such that $h_j(q) = q$). This means that ϕ deforms γ to σ. Let $r_t := |\phi(q,t)|$ and $\alpha_t := \log r_0 / \log r_t$. Then post-composition with the *radial stretch*

$$\psi(x,t) := |x|^{\alpha_t} \frac{x}{|x|}$$

yields an isotopy $\tilde{\phi}$ rel. $S^1 \cup \{p\}$ which keeps $|q|$ constant. Let $\theta_t := \arg \tilde{\phi}(q,t) - \arg q$. Post-composing with

$$\varphi\colon re^{i\theta} \longmapsto re^{i(\theta - \theta_t(1-r)/(1-|q|))}$$

yields the desired isotopy. There is a tricky point hidden here: θ_1 could be a multiple of 2π. We can however always arrange that $\theta_1 = 0$ in the following way. Let $\widetilde{\gamma}|_{[\tilde{q},q]}$ and $\tilde{\sigma}|_{[\tilde{q},q]}$ be the paths of the extensions from $\tilde{q}$ to q. By choosing the extensions $\widetilde{\gamma}$ and $\tilde{\sigma}$ in such a way that the change of argument along $\widetilde{\gamma}|_{[\tilde{q},q]}$ and $\tilde{\sigma}|_{[\tilde{q},q]}$ is equal, it follows that

$$\theta_1 = 0. \qquad \square$$

The following is due to Epstein–Zieschang, see [Bu, Theorem A.5].

THEOREM 5.4. (Isotopy rel. post) *Let $\mathcal{C}, \gamma \subset S^2$ be two Jordan curves going through the post-critical points $p_0, \ldots, p_{k-1}$ in the same cyclical order. Let $\mathcal{C}_j$ and γ_j be the arcs on $\mathcal{C}$ and γ, respectively, between p_j and p_{j+1} (indices are taken* mod k *here). Then the following conditions are equivalent*:

(1) *$\mathcal{C}_j$ and γ_j are isotopic rel.* post *for all $j = 0, \ldots, k-1$;*
(2) *$\mathcal{C}$ and γ are isotopic rel.* post.

Combining the previous with Theorem 5.1, we obtain the following result.

THEOREM 5.5. *With notation as in the previous theorem assume that*

$$\mathcal{C}_i \cap \gamma_j \neq \varnothing \quad \text{only for } j = i-1, i, i+1.$$

Then $\mathcal{C}$ and γ are isotopic rel. post.

6. Connections

In this and the following section the initial pseudo-isotopy H^0 is constructed. This was used to define the first approximation γ^1 of the Peano curve. Recall that γ^1 is an Eulerian circuit of 1-edges. Thus γ^1 is given by the following construction. For each 1-edge E ending at a 1-vertex v we have to define a *succeeding* 1-edge $E' \ni v$. Since γ^1 will be non-crossing, there will be an even number of 1-edges in the sector between E and E' (as well as in the sector between E' and E). Let E be contained in the white 1-tile X, and E' be contained in the white 1-tile X'. From the above it follows that if γ^1 traverses E positively (as boundary of X), then it traverses X' positively (as boundary of X').

Since γ^1 is non-crossing, it is possible to "distort the picture" in a neighborhood of v slightly, so that the resulting curves are simple. In this distorted picture the 1-tiles X and X' are *connected at v*. See Figure 4 for an illustration.

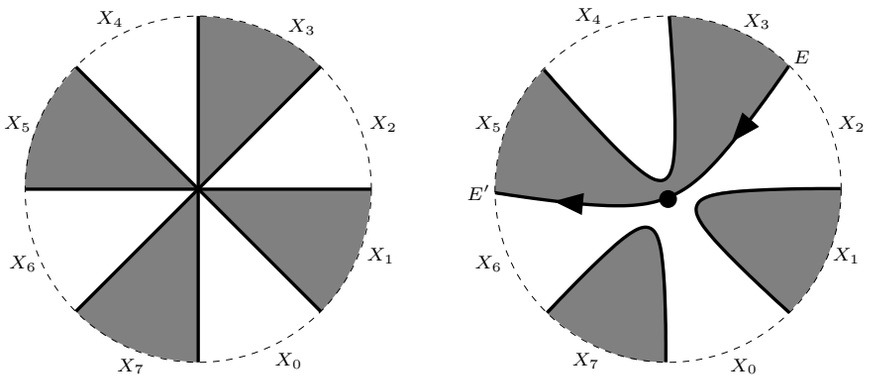

Figure 4. Connection at a vertex.

Formally we will do the reverse of the description above. Namely at each 1-vertex we will define a *connection*, which is an assignment of which 1-tiles are connected. This will be done in a *non-crossing* manner. The approximation γ^1 and the pseudo-isotopy H^0 are constructed from the connection of (all) 1-tiles.

6.1. Non-crossing partitions

Recall that a *partition* of the set $[n]{:=}\{0,...,n{-}1\}$ is a set $\pi{=}\{b_1,...,b_N\}$ of pairwise disjoint subsets (called *blocks*) of $[n]$, whose union is $[n]$. It is *crossing* if and only if it contains distinct blocks b_i and b_j with $a,c{\in}b_i$ and $b,d{\in}b_j$ such that

$$0\leqslant a<b<c<d\leqslant n-1;$$

otherwise it is *non-crossing*.

It is easy to see that the partition $\pi{=}\{b_1,...,b_N\}$ of $[n]$ is non-crossing if and only if the sets $B_{j'}{:=}\{e_m{:}m{\in}b_{j'}\}$, where $e_m{:=}e^{2\pi im/n}$, have the property that each $B_{j'}$ lies in one component of $S^1{\setminus}B_j$ (for $j'{\neq}j$).

With this description in mind let, for $i,j{\in}[n]$,

$$\begin{aligned}[i,j]&:=\begin{cases}\{i,...,j\}, & \text{if } i\leqslant j,\\ \{i,...,n{-}1\}{\cup}\{0,...,j\}, & \text{if } i>j,\end{cases}\\ (i,j)&:=[i,j]{\setminus}\{i,j\}.\end{aligned}\tag{6.1}$$

Let $b{=}\{j_0,...,j_m\}{\subset}[n]$, where $j_0{<}...{<}j_m$. Then a *component* of $[n]{\setminus}b$ is defined to be one of the sets

$$(j_0,j_1),\quad ...,\quad (j_{m-1},j_m),\quad (j_m,j_0).$$

The partition $\pi=\{b_1,...,b_N\}$ is non-crossing if and only if each b_i lies in one component of $[n]\setminus b_j$ for all $i\neq j$.

The set of non-crossing partitions of $[n]$ is partially ordered by refinement. Namely, for two partitions π and σ, one defines $\sigma\leqslant\pi$ if and only if every block in π is the union of blocks in σ. Equipped with this partial ordering the non-crossing partitions (of $[n]$) form a *lattice*, i.e., *meet* and *join* are well defined. The meet of (non-crossing) partitions $\pi_1,...,\pi_m$ is

$$\bigwedge_{i=1}^{m}\pi_i:=\{b_1\cap...\cap b_m : b_i\in\pi_i, i=1,...,m\}. \tag{6.2}$$

It is the biggest (non-crossing) partition smaller than any π_i. The join is the smallest non-crossing partition bigger than any π_i (the description is slightly more difficult).

Non-crossing partitions were introduced in [Kr], see [Si] for a recent survey. The number of non-crossing partitions of $[n]$ is equal to the nth *Catalan number*

$$C_n:=\frac{1}{n+1}\binom{2n}{n}.$$

Consider now

$$\text{even}=\text{even}_n=\{2m : m=0,...,n-1\} \quad \text{and} \quad \text{odd}=\text{odd}_n=\{2m+1 : m=0,...,n-1\},$$

so that $[2n]=\text{even}\cup\text{odd}$.

Non-crossing partitions of even and odd are defined as before. We denote by π_w a non-crossing partition of even, and by π_b a non-crossing partition of odd. They will describe how white (black) tiles are connected at a vertex v; see again Figure 4 for an illustration, and Figure 9 for a more complicated example.

LEMMA 6.1. *Let π_w be a non-crossing partition of* even_n. *Then there is a unique maximal non-crossing partition $\pi_b=\pi_b(\pi_w)$ of* odd_n *such that $\pi_w\cup\pi_b$ is a non-crossing partition of* $[2n]$.

Proof. Fix a block $b_i\in\pi_w=\{b_1,...,b_N\}$. Let $c_1,...,c_M$ be the components of $[2n]\setminus b_i$. Let

$$a_j:=\text{odd}\cap c_j, \quad j=1,...M.$$

Then $\pi_b(b_i):=\{a_1,...,a_M\}$. This is a non-crossing partition of odd. We now define (see (6.2))

$$\pi_b:=\bigwedge_{i=1}^{N}\pi_b(b_i);$$

this is a non-crossing partition of odd. Also $\pi_w\cup\pi_b$ is a non-crossing partition of $[2n]$.

Let σ_b be any non-crossing partition of odd such that $\pi_w\cup\sigma_b$ is a non-crossing partition of $[2n]$. Then $\sigma_b\leqslant\pi_b(b_i)$ for all i. Thus $\sigma_b\leqslant\pi_b$. □

The partition $\pi_b{=}\pi_b(\pi_w)$ is called the *partition complementary* to π_w. We mention some more facts which can be found in [Kr, §3].

LEMMA 6.2. (Properties of complementary partitions) *Complementary partitions have the following properties*:

- *Two blocks a and b are called* adjacent *if there are $i{\in}a$ and $j{\in}b$ such that $i{+}1{\in}b$ and $j{+}1{\in}a$. The partition $\pi_w{\cup}\pi_b$ has the property that the two blocks containing i and $i{+}1$ are adjacent for all i. This characterizes π_b, meaning that it is the unique non-crossing partition of* odd, *such that $\pi_w{\cup}\pi_b$ is non-crossing, with this property.*
- *One may define $\pi_w{=}\pi_w(\pi_b)$, the partition (of* even*) complementary to the partition π_b (of* odd*) as before. Then the previous characterization shows that*

$$\pi_w(\pi_b(\pi_w)) = \pi_w.$$

Thus we simply say that the partitions π_w and π_b are complementary.

- *It is possible to define a* graph, *where the vertices are the blocks of $\pi_w{\cup}\pi_b$, connected by edges if and only if they are adjacent. It is not very hard to show that this is a* tree *with n edges. Thus $\pi_w{\cup}\pi_b$ contains exactly $n{+}1$ blocks.*

From now on, we write *cnc-partition* for complementary non-crossing partitions $\pi_w{\cup}\pi_b$ as above.

We next proceed to construct a *geometric realization* of a given cnc-partition; see again Figure 4.

Divide the unit disk into $n{+}1$ (simply connected) domains $D_1, ..., D_{n+1}$ by n disjoint Jordan arcs $g_1, ..., g_n{\subset}\overline{\mathbb{D}}$. More precisely, the (distinct) endpoints of each g_j lie in $S^1{=}\partial\mathbb{D}$, the interior of g_j in $\mathbb{D}$. The arcs g_m divide S^1 into $2n$ circular arcs $a_0, ... a_{2n-1}{\subset}S^1$ (labeled positively on S^1). A partition $\pi(\{g_m\}_{m=1}^n)$ of $[2n]$ is obtained as follows:

$$\begin{gathered} i, j \in [2n] \text{ are in the same block of } \pi(\{g_m\}_{m=1}^n) \text{ if and only if} \\ a_i \text{ and } a_j \text{ are in the boundary of the same component } D_l. \end{gathered} \tag{6.3}$$

So, for each component D_l of $\mathbb{D}\backslash\bigcup_{j=1}^n g_j$, there is exactly one block $b_l{\in}\pi(\{g_m\}_{m=1}^n)$.

LEMMA 6.3. *The partition $\pi(\{g_m\}_{m=1}^n)$ is a* cnc-partition. *Conversely each cnc-partition of $[2n]$ is obtained in this way.*

Furthermore $\overline{D}_k$ and $\overline{D}_l$ are not disjoint *if and only if the (corresponding) blocks b_k and b_l are* adjacent. *In this case the intersection of $\overline{D}_k$ and $\overline{D}_l$ is one arc g_m. Conversely, each g_m is the intersection of the closure of two components $\overline{D}_k$ and $\overline{D}_l$.*

Proof. We first show that $\pi(\{g_m\}_{m=1}^n)$ is non-crossing. Consider distinct components D_k and D_l. Then there is a Jordan arc $g_m{\subset}\partial D_k$ that separates D_k from D_l. Let

$\alpha, \beta \in S^1$ be the endpoints of g_m. Let $a_i, a_{i+1} \subset S^1$ and $a_j, a_{j+1} \subset S^1$ be the circular arcs containing α and β. We may assume that $a_i \subset \partial D_k$, and thus $a_{j+1} \subset \partial D_k$. Then all arcs in the boundary of D_l are contained in $a_{i+1}, ..., a_j$. This means that $b_l \subset [i+1, j]$, which is one component of $[2n] \setminus b_k$ (recall that b_k is the block corresponding to D_k, while b_l is the block corresponding to D_l; see (6.1) for notation). This shows that $\pi(\{g_m\}_{m=1}^n)$ is non-crossing.

If $\partial D_l \supset a_{i+1}$ (equivalently, $i+1 \in b_l$) it follows that $g_m \subset \partial D_l$. Thus $a_j \subset \partial D_l$ (equivalently, $j \in b_l$). Hence $i, j+1 \in b_k$ and $i+1, j \in b_l$, i.e., b_k and b_l are adjacent. This shows that the partition $\pi(\{g_m\}_{m=1}^n)$ is a cnc-partition.

Furthermore, it is clear that b_k and b_l are adjacent if and only if $\overline{D}_k$ and $\overline{D}_l$ intersect.

It remains to show that each cnc-partition is obtained in this geometric fashion. Identify each $j \in [2n]$ with the circular arc $a_j = [e_j, e_{j+1}] \subset S^1$ (where $e_j = e^{2\pi i j/2n}$). For each block $b_l \in \pi_w \cup \pi_b$ the domain D_l is the hyperbolic polygon whose boundary intersects S^1 in $\bigcup_{j \in b} a_j$.

To be more precise, for each two adjacent blocks $b \ni i, j+1$ and $b' \ni i+1, j$, we connect e_{i+1} and e_{j+1} by a hyperbolic geodesic. Since every block distinct from b is contained in one component of $[2n] \setminus \{i, j+1\}$, the Jordan arcs g_m thus obtained are disjoint. □

How 1-tiles are connected at a 1-vertex v will be described by complementary non-crossing partitions. Additional data is needed however, to make the construction well defined. Namely, if $v = p$ is a post-critical point, we need to declare where p lies in the "distorted picture" (in the *geometric representation* of the complementary connections, see below).

Definition 6.4. (Marking) A cnc-partition $\pi_w \cup \pi_b$ is *marked* by singling out a pair of *adjacent blocks* $b, c \in \pi_w \cup \pi_b$. Equivalently, this means that if the cnc-partition $\pi_w \cup \pi_b$ is given geometrically as above in Lemma 6.3, we mark one of the arcs g_m. In Figure 4 the marked arc g_m is indicated by the big dot.

Given a marked cnc-partition, we always assume that the geometric realization from Lemma 6.3 was chosen in such a way that *the marked arc g_m contains the origin.*

A third equivalent way to mark a connection is given in Corollary 6.14.

Assume now that the circular arcs from Lemma 6.3 are of the form $a_j = [e_j, e_{j+1}] \subset S^1$ (where $e_j = e^{2\pi i j/2n}$). Color the set D_l white if the corresponding block $b_l \in \pi_w$, and black otherwise. Thus we obtain a "checkerboard tiling" of the unit disk, where sets which share a side g_m have different color.

Definition 6.5. (Geometric representation of cnc-partition) The decomposition of the closed unit disk into black and white sets as above is called a *geometric representation*

of the cnc-partition $\pi_w \cup \pi_b$, and is denoted by $\overline{\mathbb{D}}(\pi_w \cup \pi_b)$. The union of the white sets $\overline{D}_l$ is denoted by $\overline{\mathbb{D}}_w = \overline{\mathbb{D}}_w(\pi_w \cup \pi_b)$, and the union of the black sets $\overline{D}_l$ by $\overline{\mathbb{D}}_b = \overline{\mathbb{D}}_b(\pi_w \cup \pi_b)$.

Denote by S_j, $j=0, ..., 2n-1$, a sector in $\overline{\mathbb{D}}$, namely

$$S_j := \left\{ re^{2\pi i\theta} : \frac{j}{2n} \leqslant \theta \leqslant \frac{j+1}{2n} \text{ and } 0 \leqslant r \leqslant 1 \right\}. \tag{6.4}$$

LEMMA 6.6. (Deforming $\overline{\mathbb{D}}(\pi_w \cup \pi_b)$) *Let the geometric representation $\overline{\mathbb{D}}(\pi_w \cup \pi_b)$ be as above. Then there is a pseudo-isotopy H of $\overline{\mathbb{D}}$ rel. $\partial\overline{\mathbb{D}} \cup \{0\}$ satisfying the following properties*:

- *H deforms $\overline{\mathbb{D}}(\pi_w \cup \pi_b)$ to sectors. More precisely,*

$$H_1(\overline{\mathbb{D}}_w) = \bigcup_{j \text{ even}} S_j \quad \textit{and} \quad H_1(\overline{\mathbb{D}}_b) = \bigcup_{j \text{ odd}} S_j.$$

- *The pseudo-isotopy H "freezes" outside of a neighborhood of 0. By this we mean that, for $\varepsilon < \frac{1}{2}$,*

$$H : \overline{\mathbb{D}} \times [1-\varepsilon, 1] \longrightarrow \overline{\mathbb{D}} \quad \textit{is a pseudo-isotopy rel. } \overline{\mathbb{D}} \setminus B_\varepsilon,$$

where $B_\varepsilon = \{z : |z| < \varepsilon\}$.

- *Only one point on each arc g_m is deformed to 0 by H.*

Proof. This follows from the Schönflies theorem (Theorem 5.1). □

6.2. Connections

Let v be a 1-vertex. A *connection* at v consists of an assignment of which white and black 1-tiles are connected at v. The objective is to "cut" tiles at vertices, so that the boundary of the "white (or black) component" is a Jordan curve.

Let $n = \deg_F v$ be the degree of F at v, let $X_0, ..., X_{2n-1}$ be the 1-tiles containing v, labeled positively around v, such that white 1-tiles have even index and black 1-tiles have odd index.

Definition 6.7. (Connection at a vertex) A *connection* at a 1-vertex v consists of a labeling of 1-tiles containing v as above and cnc-partitions $\pi_w = \pi_w(v)$ and $\pi_b = \pi_b(v)$ of even_n (representing white 1-tiles) and odd_n (representing black 1-tiles). The 1-tiles X_i and X_j (of the same color) are said to be *connected* at v if i and j are contained in the same block of $\pi_w \cup \pi_b$, 1-tiles of different color are never connected. The 1-tile X_i is *incident* (at v) to the block $b \in \pi_w \cup \pi_b$ containing i. By Lemma 6.1, it is enough to define $\pi_w(v)$, then $\pi_b(v)$ will always be the complementary partition.

If $v{=}p$ is a post-critical point the connection at p is *marked* in addition (see Definition 6.4). Recall that the *marked arc* of a geometric representation $\overline{\mathbb{D}}(\pi_w{\cup}\pi_b)$ (of the connection at the post-critical point p, Definition 6.5) is assumed to *contain the origin.*

The connection illustrated in Figure 4 is given by

$$\pi_w = \{\{0,2,6\},\{4\}\} \quad \text{and} \quad \pi_b = \{\{1\},\{3,5\},\{7\}\}.$$

The marked arc is indicated by the dot.

When talking about 1-tiles X_j and cnc-partitions at the same time, it is always assumed without mention that the indices of the X_j are as above.

Let v be a 1-vertex, and $n{=}\deg_v F$. Let $X_0,...,X_{2n-1}$ be the 1-tiles containing v, labeled positively around v (white tiles have even index, black ones odd index as before). Every such 1-vertex v has arbitrarily small neighborhoods $U{=}U(v)$, that are closed and homeomorphic to the closed disk $\overline{\mathbb{D}}$, such that there is a homeomorphism

$$h = h_v \colon U \longrightarrow \overline{\mathbb{D}} \tag{6.5}$$

that maps tiles to sectors (see (6.4)),

$$h(X_j{\cap}U) = S_j$$

for $j{=}0,...,2n{-}1$. In particular, $h(v){=}0$. We require that the neighborhoods $U(v)$ and $U(v')$ have disjoint closures for distinct 1-vertices v and v'. The reader should think of the neighborhood U as a "blowup" of the point v.

Definition 6.8. (Geometric representation of a connection) Let a connection at v be given, with cnc-partition $\pi_w{\cup}\pi_b$, geometrically represented by $\overline{\mathbb{D}}(\pi_w{\cup}\pi_b)$ as in Definition 6.5. Let h and $U{=}U(v)$ be as above. A *geometric representation of the connection at* v is given by replacing U by $h^{-1}(\overline{\mathbb{D}}(\pi_w{\cup}\pi_b))$.

More precisely, the white 1-tiles in U, i.e. $(X_0{\cup}X_2{\cup}...\,X_{2n-2}){\cap}U$, are replaced by $h^{-1}(\overline{\mathbb{D}}_w)$ (see Definition 6.5). Note that this set is colored white. Similarly we replace the black 1-tiles in U, i.e. $(X_1{\cup}X_3{\cup}...\,X_{2n-1}){\cap}U$, by $h^{-1}(\overline{\mathbb{D}}_b)$. This set is colored black.

Let $v{=}p$ be a post-critical point and the connection at p be marked by the arc g_m. More precisely, in the geometric representation $\overline{\mathbb{D}}(\pi_w{\cup}\pi_b)$ of the connection $\pi_w{\cup}\pi_b$ at p, the marking corresponds to the arc $g_m{\subset}\overline{\mathbb{D}}(\pi_w{\cup}\pi_b)$. Since the marked arc was chosen to contain 0, it follows that in this case $p{\in}h^{-1}(g_m)$, and thus *the geometric representation of the marked arc contains* p. This is the purpose of the marking, namely to keep track of where in the geometric representation of the connection the post-critical point is located.

Definition 6.9. (Connection) A *connection of* 1*-tiles* is an assignment of a connection at every 1-vertex. Representing the connection at each 1-vertex geometrically as above gives a *geometric representation* of this connection of 1-tiles. Objects arising from a geometric representation will be denoted with an ε-subscript.

Assume a geometric representation of a connection of 1-tiles is given. From the construction it follows that each boundary component of some white or black component is a Jordan curve. Let X be a 1-tile with 1-vertices $v_0, ..., v_{k-1}$. Then the geometric representation of X is $X_\varepsilon := X \setminus \bigcup_{j=0}^{k-1} U(v_j)$, where the neighborhood $U(v_j)$ of v_j is as in (6.5). Note that by construction two 1-tiles X and Y (of the same color) are connected at a 1-vertex v if and only if their geometric representations X_ε and Y_ε are connected in $U(v)$. This means that X_ε and Y_ε can be joined by a path in $U(v)$ that does not intersect any boundary of some white or black component.

6.3. The connection graph

Given a connection of 1-tiles, we construct the *white* and *black connection graphs*.

Definition 6.10. (Connection graph) The *white connection graph* is constructed as follows. For each white 1-tile X there is a vertex $c(X)$ (thought of as the *center* of the 1-tile X). For each 1-vertex v and block $b \in \pi_w(v)$ there is a vertex $c(v, b)$. The vertex $c(X)$ is connected to $c(v, b)$ by an edge if and only if X is incident to b at v.

The *black connection graph* is constructed in the same manner from black 1-tiles and their connections.

We will identify a 1-tile X with (the vertex of the white connection graph) $c(X)$. For example we will say that two white 1-tiles X and Y are connected (given a connection of 1-tiles) if $c(X)$ and $c(Y)$ lie in the same component of the white connection graph.

Definition 6.11. (Cluster) A white (black) *cluster* K is one component of the white (black) connection graph. Using the previous identification, we say that K contains a 1-tile X (and write $X \subset K$), if $c(X) \in K$. This means that we identify K with the union of 1-tiles "contained" in it. Similarly, a 1-edge E (resp. 1-vertex v) is said to be contained in K if $E \subset X \subset K$ (resp. $v \in X \subset K$) for some 1-tile X. Each 1-tile is contained in exactly one cluster (of the same color), each 1-edge is contained in exactly two clusters (one black and one white). A 1-vertex v may be contained in several clusters (in fact at most $n+1$, where $n = \deg_F v$).

Assume a geometric representation of the connection has been given. Let X be a 1-tile contained in the cluster K. Then there is a unique component K_ε (of the same color as X) containing (the geometric representation) X_ε. Recall that some 1-tile Y is connected

to X at a 1-vertex v if and only if they are connected at v in a geometric representation of the connection. Thus one obtains inductively that any 1-tile Z is contained in K if and only if $Z_\varepsilon \subset K_\varepsilon$. Thus each white (black) cluster K corresponds to one white (black) component K_ε (of a geometric representation of the connection) and vice versa. We call K_ε a *geometric representation* of the cluster K.

A cluster K is a *tree* if the underlying component of the connection graph is a tree, i.e., contains no cycles. The white cluster K is a *spanning tree*, if it is a tree and contains all white 1-tiles.

In the next section the connection of 1-tiles will be constructed such that the white 1-tiles form a spanning tree in "the right homotopy class".

Remark. Assume that all white 1-tiles are connected at each 1-vertex. Of course we can extract a spanning tree (in the standard sense) from the resulting white connection graph. This spanning tree however will have only one vertex for each 1-vertex v. Thus not all spanning trees in the sense of the previous definition can be obtained in this way. See Corollary 6.20 for an inductive way to construct trees in the connection graph.

The first approximation of the Peano curve γ^1 will be constructed as "the outline" of the spanning tree. One should think of the construction as follows. A geometric representation of this (white) spanning tree will be a Jordan domain. The positively oriented boundary of this domain "is" the first approximation γ^1.

6.4. Succeeding edges

Let a connection of 1-tiles be given. Let E be a 1-edge contained in the white 1-tile X_i, positively oriented (as boundary of X_i) with terminal point v.

Since 1-tiles are cyclically ordered around v, the 1-tiles that are connected at v with X_i are cyclically ordered as well.

Let X_j be the cyclical successor (in positive order around v) of X_i among 1-tiles connected to X_i at v. If no other 1-tile is connected to X_i at v, we let $X_j = X_i$.

Formally i and j are contained in the same block of π_w, and none of the numbers in $[i+1, j-1]$ are contained in this block.

Note that X_j is a *white* 1-tile. Thus an oriented 1-edge $E' \subset X_j$ is positively oriented if and only if it is positively oriented as boundary of X_j.

Definition 6.12. (Successor) Let v and E, as well as X_i and X_j, be as above. The *successor* to E (at v) is the positively oriented 1-edge $E' \subset X_j$ with initial point v. Note that each 1-edge E' is the successor to exactly one 1-edge E.

See Figure 4 for an illustration. For each 1-edge E with initial point v and terminal point w, let $E_\varepsilon := E \setminus (U(v) \cup U(w))$. Here $U(v)$ and $U(w)$ are the neighborhoods of v and w from (6.5). Recall from Lemma 6.3 how a cnc-partition was geometrically represented by dividing the disk by arcs g_m. We call such an arc g_m *positively oriented* if it is positively oriented as boundary arc of a *white* set D_l.

LEMMA 6.13. (Equivalent formulations for succeeding edges) *Consider white 1-tiles $X_i \supset E$ and $X_j \supset E'$, where E and E' are positively oriented 1-edges containing a 1-vertex v. The following statements are equivalent*:

(1) *E' is the successor to E at v.*

(2) *E'_ε is succeeding E_ε on ∂K_ε, where K_ε is a geometric representation of the white cluster K containing E. This means that, when ∂K_ε is positively oriented (as boundary of K_ε) there is no (geometric representation of a 1-edge $\widetilde{E}$) $\widetilde{E}_\varepsilon \subset \partial K_\varepsilon$ on the positively oriented arc from E_ε to E'_ε.*

(3) *Represent the connection at v geometrically as in Lemma 6.3. Using the notation from this lemma, there is a (positively oriented) arc g_m that connects the right endpoint of the arc $a_i \subset S^1$ to the left endpoint of the arc $a_j \subset S^1$.*

(4) *There are adjacent blocks $b \in \pi_w(v)$ and $c \in \pi_b(v)$ such that*

$$i, j \in b \quad \text{and} \quad i+1, j-1 \in c.$$

The proof is clear from the proof of Lemma 6.3.

COROLLARY 6.14. (Marked connection) *A marking of a connection at a post-critical point p may be given*

(1) *by* marking an arc *g_m from a geometric representation of the connection at p*;

(2) *" or equivalently by* marking *a pair of* succeeding 1-edges *E and E' at p*;

(3) *or equivalently by* marking *a pair of* adjacent blocks *$b \in \pi_w(p)$ and $c \in \pi_b(p)$.*

The precise correspondences (i.e., which marked arc corresponds to which marked pair of succeeding edges, corresponds to which marked pair of adjacent blocks) is given by Lemma 6.13.

The 1-tiles containing the successors E and E' are connected at v. If on the other hand the 1-tiles X and Y are connected at v, we can find a chain of succeeding 1-edges.

LEMMA 6.15. *Two 1-tiles X and Y (of the same color) are connected at the 1-vertex v if and only if there is a chain*

$$X = X_1,\ E_1,\ E'_2,\ X_2,\ \ldots,\ X_{m-1},\ E_{m-1},\ E'_m,\ X_m = Y.$$

Here $X_j \ni v$ are 1-tiles of the same color as X and Y, $E_j, E'_j \subset X_j$ are 1-edges and E'_{j+1} succeeds E_j at v.

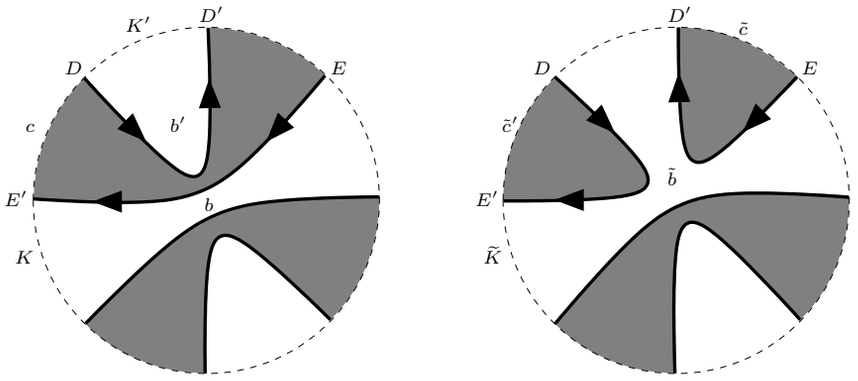

Figure 5. Adding clusters.

Note that, above, the labeling of the white 1-tiles is *not* the one used in the definition of the connection at v (there are some white 1-tiles with odd index).

Proof. If the 1-tiles in the lemma are white, the cyclical order of 1-tiles connected to X at v from $X{=}X_1$ to $Y{=}X_m$ is given by $X_1, ..., X_m$. If the 1-tiles are black this gives the anti-cyclical order. Clearly going (anti-)cyclically around v among 1-tiles connected to X gives all such 1-tiles. □

6.5. Adding clusters

The spanning tree will be built successively by adding more "secondary clusters" to a "main cluster".

Let the connection at a 1-vertex v be given by the cnc-partition $\pi_w \cup \pi_b$ (of $[2n]$, where $n{=}\deg_F(v)$), and K and K' be two white clusters containing v. Let $b{\in}\pi_w$ be a block with indices of 1-tiles in K ($X_j{\subset}K$ if $j{\in}b$), and $b'{\in}\pi_w$ be a block with indices of 1-tiles in K'. We *add* the cluster K' to K at v by replacing b and b' in π_w by $\tilde{b}{:=}b{\cup}b'$. The resulting partition $\tilde{\pi}_w$ may however not be non-crossing anymore.

LEMMA 6.16. (Adding clusters) *The partition $\tilde{\pi}_w$ is non-crossing if and only if there is a block $c{\in}\pi_b$ that is adjacent to both b and b' (see Lemma 6.2).*

In this case, let $\widetilde{K}$ be the cluster in the new connection graph that contains K and K'. If K and K' are trees, then $\widetilde{K}$ is a tree as well.

The situation is illustrated in Figure 5.

Proof. We show the equivalence first.

($\Leftarrow$) Assume that $\tilde{\pi}_w$ is crossing. Then there is a block $\hat{b}{\in}\pi_w$ such that there are

$a, a' \in \hat{b}$, $d \in b$ and $d' \in b'$ satisfying

$$a < d < a' < d'.$$

This means that b and b' have to be contained in different components of $[2n]\setminus\{a, a'\}$. Thus every block $c \in \pi_b$ adjacent to b has to be in a different component of $[2n]\setminus\{a, a'\}$ than every block $c' \in \pi_b$ adjacent to b'. Thus there is no block $c \in \pi_b$ adjacent to both b and b'.

($\Rightarrow$) Assume now that there is no $c \in \pi_b$ adjacent to both b and b'. Let

$$b = \{b_1, ..., b_N\} \quad \text{and} \quad b' = \{b'_1, ..., b'_M\},$$

where $b_1 < ... < b_N$ and $b'_1 < ... < b'_M$. Since π_w is non-crossing, b and b' are in disjoint intervals, meaning that we may assume that, for some j,

$$b_j < b'_1 < b'_M < b_{j+1}.$$

Since π_b is complementary to π_w, there are blocks $c, c' \in \pi_b$ such that

$$b_j + 1, b_{j+1} - 1 \in c \quad \text{and} \quad b'_1 - 1, b'_M + 1 \in c',$$

by Lemma 6.2. The blocks c and c' are distinct by assumption. Let $c'_1 := \min\{c'_j \in c'\}$ and $c'_2 := \max\{c'_j \in c'\}$. The numbers $c'_1 - 1$ and $c'_2 + 1$ are in the same block $\hat{b} \in \pi_w$ (since π_w and π_b are complementary). Thus we have the following ordering:

$$\underbrace{b_j}_{\in b} < \underbrace{b_j+1}_{\in c} < \underbrace{c'_1-1}_{\in \hat{b}} < \underbrace{c'_1}_{\in c'} < \underbrace{b'_1 < b'_M}_{\in b'} < \underbrace{c'_2}_{\in c'} < \underbrace{c'_2+1}_{\in \hat{b}} < \underbrace{b_{j+1}-1}_{\in c} < \underbrace{b_{j+1}}_{\in b}.$$

Clearly $b \cup b'$ and $\hat{b}$ are crossing, which finishes this implication.

We now show the second statement. Recall that in the white connection graph the block $b \in \pi_w$ is represented by a vertex $c(v, b)$ and $b' \in \pi_w$ is represented by a (different) vertex $c(v, b')$. The new white connection graph (where the connection at v is given by $\tilde{\pi}_w$) is obtained by identifying $c(v, b)$ and $c(v, b')$; this yields the vertex $c(v, \tilde{b})$. Then $\widetilde{K}$ is the component (of the new white connection graph) containing $c(v, \tilde{b})$. If K and K' are trees, then clearly $\widetilde{K}$ is a tree as well. □

Assume that c is adjacent to both b and b', i.e., that we can add K' to K at v in this fashion. Let the notation be as in the previous proof, i.e., $b = \{b_1, ..., b_N\}$ and $b' = \{b'_1, ..., b'_M\}$, where

$$b_1 < ... < b_N, \quad b'_1 < ... < b'_M \quad \text{and} \quad b_j < b'_1 < b'_M < b_{j+1}. \tag{6.6}$$

Then the complementary partition $\tilde{\pi}_b$ to $\tilde{\pi}_w$ is given by replacing $c \in \pi_b$ by the two blocks

$$\tilde{c} = c \cap [b_j, b'_1] \quad \text{and} \quad \tilde{c}' = c \cap [b'_M, b_{j+1}]. \tag{6.7}$$

These two blocks are both adjacent to $\tilde{b} = b \cup b' \in \tilde{\pi}_b$.

If we add a cluster K' to a cluster K as above at a post-critical point p, we need to specify the marking (see Definition 6.4) of the new connection at p.

Definition 6.17. (Marking of new connection) Let $\pi_w \cup \pi_b$ be a marked cnc-partition, i.e., a connection at a post-critical point p. Then the marking of the cnc-partition $\tilde{\pi}_w \cup \tilde{\pi}_b$ from the previous lemma is given as follows (notation is as before). Let the marked adjacent blocks in $\pi_w \cup \pi_b$ be

- b, c or b', c; then (in both cases) we can pick $\tilde{b}, \tilde{c}$ or $\tilde{b}, \tilde{c}'$, as the marked adjacent blocks in $\tilde{\pi}_w \cup \tilde{\pi}_b$;
- d, c, where $d \in \pi_w \setminus \{b, b'\}$; then d is adjacent to either $\tilde{c}$ or $\tilde{c}'$, which are the marked adjacent blocks in $\tilde{\pi}_w \cup \tilde{\pi}_b$;
- b, e or b', e, where $e \in \pi_b \setminus \{c\}$; then $\tilde{b}, e$ are the marked adjacent blocks in $\tilde{\pi}_w \cup \tilde{\pi}_b$;
- d, e, where $d \in \pi_c \setminus \{b, b'\}$ and $e \in \pi_b \setminus \{c\}$; then d, e are the marked adjacent blocks in $\tilde{\pi}_w \cup \tilde{\pi}_b$.

LEMMA 6.18. *Assume that a white cluster K' can be added to a white cluster K at a 1-vertex v as in Lemma 6.16 to form a cluster $\widetilde{K}$. Then there exist (uniquely) succeeding 1-edges at v*

$$E, E' \subset K \quad \textit{as well as} \quad D, D' \subset K'$$

such that

$$E \textit{ and } D' \quad \textit{as well as} \quad D \textit{ and } E'$$

are succeeding in $\widetilde{K}$.

The situation is again illustrated in Figure 5.

Proof. Consider the blocks $b, b' \in \pi_w(v)$ which are both adjacent to the block $c \in \pi_b(v)$ as in Lemma 6.16 (here b contains indices of 1-tiles in K, while b' contains indices of 1-tiles in K'). The succeeding 1-edges $E, E' \subset K$ and $D, D' \subset K'$ are the ones corresponding to these adjacencies according to Lemma 6.13. Using the notation from (6.6), we obtain that these 1-edges are contained in the following (white) 1-tiles. In K and K', we have

$$E \subset X_{b_j}, \quad E' \subset X_{b_{j+1}}, \quad D \subset X_{b'_M} \quad \text{and} \quad D' \subset X_{b'_1}.$$

Recall the description of the blocks $\tilde{c}, \tilde{c}' \in \tilde{\pi}_b$ from (6.7). They are both adjacent to $\tilde{b} = b \cup b' \in \tilde{\pi}_w$. Then $b_j + 1, b'_1 - 1 \in \tilde{c}$ and $b'_M + 1, b_{j+1} - 1 \in \tilde{c}'$. Therefore (using Lemma 6.13 again) we obtain that E, D' and D, E' are succeeding in $\widetilde{K}$. □

We will often be in the following specific situation. Consider a white cluster K. Assume that the only white 1-tiles that are possibly connected at a 1-vertex v are in K. Put differently, this means that all distinct white 1-tiles $Y, Y' \ni v$ not in K are disconnected at v. Let $X_i \ni v$ be a white 1-tile not contained in K. The following lemma means that we can *add* X_i, or the cluster containing X_i, to K at v.

LEMMA 6.19. *In the situation as above, there is a block $b \in \pi_w$ containing indices of white 1-tiles in K ($X_j \subset K$ if $j \in b$), such that the partition $\tilde{\pi}_w$, obtained by replacing $b, \{i\} \in \pi_w$ by $\tilde{b} = b \cup \{i\}$, is non-crossing.*

Furthermore, if K and the cluster containing X_i are trees, the resulting cluster $\widetilde{K}$ ($\supset K \cup X$) is a tree as well.

Proof. Consider the graph Γ representing $\pi_w \cup \pi_b$ from Lemma 6.2 (this is *neither* the white connection graph *nor* the graph $\bigcup \mathbf{E}^1$).

Let $X_j \ni v$ be a white 1-tile not contained in K. Since X_j is not connected to any other 1-tile at v the singleton $\{j\}$ is a block of π_w. This block is adjacent to a single block (in π_b), and thus $\{j\}$ is a leaf of Γ (incident to a single edge).

Consider the block $c \in \pi_b$ adjacent to $\{i\} \in \pi_w$. Since Γ is connected, c has to be connected to a block $b \in \pi_w$ containing indices corresponding to 1-tiles in K. This means that b and c are adjacent blocks. The result now follows from Lemma 6.16. □

We record the following corollary (see also Lemma 2.2).

COROLLARY 6.20. (Trees in connection graphs) *A (cluster that is a) tree in the white (black) connection graph may be constructed inductively by adding one 1-tile to a cluster at a time. Every tree in the white (black) connection graph (in a cluster) is obtained in such a way.*

6.6. Boundary circuits

The first approximation of the Peano curve γ^1 will be given as the *boundary circuit* of a (cluster that is a) spanning tree (in the white connection graph).

Definition 6.21. (Boundary circuit of a cluster) Consider a cluster K. A *boundary circuit* $\mathcal{E}$ of K is a circuit $E_0, ..., E_{M-1}$ of positively oriented 1-edges in K such that E_{j+1} is the successor of E_j for each j (indices are taken mod M, in particular E_0 succeeds E_{M-1}); furthermore, no 1-edge appears twice in $\mathcal{E}$.

Recall that every 1-edge has exactly one successor and one predecessor. Thus it is clear that starting from any 1-edge $E_0 \subset K$ and following succeeding 1-edges will yield a boundary circuit.

We note the following, which is an immediate consequence of Lemma 6.13 and Corollary 6.14, see also the discussion after Definition 6.8.

LEMMA 6.22. (K_ε contains p) *Let K be a cluster and p be a post-critical point. A boundary circuit of K contains the marked succeeding 1-edges at p if and only if $p \in K_\varepsilon$ for any geometric representation K_ε of K.*

LEMMA 6.23. *Consider a cluster K. The following are equivalent*:

(1) *the cluster K is a tree*;

(2) *K has only a single boundary circuit*;

(3) *each geometric representation K_ε of K is a Jordan domain.*

In this case the single boundary circuit $\mathcal{E}$ of K is an Eulerian circuit in K. This means that each of the km 1-edges in K appears exactly once in $\mathcal{E}$. Here m is the number of 1-tiles in K ($k = \#\,\mathrm{post} = \#0$-edges).

Proof. Assume without loss of generality that the cluster K is white.

(1) $\Rightarrow$ (2) Recall from Corollary 6.20 that every tree can be obtained inductively by adding more 1-tiles to one cluster in the connection graph. Start with a white tile graph that is totally disconnected, meaning that no two white 1-tiles are connected (at any 1-vertex). Consider one white 1-tile X_0 and a 1-edge $E_0 \subset X_0$. Clearly E_0 is contained in an Eulerian circuit in X_0 of length k (containing all 1-edges in ∂X_0). So the statement is true for the cluster $K_0 = X_0$ (consisting of a single 1-tile).

Let the white connection graph be given such that all clusters except one cluster K_{j-1} contain a single 1-tile, i.e., as in Lemma 6.19. Assume that $E_0 \subset K_{j-1}$. Furthermore we assume that $E_0, ..., E_{kj-1}$ is an Eulerian circuit in K_{j-1} containing all 1-edges in K_{j-1}, where j is the number of 1-tiles in K_{j-1}.

Add a 1-tile X to K_{j-1} at a 1-vertex $v \in K_{j-1}$ as in Lemma 6.19 to form a new component K_j. The above procedure then yields as a path

$$E_0,\ ...,\ E_i,\ E_1^X,\ ...,\ E_k^X,\ E_{i+1},\ ...,\ E_{kj-1},$$

see Lemma 6.18. Here $E_1^X, ..., E_k^X$ are the 1-edges in X, positively oriented, starting at v.

This is an Eulerian circuit in K_j. The construction ends when $K = K_j$. Since the constructed circuit contains all 1-edges in K, there is only a single boundary circuit.

(2) $\Rightarrow$ (3) Consider a neighborhood U of a 1-vertex $v \in K$ as in Definition 6.8. The boundary of K_ε is constructed from boundary circuits by replacing E_j and $E_{j+1} \cap U$ by $h^{-1}(g_m)$. Thus ∂K_ε is a single Jordan curve.

(3) $\Rightarrow$ (1) Assume that K is not a tree. Then there exists a circuit in K. This means there are 1-tiles $X_0, ..., X_{N-1}$ in K such that X_j is connected to X_{j+1} at a 1-vertex v_j

(indices mod N), where all 1-vertices v_j are distinct. Then in the interior of any geometric representation K_ε we can find a Jordan curve following this circuit (connecting $X_{0,\varepsilon}$ to $X_{1,\varepsilon}$ at $v_{0,\varepsilon}$ and so on). This Jordan curve divides K_ε into two components. Note that both components contain boundary of K_ε, namely the (geometric representations of the) two arcs on ∂X_j between v_{j-1} and v_j lie in different components. Thus K_ε is not a Jordan domain. □

We record the following, which is an easy corollary.

LEMMA 6.24. (Boundary circuit of added trees) *Consider trees K and K' with boundary circuits $\mathcal{E}=E_0,...,E_{N-1}$ and $\mathcal{E}'=D_0,...,D_{M-1}$. Assume that we can add them at a 1-vertex v as in §6.5 to form a tree $\widetilde{K}$. Then the boundary circuit $\widetilde{\mathcal{E}}$ of $\widetilde{K}$ is*

$$E_0,\ ...,\ E_i,\ D_{j+1},\ ...,\ D_{M-1},\ D_0,\ ...,\ D_j,\ E_{i+1},\ ...,\ E_{N-1}.$$

Proof. This is clear from Lemma 6.18, where $E_i, E_i{+}1 \subset K$ and $D_j, D_{j+1} \subset K'$ are the succeeding 1-edges associated with adding K to K'. □

We next show that adding a tree K' which "does not contain a post-critical point" to another tree K does not change the "homotopy type" of ∂K_ε.

Definition 6.25. (Trivial tree) A cluster K' that is a tree is called *trivial* if a (and thus any) geometric representation K'_ε does not contain a post-critical point. Equivalently the boundary circuit of K' does not contain the *marked successors* $E{=}E(p)$ and $E'{=}E'(p)$ at p for any post-critical point p (see Corollary 6.14).

LEMMA 6.26. (Adding a trivial tree does not change homotopy type) *Consider a cluster K that is a tree, and a trivial tree K' as above. Assume that it is possible to add K' to K at some 1-vertex v as in Lemma 6.16, to obtain the tree $\widetilde{K}$.*

Then, if ∂K_ε is isotopic to a Jordan curve $\mathcal{C}$ rel. post, *we have that $\partial\widetilde{K}_\varepsilon$ is isotopic to $\mathcal{C}$ rel.* post *as well (for any geometric representations K_ε and $\widetilde{K}_\varepsilon$ of K and $\widetilde{K}$).*

Proof. Let $U{=}U(v)$ be as in Definition 6.8. We consider a neighborhood V of "$K'_\varepsilon \subset \widetilde{K}_\varepsilon$". More precisely, V satisfies the following:

- V is a Jordan domain;
- V contains no post-critical point;
- V is a neighborhood of $K'_\varepsilon \setminus U$;
- ∂V intersects $\partial\widetilde{K}_\varepsilon$ exactly twice, where $\partial V \cap \partial\widetilde{K}_\varepsilon = \{w_1, w_2\} \subset U$.

The arc $\partial\widetilde{K}_\varepsilon \setminus \{w_1, w_2\}$ contained in V is now deformed to one contained in U by an isotopy rel. ∂V as in Theorem 5.1. This isotopy deforms $\widetilde{K}_\varepsilon$ to K_ε. □

7. Construction of H^0

The 0-th pseudo-isotopy H^0 as required in §3 is constructed here, and thus the first approximation γ^1 of the Peano curve.

Consider two oriented Jordan curves $\mathcal{C}, \mathcal{C}' \subset S^2$. We say that $\mathcal{C}$ and $\mathcal{C}'$ are *orientation-preserving isotopic rel. A* if there is an isotopy $H\colon S^2 \times [0,1] \to S^2$ rel. A, with $H_0 = \mathrm{id}_{S^2}$, such that H_1 maps $\mathcal{C}$ orientation preserving to $\mathcal{C}'$.

We construct a connection of 1-tiles with the following properties.

Definition 7.1. (Properties of connections)

(C1) The associated white connection graph (§6.3) is a spanning tree K.

(C2) The Jordan curve ∂K_ε is orientation-preserving isotopic to $\mathcal{C} = \gamma^0$ rel. post, where K_ε is a geometric representation of K, see Lemma 6.23.

Here ∂K_ε is positively oriented as boundary of K_ε, recall that $\mathcal{C}$ is positively oriented as boundary of the white 0-tile X^0_w.

LEMMA 7.2. *A connection of* 1*-tiles satisfies properties* (C1) *and* (C2) *if and only if there exists a pseudo-isotopy H^0 as in Definition* 3.2.

Proof. ($\Rightarrow$) Concatenate an isotopy $\widetilde{H}$ rel. post that deforms $\mathcal{C}$ to ∂K_ε (orientation preserving) with a pseudo-isotopy rel. post that deforms ∂K_ε in a neighborhood $U(v)$ (as in (6.5)) of each 1-vertex as in Lemma 6.6. This yields a pseudo-isotopy rel. post that clearly satisfies $(H^0\,1)$–$(H^0\,4)$. Since $\widetilde{H}_1$ maps $\mathcal{C}$ orientation preserving to ∂K_ε, it follows that every 1-edge in the first approximation γ^1 (constructed via H^0 as in §3.3) is positively oriented. It follows from Lemma 3.11 that $(H^0\,5)$ is satisfied.

($\Leftarrow$) Let $\gamma^1 = H^0_1(\gamma^0)$ be the Eulerian circuit constructed from H^0 as in §3.3. By Lemma 6.15 we can reconstruct the connection at each 1-vertex from γ^1. It is a cnc-partition by Lemma 6.3. Since γ^1 contains all 1-edges, all white 1-tiles are connected. Furthermore $\gamma^1_\varepsilon := H^0_{1-\varepsilon}(\gamma^0)$ is a Jordan curve, and thus it follows from Lemma 6.23 that the white connection graph is a spanning tree, i.e., (C1). Finally γ^1_ε is clearly isotopic to γ^0 rel. post. From $(H^0\,5)$ and Lemma 3.11 it follows that the orientation on γ^1_ε induced by $\mathcal{C}$ and $H^0_{1-\varepsilon}$ agrees with the orientation of γ^1_ε as boundary of (a geometric representation of the white spanning tree) K_ε. Thus (C2) holds. □

Let us note the following immediate consequence.

THEOREM 7.3. *Let $F\colon S^2 \to S^2$ be an expanding Thurston map. The following two equivalent conditions are* sufficient *for the existence of an* (*onto*) *invariant Peano curve $\gamma\colon S^1 \to S^2$ as in Theorem* 1.1:

(1) *There is a Jordan curve $\mathcal{C} \supset$ post and a pseudo-isotopy H^0 in Definition* 3.2.

(2) *There is a Jordan curve* $\mathcal{C} \supset \text{post}$ *and a connection of* 1*-tiles satisfying the properties from Definition* 7.1.

In [Me1] it will be shown that the same conditions are sufficient to ensure that F arises as a mating. Furthermore, the polynomials p_1 and p_2 into which F unmates, may then be obtained by an explicit algorithm. More precisely, the *critical portraits* of p_1 and p_2 may be obtained from the vector l considered in §4.1, see [Me2].

The proof of Theorem 1.1 will be finished by constructing the white connection as in Definition 7.1.

Let us first note the following, which is an immediate consequence of the proof of the previous lemma. Assume that a connection of 1-tiles satisfying (C1) and (C2) is given. Let H^0 be a corresponding pseudo-isotopy from Lemma 7.2.

LEMMA 7.4. *The first approximation* γ^1 (*viewed as an Eulerian circuit*) *constructed from* H^0 *as in* §3.3 *is equal to the boundary circuit of the* (*white*) *spanning tree* K (*see Lemma* 6.23).

The main work in constructing the connection as desired lies in ensuring property (C2).

The starting point is to take a sufficiently high iterate $F = f^n$ such that there is an F-invariant Jordan curve $\mathcal{C} \supset \text{post}$ and the 1-tiles defined in terms of $(F, \mathcal{C})$ (i.e., closures of components of $S^2 \setminus F^{-1}(\mathcal{C})$) are sufficiently small. We require two separate conditions, since they are needed in distinct parts of the construction; they could be expressed as a single one. In fact, the second condition is only given later, when the suitable description becomes available.

LEMMA 7.5. *For each sufficiently high* $n \in \mathbb{N}$ *there is a Jordan curve* $\mathcal{C}$ *with* $\text{post} \subset \mathcal{C}$ *satisfying the following condition*:

- $\mathcal{C}$ *is* invariant *for the iterate* $F = f^n$. *This means that* $F(\mathcal{C}) \subset \mathcal{C}$.

The 1*-tiles for* $(F, \mathcal{C})$ *satisfy the following conditions*:

- *There is no* 1*-tile* X *that* joins opposite sides *of* $\mathcal{C}$. *This means that no* 1*-tile* X *meets disjoint* 0*-edges in the case* $\#\text{post} \geqslant 4$, *and no* 1*-tile* X *intersects all three* 0*-edges in the case* $\#\text{post} = 3$.
- *The* 1*-tiles do not form a* link *in the sense of Definition* 7.12.

This is essentially [BM, Theorem 14.2], see also [CFP3]. A proof of this lemma is given in §7.3, here we show how the arguments in [BM] are slightly adjusted to obtain the statement in the above form.

The iterate $F = f^n$ as well as the F-invariant Jordan curve $\mathcal{C}$ as above will be fixed from now on, tiles are defined in terms of $(F, \mathcal{C})$.

Let us first give a slightly incomplete outline of the construction. Recall that X^0_w and X^0_b are the white and black 0-tiles, respectively; they are both bounded by the invariant curve $\mathcal{C}$. We consider a spanning tree of white 1-tiles in X^0_w. Then we consider a spanning tree of black 1-tiles in X^0_b, the complementary white 1-tiles in X^0_b form ("homotopically") trivial trees in the sense of Definition 6.25. These (white) trivial trees (in X^0_b) are then attached to the white spanning tree in X^0_w.

This construction has to be slightly adjusted for the following reason: the white 1-tiles in X^0_w (as well as the black 1-tiles in X^0_b) need not be connected. So there are no *spanning* trees as described before.

7.1. Decomposing X^0_w

Here we decompose the *white* 0-tile X^0_w into white trees.

Consider the white 1-tiles in X^0_w. We assume in the next lemma that they are all *connected* at all 1-vertices v in the *interior* of X^0_w, and *disconnected* at all 1-vertices on $\mathcal{C}$. The resulting white connection graph may not be connected.

LEMMA 7.6. *The white connection graph in X^0_w as above has exactly one (white) cluster that intersects all sides (0-edges).*

Proof. Let K be a (white) cluster in X^0_w as above. Consider one component B (in the standard topological sense) of $X^0_w \setminus K$. We call the set $a := \partial B \cap K$ a *boundary arc* of K.

Claim 1. Every boundary arc a as above is contained in a single black 1-tile.

Clearly a is a union of 1-edges. Either a starts and ends at two distinct 1-vertices $v, w \in \mathcal{C}$, or a is a closed curve. Let $E, E' \ni v$ be two 1-edges contained in a which are consecutive in ∂B, where $v \notin \mathcal{C}$ is a 1-vertex. Note that by construction all white 1-tiles $X_j \ni v$ are connected at v. Thus E and E' are contained in the same black 1-tile. The claim follows.

Assume now that a is a closed curve. Then a is a Jordan curve in the boundary of a single black 1-tile. Thus the corresponding component B is the interior of a single black 1-tile. Hence a does not separate K from any other distinct white cluster K' in X^0_w.

We call a black 1-tile $Y \subset X^0_w$ *non-trivial* if $Y \cap \mathcal{C}$ contains at least two 1-vertices. A *complementary component* of Y is the closure of a component $X^0_w \setminus Y$.

Claim 2. Let $X, X' \subset X^0_w$ be two distinct white 1-tiles. Then X and X' are contained in distinct white clusters $K, K' \subset X^0_w$ if and only if there is a black 1-tile $Y \subset X^0_w$ such that X and X' are contained in complementary components of Y.

The implication ($\Leftarrow$) is clear. To see the other implication we note that if X' is contained in a cluster distinct from the cluster $K \supset X$, then X' has to be contained in the closure of one component of $X_w^0 \setminus K$. Such a component is separated from K by a boundary arc a. However, if a does not contain two 1-vertices $v, w \in \mathcal{C}$, this component is a single black 1-tile, meaning that it does not contain X'. Otherwise X' is separated from X by the black 1-tile Y containing a, proving the claim.

Recall that we assumed that no 1-tile joins opposite sides of $\mathcal{C}$ (see Lemma 7.5). Thus for every non-trivial black 1-tile Y there is a complementary component of Y, denoted by K_Y, that intersects all 0-edges.

We now define $\overline{K} := \bigcap_Y K_Y$, where the intersection is taken over all non-trivial black 1-tiles $Y \subset X_w^0$. Since two non-trivial black 1-tiles $Y, Y' \subset X_w^0$ do not cross, it follows that $\overline{K}$ intersects all 0-edges.

By Claim 2 it follows that all white 1-tiles contained in $\overline{K}$ are connected, i.e., belong to the same cluster denoted by K.

Assume that $\overline{K}$ intersects a given 0-edge E^0 in a 1-edge E. This cannot happen if E is contained in a black 1-tile $Y \subset X_w^0$, since Y would be non-trivial, and the corresponding set K_Y does not contain E. Thus E is contained in a white 1-tile, which is in $\overline{K}$.

If $\overline{K}$ intersects E^0 only in a 1-vertex v, there is a boundary arc $a \subset \partial \overline{K}$ containing v. Let $Y \subset X_w^0$ be the corresponding non-trivial black 1-tile containing a. Let $E \subset a$ be the 1-edge containing v. Since E is not in $\mathcal{C}$, the white 1-tile containing E is in $\overline{K}$.

This means that there is a white 1-tile in K that intersects E^0. □

In each white cluster in X_w^0 define a spanning tree (see Definition 6.11). The spanning tree in the cluster from Lemma 7.6 is called the *main tree* K_M, the spanning trees in the other clusters are called the *secondary trees* in X_w^0. The *connections* at all 1-vertices $v \in X_w^0 \setminus \mathcal{C}$ are thus *defined*, they will not be changed any more in the construction.

Let $\mathcal{E}$ be the boundary circuit of the main tree K_M (see Definition 6.21 and Lemma 6.23). Let $v_0, ..., v_{N-1}$ be the 1-vertices on $\mathcal{C}$ that $\mathcal{E}$ visits (in this order). Note that a 1-vertex v may appear several times in this list.

Notation. Given points $v, w \in \mathcal{C}$, denote by $[v, w]$ (resp. (v, w)), the closed (resp. open) positively oriented arc on $\mathcal{C}$ from v to w. Note that $(v, v) = \varnothing$.

LEMMA 7.7. *The points* $\{v_i\}_{i=0}^{N-1}$ *satisfy the following conditions* (*indices are taken* mod N *here*):

(1) *Each* (*open*) *arc* (v_i, v_{i+1}) *contains no point* v_l. *This means that the points* $\{v_i\}_{i=0}^{N-1}$ *are positively oriented on* $\mathcal{C}$.

(2) *The points* v_i *and* v_{i+1} *are not contained in disjoint* 0*-edges, in particular each* 0*-edge contains at least one point* v_i.

(3) *For all i there is a black 1-tile $Y \ni v_i, v_{i+1}$.*

(4) *Let K be a secondary tree in X_w^0. Then there is an arc $[v_i, v_{i+1}]$ such that*

$$K \cap \mathcal{C} \subset [v_i, v_{i+1}].$$

Proof. (1) Let $K_{M,\varepsilon}$ be a geometric representation of K_M as in Lemma 6.23 (3). The path γ_i on $\mathcal{E}$ between v_i and v_{i+1} is then represented by a Jordan arc $\gamma_{i,\varepsilon}$ with endpoints $v_{i,\varepsilon}$ and $v_{i+1,\varepsilon}$ such that $|v_i - v_{i,\varepsilon}|$ and $|v_{i+1} - v_{i+1,\varepsilon}|$ are arbitrarily small. Since all white 1-tiles are disconnected at every 1-vertex $v \in \mathcal{C}$, we may assume that $v_{i,\varepsilon} \in \mathcal{C}$ and $\gamma_{i,\varepsilon} \subset X_w^0$ for all i.

The arcs $\gamma_{i,\varepsilon}$ are non-crossing, and thus the points $\{v_{i,\varepsilon}\}_{i=0}^{N-1}$ are ordered cyclically or anti-cyclically on $\mathcal{C}$. Hence the points $\{v_i\}_{i=0}^{N-1}$ are ordered cyclically or anti-cyclically on $\mathcal{C}$.

The winding number of $\mathcal{E}$ around $x \notin \mathcal{E}$ is 1 if and only if x is in the interior of a white 1-tile of the main tree. This follows from an inductive argument as in Corollary 6.20.

Assume that the points $\{v_i\}_{i=0}^{N-1}$ are ordered anti-cyclically on $\mathcal{C}$. Let $\mathcal{C}_i$ be the (positively oriented) arc on $\mathcal{C}$ between v_i and v_{i+1}. Then $\gamma_i + \mathcal{C}_i$ has winding number 0 around any point x in the interior of a 1-tile of the main tree. Thus $\mathcal{E} + \mathcal{C}$ has winding number 0 around such an x. This is a contradiction.

(3) Consider v_i and v_{i+1}. Then

- either $v_i = v_{i+1}$ in which case the statement is trivial;
- or $[v_i, v_{i+1}]$ is a 1-edge, property (3) is then clear again;
- or v_i and v_{i+1} are the boundary points of a boundary arc a of K_M, as in Claim 1 from the proof of Lemma 7.6. In this case there is a black 1-tile $Y \supset a$.

(2) This follows immediately from (3) and the assumption that no 1-tile intersects disjoint 0-edges. Furthermore K_M intersects a 0-edge E if and only if it intersects it in some 1-vertex. The set of all 1-vertices in which K_M intersects $\mathcal{C}$ is equal to the set $\{v_i\}_{i=0}^{N-1}$. Thus, since K_M intersects each 0-edge, it follows that each 0-edge contains one point v_i.

(4) The reader is reminded of Claims 1 and 2 in the proof of Lemma 7.6. For every secondary component K there is an arc a contained in a (non-trivial) black 1-tile Y such that $\operatorname{int} K$ is in the component of $X_w^0 \setminus a$ not intersecting all 0-edges. Let v_i and v_{i+1} be the endpoints of a (see the discussion from (3)), then

$$K \cap \mathcal{C} \subset [v_i, v_{i+1}]. \qquad \square$$

7.2. Decomposing X_b^0

We now decompose the *black* 0-tile X_b^0. Consider the *black* 1-tiles in X_b^0. Construct clusters of black 1-tiles as before. Namely assume that all black 1-tiles are connected at each 1-vertex $v \in X_b^0 \setminus \mathcal{C}$. All (black and white) 1-tiles in X_b^0 are disconnected at each 1-vertex $v \in \mathcal{C}$. Pick a spanning tree in each cluster (of black 1-tiles in X_b^0). This *defines the connections* at all 1-vertices $v \in X_b^0 \setminus \mathcal{C}$, they will not be changed anymore in the construction. As in Lemma 7.6, there is exactly one such tree (of black 1-tiles in X_b^0) that intersects all 0-edges.

Consider now the *white* 1-tiles in X_b^0. The connections at 1-vertices $v \in X_b^0 \setminus \mathcal{C}$ are already given (they are all disconnected at each 1-vertex $v \in \mathcal{C}$).

LEMMA 7.8. *Every white cluster K in X_b^0 as above*

- *is a tree*;
- *furthermore*

$$K \cap \mathcal{C} \subset [v, w],$$

where $v, w \in \mathcal{C}$ are 1-vertices contained in a single *white 1-tile.*

Proof. Assume that K is not a tree. Then K has at least two distinct boundary circuits (see Lemma 6.23).

Claim. There is a (white) 1-tile $X \subset K$ and a 1-vertex $v \in X$ at which 1-edges $E, E' \subset X$, from distinct boundary circuits, intersect.

If the claim were not true we could partition K into 1-tiles containing 1-edges from distinct boundary circuits. These partitions, and therefore K, would not be connected by Lemma 6.15.

Let v, E and E' be as in the claim. Note that $v \notin \mathcal{C}$, since all 1-tiles are disconnected at $\mathcal{C}$.

Consider the *black* 1-tiles $Y, Y' \subset X_b^0$ that contain E and E'. Let $K_b, K_b' \subset X_b^0$ be the black clusters containing Y and Y', respectively. Since they are by assumption trees, they are distinct (again by Lemma 6.23).

On the other hand the (black) 1-tiles Y and Y' were connected at v, before spanning trees were picked. This means that they are in the same tree ($K_b = K_b'$), which is a contradiction.

The arguments from Lemmas 7.6 and 7.7 apply verbatim to X_b^0. Thus there is a unique black tree $K_{M,b} \subset X_b^0$ that intersects each 0-edge. Let $w_0, ..., w_{\widetilde{N}}$ be the 1-vertices that the boundary circuit of $K_{M,b}$ visits (in this order); note that these points are ordered positively on $\mathcal{C}$ (recall that 1-edges in a boundary circuit of a cluster were always *positively oriented* as boundary of the *white* 1-tiles they are contained in, regardless of the color of

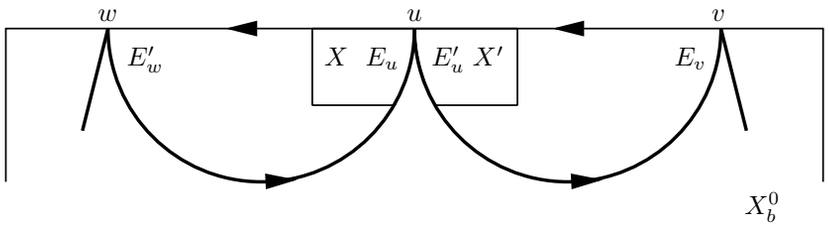

Figure 6. Illustration to Lemma 7.11.

the cluster). As in Lemma 7.7 one obtains that the endpoints w_i and w_{i+1} of each arc $[w_i, w_{i+1}]$ are contained in a single white 1-tile. Each set $K \cap \mathcal{C}$ is contained in one such arc $[w_i, w_{i+1}]$. □

We call the (white) trees from the previous lemma the *secondary trees* in X_b^0. Let us record the following immediate consequence of Lemmas 7.7 and 7.8.

LEMMA 7.9. *No secondary tree (in X_w^0 or X_b^0) intersects disjoint* 0*-edges.*

We will need to break up boundary circuits.

Definition 7.10. (Subpaths of boundary circuits) Let $\mathcal{E}$ be a boundary circuit and $D, E \subset \mathcal{E}$ be two 1-edges. Then $\mathcal{E}(D, E)$ is the positively oriented subpath (of 1-edges) of $\mathcal{E}$ with initial 1-edge D and terminal 1-edge E. Note that $\mathcal{E}(E, E) = E$.

In the next lemma we consider a secondary tree $K \subset X_b^0$ with boundary circuit $\mathcal{E}$. Consider two distinct 1-vertices $v, w \in (\mathcal{E} \cap \mathcal{C})$. Let $E_v, E'_v \subset \mathcal{E}$ and $E_w, E'_w \subset \mathcal{E}$ be succeeding 1-edges at v and w.

Given $x, y \in \mathcal{C}$, in the following we write $[x, y]_b$ for the boundary arc on $\mathcal{C} = \partial X_b^0$ between x and y that is *positively oriented* with respect to X_b^0 (and thus negatively oriented on $\mathcal{C}$).

LEMMA 7.11. *The subpath $\mathcal{E}(E'_w, E_v)$ does not intersect $[v, w]_b \setminus \{v, w\}$.*

Proof. The situation is illustrated in Figure 6. Assume that the statement is false, i.e., that $\mathcal{E}(E'_w, E_v)$ intersects $[v, w]_b \setminus \{v, w\}$ in a 1-vertex $u (\in \mathcal{C})$. Let $E_u, E'_u \subset \mathcal{E}(E'_w, E_v)$ be the succeeding 1-vertices at u. Then $\operatorname{int} K$ is divided into points bounded by (having winding number 1) $\mathcal{E}(E'_w, E_u) \cup [u, w]_b$ and $\mathcal{E}(E'_u, E_v) \cup [v, u]_b$.

Thus E_u and E'_u are contained in different white 1-tiles $X, X' \subset K$. Hence X and X' are connected at u. This contradicts the construction of K, where no 1-tiles are connected at any 1-vertex in $\mathcal{C}$. □

7.3. Connecting the trees

The secondary trees are attached to the main tree at the 1-vertices on $\mathcal{C}$.

Initially all white 1-tiles are disconnected at each 1-vertex $v\in\mathcal{C}$. To use the results from §6.5 we want the connections at all 1-vertices $v\in\mathcal{C}$ to be cnc-partitions. Thus we now assume that all black 1-tiles are connected at each 1-vertex $v\in\mathcal{C}$, and hence the connections form cnc-partitions as desired.

We first add secondary trees to ensure that all points of post are contained in the main tree. Consider the main tree K_M (in X_w^0) from §7.1. Let $v_0,...,v_{N-1}$ be the 1-vertices on $\mathcal{C}$ along the boundary circuit $\mathcal{E}$ of K_M, see Lemma 7.7.

Consider one (positively oriented) 0-edge E^0 with terminal point $p\in$post, let v_i be the last of the 1-vertices as above on E^0. Then either $v_i=p$ or $v_i\notin$post.

- If $v_i=p$, let $E_j\subset\mathcal{E}$ be the last 1-edge with terminal point v_i, and $E_{j+1}\subset\mathcal{E}$ be the succeeding 1-edge. The connection at p is now marked by E_j and E_{j+1}, see Corollary 6.14.
- If $v_i\notin$post, consider the 1-edge $E=[v_i,w]\subset E^0$ succeeding v_i in $\mathcal{C}$. Let K be the secondary cluster containing E. This means that K contains the (unique) white 1-tile containing E. Add K to the main tree K_M at v_i. Note that no white 1-tile is connected at v_i, so this is possible by Lemma 6.19. We obtain a new main tree, still denoted by K_M.
- Repeat the above procedure till the main tree contains p.

The added secondary components will only intersect the 0-edges preceding and succeeding E^0. Then we want to use the same procedure on the other 0-edges. There is one problem however: we may encounter a 1-edge E as above that belongs to a secondary component already added before (when the above procedure was applied to a *different* 0-edge $\widetilde{E}^0$). This may lead to a boundary circuit of K_M in which the post-critical points are traversed not in the same order as in $\mathcal{C}$, violating (C2).

To elaborate, let $E_1^0=E^0$, E_2^0 and E_3^0 be the 0-edges succeeding E_1^0. Let q be the terminal point of E_2^0, and v_j be the last of the points $\{v_i\}_{i=0}^{N-1}$ on E_2^0. The described problem occurs if there is a secondary component K containing a 1-edge in $[v_i,p]\subset E_1^0$ and a 1-edge in $[v_j,q]\subset E_2^0$. By Lemma 7.7 (3) and (4), as well as Lemma 7.8, this can only happen if there are white and black 1-tiles *linked* in a certain way, see Figure 7.

Definition 7.12. (Link) A *link* means that the following exist:

- a (black) 1-tile X_1 containing $v_i\in E_1^0$ and intersecting E_2^0;
- a (black) 1-tile X_2 containing $v_j\in E_2^0$ and intersecting E_3^0;
- a (white) 1-tile Y intersecting $[v_i,p]\subset E_1^0$ and $[v_j,q]\subset E_2^0$.

Thus we have given the description of the last property in Lemma 7.5.

Proof of Lemma 7.5. We essentially recall the proof of [BM, Theorem 14.2], see also [BM, Theorem 14.3] and its proof.

More precisely, we break up each 0-edge into two 0-arcs and use the same arguments as in [BM] to show that there is an f^n-invariant curve $\widetilde{\mathcal{C}}$, such that no n-tile connects

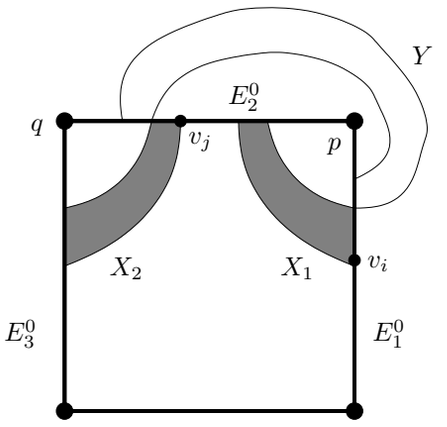

Figure 7. A link.

disjoint 0-arcs.

Let k_0 be a fixed integer such that there are at least twice as many k_0-vertices as post-critical points (recall that the number of n-vertices grows exponentially). Fix a Jordan curve $\mathcal{C}\subset S^2$ such that post$\subset\mathcal{C}$; additionally $\mathcal{C}$ has the property that each arc on $\mathcal{C}$ between two consecutive post-critical points p and q contains a k_0-vertex distinct from p and q. Let P be the set of all such k_0-vertices and post-critical points. The points in P divide divide $\mathcal{C}$ into 0-*arcs*. Each 0-edge on $\mathcal{C}$ is divided into two 0-arcs.

Consider the n-tiles given in terms of $(f,\mathcal{C})$, where $n\geqslant k_0$. Since f is expanding, n-tiles get arbitrarily small, meaning that $\max_{X\in\mathbf{X}^n}\operatorname{diam}X\to 0$ as $n\to\infty$. This implies, by [BM, Lemma 11.17], that there is an $n_0\geqslant k_0$ such for all $n\geqslant n_0$ there is a Jordan curve $\mathcal{C}'\subset f^{-n}(\mathcal{C})$ isotopic to $\mathcal{C}$ rel. P (and thus $P\subset\mathcal{C}'$). Furthermore, no n-tile joins opposite sides of $(\mathcal{C}',P)$. This means that there is no n-tile that intersects disjoint closed 0-*arcs* into which P divides the curve $\mathcal{C}'$.

Let $H\colon S^2\times[0,1]\to S^2$ be an isotopy rel. P that deforms $\mathcal{C}$ to $\mathcal{C}'$, i.e., $H_1(\mathcal{C})=\mathcal{C}'$. Then $\widehat{F}:=H_1\circ f^n$ is a Thurston map, such that $\mathcal{C}'$ is $\widehat{F}$-invariant, since

$$\widehat{F}(\mathcal{C}')=H_1(f^n(\mathcal{C}'))\subset H_1(\mathcal{C})=\mathcal{C}'.$$

The 1-tiles for $(\widehat{F},\mathcal{C}')$ are exactly the n-tiles for $(f,\mathcal{C})$. Since no 1-tile for $(\widehat{F},\mathcal{C}')$ joins opposite sides of $\mathcal{C}'$, we can choose $\widehat{F}$ to be expanding, see [BM, Corollary 13.18]. Furthermore, no 1-tile for $(\widehat{F},\mathcal{C}')$ intersects disjoint 0-arcs of $\mathcal{C}'$.

The map $\widehat{F}$ is Thurston equivalent to f^n. Since they are both expanding, they are actually topologically conjugate, i.e., there is a homeomorphism $h\colon S^2\to S^2$, such that $h\circ\widehat{F}\circ h^{-1}=f^n$ (see [BM, Theorem 11.4]). Let $\widetilde{\mathcal{C}}:=h(\mathcal{C}')$. Note that $\widetilde{\mathcal{C}}$ is f^n-invariant,

since

$$f^n(\widetilde{\mathcal{C}}) = h \circ \widehat{F} \circ h^{-1}(\widetilde{\mathcal{C}}) = h \circ \widehat{F}(\mathcal{C}') \subset h(\mathcal{C}') = \widetilde{\mathcal{C}}.$$

We call the images of 0-arcs on $\mathcal{C}'$ by h the *0-arcs of* $\widetilde{\mathcal{C}}$. The images of 1-tiles for $(\widehat{F}, \mathcal{C}')$ by h are the *n-tiles for* $(f, \widetilde{\mathcal{C}})$. It follows that no n-tile (for $(f, \widetilde{\mathcal{C}})$) intersects disjoint 0-arcs of $\widetilde{\mathcal{C}}$. Recall that each 0-edge of $\widetilde{\mathcal{C}}$ contains exactly two 0-arcs.

With this choice of $F = f^n$ and $\widetilde{\mathcal{C}}$, we will show that a link as in Definition 7.12 cannot occur. Let A_j^- and A_j^+ be the two 0-arcs in E_j^0, where A_j^+ succeeds A_j^- in $\mathcal{C}$. Then the white 1-tile Y has to intersect E_2^0 in $\operatorname{int} A_2^-$, while the black 1-tile X_2 has to intersect E_2^0 in $\operatorname{int} A_2^+$. The claim follows. □

Since we assumed that $F = f^n$ and $\mathcal{C}$ were chosen to satisfy the properties from Lemma 7.5, there are no links. Thus the following holds. Let K be a secondary cluster added (to the main tree) when considering the 0-edge E^0, and let $\widetilde{K}$ be a secondary cluster added when considering a distinct 0-edge $\widetilde{E}^0$.

COROLLARY 7.13. *The secondary clusters K and $\widetilde{K}$, given as above, are distinct.*

Thus we can apply the above procedure to each 0-edge. This yields the (new) main tree (still denoted by K_M). Note that $K_M \supset \text{post}$ by construction. More precisely, K_M contains the marked succeeding 1-edges $E(p)$ and $E'(p)$ at each post-critical point p. This means that $K_{M,\varepsilon} \supset \text{post}$ (for any geometric representation $K_{M,\varepsilon}$ of K_M), see Lemma 6.22.

7.4. The main tree is in the right homotopy class

Recall from Definition 7.10 how a boundary circuit $\mathcal{E}$ was broken up into subpaths. Assume that $\mathcal{E}$ contains the marked succeeding 1-edges $E(p)$ and $E'(p)$ at $p \in \text{post}$, as well as the marked succeeding 1-edges $E(q)$ and $E'(q)$ at $q \in \text{post}$. Then

$$\mathcal{E}(p, q) := \mathcal{E}(E'(p), E(q))$$

and, for any 1-edge $E \subset \mathcal{E}$,

$$\mathcal{E}(p, E) := \mathcal{E}(E'(p), E) \quad \text{and} \quad \mathcal{E}(E, q) := \mathcal{E}(E, E(q)).$$

Furthermore, if E and E' are succeeding in $\mathcal{E}$, we define

$$\mathcal{E}(E', E) = \varnothing.$$

We are now ready to finish the proof of Theorem 1.1. K_M is the main tree as constructed in §7.3.

LEMMA 7.14. *The main tree K_M is in the right homotopy class, i.e., satisfies* (C2).

Proof. Let $\mathcal{E}$ be the boundary circuit of K_M. Consider a 0-edge E^0 with initial and terminal points $p, q \in \text{post}$; and the subpath $\mathcal{E}(p,q) \subset \mathcal{E}$ as defined above. We will prove the following statement.

Claim 1. $\mathcal{E}(p,q)$ does not intersect any 0-edge disjoint with E^0.

The statement of the lemma follows quickly from this claim. Namely consider a geometric representation $K_{M,\varepsilon}$ of K_M, where the neighborhoods $U(v)$ from (6.5) were chosen in such a way that $U(v) \cap \mathcal{C} = \varnothing$ whenever $v \notin \mathcal{C}$. It follows from Claim 1 that the (positively oriented) arc on $\partial K_{M,\varepsilon}$ from p to q does not intersect 0-edges disjoint from E^0. Theorem 5.5 now finishes the proof.

To prove Claim 1 we go through the construction of K_M. Consider $K_{M,0}$, the main tree from §7.1 (before any secondary tree was added), with boundary circuit $\mathcal{E}_0$. Let $w_0, w_1 \in E^0$ be the first and last 1-vertices on E^0 that $\mathcal{E}_0$ visits; and $E_0, E_0' \subset \mathcal{E}_0$, as well as $E_1, E_1' \subset \mathcal{E}_0$, be the first and last succeeding 1-edges at w_0 and w_1. Consider $\mathcal{E}_0(E_0', E_1)$, and note that $\mathcal{E}(E_0', E_1) = \mathcal{E}(E_0', E_0) = \varnothing$ in the case when $\mathcal{E}_0$ intersects E^0 only once. This subpath does not intersect any 0-edge disjoint from E^0 by Lemma 7.7 (in fact it may only intersect adjacent 0-edges if $w_0 = p$ or $w_1 = q$).

Note that $\mathcal{E}_0(E_0', E_1)$ is a subpath of $\mathcal{E}(p,q)$, or $\mathcal{E}(E_0', E_1) = \mathcal{E}_0(E_0', E_1)$, which we call the *middle* subpath of $\mathcal{E}(p,q)$. The remaining subpaths of $\mathcal{E}(p,q)$ are given as follows. Let D_0 be the 1-edge preceding E_0' in $\mathcal{E}$ and D_1 be the 1-edge succeeding E_1 in $\mathcal{E}$. Then the *initial* subpath of $\mathcal{E}(p,q)$ is $\mathcal{E}(p, D_0)$ (connecting p to $\mathcal{E}(E_0', E_1)$), and the *terminal* subpath of $\mathcal{E}(p,q)$ is $\mathcal{E}(D_1', q)$ (connecting $\mathcal{E}(E_0', E_1)$ to q). Note that the initial and/or the terminal subpath may be empty. We focus our attention for now on the terminal subpath.

Let $K_1, \ldots, K_m$ be the secondary trees that were added in §7.3 to "reach" the post-critical point q. The last secondary tree K_m contains the post-critical point q by construction.

Let $K_{M,j}$ be the main tree obtained when the secondary tree K_j was added to $K_{M,j-1}$ at the 1-vertex $w_j \in E^0$. Let $E_j, E_j' \subset K_{M,j-1}$ and $D_j, D_j' \subset K_j$ be the succeeding 1-edges associated with adding K_j to $K_{M,j-1}$ by Lemma 6.18. Note that by construction the 1-vertices of $K_{M,j}$ closest to q on the 0-edge E^0 are contained in $K_j \subset K_{M,j}$. Hence K_{j+1} is attached to $K_{M,j}$ at 1-edges contained in K_j.

Thus, if we denote by $\mathcal{E}_j$ the boundary circuit of the secondary tree K_j, we have $D_j, D_j', E_{j+1}, E_{j+1}' \in \mathcal{E}_j$, and $\mathcal{E}_j$ consists of the two (non-empty) subpaths $\mathcal{E}_j(D_j', E_{j+1})$ and $\mathcal{E}_j(E_{j+1}', D_j)$ for $j = 1, \ldots, m-1$. We break $\mathcal{E}_m$ up into the (non-empty) subpaths $\mathcal{E}_m(D_m', q)$ and $\mathcal{E}_m(q, D_m)$.

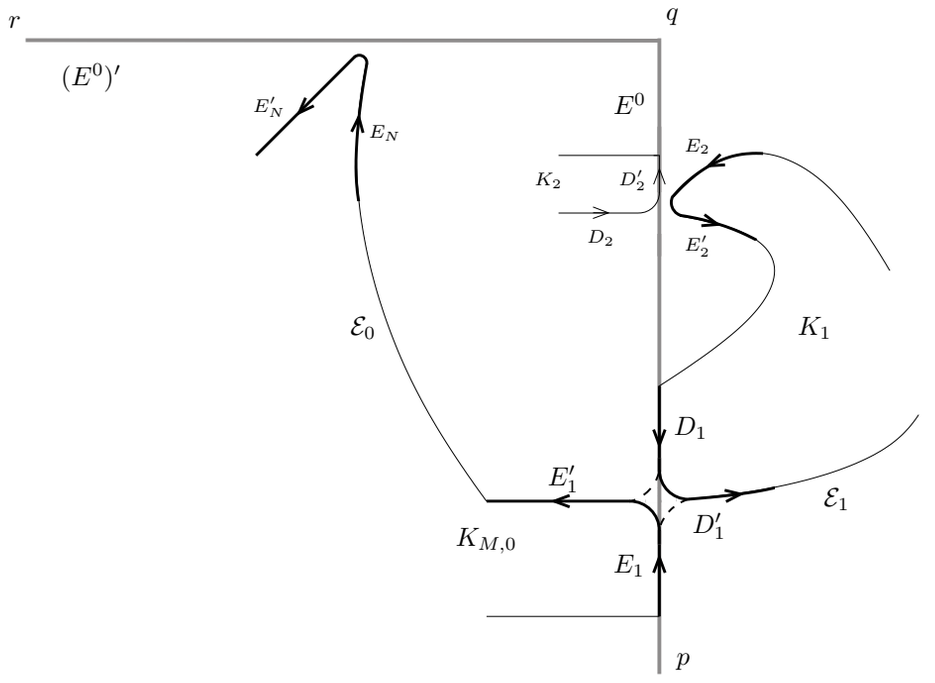

Figure 8. Adding K_j to $K_{M,j-1}$.

Lemma 6.18 implies that the terminal subpath $\mathcal{E}(D'_1, q)$ is given as the concatenation of (subpaths from the boundary circuits from the secondary trees K_j)

$$\mathcal{E}_1(D'_1, E_2), \quad \mathcal{E}_2(D'_2, E_3), \quad ..., \quad \mathcal{E}_m(D'_m, q), \tag{7.1}$$

see Figure 8. It follows from Lemma 7.9 that $\mathcal{E}(D'_1, q)$ does not intersect any 0-edge disjoint from E^0.

It remains to show that the initial subpath does not intersect a 0-edge disjoint from E^0.

Instead of looking at the initial subpath of $\mathcal{E}(p, q)$, we consider the initial subpath of $\mathcal{E}(q, r)$. Here r is the terminal point of the 0-edge $(E^0)'$ succeeding E^0. Let $E_N \subset \mathcal{E}_0$ be the first 1-edge intersecting $(E^0)'$ in a 1-vertex w_N. The initial subpath of $\mathcal{E}(q, r)$ is $\mathcal{E}(q, E_N)$; it is given as the concatenation of

$$\mathcal{E}_m(q, D_m), \quad \mathcal{E}_{m-1}(E'_m, D_{m-1}), \quad ..., \quad \mathcal{E}_1(E'_2, D_1), \quad \mathcal{E}_0(E'_1, E_N),$$

where D_j and E'_j are as above. These are the "complementary subpaths" to the ones in (7.1) (of the boundary circuits of the secondary trees K_j). See again Figure 8.

It remains to show that this path does not intersect a 0-edge disjoint from $(E^0)'$. Clearly $\mathcal{E}_0(E'_1, E_N)$ intersects $\mathcal{C}$ only at the endpoints, which are in E^0 and $(E^0)'$.

Recall that $\mathcal{E}_j(E'_{j+1}, D_j) \subset K_j$, where K_j does not intersect disjoint 0-edges. Thus $\mathcal{E}_j(E'_{j+1}, D_j)$ may only intersect E^0, $(E^0)'$, or the E^0-preceding 0-edge $\widetilde{E}^0$.

Claim 2. The subpath $\mathcal{E}_j(E'_{j+1}, D_j)$ does not intersect $\widetilde{E}^0$.

This is clear if $K_j \subset X^0_w$, since then $K_j \cap \mathcal{C} \subset [w_1, q] \cup [q, w_N]$ by Lemma 7.7 (4).

Assume now that $K_j \subset X^0_b$. Let w be the initial point of $\mathcal{E}_j(E'_{j+1}, D_j)$ and v be its terminal point. Note that, by construction, $w \in E^0$ is closer to q on E^0 than $v \in E^0$. From Lemma 7.11 it follows that $\mathcal{E}_j(E'_{j+1}, D_j) \subset [v, w] \subset E^0 \setminus \{p\}$. Claim 2 follows.

The argument that the initial subpath $\mathcal{E}(p, D_0)$ does not intersect 0-edges disjoint from E^0 is completely analogous. This completes the proof of Claim 1, and thus the proof of the lemma. □

We finish the construction of the main tree, i.e., of the connection of 1-tiles by adding the remaining secondary trees to the main tree arbitrarily, to form the spanning tree K_M. The previous lemma, together with Lemma 6.26, implies that K_M satisfies properties (C1) and (C2). Thus there is a pseudo-isotopy H^0 as required in Definition 3.2, by Lemma 7.2. This yields the invariant Peano curve by §3 and §4. The proof of Theorem 1.1 is thus finished.

8. Combinatorial construction of γ^n

The $(n+1)$-th approximation γ^{n+1} of the invariant Peano curve γ was constructed as a deformation of γ^n by H^n. Here H^n was the lift of the "initial pseudo-isotopy" H^0 by F^n. In this section we give an *alternative* way to construct γ^{n+1} from γ^n, namely in a purely *combinatorial* fashion.

Recall from Lemma 7.4 that the first approximation γ^1 may be obtained as the *boundary circuit* of the white spanning tree, defined via the *connection of* 1*-tiles*. Here we construct the *connection of* n*-tiles* (which will again satisfy (C1) and (C2)) in such a way that γ^n is the boundary circuit of the white tree of n-tiles. See Figure 2 for an illustration of the desired connections of n-tiles.

The connections of n-tiles *could* be constructed from the approximations γ^n (using Lemma 6.15). We do however take the opposite route here, namely we construct the connections inductively and show that their boundary circuits are the approximations as defined before.

8.1. Connection of n-tiles

We give the (inductive) description of the connection of n-tiles first, before showing that it has the desired properties.

Fix $n\geqslant 1$. Assume that the connection of n-tiles is given. This means that at each n-vertex v a cnc-partition $\pi_w^n(v)\cup\pi_b^n(v)$ is defined; if $v=p\in\mathrm{post}$, it is marked (see Definition 6.7). The connection satisfies properties (C1) and (C2), and the (single) boundary circuit is equal to the nth approximation γ^n (viewed as an Eulerian circuit).

Consider now an $(n+1)$-vertex v. The connection of $(n+1)$-tiles at v is defined as follows.

Case 1. v is not an n-vertex.

Note that this implies that v is *not a critical point*. Thus we can define the connection at v as the "pull-back" of the connection at $F(v)$.

More precisely, let $w:=F(v)$ $(\in\mathbf{V}^n)$. Let $X_0^n,\dots,X_{2m-1}^n$ be the n-tiles around w (labeled positively around w). Label the $(n+1)$-tiles $X_0^{n+1},\dots,X_{2m-1}^{n+1}$ around v in such a way that $F(X_j^{n+1})=X_j^n$, $j=0,\dots,2m-1$. Then

$$X_i^{n+1}\text{ and }X_j^{n+1}\text{ are connected at }v\iff X_i^n\text{ and }X_j^n\text{ are connected at }w. \tag{8.1}$$

In other words, the connection (of $(n+1)$-tiles) at v is defined by

$$\pi_w^{n+1}(v)\cup\pi_b^{n+1}(v):=\pi_w^n(w)\cup\pi_b^n(w).$$

Case 2. v is an n-vertex $(v\in\mathbf{V}^{n+1}\cap\mathbf{V}^n)$.

Then $p:=F^n(v)\in\mathrm{post}=\mathbf{V}^0$. Consider two white $(n+1)$-tiles $X^{n+1},Y^{n+1}\ni v$. They are connected (at v) if and only if

- either they are contained in the image of the *same* (white) n-tile X^n by the pseudo-isotopy H^n, namely
$$X^{n+1},Y^{n+1}\subset H_1^n(X^n),$$
and their images by F^n are connected, i.e., the 1-tiles $F^n(X^{n+1})$ and $F^n(Y^{n+1})$ are connected at p;
- or X^{n+1} and Y^{n+1} are contained in the images of *connected* n-tiles $X^n,Y^n\ni v$,
$$X^{n+1}\subset H_1^n(X^n)\quad\text{and}\quad Y^{n+1}\subset H_1^n(Y^n),$$
X^n and Y^n are connected at v, and X^{n+1} and Y^{n+1} both map to 1-tiles that are "connected to the marked succeeding 1-edges", i.e., the 1-tiles $F^n(X^{n+1})$ and $F^n(Y^{n+1})$ are connected at p to the white 1-tiles X^1 and $\widetilde{X}^1$ that *contain* the *marked* succeeding 1-edges E^1 and $\widetilde{E}^1$.

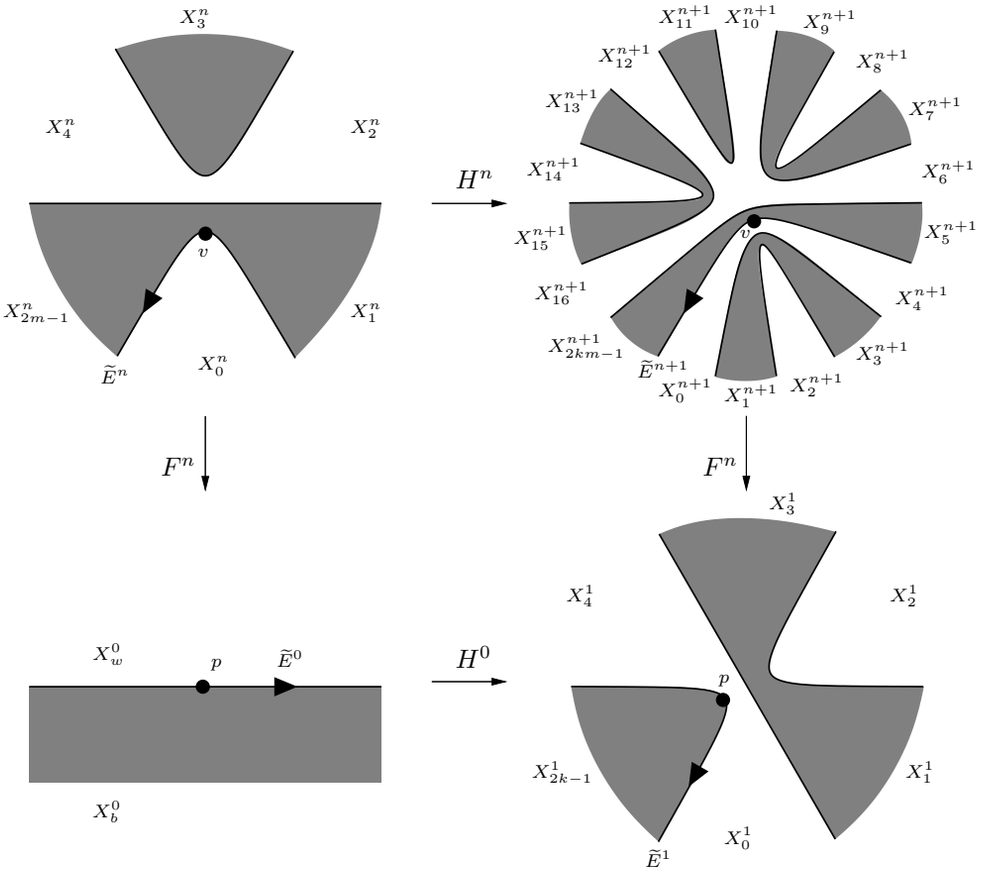

Figure 9. Inductive construction of connections.

The connection of black $(n+1)$-tiles at v is defined analogously to the above.

We will *formalize* the description above. To do this, we will first have to *label* the involved 1-tiles, n-tiles and $(n+1)$-tiles in a *consistent manner*. See Figure 9 for an illustration.

Recall from Lemma 3.6 that for each $(j+1)$-edge E^{j+1} there is a unique arc A^j contained in a j-edge E^j that is deformed by the pseudo-isotopy H^j to E^{j+1}. Since we will often want to keep track of where such an E^{j+1}-edge "comes from", in this case we use the *notation*

$$H^j\colon A^j \subset E^j \longrightarrow E^{j+1}.$$

We will single out one 0-, 1-, n- and $(n+1)$-edge. Let $\widetilde{E}^0$ be the 0-edge with initial point p ($\widetilde{E}^0$ is positively oriented as boundary of the white 0-tile X_w^0). The 1-edge $\widetilde{E}^1$ is the *marked* one with initial point p. Thus, there is an arc $\tilde{A}^0 \ni p$ such that

$$H^0\colon \tilde{A}^0 \subset \widetilde{E}^0 \longrightarrow \widetilde{E}^1.$$

We choose (arbitrarily) one n-edge $\widetilde{E}^n \ni v$ such that $F^n(\widetilde{E}^n)=\widetilde{E}^0$. Finally we choose the $(n+1)$-edge $\widetilde{E}^{n+1}\ni v$ such that there is an n-arc $\tilde{A}^n \ni v$ satisfying

$$H^n\colon \tilde{A}^n \subset \widetilde{E}^n \longrightarrow \widetilde{E}^{n+1}.$$

Let $2m$ be the number of n-tiles containing v (this means that $m=\deg_{F^n}(v)$) and $2k$ be the number of 1-tiles containing p. Then the number of $(n+1)$-tiles containing v is $2km$.

The 1-tiles $X_0^1,\dots,X_{2k-1}^1$ around p, the n-tiles $X_0^n,\dots,X_{2m-1}^n$ around v and the $(n+1)$-tiles $X_0^{n+1},\dots,X_{2km-1}^{n+1}$ around v are labeled positively (around p and v, respectively) and such that $\widetilde{E}^1\subset X_0^1$, $\widetilde{E}^n\subset X_0^n$ and $\widetilde{E}^{n+1}\subset X_0^{n+1}$.

Recall that white tiles are always labeled by even, and black tiles by odd indices. Thus X_0^1, X_0^n and X_0^{n+1} are all *white* tiles. This finishes the labeling.

The blocks b^{n+1} of the cnc-partition $\pi_w^{n+1}(v)\cup\pi_b^{n+1}(v)$ are defined as follows. For each block $b^1\in\pi_w^1(v)\cup\pi_b^1(v)$ and each $j=0,\dots,m-1$ there is a block

$$b^{n+1}=b_j^{n+1}(b^1)=b^1+2kj=\{i+2kj : i\in b^1\}. \tag{8.2}$$

This corresponds to the first part of the description above.

Now let $b_\star^1\in\pi_w^1(p)$ be the block containing 0; it contains indices of white 1-tiles that are connected to the marked succeeding 1-edges at p. The sets $b_j^{n+1}(b_\star^1)=b_\star^1+2kj$ are defined as in (8.2), they contain indices of $(n+1)$-tiles that are mapped to (1-tiles with indices in) $b_\star^1$ by F^n. For each block $b^n\in\pi_w^n(v)$ there is a block $b_\star^{n+1}\in\pi_w^{n+1}(v)$ given by

$$b_\star^{n+1}=b_\star^{n+1}(b^n):=\bigcup\{b_\star^1+2kj : 2j\in b^n\}. \tag{8.3}$$

This is the formal description of the second part described above.

In the same fashion, let $c_\star^1\in\pi_b^1(p)$ be the block containing $2k-1$. It contains indices of black 1-tiles connected to the marked succeeding 1-edges at p. For each block $c^n\in\pi_b^n(v)$ there is a block $c_\star^{n+1}\in\pi_b^{n+1}(v)$ given by

$$c_\star^{n+1}=c_\star^{n+1}(c^n):=\bigcup\{c_\star^1+2kj : 2j+1\in c^n\}. \tag{8.4}$$

The cnc-partition $\pi_w^{n+1}(v)\cup\pi_b^{n+1}(v)$ consists of all blocks $b_j^{n+1}(b^1)$ as in (8.2), with $b^1\neq b_\star^1$ and $b^1\neq c_\star^1$, as well as all blocks $b_\star^{n+1}=b^{n+1}(b^n)$ and $c_\star^{n+1}=c_\star^{n+1}(c^n)$ as above.

Case 3. $v\in$post.

Note that post$=\mathbf{V}^0\subset\mathbf{V}^n$. This case is thus a subcase of case 2. The cnc-partition $\pi_w^{n+1}(v)\cup\pi_b^{n+1}(v)$ is thus already constructed in case 2. It remains to *mark* it. Recall

that in case 2 the n-edge $\widetilde{E}^n$ with $F^n(\widetilde{E}^n)=\widetilde{E}^0$ was chosen *arbitrarily*. Now, however, we let $\widetilde{E}^n$ be the *marked* n-edge with initial point v.

The marked $(n+1)$-edge with initial point v is $\widetilde{E}^{n+1}$ (recall that there is an arc $\tilde{A}^n \ni v$ such that $H^n\colon \tilde{A}^n \subset \widetilde{E}^n \to \widetilde{E}^{n+1}$).

Alternatively, consider the blocks

$$b^{n+1}=b^{n+1}(0)\in\pi_w^{n+1}(v) \quad \text{and} \quad c^{n+1}=c^{n+1}(2km-1)\in\pi_b^{n+1}(v)$$

such that $0\in b^{n+1}$ and $2km-1\in c^{n+1}$. These two adjacent blocks mark the connection of $(n+1)$-tiles at p (see Corollary 6.14).

8.2. Properties of connections

Here we prove that the connections of n-tiles defined above have the desired properties.

PROPOSITION 8.1. *The connection of n-tiles as defined in* §8.1 *satisfies the following properties*:

(1) *each* $\pi_w^n(v)\cup\pi_b^n(v)$ *is a* cnc-partition;

(2) *the connection of n-tiles satisfies properties* (C1) and (C2) *from Definition* 7.1;

(3) *the* (*single*) boundary circuit *of the cluster of white n-tiles is* equal *to the* nth approximation γ^n (*viewed as an Eulerian circuit*).

Proof. To be able to keep the notation from §8.1, we will prove the statements for the connection of $(n+1)$-tiles.

(1) The statement will be proved by induction. Thus we assume that $\pi_w^n(w)\cup\pi_b^n(w)$ is a cnc-partition for each n-vertex w. Consider now an arbitrary $(n+1)$-vertex v. We want to show that $\pi_w^{n+1}(v)\cup\pi_b^{n+1}(v)$ is a cnc-partition. This is trivial in case 1 (i.e., if v is not an n-vertex). Thus assume that we are in case 2, i.e., that $v\in\mathbf{V}^{n+1}\cap\mathbf{V}^n$.

(1a) We first prove that $\pi_w^{n+1}(v)\cup\pi_b^{n+1}(v)$ is *non-crossing*. Consider first two blocks

$$b^{n+1}=b_i^{n+1}(b^1),\ c^{n+1}=b_j^{n+1}(c^1)\in\pi_w^{n+1}(v)\cup\pi_b^{n+1}(v)$$

as in (8.2), where $i,j=0,\dots,m-1$ and $b^1,c^1\in\pi_w^1(p)\cup\pi_b^1(p)\setminus\{b_\star^1,c_\star^1\}$. If $i\neq j$, the blocks b^{n+1} and c^{n+1} are non-crossing, since b^{n+1} and c^{n+1} are contained in disjoint intervals, namely $b^{n+1}\subset[2ki,2k(i+1)-1]$ and $c^{n+1}\subset[2kj,2k(j+1)-1]$.

If $i=j$, the blocks b^{n+1} and c^{n+1} are non-crossing, since the blocks b^1 and c^1 are.

(1b) Now let $b^{n+1}=b_i^{n+1}(b^1)$ be as before, and

$$b_\star^{n+1}=b_\star^{n+1}(b^n)=\bigcup\{b_\star^1+2kj : 2j\in b^n\}$$

be as in (8.3) (where $b^n \in \pi_w^n(v)$). Assume without loss of generality that $i=0$. Then b^{n+1} is contained in one component of $[0,2k-1]\setminus b_\star^1$. Each set $b_\star^1+2kj$ distinct from $b_\star^1$ is contained in an interval distinct from $[0,2k-1]$. It follows that b^{n+1} and $b_\star^{n+1}$ are non-crossing.

That b^{n+1} and $c_\star^{n+1}$ (as in (8.4)) are non-crossing is shown by the same argument.

(1c) Now let $b_\star^{n+1}=b_\star^{n+1}(b^n)$ be as before and $\tilde{b}_\star^{n+1}=b_\star^{n+1}(\tilde{b}^n)$ be a distinct set as in (8.3), meaning that the block $\tilde{b}^n \in \pi_w^n(v)$ is distinct from b^n. Since b^n and $\tilde{b}^n$ are non-crossing, it follows that $b_\star^{n+1}$ and $\tilde{b}_\star^{n+1}$ are non-crossing. The same argument shows that distinct $c_\star^{n+1}$ and $\tilde{c}_\star^{n+1}$ as in (8.4) are non-crossing.

(1d) Consider now two sets $b_\star^{n+1}=b_\star^{n+1}(b^n)$ and $c_\star^{n+1}=c_\star^{n+1}(c^n)$ as in (8.3) and (8.4) ($b^n \in \pi_w^n(v)$ and $c^n \in \pi_b^n(v)$). Recall that $\pi_w^n(v)\cup\pi_b^n(v)$ is a cnc-partition, by inductive hypothesis. Assume first that b^n and c^n are non-adjacent (see Lemma 6.2), i.e., they do not contain indices i and $i+1$, respectively. Then, from the fact that b^n and c^n are non-crossing, it follows that $b_\star^{n+1}$ and $c_\star^{n+1}$ are non-crossing.

(1e) Now let b^n and c^n be adjacent. Recall that $0 \in b_\star^1$ and $2k-1 \in c_\star^1$. Thus, there is an index $i^1 \in b_\star^1$ such that $i^1+1 \in c_\star^1$, since $\pi_w^1(p)\cup\pi_b^1(p)$ is a cnc-partition. This means that

$$b_\star^1 \subset [0,i^1] \quad \text{and} \quad c_\star^1 \subset [i^1+1, 2k-1].$$

Similarly, since b^n and c^n are adjacent, there are indices $i^n, j^n \in b^n$ such that

$$i^n+1, j^n-1 \in c^n,$$

meaning that

$$b^n \subset [j^n, i^n] \quad \text{and} \quad c^n \subset [i^n+1, j^n-1].$$

Here we are using the notation from (6.1). From this we obtain the smallest and biggest elements in $b_\star^{n+1}=b_\star^{n+1}(b^n)$ and $c_\star^{n+1}=c_\star^{n+1}(c^n)$ according to (8.3) and (8.4), namely

$$b_\star^{n+1} \subset [j^n k, i^1+i^n k] \quad \text{and} \quad c_\star^{n+1} \subset [i^1+i^n k+1, j^n k-1].$$

Thus, $b_\star^{n+1}$ and $c_\star^{n+1}$ are non-crossing.

We now prove that $\pi_w^{n+1}(v)$ and $\pi_b^{n+1}(v)$ are *complementary*. Let $i^{n+1}=0,\dots,2km-1$ be arbitrary. We have to show that the two blocks of $\pi_w^{n+1}(v)\cup\pi_b^{n+1}(v)$ containing i^{n+1} and $i^{n+1}+1$ are adjacent.

If we are in case (1a), i.e., if $i^{n+1} \in b^{n+1}=b_i^{n+1}(b^1)$ and $i^{n+1}+1 \in c^{n+1}=b_j^{n+1}(c^1)$, where $b^1, c^1 \in \pi_w^1(p)\cup\pi_b^1(p)\setminus\{b_\star^1, c_\star^1\}$, it follows that $i=j$. Then b^1 and c^1 are adjacent, which implies that b^{n+1} and c^{n+1} are adjacent.

When we are in case (1b), it follows that b^1 and $b_\star^1$ are adjacent. This implies that b^{n+1} and $b_\star^{n+1}$ are adjacent.

Cases (1c) and (1d) cannot happen.

In case (1e), it is clear from the description that

$$j^n k, i^1+i^n k \in b_\star^{n+1} \quad \text{and} \quad i^1+i^n k+1, j^n k-1 \in c_\star^{n+1}.$$

Thus, $b_\star^{n+1}$ and $c_\star^{n+1}$ are adjacent.

(3) Let D^{n+1} and $\widetilde{D}^{n+1}$ be two $(n+1)$-edges. We have to show that D^{n+1} and $\widetilde{D}^{n+1}$ are succeeding in γ^{n+1} if and only if they are succeeding with respect to the connection of $(n+1)$-tiles.

We keep the notation from §8.1. Case (1) is again clear. Thus we assume that we are in case (2), meaning that $v \in \mathbf{V}^{n+1} \cap \mathbf{V}^n$. Recall that $\widetilde{E}^0$ is the 0-edge with initial point $p=F^n(v)$ and $\widetilde{E}^1 \ni p$ is the marked 1-edge (some arc $\tilde{A}^0 \subset \widetilde{E}^0$ containing p is deformed by H^0 to $\widetilde{E}^1$).

Let $\widetilde{E}^0 = \widetilde{E}^n_0, ..., \widetilde{E}^n_{m-1} \ni v$ be all n-edges such that $F^n(\widetilde{E}^n_j) = \widetilde{E}^0$ (labeled positively around v).

Consider the $(n+1)$-edges $\widetilde{E}^{n+1}_j$ such that $H^n\colon \tilde{A}^n_j \subset \widetilde{E}^n_j \to \widetilde{E}^{n+1}_j$, for some arc $\tilde{A}^n_j \ni v$. These $(n+1)$-edges $\widetilde{E}^{n+1}_0, ..., \widetilde{E}^{n+1}_{m-1}$ are again labeled positively around v. Note that these are not all of the $(n+1)$-edges containing v.

Claim. $F^n(\widetilde{E}^{n+1}_j) = \widetilde{E}^1$ for all $j=0, ..., m-1$.

To prove the claim, we first note that $F^n(\widetilde{E}^{n+1}_j)$ is a 1-edge which we denote by $\widetilde{D}^1$. Since $\tilde{A}^n_j \subset \widetilde{E}^n_j$, the arc $\widetilde{B}^0 := F^n(\tilde{A}^n_j)$ is contained in $\widetilde{E}^0 = F^n(\widetilde{E}^n_j)$, with initial point $p=F^n(v)$. As H^n is the lift of H^0 by F^n, we have

$$\widetilde{D}^1 = F^n(\widetilde{E}^{n+1}_j) = F^n(H^n_1(\tilde{A}^n_j)) = H^0_1(F^n(\tilde{A}^n_j)) = H^0_1(\widetilde{B}^0).$$

The unique arc in $\widetilde{E}^0$ with initial point p that is deformed to a 1-edge is $\tilde{A}^0$. Therefore $\widetilde{B}^0 = \tilde{A}^0$, and hence $\widetilde{D}^1 = \widetilde{E}^1$, proving the claim.

Note that a sector of sufficiently small radius between $\widetilde{E}^{n+1}_j$ and $\widetilde{E}^{n+1}_{j+1}$ is mapped *bijectively* by F^n to some neighborhood of p with $\widetilde{E}^1$ removed.

Assume now that the $(n+1)$-edges D^{n+1} and $\widetilde{D}^{n+1}$ are succeeding in γ^{n+1} at the $(n+1)$-vertex v. This is the case if and only if there are distinct arcs $A^n, \tilde{A}^n \ni x$ such that

$$H^n\colon A^n \subset D^n \longrightarrow D^{n+1} \quad \text{and} \quad H^n\colon \tilde{A}^n \subset \widetilde{D}^n \longrightarrow \widetilde{D}^{n+1}$$

(where $D^n, \widetilde{D}^n \in \mathbf{E}^n$).

Case 1. A^n and $\tilde{A}^n$ are contained in the *same* n-edge, or equivalently $x \notin \mathbf{V}^n$. Note that

$$\widetilde{D}^{n+1} \neq \widetilde{E}^{n+1}_j \quad \text{for all } j=0, ..., m-1,$$

and that $H_1^n(x)=v$. If $D^{n+1}=\widetilde{E}_j^{n+1}$ for a $j=0,...,m-1$, it would follow that both endpoints of $\widetilde{E}_j^{n+1}$ are equal to v, which is impossible.

It follows that D^{n+1} and $\widetilde{D}^{n+1}$ are contained in one sector between $\widetilde{E}_j^{n+1}$ and $\widetilde{E}_{j+1}^{n+1}$, since H^n is a pseudo-isotopy.

Case 2. $x=v$, and thus A^n and $\tilde{A}^n$ are contained in n-edges that succeed at v. Then

$$\widetilde{D}^{n+1}=\widetilde{E}_j^{n+1} \quad \text{for some } j=0,...,m-1$$

in this case.

Consider two $(n+1)$-edges $D^{n+1}, \widetilde{D}^{n+1}\ni v$, with $\widetilde{D}^{n+1}\neq\widetilde{E}_j^{n+1}$ (for all $j=0,...,m-1$). They are succeeding in γ^{n+1} at v if and only if they are contained in one sector between $\widetilde{E}_j^{n+1}$ and $\widetilde{E}_{j+1}^{n+1}$, and the 1-edges $F^n(D^{n+1})$ and $F^n(\widetilde{D}^{n+1})$ are succeeding in γ^1 (since F^n is bijective on this sector). This happens if and only if D^{n+1} and $\widetilde{D}^{n+1}$ are succeeding with respect to $\pi_w^{n+1}(v)\cup\pi_b^{n+1}(v)$ by definition (see (8.2)).

Let $E^0\ni p$ be the 0-edge with terminal point p, i.e., the one preceding $\widetilde{E}^0$. Let $E_0^n,...,E_{m-1}^n$ be all n-edges such that $F^n(E_j^n)=E^0$, labeled such that E_j^n lies between $\widetilde{E}_j^n$ and $\widetilde{E}_{j+1}^n$. Then $\widetilde{E}_j^n$ and E_j^n are both contained in the same white n-tile X_j^n. Thus E_i^n and $\widetilde{E}_j^n$ are succeeding (at v) if and only if i and j are succeeding indices of a block $b^n\in\pi_w^n(v)$.

Consider the 1-edge E^1 such that $H^0\colon A^0\subset E^0\to E^1$ for an arc $A^0\ni p$. Let X_l^1 be the white 1-tile containing E^1. Now consider the $(n+1)$-edge E_j^{n+1} such that

$$H^n\colon A_j^n\subset E_j^n\longrightarrow E_j^{n+1}$$

for an arc $A_j^n\ni v$. Since H^n is a pseudo-isotopy, it follows that E_j^{n+1} is in the sector between $\widetilde{E}_j^{n+1}$ and $\widetilde{E}_{j+1}^{n+1}$; indeed it follows that $E_j^{n+1}\subset X_{2kj+l}^{n+1}$, since the diagram in Figure 9 commutes (recall that $\widetilde{E}_j^{n+1}\subset X_{2kj}^{n+1}$).

Consider now two $(n+1)$-edges $D^{n+1}\ni v$ and $\widetilde{D}^{n+1}=\widetilde{E}_j^{n+1}\ni v$. They are succeeding in γ^{n+1} if and only if $D^{n+1}=E_i^{n+1}\subset X_{2ki+l}^{n+1}$, where i and j are succeeding indices of a block $b^n\in\pi_w^n(v)$. This happens if and only if they are succeeding with respect to $\pi_w^{n+1}(v)\cup\pi_b^{n+1}(v)$ by definition (see (8.3)) (in the notation from (1e), $i=i^n$, $j=j^n$ and $l=i^1$).

(2) follows as in §4.4. □

9. Invariant Peano curve implies expansion

In this section we prove Theorem 1.2. Thus we assume that for some iterate $F=f^n$ there is a Peano curve $\gamma\colon S^1\to S^2$ (onto) such that $F(\gamma(z))=\gamma(z^d)$ for all $z\in S^1$ (where $d=\deg F$). We want to show that f is expanding.

The following is [BM, Lemma 6.3].

LEMMA 9.1. *Let f be a Thurston map and $F=f^n$, where $n\in\mathbb{N}$. Then f is expanding if and only if F is expanding.*

We will use the following equivalent formulation of "expanding" due to Haïssinsky–Pilgrim [HP]. For a proof of the following lemma we refer the reader to [BM, Proposition 6.2].

LEMMA 9.2. *A Thurston map F is* expanding *if and only if there exists a finite open cover $\mathcal{U}^0$ of S^2 by connected sets such that the following holds.*

Denote by $\mathcal{U}^n$ the set of connected components of $F^{-n}(U)$ for all $U\in\mathcal{U}^0$. Then

$$\operatorname{mesh}\mathcal{U}^n\to 0 \quad \textit{as } n\to\infty.$$

Here $\operatorname{mesh}\mathcal{U}^n$ *denotes the biggest diameter of a set in $\mathcal{U}^n$.*

Proof of Theorem 1.2. Let $\gamma\colon S^1\to S^2$ be a Peano curve (onto) such that

$$F(\gamma(z))=\gamma(z^d) \quad \text{for all } z\in S^1 \quad (\text{where } d=\deg F). \tag{9.1}$$

Fix a point $x^0\in S^2$. Let $W(x^0)\subset S^2$ be an open neighborhood of x^0 that is a Jordan domain. Furthermore we assume that $W(x^0)$ is so small that each component of $F^{-1}(W(x^0))$ contains exactly one point of $F^{-1}(x^0)$.

Consider $\gamma^{-1}(W(x^0))=:\mathcal{I}(x^0)=\bigcup_j I_j\subset S^1$; this is a (countable) union of open arcs I_j. Let

$$\begin{aligned}\mathcal{J}(x^0)&:=\bigcup\{I_j : \gamma(I_j)\ni x^0\}\subset S^1,\\ V(x^0)&:=\gamma(\mathcal{J}(x^0))\subset S^2.\end{aligned}$$

Note that $\gamma(S^1\setminus\mathcal{J}(x^0))$ is a compact set that does not contain x^0. Thus $V(x^0)$ is a neighborhood of x^0.

Fix an $x^n\in F^{-n}(x^0)$. Let $V^n(x^n)\subset S^2$ be the path component of $F^{-n}(V(x^0))$ containing x^n.

As before we view the circle as $\mathbb{R}/\mathbb{Z}$, the map $z\mapsto z^d$ is then given as

$$\begin{aligned}\phi_d\colon \mathbb{R}/\mathbb{Z}&\longrightarrow \mathbb{R}/\mathbb{Z},\\ t&\longmapsto dt \pmod 1.\end{aligned}$$

Let $\mathcal{J}^n:=\phi_{d^n}^{-1}(\mathcal{J}(x^0))$. Note that $\mathcal{J}^n=\bigcup_j J_j^n$ is a (countable) union of open intervals, each of which has length $\leqslant d^{-n}$. Thus uniform continuity of γ implies that

$$\operatorname{diam}\gamma(J_j^n)\leqslant\omega(d^{-n})\to 0 \quad \text{as } n\to\infty,$$

where ω is the modulus of continuity of γ.

From (9.1) it follows that each set $\gamma(J_j^n)$ contains a point $x_j^n \in F^{-n}(x^0)$. If $x_j^n \neq x^n$ then $\gamma(J_j^n)$ is contained in a component of $F^{-n}(W(x^0))$ distinct from the one containing x^n, and thus $\gamma(J_j^n) \cap V^n(x^n) = \varnothing$. It follows that

$$\gamma^{-1}(V^n(x^n)) = \bigcup \{J_j^n : \gamma(J_j^n) \ni x^n\} =: \mathcal{J}^n(x^n).$$

Since $\gamma(J_i^n) \cap \gamma(J_j^n) \ni x^n$ for $J_i^n, J_j^n \subset \mathcal{J}^n(x^n)$, it follows that

$$\operatorname{diam} V^n(x^n) \leqslant 2\omega(d^{-n}).$$

The sets $V^0(x^0)$ are not necessarily open, and $\operatorname{int} V^0(x^0)$ is not necessarily connected. Let $U(x^0) \subset V^0(x^0)$ be an open connected set containing x^0. Pick a finite subcover $\mathcal{U}^0$ of $\{U(x^0) : x^0 \in S^2\}$. From the above it follows that $\operatorname{mesh} \mathcal{U}^n \to 0$ as $n \to \infty$. Thus F is expanding by Lemma 9.2, and hence f is expanding by Lemma 9.1. □

10. An example

The obvious question to ask is whether an iterate $F = f^n$ is necessary in Theorem 1.1 (or whether one may choose $n = 1$). None of the assumptions in §7 seem to be necessary. It is possible to show (similarly as in [BM, Example 14.12]) that the map f for which Milnor constructs an invariant Peano curve in [Mi1] does not have an invariant Jordan curve $\mathcal{C} \supset \operatorname{post}$; also the 1-tiles do intersect disjoint 0-edges.

In this section we consider an example of an expanding Thurston map h, where no pseudo-isotopy H^0 as desired exists. This means that for any Jordan curve $\mathcal{C} \supset \operatorname{post}$ (not necessarily invariant) there is no pseudo-isotopy H^0 rel. $\operatorname{post}(h)$ as in Definition 3.2 such that $H_1^0(\mathcal{C}) = \bigcup \mathbf{E}^1 = h^{-1}(\mathcal{C})$.

Thus, one has to take an iterate (in fact h^2 will do) in our construction. Of course there could be a Peano curve γ which semi-conjugates z^d to h, but a substantially different proof would be required.

The map h is a *Lattès map* as the map g from §1.5. Start with the square

$$\left[0, \tfrac{1}{2}\sqrt{2}\,\right] \times [0, 1],$$

which is mapped by a Riemann map to the upper half-plane. This extends to a meromorphic map $\wp = \wp_L : \mathbb{C} \to \widehat{\mathbb{C}}$, which is periodic with respect to the lattice $L = \sqrt{2}\,\mathbb{Z} \times 2\mathbb{Z}$. Consider the map (here i denotes the imaginary unit)

$$\begin{aligned} \psi : \mathbb{C} &\longrightarrow \mathbb{C}, \\ z &\longmapsto \sqrt{2} i z \end{aligned} \tag{10.1}$$

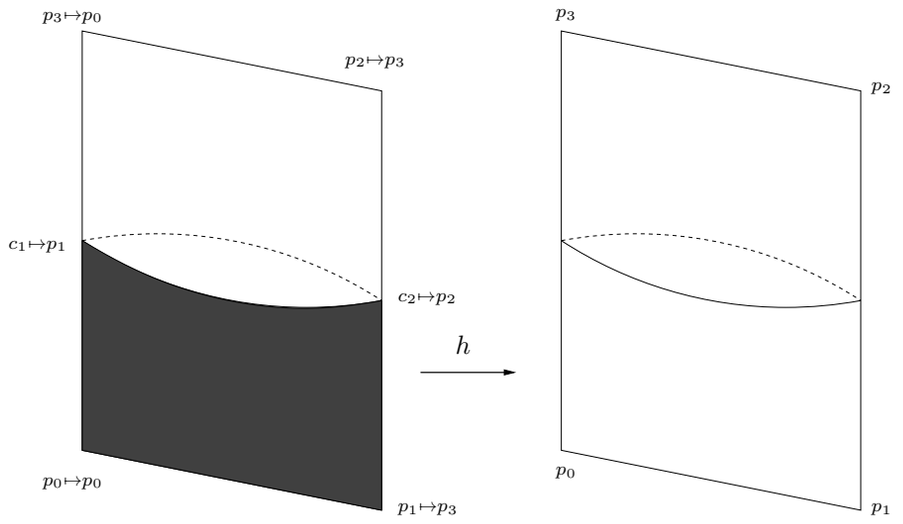

Figure 10. The map h.

Note that $\psi(L) \subset L$. The map h is the one that makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\psi} & \mathbb{C} \\ {\scriptstyle\wp}\downarrow & & \downarrow{\scriptstyle\wp} \\ S^2 & \xrightarrow[h]{} & S^2. \end{array}$$

The degree of h is 2. Again one may use $\wp$ to push the Euclidean metric from $\mathbb{C}$ to the sphere S^2. In this metric the upper and lower half-planes are both isometric to the rectangle $\left[0, \frac{1}{2}\sqrt{2}\right] \times [0,1]$. Two such rectangles glued together along their boundaries form a *pillow* as before. Divide each rectangle horizontally in two. The small rectangles are similar to the big ones. The map h is given by mapping each small rectangle (they are the 1-tiles) to big ones (the 0-tiles) as indicated in Figure 10. The critical points are c_1 and c_2, the post-critical points are p_0, p_1, p_2 and p_3; they are mapped as follows (this is known as the *ramification portrait*):

$$\begin{array}{ccccc} c_1 & \xrightarrow{2:1} & p_1 & & \\ & & & \searrow & \\ & & & & p_3 \longrightarrow p_0 \circlearrowright \\ & & & \nearrow & \\ c_2 & \xrightarrow{2:1} & p_2 & & \end{array} \tag{10.2}$$

LEMMA 10.1. *Let $\gamma^0 = \mathcal{C} \supset \mathrm{post}(h)$ be (any such) Jordan curve and γ^1 be an Eulerian circuit in $h^{-1}(\mathcal{C})$ such that $h\colon \gamma^1 \to \gamma^0$ is a d-fold cover. Then there is no pseudo-isotopy H^0 rel.* $\mathrm{post}(h)$ *as in Definition 3.2 that deforms γ^0 to γ^1.*

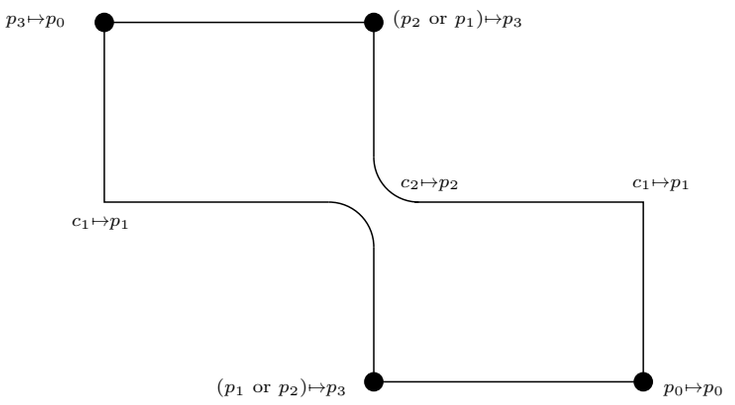

Figure 11. An Eulerian circuit in $h^{-1}(\mathcal{C})$ (case 1).

Sketch of proof. The proof is a (rather tedious) case-by-case analysis. There are however only two cases that are essentially different. One of each is presented.

Case 1. The curve $\mathcal{C}$ goes through p_0, p_1, p_2 and p_3 (in this cyclical order).

We fix an orientation of $\mathcal{C}$. Let U_w and U_b be the two components of $S^2 \backslash \mathcal{C}$, where the positively oriented boundary of U_w is $\mathcal{C}$. The closures of U_w and U_b are the white and black 0-tiles $X_w^0 = U_w \cup \mathcal{C}$ and $X_b = U_b \cup \mathcal{C}$ as before. Similarly, we define the (white) 1-tiles as closures of components of $h^{-1}(U_w)$.

Since the degree of h is 2, there are two white 1-tiles. They intersect at the critical points c_1 and c_2. The boundary of each 1-tile contains four points that are mapped to p_0, p_1, p_2 and p_3 (in this cyclical order). There are two different Eulerian circuits γ^1 in $h^{-1}(\mathcal{C})$ such that $h\colon \gamma^1 \to \gamma^0$ is a 2-fold cover. They correspond to connecting the two 1-tiles either at c_1 or at c_2. One situation (connection at c_2) is shown in Figure 11. Note that the cyclical ordering of the post-critical points (shown as dots) is different from the one on $\mathcal{C}$. Thus there is no pseudo-isotopy H^0 as desired that deforms $\mathcal{C} = \gamma^0$ to γ^1.

When $\mathcal{C}$ goes through the post-critical points in the order

$$(p_0, p_2, p_1, p_3), \quad (p_0, p_3, p_1, p_2) \quad \text{or} \quad (p_0, p_3, p_2, p_1),$$

the same argument works.

Case 2. The curve $\mathcal{C}$ goes through p_0, p_1, p_3 and p_2 (in this cyclical order). The 0- and 1-tiles are defined and colored as before (see §2).

As before, there are two different Eulerian circuits γ^1 in $h^{-1}(\mathcal{C})$, such that $h\colon \gamma^1 \to \gamma^0$ is a 2-fold cover. They correspond to whether the white 1-tiles are connected at c_1 or c_2. Assume that they are connected at c_2. The argument when they are connected at c_1 is again completely analogous.

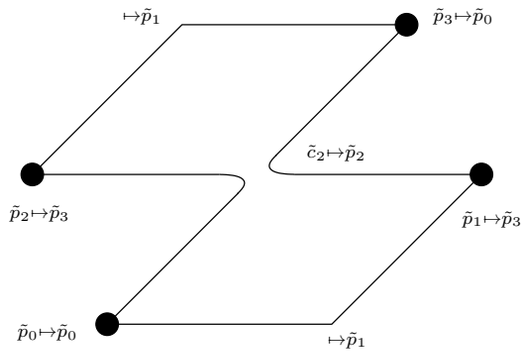

Figure 12. Eulerian circuit (case 2).

Assume that the pseudo-isotopy H^0 is as in Definition 3.2. Then H^0 deforms (the white 0-tile) X_w^0 into the two 1-tiles.

In the following we work in the (orbifold) covering. Recall that $X_w^0, X_b^0 \subset S^2$ are the white and black 0-tiles (given by $\mathcal{C}$). Pull this tiling back by $\wp$ to a tiling of $\mathbb{C}$. More precisely, a 0-tile $\widetilde{X} \subset \mathbb{C}$ is the closure of one component of $\wp^{-1}(U_{w,b})$. Similarly as in the proof of (2.1) one shows that $\wp\colon \widetilde{X} \to X_{w,b}$ is a homeomorphism. We color one such 0-tile $\widetilde{X} \subset \mathbb{C}$ white or black if it is the preimage of X_w^0 or X_b^0, respectively. This gives a tiling of the plane $\mathbb{C}$ into white and black 0-tiles.

Recall that the ramification points of $\wp$ are the points in $\frac{1}{2}\sqrt{2}\,\mathbb{Z}\times\mathbb{Z}$. At each such ramified point $c \in \frac{1}{2}\sqrt{2}\,\mathbb{Z}\times\mathbb{Z}$, two white and two black tiles intersect. Furthermore, the map $\wp$ is symmetric with respect to each such point. This means that $\wp(c+z)=\wp(c-z)$ for all $z\in\mathbb{C}$. Thus the tiling of $\mathbb{C}$ is pointwise symmetric with respect to each such point c.

We now define the 1-tiles in $\mathbb{C}$. They may be obtained in two different ways; either as preimages of 1-tiles in S^2 by $\wp$, or as preimages of 0-tiles $\widetilde{X} \subset \mathbb{C}$ by ψ (10.1).

Fix one white 0-tile $\widetilde{X} \subset \mathbb{C}$. Note that $\widetilde{X}$ has four vertices $\tilde{p}_0, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3 \in \frac{1}{2}\sqrt{2}\,\mathbb{Z}\times\mathbb{Z}$, they are mapped by $\wp$ to p_0, p_1, p_2 and p_3, respectively. We may assume that $\tilde{p}_0=0$.

As in Lemma 3.4, the pseudo-isotopy H^0 lifts to a pseudo-isotopy (rel. $\frac{1}{2}\sqrt{2}\,\mathbb{Z}\times\mathbb{Z}$) $\widetilde{H}^0\colon \mathbb{C}\times[0,1]\to\mathbb{C}$. Note that $\widetilde{H}^0$ deforms $\widetilde{X}$ into two 1-tiles (in $\mathbb{C}$) connected at a point $\tilde{c}_2$. Here $\wp(\tilde{c}_2)=c_2$.

The ordering of the post-critical points along $\mathcal{C}$ together with (10.2) implies that the situation looks as in Figure 12. Here "$\mapsto\tilde{p}_j$" labels a point $\tilde{z}$ that satisfies $h(\wp(\tilde{z}))=p_j$.

The symmetry of the 1-tiles with respect to the point $\tilde{c}_2$ implies that

$$2\tilde{c}_2 = \tilde{p}_3 = \tilde{p}_1 + \tilde{p}_2.$$

Note that $\tilde{c}_2$ and $\tilde{p}_1$ are contained in the same 1-tile $\widetilde{X}^1$, which contains $\tilde{p}_0=0$. There are two 0-tiles containing $\tilde{p}_0$, symmetric with respect to the origin. Thus (here i denotes

the imaginary unit)

$$\pm\psi(\widetilde{X}^1) = \pm\sqrt{2}i\widetilde{X}^1 = \widetilde{X}.$$

Therefore

$$\pm\sqrt{2}i\tilde{c}_2 = \tilde{p}_2 \quad \text{and} \quad \pm\sqrt{2}i\tilde{p}_1 = \tilde{p}_3.$$

Combining these three equations yields

$$\tilde{p}_2 = \pm\sqrt{2}i\tilde{c}_2 = \pm\tfrac{1}{2}\sqrt{2}i\tilde{p}_3 = \pm\tfrac{1}{2}\sqrt{2}i(\pm\sqrt{2}i\tilde{p}_1) = -\tilde{p}_1.$$

Thus,

$$\tilde{p}_3 = \tilde{p}_1 + \tilde{p}_2 = 0,$$

which is a contradiction.

If $\mathcal{C}$ goes through the post-critical points in the cyclical order p_0, p_2, p_3 and p_1, the argument is completely analogous to the one above. □

11. Open problems and concluding remarks

A rational map of degree d can naturally be viewed as a point in $\mathbb{C}^{2d+1}$ via its coefficients. Consider a post-critically finite rational map f without periodic critical points. This is an expanding Thurston map in our sense, the Julia set is all of S^2. Rees [Re1] has shown that such a map can be disturbed in a set of positive measure (in $\mathbb{C}^{2d+1}$) such that the Julia set stays S^2.

Open problem 1. Let f be a rational map with Julia set S^2. Does Theorem 1.1 hold in this case?

On the other hand one may ask if the theorem continues to hold if the Julia set is not the whole sphere. This however is false. Namely, Kameyama [Ka, §4] gives an example of a post-critically finite rational map where no such semi-conjugacy exists.

Finally one can ask if a corresponding result holds in the group case.

Open problem 2. Let Γ be a Gromov-hyperbolic group whose boundary at infinity is S^2. Is there a Peano curve $\gamma\colon S^1 \to S^2$ which is invariant under a non-trivial normal subgroup of Γ?

A positive answer might conceivably open another line of attack on *Cannon's conjecture*.

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Received November 4, 2010
Received in revised form May 25, 2012