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Complete monotonicity for inverse powers of some combinatorially defined polynomials

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1. Introduction

If P is a univariate or multivariate polynomial with real coefficients and strictly positive constant term, and β is a positive real number, it is sometimes of interest to know whether $P^{-\beta}$ has all non-negative (or even strictly positive) Taylor coefficients. A problem of this type arose in the late 1920s in Friedrichs and Lewy's study of the discretized time-dependent wave equation in two space dimensions: they needed the answer for the case $P(y_1, y_2, y_3) = (1-y_1)(1-y_2)+(1-y_1)(1-y_3)+(1-y_2)(1-y_3)$ and $\beta=1$. Lewy contacted Gabor Szegő, who proceeded to solve a generalization of this problem: Szegő [113] showed that for any $n \geqslant 1$, the polynomial

$$P_n(y_1, ..., y_n) = \sum_{i=1}^{n} \prod_{j \neq i} (1-y_j) \tag{1.1}$$

has the property that $P_n^{-\beta}$ has non-negative Taylor coefficients for all $\beta \geqslant \frac{1}{2}$. (The cases $n=1, 2$ are of course trivial; the interesting problem is for $n \geqslant 3$.) Szegő's proof was surprisingly indirect, and exploited the Gegenbauer–Sonine addition theorem for Bessel functions together with Weber's first exponential integral.([1]) Shortly thereafter, Kaluza [72] provided an elementary (albeit rather intricate) proof, but only for $n=3$ and $\beta=1$. In the early 1970s, Askey and Gasper [8] gave a partially alternative proof, using Jacobi polynomials in place of Bessel functions. Finally, Straub [112] has very recently produced simple and elegant proofs for the cases $n=3, 4$ and $\beta=1$, based on applying a positivity-preserving operator to another rational function whose Taylor coefficients are known (by a different elementary argument) to be non-negative (indeed strictly positive).

Askey and Gasper, in discussing both Szegő's problem and a related unsolved problem of Lewy and Askey, expressed the hope that "there should be a combinatorial interpretation of these results" and observed that "this might suggest new methods" [8, p. 340]. The purpose of the present paper is to provide such a combinatorial interpretation, together with new and elementary (but we think powerful) methods of proof. As a consequence we are able to prove a far-reaching generalization of Szegő's original result, which includes as a special case an affirmative solution to the problem of Lewy and Askey. Indeed, we give *two different* proofs for the Lewy–Askey problem, based on viewing it

([1]) These formulae for Bessel functions can be found in [122, p. 367, eq. 11.41 (17)] and [122, p. 394, eq. 13.3 (4)], respectively. For the special case $n=3$, Szegő also gave a version of the proof using Sonine's integral for the product of three Bessel functions [122, p. 411, eq. 13.46 (3)]. Szegő commented in his introduction [113, p. 674] that "Die angewendeten Hilfsmittel stehen allerdings in keinem Verhältnis zu der Einfachheit des Satzes." ("The tools used are, however, disproportionate to the simplicity of the result.")

Szegő in fact proved the *strict* positivity of the Taylor coefficients for all n when $\beta=1$, and for $n > 4\beta/(2\beta-1)$ when $\beta > \frac{1}{2}$ [113, Sätze I–III]. In this paper we shall concentrate on non-negativity and shall not worry about whether strict positivity holds or not. But see Remark 1 after Theorem 2.2.

as a member of two different families of generalizations of the $n=3$ Szegő problem. Our methods turn out to be closely connected with harmonic analysis on Euclidean Jordan algebras (or equivalently on symmetric cones) [52].

1.1. Spanning-tree polynomials and series-parallel graphs

From a combinatorial point of view, one can see that Szegő's polynomial (1.1) is simply the spanning-tree generating polynomial $T_G(\mathbf{x})$ for the n-cycle $G=C_n$,

$$T_{C_n}(x_1,...,x_n)=\sum_{i=1}^{n}\prod_{j\neq i}x_j, \tag{1.2}$$

after the change of variables $x_i=1-y_i$. This suggests that an analogous result might hold for the spanning-tree polynomials of some wider class of graphs.([2]) This conjecture is indeed true, as we shall show. Moreover (and this will turn out to be quite important in what follows), the change of variables $x_i=1-y_i$ can be generalized to $x_i=c_i-y_i$ for constants $c_i>0$ that are not necessarily equal. We shall prove the following result.

THEOREM 1.1. *Let $G=(V,E)$ be a connected series-parallel graph, and let $T_G(\mathbf{x})$ be its spanning-tree polynomial in the variables $\mathbf{x}=\{x_e\}_{e\in E}$. Then, for all $\beta\geqslant\frac{1}{2}$ and all choices of strictly positive constants $\mathbf{c}=\{c_e\}_{e\in E}$, the function $T_G(\mathbf{c}-\mathbf{y})^{-\beta}$ has non-negative Taylor coefficients in the variables $\mathbf{y}$.*

Conversely, if G is a connected graph and there exists $\beta\in(0,1)\setminus\{\frac{1}{2}\}$ such that $T_G(\mathbf{c}-\mathbf{y})^{-\beta}$ has non-negative Taylor coefficients (in the variables $\mathbf{y}$) for all $\mathbf{c}>\mathbf{0}$, then G is series-parallel.

The proof of the direct half of Theorem 1.1 is completely elementary (and indeed quite simple). The converse relies on a deep result from harmonic analysis on Euclidean Jordan algebras [58], [52, Chapter VII], [70], for which, however, there now exist two different elementary proofs [106], [31], [110].

Let us recall that a C^∞ function $f(x_1,...,x_n)$ defined on $(0,\infty)^n$ is termed *completely monotone* if its partial derivatives of all orders alternate in sign, i.e.

$$(-1)^k\frac{\partial^k f}{\partial x_{i_1}\dots\partial x_{i_k}}\geqslant 0 \tag{1.3}$$

everywhere on $(0,\infty)^n$, for all $k\geqslant 0$ and all choices of indices $i_1,...,i_k$. Theorem 1.1 can then be rephrased as follows.

([2]) See (1.5) and (7.1) below for the general definition of the spanning-tree polynomial $T_G(\mathbf{x})$ for a connected graph G [37], [109].

THEOREM 1.1′ *Let $G=(V,E)$ be a connected series-parallel graph, and let $T_G(\mathbf{x})$ be its spanning-tree polynomial. Then $T_G^{-\beta}$ is completely monotone on $(0,\infty)^E$ for all $\beta \geqslant \frac{1}{2}$.*

Conversely, if $G=(V,E)$ is a connected graph and there exists $\beta \in (0,1)\setminus\{\frac{1}{2}\}$ such that $T_G^{-\beta}$ is completely monotone on $(0,\infty)^E$, then G is series-parallel.

Allowing arbitrary constants $\mathbf{c}>\mathbf{0}$ thus allows the result to be formulated in terms of complete monotonicity, and leads to a characterization that is both necessary and sufficient. Szegő's result (or rather, its generalization to arbitrary $\mathbf{c}$) extends to series-parallel graphs and no farther.

1.2. Determinants

But this is not the end of the matter: we can go far beyond series-parallel graphs if we relax our demands about the set of β for which $T_G^{-\beta}$ is asserted to be completely monotone. The key here is Kirchhoff's matrix-tree theorem [74], [28], [86], [83], [35], [34], [33], [126], [84], [1], which shows how spanning-tree polynomials can be written as determinants. This line of thought suggests that complete monotonicity of $P^{-\beta}$ might hold more generally for the homogeneous multiaffine polynomials arising from determinants of the type studied in [37, §8.1]. This too is true; in fact, such a result holds for a slightly more general class of polynomials that need not be multiaffine. We shall prove, once again by elementary methods, the following result.

THEOREM 1.2. *Let $A_1,\ldots,A_n$, $n \geqslant 1$, be $m\times m$ real or complex matrices or Hermitian quaternionic matrices, and let us form the polynomial*

$$P(x_1,\ldots,x_n) = \det\left(\sum_{i=1}^n x_i A_i\right) \tag{1.4}$$

in the variables $\mathbf{x}=(x_1,\ldots,x_n)$. (In the quaternionic case, det *denotes the Moore determinant; see* [105, Appendix A].*) Assume further that there exists a linear combination of $A_1,\ldots,A_n$ that has rank m (so that $P \not\equiv 0$).*

(a) *If $A_1,\ldots,A_n$ are real symmetric positive-semidefinite matrices, then $P^{-\beta}$ is completely monotone on $(0,\infty)^n$ for $\beta = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and for all real $\beta \geqslant \frac{1}{2}(m-1)$.*

(b) *If $A_1,\ldots,A_n$ are complex Hermitian positive-semidefinite matrices, then $P^{-\beta}$ is completely monotone on $(0,\infty)^n$ for $\beta = 0,1,2,3,\ldots$ and for all real $\beta \geqslant m-1$.*

(c) *If $A_1,\ldots,A_n$ are quaternionic Hermitian positive-semidefinite matrices, then $P^{-\beta}$ is completely monotone on $(0,\infty)^n$ for $\beta = 0,2,4,6,\ldots$ and for all real $\beta \geqslant 2m-2$.*

These curious conditions on β are not just an artifact of our method of proof; they really are best possible. They can be better understood if we take a slightly more general perspective, and define complete monotonicity for functions on an arbitrary open convex cone C in a finite-dimensional real vector space V (see §2). We then have the following result that "explains" Theorem 1.2.

THEOREM 1.3. (a) *Let V be the real vector space $\mathrm{Sym}(m,\mathbb{R})$ of real symmetric $m\times m$ matrices, and let $C\subset V$ be the cone $\Pi_m(\mathbb{R})$ of positive-definite matrices. Then the map $A\mapsto(\det A)^{-\beta}$ is completely monotone on C if and only if $\beta\in\{0,\frac{1}{2},1,\frac{3}{2},\dots\}\cup[\frac{1}{2}(m-1),\infty)$. Indeed, if $\beta\notin\{0,\frac{1}{2},1,\frac{3}{2},\dots\}\cup[\frac{1}{2}(m-1),\infty)$, then the map $A\mapsto(\det A)^{-\beta}$ is not completely monotone on any non-empty open convex subcone $C'\subseteq C$.*

(b) *Let V be the real vector space $\mathrm{Herm}(m,\mathbb{C})$ of complex Hermitian $m\times m$ matrices, and let $C\subset V$ be the cone $\Pi_m(\mathbb{C})$ of positive-definite matrices. Then the map $A\mapsto(\det A)^{-\beta}$ is completely monotone on C if and only if $\beta\in\{0,1,2,3,\dots\}\cup[m-1,\infty)$. Indeed, if $\beta\notin\{0,1,2,3,\dots\}\cup[m-1,\infty)$, then the map $A\mapsto(\det A)^{-\beta}$ is not completely monotone on any non-empty open convex subcone $C'\subseteq C$.*

(c) *Let V be the real vector space $\mathrm{Herm}(m,\mathbb{H})$ of quaternionic Hermitian $m\times m$ matrices, and let $C\subset V$ be the cone $\Pi_m(\mathbb{H})$ of positive-definite matrices. Then the map $A\mapsto(\det A)^{-\beta}$ is completely monotone on C if and only if $\beta\in\{0,2,4,6,\dots\}\cup[2m-2,\infty)$. Indeed, if $\beta\notin\{0,2,4,6,\dots\}\cup[2m-2,\infty)$, then the map $A\mapsto(\det A)^{-\beta}$ is not completely monotone on any non-empty open convex subcone $C'\subseteq C$.*

In particular, if the matrices $A_1,\dots,A_n$ together span $\mathrm{Sym}(m,\mathbb{R})$, $\mathrm{Herm}(m,\mathbb{C})$ or $\mathrm{Herm}(m,\mathbb{H})$ (so that the convex cone they generate has non-empty interior), then the determinantal polynomial (1.4) has $P^{-\beta}$ completely monotone on $(0,\infty)^n$ if and *only if* β belongs to the set enumerated in Theorem 1.2.([3])

The proof of the "if" part of Theorem 1.3 is completely elementary, but the "only if" part again relies on a deep result from harmonic analysis on Euclidean Jordan algebras, namely, the characterization of parameters for which the Riesz distribution is a positive measure (Theorem 4.8 below; but see [106], [31], [110] and [105, Appendix B] for elementary proofs). In fact, when Theorem 1.3 is rephrased in this latter context, it takes on the following unified form.

THEOREM 1.4. *Let V be a simple Euclidean Jordan algebra of dimension n and rank r, with $n=r+\frac{1}{2}dr(r-1)$, let $\Omega\subset V$ be the positive cone and let $\Delta:V\to\mathbb{R}$ be the Jordan determinant. Then the map $x\mapsto\Delta(x)^{-\beta}$ is completely monotone on Ω if and only if*

([3]) Brändén [25] has recently used this latter fact to determine the exact set of $\alpha\in\mathbb{R}$ for which the α-permanent [116] is non-negative on real symmetric (resp. complex Hermitian) positive-semidefinite matrices.

$\beta \in \{0, \frac{1}{2}d, ..., \frac{1}{2}d(r-1)\}$ *or* $\beta > \frac{1}{2}d(r-1)$. *Indeed, if*

$$\beta \notin \{0, \tfrac{1}{2}d, ..., \tfrac{1}{2}d(r-1)\} \cup (\tfrac{1}{2}d(r-1), \infty),$$

then the map $x \mapsto \Delta(x)^{-\beta}$ *is not completely monotone on any non-empty open convex subcone* $\Omega' \subseteq \Omega$.

We shall see that Theorem 1.4 is essentially *equivalent* to the characterization of parameters for which the Riesz distribution is a positive measure. The set of values of β described in Theorem 1.4 is known as the *Gindikin–Wallach set* and arises in a number of contexts in representation theory [16], [58], [101], [121], [77], [50], [51], [52].

A special case of the construction (1.4) arises [37, §8.1] when B is an $m \times n$ real or complex matrix of rank m, and we set $P(\mathbf{x}) = \det(BXB^*)$, where $X = \mathrm{diag}(x_1, ..., x_n)$ and * denotes Hermitian conjugate. Then the matrix A_i in (1.4) is simply the outer product of the ith column of B with its complex conjugate, and so is of rank at most 1; as a consequence, the polynomial P is multiaffine (i.e., of degree at most 1 in each variable separately).(4)

In particular, let $G = (V, E)$ be a connected graph, and define its spanning-tree polynomial $T_G(\mathbf{x})$ by

$$T_G(\mathbf{x}) = \sum_{T \in \mathcal{T}(G)} \prod_{e \in T} x_e, \tag{1.5}$$

where $\mathbf{x} = \{x_e\}_{e \in E}$ is a family of indeterminates indexed by the edges of G, and $\mathcal{T}(G)$ denotes the family of edge sets of spanning trees in G. Now let B be the directed vertex-edge incidence matrix for an arbitrarily chosen orientation of G, with one row (corresponding to an arbitrarily chosen vertex of G) deleted; then the matrix-tree theorem [74], [28], [86], [83], [35], [34], [33], [126], [84], [1], [37] tells us that $T_G(\mathbf{x}) = \det(BXB^{\mathrm{T}})$. Applying Theorem 1.2 (a), we obtain the following result.

COROLLARY 1.5. *Let* $G = (V, E)$ *be a connected graph with* p *vertices, and let* $T_G(\mathbf{x})$ *be its spanning-tree polynomial. Then* $T_G^{-\beta}$ *is completely monotone on* $(0, \infty)^E$ *for* $\beta = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$ *and for all real* $\beta \geqslant \frac{1}{2}(p-2)$.

Likewise, we can apply Theorem 1.2 (b) to the elementary symmetric polynomial

$$E_{2,4}(x_1, x_2, x_3, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4, \tag{1.6}$$

which can be represented in the form (1.4) with

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} 1 & e^{-i\pi/3} \\ e^{i\pi/3} & 1 \end{pmatrix}, \tag{1.7}$$

(4) See Proposition 4.3 below.

or equivalently as $E_{2,4}(\mathbf{x})=\det(BXB^*)$ with

$$B=\begin{pmatrix}1 & 0 & 1 & 1\\ 0 & 1 & 1 & e^{i\pi/3}\end{pmatrix}.$$

We obtain the following result.

COROLLARY 1.6. *The function $E_{2,4}^{-\beta}$ is completely monotone on $(0,\infty)^4$ if and only if $\beta=0$ or $\beta\geqslant 1$. In particular, the function*

$$\Bigg(\sum_{1\leqslant i<j\leqslant 4}(1-y_i)(1-y_j)\Bigg)^{-\beta} \tag{1.8}$$

has non-negative Taylor coefficients for all $\beta\geqslant 1$.

Indeed, the "if" part can be corroborated by an explicit Laplace-transform formula for $E_{2,4}^{-\beta}$ for $\beta>1$; see (4.18) below. The "only if" follows from the observation made after Theorem 1.3, since the matrices $A_1,\dots,A_4$ in (1.7) span $\mathrm{Herm}(2,\mathbb{C})$.

The second sentence of Corollary 1.6 answers in the affirmative a question posed long ago by Lewy [8, p. 340], of which Askey remarks that it "has caused me many hours of frustration" [7, p. 56].([5]) (See also the recent discussion in [73].) Indeed, Lewy's question concerned only $\beta=1$, and made the weaker conjecture that the function (1.8) multiplied by $(4-y_1-y_2-y_3-y_4)^{-1}$ has non-negative Taylor coefficients. This latter factor is now seen to be unnecessary.([6])

Similarly, Theorem 1.2 (c) applied to the quaternionic determinant

$$\det\begin{pmatrix}a & q\\ \bar{q} & b\end{pmatrix}=ab-q\bar{q}$$

([5]) Askey [7, p. 56] comments that, in his view, "So far the most powerful method of treating problems of this type is to translate them into another problem involving special functions and then use the results and methods which have been developed for the last two hundred years to solve the special function problem. So far I have been unable to make a reduction in [Lewy's problem] and so have no place to start." But he immediately adds, wisely, that "it is possible to solve some problems without using special functions, so others should not give up on [Lewy's problem]".

([6]) Ismail and Tamhankar [71, p. 483] mistakenly asserted that "the early coefficients in the power series expansion of

$$[(1-r)(1-s)+(1-r)(1-t)+(1-r)(1-u)+(1-s)(1-t)+(1-s)(1-u)+(1-t)(1-u)]^{-1}$$

are positive but the later coefficients do change sign", arguing that this is "because Huygen's [sic] principle holds in three-space". Huygens' principle indeed suggests that the coefficients approach zero, as Askey and Gasper [8, p. 340] observed; but this in no way contradicts the non-negativity of those coefficients.

for $a, b \in \mathbb{R}$ and $q \in \mathbb{H}$, with $A_1, ..., A_4$ as above, and[7]

$$A_5 = \begin{pmatrix} 1 & q_5 \\ \bar{q}_5 & 1 \end{pmatrix}, \qquad A_6 = \begin{pmatrix} 1 & q_6 \\ \bar{q}_6 & 1 \end{pmatrix}, \tag{1.9}$$

$$q_5 = \frac{1}{2} - \frac{\sqrt{3}}{6} i - \frac{\sqrt{6}}{3} j, \qquad q_6 = \frac{1}{2} - \frac{\sqrt{3}}{6} i - \frac{\sqrt{6}}{12} j - \frac{\sqrt{10}}{4} k, \tag{1.10}$$

yields an analogous result for the elementary symmetric polynomial

$$E_{2,6}(x_1, ..., x_6) = x_1 x_2 + x_1 x_3 + ... + x_5 x_6 \tag{1.11}$$

of degree 2 in six variables.[8]

COROLLARY 1.7. *The function $E_{2,6}^{-\beta}$ is completely monotone on $(0, \infty)^6$ if and only if $\beta = 0$ or $\beta \geqslant 2$.*

Corollaries 1.5 and 1.6 are in fact special cases of a much more general result concerning the basis generating polynomials $B_M(\mathbf{x})$ of certain classes of matroids. (We stress that no knowledge of matroid theory is needed to understand the main arguments of this paper; readers allergic to matroids, or simply unfamiliar with them, can skip all references to them without loss of logical continuity. Still, we think that the matroidal perspective is fruitful and we would like to make some modest propaganda for it.[9]) So let M be a matroid with ground set E, and let $\mathcal{B}(M)$ be its set of bases; then the *basis generating polynomial* of M is, by definition,

$$B_M(\mathbf{x}) = \sum_{S \in \mathcal{B}(M)} x^S,$$

where $\mathbf{x} = \{x_e\}_{e \in E}$ is a family of indeterminates indexed by the elements of M, and we have used the shorthand $x^S = \prod_{e \in S} x_e$.

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[7] Here i, j and k are the quaternionic units.

[8] If we define $q_3 = 1$ and $q_4 = e^{-i\pi/3}$ (cf. (1.7)), then q_3, q_4, q_5 and q_6 are quaternions satisfying

$$\mathrm{Re}(q_i \bar{q}_j) = \begin{cases} 1, & \text{if } i = j, \\ \frac{1}{2}, & \text{if } i \neq j, \end{cases}$$

or equivalently $|q_i|^2 = 1$ and $|q_i - q_j|^2 = 1$ for all $i \neq j$. From this it easily follows that

$$\det \begin{pmatrix} x_1 + x_3 + x_4 + x_5 + x_6 & x_3 q_3 + x_4 q_4 + x_5 q_5 + x_6 q_6 \\ x_3 \bar{q}_3 + x_4 \bar{q}_4 + x_5 \bar{q}_5 + x_6 \bar{q}_6 & x_2 + x_3 + x_4 + x_5 + x_6 \end{pmatrix} = E_{2,6}(x_1, ..., x_6).$$

[9] See [90] for background on matroid theory, and [37] for background on basis generating polynomials. In interpreting Corollary 1.8 below, please note that if $G = (V, E)$ is a graph with k connected components, then the graphic matroid $M(G)$ has rank $|V| - k$, while the cographic matroid $M^*(G)$ has rank $|E| - |V| + k$. Note also that the equivalence of "complex-unimodular matroid" with "sixth-root-of-unity matroid" is proven in [37, Theorem 8.9].

Now let B be an arbitrary $m\times n$ real or complex matrix of rank m, and define $P(\mathbf{x})=\det(BXB^*)$. Then, as discussed previously, Theorem 1.2 ((a) or (b)) applies to P and gives a sufficient condition for $P^{-\beta}$ to be completely monotone. On the other hand, the Cauchy–Binet formula gives

$$P(\mathbf{x}) = \det(BXB^*) = \sum_{\substack{S\subseteq[n]\\|S|=m}} |\det B_{\star S}|^2 x^S, \tag{1.12}$$

where $B_{\star S}$ denotes the submatrix of B with columns S. Since $\det B_{\star S}\neq 0$ if and only if the columns S of B are linearly independent, we see that P is a *weighted* version of the basis generating polynomial for the matroid $M=M[B]$ that is represented by B (this matroid has rank m). In particular, a matroid is said to be *real-unimodular* (resp. *complex-unimodular*) if it has a real (resp. complex) representing matrix B, with a number of rows equal to its rank, such that $|\det B_{\star S}|^2\in\{0,1\}$ for all S.([10]) In this case the basis generating polynomial is precisely $B_M(\mathbf{x})=\det(BXB^*)$. We thereby obtain from Theorem 1.2 (a) and (b) the following result.

COROLLARY 1.8. *Let M be a matroid of rank r on the ground set E, and let $B_M(\mathbf{x})$ be its basis generating polynomial.*

(a) *If M is a regular (i.e. real-unimodular) matroid, then $B_M^{-\beta}$ is completely monotone on $(0,\infty)^E$ for $\beta=0,\frac{1}{2},1,\frac{3}{2},\ldots$ and for all real $\beta\geqslant\frac{1}{2}(r-1)$. (This holds in particular if M is a graphic or cographic matroid, i.e. for the spanning-tree or complementary-spanning-tree polynomial of a connected graph.)*

(b) *If M is a complex-unimodular matroid (equivalently, a sixth-root-of-unity matroid), then $B_M^{-\beta}$ is completely monotone on $(0,\infty)^E$ for $\beta=0,1,2,3,\ldots$ and for all real $\beta\geqslant r-1$.*

In particular, by specializing (a) to a graphic matroid $M(G)$ we recover Corollary 1.5, and by specializing (b) to the uniform matroid $U_{2,4}$ we recover Corollary 1.6.

We have also proven a (very) partial converse to Corollary 1.8, which concerns the cases of rank-r n-element simple matroids in which the matrices $A_1,\ldots,A_n$ together span $\mathrm{Sym}(r,\mathbb{R})$ or $\mathrm{Herm}(r,\mathbb{C})$; see Proposition 7.9 below.

Remark. There is also an analogue of Corollary 1.8 in the quaternionic case. Recall first the quaternionic analogue of the Cauchy–Binet formula (1.12) [105, Proposition A.3 (g)]: if B is an $m\times n$ quaternionic matrix, then

$$P(x_1,\ldots,x_n) = \det(BXB^*) = \det\left(\sum_{i=1}^n x_i A_i\right)$$

([10]) This is not the usual definition of real-unimodular/complex-unimodular, but it is equivalent to the usual definition by virtue of [37, Proposition 8.6].

is well defined for $X=\mathrm{diag}(x_1,...,x_n)$ with $x_1,...,x_n$ *real*, and equals the polynomial

$$\sum_{\substack{S\subseteq[n]\\|S|=m}} \det[B_{\star S}(B_{\star S})^*]\, x^S.$$

(Note, by contrast, that $\det B_{\star S}$ is in general meaningless because $B_{\star S}$ need not be Hermitian.) We can then define a matroid M to be *quaternionic-unimodular* if its basis generating polynomial can be represented in this way, i.e. if it has a quaternionic representing matrix B, with a number of rows equal to its rank, such that $\det[B_{\star S}(B_{\star S})^*]\in\{0,1\}$ for all S. For such matroids M, Theorem 1.2 (c) implies that $B_M^{-\beta}$ is completely monotone on $(0,\infty)^E$ for $\beta=0,2,4,6,...$ and for all real $\beta\geqslant 2r-2$.

A deeper study of quaternionic-unimodular matroids would be of interest. For instance, is the class of quaternionic-unimodular matroids closed under duality? Or even under contraction? (The class is obviously closed under deletion.) Which uniform matroids $U_{r,n}$ are quaternionic-unimodular?

A different notion of "quaternionic-unimodular matroid" has been introduced recently by Pendavingh and van Zwam [96]. It is not clear to us what is the relation between their notion and ours.

1.3. Quadratic forms

Of course, $E_{2,4}$ and $E_{2,6}$ are quadratic forms in the variables $\mathbf{x}$, as is the polynomial $E_{2,3}$ arising in the $n=3$ Szegő problem. This suggests that it might be fruitful to study more general quadratic forms. We shall prove, by elementary methods, the following result.

THEOREM 1.9. *Let V be a finite-dimensional real vector space, let B be a symmetric bilinear form on V having inertia (n_+,n_-,n_0), and define the quadratic form $Q(x)=B(x,x)$. Let $C\subset V$ be a non-empty open convex cone with the property that $Q(x)>0$ for all $x\in C$. Then $n_+\geqslant 1$, and moreover the following are true*:

(a) *If $n_+=1$ and $n_-=0$, then $Q^{-\beta}$ is completely monotone on C for all $\beta\geqslant 0$. For all other values of β, $Q^{-\beta}$ is not completely monotone on any non-empty open convex subcone $C'\subseteq C$.*

(b) *If $n_+=1$ and $n_-\geqslant 1$, then $Q^{-\beta}$ is completely monotone on C for $\beta=0$ and for all $\beta\geqslant\frac{1}{2}(n_--1)$. For all other values of β, $Q^{-\beta}$ is not completely monotone on any non-empty open convex subcone $C'\subseteq C$.*

(c) *If $n_+>1$, then $Q^{-\beta}$ is not completely monotone on any non-empty open convex subcone $C'\subseteq C$ for any $\beta\neq 0$.*

Theorem 1.9 follows fairly easily from the classic work of Marcel Riesz [99] (see also [47] and [52, Chapter VII]) treating the case in which B is the Lorentz form on $\mathbb{R}^n$,

$$B(x,y) = x_1y_1 - x_2y_2 - \ldots - x_ny_n, \tag{1.13}$$

and C is the Lorentz cone (that is, forward light cone) $\left\{x \in \mathbb{R}^n : x_1 > \sqrt{x_2^2 + \ldots + x_n^2}\right\}$. We are able to give a completely elementary proof of both the sufficiency and the necessity; and we are able to give in case (b) an explicit Laplace-transform formula for $Q^{-\beta}$ (see Proposition 5.6).

Specializing Theorem 1.9 with $V = \mathbb{R}^n$ and $C = (0,\infty)^n$ to the degree-2 elementary symmetric polynomials

$$E_{2,n}(x_1, \ldots, x_n) = \sum_{1 \leqslant i < j \leqslant n} x_i x_j, \tag{1.14}$$

we obtain the following result.

COROLLARY 1.10. *The function $E_{2,n}^{-\beta}$ is completely monotone on $(0,\infty)^n$ if and only if $\beta = 0$ or $\beta \geqslant \frac{1}{2}(n-2)$.*

By this method we obtain alternative proofs of Corollaries 1.6 and 1.7—hence in particular a second solution to the Lewy–Askey problem—as well as of Szegő's [113] original result in the case $n = 3$.([11]) We also obtain an explicit Laplace-transform formula for $E_{2,n}^{-\beta}$ (see Corollary 5.8).

Remark. It is easy to see that $E_{2,n}$ is the spanning-tree polynomial of a graph only if $n = 2$ or $n = 3$: a connected graph G whose spanning-tree polynomial is of degree 2 must have precisely three vertices; if G has multiple edges, then $T_G \neq E_{2,n}$ because monomials corresponding to pairs of parallel edges are absent from T_G; so G must be either the 3-vertex path or the 3-cycle, corresponding to $E_{2,2}$ or $E_{2,3}$, respectively. But this fact can also be seen from our results: Corollary 1.5 says that $T_G^{-1/2}$ is completely monotone for all graphs G, while Corollary 1.10 says that $E_{2,n}^{-1/2}$ is not completely monotone when $n > 3$.

([11]) In fancy language—which is, however, completely unnecessary for understanding our proofs—our "determinantal" proof of Corollary 1.6 is based on harmonic analysis on the cone of positive-definite $m \times m$ complex Hermitian matrices specialized to $m = 2$, while our "quadratic form" proof is based on harmonic analysis on the Lorentz cone in $\mathbb{R}^n$ specialized to $n = 4$. The point here is that the Jordan algebra $\mathrm{Herm}(2,\mathbb{C}) \simeq \mathbb{R} \times \mathbb{R}^3$ can be viewed as a member of two different families of Jordan algebras: $\mathrm{Herm}(m,\mathbb{C})$ and $\mathbb{R} \times \mathbb{R}^{n-1}$ [52, p. 98]. Likewise, our "determinantal" proof of Corollary 1.7 is based on harmonic analysis on the cone of positive-definite $m \times m$ quaternionic Hermitian matrices specialized to $m = 2$, while our "quadratic form" proof is based on harmonic analysis on the Lorentz cone in $\mathbb{R}^n$ specialized to $n = 6$; and we have the isomorphism of Jordan algebras $\mathrm{Herm}(2,\mathbb{H}) \simeq \mathbb{R} \times \mathbb{R}^5$ [52, p. 98]. And finally, our "determinantal" proof of the $n = 3$ Szegő result is based on harmonic analysis on the cone of positive-definite $m \times m$ real symmetric matrices specialized to $m = 2$, while our "quadratic form" proof is based on harmonic analysis on the Lorentz cone in $\mathbb{R}^n$ specialized to $n = 3$; and we have $\mathrm{Sym}(2,\mathbb{R}) \simeq \mathbb{R} \times \mathbb{R}^2$ [52, p. 98].

Corollaries 1.6, 1.7 and 1.10 lead naturally to the following question: If we write $E_{r,n}$ for the elementary symmetric polynomial of degree r in n variables,

$$E_{r,n}(x_1, ..., x_n) = \sum_{1 \leqslant i_1 < i_2 < ... < i_r \leqslant n} x_{i_1} x_{i_2} \dots x_{i_r} \tag{1.15}$$

(where we set $E_{0,n} \equiv 1$), then for which $\beta > 0$ is $E_{r,n}^{-\beta}$ completely monotone on $(0,\infty)^n$? The cases $r=0$, $r=1$ and $r=n$ are trivial: we have complete monotonicity for all $\beta \geqslant 0$. Our results for the cases $r=n-1$ (Theorem 1.1′ specialized to cycles C_n) and $r=2$ (Corollary 1.10), as well as numerical experiments for $(r,n)=(3,5)$, $(r,n)=(3,6)$ and $(r,n)=(4,6)$, suggest the following conjecture.

CONJECTURE 1.11. *Let $2 \leqslant r \leqslant n$. Then $E_{r,n}^{-\beta}$ is completely monotone on $(0,\infty)^n$ if and only if $\beta = 0$ or $\beta \geqslant \frac{1}{2}(n-r)$.*

However, we have been unable to find a proof of either the necessity or the sufficiency.

We remark that the elementary symmetric polynomial $E_{r,n}$ is the basis generating polynomial of the uniform matroid $U_{r,n}$. So Corollary 1.10 and Conjecture 1.11 concern the same general subject as Corollary 1.8, namely, complete monotonicity for inverse powers of the basis generating polynomials of matroids.

1.4. Discussion

In summary, we have two ab-initio methods for proving, given a polynomial P and a positive real number β, that $P^{-\beta}$ is completely monotone on $(0,\infty)^n$ (or more generally on a convex cone C):

(a) The determinantal method (Theorems 1.2–1.4; see §4).

(b) The quadratic-form method (Theorem 1.9; see §5).

Interestingly, these two methods can be viewed as versions of the *same* construction, involving the determinant on a Euclidean Jordan algebra and the Laplace-transform representation of its inverse powers [52, Chapters II–VII]. We discuss this connection in §4.3 and §5.2.

In addition to these two ab-initio methods, we have a variety of constructions that, given such polynomials, can create other ones with the same property (see §3). Among these are algebraic analogues of the graph (or matroid) operations of deletion, contraction, direct sum([12]), parallel connection, series connection and 2-sum (but *not* duality). By combining these operations with our ab-initio proofs, we are able to prove the complete monotonicity of $T_G^{-\beta}$ for some values of β beyond those covered by Corollary 1.5; we do it in the following proposition.

([12]) By "direct sum" of graphs we mean either disjoint union ("0-sum") or gluing at a cut vertex ("1-sum"). Both of these operations correspond to the direct sum of matroids.

PROPOSITION 1.12. *Fix $p \geqslant 2$, and let $G=(V,E)$ be any graph that can be obtained from copies of the complete graph K_p by parallel connection, series connection, direct sum, deletion and contraction. Then $T_G^{-\beta}$ is completely monotone on $(0,\infty)^E$ for $\beta = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and for all real $\beta \geqslant \frac{1}{2}(p-2)$.*

In particular, the case $p=3$ covers series-parallel graphs; this is essentially our proof of the direct half of Theorem 1.1′. We also have versions of this proposition for matroids: see Propositions 7.14 and 7.15 below. Finally, in Propositions 7.16 and 7.17 we give excluded-minors characterizations of the class of graphs/matroids handled by Propositions 1.12 and 7.14, respectively.

But even in the graphic case, we are still far from having a complete answer to the following fundamental problem.

Problem 1.13. Given a graph $G=(V,E)$, for which real numbers $\beta>0$ is the function $T_G^{-\beta}$ completely monotone on $(0,\infty)^E$?

This question can be rephrased usefully as follows:

Problem 1.13′. For each $\beta>0$, characterize the class $\mathcal{G}_\beta$ of graphs for which $T_G^{-\beta}$ is completely monotone on $(0,\infty)^E$.

We will show in §7.1 that the class $\mathcal{G}_\beta$ is closed under minors—so that it can be characterized by listing the excluded minors—and under parallel connection. Furthermore, it is closed under series connection when (but only when) $\beta \geqslant \frac{1}{2}$.

In this paper we have solved Problem 1.13′ in a few cases:

(i) For $\beta \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$, $\mathcal{G}_\beta = \{\text{all graphs}\}$. See Corollary 1.8 (a).

(ii) For $\beta \in (0, \frac{1}{2})$, $\mathcal{G}_\beta =$ graphs obtained from forests by parallel extension of edges (that is, graphs with no K_3 minor). See Theorem 7.10.

(iii) For $\beta \in (\frac{1}{2}, 1)$, $\mathcal{G}_\beta = \{\text{series-parallel graphs}\}$ (that is, graphs with no K_4 minor). See Theorems 1.1′ and 7.12.

So the first unsolved cases are $\beta \in (1, \frac{3}{2})$: Might it be true that $\mathcal{G}_\beta = \{\text{all graphs with no } K_5 \text{ minor}\}$? Or might there exist, alternatively, other excluded minors? We have been thus far unable to determine the complete monotonicity of $T_G^{-\beta}$ for the cases $G=W_4$ (the wheel with four spokes) and $G=K_5-e$ (the complete graph K_5 with one edge deleted). Indeed, for $k \geqslant 2$ we do not even know the answer to the following question.

Question 1.14. Fix an integer $k \geqslant 0$. Must we have $\mathcal{G}_\beta = \mathcal{G}_{\beta'}$ whenever

$$\beta, \beta' \in \left(\tfrac{1}{2}k, \tfrac{1}{2}(k+1)\right)?$$

Let us mention, finally, an alternative approach to "converse" results that we have not pursued, for lack of competence. When $P^{-\beta}$ does *not* have all non-negative Taylor

coefficients, this fact should in most cases be provable either by explicit computation of low-order coefficients or by asymptotic computation of suitable families of high-order coefficients (computer experiments can usually suggest which families to focus on). This type of multivariate asymptotic calculation has been pioneered recently by Pemantle and collaborators [93], [94], [95], [92], [15] and involves some rather non-trivial algebraic geometry/topology. In fact, Baryshnikov and Pemantle [15, §4.4] have recently used their method to study the asymptotics of the Taylor coefficients of $P_n^{-\beta}$ for the Szegő polynomial (1.1) with $n=3$, but thus far only for $\beta>\frac{1}{2}$.([13]) It would be interesting to know whether this analysis can be extended to the case $\beta<\frac{1}{2}$, thereby providing an explicit proof that some of the Taylor coefficients are asymptotically negative. More generally, one might try to study the elementary symmetric polynomials $E_{r,n}$: after the $n=3$ Szegő case $E_{2,3}$, the next simplest would probably be the Lewy–Askey case $E_{2,4}$ (i.e., (1.8)).

1.5. Some further remarks

The half-plane property. Let us recall that a polynomial P with complex coefficients is said to have the *half-plane property* [37], [36], [117], [24], [23], [120], [26], [119] if either $P\equiv 0$ or else $P(x_1,\ldots,x_n)\neq 0$ whenever $x_1,\ldots,x_n$ are complex numbers with strictly positive real part.([14]) We shall show (Corollary 2.3 below) that if P is a polynomial with real coefficients that is strictly positive on $(0,\infty)^n$ and such that $P^{-\beta}$ is completely monotone on $(0,\infty)^n$ for at least one $\beta>0$, then P necessarily has the half-plane property (but not conversely). The complete monotonicity of $P^{-\beta}$ can therefore be thought of as a strong quantitative form of the half-plane property. In particular, it follows that the determinantal polynomials considered in Theorem 1.2 have the half-plane property—a fact that can easily be proven directly (Corollary 4.5 below). The same is true for the quadratic polynomials considered in Theorem 1.9; see [37, Theorem 5.3] and Theorem 5.4 below.

The Rayleigh property. Complete monotonicity is also connected with the Rayleigh property [38] for matroids and, more generally, for multiaffine polynomials. Let us say that a function f is *completely monotone of order* K if the inequalities (1.3) hold for $0\leqslant k\leqslant K$. Thus, a function is completely monotone of order 0 (resp. 1) if and only if it is non-negative (resp. non-negative and decreasing). A function is completely monotone of

([13]) The formula in their Theorem 4.4 has a misprint: the power $-\frac{1}{2}$ should be $\beta-\frac{3}{2}$.
([14]) A polynomial $P\not\equiv 0$ with the half-plane property is also termed *Hurwitz stable*.

order 2 if, in addition, $\partial^2 f/\partial x_i \partial x_j \geqslant 0$ for all i and j. Specializing this to $f = P^{-\beta}$, where P is a polynomial, we obtain

$$P \frac{\partial^2 P}{\partial x_i \partial x_j} \leqslant (\beta+1) \frac{\partial P}{\partial x_i} \frac{\partial P}{\partial x_j} \quad \text{for all } i \text{ and } j. \tag{1.16}$$

If P is multiaffine, then $\partial^2 P/\partial x_i^2 = 0$, so it suffices to consider the cases $i \neq j$. The inequality (1.16) is then a generalization of the Rayleigh (or negative-correlation) inequality in which an extra constant $C = \beta+1$ is inserted on the right-hand side. (The ordinary Rayleigh property corresponds to $\beta \downarrow 0$, and hence to taking $f = -\log P$ and omitting the $k=0$ condition.) It would be interesting to know whether the combinatorial consequences of the Rayleigh property—such as the matroidal-support property [118]—extend to the C-Rayleigh property for arbitrary $C < \infty$. It would also be interesting to extend the results of the present paper, which address complete monotonicity of order ∞, to complete monotonicity of finite orders K. In what way do the conditions on β become K-dependent?

Connected-spanning-subgraph polynomials. Let us remark that the literature contains some other examples of multivariate polynomials P for which $P^{-\beta}$ has all nonnegative Taylor coefficients, for some specified set of numbers β. For instance, Askey and Gasper [9] showed that this is the case for

$$P(x,y,z) = 1 - \tfrac{1}{2}(x+y+z) + \tfrac{1}{2}xyz \tag{1.17}$$

whenever $\beta \geqslant \frac{1}{2}(\sqrt{17}-3) \approx 0.561553$; Gillis, Reznick and Zeilberger [57] later gave an elementary proof. Likewise, Koornwinder [75] proved this for

$$P(x,y,z,u) = 1 - \tfrac{1}{2}(x+y+z+u) + \tfrac{1}{2}(xyz+xyu+xzu+yzu) - xyzu \tag{1.18}$$

whenever $\beta \geqslant 1$; an elementary proof later emerged from the combined work of Ismail and Tamhankar [71] and Gillis–Reznick–Zeilberger [57].

It turns out that these two examples also have a combinatorial interpretation: not in terms of the spanning-tree polynomial $T_G(\mathbf{x})$, but rather in terms of the connected-spanning-subgraph polynomial [102], [109]

$$C_G(\mathbf{v}) = \sum_{\substack{A \subseteq E \\ (V,A) \text{ connected}}} \prod_{e \in A} v_e, \tag{1.19}$$

which has $T_G(\mathbf{x})$ as a limiting case:

$$T_G(\mathbf{x}) = \lim_{\lambda \to 0} \lambda^{-(|V|-1)} C_G(\lambda \mathbf{x}). \tag{1.20}$$

If we specialize to $G=C_n$ and make the change of variables $v_i=-\lambda(1-z_i)$ with $0<\lambda<n$, thus defining $P_{G,\lambda}(\mathbf{z})=T_G(-\lambda(\mathbf{1}-\mathbf{z}))/T_G(-\lambda\mathbf{1})$, it then turns out that the Askey–Gasper polynomial (1.17) corresponds to the case $n=3$, $\lambda=1$, while the Koornwinder polynomial (1.18) corresponds to the case $n=4$, $\lambda=2$. On the other hand, in the limit $\lambda\to 0$ we recover a multiple of the Szegő polynomial (1.1); this is simply a special case of (1.20).

In the same way that the complete monotonicity of $T_G^{-\beta}$ is a strong quantitative form of the half-plane property, it turns out that the non-negativity of Taylor coefficients of $P_{G,\lambda}^{-\beta}$ in these examples is a strong quantitative form of the multivariate Brown–Colbourn property (or more precisely, the multivariate property BC_λ) discussed in [102], [109]. But it seems to be a difficult problem to determine the set of pairs (λ,β) for which $P_{G,\lambda}^{-\beta}$ has non-negative Taylor coefficients, even in the simplest case $G=C_3$. We have some partial results on this problem, but we leave these for a future paper.

1.6. Plan of this paper

The plan of this paper is as follows: In §2 we define complete monotonicity on cones and recall the Bernstein–Hausdorff–Widder–Choquet theorem; we also prove a general result showing that complete monotonicity of $P^{-\beta}$ on a cone $C\subset V$ implies the non-vanishing of P in the complex tube $C+iV$.([15]) In §3 we discuss some general constructions by which new completely monotone polynomials P with $P^{-\beta}$ can be obtained from old ones. In §4 we present the determinantal construction and prove Theorems 1.2–1.4. In §5 we present the quadratic-form construction and prove Theorem 1.9. In §6 we present briefly the theory of positive-definite functions (in the semigroup sense) on convex cones—which is a close relative of the theory of completely monotone functions—and its application to the class of cones treated here. Finally, in §7 we apply the results of §§2–5 to the spanning-tree polynomials of graphs and the basis generating polynomials of matroids; in particular we analyze the series-parallel case and prove Theorem 1.1.

In the arXiv version of this paper [105] we include two appendices that are being omitted from this journal version due to space constraints: Appendix A reviewing the definition and main properties of the Moore determinant for Hermitian quaternionic matrices, and Appendix B explaining an elementary proof of Gindikin's characterization of parameters for which the Riesz distribution is a positive measure.

We have tried hard to make this paper comprehensible to the union (not the intersection!) of combinatorialists and analysts. We apologize in advance to experts in each of these fields for boring them every now and then with overly detailed explanations of elementary facts.

([15]) Here, and in similar situations, i denotes the imaginary unit.

2. Complete monotonicity on cones

In the introduction we defined complete monotonicity for functions on $(0,\infty)^n$. For our later needs (see §4 and §5), it turns out to be natural to consider complete monotonicity on more general open convex cones $C \subset \mathbb{R}^n$. This is a genuine generalization, because for $n \geqslant 3$, an open convex cone is not necessarily the affine image of a (possibly higher-dimensional) orthant, i.e. it need not have "flat sides": an example is the Lorentz cone $\{\mathbf{x} \in \mathbb{R}^n : x_1 > \sqrt{x_2^2 + \ldots + x_n^2}\}$ in dimension $n \geqslant 3$.

Definition 2.1. Let V be a finite-dimensional real vector space, and let C be an open convex cone in V. Then a C^∞ function $f : C \to \mathbb{R}$ is termed *completely monotone* if for all $k \geqslant 0$, all choices of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k \in C$, and all $x \in C$, we have

$$(-1)^k D_{\mathbf{u}_1} \ldots D_{\mathbf{u}_k} f(x) \geqslant 0, \tag{2.1}$$

where $D_{\mathbf{u}}$ denotes a directional derivative. A function f is termed *conditionally completely monotone* if the inequality (2.1) holds for all $k \geqslant 1$ but not necessarily for $k = 0$.([16])

Of course, if the inequality (2.1) holds for all $\mathbf{u}_1, \ldots, \mathbf{u}_k$ in some set S, then by linearity and continuity it holds also for all $\mathbf{u}_1, \ldots, \mathbf{u}_k$ in the closed convex cone generated by S. This observation also shows the equivalence of Definition 2.1, specialized to the case $V = \mathbb{R}^n$ and $C = (0,\infty)^n$, with the definition given in the introduction.

If $T : (V_1, C_1) \to (V_2, C_2)$ is a positive linear map (i.e., a linear map $T : V_1 \to V_2$ satisfying $T[C_1] \subseteq C_2$) and $f : C_2 \to \mathbb{R}$ is completely monotone, then it is easily seen that $f \circ T : C_1 \to \mathbb{R}$ is completely monotone. Conversely, if $f \circ T$ is completely monotone for *all* positive linear maps $T : (\mathbb{R}^n, (0,\infty)^n) \to (V_2, C_2)$ *for arbitrarily large* n, then f is completely monotone: for if (2.1) fails for some k, then we can take $n = k$ and $T\mathbf{e}_i = \mathbf{u}_i$ (where $\mathbf{e}_i$ is the ith coordinate unit vector in $\mathbb{R}^n$) and $f \circ T$ will fail one of the kth-order complete-monotonicity inequalities.

Let us next recall some elementary facts. If f is completely monotone, then $f(0^+) = \lim_{x \to 0, x \in C} f(x)$ exists and equals $\sup_{x \in C} f(x)$, but it might be $+\infty$. The product of two

([16]) The terminology "conditionally completely monotone" is new, but we think it felicitous: it is chosen by analogy with "conditionally positive-definite matrix" [18], [14], with which this concept is in fact closely related [66], [67], [68], [18], [17]. (*Warning*: The book [18] uses the term "negative-definite" for what we would call "conditionally negative-definite".)

Please note that if f is bounded below, then f is conditionally completely monotone if and only if there exists a constant c such that $f + c$ is completely monotone. But there also exist conditionally completely monotone functions that are unbounded below (and hence for which such a constant c cannot exist): examples on $(0,\infty)$ are $f(x) = -(a+x)^\alpha$ and $f(x) = -\log(a+x)$ with $a \geqslant 0$ and $0 < \alpha \leqslant 1$.

Of course, it follows immediately from the definition that f is conditionally completely monotone if and only if $-D_{\mathbf{u}} f$ is completely monotone for all vectors $\mathbf{u} \in C$. In the multidimensional case this seems rather difficult to work with; but in the 1-dimensional case $C = (0,\infty)$ it says that f is conditionally completely monotone if and only if $-f'$ is completely monotone.

completely monotone functions is completely monotone. If f is completely monotone and $\Phi\colon [0,\infty)\to[0,\infty)$ is absolutely monotone (i.e. its derivatives of all orders are everywhere non-negative), then $\Phi\circ f$ is completely monotone. If f is conditionally completely monotone and $\Phi\colon (-\infty,\infty)\to[0,\infty)$ is absolutely monotone, then $\Phi\circ f$ is completely monotone. (In particular, this occurs when Φ is the exponential function.) Finally, a locally uniform limit of a sequence of completely monotone functions is completely monotone.

The fundamental fact in the theory of completely monotone functions on $(0,\infty)$ is the Bernstein–Hausdorff–Widder theorem [124]: A function f defined on $(0,\infty)$ is completely monotone if and only if it can be written in the form

$$f(x) = \int_0^\infty e^{-tx}\, d\mu(t), \tag{2.2}$$

where μ is a non-negative Borel measure on $[0,\infty)$. We shall need a multidimensional version of the Bernstein–Hausdorff–Widder theorem, valid for arbitrary cones. Such a result was proven by Choquet [39].([17])

THEOREM 2.2. (Bernstein–Hausdorff–Widder–Choquet theorem) *Let V be a finite-dimensional real vector space, let C be an open convex cone in V, and let*

$$C^* = \{\ell \in V^* : \langle \ell, x\rangle \geqslant 0 \text{ for all } x \in C\}$$

be the closed dual cone. Then a function $f\colon C\to\mathbb{R}$ is completely monotone if and only if there exists a positive measure μ on C^* satisfying

$$f(x) = \int_{C^*} e^{-\langle \ell, x\rangle}\, d\mu(\ell). \tag{2.3}$$

In this case, $\mu(C^*)=f(0^+)$; in particular, μ is finite if and only if f is bounded.

Consequently, if f is completely monotone on C, then it is extendible (using (2.3)) to an analytic function on the complex tube $C+iV$ satisfying

$$|D_{\mathbf{u}_1} \ldots D_{\mathbf{u}_k} f(x+iy)| \leqslant (-1)^k D_{\mathbf{u}_1} \ldots D_{\mathbf{u}_k} f(x) \tag{2.4}$$

for all $k\geqslant 0$, $x\in C$, $y\in V$ and $\mathbf{u}_1, \ldots, \mathbf{u}_k \in C$.

Remarks. (1) Since C is non-empty and open, it is not hard to see that $\ell\in C^*\setminus\{0\}$ implies $\langle \ell, x\rangle > 0$ for all $x\in C$. It then follows from (2.3) that either

(a) μ is supported on $\{0\}$, in which case f is constant, or else

(b) we have the *strict* inequality

$$(-1)^k D_{\mathbf{u}_1} \ldots D_{\mathbf{u}_k} f(x) > 0 \tag{2.5}$$

for all $k\geqslant 0$, all $\mathbf{u}_1, \ldots, \mathbf{u}_k \in C$, and all $\mathbf{x}\in C$.

([17]) See also Nussbaum [87], Devinatz and Nussbaum [43], Hirsch [64, §VII.2], Glöckner [59, §16] and Thomas [115].

Furthermore, for $\mathbf{u}_1, ..., \mathbf{u}_k$ in the *closure* of C, the left-hand side of (2.5) is either strictly positive for all $x \in C$ or else identically zero on C. Of course, the latter case can occur: e.g. $f(x_1, x_2) = e^{-x_1}$ on $(0, \infty)^2$.

(2) In our definition of complete monotonicity, the cone C plays two distinct roles: it is the domain on which f is defined, and it provides the direction vectors $\mathbf{u}_i$ for which the inequalities (2.1) hold. Choquet [39] elegantly separates these roles, and considers functions on an arbitrary open set $\Omega \subseteq V$ that are completely monotone with respect to the cone C. He then proves the integral representation (2.3) under the hypothesis $\Omega + C \subseteq \Omega$. This is a beautiful generalization, but we shall not need it.

By virtue of Theorem 2.2, one way to test a function f for complete monotonicity is to compute its inverse Laplace transform and ask whether it is non-negative and supported on C^*. Of course, this procedure is not necessarily well defined, because the inverse Laplace transform need not exist; moreover, if it does exist, it may need to be understood as a distribution in the sense of Schwartz [104] rather than as a pointwise-defined function. But we can say this: If $f: C \to \mathbb{R}$ is the Laplace transform of a distribution T on V^*, then f is completely monotone if and only if T is positive (and hence a positive measure) and supported on C^*. This follows from the injectivity of the Laplace transform on the space $\mathcal{D}'(\mathbb{R}^n)$ of distributions [104, p. 306]. Note that the complete monotonicity of f can fail either because T fails to be positive, or because T fails to be supported on C^*. In the former case, we can conclude that f is not completely monotone on *any* non-empty open convex subcone $C' \subseteq C$. In the latter case, T might possibly be supported on some larger proper cone; but if it is not (e.g. if the smallest convex cone containing the support of T is all of V^*), then once again we can conclude that f is not completely monotone on any non-empty open convex subcone $C' \subseteq C$. And finally, if f is *not* the Laplace transform of any distribution on V^*, then it is certainly not the Laplace transform of a positive measure, and hence is not completely monotone on any non-empty open convex subcone $C' \subseteq C$.

In the applications to be made in this paper, the functions f will typically be of the form $F^{-\beta}$, where F is a function (usually a polynomial) that is strictly positive on the cone C and has an analytic continuation to the tube $C + iV$ (for polynomials this latter condition of course holds trivially). The following corollary of Theorem 2.2 shows that the complete monotonicity of $F^{-\beta}$ on the real cone C implies the absence of zeros of F in the complex tube $C + iV$.

COROLLARY 2.3. *Let V be a finite-dimensional real vector space and let C be an open convex cone in V. Let F be an analytic function on the tube $C + iV$ that is real and strictly positive on C. If $F^{-\beta}$ is completely monotone on C for at least one $\beta > 0$,*

then F is non-vanishing on $C+iV$. (In particular, when $V=\mathbb{R}^n$ and $C=(0,\infty)^n$, the function F has the half-plane property.)

Proof. Suppose that $G=F^{-\beta}$ is completely monotone on C; then by Theorem 2.2 it has an analytic continuation to $C+iV$ (call it also G). Now suppose that

$$S=\{z\in C+iV : F(z)=0\}$$

is non-empty. Choose a simply connected domain $D\subset(C+iV)\setminus S$ such that $D\cap C\neq\varnothing$ and $\overline{D}\cap S\neq\varnothing$.([18]) Then $H=F^{-\beta}$ is a well-defined analytic function on D (we take the branch that is real and positive on $D\cap C$). On the other hand, H coincides with G on the real environment $D\cap C$, so it must coincide with G everywhere in D. But $\lim_{z\to z_0}|H(z)|=\infty$ for all $z_0\in\overline{D}\cap S$, which contradicts the analyticity of G on $C+iV$. □

Remarks. (1) It also follows that the analytic function $G=F^{-\beta}$ defined on $C+iV$ is non-vanishing there.

(2) The hypothesis that F have an analytic continuation to $C+iV$ is essential; it cannot be derived as a consequence of the complete monotonicity of $F^{-\beta}$ on C. To see this, take $V=\mathbb{R}$ and $C=(0,\infty)$ and consider $F(x)=\bigl(1+\frac{1}{2}e^{-x}\bigr)^{-1/\beta}$ with any $\beta>0$.

(3) The converse of Corollary 2.3 is easily seen to be false: for instance, the univariate polynomial $P(x)=1+x^2$ has the half-plane property (i.e. is non-vanishing for $\operatorname{Re}x>0$), but $P^{-\beta}$ is not completely monotone on $(0,\infty)$ for any $\beta>0$. The same holds for the bivariate multiaffine polynomial $P(x_1,x_2)=1+x_1x_2$. So the complete monotonicity of $P^{-\beta}$ for some $\beta>0$ is strictly stronger than the half-plane property. It would be interesting to know whether similar counterexamples can be found if P is required to be homogeneous, or homogeneous and multiaffine.

In this paper we will typically consider a polynomial P that is strictly positive on an open convex cone C, and we will ask for which values of β the function $P^{-\beta}$ is completely monotone. We begin with a trivial observation: If P is a non-constant polynomial, then $P^{-\beta}$ cannot be completely monotone on any non-empty open convex cone for any $\beta<0$ (because P grows at infinity in all directions except at most a variety of codimension 1); and $P^{-\beta}$ is trivially completely monotone for $\beta=0$. So we can restrict attention to $\beta>0$.

([18]) For instance, let Ω be a simply connected open subset of C whose closure is a compact subset of C and which satisfies $(\Omega+iV)\cap S\neq\varnothing$, fix a norm $\|\cdot\|$ on V and let

$$R=\inf_{\substack{x\in\Omega\\ y\in V\\ x+iy\in S}}\|y\|.$$

By compactness we must have $R>0$ (for otherwise we would have $\overline{\Omega}\cap S\neq\varnothing$, contrary to the hypothesis that $F>0$ on C). Now take $D=\Omega+iB_R$, where $B_R=\{y\in V:\|y\|<R\}$.

Given a function $F: C \to (0,\infty)$—for instance, a polynomial—we can ask about the set

$$\mathcal{B}_F = \{\beta > 0 : F^{-\beta} \text{ is completely monotone on } C\}. \tag{2.6}$$

Clearly $\mathcal{B}_F$ is a closed additive subset of $(0,\infty)$. In particular, we either have $\mathcal{B}_F \subseteq [\varepsilon,\infty)$ for some $\varepsilon>0$ or else $\mathcal{B}_F = (0,\infty)$. The following easy lemma [66], [17] characterizes the latter case.

LEMMA 2.4. *Let V be a finite-dimensional real vector space, let C be an open convex cone in V, and let $F: C \to (0,\infty)$. Then the following are equivalent*:

(a) *$F^{-\beta}$ is completely monotone on C for all $\beta>0$;*

(b) *$F^{-\beta_i}$ is completely monotone on C for a sequence $\{\beta_i\}_{i=1}^{\infty}$ of strictly positive numbers converging to* 0;

(c) *$-\log F$ is conditionally completely monotone on C.*

Proof. (a) $\Rightarrow$ (b) is trivial, and (b) $\Rightarrow$ (c) follows from

$$-\log F = \lim_{\beta \downarrow 0} \frac{F^{-\beta}-1}{\beta} \tag{2.7}$$

and its derivatives with respect to x. Finally, (c) $\Rightarrow$ (a) follows from $F^{-\beta} = \exp(-\beta \log F)$ and the fact that exp is absolutely monotone (i.e. has all derivatives non-negative) on $(-\infty,\infty)$. □

Already for $C=(0,\infty)$ it seems to be a difficult problem to characterize in a useful way the functions F described in Lemma 2.4, or even the subclass consisting of polynomials P.[19] For polynomials $P(x) = \prod_i (1+x/x_i)$, a *necessary* condition from Corollary 2.3 is that P have the half-plane property, i.e. $\operatorname{Re} x_i \geqslant 0$ for all i. A *sufficient* condition is that all x_i be real and positive; and for quadratic polynomials this condition is necessary as well. But already for quartic polynomials the situation becomes more complicated: for instance, we can take $x_1 = a+bi$, $x_2 = a-bi$ and $x_3 = x_4 = c$ with $0 < c \leqslant a$ and $b \in \mathbb{R}$, and it is not hard to see that $-\log P$ is conditionally completely monotone on $(0,\infty)$.[20]

It also seems to be a difficult problem to characterize the closed additive subsets $S \subseteq (0,\infty)$ that can arise as $S = \mathcal{B}_F$.

Example 2.5. Fix $a>0$, and let $F(x) = (1+ae^{-x})^{-1}$. Then the function $F(x)^{-\beta} = (1+ae^{-x})^{\beta}$ is obviously completely monotone on $(0,\infty)$ whenever $\beta \in \{0,1,2,3,\ldots\}$. On

(19) Functions $f = F^{-1}$ for which f^{β} is completely monotone for all $\beta>0$ are sometimes called *logarithmically completely monotone* [17].

(20) The function $-\log P$ is conditionally completely monotone on $(0,\infty)$ if and only if $(\log P)' = \sum_i (x+x_i)^{-1}$ is completely monotone on $(0,\infty)$; and this happens if and only if its inverse Laplace transform, which is $g(t) = \sum_i e^{-tx_i}$, is non-negative on $[0,\infty)$.

the other hand, if $\beta \notin \{0,1,2,3,\dots\}$ we claim that $F^{-\beta}$ is *not* completely monotone. Indeed, for $0<a\leqslant 1$ the convergent binomial expansion

$$F(x)^{-\beta} = \sum_{k=0}^{\infty} a^k \binom{\beta}{k} e^{-kx} \tag{2.8}$$

shows that $F^{-\beta}$ is the Laplace transform of the signed measure

$$\sum_{k=0}^{\infty} a^k \binom{\beta}{k} \delta_k,$$

which is non-negative if and only if $\beta \in \{0,1,2,3,\dots\}$. On the other hand, for $a>1$ the function $F^{-\beta}$ has singularities in the right half-plane at $x = \log a \pm i\pi$ whenever $\beta \notin \{0,1,2,3,\dots\}$, so it is not the Laplace transform of *any* distribution.

Example 2.6. It is an interesting problem [6], [10], [54], [56], [81], [125] to determine the pairs $(\mu,\lambda) \in \mathbb{R}^2$ for which the function

$$F_{\mu,\lambda}(x) = x^{-\mu}(x^2+1)^{-\lambda} \tag{2.9}$$

is completely monotone on $(0,\infty)$. It is easy to show that there is a function $\mu_\star(\lambda)$ such that $F_{\mu,\lambda}$ is completely monotone if and only if $\mu \geqslant \mu_\star(\lambda)$; furthermore, the function $\mu_\star$ is subadditive. The state of current knowledge about $\mu_\star$ seems to be:

$$\mu_\star(\lambda) = -2\lambda \qquad \text{for } \lambda \leqslant 0, \tag{2.10}$$

$$\lambda < \mu_\star(\lambda) \leqslant \min\{2\lambda, 1\} \qquad \text{for } 0 < \lambda < 1, \tag{2.11}$$

$$\mu_\star(\lambda) = 2\lambda + o(\lambda) \qquad \text{for } \lambda \downarrow 0 \tag{2.12}$$

$$\mu_\star(\lambda) = \lambda \qquad \text{for } \lambda \geqslant 1. \tag{2.13}$$

It seems to be an open problem even to prove that $\mu_\star$ is continuous.

3. Constructions

In this section we discuss some general constructions by which new polynomials P with $P^{-\beta}$ completely monotone can be obtained from old ones. In the situations we have in mind, the vector space V decomposes as a direct sum $V = V_1 \oplus V_2$ and the cone C is a product cone $C = C_1 \times C_2$ (with $C_1 \subset V_1$ and $C_2 \subset V_2$). Since we shall be using the letters A, B, C and D in this section to denote functions, we shall write our cones as $\mathcal{C}$.

Let us begin with a trivial fact: a function $f(x,y)$ that is completely monotone on $\mathcal{C}_1 \times \mathcal{C}_2$ can be specialized by fixing y to a specific point in $\mathcal{C}_2$, and the resulting function

will be completely monotone on $\mathcal{C}_1$. In particular, this fixed value can then be taken to zero or infinity, and if the limit exists—possibly with some rescaling—then the limiting function is also completely monotone on $\mathcal{C}_1$. Rather than stating a general theorem of this kind, let us just give the special case that we will need, which concerns functions of the form

$$f(x,y) = [A(x)+B(x)\,y]^{-\beta} \tag{3.1}$$

with $V_2=\mathbb{R}$ and $\mathcal{C}_2=(0,\infty)$.

LEMMA 3.1. *Let V be a finite-dimensional real vector space and let $\mathcal{C}$ be an open convex cone in V. Fix $\beta>0$, and let $A,B:\mathcal{C}\to(0,\infty)$. If $(A+By)^{-\beta}$ is completely monotone on $\mathcal{C}\times(0,\infty)$, then $A^{-\beta}$ and $B^{-\beta}$ are completely monotone on $\mathcal{C}$.*

Proof. Restrict to fixed $y\in(0,\infty)$ and then take $y\downarrow 0$; this proves that $A^{-\beta}$ is completely monotone. Restrict to fixed $y\in(0,\infty)$, multiply by y^β and then take $y\uparrow\infty$; this proves that $B^{-\beta}$ is completely monotone. □

As we shall see later, this trivial lemma is an analytic version of deletion ($y\to 0$) or contraction ($y\to\infty$) for graphs or matroids.

Let us also observe a simple but important fact about complete monotonicity for functions defined on a product cone $\mathcal{C}_1\times\mathcal{C}_2$.

LEMMA 3.2. *For $i=1,2$, let V_i be a finite-dimensional real vector space and let $\mathcal{C}_i$ be an open convex cone in V_i. Let $f:\mathcal{C}_1\times\mathcal{C}_2\to\mathbb{R}$. Then the following are equivalent*:

(a) *f is completely monotone on $\mathcal{C}_1\times\mathcal{C}_2$.*

(b) *For all $k\geqslant 0$, all $y\in\mathcal{C}_2$, and all choices of vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k\in\mathcal{C}_2$, the function*

$$F_{k,y,\mathbf{u}_1,\ldots,\mathbf{u}_k}(x) \equiv (-1)^k \frac{\partial^k}{\partial t_1\ldots\partial t_k} f(x, y+t_1\mathbf{u}_1+\ldots+t_k\mathbf{u}_k)\Big|_{t_1=\ldots=t_k=0} \tag{3.2}$$

is completely monotone on $\mathcal{C}_1$.

(c) *For all $k\geqslant 0$, all $y\in\mathcal{C}_2$, and all choices of vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k\in\mathcal{C}_2$, there exists a positive measure $\mu_{k,y,\mathbf{u}_1,\ldots,\mathbf{u}_k}$ on $\mathcal{C}_1^*$ such that*

$$F_{k,y,\mathbf{u}_1,\ldots,\mathbf{u}_k}(x) = \int_{\mathcal{C}_1^*} e^{-\langle\ell,x\rangle}\, d\mu_{k,y,\mathbf{u}_1,\ldots,\mathbf{u}_k}(\ell). \tag{3.3}$$

In particular, when $V_2=\mathbb{R}$ and $\mathcal{C}_2=(0,\infty)$, (b) reduces to the statement that the functions $F_{k,y}(x)=(-1)^k\partial^k f/\partial y^k$ are completely monotone on $\mathcal{C}_1$ for all $k\geqslant 0$ and all $y>0$, and (c) reduces analogously.

Proof. The equivalence (a) $\Leftrightarrow$ (b) is a trivial consequence of the definition (2.1), while (b) $\Leftrightarrow$ (c) follows immediately from the the Bernstein–Hausdorff–Widder–Choquet theorem (Theorem 2.2). □

Statement (c) can be rephrased loosely as saying that the inverse Laplace transform of $f(x,y)$ with respect to x is a completely monotone function of $y \in \mathcal{C}_2$. (To make this more precise, one should add the same qualifications as in the paragraph after Theorem 2.2.)

One important application of this lemma concerns functions of the form (3.1).

LEMMA 3.3. *Let V be a finite-dimensional real vector space and let $\mathcal{C}$ be an open convex cone in V. Fix $\beta > 0$, and let $A, B : \mathcal{C} \to (0,\infty)$. Then the following are equivalent*:

(a) $(A+By)^{-\beta}$ *is completely monotone on* $\mathcal{C} \times (0,\infty)$.

(b) $B^{-\beta} \exp(-tA/B)$ *is completely monotone on* $\mathcal{C}$ *for all* $t \geqslant 0$.

(c) $B^{\varkappa}(A+zB)^{-(\beta+\varkappa)}$ *is completely monotone on* $\mathcal{C}$ *for all* $\varkappa \geqslant -\beta$ *and all* $z \geqslant 0$.

(d) $B^k(A+zB)^{-(\beta+k)}$ *is completely monotone on* $\mathcal{C}$ *for all integers* $k \geqslant 0$ *and all* $z \geqslant 0$.

Proof. We have the Laplace-transform formula

$$(A+By)^{-\beta} = \int_0^\infty e^{-ty} \frac{t^{\beta-1}}{\Gamma(\beta)} B^{-\beta} \exp\left(-\frac{tA}{B}\right) dt. \tag{3.4}$$

Therefore, Lemma 3.2 (a) $\Leftrightarrow$ (c) with $\mathcal{C}_1 = (0,\infty)$ and $\mathcal{C}_2 = \mathcal{C}$ proves the equivalence of (a) and (b).

Now assume that $B^{-\beta} \exp(-tA/B)$ is completely monotone on $\mathcal{C}$ for all $t \geqslant 0$. Then we can multiply by $e^{-zt} t^{p-1}/\Gamma(p)$ for any $p > 0$ and integrate over $t \in (0,\infty)$, and the result will be completely monotone. This (together with a trivial evaluation at $t=0$ to handle $p=0$) shows that (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (d) is trivial.

The equivalence of (a) and (d) follows from Lemma 3.2 (a) $\Leftrightarrow$ (b), used with $\mathcal{C}_1 = \mathcal{C}$ and $\mathcal{C}_2 = (0,\infty)$. □

COROLLARY 3.4. *Let V be a finite-dimensional real vector space, let $\mathcal{C}$ be an open convex cone in V, and let $A, B : \mathcal{C} \to (0,\infty)$. Define*

$$\mathcal{B}_B = \{\beta > 0 : B^{-\beta} \text{ is completely monotone on } \mathcal{C}\}, \tag{3.5}$$

$$\mathcal{B}_{A+By} = \{\beta > 0 : (A+By)^{-\beta} \text{ is completely monotone on } \mathcal{C} \times (0,\infty)\}. \tag{3.6}$$

Then $\mathcal{B}_{A+By} + \mathcal{B}_B \subseteq \mathcal{B}_{A+By}$.

In particular, if $\mathcal{B}_B = (0,\infty)$, then $\mathcal{B}_{A+By}$ is either the empty set or all of $(0,\infty)$ or a closed interval $[\beta_0,\infty)$ with $\beta_0 > 0$.

Proof. This follows immediately from Lemma 3.3 (a) $\Leftrightarrow$ (b): for if $\beta \in \mathcal{B}_{A+By}$ and $\lambda \in \mathcal{B}_B$, then $B^{-\beta} \exp(-tA/B)$ and $B^{-\lambda}$ are both completely monotone on $\mathcal{C}$, hence so is their product, and thus $\beta + \lambda \in \mathcal{B}_{A+By}$. □

Lemma 3.3 leads to the following extremely important result, which (as we shall see later) is an analytic version of parallel connection for graphs or matroids.

PROPOSITION 3.5. *Let V be a finite-dimensional real vector space and let $\mathcal{C}$ be an open convex cone in V. Fix $\beta>0$, and let $A, B, C, D: \mathcal{C} \to (0,\infty)$. Suppose that*

$$(A+By)^{-\beta} \quad \text{and} \quad (C+Dy)^{-\beta}$$

are completely monotone on $\mathcal{C}\times(0,\infty)$. Then the same is true of $(AD+BC+BDy)^{-\beta}$.

Proof. By Lemma 3.3, we have that $B^{-\beta}\exp(-tA/B)$ and $D^{-\beta}\exp(-tC/D)$ are completely monotone on $\mathcal{C}$ for all $t\geqslant 0$. Hence the same is true of their product, which is $(BD)^{-\beta}\exp[-t(AD+BC)/BD]$. But then using Lemma 3.3 again (this time in the reverse direction), we conclude that $(AD+BC+BDy)^{-\beta}$ is completely monotone on $\mathcal{C}\times(0,\infty)$. □

We also have an analytic version of series connection for graphs or matroids, but only for $\beta\geqslant\frac{1}{2}$.

PROPOSITION 3.6. *Let V be a finite-dimensional real vector space and let $\mathcal{C}$ be an open convex cone in V. Fix $\beta\geqslant\frac{1}{2}$, and let $A, B, C, D: \mathcal{C} \to (0,\infty)$. Suppose that*

$$(A+By)^{-\beta} \quad \text{and} \quad (C+Dy)^{-\beta}$$

are completely monotone on $\mathcal{C}\times(0,\infty)$. Then the same is true of $[AC+(AD+BC)y]^{-\beta}$.

To prove Proposition 3.6, we begin with a lemma that we think is of independent interest; both the sufficiency and the necessity will play important roles for us.

LEMMA 3.7. *For $\beta\in\mathbb{R}$ and $\lambda>0$, the function*

$$F_{\beta,\lambda}(u,v) = (u+v)^{-\beta}\,\exp\Big(-\lambda\frac{uv}{u+v}\Big) \tag{3.7}$$

is completely monotone on $(0,\infty)^2$ if and only if $\beta\geqslant\frac{1}{2}$.

In particular, for $\beta\geqslant\frac{1}{2}$ there exists a positive measure $\mu_{\beta,\lambda}$ on $[0,\infty)^2$ such that

$$(u+v)^{-\beta}\exp\Big(-\lambda\frac{uv}{u+v}\Big) = \int_{[0,\infty)^2} e^{-t_1u-t_2v}\,d\mu_{\beta,\lambda}(t_1,t_2). \tag{3.8}$$

Proof. "If": Since $(u+v)^{-(\beta-1/2)}$ is completely monotone when $\beta\geqslant\frac{1}{2}$, it suffices to prove the complete monotonicity for $\beta=\frac{1}{2}$. But this follows immediately from the identity

$$(u+v)^{-1/2}\exp\Big(-\lambda\frac{uv}{u+v}\Big) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left[-\left(s+\frac{\sqrt{\lambda}}{2}\right)^2u-\left(s-\frac{\sqrt{\lambda}}{2}\right)^2v\right]ds, \tag{3.9}$$

which is easily verified by completing the square in the Gaussian integral. The statement about the measure $\mu_{\beta,\lambda}$ then follows from the Bernstein–Hausdorff–Widder–Choquet theorem (Theorem 2.2).

"Only if": If $F_{\beta,\lambda}$ is completely monotone, then so is $F_{\beta',\lambda}$ for all $\beta'>\beta$; so it suffices to prove the failure of complete monotonicity for $0<\beta<\frac{1}{2}$. Now, by Lemma 3.2, $F_{\beta,\lambda}$ is completely monotone on $(0,\infty)^2$ if and only if the functions

$$F_{\beta,\lambda;k,v}(u) = (-1)^k \frac{\partial^k}{\partial v^k} F_{\beta,\lambda}(u,v) \tag{3.10}$$

are completely monotone on $(0,\infty)$ for all $k\geqslant 0$ and all $v>0$, or equivalently if their inverse Laplace transforms with respect to u,

$$G_{\beta,\lambda;k,v}(t) = (-1)^k \frac{\partial^k}{\partial v^k} \left[\left(\frac{t}{\lambda}\right)^{(\beta-1)/2} e^{-(t+\lambda)v} v^{1-\beta} I_{\beta-1}\bigl(2v\sqrt{\lambda t}\,\bigr) \right] \tag{3.11}$$

(see [48, p. 245, eq. 5.6(35)]), are non-negative for all $k\geqslant 0$ and all $t,v>0$ (here $I_{\beta-1}$ is the modified Bessel function). For $k=0$ this manifestly holds for all $\beta\geqslant 0$; but let us now show that for $k=1$ it holds only for $\beta\geqslant\frac{1}{2}$. We have

$$\begin{aligned} G_{\beta,\lambda;1,v}(t) = {} & \left(\frac{t}{\lambda}\right)^{(\beta-1)/2} e^{-(t+\lambda)v} v^{1-\beta} \\ & \times \left[\left(t+\lambda+\frac{\beta-1}{v}\right) I_{\beta-1}\bigl(2v\sqrt{\lambda t}\,\bigr) - 2\sqrt{\lambda t}\, I'_{\beta-1}\bigl(2v\sqrt{\lambda t}\,\bigr) \right], \end{aligned} \tag{3.12}$$

and we need the term in square brackets to be non-negative for all $t,v>0$. Write $x=2v\sqrt{\lambda t}$ and eliminate v in favor of x; we need

$$t + 2\sqrt{\lambda t}\left[\frac{\beta-1}{x} - \frac{I'_{\beta-1}(x)}{I_{\beta-1}(x)}\right] + \lambda \geqslant 0 \tag{3.13}$$

for all $t,x>0$. This quadratic in $\sqrt{t}$ is non-negative for all $t>0$ if and only if

$$\frac{\beta-1}{x} - \frac{I'_{\beta-1}(x)}{I_{\beta-1}(x)} \geqslant -1. \tag{3.14}$$

But using the large-x asymptotic expansion

$$\frac{d}{dx}\log I_{\beta-1}(x) = 1 - \frac{1}{2x} + O\left(\frac{1}{x^2}\right), \tag{3.15}$$

we see that

$$\frac{\beta-1}{x} - \frac{I'_{\beta-1}(x)}{I_{\beta-1}(x)} = -1 + \frac{\beta-\frac{1}{2}}{x} + O\left(\frac{1}{x^2}\right), \tag{3.16}$$

which is <-1 for all sufficiently large x whenever $\beta<\frac{1}{2}$. □

Remarks. (1) It is obvious by rescaling of u and v that, for any given β, the functions $F_{\beta,\lambda}$ are either completely monotone for all $\lambda>0$ or for none.

(2) The appeal to Bernstein–Hausdorff–Widder–Choquet's theorem can be avoided: for $\beta=\frac{1}{2}$, the integral representation (3.9) already provides the desired measure $\mu_{1/2,\lambda}$; and for $\beta>\frac{1}{2}$, (3.9) together with

$$(u+v)^{-(\beta-1/2)} = \int_0^\infty \frac{t^{\beta-3/2}}{\Gamma\left(\beta-\frac{1}{2}\right)} e^{-t(u+v)}\, dt \tag{3.17}$$

represents $\mu_{\beta,\lambda}$ as the convolution of two positive measures. Indeed, multiplying (3.9) by (3.17), one obtains after a straightforward change of variables the explicit formula

$$\mu_{\beta,\lambda}(t_1,t_2) = \frac{(4\lambda)^{1-\beta}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\beta-\frac{1}{2}\right)} P(t_1,t_2,\lambda)^{\beta-3/2}\chi(t_1,t_2,\lambda), \tag{3.18}$$

where

$$P(t_1,t_2,\lambda) = 2(t_1t_2+\lambda t_1+\lambda t_2)-(t_1^2+t_2^2+\lambda^2) \tag{3.19}$$

and

$$\chi(t_1,t_2,\lambda) = \begin{cases} 1, & \text{if } t_1,t_2\geqslant 0 \text{ and } |\sqrt{t_1}-\sqrt{t_2}|\leqslant\sqrt{\lambda}\leqslant\sqrt{t_1}+\sqrt{t_2}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.20}$$

The constraint $\chi(t_1,t_2,\lambda)\neq 0$ states simply that $\sqrt{t_1}$, $\sqrt{t_2}$ and $\sqrt{\lambda}$ form the sides of a triangle; and $P(t_1,t_2,\lambda)$ is precisely 16 times the square of the area of this triangle (Heron's formula [41, §3.2]).([21]) In view of these explicit formulae, the proof of Lemma 3.7 is in fact completely elementary.

(3) It would be interesting to know whether Lemma 3.7 can be generalized to other ratios of elementary symmetric polynomials, e.g. $E_{r,n}^{-\beta}\exp(-\lambda E_{r+1,n}/E_{r,n})$. By Lemma 3.3 this would determine the complete monotonicity of $E_{r+1,n+1}^{-\beta}$.

Proof of Proposition 3.6. By Lemma 3.3, $B^{-\beta}\exp(-t_1A/B)$ and $D^{-\beta}\exp(-t_2C/D)$ are completely monotone on $\mathcal{C}$ for all $t_1,t_2\geqslant 0$. Hence the same is true of their product for any choice of $t_1,t_2\geqslant 0$. We now use the identity (3.8), multiplied on both sides by $(BD)^{-\beta}$, with $u=A/B$ and $v=C/D$. This shows that $(AD+BC)^{-\beta}\exp[-\lambda AC/(AD+BC)]$ is completely monotone on $\mathcal{C}$ for all $\lambda\geqslant 0$. But then using Lemma 3.3 again (this time in the reverse direction), we conclude that $[AC+(AD+BC)y]^{-\beta}$ is completely monotone on $\mathcal{C}\times(0,\infty)$. □

([21]) It is curious that similar expressions, involving the area of a triangle in terms of its sides, arise also in Sonine's integral for the product of three Bessel functions [122, p. 411, eq. 13.46 (3)], [7, p. 36, eq. (4.39) and p. 40]—a formula that Szegő [113] employed in one version of his non-negativity proof for (1.1) in the case $n=3$. Probably this is not an accident; it would be interesting to understand the precise connection between Sonine's formula and (3.8)/(3.18) (cf. also (3.21) below).

Using Lemma 3.7 we can also show that the spanning-tree polynomial of the 3-cycle — or equivalently, the elementary symmetric polynomial of degree 2 in three variables— has the property that $P^{-\beta}$ is completely monotone on $(0,\infty)^3$ *if and only if* $\beta \geqslant \frac{1}{2}$.

PROPOSITION 3.8. *The function* $F(x_1,x_2,x_3)=(x_1x_2+x_1x_3+x_2x_3)^{-\beta}$ *is completely monotone on* $(0,\infty)^3$ *if and only if* $\beta=0$ *or* $\beta \geqslant \frac{1}{2}$.

Proof. As always, F is completely monotone for $\beta=0$ and not completely monotone for $\beta<0$, so it suffices to consider $\beta>0$. Using Lemma 3.3 with $A=x_1x_2$, $B=x_1+x_2$ and $y=x_3$, we see that F is completely monotone on $(0,\infty)^3$ if and only if the function $F_{\beta,\lambda}$ defined in (3.7) is completely monotone for all $\lambda \geqslant 0$. But by Lemma 3.7, this occurs if and only if $\beta \geqslant \frac{1}{2}$. □

Remarks. (1) We shall later give two further independent proofs of Proposition 3.8: one based on harmonic analysis on the cone of positive-definite $m\times m$ real symmetric matrices specialized to $m=2$ (Corollary 1.5, which follows from results to be proved in §4, together with Proposition 7.9 (a)), and one based on harmonic analysis on the Lorentz cone in $\mathbb{R}^n$ specialized to $n=3$ (Corollary 1.10, which follows from results to be proved in §5). The point here is that the Jordan algebra $\mathrm{Sym}(2,\mathbb{R}) \simeq \mathbb{R}\times\mathbb{R}^2$ can be viewed as a member of two different families of Jordan algebras: $\mathrm{Sym}(m,\mathbb{R})$ and $\mathbb{R}\times\mathbb{R}^{n-1}$ [52, p. 98].

(2) Proposition 3.8 implies that the property stated in Proposition 3.6 does *not* hold for $0<\beta<\frac{1}{2}$. Indeed, it suffices to take $\mathcal{C}=(0,\infty)^2$, $A=x_1$, $B=1$, $C=x_2$ and $D=1$, leading to the function $(x_1x_2+x_1y+x_2y)^{-\beta}$.

(3) Proposition 3.8 is, of course, just Theorem 1.1′ restricted to the 3-cycle $G=K_3$. In particular it implies Szegő's [113] result (except the strict positivity) for the polynomial (1.1) in the special case $n=3$.

(4) Combining (3.4) with (3.8)/(3.18), we obtain for $\beta>\frac{1}{2}$ the formula

$$(x_1x_2+x_1x_3+x_2x_3)^{-\beta} = \frac{4^{1-\beta}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma(\beta)} \times \int_0^\infty\int_0^\infty\int_0^\infty e^{-t_1x_1-t_2x_2-t_3x_3} P(t_1,t_2,t_3)_+^{\beta-3/2}\, dt_1\, dt_2\, dt_3, \tag{3.21}$$

which provides an explicit elementary proof of the direct ("if") half of Proposition 3.8. See also Remark 1 in §4.3 for an alternative derivation of (3.21), and see Corollary 5.8 for a generalization from $E_{2,3}$ to $E_{2,n}$.

The following generalization of Lemma 3.7 is an analytic version of series extension of a single edge.

LEMMA 3.9. *Let V be a finite-dimensional real vector space and let $\mathcal{C}$ be an open convex cone in V. Let f be completely monotone on $\mathcal{C}\times(0,\infty)$, and let $\beta\geqslant\frac{1}{2}$. Then the function*

$$f_\beta^\#(x,y,z)=(y+z)^{-\beta}f\left(x,\frac{yz}{y+z}\right) \tag{3.22}$$

is completely monotone on $\mathcal{C}\times(0,\infty)^2$.

Proof. By the Bernstein–Hausdorff–Widder–Choquet theorem (Theorem 2.2) and linearity, it suffices to prove the lemma for $f(x,y)=\exp(-\langle\ell,x\rangle-\lambda y)$ with $\ell\in\mathcal{C}^*$ and $\lambda\geqslant 0$. The variable x now simply goes for the ride, so that the claim follows immediately from Lemma 3.7. □

4. Determinantal polynomials

In this section we consider polynomials defined by determinants as in (1.4). We begin with some preliminary algebraic facts about such determinantal polynomials. After this, we turn to the analytic results that are our principal concern. We first prove a simple abstract version of the half-plane property for determinantal polynomials. Then we turn to the main topic of this section, namely, the proofs of Theorems 1.2–1.4.

4.1. Algebraic preliminaries

First, some notation: We write $[m]=\{1,\dots,m\}$. If $A=(a_{ij})_{i,j=1}^m$ is an $m\times m$ matrix and $I,J\subseteq[m]$, we denote by A_{IJ} the submatrix of A corresponding to the rows I and the columns J, all kept in their original order. We write I^c to denote the complement of I in $[m]$. Then $\varepsilon(I,J)=(-1)^{\sum_{i\in I}i+\sum_{j\in J}j}$ is the sign of the permutation that takes II^c into JJ^c.

We begin with a simple formula for the determinant of the sum of two matrices, which ought to be found in every textbook of matrix theory but seems to be surprisingly little known.([22])

([22]) This formula can be found in [78, pp. 162–163, Exercise 6] and [76, pp. 221–223]. It can also be found—albeit in an ugly notation that obscures what is going on—in [78, pp. 145–146 and 163–164] [103, pp. 31–33] [97, pp. 281–282]; and in an even more obscure notation in [3, p. 102, item 5]. We would be grateful to readers who could supply additional references. The proof here is taken from [29, Lemma A.1].

We remark that, by the same method, one can prove a formula analogous to (4.1) in which all three occurrences of determinant are replaced by permanent and the factor $\varepsilon(I,J)$ is omitted.

LEMMA 4.1. *Let A and B be $m\times m$ matrices with elements in a commutative ring R. Then*[23]

$$\det(A+B) = \sum_{\substack{I,J\subseteq[m]\\ |I|=|J|}} \varepsilon(I,J)(\det A_{IJ})(\det B_{I^cJ^c}). \tag{4.1}$$

Proof. Using the definition of determinant and expanding the products, we have

$$\det(A+B) = \sum_{\pi\in\mathcal{S}_m} \operatorname{sgn}(\pi) \sum_{I\subseteq[m]} \prod_{i\in I} a_{i\pi(i)} \prod_{i'\in I^c} b_{i'\pi(i')}, \tag{4.2}$$

where the outermost sum runs over all permutations π of $[m]$. Define now $J=\pi[I]$. Then we can interchange the order of summation:

$$\det(A+B) = \sum_{\substack{I,J\subseteq[m]\\ |I|=|J|}} \sum_{\substack{\pi\in\mathcal{S}_m\\ \pi[I]=J}} \operatorname{sgn}(\pi) \prod_{i\in I} a_{i\pi(i)} \prod_{i'\in I^c} b_{i'\pi(i')}. \tag{4.3}$$

Suppose now that $|I|=|J|=k$, and let us write $I=\{i_1,\ldots,i_k\}$ and $J=\{j_1,\ldots,j_k\}$ where the elements are written in increasing order, and likewise $I^c=\{i'_1,\ldots,i'_{m-k}\}$ and $J=\{j'_1,\ldots,j'_{m-k}\}$. Let $\pi'\in\mathcal{S}_k$ and $\pi''\in\mathcal{S}_{m-k}$ be the permutations defined so that

$$\pi'(\alpha)=\beta \longleftrightarrow \pi(i_\alpha)=j_\beta, \tag{4.4}$$

$$\pi''(\alpha)=\beta \longleftrightarrow \pi(i'_\alpha)=j'_\beta. \tag{4.5}$$

It is easy to see that $\operatorname{sgn}(\pi)=\operatorname{sgn}(\pi')\operatorname{sgn}(\pi'')\varepsilon(I,J)$. The formula (4.1) then follows by using twice again the definition of determinant. □

The following special case is frequently useful.

COROLLARY 4.2. *Let A and B be $m\times m$ matrices with elements in a commutative ring R, with at least one of them being a diagonal matrix. Then*

$$\det(A+B) = \sum_{I\subseteq[m]} (\det A_{II})(\det B_{I^cI^c}). \tag{4.6}$$

Proof. If A is diagonal, then $\det A_{IJ}=0$ whenever $I\neq J$, and likewise for B. □

Remark. We show in [105, Appendix A] that Corollary 4.2 generalizes to quaternions, but Lemma 4.1 does not; see [105, Proposition A.5] and the following remark.

---

[23] The determinant of an empty matrix is of course defined to be 1. This makes sense in the present context even if the ring R lacks an identity element: the term $I=J=\varnothing$ contributes $\det B$ to the sum (4.1), while the term $I=J=[m]$ contributes $\det A$.

Iterating Lemma 4.1, we obtain a formula for the determinant of a sum of n matrices:

$$\det\left(\sum_{k=1}^{n} A_k\right) = \sum_{\mathbf{I},\mathbf{J}} \varepsilon(\mathbf{I},\mathbf{J}) \prod_{k=1}^{n} \det[(A_k)_{I_k J_k}], \tag{4.7}$$

where the sum runs over ordered partitions $\mathbf{I}=(I_1,...,I_n)$ and $\mathbf{J}=(J_1,...,J_n)$ of $[m]$ into n *possibly empty* blocks satisfying $|I_k|=|J_k|$ for all k; here $\varepsilon(\mathbf{I},\mathbf{J})$ is the sign of the permutation taking $I_1 I_2 \ldots I_n$ into $J_1 J_2 \ldots J_n$.

We can now say something about determinantal polynomials of the type (1.4). Recall that if A is a (not necessarily square) matrix with elements in a commutative ring R, then the (*determinantal*) *rank* of A is defined to be the largest integer r such that A has a non-zero $r\times r$ minor; if no such minor exists (i.e., $A=0$), we say that A has rank 0.

PROPOSITION 4.3. *Let $A_1,...,A_n$ be $m\times m$ matrices with elements in a commutative ring R, and let $x_1,...,x_n$ be indeterminates. Then*

$$P(x_1,...,x_n) = \det\left(\sum_{i=1}^{n} x_i A_i\right) \tag{4.8}$$

is a homogeneous polynomial of degree m with coefficients in R. Furthermore, the degree of P in the variable x_i is $\leqslant \mathrm{rank}(A_i)$. (In particular, if each A_i is of rank at most 1, then P is multiaffine.)

Proof. Both assertions about P are immediate consequences of (4.7). □

We are grateful to Andrea Sportiello for drawing our attention to Lemma 4.1 and its proof, and for showing us this elegant proof of Proposition 4.3.

An analogue of Proposition 4.3 holds for Hermitian quaternionic matrices, albeit with a different proof: here the determinant is the Moore determinant, and "rank" means left row rank (= right column rank); moreover, the polynomial $P(x_1,...,x_n)$ is defined initially by letting $x_1,...,x_n$ be *real* numbers. See [105, Proposition A.8].

4.2. The half-plane property

Now we take an analytic point of view, so that the commutative ring R will be either $\mathbb{R}$ or $\mathbb{C}$.

In this subsection we make a slight digression from our main theme, by showing that if the A_i are complex Hermitian positive-semidefinite matrices (this of course includes the special case of real symmetric positive-semidefinite matrices), then the determinantal polynomial (4.8) has the half-plane property. This turns out to be an easy extension of the proof of [37, Theorem 8.1 (a)]. Indeed, we can go farther, by first stating the result in

a clean abstract way, and then deducing the half-plane property for (4.8) as an immediate corollary.

PROPOSITION 4.4. *Let V be the real vector space* $\mathrm{Herm}(m,\mathbb{C})$ *of complex Hermitian $m\times m$ matrices, and let $C\subset V$ be the cone* $\Pi_m(\mathbb{C})$ *of positive-definite matrices. Then the polynomial $P(A)=\det A$ is non-vanishing on the tube $C+iV$.*

(*Of course, by restriction the same result holds when V is the real vector space* $\mathrm{Sym}(m,\mathbb{R})$ *of real symmetric $m\times m$ matrices and $C\subset V$ is the cone* $\Pi_m(\mathbb{R})$ *of positive-definite matrices.*)

COROLLARY 4.5. *Let $A_1,\ldots,A_n$ be complex Hermitian positive-semidefinite $m\times m$ matrices. Then the polynomial*

$$P(x_1,\ldots,x_n)=\det\left(\sum_{i=1}^n x_iA_i\right) \tag{4.9}$$

has the half-plane property, that is, either $P\equiv 0$ or $P(x_1,\ldots,x_n)\neq 0$ whenever $\mathrm{Re}\,x_i>0$ for all i.

First proof of Proposition 4.4. Let $A\in C+iV$, and let ψ be a non-zero vector in $\mathbb{C}^m$. Then the Hermitian form $\psi^*A\psi=\sum_{j,k=1}^m \bar{\psi}_jA_{jk}\psi_k$ has strictly positive real part, and in particular is non-zero; it follows that $A\psi\neq 0$. Since this is true for every non-zero $\psi\in\mathbb{C}^m$, we conclude that $\ker A=\{0\}$, i.e. A is non-singular; and this implies that (and is in fact equivalent to) $\det A\neq 0$. □

Second proof of Proposition 4.4. [24, Lemma 4.1]. Write $A=P+iQ$ with P positive-definite and Q Hermitian. Then P has a positive-definite square root $P^{1/2}$, and we have

$$\det(P+iQ)=\det[P^{1/2}(I+iP^{-1/2}QP^{-1/2})P^{1/2}] \tag{4.10}$$

$$=(\det P)\det(I+iP^{-1/2}QP^{-1/2}). \tag{4.11}$$

This is non-zero because all the eigenvalues of $I+iP^{-1/2}QP^{-1/2}$ have real part equal to 1. □

Proof of Corollary 4.5. If at least one of the matrices $A_1,\ldots,A_n$ is strictly positive-definite, then $\sum_{j=1}^n x_jA_j$ lies in $C+iV$ whenever $\mathrm{Re}\,x_j>0$ for all j. Proposition 4.4 then implies that $P(x_1,\ldots,x_n)\neq 0$.

The general case can be reduced to this one by replacing A_j by $A_j+\varepsilon I$ ($\varepsilon>0$) and taking $\varepsilon\downarrow 0$. By Hurwitz's theorem, the limiting function is either non-vanishing on the product of open right half-planes or else is identically zero. □

Remarks. (1) The first proof of Proposition 4.4 is an abstraction of the proof of Choe, Oxley, Sokal and Wagner [37, Theorem 8.1 (a)]; it can also be found in Faraut and Korányi [52, Lemma X.1.2] and, for $V=\mathrm{Sym}(m,\mathbb{R})$, in Hörmander [65, p. 85]. Faraut and Korányi furthermore observe that $A\mapsto A^{-1}$ is an involutive holomorphic automorphism of the tube $C+iV$, having I as its unique fixed point; and these facts hold not only for $\mathrm{Herm}(m,\mathbb{C})$ but in fact for any simple Euclidean Jordan algebra [52, Theorem X.1.1].

(2) It is natural to ask whether there is an analogous result for permanents, which would extend [37, Theorem 10.2] in the same way that Proposition 4.4 extends [37, Theorem 8.1 (a)]. The most obvious such extension would be: Let V be the vector space $\mathbb{R}^{m\times n}$ of real $m\times n$ matrices ($m\leqslant n$) and let $C\subset V$ be the cone $(0,\infty)^{m\times n}$ of matrices with strictly positive entries. Then the polynomial $Q(A)=\mathrm{per}\,A$ is non-vanishing on the tube $C+iV$. But this is simply *false*: for instance, for $m=n=2$ we have

$$\mathrm{per}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} = 0.$$

4.3. Proofs of Theorems 1.2–1.4

Let us now turn to the main results of this section, namely, Theorems 1.3 and 1.4 and their consequence, Theorem 1.2. The proof of all these results turns out to be surprisingly easy; and all but the "only if" part of Theorems 1.3 and 1.4 is completely elementary.

In order to make the elementary nature of the proof as transparent as possible, we proceed as follows: First we prove the direct ("if") half of Theorem 1.3 (a) and (b) by completely elementary methods, without reference to Euclidean Jordan algebras. Then we prove Theorem 1.4: we will see that the proof of the direct half of this theorem is a straightforward abstraction of the preceding elementary proof in the concrete cases; only the converse ("only if") half is really deep.

Proof of the direct half of Theorem 1.3 (a) *and* (b). Let us begin with the real case and $\beta=\frac{1}{2}$. We use the Gaussian integral representation

$$(\det A)^{-1/2} = \int_{\mathbb{R}^m} \exp(-\mathbf{x}^{\mathrm{T}} A\mathbf{x}) \prod_{j=1}^{m} \frac{dx_j}{\sqrt{\pi}}, \tag{4.12}$$

where A is a real symmetric positive-definite $m\times m$ matrix and we have written $\mathbf{x}=(x_1,\ldots,x_m)\in\mathbb{R}^m$. This exhibits $(\det A)^{-1/2}$ as the Laplace transform of a positive measure on $\Pi_m(\mathbb{R})^*=\overline{\Pi_m(\mathbb{R})}$, namely, the push-forward of Lebesgue measure $d\mathbf{x}/\pi^{m/2}$ on $\mathbb{R}^m$ by the map $\mathbf{x}\mapsto\mathbf{x}\mathbf{x}^{\mathrm{T}}$. (We remark that this measure is supported on positive-semidefinite matrices of rank 1.) Alternatively, one can see directly, by differentiating

under the integral sign in (4.12), that the k-fold directional derivative of $(\det A)^{-1/2}$ in directions $B_1, ..., B_k \in \Pi_m(\mathbb{R})$ has sign $(-1)^k$, because each derivative brings down a factor $-\mathbf{x}^{\mathrm{T}} B_i \mathbf{x} \leqslant 0$.

Since a product of completely monotone functions is completely monotone, it follows immediately that $(\det A)^{-N/2}$ is completely monotone for all positive integers N. We remark that this can alternatively be seen from the Gaussian integral representation

$$(\det A)^{-N/2} = \int_{(\mathbb{R}^m)^N} \exp\Bigg(-\sum_{\alpha=1}^{N} \mathbf{x}^{(\alpha)\mathrm{T}} A \mathbf{x}^{(\alpha)} \Bigg) \prod_{\alpha=1}^{N} \prod_{j=1}^{m} \frac{dx_j^{(\alpha)}}{\sqrt{\pi}}, \tag{4.13}$$

where we have introduced vectors $\mathbf{x}^{(\alpha)} \in \mathbb{R}^m$ for $\alpha = 1, ..., N$. If we assemble these vectors into an $m \times N$ matrix X, then we have exhibited $(\det A)^{-N/2}$ as the Laplace transform of a positive measure on $\overline{\Pi_m(\mathbb{R})}$, namely, the push-forward of Lebesgue measure $dX/\pi^{mN/2}$ on $\mathbb{R}^{m \times N}$ by the map $X \mapsto XX^{\mathrm{T}}$. This measure is supported on positive-semidefinite matrices of rank $\min(N, m)$.

Finally, for real values of $\beta > \frac{1}{2}(m-1)$, we use the integral representation[24]

$$(\det A)^{-\beta} = \left[\pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma\Big(\beta - \frac{j}{2}\Big) \right]^{-1} \int_{B>0} e^{-\operatorname{tr}(AB)} (\det B)^{\beta-(m+1)/2} \, dB, \tag{4.14}$$

where the integration runs over real symmetric positive-definite $m \times m$ matrices B, with measure $dB = \prod_{1 \leqslant i \leqslant j \leqslant m} dB_{ij}$. This exhibits $(\det A)^{-\beta}$ as the Laplace transform of a positive measure on $\Pi_m(\mathbb{R})$.

The proof in the complex case is completely analogous. For $\beta = 1$ we use the Gaussian integral representation

$$(\det A)^{-1} = \int_{\mathbb{C}^m} \exp(-\bar{\mathbf{z}}^{\mathrm{T}} A \mathbf{z}) \prod_{j=1}^{m} \frac{(d \operatorname{Re} z_j)(d \operatorname{Im} z_j)}{\pi}, \tag{4.15}$$

where A is a complex Hermitian positive-definite matrix, $\mathbf{z} = (z_1, ..., z_m) \in \mathbb{C}^m$ and $\bar{}$ denotes complex conjugation. For real values of $\beta > m-1$, we use the integral representation[25]

$$(\det A)^{-\beta} = \left[\pi^{m(m-1)/2} \prod_{j=0}^{m-1} \Gamma(\beta - j) \right]^{-1} \int_{B>0} e^{-\operatorname{tr}(AB)} \, (\det B)^{\beta-m} \, dB, \tag{4.16}$$

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[24] See e.g. [69], [108, pp. 585–586], [114, p. 41], [49, Lemma 1], or [5, Theorem 7.2.2 and Corollary 7.2.4].

[25] See e.g. [60, eqq. (5.13)–(5.40)]. Please note that [5, Problem 7.9] and [61, eq. (1.2)] appear to contain errors in the normalization constant.

where the integration runs over complex Hermitian positive-definite $m \times m$ matrices B, with measure

$$dB = \left(\prod_{i=1}^{m} dB_{ii} \right) \prod_{1 \leqslant i < j \leqslant m} (d \operatorname{Re} B_{ij})(d \operatorname{Im} B_{ij}). \tag{4.17}$$

□

Remarks. (1) By applying (4.14) for the case $m=2$ to

$$A = \begin{pmatrix} x_1+x_3 & x_3 \\ x_3 & x_2+x_3 \end{pmatrix},$$

we obtain after a bit of algebra an alternative derivation of the formula (3.21) for

$$(x_1x_2+x_1x_3+x_2x_3)^{-\beta},$$

valid for $\beta > \frac{1}{2}$. In particular it implies the direct ("if") half of Proposition 3.8, and hence also Szegő's [113] result (except the strict positivity) for the polynomial (1.1) in the special case $n=3$.

(2) Similarly, by combining (4.16)/(4.17) for the case $m=2$ with (1.6)/(1.7), we obtain after some algebra an explicit formula for $E_{2,4}(\mathbf{x})^{-\beta}$, valid for $\beta > 1$,

$$\begin{aligned} (x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)^{-\beta} &= \frac{3^{3/2-\beta}}{2\pi\Gamma(\beta-1)\Gamma(\beta)} \\ \times \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty & e^{-t_1x_1-t_2x_2-t_3x_3-t_4x_4} Q(t_1,t_2,t_3,t_4)_+^{\beta-2} \, dt_1 \, dt_2 \, dt_3 \, dt_4, \end{aligned} \tag{4.18}$$

where

$$Q(t_1,t_2,t_3,t_4) = (t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4) - (t_1^2+t_2^2+t_3^2+t_4^2). \tag{4.19}$$

This formula provides an explicit elementary proof of the direct half of Corollary 1.6—and in particular solves the Lewy–Askey problem—in the same way that (3.21) provides an explicit elementary proof of the direct half of Proposition 3.8. Note also that by setting $x_4=0$ in (4.18) and performing the integral over t_4, we obtain (3.21). Finally, see Corollary 5.8 for a generalization from $E_{2,3}$ and $E_{2,4}$ to $E_{2,n}$.

As preparation for the proof of Theorem 1.4, let us review some facts from the theory of harmonic analysis on Euclidean Jordan algebras (see [52] for definitions and further background).

Let V be a simple Euclidean (real) Jordan algebra of dimension n and rank r, with Peirce subspaces V_{ij} of dimension d; recall that $n = r + \frac{1}{2}dr(r-1)$. It is illuminating (though not logically necessary) to know that there are precisely five cases [52, p. 97]:

(a) $V = \mathrm{Sym}(m, \mathbb{R})$, the space of $m \times m$ real symmetric matrices ($d=1$, $r=m$);

(b) $V=\mathrm{Herm}(m,\mathbb{C})$, the space of $m\times m$ complex Hermitian matrices ($d=2$, $r=m$);

(c) $V=\mathrm{Herm}(m,\mathbb{H})$, the space of $m\times m$ quaternionic Hermitian matrices ($d=4$, $r=m$);

(d) $V=\mathrm{Herm}(3,\mathbb{O})$, the space of 3×3 octonionic Hermitian matrices ($d=8$, $r=3$); and

(e) $V=\mathbb{R}\times\mathbb{R}^{n-1}$ ($d=n-2$, $r=2$).

We denote by $(x|y)=\mathrm{tr}(xy)$ the inner product on V, where tr is the Jordan trace and xy is the Jordan product.([26])

Let $\Omega\subset V$ be the positive cone (i.e. the interior of the set of squares in V, or equivalently the set of invertible squares in V); it is open, convex and self-dual.([27]) We denote by $\Delta(x)$ the Jordan determinant on V: it is a homogeneous polynomial of degree r on V, which is strictly positive on Ω and vanishes on $\partial\Omega$.([28]) We have the following fundamental Laplace-transform formula [52, Corollary VII.1.3]: for $y\in\Omega$ and $\mathrm{Re}\,\alpha>(r-1)\frac{1}{2}d=n/r-1$,

$$\int_\Omega e^{-(x|y)}\Delta(x)^{\alpha-n/r}\,dx=\Gamma_\Omega(\alpha)\Delta(y)^{-\alpha}, \tag{4.20}$$

where([29])

$$\Gamma_\Omega(\alpha)=(2\pi)^{(n-r)/2}\prod_{j=0}^{r-1}\Gamma\left(\alpha-j\frac{d}{2}\right). \tag{4.21}$$

Thus, for $\mathrm{Re}\,\alpha>(r-1)\frac{1}{2}d$, the function $\Delta(x)^{\alpha-n/r}/\Gamma_\Omega(\alpha)$ is locally integrable on $\overline{\Omega}$ and polynomially bounded; it therefore defines a tempered distribution $\mathcal{R}_\alpha$ on V by the

([26]) In cases (a) and (b), we have $(A|B)=\mathrm{tr}(AB)$; in case (c) we have $(A|B)=\frac{1}{2}\,\mathrm{tr}(AB+BA)$; and in case (e) we have $((x_0,\mathbf{x})|(y_0,\mathbf{y}))=2(x_0y_0+\mathbf{x}\cdot\mathbf{y})$.

([27]) In cases (a)–(d) the positive cone Ω is the cone of positive-definite matrices; in case (e) it is the Lorentz cone $\{(x_0,\mathbf{x}):x_0>\sqrt{\mathbf{x}^2}\}$.

([28]) In cases (a) and (b), the Jordan determinant is the ordinary determinant; in case (c) it is the Moore determinant (see [105, Appendix A]); in case (d) it is the Freudenthal determinant [55], [45], [13], [82]; and in case (e) it is the Lorentz quadratic form $\Delta(x_0,\mathbf{x})=x_0^2-\mathbf{x}^2$.

([29]) Here dx is Lebesgue measure on the Euclidean space V with inner product $(\cdot\,|\,\cdot)$. Thus in case (a) we have $(B|B)=\sum_{i=1}^m B_{ii}^2+2\sum_{1\leqslant i<j\leqslant m}B_{ij}^2$, so that

$$dx=\left(\prod_{i=1}^m dB_{ii}\right)\prod_{1\leqslant i<j\leqslant m}\sqrt{2}dB_{ij}=2^{m(m-1)/4}\,dB,$$

showing that (4.20)/(4.21) agrees with (4.14). Likewise, in case (b) we have

$$(B\,|\,B)=\sum_{i=1}^m B_{ii}^2+2\sum_{1\leqslant i<j\leqslant m}|B_{ij}|^2,$$

so that

$$dx=\left(\prod_{i=1}^m dB_{ii}\right)\prod_{1\leqslant i<j\leqslant m}(\sqrt{2}d\,\mathrm{Re}\,B_{ij})(\sqrt{2}d\,\mathrm{Im}\,B_{ij})=2^{m(m-1)/2}\,dB,$$

showing that (4.20)/(4.21) agrees with (4.16).

usual formula

$$\mathcal{R}_\alpha(\varphi) = \frac{1}{\Gamma_\Omega(\alpha)} \int_\Omega \varphi(x)\, \Delta(x)^{\alpha - n/r}\, dx \quad \text{for } \varphi \in \mathcal{S}(V). \tag{4.22}$$

Using (4.20), a beautiful argument—which is a special case of I. Bernstein's general method for analytically continuing distributions of the form $\mathcal{P}_\Omega^\lambda$ [19], [20]—shows that the distributions $\mathcal{R}_\alpha$ can be analytically continued to the whole complex α-plane, leading to the following result.

THEOREM 4.6. ([52, Theorem VII.2.2 et seq.]) *The distributions $\mathcal{R}_\alpha$ can be analytically continued to the whole complex α-plane as a tempered-distribution-valued entire function of α. The distributions $\mathcal{R}_\alpha$ have support contained in $\overline{\Omega}$ and have the following properties*:

$$\mathcal{R}_0 = \delta, \tag{4.23}$$

$$\mathcal{R}_\alpha * \mathcal{R}_\beta = \mathcal{R}_{\alpha+\beta}, \tag{4.24}$$

$$\Delta(\partial/\partial x)\mathcal{R}_\alpha = \mathcal{R}_{\alpha-1}, \tag{4.25}$$

$$\Delta(x)\mathcal{R}_\alpha = \left(\prod_{j=0}^{r-1} \left(\alpha - j\frac{d}{2} \right) \right) \mathcal{R}_{\alpha+1} \tag{4.26}$$

(*here δ denotes the Dirac measure at* 0). *Finally, the Laplace transform of $\mathcal{R}_\alpha$ is*

$$(\mathcal{L}\mathcal{R}_\alpha)(y) = \Delta(y)^{-\alpha} \tag{4.27}$$

for y in the complex tube $\Omega + iV$.([30])

The distributions $\{\mathcal{R}_\alpha\}_{\alpha\in\mathbb{C}}$ constructed in Theorem 4.6 are called the *Riesz distributions* on the Euclidean Jordan algebra V.

It is fairly easy to find a *sufficient* condition for a Riesz distribution to be a positive measure, as is stated in the following proposition.

PROPOSITION 4.7. ([52, Proposition VII.2.3]; see also [77], [22] and [63, §3.2])

(a) *For $\alpha = \frac{1}{2}kd$ with $k = 0, 1, \ldots, r-1$, the Riesz distribution $\mathcal{R}_\alpha$ is a positive measure that is supported on the set of elements of $\overline{\Omega}$ of rank exactly k (which is a subset of $\partial\Omega$).*

(b) *For $\alpha > \frac{1}{2}(r-1)d$, the Riesz distribution $\mathcal{R}_\alpha$ is a positive measure that is supported on Ω and given there by a density (with respect to Lebesgue measure) that lies in $L^1_{\mathrm{loc}}(\overline{\Omega})$.*

([30]) The property (4.26) is not explicitly stated in [52], but for $\operatorname{Re}\alpha > \frac{1}{2}(r-1)d$ it is an immediate consequence of (4.21) and (4.22), and then for other values of α it follows by analytic continuation (see also [63, Proposition 3.1 (iii) and Remark 3.2]).

Indeed, part (b) is immediate from the definition (4.22), while part (a) follows by reasoning that abstracts the constructions given in (4.12)/(4.13) and (4.15) above for the special cases of real symmetric and complex Hermitian matrices.([31])

It is a highly non-trivial fact that the converse of Proposition 4.7 also holds.

THEOREM 4.8. ([52, Theorem VII.3.1]) *The Riesz distribution $\mathcal{R}_\alpha$ is a positive measure if and only if $\alpha = 0, \frac{1}{2}d, \ldots, \frac{1}{2}(r-1)d$ or $\alpha > \frac{1}{2}(r-1)d$.*

This fundamental fact was first proven by Gindkin [58] (see also [16] and [70]) and is generally considered to be deep. However, there now exist two elementary proofs: one that is a fairly simple but clever application of Theorem 4.6 and Proposition 4.7 [106], [31], [110, Appendix], and another that analyzes the integrability of $\Delta(x)^{\alpha - n/r}$ near $\partial\Omega$ and characterizes those $\alpha \in \mathbb{C}$ for which $\mathcal{R}_\alpha$ is a locally finite complex measure [110]. In [105, Appendix B] we give the first of these proofs; it is amazingly short.

Using Proposition 4.7 for the case $d=4$, we can prove the direct half of Theorem 1.3 (c) [i.e. the quaternionic case] by complete analogy with the elementary proofs given above for the real and complex cases. Moreover, with Theorem 4.8 in hand, the proof of Theorem 1.4 (and hence also of the converse half of Theorem 1.3) becomes utterly trivial.

Proof of Theorem 1.4. Equation (4.27) writes $\Delta(y)^{-\alpha}$ as the Laplace transform of the distribution $\mathcal{R}_\alpha$, which is supported on $\overline{\Omega}$ (this is the closed dual cone of Ω since Ω is self-dual). By the Bernstein–Hausdorff–Widder–Choquet theorem (Theorem 2.2), it follows that the map $y \mapsto \Delta(y)^{-\alpha}$ is completely monotone on Ω if and only if $\mathcal{R}_\alpha$ is a positive measure; moreover, if $\mathcal{R}_\alpha$ is not a positive measure, then this map is not completely monotone on any non-empty open convex subcone $\Omega' \subseteq \Omega$. So Theorem 1.4 is an immediate consequence of Theorem 4.8. □

Remark. The formulae in this section arise in multivariate statistics in connection with the Wishart distribution [85], [5, Chapter 7]; in recent decades some statisticians have introduced the formalism of Euclidean Jordan algebras as a unifying device [22], [31], [79], [32], [80]. These formulae also arise in quantum field theory in studying the analytic continuation of Feynman integrals to "complex space-time dimension" [111], [21], [49], [30].

([31]) Thus, for integer $N \geqslant 0$ in the real symmetric (resp. complex Hermitian) case, the positive measure $R_{N/2}$ is supported on the positive-semidefinite matrices of rank $\min(N, m)$ [52, Proposition VII.2.3] and is nothing other than the push-forward of Lebesgue measure on $\mathbb{R}^{m \times N}$ (resp. $\mathbb{C}^{m \times N}$) by the map $X \mapsto XX^{\mathrm{T}}$ (resp. $X \mapsto XX^*$), as discussed above during the proof of the direct half of Theorem 1.3 (a). For $N \geqslant m$ this is a straightforward calculation [111], [21], [85], [49], [5, Chapter 7], which shows the equivalence of (4.13) and (4.14) (or the corresponding formulae in the complex case); for $0 \leqslant N \leqslant m-1$ it follows by comparing (4.13) with (4.27) and invoking the injectivity of the Laplace transform on the space of distributions [104, p. 306]; see [52, Proposition VII.2.4].

Theorem 1.3 is an immediate consequence of Theorem 1.4, once we use the fact that the Jordan determinant coincides with the ordinary determinant on $\mathrm{Sym}(m,\mathbb{R})$ and $\mathrm{Herm}(m,\mathbb{C})$ and with the Moore determinant (see [105, Appendix A]) on $\mathrm{Herm}(m,\mathbb{H})$. Theorem 1.2 is in turn an easy consequence of Theorem 1.3.

Proof of Theorem 1.2. If $A_1,...,A_n$ are strictly positive-definite, Theorem 1.2 is an immediate corollary of Theorem 1.3 and the definition of complete monotonicity on cones (Definition 2.1). The general case is reduced to this one by replacing A_i by $A_i+\varepsilon I$ ($\varepsilon>0$) and taking $\varepsilon\downarrow 0$. □

Conversely, if β does *not* lie in the set described in Theorem 1.3, then the map $A\mapsto(\det A)^{-\beta}$ is *not* completely monotone on *any* non-empty open convex subcone of the cone of positive-definite matrices. This means, in particular, that if the matrices $A_1,...,A_n$ span $\mathrm{Sym}(m,\mathbb{R})$ or $\mathrm{Herm}(m,\mathbb{C})$ (so that the convex cone they generate is open), then the determinantal polynomial (4.9) does *not* have $P^{-\beta}$ completely monotone on $(0,\infty)^n$. See §7.3 for an application of this idea to graphs and matroids.

5. Quadratic forms

In this section we consider quadratic forms (= homogeneous polynomials of degree 2). We begin by proving an abstract theorem giving a necessary and sufficient condition for a quadratic form to be non-vanishing in a complex tube $C+iV$; in the special case $C=(0,\infty)^n$ this corresponds to the half-plane property. We then employ these results as one ingredient in our proof of Theorem 1.9.

5.1. The half-plane property

In this subsection we proceed in three steps. First we study the analytic geometry associated with a symmetric bilinear form B on a finite-dimensional real vector space V (Lemma 5.1). Next we extend the quadratic form $Q(x)=B(x,x)$ to the complexified space $V+iV$ and study the values it takes (Proposition 5.2). Finally we introduce the additional structure of an open convex cone $C\subseteq V$ on which Q is assumed strictly positive (Corollary 5.3 and Theorem 5.4).

So let V be a finite-dimensional real vector space, and let $B:V\times V\to\mathbb{R}$ be a symmetric bilinear form having inertia (n_+,n_-,n_0). Define $\mathcal{S}_+=\{x:B(x,x)>0\}$ and $\mathcal{S}_-=\{x:B(x,x)<0\}$. Clearly $\mathcal{S}_+$ and $\mathcal{S}_-$ are open cones (not in general convex or even connected). Indeed, $\mathcal{S}_+$ and $\mathcal{S}_-$ are *never* convex (except when they are empty) because $x\in\mathcal{S}_\pm$ implies $-x\in\mathcal{S}_\pm$ but manifestly $0\notin\mathcal{S}_\pm$.

Many of our proofs will involve choosing a basis in V (and hence identifying V with $\mathbb{R}^n$) in such a way that B takes the form

$$B(x,y) = \sum_{i=1}^{n_+} x_i y_i - \sum_{i=n_++1}^{n_++n_-} x_i y_i. \tag{5.1}$$

Moreover, whenever $\mathcal{S}_+ \neq \varnothing$ we can choose the basis in this construction such that the first coordinate direction lies along any desired vector $x \in \mathcal{S}_+$: that this can be done follows from the standard Gram–Schmidt proof of the canonical form (5.1).

Elementary analytic geometry gives the following result.

LEMMA 5.1. *Let V be a finite-dimensional real vector space, and let B be a symmetric bilinear form on V having inertia (n_+, n_-, n_0).*

(a) *If $n_+ = 0$, then $\mathcal{S}_+ = \varnothing$.*

(b) *If $n_+ = 1$, then $\mathcal{S}_+$ is a non-empty disconnected open non-convex cone, and there is a non-empty open convex cone $\mathcal{C}$ such that $\mathcal{S}_+ = \mathcal{C} \cup (-\mathcal{C})$ and $\mathcal{C} \cap (-\mathcal{C}) = \varnothing$. (Clearly $\mathcal{C}$ is uniquely determined modulo a sign.) Moreover, $B(x,y) \geqslant B(x,x)^{1/2} B(y,y)^{1/2} > 0$ whenever $x, y \in \mathcal{C}$; and $B(x,y)^2 \geqslant B(x,x)B(y,y)$ whenever $x \in \overline{\mathcal{S}}_+$ and $y \in V$ (or vice versa).*

(c) *If $n_+ \geqslant 2$, then $\mathcal{S}_+$ is a non-empty connected open non-convex cone. Moreover, for each $x \in \mathcal{S}_+$, the set*

$$\mathcal{T}_+(x) = \{y \in V : B(x,y)^2 < B(x,x)B(y,y)\} \tag{5.2}$$

is a non-empty open non-convex cone that is contained in $\mathcal{S}_+$ and has a non-empty intersection with every neighborhood of x; moreover we can write

$$\mathcal{T}_+(x) = \{y \in V : \mathbb{R}x + \mathbb{R}y \text{ is a 2-dimensional subspace contained within } \mathcal{S}_+ \cup \{0\}\} \tag{5.3}$$

(that is, it is a 2-dimensional subspace on which B is positive-definite).

Analogous statements hold for $\mathcal{S}_-$ when $n_- = 0$, $n_- = 1$ or $n_- \geqslant 2$.

Proof. (a) This is trivial.

(b) Assume that B takes the canonical form (5.1) with $n_+ = 1$, and define $\mathcal{C}$ to be the "forward light cone"

$$\mathcal{C} = \{x \in \mathbb{R}^n : x_1^2 - x_2^2 - \ldots - x_{n_-+1}^2 > 0 \text{ and } x_1 > 0\}. \tag{5.4}$$

It is immediate that $\mathcal{S}_+ = \mathcal{C} \cup (-\mathcal{C})$ and $\mathcal{C} \cap (-\mathcal{C}) = \varnothing$, and the statements about $\mathcal{S}_+$ follow easily.

Now consider any $x, y \in \mathcal{C}$ and define

$$g(\alpha) = B(x+\alpha y, x+\alpha y) = B(x,x) + 2\alpha B(x,y) + \alpha^2 B(y,y). \tag{5.5}$$

We have $g(0)=B(x,x)>0$; but for the special value $\alpha_\star=-x_1/y_1$ the vector $x+\alpha_\star y$ has its first component equal to zero and hence $g(\alpha_\star)=B(x+\alpha_\star y, x+\alpha_\star y)\leqslant 0$. So the quadratic equation $g(\alpha)=0$ has a real solution, which implies that its discriminant is non-negative, i.e. that $B(x,y)^2\geqslant B(x,x)\,B(y,y)$.

Next assume that $x\in\bar{\mathcal{S}}_+$ and $y\in V$. If $B(x,x)=0$ or $B(y,y)\leqslant 0$, the assertion is trivial; so we may assume that $x,y\in\mathcal{S}_+$. By the replacements $x\mapsto\pm x$ and $y\mapsto\pm y$ (which do not affect the desired conclusion) we may assume that $x,y\in\mathcal{C}$. But in this case the desired inequality has already been proven.

Finally, using $B(x,y)>0$ for $x,y\in\mathcal{C}$ it is easily checked that $\mathcal{C}$ is convex.

(c) Clearly $\mathcal{S}_+$ is non-empty; and as explained earlier it is non-convex. To prove that $\mathcal{S}_+$ is connected, we may assume that B takes the form (5.1) with $n_+\geqslant 2$. It is now sufficient to find a path in $\mathcal{S}_+$ from an arbitrary vector $x\in\mathcal{S}_+$ to the vector $e_1=(1,0,...,0)$. But this is easy: first move coordinates x_i with $i>n_+$ monotonically to zero (this increases $B(x,x)$ monotonically and hence stays in $\mathcal{S}_+$); then rotate and scale inside the subspace spanned by the first n_+ coordinates to obtain e_1.

Now assume that B takes the canonical form (5.1) with $n_+\geqslant 2$ and with the given vector $x\in\mathcal{S}_+$ lying along the first coordinate direction. Then an easy computation shows that $y=(y_1,y_2,...,y_n)$ belongs to $\mathcal{T}_+(x)$ if and only if $y'\equiv(0,y_2,...,y_n)$ belongs to $\mathcal{S}_+$. Therefore, the preceding results (b) and (c) applied with n_+ replaced by n_+-1 show that $\mathcal{T}_+(x)$ is a non-empty open non-convex cone, which is obviously contained in $\mathcal{S}_+$; and by taking y' small we see that $\mathcal{T}_+(x)$ meets every neighborhood of x. Moreover, $\mathbb{R}x+\mathbb{R}y=\mathbb{R}x+\mathbb{R}y'$ is a 2-dimensional subspace if and only if $y'\neq 0$ (i.e. $y\notin\mathbb{R}x$); and since $B(x,y')=0$, we have $\mathbb{R}x+\mathbb{R}y'\subseteq\mathcal{S}_+\cup\{0\}$ if and only if $y'\in\mathcal{S}_+\cup\{0\}$. □

Remark. Note the sharp contrast between parts (b) and (c): in the latter case, given any $x\in\mathcal{S}_+$ there is a non-empty open cone of vectors y satisfying the Schwarz inequality (strictly) with x; while in the former case all vectors $y\in V$ satisfy the *reverse* Schwarz inequality with x.

We now consider the quadratic form $Q(x)=B(x,x)$, extended to the complexified space $V+iV$ in the obvious way: $Q(x+iy)=B(x,x)-B(y,y)+2iB(x,y)$. We want to study the values taken by Q in the complex tubes $\mathcal{S}_++iV$ and $\mathcal{S}_-+iV$, and in particular the presence or absence of zeros. We write $\mathbb{H}$ to denote the open right half-plane

$$\{\zeta\in\mathbb{C}:\operatorname{Re}\zeta>0\}.$$

PROPOSITION 5.2. *Let V be a finite-dimensional real vector space, let B be a symmetric bilinear form on V having inertia (n_+,n_-,n_0), and let Q be the associated qua-*

dratic form extended to $V+iV$.

(a) *If $n_+=1$, then for every $x\in\mathcal{S}_+$ we have $Q[\mathbb{H}x]=Q[\mathcal{S}_+ +iV]=\mathbb{C}\setminus(-\infty,0]$. In particular, Q is non-vanishing on $\mathcal{S}_+ +iV$.*

(b) *If $n_+\geqslant 2$, then for every $x\in\mathcal{S}_+$ and $y\in\mathcal{T}_+(x)$ (recall that $\mathcal{T}_+(x)$ is non-empty) we have $Q[[1,\infty)x+i(\mathbb{R}x+\mathbb{R}y)]=\mathbb{C}$. In particular, for each $x\in\mathcal{S}_+$ and $y\in\mathcal{T}_+(x)$ there exists $z\in\mathbb{R}x+\mathbb{R}y$ such that $Q(x+iz)=0$.*

Analogous statements hold for $\mathcal{S}_-$ when $n_-=1$ or $n_-\geqslant 2$.

Proof. (a) If $x\in\mathcal{S}_+$, then $Q(\zeta x)=\zeta^2 Q(x)$ can take any value in $\mathbb{C}\setminus(-\infty,0]$ as ζ ranges over $\mathbb{H}$. On the other hand, if $x\in\mathcal{S}_+$ and $y\in V$, then

$$Q(x+iy)=B(x,x)-B(y,y)+2iB(x,y)$$

cannot take a value in $(-\infty,0]$, as $B(x,y)=0$ implies by Lemma 5.1 (b) that $y\notin\mathcal{S}_+$, i.e. $B(y,y)\leqslant 0$.

(b) Given $x\in\mathcal{S}_+$ and $y\in\mathcal{T}_+(x)$, the vector $y'=y+\mu x$ with $\mu=-B(x,y)/B(x,x)$ satisfies $B(x,y')=0$ and $Q(y')>0$. Therefore

$$Q(\lambda x+i(\alpha x+\beta y'))=(\lambda+i\alpha)^2 Q(x)-\beta^2 Q(y'), \tag{5.6}$$

and this is easily seen to take all complex values as λ ranges over $[1,\infty)$, α over $\mathbb{R}$, and β over $(0,\infty)$. □

Now we introduce the additional structure of an open convex cone $C\subseteq V$ on which Q is assumed strictly positive (i.e. $C\subseteq\mathcal{S}_+$). The hypotheses in the following result are identical to those of Theorem 1.9.

Corollary 5.3. *Let V be a finite-dimensional real vector space, let B be a symmetric bilinear form on V having inertia (n_+,n_-,n_0), and define the quadratic form $Q(x)=B(x,x)$. Let $C\subset V$ be a non-empty open convex cone with the property that $Q(x)>0$ for all $x\in C$. Then $n_+\geqslant 1$, and moreover*

(a) *if $n_+=1$, then either $C\subseteq\mathcal{C}$ or $C\subseteq-\mathcal{C}$ (where $\mathcal{C}$ is defined as in Lemma 5.1 (b)), and we have $B(x,y)>0$ for all $x,y\in C$;*

(b) *if $n_+\geqslant 2$, then for every $x\in C$ we have $Q[[1,\infty)x+iV]=\mathbb{C}$, and in particular there exists $z\in V$ such that $Q(x+iz)=0$.*

Proof. (a) Let us use the canonical form (5.1) with $n_+=1$ and $\mathcal{C}$ defined by (5.4). By hypothesis the cone C is contained within $\mathcal{S}_+=\mathcal{C}\cup(-\mathcal{C})$; but since C is convex we must in fact have either $C\subseteq\mathcal{C}$ or $C\subseteq-\mathcal{C}$: for otherwise C would contain a point with

$x_1=0$ and hence $B(x,x)\leqslant 0$.([32]) The remaining statements follow from Lemma 5.1 (b) and Proposition 5.2 (a).

(b) Since $C\subseteq\mathcal{S}_+$, the statements follow from Lemma 5.1 (c) and Proposition 5.2 (b). □

We can also summarize our results in a way that extends [37, Theorem 5.3] from $C=(0,\infty)^n$ to general open convex cones C.

THEOREM 5.4. *Let V be a finite-dimensional real vector space, let B be a symmetric bilinear form on V having inertia (n_+,n_-,n_0), and define the quadratic form $Q(x)=B(x,x)$. Let $C\subset V$ be a non-empty open convex cone with the property that $Q(x)>0$ for all $x\in C$. Then $n_+\geqslant 1$, and the following are equivalent*:

(a) $n_+=1$;

(b) *Q is non-vanishing on the tube $C+iV$*;

(c) $Q[C+iV]=\mathbb{C}\setminus(-\infty,0]$;

(d) *If $x,y\in V$ with $Q(x)\geqslant 0$, then $B(x,y)^2\geqslant B(x,x)B(y,y)$*;

(e) *If $x,y\in C$, then $B(x,y)^2\geqslant B(x,x)B(y,y)$.*

Proof. This follows immediately by putting together the statements in Lemma 5.1, Proposition 5.2 and Corollary 5.3: if $n_+=1$, then (b)–(e) are all true; and if $n_+\geqslant 2$, then (b)–(e) are all false. □

Remark. See [37, Remark 1 after the proof of Theorem 5.3] for some of the history of this result in the traditional case $C=(0,\infty)^n$.

5.2. Proof of Theorem 1.9

Proof of Theorem 1.9. We are concerned with the complete monotonicity of $Q^{-\beta}$, where $Q(x)=B(x,x)$. As always, it suffices to consider $\beta>0$, because complete monotonicity trivially holds when $\beta=0$ and never holds when $\beta<0$ (because Q grows at infinity).

We assume that B takes the canonical form (5.1) on $V=\mathbb{R}^n$, and we consider separately the three cases:

(a) The case $n_+=1$, $n_-=0$ is trivial: we have $Q(x)=x_1^2$, and the convex cone C must be contained in one of the half-spaces $\{x:x_1>0\}$ or $\{x:x_1<0\}$. The map $x\mapsto Q(x)^{-\beta}=x_1^{-2\beta}$ is clearly completely monotone on each of these two half-spaces.

([32]) Alternatively, we can argue (assuming for simplicity that $n_0=0$) that a convex set S contained in $\overline{\mathcal{S}}_+$ must either lie on a single line through the origin (which is obviously impossible if S is open) or else be contained in either $\overline{\mathcal{C}}$ or $-\overline{\mathcal{C}}$. For if we had $x\in S\cap\overline{\mathcal{C}}$ and $y\in S\cap-\overline{\mathcal{C}}$ with x and y not on the same line through the origin, then there would exist $\lambda\in(0,1)$ such that $z=\lambda x+(1-\lambda)y$ has $z_1=0$ and $z\neq 0$, and hence $z\notin\overline{\mathcal{S}}_+$.

(b) Next consider the case $n_+=1$, $n_-\geqslant 1$: here (5.1) is the Lorentz form in one "timelike" variable and n_- "spacelike" variables, and we have $\mathcal{S}_+=\mathcal{C}\cup(-\mathcal{C})$, where $\mathcal{C}$ is the forward light cone (5.4). By Corollary 5.3 (a) we have either $C\subseteq\mathcal{C}$ or $C\subseteq-\mathcal{C}$; let us suppose the former.

Let us now show that if $\beta\geqslant\frac{1}{2}(n_--1)$, then the map $x\mapsto Q(x)^{-\beta}$ is completely monotone on $\mathcal{C}$. The variables $x_{n_-+2},\ldots,x_n$ play no role in this, so we may assume without loss of generality that $n_0=0$, i.e. $n=n_-+1$. For $\beta>\frac{1}{2}(n_--1)=\frac{1}{2}(n-2)$, the desired complete monotonicity then follows from the integral representation [99, pp. 31–34], [47, eqq. (24)–(28)]

$$\begin{aligned}(x_1^2-x_2^2-\ldots-x_n^2)^{-\beta} = \left[\pi^{(n-2)/2}2^{2\beta-1}\Gamma(\beta)\Gamma\left(\beta-\frac{n-2}{2}\right)\right]^{-1} \\ \times\int_{\mathcal{C}} e^{-(x_1y_1-x_2y_2-\ldots-x_ny_n)}(y_1^2-y_2^2-\ldots-y_n^2)^{\beta-n/2}\,dy\end{aligned} \tag{5.7}$$

valid for $\beta>\frac{1}{2}(n-2)$, which explicitly represents $Q(x)^{-\beta}$ as the Laplace transform of a positive measure supported on the closed forward light cone $\mathcal{C}^*=\bar{\mathcal{C}}$. For $\beta=\frac{1}{2}(n-2)$ the result follows by taking limits.

Conversely, let us show that if $\beta\notin\{0\}\cup\left[\frac{1}{2}(n_--1),\infty\right)$, then the map $x\mapsto Q(x)^{-\beta}$ is not completely monotone on any non-empty open convex subcone $C'\subseteq\mathcal{C}$. For suppose that this map *is* completely monotone on C' for some such β: then by the Bernstein–Hausdorff–Widder–Choquet theorem (Theorem 2.2), we must have (assuming again without loss of generality that $n_0=0$)([33])

$$(x_1^2-x_2^2-\ldots-x_n^2)^{-\beta} = \int_{(C')^*} e^{-(x_1y_1-x_2y_2-\ldots-x_ny_n)}\,d\mu_\beta(y) \tag{5.8}$$

for some positive measure μ_β supported on $(C')^*$. Now, any such measure must clearly be Lorentz-invariant and homogeneous of degree $2\beta-n$ (this follows from the injectivity of the Laplace transform). Furthermore, μ_β must be supported on $\bar{\mathcal{C}}$, for otherwise the support would contain a spacelike hyperboloid $\{y\in\mathbb{R}^n: y_1^2-y_2^2-\ldots-y_n^2=\lambda\}$ for some $\lambda<0$, whose convex hull is all of $\mathbb{R}^n$ and hence not contained in the proper cone $(C')^*$. On the other hand, every Lorentz-invariant locally finite positive measure on $\mathbb{R}^n$ that is supported on $\bar{\mathcal{C}}$ is of the form [98, Theorem IX.33, pp. 70–76]

$$\mu = c\delta_0 + \int_{m\geqslant 0} d\Omega_m\,d\varrho(m), \tag{5.9}$$

([33]) If the map $x\mapsto Q(x)^{-\beta}$ is completely monotone on C', then it is also completely monotone on the cone C'' obtained by projecting C' onto the first n_++n_- coordinates (since $Q(x)$ is independent of the last n_0 coordinates).

where $c \geqslant 0$, δ_0 denotes the point mass at the origin, ϱ is a positive measure on $[0, \infty)$, and $d\Omega_m = \delta(y^2 - m^2)\, dy$ is the unique (up to a constant multiple) Lorentz-invariant measure on the "mass hyperboloid"

$$H_m = \{y \in \mathbb{R}^n : y_1^2 - y_2^2 - \ldots - y_n^2 = m^2 \text{ and } y_1 > 0\}. \tag{5.10}$$

(When $n=2$, we consider $m>0$ only, as there is no locally finite Lorentz-invariant measure on $\mathbb{R}^2$ that is supported on H_0.) A measure μ of the form (5.9) is homogeneous in precisely three cases:

(i) $c=0$, $d\varrho(m) = \text{const} \times m^{\lambda-1}\, dm$ with $\lambda > 0$: here μ is homogeneous of degree $\lambda - 2$.

(ii) $c=0$, $\varrho = \text{const} \times \delta_0$ (for $n>2$ only): here μ is homogeneous of degree -2.

(iii) $c \geqslant 0$, $\varrho = 0$: here μ is homogeneous of degree $-n$.

This proves that a positive measure μ_β can exist *only if* $\beta = 0$ or $\beta \geqslant \frac{1}{2}(n-2)$.

(c) Finally, consider the case $n_+ > 1$. By Corollary 5.3 (b), Q has zeros in the tube $C' + iV$ for every non-empty open convex subcone $C' \subseteq \mathcal{S}_+$ (indeed, for every non-empty *subset* $C' \subseteq \mathcal{S}_+$). We conclude by Corollary 2.3 that $Q^{-\beta}$ cannot be completely monotone on C' for any $\beta > 0$. □

Remark. More can be said about the integral representation (5.7). It turns out that the quantity[34]

$$R_\beta = \left[\pi^{(n-2)/2}\, 2^{2\beta-1} \Gamma(\beta) \Gamma\left(\beta - \tfrac{1}{2}(n-2)\right)\right]^{-1} (y_1^2 - y_2^2 - \ldots - y_n^2)^{\beta - n/2} \mathbf{1}_{y \in \mathcal{C}}, \tag{5.11}$$

which is initially defined for $\beta > \frac{1}{2}(n-2)$ as a positive measure (or for $\operatorname{Re}\beta > \frac{1}{2}(n-2)$ as a complex measure) on $\mathbb{R}^n$ (and which is of course supported on $\mathcal{C}$), can be analytically continued as a tempered-distribution-valued entire function of β [47], [52, Theorem VII.2.2]: this is the *Riesz distribution* $\mathcal{R}_\beta$ on the Euclidean Jordan algebra $\mathbb{R} \times \mathbb{R}^{n-1}$.[35] The integral representation

$$(x_1^2 - x_2^2 - \ldots - x_n^2)^{-\beta} = \int_{\mathcal{C}} e^{-(x_1 y_1 - x_2 y_2 - \ldots - x_n y_n)}\, \mathcal{R}_\beta(y)\, dy, \tag{5.12}$$

where $x \in \mathcal{C}$, then holds for all complex β, by analytic continuation. However, the distribution $\mathcal{R}_\beta$ is a positive measure if and only if either $\beta = 0$ or $\beta \geqslant \frac{1}{2}(n-2)$. This follows from general results of harmonic analysis on Euclidean Jordan algebras (i.e. Theorem 4.8), but

[34] Here $\mathbf{1}_p$ denotes the characteristic function of the proposition p.

[35] A slightly different normalization is used in [52], arising from the fact that the Jordan inner product on $\mathbb{R} \times \mathbb{R}^{n-1}$ is $((x_0, \mathbf{x})|(y_0, \mathbf{y})) = 2(x_0 y_0 + \mathbf{x} \cdot \mathbf{y})$: this has the consequence that $dx_{\text{Jordan}} = 2^{n/2}\, dx_{\text{ordinary}}$, and also the Laplace transform is written with an extra factor 2 in the exponential. The change of sign from $x_0 y_0 - \mathbf{x} \cdot \mathbf{y}$ to $x_0 y_0 + \mathbf{x} \cdot \mathbf{y}$ is irrelevant, because the Riesz distribution $\mathcal{R}_\beta(y)$ is invariant under the reflections $y_i \mapsto -y_i$ for $2 \leqslant i \leqslant n$.

we have given here a direct elementary proof. Indeed, once one has in hand the fundamental properties of the Riesz distribution $\mathcal{R}_\beta$, one obtains an even simpler elementary proof by observing that (5.11) is not locally integrable near the boundary of the cone $\mathcal{C}$ when $\operatorname{Re}\beta \leqslant \frac{1}{2}(n-2)$ and $\beta \neq \frac{1}{2}(n-2)$, and hence [110, Lemma 2.1] the distribution $\mathcal{R}_\beta$ is not a locally finite complex measure in these cases.

5.3. Explicit Laplace-transform formula for inverse powers of a quadratic form

By a simple change of variables, we can generalize (5.7) to replace the Lorentz matrix $L_n = \operatorname{diag}(1,-1,\ldots,-1)$ by an arbitrary real symmetric matrix A of inertia $(n_+,n_-,n_0) = (1,n-1,0)$. This will provide, among other things, an explicit Laplace-transform formula for $E_{2,n}(\mathbf{x})^{-\beta}$ that generalizes the formulae (3.21) and (4.18) obtained previously for $n=3,4$, respectively.

So let A be a real symmetric $n\times n$ matrix with one positive eigenvalue, $n-1$ negative eigenvalues, and no zero eigenvalues. We first need a slight refinement of Lemma 5.1(b) to take advantage of the fact that we now have $n_0=0$; for simplicity we state it in the "concrete" situation $V=\mathbb{R}^n$.

LEMMA 5.5. *Fix $n\geqslant 2$, and let A be a real symmetric $n\times n$ matrix with one positive eigenvalue, $n-1$ negative eigenvalues, and no zero eigenvalues. Then there exists a nonempty open convex cone $\mathcal{C}\subset\mathbb{R}^n$ (which is uniquely determined modulo a sign) such that*

$$\mathcal{C}\cap(-\mathcal{C}) = \varnothing, \tag{5.13}$$

$$\bar{\mathcal{C}}\cap(-\bar{\mathcal{C}}) = \{0\}, \tag{5.14}$$

$$\mathcal{C}^{\wedge}\cap(-\mathcal{C}^{\wedge}) = \varnothing, \tag{5.15}$$

$$\overline{\mathcal{C}^{\wedge}}\cap(-\overline{\mathcal{C}^{\wedge}}) = \{0\}, \tag{5.16}$$

$$\{\mathbf{y}:\mathbf{y}^{\mathrm{T}}A\mathbf{y}>0\} = \mathcal{C}\cup(-\mathcal{C}), \tag{5.17}$$

$$\{\mathbf{x}:\mathbf{x}^{\mathrm{T}}A^{-1}\mathbf{x}>0\} = \mathcal{C}^{\wedge}\cup(-\mathcal{C}^{\wedge}), \tag{5.18}$$

where

$$\mathcal{C}^{\wedge} = \{\mathbf{x}:\mathbf{x}^{\mathrm{T}}\mathbf{y}>0 \text{ for all } \mathbf{y}\in\bar{\mathcal{C}}\setminus\{0\}\} \tag{5.19}$$

is the open dual cone to $\mathcal{C}$.

Proof. We can write $A=S^{\mathrm{T}}L_nS$, where S is a non-singular real matrix. Then the claims follow easily from the corresponding properties of the Lorentz quadratic form. □

PROPOSITION 5.6. *Fix $n \geqslant 2$, and let A be a real symmetric $n \times n$ matrix with one positive eigenvalue, $n-1$ negative eigenvalues, and no zero eigenvalues; and let $\mathcal{C}$ be the open convex cone from Lemma 5.5. Then for $\beta > \frac{1}{2}(n-2)$ we have*

$$(\mathbf{x}^{\mathrm{T}} A^{-1} \mathbf{x})^{-\beta} = \left[\pi^{(n-2)/2} 2^{2\beta-1} \Gamma(\beta) \Gamma\left(\beta - \frac{n-2}{2}\right) \right]^{-1} |\det A|^{1/2} \int_{\mathcal{C}} e^{-\mathbf{x}^{\mathrm{T}}\mathbf{y}} (\mathbf{y}^{\mathrm{T}} A \mathbf{y})^{\beta - n/2} \, d\mathbf{y} \tag{5.20}$$

for $\mathbf{x} \in \hat{\mathcal{C}}$.

Proof. Write $A = S^{\mathrm{T}} L_n S$, where S is a non-singular real matrix, and make the changes of variable $\mathbf{y} = S\mathbf{y}'$ and $\mathbf{x} = L_n S^{-\mathrm{T}} \mathbf{x}'$ in (5.7). Then $d\mathbf{y} = |\det S| \, d\mathbf{y}'$, where $|\det S| = |\det A|^{1/2}$; and the formula (5.20) follows immediately after dropping primes. □

Let us now specialize to matrices A of the form $A = \lambda E_n - \mu I_n$, where I_n is the $n \times n$ identity matrix and E_n is the $n \times n$ matrix with all entries 1. Then A has eigenvalues $n\lambda - \mu, -\mu, \ldots, -\mu$, and hence has inertia $(n_+, n_-, n_0) = (1, n-1, 0)$ provided that $\mu > 0$ and $\lambda > \mu/n$; and in that case we have $A^{-1} = \lambda' E_n - \mu' I_n$, where

$$\lambda' = \frac{\lambda}{\mu(n\lambda - \mu)} \quad \text{and} \quad \mu' = \frac{1}{\mu}. \tag{5.21}$$

(The map $(\lambda, \mu) \mapsto (\lambda', \mu')$ is of course involutive.) Furthermore, we have

$$\mathbf{y}^{\mathrm{T}} A \mathbf{y} = 2\lambda E_{2,n}(\mathbf{y}) + (\lambda - \mu) \|\mathbf{y}\|^2, \tag{5.22}$$

where

$$E_{2,n}(\mathbf{y}) = \sum_{1 \leqslant i < j \leqslant n} y_i y_j \quad \text{and} \quad \|\mathbf{y}\|^2 = \sum_{i=1}^{n} y_i^2,$$

and analogously for $\mathbf{x}^{\mathrm{T}} A^{-1} \mathbf{x}$.

COROLLARY 5.7. *Fix $n \geqslant 2$, $\mu > 0$ and $\lambda > \mu/n$. Then for $\beta > \frac{1}{2}(n-2)$ we have*

$$\begin{aligned} &\left(\frac{2\lambda}{\mu(n\lambda - \mu)} E_{2,n}(\mathbf{x}) - \frac{(n-1)\lambda - \mu}{\mu(n\lambda - \mu)} \|\mathbf{x}\|^2 \right)^{-\beta} \\ &\quad = \left[\pi^{(n-2)/2} 2^{2\beta-1} \Gamma(\beta) \Gamma\left(\beta - \frac{n-2}{2}\right) \right]^{-1} \mu^{(n-1)/2} (n\lambda - \mu)^{1/2} \\ &\qquad \times \int_{\mathcal{C}} e^{-\mathbf{x}^{\mathrm{T}}\mathbf{y}} (2\lambda E_{2,n}(\mathbf{y}) + (\lambda - \mu)\|\mathbf{y}\|^2)^{\beta - n/2} \, d\mathbf{y} \end{aligned} \tag{5.23}$$

for $\mathbf{x} \in \hat{\mathcal{C}}$.

Specializing further to the case $\lambda = 2/(n-1)$, $\mu = 2$ (corresponding to $\lambda' = \mu' = \frac{1}{2}$), we obtain the following result.

COROLLARY 5.8. (Laplace-transform formula for $E_{2,n}^{-\beta}$) *Fix an integer $n \geqslant 2$. Then for $\beta > \frac{1}{2}(n-2)$ we have*

$$E_{2,n}(\mathbf{x})^{-\beta} = \frac{(n-1)^{(n-1)/2-\beta}}{(2\pi)^{(n-2)/2}\Gamma(\beta)\Gamma\big(\beta - \frac{1}{2}(n-2)\big)} \int_{\mathcal{C}} e^{-\mathbf{x}^{\mathrm{T}}\mathbf{y}} \left(E_{2,n}(\mathbf{y}) + \frac{n-2}{2}\|\mathbf{y}\|^2 \right)^{\beta-n/2} d\mathbf{y} \tag{5.24}$$

for $\mathbf{x} \in \mathcal{C}\hat{} \supseteq (0,\infty)^n$.

For $n=2$ this is elementary; for $n=3$ it reduces to (3.21); and for $n=4$ it reduces to (4.18). The formula (5.24) provides, in particular, an explicit elementary proof of the direct ("if") half of Corollary 1.10.

6. Positive-definite functions on cones

In this section we recall briefly the theory of positive-definite functions (in the semigroup sense) on convex cones, which closely parallels the theory of completely monotone functions developed in §2 and indeed can be considered as a natural extension of it. We then apply this theory to powers of the determinant on a Euclidean Jordan algebra, and derive (in Theorem 6.5) a strengthening of Theorem 1.4. As an application of this latter result, we disprove (in Example 6.6) a recent conjecture of Gurau, Magnen and Rivasseau [62].

This section is not required for the application to graphs and matroids (§7).

6.1. General theory

Here we summarize the basic definitions and results from the theory of positive-definite functions on convex cones and, more generally, on convex sets. A plethora of useful additional information concerning positive-definite (and related) functions on semigroups can be found in the monograph by Berg, Christensen and Ressel [18].

Definition 6.1. Let V be a real vector space, and let C be a convex cone in V. Then a function $f: C \to \mathbb{R}$ is termed *positive-definite in the semigroup sense* if for all $n \geqslant 1$ and all $x_1, \ldots, x_n \in C$, the matrix $\{f(x_i + x_j)\}_{i,j=1}^n$ is positive-semidefinite; or in other words, if for all $n \geqslant 1$, all $x_1, \ldots, x_n \in C$ and all $c_1, \ldots, c_n \in \mathbb{C}$ we have

$$\sum_{i,j=1}^n \bar{c}_i c_j f(x_i + x_j) \geqslant 0. \tag{6.1}$$

Similarly, a function $f: C + iV \to \mathbb{C}$ is termed *positive-definite in the involutive-semigroup sense* if for all $n \geqslant 1$ and all $x_1, \ldots, x_n \in C + iV$, the matrix $\{f(x_i + \bar{x}_j)\}_{i,j=1}^n$ is positive-semidefinite.

THEOREM 6.2. *Let V be a finite-dimensional real vector space, let C be an* open *convex cone in V, and let $f: C \to \mathbb{R}$. Then the following are equivalent*:

(a) *f is continuous and positive-definite in the semigroup sense*;

(b) *f extends to an analytic function on the tube $C+iV$ that is positive-definite in the involutive-semigroup sense*;

(c) *there exists a positive measure μ on V^* satisfying*

$$f(x) = \int_{V^*} e^{-\langle \ell, x \rangle} \, d\mu(\ell) \tag{6.2}$$

for all $x \in C$.

Moreover, in this case the measure μ is unique, and the analytic extension to $C+iV$ is given by (6.2).

Please note that the completely monotone functions (Theorem 2.2) correspond to the subset of positive-definite functions that are bounded at infinity (in the sense that f is bounded on the set $x+C$ for each $x \in C$), or equivalently decreasing (with respect to the order induced by the cone C), or equivalently for which the measure μ is supported on the closed dual cone C^* (rather than on the whole space V^* as in Theorem 6.2 (c)). See [87, Lemma 1, p. 579] for a direct proof that complete monotonicity implies positive-definiteness.

We remark that the hypothesis of continuity (or at least something weaker, such as measurability or local boundedness) in Theorem 6.2 (a) is essential, even in the simplest case $V = \mathbb{R}$ and $C = (0, \infty)$. Indeed, using the axiom of choice it can easily be shown [2, pp. 35–36, 39] that there exist discontinuous solutions to the functional equation

$$\varrho(x+y) = \varrho(x)\varrho(y) \quad \text{for } x, y \in (0, \infty),$$

and any such function is automatically positive-definite in the semigroup sense. However, any such function is necessarily non-Lebesgue-measurable and everywhere locally unbounded [2, pp. 34–35, 37–39].

Theorem 6.2 is actually a special case of a more general theorem for open convex *sets* that need not be cones. We begin with the relevant definition [59, p. x].

Definition 6.3. Let V be a real vector space. If $C \subseteq V$ is a convex set, then a function $f: C \to \mathbb{R}$ is termed *positive-definite in the convex-set sense* if for all $n \geqslant 1$ and all $x_1, \ldots, x_n \in C$, the matrix $\big\{f\big(\frac{1}{2}(x_i + x_j)\big)\big\}_{i,j=1}^n$ is positive-semidefinite. More generally, if $C \subseteq V+iV$ is a conjugation-invariant convex set, then a function $f: C \to \mathbb{C}$ is termed *positive-definite in the involutive-convex-set sense* if for all $n \geqslant 1$ and all $x_1, \ldots, x_n \in C$, the matrix $\big\{f\big(\frac{1}{2}(x_i + \bar{x}_j)\big)\big\}_{i,j=1}^n$ is positive-semidefinite.

Note that if C is in fact a convex *cone*, then a function $f: C \to \mathbb{R}$ is positive-definite in the convex-set sense if and only if it is positive-definite in the semigroup sense. So this concept is a genuine generalization of the preceding one.

THEOREM 6.4. *Let V be a finite-dimensional real vector space, let $C \subseteq V$ be an* open *convex set, and let $f: C \to \mathbb{R}$. Then the following are equivalent*:

(a) *f is continuous and positive-definite in the convex-set sense*;

(b) *f extends to an analytic function on the tube $C+iV$ that is positive-definite in the involutive-convex-set sense*;

(c) *there exists a positive measure μ on V^* satisfying*

$$f(x) = \int_{V^*} e^{-\langle \ell, x \rangle} \, d\mu(\ell) \tag{6.3}$$

for all $x \in C$.

Moreover, in this case the measure μ is unique, and the analytic extension to $C+iV$ is given by (6.3).

Theorem 6.4 was first proven by Devinatz [42], using the spectral theory of commuting unbounded self-adjoint operators on Hilbert space (he gives details for $\dim V = 2$ but states that the methods work in any finite dimension); see also Akhiezer [4, pp. 229–231] for the special case in which C is a Cartesian product of open intervals. A detailed alternative proof, based on studying positive-definiteness on convex sets of *rational* numbers as an intermediate step [11], has been given by Glöckner [59, Proposition 18.7 and Theorem 18.8], who also gives generalizations to infinite-dimensional spaces V and to operator-valued positive-definite functions. See also Shucker [107, Theorem 4 and Corollary] and Glöckner [59, Theorem 18.8] for the very interesting extension to convex sets C that are not necessarily open (but have non-empty interior): in this latter case the representation (6.3) does *not* imply the continuity of f on C, but only on line segments (or more generally, closed convex hulls of finitely many points) within C. But with this modification the equivalence (a′) $\Leftrightarrow$ (c) holds.

Surprisingly, we have been unable to find in the literature any complete proof of Theorem 6.2 except as a corollary of the more general Theorem 6.4. But see [59, Theorem 16.6] for a version of Theorem 6.2 for the subclass of positive-definite functions that are α-bounded with respect to a "tame" absolute value α.

It would be interesting to try to find simpler proofs of Theorems 6.2 and 6.4.

6.2. Powers of the determinant on a Euclidean Jordan algebra

We can now deduce analogues of Theorems 1.3 and 1.4 in which complete monotonicity is replaced by positive-definiteness in the semigroup sense. For brevity we state only the

abstract result in terms of Euclidean Jordan algebras. The "converse" half of this result constitutes an interesting strengthening of the corresponding half of Theorem 1.4; we will apply it in Example 6.6.

THEOREM 6.5. *Let V be a simple Euclidean Jordan algebra of dimension n and rank r, with $n=r+\frac{1}{2}dr(r-1)$; let $\Omega\subset V$ be the positive cone; let $\Delta: V\to\mathbb{R}$ be the Jordan determinant; and let $\beta\in\mathbb{R}$. Then the following are equivalent:*

(a) *the map $x\mapsto\Delta(x)^{-\beta}$ is positive-definite in the semigroup sense on Ω;*

(b) *the map $x\mapsto\Delta(x)^{-\beta}$ is positive-definite in the semigroup sense on some nonempty open convex subcone $\Omega'\subseteq\Omega$;*

(c) $\beta\in\{0, \frac{1}{2}d, ..., \frac{1}{2}d(r-1)\}\cup\big(\frac{1}{2}d(r-1), \infty\big)$.

Theorem 6.5 is a direct consequence of facts about Riesz distributions—namely, the Laplace-transform formula (4.27) and Theorem 4.8—together with Theorem 6.2. Indeed, the proof of Theorem 6.5 is essentially *identical* to that of Theorem 1.4, but using Theorem 6.2 in place of the Bernstein–Hausdorff–Widder–Choquet theorem. The point, quite simply, is that our proof of the "converse" half of Theorem 1.4 used only the failure of positivity of the Riesz distribution, not any failure to be supported on the closed dual cone (indeed, it is *always* supported there); so it proves Theorem 6.5 as well.

Example 6.6. Let V be the real vector space $\mathrm{Sym}(m,\mathbb{R})$ of real symmetric $m\times m$ matrices, let $\Pi_m(\mathbb{R})\subset V$ be the cone of positive-definite matrices, and let $C\subset\Pi_m(R)$ be the subcone consisting of matrices that are also elementwise strictly positive. It follows from Theorem 6.5 that the map $A\mapsto(\det A)^{-\beta}$ is positive-definite in the semigroup sense on C if and only if it is positive-definite in the semigroup sense on $\Pi_m(\mathbb{R})$, and this happens if and only if $\beta\in\{0, \frac{1}{2}, 1, \frac{3}{2}, ...\}\cup\big[\frac{1}{2}(m-1), \infty\big)$. This disproves the conjecture of Gurau, Magnen and Rivasseau [62, §7, Conjecture 1] that the map $A\mapsto(\det A)^{-\beta}$ would be positive-definite in the semigroup sense on C for all $\beta\geqslant 0$.

7. Application to graphs and matroids

7.1. Graphs

Let $G=(V,E)$ be a finite undirected graph with vertex set V and edge set E; in this paper all graphs are allowed to have loops and multiple edges unless explicitly stated otherwise. Now let $\mathbf{x}=\{x_e\}_{e\in E}$ be a family of indeterminates indexed by the edges of G. If G is a *connected* graph, we denote by $T_G(\mathbf{x})$ the generating polynomial of spanning trees in G, namely

$$T_G(\mathbf{x}) = \sum_{T\in\mathcal{T}(G)} \prod_{e\in T} x_e, \tag{7.1}$$

where $\mathcal{T}(G)$ denotes the family of edge sets of spanning trees in G. If G is *disconnected*, we define $T_G(\mathbf{x})$ to be the product of the spanning-tree polynomials of its connected components. Otherwise put, $T_G(\mathbf{x})$ is in all cases the generating polynomial of *maximal spanning forests* in G. This is a slightly non-standard definition (the usual definition would put $T_G \equiv 0$ if G is disconnected), but it is convenient for our purposes and is natural from a matroidal point of view (see below). In order to avoid any possible misunderstanding, we have inserted in Theorems 1.1 and 1.1′ and Corollary 1.8 the word "connected", so that the claims made in the introduction will be true on either interpretation of T_G. Please note that, in our definition, T_G is always strictly positive on $(0,\infty)^E$, because the set of maximal spanning forests is non-empty. Note also that, on either definition of T_G, loops in G (if any) play no role in T_G. And it goes without saying that T_G is a *multiaffine* polynomial, i.e. of degree at most 1 in each x_e separately. If e is an edge of G, the spanning-tree polynomial of G can be related to that of the deletion $G\backslash e$ and the contraction G/e:

$$T_G(\mathbf{x}) = \begin{cases} T_{G\backslash e}(\mathbf{x}_{\neq e}) + x_e T_{G/e}(\mathbf{x}_{\neq e}), & \text{if } e \text{ is neither a bridge nor a loop,} \\ x_e T_{G\backslash e}(\mathbf{x}_{\neq e}) = x_e T_{G/e}(\mathbf{x}_{\neq e}), & \text{if } e \text{ is a bridge,} \\ T_{G\backslash e}(\mathbf{x}_{\neq e}) = T_{G/e}(\mathbf{x}_{\neq e}), & \text{if } e \text{ is a loop,} \end{cases} \tag{7.2}$$

where $\mathbf{x}_{\neq e}$ denotes $\{x_f\}_{f \in E\backslash\{e\}}$. The fact that $T_{G\backslash e}$ equals $T_{G/e}$ (rather than equalling zero) when e is a bridge is a consequence of our peculiar definition of T_G.

Now let us take an analytic point of view, so that the indeterminates x_e will be interpreted as real or complex variables.

Definition 7.1. For each $\beta > 0$, we denote by $\mathcal{G}_\beta$ the class of graphs G for which $T_G^{-\beta}$ is completely monotone on $(0,\infty)^E$.

The naturality of the classes $\mathcal{G}_\beta$ is illustrated by the following easy but fundamental result.

PROPOSITION 7.2. *Each class $\mathcal{G}_\beta$ is closed under taking minors (i.e. under deletion and contraction of edges and deletion of isolated vertices), under disjoint unions, and under gluing at a cut vertex.*

Proof. Deletion of a non-bridge edge e corresponds to taking $x_e \downarrow 0$. Contraction of a non-loop edge e corresponds to dividing by x_e and taking $x_e \uparrow \infty$. Both of these operations preserve complete monotonicity. Deletion of a bridge has the same effect as contracting it, in our peculiar definition of T_G. Contraction of a loop is equivalent to deleting it (but loops play no role in T_G anyway). Isolated vertices play no role in T_G. This proves closure under taking minors.

If G is obtained from G_1 and G_2 either by disjoint union or by gluing at a cut vertex, then $T_G = T_{G_1} T_{G_2}$ (on disjoint sets of variables) in our definition of T_G; this again preserves complete monotonicity. □

Proposition 7.2 illustrates the principal reason for allowing arbitrary constants $\mathbf{c} > \mathbf{0}$ (rather than just $\mathbf{c} = \mathbf{1}$) in Theorem 1.1 and subsequent results: it leads to a minor-closed class of graphs. This, in turn, allows for characterizations that are necessary as well as sufficient. A similar situation arises in studying the negative-correlation property for a randomly chosen basis of a matroid. If only the "uniformly-at-random" situation is considered (i.e., element weights $\mathbf{x} = \mathbf{1}$), then the resulting class of matroids is not minor-closed, and closure under minors has to be added by hand, leading to the class of so-called *balanced matroids* [53]. But it then turns out that the class of balanced matroids is *not* closed under taking 2-sums [38]. If, by contrast, one demands negative correlation for *all* choices of element weights $\mathbf{x} > \mathbf{0}$, then the resulting class—the so-called *Rayleigh matroids*—is automatically closed under taking minors (by the same $x_e \to 0$ and $x_e \to \infty$ argument as in Proposition 7.2). Moreover, it turns out to be closed under 2-sums as well [38].

The very important property of closure under 2-sums holds also in our context. To see this, let us first recall the definitions of parallel connection, series connection and 2-sum of graphs [90, §7.1], and work out how T_G transforms under these operations.

For $i = 1, 2$, let $G_i = (V_i, E_i)$ be a graph and let e_i be an edge of G_i; it is convenient (though not absolutely necessary) to assume that e_i is neither a loop nor a bridge in G_i. Let us furthermore choose an orientation $\vec{e}_i = \overrightarrow{x_i y_i}$ for the edge e_i. (To avoid notational ambiguity, it helps to assume that the sets V_1, V_2, E_1 and E_2 are all disjoint.) Then the *parallel connection* of $(G_1, \vec{e}_1)$ with $(G_2, \vec{e}_2)$ is the graph $(G_1, \vec{e}_1) \| (G_2, \vec{e}_2)$ obtained from the disjoint union $G_1 \cup G_2$ by identifying x_1 with x_2, y_1 with y_2, and e_1 with e_2. (Equivalently, it is obtained from the disjoint union $(G_1 \setminus e_1) \cup (G_2 \setminus e_2)$ by identifying x_1 with x_2 (call the new vertex x), y_1 with y_2 (call the new vertex y), and then adding a new edge e from x to y.) The *series connection* of $(G_1, \vec{e}_1)$ with $(G_2, \vec{e}_2)$ is the graph $(G_1, \vec{e}_1) \bowtie (G_2, \vec{e}_2)$ obtained from the disjoint union $(G_1 \setminus e_1) \cup (G_2 \setminus e_2)$ by identifying x_1 with x_2 and adding a new edge e from y_1 to y_2. The 2-*sum* of $(G_1, \vec{e}_1)$ with $(G_2, \vec{e}_2)$ is the graph $(G_1, \vec{e}_1) \oplus_2 (G_2, \vec{e}_2)$ obtained from the parallel connection $(G_1, \vec{e}_1) \| (G_2, \vec{e}_2)$ by deleting the edge e that arose from identifying e_1 with e_2, or equivalently from the series connection $(G_1, \vec{e}_1) \bowtie (G_2, \vec{e}_2)$ by contracting the edge e.

To calculate the spanning-tree polynomial of a parallel connection, series connection or 2-sum, it is convenient to change slightly the notation and suppose that e_1 and e_2 have already been identified (let us call this common edge e), so that $E_1 \cap E_2 = \{e\}$. It is then not difficult to see [90, Proposition 7.1.13] that the spanning-tree polynomial of a

parallel connection $G_1 \|_e G_2$ is given by

$$T_{G_1\|_e G_2}(\mathbf{x}) = T_{G_1\setminus e}T_{G_2/e} + T_{G_1/e}T_{G_2\setminus e} + x_e T_{G_1/e}T_{G_2/e}, \tag{7.3}$$

while that of a series connection $G_1 \bowtie_e G_2$ is

$$T_{G_1\bowtie_e G_2}(\mathbf{x}) = T_{G_1\setminus e}T_{G_2\setminus e} + x_e T_{G_1\setminus e}T_{G_2/e} + x_e T_{G_1/e}T_{G_2\setminus e}. \tag{7.4}$$

(All the spanning-tree polynomials on the right-hand sides are of course evaluated at $\mathbf{x}_{\neq e}$.) The spanning-tree polynomial of a 2-sum $G_1 \oplus_{2,e} G_2$ is therefore

$$T_{G_1\oplus_{2,e} G_2}(\mathbf{x}) = T_{G_1\setminus e}T_{G_2/e} + T_{G_1/e}T_{G_2\setminus e}. \tag{7.5}$$

PROPOSITION 7.3. *Each class $\mathcal{G}_\beta$ is closed under parallel connection and under 2-sums.*

Proof. Closure under parallel connection is an immediate consequence of Proposition 3.5 and the formula (7.3) for parallel connection. Since the 2-sum is obtained from the parallel connection by deletion, closure under 2-sum then follows from Proposition 7.2. □

PROPOSITION 7.4. *The class $\mathcal{G}_\beta$ is closed under series connection for $\beta \geqslant \frac{1}{2}$ but not for $0 < \beta < \frac{1}{2}$.*

Proof. For $\beta \geqslant \frac{1}{2}$, closure under series connection is an immediate consequence of Proposition 3.6 and the formula (7.4) for series connection. For $0 < \beta < \frac{1}{2}$, non-closure under series connection follows immediately from the observation that the 2-cycle $C_2 = K_2^{(2)}$ (a pair of vertices connected by two parallel edges) lies in $\mathcal{G}_\beta$ for all $\beta > 0$, but the series connection of a 2-cycle with another 2-cycle is a 3-cycle, which lies in $\mathcal{G}_\beta$ only for $\beta \geqslant \frac{1}{2}$ (by Proposition 3.8). □

Remarks. (1) Unlike the situation for the half-plane and Rayleigh properties, the classes $\mathcal{G}_\beta$ are *not* in general closed under duality for planar graphs. For instance, the graph $C_3^* = K_2^{(3)}$ (a pair of vertices connected by three parallel edges) lies in $\mathcal{G}_\beta$ for all $\beta \geqslant 0$; but its dual C_3 lies in $\mathcal{G}_\beta$ only for $\beta \geqslant \frac{1}{2}$ (by Proposition 3.8).

However, the class $\mathcal{G}_\beta$ *is* duality-invariant for $\beta \in \left(\frac{1}{2}, 1\right)$, as it consists of all series-parallel graphs (Theorem 7.12 below). And since $\mathcal{G}_\beta = \{\text{all graphs}\}$ for $\beta \in \left\{\frac{1}{2}, 1, \frac{3}{2}, \ldots\right\}$, these classes $\mathcal{G}_\beta \cap \mathcal{P}$ (where $\mathcal{P} = \{\text{planar graphs}\}$) are also duality-invariant. We do not know whether $\mathcal{G}_\beta \cap \mathcal{P}$ is duality-invariant for $\beta \in (1, \infty) \setminus \left\{\frac{3}{2}, 2, \frac{5}{2}, \ldots\right\}$. In any case, the duality question is most naturally posed for matroids rather than for graphs.

(2) Since $\mathcal{G}_\beta$ is closed under 0-sums (disjoint unions), 1-sums (gluing at a cut vertex) and 2-sums (essentially gluing at an edge), it is natural to ask whether it is also closed

under 3-sums (gluing along triangles). We do not know the answer. In particular, $K_4 \in \mathcal{G}_\beta$ for all $\beta \geqslant 1$ by Corollary 1.5, but as noted in the discussion after Problem 1.13′, we do not know whether $K_5 - e = K_4 \oplus_3 K_4$ belongs to $\mathcal{G}_\beta$ for $\beta \in \left(1, \frac{3}{2}\right)$.

It is a well-known (and easy) result that any minor-closed class of graphs is of the form

$$\mathrm{Ex}(\mathcal{F}) = \{G : G \text{ does not contain any minor from } \mathcal{F}\} \tag{7.6}$$

for some family $\mathcal{F}$ of "excluded minors"; indeed, the minimal choice of $\mathcal{F}$ consists of those graphs that do not belong to the class in question but whose proper minors all do belong to the class. (Here we consider isomorphic graphs to be identical, or alternatively take only one representative from each isomorphism class.)

In one of the deepest and most difficult theorems of graph theory, Roberston and Seymour [100] sharpened this result by proving that any minor-closed class of graphs is of the form $\mathrm{Ex}(\mathcal{F})$ for some *finite* family $\mathcal{F}$. Therefore, each of our classes $\mathcal{G}_\beta$ can be characterized by a finite family of excluded minors. One of the goals of this paper—alas, incompletely achieved—is to determine these excluded minors.

7.2. Matroids

The foregoing considerations have an immediate generalization to matroids. (Readers unfamiliar with matroids can skip this subsection without loss of logical continuity.) Let M be a matroid with ground set E, and let $\mathbf{x} = \{x_e\}_{e \in E}$ be a family of indeterminates indexed by the elements of M. We denote by $B_M(\mathbf{x})$ the basis generating polynomial of M, namely

$$B_M(\mathbf{x}) = \sum_{B \in \mathcal{B}(M)} \prod_{e \in B} x_e, \tag{7.7}$$

where $\mathcal{B}(M)$ denotes the family of bases of M. Please note that loops in M (if any) play no role in B_M. Note also that if M is the graphic matroid $M(G)$ associated with a graph G, we have $B_{M(G)}(\mathbf{x}) = T_G(\mathbf{x})$. This identity would not hold for disconnected G if we had taken the standard definition of T_G.

Definition 7.5. For each $\beta > 0$, we denote by $\mathcal{M}_\beta$ the class of matroids M for which $B_M^{-\beta}$ is completely monotone on $(0, \infty)^E$.

Once again we have the following result.

PROPOSITION 7.6. *Each class $\mathcal{M}_\beta$ is closed under taking minors (i.e. deletion and contraction) and direct sums.*

Proof. The proof is identical to that of Proposition 7.2 if one substitutes "element" for "edge", "coloop" for "bridge", and "direct sum" for either form of union. □

We refer to [90, §7.1] for the definitions of parallel connection, series connection and 2-sum of matroids, which generalize those for graphs. The upshot [90, Proposition 7.1.13] is that the formulae (7.3)–(7.5) for the spanning-tree polynomials of graphs extend unchanged to the basis generating polynomials of matroids. We therefore have the following result.

PROPOSITION 7.7. *Each class $\mathcal{M}_\beta$ is closed under parallel connection and under 2-sums.*

PROPOSITION 7.8. *The class $\mathcal{M}_\beta$ is closed under series connection for $\beta \geqslant \frac{1}{2}$ but not for $0 < \beta < \frac{1}{2}$.*

Since each $\mathcal{M}_\beta$ is a minor-closed class, we can once again seek a characterization of $\mathcal{M}_\beta$ by excluded minors. However, in this case no analogue of the Robertson–Seymour theorem exists, so we have no *a priori* guarantee of finiteness of the set of excluded minors. Indeed, there exist minor-closed classes of matroids having an infinite family of excluded minors [90, Exercise 6.5.5 (g)]; and in fact, for any infinite field F, the class of F-representable matroids has infinitely many excluded minors [90, Theorem 6.5.17].

We suspect that the classes $\mathcal{M}_\beta$ are *not* closed under duality in general. For instance, Corollary 1.10 shows that $U_{2,5} \in \mathcal{M}_\beta$ if and only if $\beta \geqslant \frac{3}{2}$; but we suspect (Conjecture 1.11) that $U_{3,5} \in \mathcal{M}_\beta$ if and only if $\beta \geqslant 1$. On the other hand, we shall show in Theorem 7.13 that $\mathcal{M}_\beta$ for $\frac{1}{2} < \beta < 1$ consists precisely of the graphic matroids of series-parallel graphs—a class that *is* closed under duality.

7.3. Partial converse to Corollary 1.8

It was remarked at the end of §4.3 that if β does not lie in the set described in Theorem 1.3, then the map $A \mapsto (\det A)^{-\beta}$ is not completely monotone on any non-empty open convex subcone of the cone of positive-definite matrices; and in particular, if the matrices $A_1, \ldots, A_n$ span $\mathrm{Sym}(m, \mathbb{R})$ or $\mathrm{Herm}(m, \mathbb{C})$, then the determinantal polynomial (1.4)/(4.9) does not have $P^{-\beta}$ completely monotone on $(0, \infty)^n$. The spanning-tree polynomial of the complete graph K_{m+1} provides an example of this situation; and it turns out that there are two other cases arising from complex-unimodular matroids. The following result thus provides a (very) partial converse to Corollary 1.8, where part (a) concerns regular (=real-unimodular) matroids and part (b) concerns complex-unimodular matroids.

PROPOSITION 7.9. *Let M be a matroid on the ground set E, and let $B_M(\mathbf{x})$ be its basis generating polynomial.*

(a) *If $M=M(K_p)$ (the graphic matroid associated with the complete graph K_p) and $\beta\notin\{0,\frac{1}{2},1,\frac{3}{2},\dots\}\cup\big[\frac{1}{2}(p-2),\infty\big)$, then $B_M^{-\beta}$ is* not *completely monotone on any non-empty open convex subcone of $(0,\infty)^E$.*

($\mathrm{b}_{r=2}$) *If $M=U_{2,4}$ (the uniform matroid of rank 2 on four elements) and $\beta\notin\{0\}\cup[1,\infty)$, then $B_M^{-\beta}$ is* not *completely monotone on any non-empty open convex subcone of $(0,\infty)^E$.*

($\mathrm{b}_{r=3}$) *If $M=\mathrm{AG}(2,3)$ (the ternary affine plane) and $\beta\notin\{0,1\}\cup[2,\infty)$, then $B_M^{-\beta}$ is* not *completely monotone on any non-empty open convex subcone of $(0,\infty)^E$.*

Proof. When M is a real-unimodular (resp. complex-unimodular) matroid of rank r with n elements, we let B be an $r\times n$ real-unimodular (resp. complex-unimodular) matrix that represents M; we then define A_i ($1\leqslant i\leqslant n$) to be the outer product of the ith column of B with its complex conjugate. We shall show that in the cases enumerated above, we can choose B so that the matrices $A_1,\dots,A_n$ span $\mathrm{Sym}(r,\mathbb{R})$ (resp. $\mathrm{Herm}(r,\mathbb{C})$). The result then follows from Theorem 1.3 and the observation made immediately after it.

(a) Let H_p be the directed vertex-edge incidence matrix for K_p with an arbitrarily chosen orientation of the edges; it is of size $p\times\binom{p}{2}$ and is real-unimodular. Then K_p is represented over $\mathbb{R}$ by the matrix H'_p obtained from H_p by deleting one of the rows. But we can reorder the columns of H'_p to get $H''_p=(I_{p-1}|H_{p-1})$, where I_{p-1} is the $(p-1)\times(p-1)$ identity matrix (and H_{p-1} is defined using the orientation of K_{p-1} inherited from K_p). The corresponding matrices $A_1,\dots,A_{\binom{p}{2}}$, obtained by taking outer products of the columns of H''_p with their complex conjugates, are easily seen to span $\mathrm{Sym}(p-1,\mathbb{R})$.

($\mathrm{b}_{r=2}$) The matrices $A_1,\dots,A_4$ defined in (1.7) are easily seen to span $\mathrm{Herm}(2,\mathbb{C})$.

($\mathrm{b}_{r=3}$) Write $\omega=e^{\pm 2\pi i/3}$; then the matrix [123, p. 597]

$$B=\begin{pmatrix}1&0&0&1&0&1&1&1&1\\0&1&0&1&1&0&1+\omega&1&1+\omega\\0&0&1&0&1&\omega&\omega&1+\omega&1+\omega\end{pmatrix}\tag{7.8}$$

is easily seen to be complex-unimodular and to represent $\mathrm{AG}(2,3)$. A tedious computation (or an easy one using MATHEMATICA or MAPLE) now shows that the matrices $A_1,\dots,A_9$ are linearly independent, hence span the 9-dimensional space $\mathrm{Herm}(3,\mathbb{C})$. □

Let us remark that the cases enumerated in Proposition 7.9 *exhaust* the list of regular or complex-unimodular matroids (of rank $r\geqslant 2$) for which the matrices $A_1,\dots,A_n$ span $\mathrm{Sym}(r,\mathbb{R})$ or $\mathrm{Herm}(r,\mathbb{C})$, respectively. Indeed, it is known that a simple rank-r matroid that is *regular* (or, more generally, is binary with no F_7 minor) can have at most

$\frac{1}{2}r(r+1)$ elements; furthermore, the unique matroid attaining this bound is $M(K_{r+1})$ [90, Proposition 14.10.3]. See also [12] for an intriguing proof that uses the matrices $A_1, ..., A_n$ (but over GF(2) rather than $\mathbb{C}$). Likewise, it is known [91, Theorem 2.1] that a simple rank-r matroid that is *complex-unimodular* can have at most $\frac{1}{2}(r^2+3r-2)$ elements if $r \neq 3$, or 9 elements if $r=3$; furthermore, the unique matroid attaining this bound is T_r (defined in [91]) when $r \neq 3$, or AG(2, 3) when $r=3$. The only cases in which this size reaches $\dim \mathrm{Herm}(r, \mathbb{C}) = r^2$ are thus $r=1, 2, 3$, yielding $T_1 = U_{1,1}$, $T_2 = U_{2,4}$ and AG(2, 3), respectively.

7.4. Series-parallel graphs (and matroids): Proof of Theorem 1.1′

Before proving Theorem 1.1′, let us prove a similar but simpler theorem concerning the interval $0 < \beta < \frac{1}{2}$.

THEOREM 7.10. *Let G be a graph, and let $\beta \in \left(0, \frac{1}{2}\right)$. Then the following are equivalent*:

(a) $G \in \mathcal{G}_\beta$;

(b) *G can be obtained from a forest by parallel extensions of edges (i.e., replacing an edge by several parallel edges) and additions of loops*;

(c) *G has no K_3 minor.*

Moreover, these equivalent conditions imply that $G \in \mathcal{G}_{\beta'}$ for all $\beta' > 0$.

Proof. The equivalence of (b) and (c) is an easy graph-theoretic exercise.

If G is obtained from a forest by parallel extensions of edges and additions of loops, then $T_G(\mathbf{x})$ is a product of factors of the form $x_{e_1} + ... + x_{e_k}$ (where $e_1, ..., e_k$ are a set of parallel edges in G), so that $T_G^{-\beta}$ is completely monotone on $(0, \infty)^E$ for all $\beta \geqslant 0$. Therefore (b) $\Rightarrow$ (a).

Conversely, Proposition 3.8 tells us that $K_3 \notin \mathcal{G}_\beta$ for $\beta \in \left(0, \frac{1}{2}\right)$. Since $\mathcal{G}_\beta$ is a minor-closed class, this proves that (a) $\Rightarrow$ (c). □

Dave Wagner has pointed out to us that Theorem 7.10 extends easily to matroids, as we state in the following theorem.

THEOREM 7.11. *Let M be a matroid, and let $\beta \in \left(0, \frac{1}{2}\right)$. Then the following are equivalent*:

(a) $M \in \mathcal{M}_\beta$;

(b) *M is the graphic matroid $M(G)$ for a graph G that can be obtained from a forest by parallel extensions of edges and additions of loops*;

(c) *G has no $M(K_3)$ or $U_{2,4}$ minor.*

Moreover, these equivalent conditions imply that $M \in \mathcal{M}_{\beta'}$ for all $\beta' > 0$.

Proof. Tutte has proven [90, Theorem 10.3.1] that a matroid is graphic if and only if it has no minor isomorphic to $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$ or $M^*(K_{3,3})$. Since $M(K_3)$ is a minor of the last four matroids on this list, the equivalence of (b) and (c) follows from the graphic case.

(b) $\Rightarrow$ (a) has already been proven in Theorem 7.10.

Finally, Proposition 3.8 tells us that $M(K_3) \notin \mathcal{M}_\beta$ for $\beta \in \left(0, \frac{1}{2}\right)$, and Corollary 1.6 (or Corollary 1.10) tells us that $U_{2,4} \notin \mathcal{M}_\beta$ for $\beta \in (0,1)$. Since $\mathcal{M}_\beta$ is a minor-closed class, this proves that (a) $\Rightarrow$ (c). □

Let us now prove the corresponding characterization for $\frac{1}{2} < \beta < 1$, which is a rephrasing of Theorem 1.1′ and concerns series-parallel graphs. Unfortunately, there seems to be no completely standard definition of "series-parallel graph"; a plethora of slightly different definitions can be found in the literature [46], [88], [40], [27], [90]. So let us be completely precise about our own usage: we shall call a loopless graph *series-parallel* if it can be obtained from a forest by a finite sequence of series and parallel extensions of edges (i.e. replacing an edge by two edges in series or two edges in parallel). We shall call a general graph (allowing loops) series-parallel if its underlying loopless graph is series-parallel.([36])

So we need to understand how the spanning-tree polynomial $T_G(\mathbf{x})$ behaves under series and parallel extensions of edges. Parallel extension is easy: if G' is obtained from G by replacing the edge e by a pair of edges e_1 and e_2 in parallel, then

$$T_{G'}(\mathbf{x}_{\neq e}, x_{e_1}, x_{e_2}) = T_G(\mathbf{x}_{\neq e}, x_{e_1} + x_{e_2}). \tag{7.9}$$

In other words, two parallel edges with weights x_{e_1} and x_{e_2} are equivalent to a single edge with weight $x_{e_1} + x_{e_2}$. This is because the spanning trees of G' are in correspondence with the spanning trees T of G as follows: if T does not contain e, then leave T as is (it is a spanning tree of G'); if T does contain e, then adjoin to $T \setminus e$ one but not both of the edges e_1 and e_2.

Series extension is only slightly more complicated: if G' is obtained from G by replacing the edge e by a pair of edges e_1 and e_2 in series, then

$$T_{G'}(\mathbf{x}_{\neq e}, x_{e_1}, x_{e_2}) = (x_{e_1} + x_{e_2})\, T_G\left(\mathbf{x}_{\neq e}, \frac{x_{e_1} x_{e_2}}{x_{e_1} + x_{e_2}}\right). \tag{7.10}$$

In other words, two series edges with weights x_{e_1} and x_{e_2} are equivalent to a single edge with weight $x_{e_1} x_{e_2} / (x_{e_1} + x_{e_2})$ together with a prefactor that clears the resulting

([36]) Some authors write "obtained from a tree", "obtained from K_2" or "obtained from C_2" in place of "obtained from a forest"; in our terminology these definitions yield, respectively, all *connected* series-parallel graphs, all connected series-parallel graphs whose blocks form a path, or all 2-*connected* series-parallel graphs. See [27, §11.2] for a more extensive bibliography.

denominator. This is because the spanning trees of G' are in correspondence with the spanning trees T of G as follows: if T does not contain e, then adjoin to $T\backslash e$ one but not both of the edges e_1 and e_2; if T does contain e, then adjoin to $T\backslash e$ both of the edges e_1 and e_2. Since $T_G(\mathbf{x}) = T_{G\backslash e}(\mathbf{x}_{\neq e}) + x_e T_{G/e}(\mathbf{x}_{\neq e})$, where $T_{G\backslash e}$ (resp. $T_{G/e}$) counts the spanning trees of G that do not (resp. do) contain e (the latter without the factor x_e), this proves (7.10).

Let us remark that the parallel and series laws for $T_G(\mathbf{x})$ are precisely the laws for combining electrical conductances in parallel or series. This is no accident, because as Kirchhoff [74] showed a century-and-a-half ago, the theory of linear electrical circuits can be written in terms of spanning-tree polynomials (see e.g. [35] for a modern treatment). Let us also remark that the parallel and series laws for $T_G(\mathbf{x})$ are limiting cases of the parallel and series laws for the multivariate Tutte polynomial $Z_G(q, \mathbf{v})$, obtained when $q \to 0$ and $\mathbf{v}$ is infinitesimal; see [109, §§4.4–4.7] for a detailed explanation.

We are now ready to state and prove the main result of this section.

THEOREM 7.12. *Let G be a graph, and let $\beta \in \left(\frac{1}{2}, 1\right)$. Then the following are equivalent*:

(a) $G \in \mathcal{G}_\beta$;

(b) *G is series-parallel*;

(c) *G has no K_4 minor.*

Moreover, these equivalent conditions imply that $G \in \mathcal{G}_{\beta'}$ for all $\beta' \geqslant \frac{1}{2}$.

Please note that Theorems 7.10 and 7.12 together imply Theorem 1.1′.

Proof. The equivalence of statements (b) and (c) is a well-known graph-theoretic result [44, Exercise 7.30 and Proposition 1.7.4] (see also [46] and [88]).

Now let G be a series-parallel graph: this means that G can be obtained from a forest by series and parallel extensions of edges and additions of loops. As shown in Theorem 7.10, if G is a forest, then $T_G^{-\beta}$ is completely monotone for all $\beta \geqslant 0$. Parallel extension (7.9) obviously preserves complete monotonicity. By Lemma 3.9, series extension (7.10) preserves complete monotonicity whenever $\beta \geqslant \frac{1}{2}$. Finally, additions of loops do not affect T_G. Therefore, every series-parallel graph G belongs to the class $\mathcal{G}_\beta$ for all $\beta \geqslant \frac{1}{2}$.

Conversely, Proposition 7.9 (a) tells us that $K_4 \notin \mathcal{G}_\beta$ for $\beta \in \left(\frac{1}{2}, 1\right)$. Since $\mathcal{G}_\beta$ is a minor-closed class, this proves (a) $\Rightarrow$ (c). □

Remarks. (1) Instead of using series and parallel extension of edges (equations (7.9)/(7.10) and Lemma 3.9), we could equally well have written this proof in terms of the more general concept of series and parallel connection of graphs (equations (7.3)/(7.4) and Propositions 3.5 and 3.6).

(2) It ought to be possible to give an "elementary" proof of the fact that $K_4 \notin \mathcal{G}_\beta$ for $\beta \in (\frac{1}{2}, 1)$—and more generally of the fact that the derivatives of $T_{K_4}^{-\beta}$ at $\mathbf{c}=\mathbf{1}$ do not all have sign $(-1)^k$—by asymptotic calculation of coefficients à la Pemantle–Wilson–Baryshnikov [93], [94], [95], [92], [15].

Once again, Dave Wagner has pointed out to us that Theorem 7.12 extends immediately to matroids, as we state in the following theorem.

THEOREM 7.13. *Let M be a matroid, and let $\beta \in (\frac{1}{2}, 1)$. Then the following are equivalent*:

(a) $M \in \mathcal{M}_\beta$;

(b) *M is the graphic matroid $M(G)$ (or equivalently the cographic matroid $M^*(G)$) of a series-parallel graph G*;

(c) *G has no $M(K_4)$ or $U_{2,4}$ minor.*

Moreover, these equivalent conditions imply that $M \in \mathcal{M}_{\beta'}$ for all $\beta' \geqslant \frac{1}{2}$.

The proof is completely analogous to that of Theorem 7.11 (see also [90, Corollary 12.2.14] for an alternative proof of (b) $\Leftrightarrow$ (c) in this case).

Remark. Corollary 1.8 (a) implies that $\mathcal{G}_0 = \mathcal{G}_{1/2} = \mathcal{G}_1 = \mathcal{G}_{3/2} = \ldots = \{\text{all graphs}\}$. But what is the story for matroids? Does $\mathcal{M}_{1/2}$ contain *only* regular matroids? Does $\mathcal{M}_1$ contain *only* complex-unimodular matroids? We suspect that the answer to this last question is *no*, since we suspect that $U_{n-2,n} \in \mathcal{M}_1$ for all $n \geqslant 2$ (Conjecture 1.11).

7.5. Combining the determinantal method with constructions

Let us now combine the ab-initio results from the determinantal method (Theorems 1.2–1.4 and their corollaries) with the constructions from §3, §7.1 and §7.2 (deletion, contraction, direct sum, parallel connection and series connection). For graphs we have the following proof.

Proof of Proposition 1.12. For $p=2$ the result is trivial, as the graphs concerned are precisely those that can be obtained from a forest by parallel extensions of edges and additions of loops (Theorem 7.12). For $p \geqslant 3$ (and hence $\beta \geqslant \frac{1}{2}$), the result is an immediate consequence of Corollary 1.5 and Propositions 7.2–7.4. □

The matroid generalization of Proposition 1.12 is the following result.

PROPOSITION 7.14. *Fix $r \geqslant 1$, and let M be any matroid (on the ground set E) that can be obtained from regular matroids of rank at most r by parallel connection, series connection, direct sum, deletion and contraction. Then $B_M^{-\beta}$ is completely monotone on $(0,\infty)^E$ for $\beta = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and for all real $\beta \geqslant \frac{1}{2}(r-1)$.*

The proof is essentially identical to the previous one, but uses Corollary 1.8 (a) in place of Corollary 1.5 and Propositions 7.6–7.8 in place of Propositions 7.2–7.4.

And we also have the following result.

PROPOSITION 7.15. *Fix $r \geqslant 1$, and let M be any matroid (on the ground set E) that can be obtained from regular matroids of rank at most $2r-1$ and complex-unimodular matroids of rank at most r by parallel connection, series connection, direct sum, deletion and contraction. Then $B_M^{-\beta}$ is completely monotone on $(0,\infty)^E$ for $\beta = 0, 1, 2, 3, \ldots$ and for all real $\beta \geqslant r-1$.*

The proof is again identical, but uses both parts of Corollary 1.8 instead of only part (a).

Propositions 1.12, 7.14 and 7.15 give a rather abstract characterization of the class of graphs or matroids that they handle, and so it is of interest to give a more explicit characterization. Let us start with the graph case. We say that a graph G is *minimally 3-connected* if G is 3-connected but, for all edges e of G, the graph $G \backslash e$ is not 3-connected. We then have the following result.

PROPOSITION 7.16. *Let $\mathcal{G}_p$ be the class of graphs obtained from K_p by disjoint union, gluing at a cut vertex, series and parallel connection, deletion and contraction of edges, and deletion of vertices. Then, for $p \geqslant 3$, $\mathcal{G}_p$ is minor-closed, and the minimal excluded minors for $\mathcal{G}_p$ are the minimally 3-connected graphs on $p+1$ vertices.*

Proof. Let $\mathcal{H}_p$ be the class of graphs with no minimally 3-connected minor on $p+1$ vertices. It is clear that $\mathcal{G}_p$ is minor-closed, so our aim is prove that $\mathcal{G}_p = \mathcal{H}_p$.

We first show that $\mathcal{G}_p \subseteq \mathcal{H}_p$. It is clear that $\mathcal{H}_p$ is minor-closed, and is closed under disjoint union ("0-sum") and gluing at a cut vertex ("1-sum"). It is easily checked that if a graph G is obtained by parallel connection of graphs G_1 and G_2, then any 3-connected minor of G is a minor of either G_1 or G_2; it follows that $\mathcal{H}_p$ is closed under parallel connection. Since series connection can be obtained by combining the other operations (exploiting $K_3 \in \mathcal{H}_p$), we conclude that $\mathcal{H}_p$ is also closed under series connection. Finally, we note that $K_p \in \mathcal{H}_p$, and as $\mathcal{G}_p$ is the closure of $\{K_p\}$ under these operations, we see that $\mathcal{G}_p \subseteq \mathcal{H}_p$.

We now show that $\mathcal{H}_p \subseteq \mathcal{G}_p$. For suppose otherwise, and choose $G \in \mathcal{H}_p \backslash \mathcal{G}_p$ with the minimal number of vertices. Clearly G is 2-connected and has at least $p+1$ vertices. If G is not 3-connected, then G has a cutset $\{x, y\}$ and there is a decomposition $G = G_1 \cup G_2$, where G_1 and G_2 are connected graphs with at least three vertices and $V(G_1) \cap V(G_2) = \{x, y\}$. Now let G_1' and G_2' be the graphs obtained from G_1 and G_2, respectively, by adding the edge xy if not present. Then G_1' and G_2' are both minors of G (obtained by contracting the other side to a single edge). As $\mathcal{H}_p$ is minor-closed, we have $G_1', G_2' \in \mathcal{H}_p$.

Therefore, by minimality of G, we have $G'_1, G'_2 \in \mathcal{G}_p$. But G can be obtained by taking a parallel connection of G'_1 and G'_2 along xy and then deleting the edge xy if necessary, yielding $G \in \mathcal{G}_p$, contrary to hypothesis. We conclude that G must be 3-connected.

We now use the fact that every 3-connected graph other than K_4 has an edge that can be contracted to produce another 3-connected graph [44, Lemma 3.2.4]. Contracting suitable edges, we see that G has a 3-connected minor on $p+1$ vertices, which in turn (deleting edges if necessary) contains a minimally 3-connected minor on $p+1$ vertices. But this contradicts $G \in \mathcal{H}_p$. □

The matroid case is analogous. We refer to [90, Chapter 8] for the definitions of 3-connectedness and minimal 3-connectedness for matroids. The following result and its proof are due to Oxley [89].

PROPOSITION 7.17. *Let $\mathcal{F}$ be a class of matroids that is closed under minors, direct sums and 2-sums. Let $r \geqslant 1$, let $\mathcal{F}_r = \{M \in \mathcal{F} : \mathrm{rank}(M) \leqslant r\}$, and let $\mathcal{F}'_r$ denote the class of matroids that can be obtained from $\mathcal{F}_r$ by direct sums and 2-sums. Then $\mathcal{F}'_r$ is closed under minors (and of course also under direct sums and 2-sums), and it consists of all matroids in $\mathcal{F}$ that have no $\mathcal{F}^\circ_{r+1}$ minor, where*

$$\mathcal{F}^\circ_{r+1} = \{M \in \mathcal{F} : M \text{ is minimally 3-connected and } \mathrm{rank}(M) = r+1\}. \tag{7.11}$$

Proof. Consider first the case $r=1$. The unique minimally 3-connected matroid of rank 2 is $U_{2,3}$. If $U_{2,3} \in \mathcal{F}$, then the result clearly holds; if $U_{2,3} \notin \mathcal{F}$, then $\mathcal{F}'_1 = \mathcal{F}$ and the result again holds.

Now assume $r \geqslant 2$, and let $\mathcal{H}_r$ denote the class of matroids in $\mathcal{F}$ that have no $\mathcal{F}^\circ_{r+1}$ minor. Clearly $\mathcal{H}_r$ is minor-closed, and our goal is to show that $\mathcal{F}'_r = \mathcal{H}_r$.

We first show that $\mathcal{F}'_r \subseteq \mathcal{H}_r$. It is easy to see [90, Propositions 4.2.20 and 8.3.5] that if a matroid M is either a direct sum or a 2-sum of matroids M_1 and M_2, then any 3-connected minor of M is isomorphic to a minor of either M_1 or M_2; it follows that $\mathcal{H}_r$ is closed under direct sum and 2-sum. Since $\mathcal{F}_r \subseteq \mathcal{H}_r$, and $\mathcal{F}'_r$ is the closure of $\mathcal{F}_r$ under direct sum and 2-sum, we see that $\mathcal{F}'_r \subseteq \mathcal{H}_r$.

We now show that $\mathcal{H}_r \subseteq \mathcal{F}'_r$. For suppose otherwise, and choose $M \in \mathcal{H}_r \setminus \mathcal{F}'_r$ with the minimal number of elements. It is not hard to see that M must be 3-connected.([37]) Note also that $\mathrm{rank}(M) > r$ since $M \in \mathcal{F} \setminus \mathcal{F}'_r$.

We now use Tutte's wheels and whirls theorem [90, Theorem 8.8.4], which says that every 3-connected matroid N that is not a wheel or a whirl has an element e such that

([37]) If M were a direct sum or a 2-sum of matroids M_1 and M_2, each having at least one element, then M_1 and M_2 would be minors of M [90, Proposition 7.1.21], and hence $M_1, M_2 \in \mathcal{H}_r$ because $\mathcal{H}_r$ is minor-closed. But M_1 and M_2 cannot both belong to $\mathcal{F}'_r$ because $\mathcal{F}'_r$ is closed under direct sum and 2-sum and $M \notin \mathcal{F}'_r$. Therefore M_1 or M_2 would be a counterexample to the minimality of M.

either $N\backslash e$ or N/e is 3-connected (or both). On the other hand, if N is a wheel or a whirl, then by contracting a rim element and deleting one of the spokes adjacent to that rim element, we obtain a 3-connected minor N' of N such that $\mathrm{rank}(N')=\mathrm{rank}(N)-1$.

So we apply this argument repeatedly to M until we obtain a 3-connected minor M' of M with $\mathrm{rank}(M')=r+1$. We then delete elements from M' while maintaining 3-connectedness until we arrive at a minimally 3-connected matroid M''. Therefore M has a minor $M''\in\mathcal{F}^{\circ}_{r+1}$, contradicting the hypothesis that $M\in\mathcal{H}_r$. □

Proposition 7.17 with $\mathcal{F}=\{\text{regular matroids}\}$ gives an excluded-minor characterization of the class of matroids handled by Proposition 7.14. We leave it as a problem for readers more expert in matroid theory than ourselves to provide an analogous characterization for Proposition 7.15.

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