

# Skinning maps are finite-to-one

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## 1. Introduction

Skinning maps were introduced by William Thurston in the proof of the geometrization theorem for Haken 3-manifolds (see [Ot]). At a key step in the proof one has a compact 3-manifold  $M$  with non-empty boundary whose interior admits a hyperbolic structure. The interplay between deformations of the hyperbolic structure and the topology of  $M$  and  $\partial M$  determines a holomorphic map of Teichmüller spaces, the *skinning map*

$$\sigma_M: \mathcal{T}(\partial_0 M) \longrightarrow \mathcal{T}(\overline{\partial_0 M}),$$

where  $\partial_0 M$  is the union of the non-torus boundary components and  $\overline{\partial_0 M}$  denotes the boundary with the opposite orientation. The problem of finding a hyperbolic structure on a related closed manifold is solved by showing that the composition of  $\sigma_M$  with a certain isometry  $\tau: \mathcal{T}(\overline{\partial_0 M}) \rightarrow \mathcal{T}(\partial_0 M)$  has a fixed point.

Thurston's original approach to the fixed point problem involved extending the skinning map of an acylindrical manifold to a continuous map  $\hat{\sigma}_M: \text{AH}(M) \rightarrow \mathcal{T}(\overline{\partial_0 M})$  defined on the compact space  $\text{AH}(M)$  of hyperbolic structures on  $M$ . (A proof of this bounded image theorem can be found in [Ke, §9].) McMullen provided an alternative approach based on an analytic study of the differential of the skinning map [M1], [M2]. In each case there are additional complications when  $M$  has essential cylinders.

More recently, Kent studied the diameter of the image of the skinning map (in cases when it is finite), producing examples where this diameter is arbitrarily large or small and relating the diameter to hyperbolic volume and the depth of an embedded collar around the boundary [Ke]. However, beyond the contraction and boundedness properties used to solve the fixed point problem—and the result of Kent and the author that skinning maps are never constant [DK2]—little is known about skinning maps in general.

Our main theorem concerns the fibers of skinning maps.

**THEOREM A.** *Skinning maps are finite-to-one. That is, let  $M$  be a compact oriented 3-manifold whose boundary is non-empty and not a union of tori. Suppose that the interior of  $M$  admits a complete hyperbolic structure without accidental parabolics, so that  $M$  has an associated skinning map  $\sigma_M$ . Then, for each  $X \in \mathcal{J}(\overline{\partial_0 M})$ , the preimage  $\sigma_M^{-1}(X)$  is finite.*

As a holomorphic map with finite fibers, it follows from Theorem A that skinning maps are open (answering a question in [DK2]) and locally biholomorphic away from the analytic hypersurface defined by the vanishing of the Jacobian determinant. Thus our results give strong non-degeneracy properties for all skinning maps.

Our proof of Theorem A does not bound the size of the finite set  $\sigma_M^{-1}(X)$ ; instead, we show that each fiber of the skinning map is both compact and discrete. In particular, it is not clear if the number of preimages of a point is uniformly bounded over  $\mathcal{J}(\overline{\partial_0 M})$ .

In case  $M$  is acylindrical, we also show that Thurston's extension of the skinning map to  $\text{AH}(M)$  is finite-to-one; this result appears as Theorem 10.1 below.

### The intersection problem

To study the fibers of the skinning map we translate the problem to one of intersections of certain subvarieties of the  $\text{SL}_2\mathbb{C}$ -character variety of  $\partial_0 M$ . The same reduction to an intersection problem is used in [DK2]. The relevant subvarieties are the following:

- the *extension variety*  $\mathcal{E}_M$ , which is the smallest closed algebraic subvariety containing the characters of all homomorphisms  $\pi_1(\partial_0 M) \rightarrow \text{SL}_2\mathbb{C}$  that can be extended to  $\pi_1 M \rightarrow \text{SL}_2\mathbb{C}$ ;
- the *holonomy variety*  $\mathcal{H}_X$ , which is the analytic subvariety consisting of characters of the  $\text{SL}_2\mathbb{C}$ -lifts of holonomy representations of  $\mathbb{CP}^1$ -structures on  $\overline{\partial_0 M}$  compatible with the complex structure  $X \in \mathcal{J}(\overline{\partial_0 M})$ .

Precise definitions of these objects are provided in §6, with additional details of the disconnected boundary case in §9.

The main theorem is derived from the following result about the intersections of holonomy and extension varieties, the proof of which occupies most of the paper.

**THEOREM B.** (Intersection theorem) *Let  $M$  be an oriented 3-manifold with non-empty boundary that is not a union of tori. Let  $X$  be a marked Riemann surface structure on  $\partial_0 M$ . Then the intersection  $\mathcal{H}_X \cap \mathcal{E}_M$  is a discrete subset of the character variety.*

This theorem applies in a more general setting than the specific intersection problem arising from skinning maps. For example, while skinning maps are defined for manifolds

with incompressible boundary, such a hypothesis is not needed in Theorem B.

While the theorem above involves an oriented manifold  $M$ , the set  $\mathcal{E}_M$  is independent of the orientation. Thus we also obtain discreteness of intersections  $\mathcal{H}_{\bar{X}} \cap \mathcal{E}_M$ , where  $\bar{X}$  is a Riemann surface structure on  $\partial_0 M$  that induces an orientation opposite that of the boundary orientation of  $M$ .

### Steps to the intersection theorem

Our study of  $\mathcal{H}_X \cap \mathcal{E}_M$  is based on the parametrization of the irreducible components of  $\mathcal{H}_X$  by the vector space  $Q(X)$  of holomorphic quadratic differentials; this parametrization is the *holonomy map* of  $\mathbb{C}\mathbb{P}^1$ -structures, denoted by “hol”. The overall strategy is to show that the preimage of  $\mathcal{E}_M$ , i.e. the set

$$\mathcal{V}_M = \text{hol}^{-1}(\mathcal{E}_M),$$

is a complex-analytic subvariety of  $Q(X)$  that is subject to certain constraints on its behavior at infinity, and ultimately to show that only a discrete set can satisfy these.

We now sketch the main steps of the argument and state some intermediate results of independent interest. In this sketch we restrict attention to the case of a 3-manifold  $M$  with *connected* boundary  $S$ . Let  $X \in \mathcal{T}(S)$  be a marked Riemann surface structure on the boundary.

*Step 1. Construction of an isotropic cone in the space of measured foliations.*

The defining property of this cone in  $\mathcal{MF}(S)$  is that it determines which quadratic differentials  $\phi \in Q(X)$  have dual trees  $T_\phi$  that admit “nice” equivariant maps into trees on which  $\pi_1 M$  acts by isometries. Because of the way this cone is used in a later step of the argument, here we must consider actions of  $\pi_1 M$  not only on  $\mathbb{R}$ -trees but also on  $\Lambda$ -trees, where  $\Lambda$  is a more general ordered abelian group. And while an isometric embedding of  $T_\phi$  into an  $\mathbb{R}$ -tree on which  $\pi_1 M$  acts is the prototypical example of a nice map, we must also consider the *straight maps* defined in [D] and certain partially defined maps arising from non-isometric trees with the same length function.

In the case of straight maps, we show the following result.

**THEOREM C.** *There is an isotropic piecewise linear cone  $\mathcal{L}_{M,X} \subset \mathcal{MF}(S)$  with the following property: If  $\pi_1 M$  acts on a  $\Lambda$ -tree  $T$  by isometries, and if  $\phi \in Q(X)$  is a quadratic differential whose dual tree  $T_\phi$  admits an equivariant straight map  $T_\phi \rightarrow T$ , then the horizontal foliation of  $\phi$  lies in  $\mathcal{L}_{M,X}$ .*

This result is stated precisely in Theorem 4.1 below, and Theorem 4.4 presents a further refinement that is used in the proof of the main theorem.

*Step 2. Limit points of  $\mathcal{V}_M$  have foliations in the isotropic cone.*

In [D] we analyzed the large-scale behavior of the holonomy map, showing that straight maps arise naturally when comparing limits in the Morgan–Shalen compactification—which are represented by actions of  $\pi_1 S$  on  $\mathbb{R}$ -trees—to the dual trees of limit quadratic differentials. The main result can be summarized as follows.

**THEOREM.** *If a divergent sequence in  $Q(X)$  can be rescaled to have limit  $\phi$ , then any Morgan–Shalen limit of the associated holonomy representations corresponds to an  $\mathbb{R}$ -tree  $T$  that admits an equivariant straight map  $T_\phi \rightarrow T$ .*

Here rescaling of the divergent sequence uses the action of  $\mathbb{R}^+$  on  $Q(X)$ . The precise limit result we use is stated in Theorem 6.4, and other related results and discussion can be found in [D].

When we restrict attention to the subset  $\mathcal{V}_M \subset Q(X)$ , the associated holonomy representations lie in  $\mathcal{E}_M$  and so they arise as compositions of  $\pi_1 M$ -representations with the map  $i_*: \pi_1 S \rightarrow \pi_1 M$  induced by the inclusion of  $S$  as the boundary of  $M$ . (Strictly speaking, this describes a Zariski open subset of  $\mathcal{E}_M$ .) We think of these as representations that “extend” from  $i_*(\pi_1 S)$  to the larger group  $\pi_1 M$ .

Note that a priori the passage from a  $\pi_1 S$ -representation to a  $\pi_1 M$ -representation could radically change the geometry of the associated action on  $\mathbb{H}^3$ , as measured for example by the diameter of the orbit of a finite generating set. This possibility, combined with the need to keep track of the limiting  $\pi_1 S$ - and  $\pi_1 M$ -dynamics simultaneously, requires us to consider  $\Lambda$ -trees more general than  $\mathbb{R}$ -trees.

Using the valuation constructions of Morgan–Shalen, we show that there is a similar extension property for the trees obtained as limits points of  $\mathcal{E}_M$ , or more precisely, for their length functions (Theorem 6.3). In the generic case of a non-abelian action, the combination of this construction with the holonomy limit theorem above gives an  $\mathbb{R}^n$ -tree  $\widehat{T}$  (where  $\mathbb{R}^n = \Lambda$  is given the lexicographical order) on which  $\pi_1 M$  acts by isometries and a straight map

$$T_\phi \longrightarrow \widehat{T},$$

where  $\phi$  is the rescaled limit of a divergent sequence in  $\mathcal{V}_M$ . These satisfy the hypotheses of Theorem C, so the horizontal foliation of  $\phi$  lies in  $\mathcal{L}_{M,X}$ . Limit points of  $\mathcal{E}_M$  that correspond to abelian actions introduce minor additional complications that are handled by Theorem 4.4.

*Step 3. The foliation map  $\mathcal{F}: Q(X) \rightarrow \mathcal{MF}(S)$  is symplectic.*

Hubbard and Masur showed that the foliation map  $\mathcal{F}: Q(X) \rightarrow \mathcal{MF}(S)$  is a homeomorphism. In order to use the isotropic cone  $\mathcal{L}_{M,X}$  to understand the set  $\mathcal{V}_M$ , we

analyze the relation between the foliation map, the complex structure of  $Q(X)$ , and the symplectic structure of  $\mathcal{MF}(S)$ .

We introduce a natural Kähler structure on  $Q(X)$  using the Weil–Petersson-type hermitian pairing

$$\langle \psi_1, \psi_2 \rangle_\phi = \int_X \frac{\psi_1 \bar{\psi}_2}{4|\phi|}.$$

Here we have a base point  $\phi \in Q(X)$  and the quadratic differentials  $\psi_j \in T_\phi Q(X) \simeq Q(X)$  are considered as tangent vectors. This integral pairing is not smooth, nor even well defined for all tangent vectors, due to singularities of the integrand coming from higher-order zeros of  $\phi$ . However, we show that the pairing does give a well-defined Kähler structure relative to a stratification of  $Q(X)$ .

We show that the underlying symplectic structure of this Kähler metric is the one pulled back from  $\mathcal{MF}(S)$  by the foliation map.

**THEOREM D.** *For any  $X \in \mathcal{T}(S)$ , the map  $\mathcal{F}: Q(X) \rightarrow \mathcal{MF}(S)$  is a real-analytic stratified symplectomorphism, where  $Q(X)$  is given the symplectic structure coming from the pairing  $\langle \psi_1, \psi_2 \rangle_\phi$  and where  $\mathcal{MF}(S)$  has the Thurston symplectic form.*

The lack of a smooth structure on  $\mathcal{MF}(S)$  means that the regularity aspect of this result must be interpreted carefully. We show that for any point  $\phi \in Q(X)$  there is a neighborhood in its stratum and a train-track chart containing  $\mathcal{F}(\phi)$  in which the local expression of the foliation map is real-analytic and symplectic. The details are given in Theorem 5.8.

*Step 4. Analytic sets with totally real limits are discrete.*

In a Kähler manifold, an isotropic submanifold is totally real. Even though the piecewise linear cone  $\mathcal{L}_{M,X}$  is not globally a manifold, Theorem D allows us to describe  $\mathcal{F}^{-1}(\mathcal{L}_{M,X})$  locally in a stratum of  $Q(X)$  in terms of totally real, real-analytic submanifolds. Since limit points of  $\mathcal{V}_M$  correspond to elements of  $\mathcal{F}^{-1}(\mathcal{L}_{M,X})$ , this gives a kind of “totally real” constraint on the behavior of  $\mathcal{V}_M$  at infinity.

To formulate this constraint we consider the set  $\partial_{\mathbb{R}} \mathcal{V}_M \subset S^{2N-1}$  of points in the unit sphere of  $Q(X) \simeq \mathbb{C}^N$  that can be obtained as  $\mathbb{R}^+$ -rescaled limits of divergent sequences in  $\mathcal{V}_M$ . Projecting this set through the Hopf fibration  $S^{2N-1} \rightarrow \mathbb{C}\mathbb{P}^{N-1}$  we obtain the set  $\partial_{\mathbb{C}} \mathcal{V}_M$  of boundary points of  $\mathcal{V}_M$  in the complex projective compactification of  $Q(X)$ . Using the results of steps 1–3 we show (in Theorem 7.2) that

- (i) in a neighborhood of some point,  $\partial_{\mathbb{C}} \mathcal{V}_M$  is contained in a totally real manifold,
- (ii) the intersection of  $\partial_{\mathbb{R}} \mathcal{V}_M$  with a fiber of the Hopf map has empty interior.

Using extension and parametrization results from analytic geometry it is not hard to show that among analytic subvarieties of  $Q(X)$ , only a discrete subset can have both of

these properties. Condition (i) forces any analytic curve in  $\mathcal{V}_M$  to extend to an analytic curve in a neighborhood of some boundary point  $p \in \mathbb{C}\mathbb{P}^{N-1}$ . Within this extension there is generically a circle of directions in which to approach the boundary point, some arc of which is realized by the original curve. Analyzing the correspondence between this circle and the Hopf fiber over  $p$ , one finds that  $\partial_{\mathbb{R}}\mathcal{V}_M$  contains an open arc of this fiber, violating condition (ii).

This contradiction shows that  $\mathcal{V}_M$  contains no analytic curves, making it a discrete set. The intersection theorem follows.

### Applications and complements

The construction of the isotropic cone in Theorem C was inspired by the work of Floyd on the space of boundary curves of incompressible,  $\partial$ -incompressible surfaces in 3-manifolds [F]. Indeed, in the incompressible boundary case, lifting such a surface to the universal cover and considering dual trees in the boundary and in the 3-manifold gives rise to an isometric embedding of  $\mathbb{Z}$ -trees. Using Theorem C, we recover Floyd’s result in this case. This connection is explained in more detail in §4.5, where we also note that the same “cancellation” phenomenon is at the core of both arguments.

Since Theorem D provides an interpretation of Thurston’s symplectic form in terms of Riemannian and Kähler geometry of a (stratified) smooth manifold, we hope that it might allow new tools to be applied to problems involving the space of measured foliations. In §5.7 we describe work of Mirzakhani in this direction, where it is shown that a certain function connected to  $\text{Mod}(S)$ -orbit counting problems in Teichmüller space is constant.

As a possible extension of Theorem B one might ask whether  $\mathcal{H}_X$  and  $\mathcal{E}_M$  always intersect transversely, or equivalently, whether skinning maps are always immersions. In the slightly more general setting of manifolds with rank-1 cusps a negative answer was recently given by Gaster [Gas]. In addition to Gaster’s example, numerical experiments conducted jointly with Richard Kent suggest critical points for some other manifolds with rank-1 cusps [DK3].

Nevertheless, it would be interesting to understand the discreteness of the intersection  $\mathcal{H}_X \cap \mathcal{E}_M$  through local or differential properties rather than the compactifications and asymptotic arguments used here. The availability of rich geometric structure on the character variety and its compatibility with the subvarieties in question offers some hope in this direction; for example, both  $\mathcal{H}_X$  and  $\mathcal{E}_M$  are Lagrangian with respect to the complex symplectic structure of the character variety (see [Kaw], [La, §7.2] and [KS, §1.7]).

## Structure of the paper

§2 recalls some definitions and basic results related to  $\Lambda$ -trees, measured foliations, and Teichmüller theory.

§§3–7 contain the proofs of the main theorems in the case of a 3-manifold with connected boundary. Working in this setting avoids some cumbersome notation and other issues related to disconnected spaces, while all essential features of the argument are present. §§3–4 are devoted to the isotropic cone construction, §5 introduces the stratified Kähler structure, and §§6–7 combine these with the results of [D] to prove Theorem A in the connected boundary case.

In §9 we adapt the definitions and results of the previous sections as necessary to handle a manifold with disconnected boundary, possibly including torus components, completing the proof of Theorem A.

Finally, in §10 we prove an analogue of Theorem A for the extended skinning map of an acylindrical 3-manifold.

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## 2. Preliminaries

### 2.1. Ordered abelian groups

An *ordered abelian group* is a pair  $(\Lambda, <)$ , where  $\Lambda$  is an abelian group and  $<$  is a translation-invariant total order on  $\Lambda$ . We often consider the order to be implicit and denote an ordered abelian group by  $\Lambda$  alone. Note that an order on  $\Lambda$  also induces an order on any subgroup of  $\Lambda$ .

The positive subset of an ordered abelian group  $\Lambda$  is the set  $\Lambda^+ = \{g \in \Lambda : g > 0\}$ . If  $x \in \Lambda$  is non-zero, then exactly one of  $x, -x$  lies in  $\Lambda^+$ , and we denote this element by  $|x|$ .

If  $x, y \in \Lambda$ , we say that  $y$  is *infinitely larger* than  $x$  if  $n|x| < |y|$  for all  $n \in \mathbb{N}$ . If neither of  $x$  and  $y$  is infinitely larger than the other, then  $x$  and  $y$  are *archimedean equivalent*. When the set of archimedean equivalence classes of non-zero elements is finite, the number of

such classes is the *rank* of  $\Lambda$ . In what follows we consider *only* ordered abelian groups of finite rank.

A subgroup  $\Lambda' \subset \Lambda$  is *convex* if whenever  $g, k \in \Lambda'$ ,  $h \in \Lambda$ , and  $g < h < k$ , we have  $h \in \Lambda'$ . The convex subgroups of a given group are ordered by inclusion. A convex subgroup is a union of archimedean equivalence classes and is uniquely determined by the largest archimedean equivalence class that it contains (which exists, since the rank is finite). In this way the convex subgroups of a given ordered abelian group are in one-to-one order-preserving correspondence with its archimedean equivalence classes.

## 2.2. Embeddings and left inverses

If we equip  $\mathbb{R}^n$  with the lexicographical order, then the inclusion  $\mathbb{R} \hookrightarrow \mathbb{R}^n$  as one of the factors is order-preserving. This inclusion has a left inverse  $\mathbb{R}^n \rightarrow \mathbb{R}$  given by projecting onto the factor. This projection is of course a homomorphism but it is *not* order-preserving.

Similarly, the following lemma shows that an order-preserving embedding of  $\mathbb{R}$  into any ordered abelian group of finite rank has a left inverse; this is used in §3.6.

LEMMA 2.1. *If  $\Lambda$  is an ordered abelian group of finite rank and  $\mathfrak{o}: \mathbb{R} \hookrightarrow \Lambda$  is an order-preserving homomorphism, then  $\mathfrak{o}$  has a left inverse. That is, there is a homomorphism  $\varphi: \Lambda \rightarrow \mathbb{R}$  such that  $\varphi \circ \mathfrak{o} = \text{Id}$ .*

To construct a left inverse we use the following structural result for ordered abelian groups; part (i) is the *Hahn embedding theorem* (see e.g. [Gr] or [KK, §II.2]).

THEOREM 2.2. *Let  $\Lambda$  be an ordered abelian group of rank  $n$ .*

(i) *There exists an order-preserving embedding  $\Lambda \hookrightarrow \mathbb{R}^n$  where  $\mathbb{R}^n$  is given the lexicographical order.*

(ii) *If  $n=1$ , the order-preserving embedding  $\Lambda \hookrightarrow \mathbb{R}$  is unique up to multiplication by a positive constant.*

*Proof of Lemma 2.1.* By Theorem 2.2, it suffices to consider the case of an order-preserving embedding  $\mathfrak{o}: \mathbb{R} \rightarrow \mathbb{R}^n$ , where  $\mathbb{R}^n$  has the lexicographical order. Since the only order-preserving self-homomorphisms of the additive group  $\mathbb{R}$  are multiplication by positive constants, it is enough to find a homomorphism  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\varphi \circ \mathfrak{o}$  is order-preserving and fixes a point.

Let  $a = \mathfrak{o}(1)$ , and write  $a = (a_1, \dots, a_n)$ . Since  $a > 0$  in  $\mathbb{R}^n$ , the first non-zero element of the tuple  $a$  is positive. Let  $k$  be the index of this element, i.e.  $k = \min\{j: a_j > 0\}$ .

We claim that for any  $t \in \mathbb{R}$ , the image  $\mathfrak{o}(t)$  has the form  $(0, \dots, 0, b_k, \dots, b_n)$ . If not, then after possibly replacing  $t$  by  $-t$  we have  $t \in \mathbb{R}_+$  such that  $\mathfrak{o}(t)$  is infinitely larger than



$\sigma(1)$ . The existence of a positive integer  $n$  such that  $t < n$  shows that this contradicts the order-preserving property of  $\sigma$ .

Similarly, we find that if  $t > 0$  then  $b = \sigma(t)$  satisfies  $b_k > 0$ : The order-preserving property of  $\sigma$  implies that  $b_k \geq 0$ , so the only possibility to rule out is  $b_k = 0$ . But if  $b_k = 0$  then  $\sigma(1)$  is infinitely larger than  $\sigma(t)$ , which is contradicted by the existence of a positive integer  $n$  such that  $1 < nt$ .

Now define  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$\varphi(x_1, \dots, x_n) = \frac{x_k}{a_k}.$$

This is a homomorphism satisfying  $\varphi(\sigma(1)) = 1$ , and the properties of  $\sigma$  derived above show that  $\varphi \circ \sigma$  is order preserving, as desired.  $\square$

We now consider the properties of embeddings  $\Lambda \rightarrow \mathbb{R}^n$ , such as those provided by Theorem 2.2, with respect to convex subgroups. First, a proper convex subgroup maps into  $\mathbb{R}^{n-1}$ .

LEMMA 2.3. *Let  $F: \Lambda \rightarrow \mathbb{R}^n$  be an order-preserving embedding, where  $\Lambda$  has rank  $n$ . If  $\Lambda' \subset \Lambda$  is a proper convex subgroup, then  $F(\Lambda') \subset \{(0, x_2, \dots, x_n) : x_2, \dots, x_n \in \mathbb{R}\}$ .*

*Proof.* Since  $\Lambda' \neq \Lambda$ , the convex subgroup  $\Lambda'$  does not contain the largest archimedean equivalence class of  $\Lambda$ . Thus there exists a positive element  $g \in \Lambda_+$  such that  $h < g$  for all  $h \in \Lambda'$ .

Suppose that there exists  $h \in \Lambda'$  such that  $F(h) = (a_1, a_2, \dots, a_n)$  with  $a_1 \neq 0$ . Then we have  $F(kh) = ka_1 > F(g)$  for some  $k \in \mathbb{Z}$ . This contradicts the order-preserving property of  $F$ , so no such  $h$  exists and  $F(\Lambda')$  has the desired form.  $\square$

Building on the previous result, the following lemma shows that in some cases the embeddings given by Hahn's theorem behave functorially with respect to rank-1 subgroups. This result is used in §6.4.

LEMMA 2.4. *Let  $\Lambda$  be an ordered abelian group of finite rank and  $\Lambda' \subset \Lambda$  be a subgroup contained in the minimal non-trivial convex subgroup of  $\Lambda$ . Then there is a commutative diagram of order-preserving embeddings*

$$\begin{array}{ccc} \Lambda' & \xrightarrow{i} & \Lambda \\ \downarrow f & & \downarrow F \\ \mathbb{R} & \xrightarrow{i_n} & \mathbb{R}^n, \end{array}$$

where  $i_n(x) = (0, \dots, 0, x)$ ,  $x \in \mathbb{R}$ , and  $n$  is the rank of  $\Lambda$ .

*Proof.* Let  $\Lambda_1 \subset \Lambda_2 \subset \dots \subset \Lambda_n = \Lambda$  be the convex subgroups of  $\Lambda$ . We may assume that  $\Lambda' = \Lambda_1$  since all other cases are handled by restricting the maps from this one.

We are given the inclusion  $i: \Lambda_1 \rightarrow \Lambda$  and the Hahn embedding theorem provides an order-preserving embedding  $F: \Lambda \rightarrow \mathbb{R}^n$ . Applying Lemma 2.3 to each step in the chain of convex subgroups of  $\Lambda$ , we find that for all  $g \in \Lambda_1$  we have

$$F(g) = (0, \dots, 0, f(g)),$$

and the induced map  $f: \Lambda_1 \rightarrow \mathbb{R}$  is order preserving. Since  $F \circ i = i_n \circ f$  by construction, these maps complete the commutative diagram.  $\square$

### 2.3. $\Lambda$ -metric spaces and $\Lambda$ -trees

We refer to the book [Chis] for general background on  $\Lambda$ -metric spaces and  $\Lambda$ -trees. Here we recall the essential definitions and fix notation.

As before let  $\Lambda$  denote an ordered abelian group. A  $\Lambda$ -metric space is a pair  $(M, d)$ , where  $M$  is a set and  $d: M \times M \rightarrow \Lambda$  is a function which satisfies the usual axioms for the distance function of a metric space. In particular, an  $\mathbb{R}$ -metric space (where  $\mathbb{R}$  has the standard order) is the usual notion of a metric space.

An isometric embedding of one  $\Lambda$ -metric space into another is defined in the natural way. Generalizing this, let  $(M, d)$  be a  $\Lambda$ -metric space and  $(M', d')$  be a  $\Lambda'$ -metric space. An *isometric embedding* of  $M$  into  $M'$  is a pair  $(f, \sigma)$  consisting of a map  $f: M \rightarrow M'$  and an order-preserving homomorphism  $\sigma: \Lambda \rightarrow \Lambda'$  such that

$$d'(f(x), f(y)) = \sigma(d(x, y)) \quad \text{for all } x, y \in M.$$

More generally we say that  $f: M \rightarrow M'$  is an *isometric embedding* if there exists an order-preserving homomorphism  $\sigma$  such that the pair  $(f, \sigma)$  satisfies this condition.

An ordered abelian group  $\Lambda$  is an example of a  $\Lambda$ -metric space, with metric

$$d(g, h) = |g - h|.$$

An isometric embedding of the subspace

$$[g, h] := \{k \in \Lambda : g \leq h \leq k\} \subset \Lambda$$

into a  $\Lambda$ -metric space is a *segment*. A  $\Lambda$ -metric space is *geodesic* if any pair of points can be joined by a segment.

A  $\Lambda$ -tree is a  $\Lambda$ -metric space  $(T, d)$  satisfying the following three conditions:

- $(T, d)$  is geodesic;

- if two segments in  $T$  share an endpoint but have no other intersection points, then their union is a segment;
- if two segments in  $T$  share an endpoint, then their intersection is a segment (or a point).

The notion of  $\delta$ -hyperbolicity for metric spaces generalizes naturally to  $\Lambda$ -metric spaces, where now  $\delta \in \Lambda$ ,  $\delta \geq 0$ . In terms of this generalization, any  $\Lambda$ -tree is 0-hyperbolic. (The converse holds under mild additional assumptions on the space.) The 0-hyperbolicity condition has various equivalent characterizations, but the one we will use in the sequel is the following condition on 4-tuples of points.

LEMMA 2.5. (0-hyperbolicity of  $\Lambda$ -trees) *If  $(T, d)$  is a  $\Lambda$ -tree, then for all  $x, y, z, t \in T$  we have*

$$d(x, y) + d(z, t) \leq \max\{d(x, z) + d(y, t), d(x, t) + d(y, z)\}.$$

For a proof and further discussion see [Chis, Lemmas 1.2.6 and 2.1.6]. By permuting a given 4-tuple  $x, y, z, t$  and considering the inequality of this lemma, we obtain the following corollary (see [Chis, p. 35]).

LEMMA 2.6. (Four points in a  $\Lambda$ -tree) *Let  $(T, d)$  be a  $\Lambda$ -tree and  $x, y, z, t \in T$ . Then among the three sums*

$$d(x, y) + d(z, t), \quad d(x, z) + d(y, t), \quad \text{and} \quad d(x, t) + d(y, z),$$

*two are equal, and these two are not less than the third one.*

Given a  $\Lambda$ -tree, there are natural constructions that associate trees with certain subgroups or extensions of  $\Lambda$ ; in what follows we require two such operations. First, let  $(T, d)$  be a  $\Lambda$ -tree and  $\Lambda' \subset \Lambda$  be a convex subgroup. For any  $x \in T$  we can consider the subset  $T_{\Lambda', x} = \{y \in T : d(x, y) \in \Lambda'\}$ . Then the restriction of  $d$  to  $T_{\Lambda', x}$  takes values in  $\Lambda'$ , and this gives  $T_{\Lambda', x}$  the structure of a  $\Lambda'$ -tree [MoS1, Proposition II.1.14].

Second, suppose that  $\sigma: \Lambda \rightarrow \Lambda'$  is an order-preserving homomorphism and that  $(T, d)$  is a  $\Lambda$ -tree. Then there is a natural *base-change* construction that produces a  $\Lambda'$ -tree  $\Lambda' \otimes_{\Lambda} T$  and an isometric embedding  $T \rightarrow \Lambda' \otimes_{\Lambda} T$  with respect to  $\sigma$  (see [Chis, Theorem 4.7] for details). Roughly speaking, if one views  $T$  as a union of segments, each identified with some interval  $[g, h] \subset \Lambda$ , then  $\Lambda' \otimes_{\Lambda} T$  is obtained by replacing each such segment with  $[\sigma(g), \sigma(h)] \subset \Lambda'$ .

## 2.4. Group actions on $\Lambda$ -trees and length functions

Every isometry of a  $\Lambda$ -tree is either *elliptic*, *hyperbolic*, or an *inversion*; see [Chis, §3.1] for a detailed discussion of this classification. Elliptic isometries are those with fixed points,

while hyperbolic isometries have an invariant axis (identified with a subgroup of  $\Lambda$ ) on which they act as a translation. An inversion is an isometry that has no fixed point but which induces an elliptic isometry after an index-2 base change; permitting such a base change allows us to make the standing assumption that *isometric group actions on  $\Lambda$ -trees that we consider are without inversions*.

The *translation length*  $\ell(g)$  of an isometry  $g: T \rightarrow T$  of a  $\Lambda$ -tree is defined as

$$\ell(g) = \begin{cases} 0, & \text{if } g \text{ is elliptic,} \\ |t|, & \text{if } g \text{ is hyperbolic and acts on its axis as } h \mapsto h+t. \end{cases}$$

Note that  $\ell(g) \in \Lambda^+ \cup \{0\}$ . It can be shown that the translation length is also given by

$$\ell(g) = \min\{d(x, g(x)) : x \in T\}.$$

When a group  $G$  acts on a  $\Lambda$ -tree by isometries, taking the translation length of each element of  $G$  defines a function  $\ell: G \rightarrow \Lambda^+ \cup \{0\}$ , the *translation-length function* (or briefly, the *length function*) of the action.

When the translation-length function takes values in a convex subgroup, one can extract a subtree whose distance function takes values in the same subgroup.

LEMMA 2.7. *Let  $G$  act on a  $\Lambda$ -tree  $T$  with length function  $\ell$ . If  $\Lambda' \subset \Lambda$  is a convex subgroup and  $\ell(G) \subset \Lambda'$ , then there is a  $\Lambda'$ -tree  $T' \subset T$  that is invariant under  $G$  and such that  $\ell$  is also the length function of the induced action of  $G$  on  $T'$ .*

This lemma is implicit in the proof of [Mor1, Theorem 3.7], which uses the structure theory of actions developed in [MoS1]. For the convenience of the reader, we reproduce the argument here.

*Proof.* Because  $\Lambda'$  is convex, there is an induced order on the quotient group  $\Lambda/\Lambda'$ . Define an equivalence relation on  $T$  by  $x \sim y$  if  $d(x, y) \in \Lambda'$ . Then the quotient  $T_0 = T/\sim$  is a  $(\Lambda/\Lambda')$ -tree, and each fiber of the projection  $T \rightarrow T_0$  is a  $\Lambda'$ -tree. The action of  $G$  on  $T$  induces an action on  $T_0$  whose length function is the composition of  $\ell$  with the map  $\Lambda \rightarrow \Lambda/\Lambda'$ , which is identically zero since  $\ell(G) \subset \Lambda'$ . It follows that the action of  $G$  on  $T_0$  has a global fixed point [MoS1, Proposition II.2.15], and thus  $G$  acts on the fiber of  $T$  over  $T_0$ , which is a  $\Lambda'$ -tree  $T'$ . By [MoS1, Proposition II.2.12], the length function of the action of  $G$  on  $\Lambda'$  is  $\ell$ .  $\square$

## 2.5. Measured foliations and train tracks

Let  $\mathcal{MF}(S)$  denote the space of measured foliations on a compact oriented surface  $S$  of genus  $g$ . Then  $\mathcal{MF}(S)$  is a piecewise linear manifold which is homeomorphic to  $\mathbb{R}^{6g-6}$ .

A point  $[\nu] \in \mathcal{MF}(S)$  is an equivalence class, up to Whitehead moves of a singular foliation  $\nu$  on  $S$ , equipped with a transverse measure of full support. For detailed discussion of measured foliations and of the space  $\mathcal{MF}(S)$  see [FLP].

Piecewise linear charts of  $\mathcal{MF}(S)$  correspond to sets of measured foliations that are carried by a train track; we will now discuss the construction of these charts in some detail. While this material is well known to experts, most standard references that discuss train-track charts use the equivalent language of measured laminations, whereas our primary interest in foliations arising from quadratic differentials makes the direct consideration of foliations preferable. Additional details of the carrying construction from this perspective can be found in [P] and [Mos].

A *train track* on  $S$  is a  $C^1$  embedded graph in which all edges incident on a given vertex share a tangent line at that point. Vertices of the train track are called *switches* and its edges are *branches*. We consider only *generic* train tracks in which each switch has three incident edges, two *incoming* and one *outgoing*, such that the union of any incoming edge and the outgoing edge forms a  $C^1$  curve.

The complement of a train track is a finite union of subsurfaces with cusps on their boundaries. In order to give a piecewise linear chart of  $\mathcal{MF}(S)$ , each complementary disk must have at least three cusps on its boundary and each complementary annulus must have at least one cusp. We will always require this of the train tracks we consider.

If  $\tau$  is such a train track, let  $W(\tau)$  denote the vector space of real-valued functions  $w$  on its set of edges that obey the relation  $w(a) + w(b) = w(c)$  for any switch with incoming edges  $\{a, b\}$  and outgoing edge  $c$ . This *switch relation* ensures that  $w$  determines a signed transverse measure, or *weight*, on the embedded train track. Within  $W(\tau)$  there is the finite-sided convex cone of non-negative weight functions, denoted  $\mathcal{MF}(\tau)$ . It is this cone which forms a chart for  $\mathcal{MF}(S)$ .

A measured foliation is *carried* by the train track  $\tau$  if the foliation can be cut open near singularities and along saddle connections and then moved by an isotopy so that all of the leaves lie in an arbitrarily small open neighborhood of  $\tau$  and are nearly parallel to its branches, as depicted in Figure 1. Here “cutting open” refers to the procedure of replacing a union of leaf segments and saddle connections coming out of singularities with a subsurface with cusps on its boundary. The result of cutting open a measured foliation is a *partial measured foliation* in which there are *non-foliated regions*, each of which has a union of leaf segments of the original foliation as a spine.

A measured foliation  $\nu$  determines a weight on any train track that carries it, as follows: For each branch  $e \subset \tau$  choose a *tie*  $r_e$ , a short closed arc that crosses  $e$  transversely at an interior point and which is otherwise disjoint from  $\tau$ . Now select an open neighborhood of  $U$  of  $\tau$  that intersects each tie  $r_e$  in a connected open interval. Let  $\nu'$

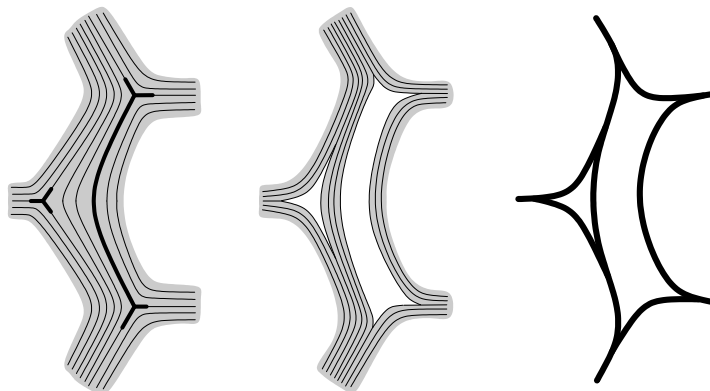


Figure 1. Local picture of carrying: Cutting a measured foliation along saddle connections and singular leaf segments and inflating these to cusped subsurfaces allows an isotopy into a small neighborhood of the train track.

be a partial measured foliation associated with  $\nu$  that has been isotoped to lie in  $U$  and to be transverse to each tie. Note that each tie  $r_e$  then has endpoints in non-foliated regions of  $\nu'$ , since the endpoints of  $r_e$  lie outside  $U$ . Let  $w(e)$  be the total transverse measure of  $r_e$  with respect to  $\nu'$ .

The resulting function  $w$  lies in  $\mathcal{MF}(\tau)$  and regarding this construction as a map  $\nu \mapsto w$  gives a one-to-one correspondence between equivalence classes of measured foliations that are carried by  $\tau$  and the convex cone  $\mathcal{MF}(\tau)$ . Furthermore, these cones in train-track weight spaces form the charts of a piecewise linear atlas on  $\mathcal{MF}(S)$ .

## 2.6. The symplectic structure of $\mathcal{MF}(S)$

The orientation of  $S$  induces a natural antisymmetric bilinear map  $\omega_{\text{Th}}: W(\tau) \times W(\tau) \rightarrow \mathbb{R}$  on the space of weights on a train track  $\tau$ . This *Thurston form* can be defined as follows (compare [PH, §3.2] and [Bon, §3]): For each switch  $v \in \tau$ , let  $a_v$  and  $b_v$  be its incoming edges and  $c_v$  be its outgoing edge, where  $a_v$  and  $b_v$  are ordered so that intersecting  $\{a_v, b_v, c_v\}$  with a small circle around  $v$  gives a positively oriented triple. Then we define

$$\omega_{\text{Th}}(w_1, w_2) = \frac{1}{2} \sum_{v \in \tau} \det \begin{pmatrix} w_1(a_v) & w_1(b_v) \\ w_2(a_v) & w_2(b_v) \end{pmatrix}. \quad (2.1)$$

If  $\tau$  defines a chart of  $\mathcal{MF}(S)$  then this form is non-degenerate, and the induced symplectic forms on train-track charts  $\mathcal{MF}(\tau)$  are compatible. This gives  $\mathcal{MF}(S)$  the structure of a piecewise-linear symplectic manifold.

The Thurston form can also be interpreted as a homological intersection number. If  $\tau$  can be consistently oriented then each weight function  $w$  defines a 1-cycle  $c_w = \sum_e w(e)\vec{e}$ ,

where  $\vec{e}$  denotes the singular 1-simplex defined by the oriented edge  $e$  of  $\tau$ . In terms of these cycles, we have  $\omega_{\text{Th}}(w_1, w_2) = c_{w_1} \cdot c_{w_2}$ . For a general train track, there is a branched double cover  $\widehat{S} \rightarrow S$  (with branching locus disjoint from  $\tau$ ) such that the preimage  $\hat{\tau} \subset \widehat{S}$  is orientable. Lifting weight functions, we obtain cycles  $\hat{c}_{w_j} \in H^1(\widehat{S}, \mathbb{R})$  such that

$$\omega_{\text{Th}}(w_1, w_2) = \frac{1}{2}(\hat{c}_{w_1} \cdot \hat{c}_{w_2}). \quad (2.2)$$

Note that if  $\bar{S}$  denotes the opposite orientation of the surface  $S$ , then there is a natural identification between measured foliation spaces  $\mathcal{MF}(S) \simeq \mathcal{MF}(\bar{S})$ , but this identification does *not* respect the Thurston symplectic forms. Rather, in corresponding local charts, we have  $\omega_{\text{Th}}^S = -\omega_{\text{Th}}^{\bar{S}}$ .

## 2.7. Dual trees

Let  $\nu$  be a measured foliation on  $S$  and  $\tilde{\nu}$  be its lift to the universal cover  $\tilde{S}$ . There is a pseudo-metric  $d$  on  $\tilde{S}$ , where  $d(x, y)$  is the minimum  $\tilde{\nu}$ -transverse measure of a path connecting  $x$  to  $y$ . The quotient metric space  $T_\nu := \tilde{S}/d^{-1}(0)$  is an  $\mathbb{R}$ -tree (see [Bow] and [MoS4]). The action of  $\pi_1 S$  on  $\tilde{S}$  by deck transformations determines an action on  $T_\nu$  by isometries. The dual tree of the zero foliation  $0 \in \mathcal{MF}(S)$  is a point.

This pseudo-metric construction can be applied to the partial measured foliation  $\nu'$  obtained by cutting  $\nu$  open along leaf segments from singularities, as when  $\nu$  is carried by a train track  $\tau$ . The result is a tree naturally isometric to  $T_\nu$ , which we identify with  $T_\nu$  from now on. Non-foliated regions of  $\tilde{\nu}'$  are collapsed to points in this quotient, so in particular each complementary region of the lift  $\tilde{\tau}$  has a well-defined image point  $T_\nu$ .

Similarly, the lift of a tie  $r_e$  of  $\tau$  to the universal cover projects to a geodesic segment in  $T_\nu$  of length  $w(e)$ ; the endpoints of this segment are the projections of the two complementary regions adjacent to the lift of the edge  $e$ .

To summarize, we have the following relation between carrying and dual trees.

**PROPOSITION 2.8.** *Let  $\nu$  be a measured foliation carried by the train track  $\tau$  with associated weight function  $w$ . Let  $\tilde{\tau}$  denote the lift of  $\tau$  to the universal cover. If  $A$  and  $B$  are complementary regions of  $\tilde{\tau}$  that are adjacent along an edge  $\tilde{e}$  of  $\tilde{\tau}$ , and if  $a$  and  $b$  are the associated points in  $T_\nu$ , then we have*

$$w(e) = d(a, b),$$

where  $d$  is the distance function of  $T_\nu$ .

## 2.8. Teichmüller space and quadratic differentials

Let  $\mathcal{T}(S)$  be the Teichmüller space of marked isomorphism classes of complex structures on  $S$  compatible with its orientation. For any  $X \in \mathcal{T}(S)$  we denote by  $Q(X)$  the set of holomorphic quadratic differentials on  $X$ , a complex vector space of dimension  $3g-3$ .

Associated with  $\phi \in Q(X)$  we have the following structures on  $X$ :

- the flat metric  $|\phi|$ , which has cone singularities at the zeros of  $\phi$ ;
- the measured foliation  $\mathcal{F}(\phi)$  whose leaves integrate the distribution  $\ker(\operatorname{Im}(\sqrt{\phi}))$ , with transverse measure given by  $|\operatorname{Im}(\sqrt{\phi})|$ ;
- the dual tree  $T_\phi := \widetilde{T_{\mathcal{F}(\phi)}}$  and the  $\pi_1 S$ -equivariant map  $\pi: \widetilde{X} \rightarrow T_\phi$  that collapses leaves of the lifted foliation  $\widetilde{\mathcal{F}(\phi)}$  to points of  $T_\phi$ .

The dual tree construction is homogeneous with respect to the action of  $\mathbb{R}^+$  on  $Q(X)$  in the sense that for any  $c \in \mathbb{R}^+$  we have

$$T_{c\phi} = c^{1/2} T_\phi,$$

where the right-hand side represents the metric space obtained from  $T_\phi$  by multiplying its distance function by  $c^{1/2}$ .

Note that the point  $0 \in Q(X)$  is a degenerate case in which there is no corresponding flat metric, and by convention  $\mathcal{F}(0)$  is the empty measured foliation whose dual tree is a point.

We say that a  $|\phi|$ -geodesic is *non-singular* if its interior is disjoint from the zeros of  $\phi$ . Choosing a local coordinate  $z$  in which  $\phi = dz^2$  (a *natural coordinate* for  $\phi$ ), a non-singular  $|\phi|$ -geodesic segment  $I$  becomes a line segment in the  $z$ -plane. The vertical variation of this segment in  $\mathbb{C}$  (i.e.  $|\operatorname{Im}(z_2 - z_1)|$ , where  $z_j$  are the endpoints) is the *height* of  $I$ .

The leaves of the foliation  $\mathcal{F}(\phi)$  are geodesics of the  $|\phi|$ -metric. Conversely, a non-singular  $|\phi|$ -geodesic  $I$  is either a leaf of  $\mathcal{F}(\phi)$  or it is transverse to  $\mathcal{F}(\phi)$ . In the latter case, the height  $h$  of  $I$  is equal to its  $\mathcal{F}(\phi)$ -transverse measure, and any lift of  $I$  to  $\widetilde{X}$  projects homeomorphically to a geodesic segment in  $T_\phi$  of length  $h$ .

## 3. The isotropic cone: Embeddings

The goal of this section is to establish the following result relating 3-manifold actions on  $\Lambda$ -trees and measured foliations.

**THEOREM 3.1.** *Let  $M$  be a 3-manifold with connected boundary  $S$ . There exists an isotropic piecewise linear cone  $\mathcal{L}_M \subset \mathcal{MF}(S)$  with the following property: If  $\nu$  is a measured foliation on  $S$  whose dual tree embeds isometrically and  $\pi_1 S$ -equivariantly into a  $\Lambda$ -tree  $T$  equipped with an isometric action of  $\pi_1 M$ , then  $[\nu] \in \mathcal{L}_M$ .*



Here a *piecewise linear cone* refers to a closed  $\mathbb{R}^+$ -invariant subset of  $\mathcal{MF}(S)$  whose intersection with any train-track chart  $\mathcal{MF}(\tau)$  is a finite union of finite-sided convex cones in linear subspaces of  $W(\tau)$ . Such a cone is *isotropic* if the linear spaces can be chosen to be isotropic with respect to the Thurston symplectic form. Since transition maps between these charts are piecewise linear and symplectic, it suffices to check these conditions in any covering of the set by train-track charts.

The first step in the proof of Theorem 3.1 will be to use the foliation and embedding to construct a weight function on the 1-skeleton of a triangulation of  $M$ . We begin with some generalities about train tracks, triangulations, and weight functions.

### 3.1. Complexes and weight functions

Let  $\Delta$  be a simplicial complex, and let  $\Delta^{(k)}$  denote its set of  $k$ -simplices. Given an abelian group  $G$ , define the space of  $G$ -valued weights on  $\Delta$  as the  $G$ -module consisting of functions  $\Delta^{(1)} \rightarrow G$ ; we denote this space by

$$W(\Delta, G) := G^{\Delta^{(1)}}.$$

The case  $G = \mathbb{R}$  will be of primary interest and so we abbreviate  $W(G) := W(G, \mathbb{R})$ . If  $w \in W(\Delta, G)$  and  $e \in \Delta^{(1)}$ , we say that  $w(e)$  is the *weight* of  $e$  with respect to  $w$ .

A homomorphism of groups  $\varphi: G \rightarrow G'$  induces a homomorphism of weight spaces  $\varphi_*: W(\Delta, G) \rightarrow W(\Delta, G')$ , and an inclusion of simplicial complexes  $i: (M, \Delta) \rightarrow (M', \Delta')$  induces a  $G$ -linear restriction map  $i^*: W(\Delta', G) \rightarrow W(\Delta, G)$ . These functorial operations commute, i.e.  $\varphi_* \circ i^* = i^* \circ \varphi_*$ .

A map  $f$  from  $\Delta^{(0)}$  to a  $\Lambda$ -metric space induces a  $\Lambda$ -valued weight on  $\Delta$  in a natural way: For each  $e \in \Delta^{(1)}$  we define its *weight* to be the distance between the  $f$ -images of its endpoints. We write  $w_f$  for the weight function defined in this way.

This construction has a natural extension to equivariant maps on regular covers. Suppose that  $\tilde{\Delta}$  and  $\Delta$  are simplicial complexes such that there is a regular covering  $\pi: \tilde{\Delta} \rightarrow \Delta$  which is also a simplicial map. Suppose also that the deck group  $\Gamma$  of this covering acts isometrically on a  $\Lambda$ -metric space  $E$ . Then, if  $f: \tilde{\Delta} \rightarrow E$  is a  $\Gamma$ -equivariant map, the resulting weight function  $\tilde{w}_f \in W(\tilde{\Delta}, \Lambda)$  is also  $\Gamma$ -invariant, hence it descends to a weight function  $w_f \in W(\Delta, \Lambda)$  on the base of the covering.

### 3.2. Extending triangulations and maps

We will now consider the space of weight functions as defined above in cases where the complex  $\Delta$  is a triangulation of a 2- or 3-manifold, possibly with boundary.

For example, let  $\tau$  be a maximal, generic train track on a surface  $S$ . Then there is a triangulation  $\Delta_\tau$  of  $S$  dual to the embedded trivalent graph underlying  $\tau$ . Each triangle of  $\Delta_\tau$  contains one switch of the train track, each edge of  $\Delta_\tau$  corresponds to an edge of  $\tau$ , and each vertex of  $\Delta_\tau$  corresponds to a complementary region of  $\tau$ . The correspondence between edges gives a natural (linear) embedding

$$W(\tau) \hookrightarrow W(\Delta_\tau).$$

Now suppose that  $\nu$  is a measured foliation on  $S$  that is carried by the train track  $\tau$ , so we consider the class  $[\nu]$  as an element of  $\mathcal{MF}(\tau) \subset W(\tau)$ . Let  $\tilde{\tau}$  denote the lift of  $\tau$  to the universal cover  $\tilde{S}$ . As in §2.7, the carrying relationship between  $\nu$  and  $\tau$  gives a map from complementary regions of  $\tau$  to the dual tree  $T_\nu$ . In terms of the dual triangulation  $\Delta := \Delta_\tau$ , this is a map

$$f: \tilde{\Delta}^{(0)} \longrightarrow T_\nu,$$

and it is immediate from the definitions above and Proposition 2.8 that the associated weight function  $w_f \in W(\Delta)$  is the image of  $[\nu]$  under the embedding  $W(\tau) \hookrightarrow W(\Delta)$ .

Let us further assume that, as in the hypotheses of Theorem 3.1, there is an equivariant isometric embedding of  $T_\nu$  into a  $\Lambda$ -tree  $T$  equipped with an action of  $\pi_1 M$ , where  $M$  is a 3-manifold with  $\partial M = S$ . Using this embedding, we can consider the map  $f$  constructed above as taking values in  $T$ . We extend the triangulation  $\Delta$  of  $S$  to a triangulation  $\Delta_M$  of  $M$ , and the map  $f$  to a  $\pi_1 M$ -equivariant map

$$F: \tilde{\Delta}_M^{(0)} \longrightarrow T.$$

Such an extension can be constructed by choosing a fundamental domain  $V$  for the  $\pi_1 M$ -action on  $\tilde{\Delta}_M^{(0)}$  and mapping elements of  $V \setminus \tilde{\Delta}_M^{(0)}$  to arbitrary points in  $T$ . Combining these with the values of  $f$  on  $\tilde{\Delta}^{(0)}$  and the free action of  $\pi_1 M$ , gives a unique equivariant extension to all of  $\tilde{\Delta}_M^{(0)}$ .

Associated with the map  $F$  is the weight function  $w_F \in W(\Delta_M, \Lambda)$ . By construction, its values on the edges of  $\Delta$  are the coordinates of  $[\nu]$  relative to the train-track chart of  $\tau$ , considered as elements of  $\Lambda$  using the embedding  $\mathfrak{o}: \mathbb{R} \rightarrow \Lambda$  that is implicit in the isometric map  $T_\nu \rightarrow T$ .

We record the constructions of this paragraph in the following proposition.

**PROPOSITION 3.2.** *Let  $\nu$  be a measured foliation on  $S = \partial M$  carried by a maximal generic train track  $\tau$ , and let  $\Delta_M$  be a triangulation of  $M$  extending the dual triangulation of  $\tau$ . Suppose that there exists a  $\Lambda$ -tree  $T$  equipped with an isometric action of  $\pi_1 M$  and an equivariant isometric embedding*

$$h: T_\nu \longrightarrow T,$$

relative to an order-preserving embedding  $\mathfrak{o}: \mathbb{R} \rightarrow \Lambda$ . Then there exists a weight function  $w \in W(\Delta_M, \Lambda)$  with the following properties:

- (i) the weight  $w$  is induced by an equivariant map  $F: \tilde{\Delta}_M^{(0)} \rightarrow T$ ;
- (ii) the restriction of  $w$  to  $\Delta_\tau$  is the image of  $[\nu] \in W(\tau)$  under the natural inclusion

$$W(\tau) \hookrightarrow W(\Delta_\tau) \xrightarrow{\mathfrak{o}^*} W(\Delta_\tau, \Lambda).$$

### 3.3. Triangle forms and the symplectic structure

In [PP], Penner and Papadopoulos relate the Thurston symplectic structure of  $\mathcal{MF}(S)$  for a punctured surface  $S$  to a certain linear 2-form on the space of weights on a “null-gon track” dual to an ideal triangulation of  $S$ . In this section we discuss a related construction for a triangulation of a compact surface dual to a train track.

Let  $\sigma$  be an oriented triangle with edges  $e$ ,  $f$ , and  $g$  (cyclically ordered according to the orientation). Let  $de$ ,  $df$ , and  $dg$  denote the corresponding linear functionals on  $\mathbb{R}^{\{e,f,g\}}$ , which evaluate a function on the given edge. We call the alternating 2-form

$$\omega_\sigma := -\frac{1}{2}(de \wedge df + df \wedge dg + dg \wedge de)$$

the *triangle form* associated with  $\sigma$ . Note that, if  $-\sigma$  represents the triangle with the opposite orientation, then  $\omega_{-\sigma} = -\omega_\sigma$ .

Given a triangulation  $\Delta$  of a compact oriented 2-manifold  $S$ , the triangle form corresponding to any  $\sigma \in \Delta^{(2)}$  (with its induced orientation) is naturally an element of  $\wedge^2 W(\Delta)^*$ . Denote the sum of these by

$$\omega_\Delta = \sum_{\sigma \in \Delta^{(2)}} \omega_\sigma. \quad (3.1)$$

This 2-form on  $W(\Delta)$  is an analogue of the Thurston symplectic form, in a manner made precise by the following.

**LEMMA 3.3.** *If  $\tau$  is a maximal generic train track on  $S$  with dual triangulation  $\Delta = \Delta_\tau$ , then the Thurston form on  $W(\tau)$  is the pullback of  $\omega_\Delta$  by the natural inclusion  $W(\tau) \rightarrow W(\Delta)$ .*

*Proof.* By direct calculation: both the Thurston form and  $\omega_\Delta$  are given as a sum of 2-forms, one for each triangle of  $\Delta$  (equivalently, switch of  $\tau$ ). The image of  $W(\tau)$  in  $W(\Delta)$  is cut out by imposing a switch condition for each triangle  $\sigma \in \Delta^{(2)}$ , which, for an appropriate labeling of the sides as  $e$ ,  $f$ , and  $g$ , can be written as

$$de + df = dg.$$

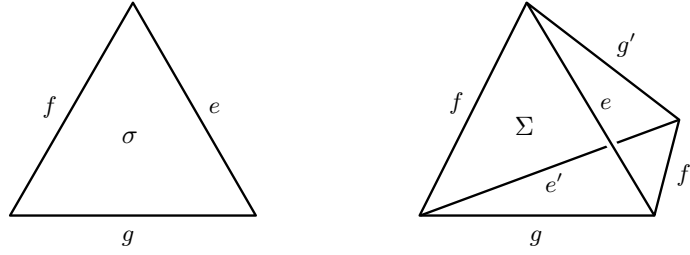


Figure 2. Labeled edges of an oriented 2-simplex  $\sigma$  and an oriented 3-simplex  $\Sigma$ . The 2-forms  $\omega_\sigma$  and  $\omega_\Sigma$  are defined in terms of these labels.

On the subspace defined by this constraint the triangle form pulls back to

$$-\frac{1}{2}(de \wedge df + df \wedge dg + dg \wedge de) = \frac{1}{2}de \wedge df,$$

which is the associated summand in the Thurston form (2.1).  $\square$

### 3.4. Tetrahedron forms

Let  $\Sigma$  be an oriented 3-simplex. Call a pair of edges of  $\Sigma$  *opposite* if they do not share a vertex. Label the edges of  $\Sigma$  by  $e, f, g, e', f',$  and  $g'$  so that the following conditions are satisfied:

- (i) the pairs  $\{e, e'\}, \{f, f'\},$  and  $\{g, g'\}$  are opposite;
- (ii) the ordering  $e, f, g$  gives the oriented boundary of one of the faces of  $\Sigma$ .

An example of such a labeling is shown in Figure 2.

Define the *tetrahedron form*  $\Omega_\Sigma \in \wedge^2 W(\Sigma)^*$  as

$$\Omega_\Sigma = -\frac{1}{2}(d(e+e') \wedge d(f+f') + d(f+f') \wedge d(g+g') + d(g+g') \wedge d(e+e')).$$

Here we abbreviate  $d(e+e') = de + de'$  and similarly for the other edges. It is easy to check that this 2-form does not depend on the labeling (as long as it satisfies the conditions above). As in the case of triangle forms,  $\Omega_\Sigma$  is naturally a 2-form on the space of weights for any oriented simplicial complex containing  $\Sigma$ .

A simple calculation using the definition above gives the following identity.

LEMMA 3.4. *The tetrahedron form is equal to the sum of the triangle forms of its oriented boundary faces, i.e.*

$$\Omega_\Sigma = \sum_{\sigma \in \partial \Sigma} \omega_\sigma.$$

Now consider a triangulation  $\Delta_M$  of an oriented 3-manifold with boundary  $S$ , and let  $\Delta_S$  denote the induced triangulation of the boundary. Denote the sum of the tetrahedron

forms by

$$\Omega_{\Delta_M} = \sum_{\Sigma \in \Delta_M^{(3)}} \Omega_{\Sigma} \in \wedge^2 W(\Delta_M)^*.$$

In fact, due to cancellation in this sum, the 2-form defined above “lives” on the boundary.

LEMMA 3.5. *The form  $\Omega_{\Delta_M}$  is equal to the pullback of  $\omega_{\Delta_S}$  under the restriction map  $W(\Delta_M) \rightarrow W(\Delta_S)$ .*

*Proof.* By Lemma 3.4, we have

$$\Omega_{\Delta_M} = \sum_{\Sigma \in \Delta_M^{(3)}} \sum_{\sigma \in \partial \Sigma} \omega_{\sigma}.$$

In this sum, each interior triangle of  $\Delta_M$  appears twice (once with each orientation) and so these terms cancel. The remaining terms are the elements of  $\Delta_S^{(2)}$  with the boundary orientation, so we are left with the sum (3.1), which defines  $\omega_{\Delta_S}$ . The result is the pullback of  $\omega_{\Delta_S}$  by the restriction map, because in the formula above, we are considering  $\omega_{\sigma}$  as an element of  $\wedge^2 W(\Delta_M)^*$  rather than  $\wedge^2 W(\Delta_S)^*$ .  $\square$

### 3.5. The four-point condition

Given four points in a  $\Lambda$ -tree, Lemma 2.6 implies that there is always a labeling  $\{p, q, r, s\}$  of these points such that the distance function satisfies

$$d(p, q) + d(r, s) = d(p, s) + d(r, q). \quad (3.2)$$

We call this the *weak four-point condition* to distinguish it from the stronger four-point condition of Lemma 2.6 which also involves an inequality.

If we think of  $p, q, r,$  and  $s$  as labeling the vertices of a 3-simplex  $\Sigma$ , then the pairwise distances give a weight function  $w: \Sigma^{(1)} \rightarrow \Lambda$ . Condition (3.2) is equivalent to the existence of opposite edge pairs  $\{e, e'\}, \{f, f'\} \subset \Sigma^{(1)}$  such that

$$w(e) + w(e') = w(f) + w(f'). \quad (3.3)$$

Given a simplicial complex  $\Delta$ , let  $W_4(\Delta, \Lambda)$  denote the set of  $\Lambda$ -valued weights such that in each 3-simplex of  $\Delta$  there exist opposite edge pairs so that (3.3) is satisfied.

The following basic properties of  $W_4(\Delta, \Lambda)$  follow immediately from the definition of this set (and the relation between the four-point condition and 4-tuples in  $\Lambda$ -trees).

LEMMA 3.6. (i) *The set  $W_4(\Delta, \Lambda)$  is a finite union of subspaces (i.e.  $\Lambda$ -submodules) of  $W(\Delta, \Lambda)$ ; each subspace corresponds to choosing opposite edge pairs in each of the 3-simplices of  $\Delta$ .*

(ii) *If  $\varphi: \Lambda \rightarrow \Lambda'$  is a homomorphism, then we have  $\varphi_*(W_4(\Delta, \Lambda)) \subset W_4(\Delta, \Lambda')$ .*

(iii) *If  $f: \tilde{\Delta}^{(0)} \rightarrow T$  is an equivariant map to a  $\Lambda$ -tree, then  $w_f \in W_4(\Delta, \Lambda)$ .*

Ultimately, the isotropic condition in Theorem 3.1 arises from the following property of the set  $W_4(\Delta)=W_4(\Delta, \mathbb{R})$ .

LEMMA 3.7. *Let  $M$  be an oriented 3-manifold and  $\Delta_M$  be a triangulation. Then  $W_4(\Delta_M)$  is a finite union of  $\Omega_{\Delta_M}$ -isotropic subspaces of  $W(\Delta_M)$ .*

*Proof.* By Lemma 3.6 (i),  $W(\Delta_M)$  is a finite union of subspaces. Let  $V \subset W(\Delta_M)$  be one of these subspaces. Then, for each  $\Sigma \in \Delta_M^{(3)}$ , we have opposite edge pairs  $\{e, e'\}$  and  $\{f, f'\}$  such that (3.3) holds, or, equivalently, on the subspace  $V$  the equation

$$d(e+e')=d(f+f')$$

is satisfied. Substituting this into the definition of the tetrahedron form  $\Omega_\Sigma$  gives zero. Since  $\Omega_{\Delta_M}$  is the sum of these forms, the subspace  $V$  is isotropic.  $\square$

### 3.6. Construction of the isotropic cone

We now combine the results on triangulations, weight functions, and the symplectic structure of  $\mathcal{MF}(S)$  with the constructions of Proposition 3.2 to prove Theorem 3.1.

*Proof of Theorem 3.1.* Let  $\Upsilon$  be a finite set of maximal, generic train tracks such that any measured foliation on  $S$  is carried by one of them. For each  $\tau \in \Upsilon$ , let  $\Delta_M^\tau$  be an extension of  $\Delta_\tau$  to a triangulation of  $M$ .

Define

$$\mathcal{L}_\tau = i^*(W_4(\Delta_M^\tau)) \cap \mathcal{ML}(\tau),$$

where  $i^*: W(\Delta_M^\tau) \rightarrow W(\Delta_\tau)$  is the restriction map (i.e. the map that restricts a weight to the edges that lie on  $S$ ). That is, an element of  $\mathcal{L}_\tau$  is a measured foliation carried by  $\tau$  whose associated weight function on  $\Delta_\tau$  can be extended to  $\Delta_M^\tau$  in such a way that it satisfies the weak four-point condition in each simplex.

Let  $\mathcal{L}_M = \bigcup_{\tau \in \Upsilon} \mathcal{L}_\tau$ . By Lemmas 3.3 and 3.7, the set  $\mathcal{L}_M$  is an isotropic piecewise linear cone in  $\mathcal{MF}(S)$ . We need only show that for  $\nu$  and  $T_\nu \hookrightarrow T$  as in the statement of the theorem, we have  $[\nu] \in \mathcal{L}_M$ .

Given such  $\nu$  and  $T_\nu \hookrightarrow T$ , let  $\tau \in \Upsilon$  carry  $\nu$  and set  $\Delta_M = \Delta_M^\tau$ . Let  $F: \tilde{\Delta}_M^{(0)} \rightarrow T$  and  $w = w_F \in W(\Delta_M, \Lambda)$  be the map and the associated weight function given by Proposition 3.2. By Lemma 3.6 (iii), we have  $w \in W_4(\Delta_M, \Lambda)$ .

Let  $\varphi: \Lambda \rightarrow \mathbb{R}$  be a left inverse to the inclusion  $\mathfrak{o}: \mathbb{R} \rightarrow \Lambda$  associated with the isometric embedding  $T_\nu \hookrightarrow T$ ; such a map exists by Lemma 2.1. Then  $\varphi_*: W(\Delta_\tau, \Lambda) \rightarrow W(\Delta_\tau)$  is correspondingly a left inverse to  $\mathfrak{o}_*: W(\Delta_\tau) \hookrightarrow W(\Delta_\tau, \Lambda)$ . Since by Proposition 3.2 (ii) we have that  $i^*(w) \in W(\Delta_\tau, \Lambda)$  is the image of  $[\nu] \in \mathcal{ML}(\tau)$  under this inclusion, it follows that  $\varphi_*(i^*(w)) = i^*(\varphi_*(w))$  also represents  $[\nu]$ .

By Lemma 3.6 (ii) we have  $\varphi_*(w) \in W_4(\Delta_M)$ , so we have shown that

$$[\nu] \in i^*(W_4(\Delta_M^\tau)) \cap \mathcal{ML}(\tau) = \mathcal{L}_\tau \subset \mathcal{L}_M,$$

as desired.  $\square$

#### 4. The isotropic cone: Straight maps and length functions

In this section we introduce refinements of Theorem 3.1 that will be used in the proof of the main theorem. These refinements replace the isometric embedding hypothesis of Theorem 3.1 with weaker conditions relating the trees carrying actions of  $\pi_1 S$  and  $\pi_1 M$ .

##### 4.1. Straight maps

We first recall (and generalize) the notion of a straight map, which is a certain type of morphism of trees.

Let  $X \in \mathcal{T}(S)$  be a marked Riemann surface structure on  $S$  and  $\phi \in Q(X)$  be a holomorphic quadratic differential. Recall that there is a dual  $\mathbb{R}$ -tree  $T_\phi$  and projection  $\pi: \tilde{X} \rightarrow T_\phi$ , and that non-singular  $|\phi|$ -geodesics in  $\tilde{X}$  project to geodesics in  $T_\phi$ . Let  $\mathcal{S}_\phi$  denote the set of all geodesics in  $T_\phi$  that arise in this way (including both segments and complete geodesics).

Let  $T$  be an  $\mathbb{R}$ -tree. Following [D], we say that a map  $f: T_\phi \rightarrow T$  is *straight* if it is an isometric embedding when restricted to any element of  $\mathcal{S}_\phi$ . Thus, for example, an isometric embedding of  $T_\phi$  is a straight map, but the converse does not hold (see e.g. [D, Lemma 6.5]).

Note that straightness of a map  $T_\phi \rightarrow T$  depends on the differential  $\phi$  and not just on the isometry type of the dual tree; Figure 3 shows an example of differentials with isometric dual trees but distinct notions of straightness.

More generally, if  $T$  is a  $\Lambda$ -tree, we say that a map  $f: T_\phi \rightarrow T$  is straight if there is an order-preserving map  $\sigma: \mathbb{R} \rightarrow \Lambda$  such that the restriction of  $f$  to each element of  $\mathcal{S}_\phi$  is an isometric embedding with respect to  $\sigma$ . As in the case of  $\mathbb{R}$ -trees, isometric embeddings (now in the sense of §2.3) are examples of straight maps.

For the degenerate case  $\phi=0$ , we make the convention that any map of the point  $T_0$  to a  $\Lambda$ -tree is straight.

##### 4.2. Isotropic cone for straight maps

In the following generalization of Theorem 3.1, we fix a Riemann surface structure on  $S = \partial M$  and consider straight maps instead of isometric embeddings.

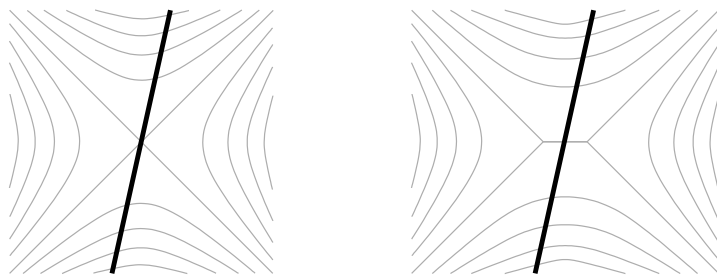


Figure 3. Quadratic differentials with isometric dual trees may induce different notions of straight mapping: The local foliation pictures shown here have isometric leaf spaces, but the indicated path is required to map isometrically by a straight map in one case (right) but not in the other (left).

**THEOREM 4.1.** *Let  $M$  be an oriented 3-manifold with connected boundary  $S$ , and let  $X \in \mathcal{J}(S)$  be a marked Riemann surface structure on  $S$ . There exists an isotropic cone  $\mathcal{L}_{M,X} \subset \mathcal{MF}(S)$  with the following property: If  $\phi \in Q(X)$  is a holomorphic quadratic differential such that there exist a  $\Lambda$ -tree  $T$  equipped with an isometric action of  $\pi_1 M$  and a  $\pi_1 S$ -equivariant straight map*

$$h: T_\phi \longrightarrow T,$$

then  $[\mathcal{F}(\phi)] \in \mathcal{L}_{M,X}$ .

In the proof of Theorem 3.1, the assumption that the map  $T_\phi \rightarrow T$  is an isometric embedding was only used through its role in the construction of §3.2: A train track carrying  $\nu$  gives a map  $f: \tilde{\Delta}_\tau^{(0)} \rightarrow T_\nu$  whose associated weight function represents  $[\nu]$ , and since  $h: T_\nu \rightarrow T$  is an isometric embedding, the composition  $h \circ f$  has the same associated weight.

Attempting to reproduce this with the weaker hypotheses of Theorem 4.1, we can again choose a train track  $\tau$  carrying  $\mathcal{F}(\phi)$  and construct a map  $f: \tilde{\Delta}_\tau^{(0)} \rightarrow T_\phi$ . We would then like to compose  $f$  with the straight map  $h: T_\phi \rightarrow T$  without changing the associated weight function. This will hold if the segments in  $T_\phi$  corresponding to the ties of  $\tau$  are mapped isometrically by  $h$ , so it is enough to know that they correspond to non-singular  $|\phi|$ -geodesic segments in  $\tilde{X}$ . To summarize, we have the following result.

**PROPOSITION 4.2.** *Let  $\tau$  be a train track that carries  $\mathcal{F}(\phi)$  such that each tie of  $\tilde{\tau}$  corresponds to a non-singular  $|\phi|$ -geodesic segment in  $\tilde{X}$ . Let  $\Delta_M$  be a triangulation of  $M$  extending the dual triangulation of  $\tau$ . Suppose that there exist a  $\Lambda$ -tree  $T$  equipped with an isometric action of  $\pi_1 M$  and a  $\pi_1 S$ -equivariant straight map*

$$h: T_\phi \longrightarrow T,$$



relative to an order-preserving embedding  $\sigma: \mathbb{R} \rightarrow \Lambda$ . Then there exists a weight function  $w \in W(\Delta_M, \Lambda)$  satisfying conditions (i) and (ii) of Proposition 3.2.

Therefore, while we used an arbitrary finite collection of train-track charts covering  $\mathcal{MF}(S)$  in the previous section, we now have a stronger condition that the carrying train track must satisfy. The existence of a suitable finite collection of train tracks that cover  $Q(X)$  is given by the following lemma.

LEMMA 4.3. *For each non-zero  $\phi \in Q(X)$  there exist a triangulation  $\Delta$  of  $X$  and a maximal train track  $\tau$  such that*

- (i) *the vertices  $\Delta$  are zeros of  $\phi$ , the edges are saddle connections of  $\phi$ , and the triangulation  $\Delta$  is dual to the train track  $\tau$  in the sense of §3.1;*
- (ii) *the foliation  $\mathcal{F}(\phi)$  is carried by  $\tau$  in such a way each edge  $e$  of  $\Delta$  becomes a tie of the corresponding edge of  $\tau$ ; in particular,*
- (iii) *the  $\phi$ -heights of the edges of  $\Delta$  give the weight function on  $\tau$  representing  $[\mathcal{F}(\phi)]$ .*

*Furthermore, there is a finite set of pairs  $(\Delta, \tau)$  such that the triangulation and train track constructed above can always be chosen to be isotopic to an element of this set.*

*Proof.* Let  $\Delta$  be a Delaunay triangulation of the singular Euclidean surface  $(X, |\phi|)$  with vertices at the zeros of  $\phi$ , as in [MaS, §4]. Such a triangulation has non-singular  $|\phi|$ -geodesic segments as edges and is defined by the condition that each triangle has a circumcircle (with respect to the singular Euclidean structure  $|\phi|$ ) which is “empty”, i.e. has no zeros of  $\phi$  in its interior. There are only finitely many Delaunay triangulations of a given singular Euclidean surface, and for generic  $X$  and  $\phi$  there is a unique one.

First suppose that this triangulation  $\Delta$  has no horizontal edges. Each triangle has two “vertically short” edges whose heights sum to that of the third edge, and we construct a train track  $\tau$  by placing a switch in each triangle so that the incoming branches at the switch are dual to the short edges of the triangle (as shown in Figure 4). The complementary regions of  $\tau$  are disk neighborhoods of the vertices of the triangulation, so  $\tau$  is maximal. Thus  $\Delta$  and  $\tau$  satisfy condition (i).

After cutting along leaf segments near singularities, an isotopy pushes the leaves of  $\mathcal{F}(\phi)$  into a small neighborhood of the train track. Throughout this isotopy the image of an edge  $e$  of the triangulation in the dual tree remains the same, and so it corresponds to the tie  $r_e$  of the train track. The length of the image segment in  $T_\phi$  is the height of the geodesic edge, so properties (ii) and (iii) follow.

It remains to consider the possibility that the Delaunay triangulation has horizontal edges. In this case we can still form a dual train track but it is not clear whether the dual to a horizontal edge should be incoming or outgoing at the switch in a given triangle. To

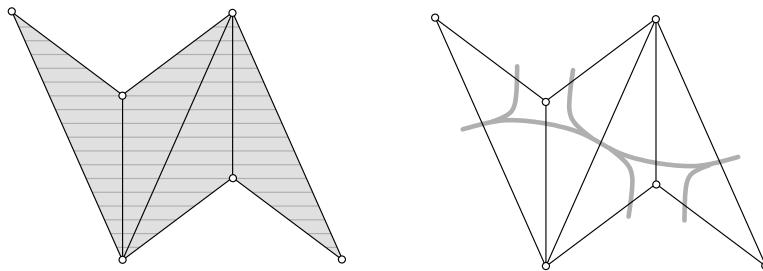


Figure 4. A geodesic triangulation for a quadratic differential with vertices at the zeros and the associated train track carrying the measured foliation.

determine this, we consider a slight deformation of  $\phi$  to a quadratic differential  $\phi'$  with the same zero structure but no horizontal saddle connections. (A generic deformation preserving the multiplicities of zeros will have this property.) For a small enough deformation, the same combinatorial triangulation can be realized geodesically for  $\phi'$ , and the heights of the previously horizontal edges determine how to form switches for  $\tau$ .

Finally we show that only finitely many isotopy classes of pairs  $(\Delta, \tau)$  arise from this construction. In fact, it suffices to consider  $\Delta$  alone since filling in the train track  $\tau$  involves only finitely many choices (incoming and outgoing edges for each switch).

The construction of  $\Delta$  is independent of scaling  $\phi$ , so we assume that  $|\phi|$  has unit area, i.e.  $\|\phi\|=1$ , where  $\|\cdot\|$  is the  $L^1$  norm. The resulting family of metrics (the unit sphere in  $Q(X)$ ) is compact, and in particular the diameters of these spaces are uniformly bounded. By [MaS, Theorem 4.4], this diameter bound also gives an upper bound,  $R$ , on the length of each edge of the Delaunay triangulation. The number of zeros of  $\tilde{\phi}$  in a ball of  $|\phi|$ -radius  $R$  in  $\tilde{X}$  is uniformly bounded (again by compactness of the family of metrics, and the fixed number of zeros of  $\phi$  on  $X$ ), so the edges that appear in  $\Delta$  belong to finitely many isotopy classes of arcs between pairs of zeros. Thus, up to isotopy, only finitely many triangulations can be constructed of these arcs.  $\square$

With these preliminaries in place, it is straightforward to generalize the proof of Theorem 3.1.

*Proof of Theorem 4.1.* Let  $\Upsilon_X$  denote the finite set of train tracks given by Lemma 4.3, and extend each dual triangulation  $\Delta_\tau$  to a triangulation  $\Delta_M^\tau$  of  $M$ . Define

$$\mathcal{L}_{M,X} = \bigcup_{\tau \in \Upsilon_X} \mathcal{L}_\tau,$$

where  $\mathcal{L}_\tau = i^*(W_4(\Delta_M^\tau)) \cap \mathcal{ML}(\tau)$ . As before, Lemmas 3.3 and 3.7 show that this set is an isotropic cone in  $\mathcal{MF}(S)$ .

If  $\phi \in Q(X)$  and  $h: T_\phi \rightarrow T$  is a straight map as in the statement of the theorem, then by Lemma 4.3 and Proposition 4.2 we have a train track  $\tau \in \Upsilon_X$  and a weight function

$w \in W(\Delta_M^\tau, \Lambda)$  such that  $i^*(\varphi_*(w)) \in W(\Delta_\tau)$  represents  $[\mathcal{F}(\phi)]$ . Here we retain the notation of the previous section, i.e.  $\varphi$  denotes a left inverse of  $\mathfrak{o}: \mathbb{R} \rightarrow \Lambda$  and

$$i^*: W(\Delta_M^\tau) \longrightarrow W(\Delta_\tau)$$

restricts a weight function to the boundary triangulation.

By Proposition 4.2, the weight  $w$  is associated with a map  $\tilde{\Delta}_M^{\tau(0)} \rightarrow T$ . As in the proof of Theorem 3.1, this implies  $w \in W_4(\Delta_M^\tau, \Lambda)$  and therefore  $i^*(\varphi_*(w)) \in \mathcal{L}_\tau$ . We conclude that  $[\mathcal{F}(\phi)] \in \mathcal{L}_{M,X}$ .  $\square$

### 4.3. Isotropic cone for length functions

In this section we introduce a further refinement to the isotropic cone construction that addresses special properties of abelian actions of groups on  $\mathbb{R}$ -trees (which are described below).

Keeping the notation of the previous section for  $M$ ,  $S$ , and  $X$ , suppose that we have  $\phi \in Q(X)$  and a  $\pi_1 S$ -equivariant straight map  $h: T_\phi \rightarrow T$  as in Theorem 4.1. Then the image  $h(T_\phi) \subset T$  is naturally an  $\mathbb{R}$ -tree preserved by  $\pi_1 S$ . Let  $\ell: \pi_1 S \rightarrow \mathbb{R}$  denote the translation-length function of this action and write  $T_\ell = h(T_\phi)$ . Then  $T_\ell$  is the intermediate step in a factorization of  $h$  as a straight map followed by an isometric embedding:

$$T_\phi \xrightarrow{\text{straight}} T_\ell \hookrightarrow T.$$

Theorem 4.1 shows that this situation forces  $[\mathcal{F}(\phi)]$  to lie in an isotropic cone.

The generalization we now consider is to replace  $T_\ell$  with a pair of trees  $T_\ell, T'_\ell$  on which  $\pi_1 S$  acts with the same length function  $\ell$ —we say that these actions are *isospectral*. We suppose that one of these is the image of a straight map while the other isometrically embeds in a  $\Lambda$ -tree  $T$  with a  $\pi_1 M$  action. From this weaker connection between  $T_\phi$  and  $T$ , i.e.

$$T_\phi \xrightarrow{\text{straight}} T_\ell \overset{\text{isospectral}}{\dashrightarrow} T'_\ell \hookrightarrow T,$$

we can still conclude that  $[\mathcal{F}(\phi)] \in \mathcal{L}_{M,X}$ . The following theorem makes this precise.

**THEOREM 4.4.** *Let  $T$  be a  $\Lambda$ -tree on which  $\pi_1 M$  acts. Let  $T_\ell$  and  $T'_\ell$  be  $\mathbb{R}$ -trees on which  $\pi_1 S$  acts minimally with length function  $\ell$ . Let  $\phi \in Q(X)$  be a holomorphic quadratic differential such that there exist a  $\pi_1 S$ -equivariant straight map  $h: T_\phi \rightarrow T_\ell$  and a  $\pi_1 S$ -equivariant isometric embedding  $k: T'_\ell \rightarrow T$ . Then  $[\mathcal{F}(\phi)] \in \mathcal{L}_{M,X}$ .*

Evidently this theorem would follow directly from Theorem 4.1 if the isospectrality condition implied the existence of an isometry  $T_\ell \simeq T'_\ell$ , for this isometry would allow  $h$

and  $k$  to be composed, giving a straight map  $T_\phi \rightarrow T$ . This approach works for some length functions but not for others, so before giving the proof we discuss the relevant dichotomy.

#### 4.4. Abelian and non-abelian actions

Recall that an isometric action of a group  $\Gamma$  on an  $\mathbb{R}$ -tree is called *abelian* if the associated length function has the form  $\ell(g) = |\chi(g)|$ , where  $\chi: \Gamma \rightarrow \mathbb{R}$  is a homomorphism; otherwise, the action (or length function) is called *non-abelian*. We have the following fundamental result of Culler and Morgan.

**THEOREM 4.5.** ([CM]) *Let  $T_\ell$  and  $T'_\ell$  be  $\mathbb{R}$ -trees equipped with minimal, isospectral actions of a group  $\Gamma$ . If the length function  $\ell$  is non-abelian, then there is an equivariant isometry  $T_\ell \rightarrow T'_\ell$ .*

As remarked above, this shows that the conclusion of Theorem 4.4 follows from Theorem 4.1 whenever the length function is non-abelian. Thus we assume from now on that  $\ell = |\chi|: \pi_1 S \rightarrow \mathbb{R}$  is an abelian length function. In this case, there may be many non-isometric trees on which  $\pi_1 S$  acts with this length function [Br].

An *end* of an  $\mathbb{R}$ -tree is an equivalence class of rays, where two rays are equivalent if their intersection is a ray. An abelian action of  $\pi_1 S$  on an  $\mathbb{R}$ -tree  $T_\ell$  has a fixed end (see [CM, Corollary 2.3] or [AB, Theorem 7.5]). The fixed end has an associated *Busemann function*  $\beta: T_\ell \rightarrow \mathbb{R}$ , that intertwines the action of  $\pi_1 S$  on  $T_\ell$  with the translation action on  $\mathbb{R}$  induced by  $\chi$  [AB, Theorem 7.6]. Furthermore the function  $\beta$  is unique up to adding a constant. Here we use the term ‘‘Busemann function’’ following e.g. [Le2, §2], which is consistent with its use in the theory of metric spaces of non-positive curvature [BH, §II.8]; the same object is called an *end map* and discussed in [Chis, §2.3], while the above-cited result in [AB] simply calls the map  $\alpha$ .

The following result from [D, §6] shows that composition with a Busemann function preserves straightness of maps from  $T_\phi$ .

**LEMMA 4.6.** *Let  $T_\ell$  be an  $\mathbb{R}$ -tree equipped with an abelian action of  $\pi_1 S$  by isometries, and let  $\beta: T \rightarrow \mathbb{R}$  denote a Busemann function of a fixed end. If  $h: T_\phi \rightarrow T_\ell$  is an equivariant straight map, then  $\beta \circ h: T_\phi \rightarrow \mathbb{R}$  is also straight.*

Effectively this result will allow us to replace  $T_\ell$  with  $\mathbb{R}$  in the hypotheses of Theorem 4.4, since any straight map can be composed with a Busemann function, preserving straightness and without changing the length function.

In the proof of Theorem 4.1, the straightness of  $h: T_\phi \rightarrow T$  was only used to conclude that the map is isometric when applied to the endpoints of each segment in  $T_\phi$  that corre-

sponds to one of the  $\phi$ -geodesic edges of a triangulation of  $X$  (furnished by Lemma 4.3). To generalize the proof to the situation of Theorem 4.4, it will therefore suffice to show that if there exists a straight map  $T_\phi \rightarrow T_\ell$ , then there also exists a partially defined map  $T_\phi \dashrightarrow T'_\ell$  that is “locally straight” in that it is an isometry when applied to the endpoints of any of these segments. Since these segments arise from lifting the finite set of edges of a triangulation of  $X$ , they lie in finitely many  $\pi_1 S$ -equivalence classes. Thus, Theorem 4.4 is reduced to the following.

**THEOREM 4.7.** *Let  $T_\ell$  and  $T'_\ell$  be  $\mathbb{R}$ -trees on which  $\pi_1 S$  acts minimally and isospectrally, with abelian length function  $\ell$ , and suppose that  $h: T_\phi \rightarrow T_\ell$  is an equivariant straight map, for some  $\phi \in Q(X)$ .*

*Let  $\mathcal{S} \subset \mathcal{S}_\phi$  be a set of segments in  $T_\phi$  that arise from non-singular  $\phi$ -geodesic segments in  $\tilde{X}$  and suppose that  $\mathcal{S}$  contains only finitely many  $\pi_1 S$ -equivalence classes. Let  $E \subset T_\phi$  be the set of endpoints of elements of  $\mathcal{S}$ .*

*Then there exists an equivariant map  $h': E \rightarrow T'_\ell$  such that for any segment  $J \in \mathcal{S}$  with endpoints  $x$  and  $y$ , we have*

$$d(h(x), h(y)) = d(h'(x), h'(y)).$$

The proof will depend on properties of a certain endomorphism of the tree  $T'_\ell$  related to the end fixed by  $\pi_1 S$ .

Given an  $\mathbb{R}$ -tree  $T$  and an end  $e$ , for any  $x \in T$  and  $s \geq 0$  let  $P_s(x)$  denote the point on the ray from  $x$  to  $e$  such that  $d(x, P_s(x)) = s$ . Then  $P_s: T \rightarrow T$  is a weakly contracting map, and if  $\pi_1 S$  acts on  $T$  fixing  $e$ , then  $P_s$  is  $\pi_1 S$ -equivariant. We call  $P_s$  the *pushing map* of distance  $s$  for the end  $e$ .

**LEMMA 4.8.** *Let  $T$  be an  $\mathbb{R}$ -tree and let  $\beta: T \rightarrow \mathbb{R}$  denote a Busemann function of an end  $e$ . Then for any  $p, q \in T$  there exists  $s_0 \geq 0$  such that  $d(P_s(p), P_s(q)) = |\beta(p) - \beta(q)|$  for all  $s \geq s_0$ , where  $P_s: T \rightarrow T$  is the pushing map for the end  $e$ .*

*Proof.* The ray from  $p$  to  $e$  and the ray from  $q$  to  $e$  overlap in a ray from  $o$  to  $e$ , where  $o$  is a point on the geodesic segment from  $p$  to  $q$ , and the Busemann function satisfies  $|\beta(p) - \beta(q)| = |d(p, o) - d(q, o)|$ . Let  $r(t)$  parameterize the ray from  $o$  to  $e$ ,  $t \geq 0$ . Then, for  $s \geq \max\{d(p, o), d(q, o)\}$ , we have  $P_s(p) = r(s - d(p, o))$  and  $P_s(q) = r(s - d(q, o))$ . Since  $r$  is an isometry onto its image, we have  $d(P_s(p), P_s(q)) = |d(p, o) - d(q, o)|$ .  $\square$

Using the pushing map, we can now give the following.

*Proof of Theorem 4.7.* Enlarging  $\mathcal{S}$  if necessary, we can take this set and its set of endpoints  $E$  to be  $\pi_1 S$ -invariant.

Let  $E_0 \subset E$  be a finite subset containing exactly one point from each  $\pi_1 S$ -orbit in  $E$ . Let  $\beta$  and  $\beta'$  be Busemann functions of the fixed ends of  $\pi_1 S$  acting on  $T_\ell$  and  $T'_\ell$ , respectively. For each  $x \in E_0$ , choose a point  $g'(x) \in T'_\ell$  such that  $\beta'(g'(x)) = \beta(h(x))$ , giving a map  $g': E_0 \rightarrow T'_\ell$ . This is possible since the map  $\beta': T' \rightarrow \mathbb{R}$  admits a section, e.g. any complete geodesic  $\mathbb{R} \rightarrow T'$  that extends a ray representing the fixed end.

Using the action of  $\pi_1 S$  on  $T_\ell$ , we extend  $g'$  to an equivariant map  $g': E \rightarrow T'$  which then satisfies  $\beta'(g'(x)) = \beta(h(x))$  for all  $x \in E$ .

For any  $s \geq 0$  let  $h'_s(x) = P'_s(g'(x))$ , where  $P'_s: T'_\ell \rightarrow T'_\ell$  is the pushing map for the fixed end of  $\pi_1 S$ . By Lemma 4.6, for any segment  $J \in \mathcal{J}$  with endpoints  $x, y \in E$  we have

$$d(h(x), h(y)) = |\beta(h(x)) - \beta(h(y))| = |\beta'(g'(x)) - \beta'(g'(y))|,$$

and by Lemma 4.8 there exists  $s_J \geq 0$  such that for all  $s \geq s_J$  we have

$$d(h'_s(x), h'_s(y)) = |\beta(g'(x)) - \beta(g'(y))| = d(h(x), h(y)).$$

Taking  $s$  larger than the maximum of  $s_J$  as  $J$  ranges over a finite set representing each  $\pi_1 S$ -orbit in  $\mathcal{J}$ , the above condition holds for each such representative, and by equivariance, for each  $J \in \mathcal{J}$ . Then  $h' = h'_s: E \rightarrow T'_\ell$  is the desired map.  $\square$

As remarked above, this completes the proof of Theorem 4.4.

#### 4.5. Application: Floyd's theorem

In this section we discuss some context for Theorems 3.1, 4.1, and 4.4. The contents of this section are not used in the sequel.

Theorem 3.1 and its refinements can be seen as generalizations of the following theorem by Floyd [F].

**THEOREM 4.9.** *Let  $M$  be a compact, irreducible 3-manifold with boundary  $S$ . Then the set of boundary curves of a two-sided incompressible,  $\partial$ -incompressible surface in  $M$  is contained in a finite union of half-dimensional piecewise linear cells in  $\mathcal{MF}(S)$ .*

Note that the correspondence between measured laminations and measured foliations on surfaces (see e.g. [Le1]) allows us to consider the boundary of a surface in  $M$  as an element of  $\mathcal{MF}(S)$ . The original statement in [F] uses the language of measured laminations.

Floyd's theorem answers a question of Hatcher, who established a similar result for manifolds with torus boundary [Hat]. Hatcher's theorem is often used through its corollary that a knot complement manifold has finitely many boundary slopes.

In both cases the half-dimensional set is constructed as an isotropic cone in the symplectic space  $\mathcal{MF}(S)$ , and these results can be compared to the more elementary (co)homological version: As a consequence of Poincaré duality, the image of the connecting map

$$H_2(M, \partial M) \xrightarrow{\delta} H_1(\partial M)$$

is isotropic with respect to the intersection pairing. Dually, the image of the map  $H^1(M) \rightarrow H^1(\partial M)$  induced by inclusion of the boundary is isotropic for the cup product.

To show the connection with our results, we derive Floyd's theorem from Theorem 3.1 under the additional assumption that the boundary  $S$  is incompressible.

*Proof of Theorem 4.9 (incompressible boundary case).* Let  $F$  be an incompressible and  $\partial$ -incompressible surface in  $M$ . The preimage  $\tilde{F}$  of  $F$  in  $\tilde{M}$  is a collection of planes, separating  $\tilde{M}$  into a countable family of complementary regions. The adjacency graph of these regions, with one vertex for each region and one unit-length edge for each plane, gives an  $\mathbb{R}$ -tree (which comes from an underlying  $\mathbb{Z}$ -tree) on which  $\pi_1 M$  acts by isometries. In this tree, the distance between two vertices is the minimum number of intersections between  $\tilde{F}$  and a path between the corresponding complementary regions in  $\tilde{M}$ .

Similarly, the boundary curves  $\partial F$  lift to a collection of lines separating  $\tilde{S}$  and give a dual tree  $T_{\partial F}$  on which  $\pi_1 S$  acts by isometries. The equivalence between laminations and foliations allows us to identify  $\partial F$  with a measured foliation class in  $\mathcal{MF}(S)$ ; under this correspondence,  $T_{\partial F}$  becomes the dual tree of that measured foliation (in the sense of §2.7).

Since the boundary is incompressible, the inclusion  $S \hookrightarrow M$  lifts to  $\tilde{S} \rightarrow \tilde{M}$  which induces a map  $T_{\partial F} \rightarrow T_F$ . This map of trees is an isometric embedding: It is weakly contracting, since minimizing the number of intersections of a path in  $\tilde{S}$  with  $\partial F$  is a more constrained problem than minimizing intersections of a path in  $\tilde{M}$  with  $\tilde{F}$ . However, if an isotopy of such a path in  $\tilde{M}$  were to decrease the number of intersections with  $\tilde{F}$  (i.e. if the map  $T_{\partial F} \rightarrow T_F$  strictly contracted any distance), then putting the isotopy in general position relative to  $\tilde{F}$  would reveal a boundary compression of  $F$ . Since  $F$  is  $\partial$ -incompressible, this is a contradiction.

Applying Theorem 3.1 to  $T_{\partial F} \rightarrow T_F$ , we conclude that  $\partial F \in \mathcal{L}_M$ . Since  $\mathcal{L}_M$  is an isotropic piecewise linear cone, the desired conclusion follows.  $\square$

Comparing Floyd and Hatcher's proofs with that of Theorem 3.1 shows that the same "cancellation" phenomenon is at work in both cases. Briefly, the connection is as follows. Floyd and Hatcher analyze weight functions on *branched surfaces* that carry all of the incompressible,  $\partial$ -incompressible surfaces in  $M$ . Weights on a branched surface satisfy a linear condition at each singular vertex. When the Thurston form is applied to a

pair of weights on the boundary train track of a branched surface, these vertex conditions lead to pairwise cancellation of terms in the Thurston form, giving an isotropic space of boundary weights.

The finite set of branched surfaces that is used in this argument comes from a construction by Floyd–Oertel [FO], which is based on normal surface theory and a triangulation of the 3-manifold. In this way, the weight conditions at the singular vertices of a branched surface are dual to the weak four-point condition (3.2) in each 3-simplex that defines the cone  $\mathcal{L}_M$  in our approach, and the role of the spaces  $W_4(\Delta_M)$  in the proof of Theorem 3.1 is analogous to that of the space of boundary weights of a branched surface in the arguments of Floyd and Hatcher.

## 5. The Kähler structure of $Q(X)$

The goal of this section is to introduce a Kähler metric on  $Q(X)$  and then to show that the foliation map  $\mathcal{F}: Q(X) \rightarrow \mathcal{MF}(S)$  identifies the underlying symplectic space with the Thurston symplectic structure on  $\mathcal{MF}(S)$ . The Kähler metric we construct has singularities but we show that it is smooth relative to a stratification of  $Q(X)$ .

### 5.1. The stratification

Let  $Z$  be a manifold. A *stratification* of  $Z$  is a locally finite collection of locally closed submanifolds  $\{Z_j: j \in I\}$  of  $Z$ , the *strata*, indexed by a set  $I$  such that

- (1)  $Z = \bigcup_{j \in I} Z_j$ ,
- (2)  $Z_j \cap \bar{Z}_k \neq \emptyset$  if and only if  $Z_j \subset \bar{Z}_k$ .

These conditions induce a partial order on  $I$ , where  $j \leq k$  if  $Z_j \subset \bar{Z}_k$ . A stratification of a complex manifold  $Z$  is a *complex-analytic stratification* if the closure and boundary of each stratum (i.e.  $\bar{Z}_j$  and  $\bar{Z}_j \setminus Z_j$ ) are complex-analytic sets.

Let  $\mathcal{Q}(S)$  denote the space of holomorphic quadratic differentials on marked Riemann surfaces diffeomorphic to  $S$ , i.e. the set of all pairs  $(X, \phi)$ , where  $X \in \mathcal{T}(S)$  and  $\phi \in \mathcal{Q}(X)$ . This space is a vector bundle over  $\mathcal{T}(S)$  isomorphic to the cotangent bundle  $T^*\mathcal{T}(S)$ . Let  $s_0: \mathcal{T}(S) \rightarrow \mathcal{Q}(S)$  denote the zero section.

There is a natural complex-analytic stratification of  $\mathcal{Q}(S)$  according to the numbers and types of zeros of the quadratic differential (see [V2] and [MaS]). Specifically, let the *symbol* of a non-zero quadratic differential  $\phi$  be the pair  $(\mathbf{n}, \varepsilon)$ , where  $\mathbf{n} = (n_1, \dots, n_k)$  is the list of multiplicities (in weakly decreasing order) of the zeros of  $\phi$ , and where  $\varepsilon = \pm 1$  according to whether  $\phi$  is the square of a holomorphic 1-form ( $\varepsilon = 1$ ) or not ( $\varepsilon = -1$ ).



Thus we have  $\sum_{j=1}^k n_j = 4g - 4$  and there are finitely many possible symbols; we denote the set of all such symbols by  $\mathcal{S}$ .

Given  $\pi \in \mathcal{S}$ , let  $\mathcal{Q}(S, \pi)$  denote the set of quadratic differentials with symbol  $\pi$ . This set is a manifold, with local charts described below (in §5.4). The stratification of  $\mathcal{Q}(S)$  is formed by the sets  $\mathcal{Q}(S, \pi)$  and the zero section  $s_0(\mathcal{T}(S))$ .

There is a related stratification of a fiber  $Q(X)$  with the following properties.

LEMMA 5.1. *For each  $X \in \mathcal{T}(S)$ , there is a complex-analytic stratification  $\{Q_j(X)\}$  of  $Q(X)$  such that*

- (i) *each stratum is a connected and  $\mathbb{C}^*$ -invariant;*
- (ii) *the symbol is constant on each stratum  $Q_j(X)$ ;*
- (iii) *if  $q \in Q_j(X)$  and  $v \in T_q Q_j(X)$ , then the meromorphic function  $v/q$  has at most simple poles.*

*Proof.* A complex-analytic stratification can always be refined so that a given complex-analytic subset becomes a union of strata (see [W] and [GM, p. 43, Theorem 1.6]), and a further refinement can be taken so that the strata are connected. Here *refinement* refers to changing the stratification in such a way that each new stratum is entirely contained in one of the old strata.

Applying this to the stratification of  $\mathcal{Q}(S)$  discussed above and the closed subvariety  $Q(X)$ , we obtain a stratification of  $\mathcal{Q}(S)$  such that the symbol is constant on each stratum and so that  $Q(X)$  is a union of strata. In particular there is an induced stratification  $\{Q_j(X)\}$  of  $Q(X)$  satisfying (ii). The original stratification of  $\mathcal{Q}(S)$  is  $\mathbb{C}^*$ -invariant, and the strata of the refinement can be constructed using finitely many operations that preserve this invariance (i.e. boolean operations and passage from a complex-analytic set to its singular locus or to an irreducible component), so property (i) also follows.

Thus the proof is completed by the following lemma, which shows that property (iii) is a consequence of property (ii).  $\square$

LEMMA 5.2. *Let  $M \subset Q(X)$  be a submanifold on which the symbol is constant. Then for any  $(q, \dot{q}) \in TM$ , the function  $\dot{q}/q$  has at most simple poles on  $X$ .*

*Proof.* Let  $q_t$  be a smooth family of quadratic differentials in  $M$  with  $q_0 = q$  and with tangent vector  $\dot{q}$  at  $t=0$ .

Let  $p \in X$  be a zero of  $q$  of order  $k > 0$ , and choose a local coordinate  $z$  in which  $z(p) = 0$  and  $q = z^k dz^2$ . Since  $q_t$  has the same symbol as  $q$  for small  $t$ , in a neighborhood of  $p$  we can write

$$q_t = \alpha_t^*(z^k dz^2),$$

where  $\alpha_t$  is a smooth family of holomorphic functions defined on the set  $\{z: |z| < \varepsilon\}$  and  $\alpha_0(z) = z$ . This is equivalent to the statement that the family of polynomial differen-

tials  $(z^k + a_{k-2}z^{k-2} + \dots + a_0)dz^2$  is a universal deformation of  $z^k dz^2$  (see [HM, Proposition 3.1]). Since  $\alpha_t^*(z^k dz^2) = \alpha_t(z)^k (\alpha_t'(z))^2 dz^2$ , a calculation gives

$$\dot{q} = z^{k-1}(k\dot{\alpha} + 2z\dot{\alpha}')dz^2,$$

and  $\dot{q}$  has a zero of order at least  $k-1$  at  $p$ . It follows that  $\dot{q}/q$  has at most simple poles.  $\square$

## 5.2. The Kähler form

The vector space  $Q(X)$  is a complex manifold with a global parallelization which identifies  $T_\phi Q(X) \simeq Q(X)$  for any  $\phi \in Q(X)$ . We consider the hermitian pairing  $\langle \cdot, \cdot \rangle_\phi$  on  $T_\phi Q(X)$  defined by

$$\langle \psi_1, \psi_2 \rangle_\phi := \int_X \frac{\psi_1 \bar{\psi}_2}{4|\phi|}. \quad (5.1)$$

Note that in this expression we consider  $\psi_1 \bar{\psi}_2 / |\phi|$  as a complex-valued quadratic form on  $TX$ , and we integrate the corresponding complexified volume form. With respect to a local choice of a holomorphic 1-form  $\sqrt{\phi}$ , the integrand can also be written as a

$$\frac{i}{2} \left( \frac{\psi_1}{2\sqrt{\phi}} \wedge \frac{\bar{\psi}_2}{2\sqrt{\phi}} \right).$$

A branched double covering of  $X$  can be used to globalize this interpretation (as in the proof of Theorem 5.8 below).

Let  $g_\phi$  and  $\omega_\phi$  be the real and imaginary parts of this hermitian pairing, i.e.

$$\langle \psi_1, \psi_2 \rangle_\phi = g_\phi(\psi_1, \psi_2) + i\omega_\phi(\psi_1, \psi_2).$$

Similarly, we write  $\|\psi\|_\phi^2 = g_\phi(\psi, \psi) = \langle \psi, \psi \rangle_\phi$ .

The pairing is not defined for all vectors because the function  $\psi_1 \bar{\psi}_2 / |\phi|$  is not necessarily integrable on  $X$ . However, it is defined on the strata  $Q_j(X)$ , as we show in the following result.

**THEOREM 5.3.** *For each stratum  $Q_j(X) \subset Q(X)$  we have*

(i) *The pairing  $\langle \psi_1, \psi_2 \rangle_\phi$  is well defined and positive definite on the tangent bundle  $TQ_j(X)$ ;*

(ii) *The alternating form  $\omega_\phi$  on  $TQ_j(X)$  can be expressed as*

$$\omega_\phi = \frac{1}{2} i\partial\bar{\partial}N,$$

where  $N: Q(X) \rightarrow \mathbb{R}$  is defined by  $N(\phi) = \|\phi\|$ . In particular,  $\omega_\phi$  is closed; and thus

(iii) *The hermitian form  $\langle \cdot, \cdot \rangle_\phi$  defines a Kähler structure on  $Q_j(X)$ .*

A formula for the second derivatives of  $N$  equivalent to (ii) above was derived by Royden (see [R, Lem. 1]) in the case of the open stratum consisting of differentials with at most simple zeros. Royden also analyzes the failure of  $N$  to be twice differentiable in certain directions transverse to the other strata, however, the fact that  $N$  is  $C^2$  when restricted to a stratum and the analogous derivative formula, follow easily by similar methods. We describe the necessary adaptation of his argument below.

*Proof.* The function  $|z|^{-1}$  is integrable in a neighborhood of 0 in  $\mathbb{C}$ , so if  $\psi_1/\phi$  has at most simple poles, then  $\langle \psi_1, \psi_2 \rangle_\phi$  is finite for all  $\psi_2 \in Q(X)$ . By part (iii) of Lemma 5.1, this holds for all  $(\phi, \psi_1) \in TQ_j(X)$ , so the pairing is well defined there. For any  $\psi \neq 0$ , the function  $|\psi|^2/|\phi|$  is positive except for finitely many zeros, and thus

$$\|\psi\|_\phi^2 = \int_X \frac{|\psi|^2}{4|\phi|} > 0.$$

Now we consider the existence of derivatives of the function  $N(\phi) = \int_X |\phi|$ . The lack of smoothness of  $|\phi|$  at the zeros of  $\phi$  is the only problem: If  $K \subset X$  is compact and contains no zeros of  $\phi$ , then  $\phi \mapsto \int_K |\phi|$  is  $C^\infty$  on a neighborhood of  $\phi$  in  $Q(X)$ , and its derivatives are obtained by differentiating inside the integral. Thus our strategy will be to determine the resulting formula for  $\partial\bar{\partial}N$  away from the zeros, and then show that the zeros give no contribution.

Fix  $\phi \in Q_j(X)$  and for  $\varepsilon > 0$  let  $X_\varepsilon$  denote an open neighborhood of the zero set of  $\phi$  such that each connected component of  $X_\varepsilon$  is a disk containing a single zero of  $\phi$ , and where each such disk admits a local holomorphic coordinate  $z$  in which the restriction of  $\phi$  is identified with  $z^k dz^2$  on the open disk  $\{z: |z| < \varepsilon\} \subset \mathbb{C}$ . Here  $k \in \mathbb{N}$  depends on the component (and is equal the multiplicity of the zero of  $\phi$  it contains). Such standard disk neighborhoods exist for all  $\varepsilon$  sufficiently small.

Then we can write  $N(\phi) = N_0^\varepsilon(\phi) + N_1^\varepsilon(\phi)$ , where

$$N_0^\varepsilon(\phi) = \int_{X_\varepsilon} |\phi| \quad \text{and} \quad N_1^\varepsilon(\phi) = \int_{X \setminus X_\varepsilon} |\phi|.$$

As explained above, the function  $N_1^\varepsilon(\phi)$  is smooth on a neighborhood of  $\phi$  in  $Q(X)$ , so it restricts to a smooth function on  $Q_j(X)$ . From now on, we consider  $N_0^\varepsilon$  and  $N_1^\varepsilon$  as functions on  $Q_j(X)$ . We claim that

(I)  $N_0^\varepsilon$  is smooth on a neighborhood of  $\phi$  in  $Q_j(X)$ ,

and that at the point  $\phi$  we have

(II)  $\partial\bar{\partial}N_1^\varepsilon \rightarrow -2i\omega_\phi$  as  $\varepsilon \rightarrow 0$ , and

(III)  $\partial\bar{\partial}N_0^\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Since (I) allows the expression  $\partial\bar{\partial}N = \partial\bar{\partial}N_1^\varepsilon + \partial\bar{\partial}N_0^\varepsilon$ , taking  $\varepsilon \rightarrow 0$  and using (II) and (III) gives the desired formula  $\frac{1}{2}i\partial\bar{\partial}N = \omega_\phi$ . Thus the theorem is reduced to these claims.

We begin with (II), which amounts to differentiating inside the integral defining  $N_1^\varepsilon$ . Let  $a: U \rightarrow Q_j(X)$  be a local holomorphic parametrization, where  $U \subset \mathbb{C}^m$  is a neighborhood of the origin and  $a(0) = \phi$ . Consider the pointwise norm  $|a(\zeta)|$  as a function of  $\zeta$ . Differentiating  $|a| = \sqrt{a\bar{a}}$ , we find that

$$\frac{\partial^2 |a|}{\partial \zeta_k \partial \bar{\zeta}_l}(0) = \frac{a_k \bar{a}_l}{4|\phi|},$$

where for brevity we have written  $a_k$  for

$$\frac{\partial a}{\partial \zeta_k}(0),$$

and this formula is valid at any point where  $\phi \neq 0$ . Since the vectors  $a_k$  form a basis for  $T_\phi Q_j(X)$ , this is equivalent to the statement that, for all  $\psi_1, \psi_2 \in T_\phi Q_j(X)$ , we have

$$\partial\bar{\partial}n(\psi_1, \psi_2) = -2i \frac{1}{4|\phi(z)|} \operatorname{Im}(\psi_1(z)\bar{\psi}_2(z)),$$

where  $n(\psi) = |\psi(z)|$  and  $z$  is any point with  $\phi(z) \neq 0$ . Integrating over the set  $X \setminus X_\varepsilon$ , which is compact and which does not contain any zeros of  $\phi$ , we have the corresponding expression

$$\partial\bar{\partial}N_1^\varepsilon(\psi_1, \psi_2) = -2i \int_{X \setminus X_\varepsilon} \frac{\operatorname{Im}(\psi_1 \bar{\psi}_2)}{4|\phi|}.$$

As  $\psi_1, \psi_2 \in T_\phi Q_j$ , the form  $\operatorname{Im}(\psi_1 \bar{\psi}_2)/4|\phi|$  is integrable on  $X$ , and by definition its integral over  $X$  is  $\omega_\phi(\psi_1, \psi_2)$ . Since the measure of  $X_\varepsilon$  goes to zero as  $\varepsilon \rightarrow 0$ , claim (II) follows.

We now consider what happens near the zeros of  $\phi$ . Here it will be essential that we are working in a stratum, so the symbol is constant. This means that the zeros move holomorphically as a function of the point in  $Q_j(X)$  and with constant multiplicity. In the same coordinate system where  $\phi$  restricted to a component of  $X_\varepsilon$  becomes  $z^k dz^2$  on  $\Delta_\varepsilon := \{z: |z| < \varepsilon\}$ , each  $\psi$  in a neighborhood  $U$  of  $\phi$  in  $Q_j(X)$  can therefore be expressed as

$$e^{h(\psi, z)}(z - u(\psi))^k dz^2,$$

where  $u: U \rightarrow \Delta_\varepsilon$  and  $h(\psi, z): U \times \Delta_\varepsilon \rightarrow \mathbb{C}$  are holomorphic functions and  $u(\phi) = 0$ . Thus  $N_0^\varepsilon(\psi)$  is the sum of finitely many terms of the form

$$\int_{|z| < \varepsilon} |e^{h(\psi, z)}(z - u(\psi))^k| |dz|^2 = \int_{|z| < \varepsilon} e^{\operatorname{Re} h(\psi, z)} |z - u(\psi)|^k |dz|^2.$$

Using the change of variable  $w=z-u(\psi)$  this integral becomes

$$\int_{|w+u(\psi)|<\varepsilon} e^{\operatorname{Re} h(\psi, w+u(\psi))} |w|^k |dw|^2. \quad (5.2)$$

In this form, the integrand is  $C^\infty$  as a function of  $\psi$  and all of its  $\psi$ -derivatives are continuous in  $w$ . Furthermore, on the boundary curve  $|w+u(\psi)|=\varepsilon$  the integrand is smooth in both  $\psi$  and  $w$  (since for  $\psi$  near  $\phi$  this curve avoids the locus  $w=0$  where the integrand may fail to be smooth).

These conditions are exactly what we need to differentiate under the integral in computing derivatives of (5.2), and since  $N_0^\varepsilon$  is a sum of finitely many terms of this form, we find that it is smooth. This establishes (II).

Finally, we must estimate the derivative of  $N_0^\varepsilon$  using (5.2). The second  $\psi$ -derivatives of the integral at  $\psi=\phi$  split into an interior term (an integral over  $|w|<\varepsilon$ ) and boundary terms (integrals over  $|w|=\varepsilon$ ). The boundary terms involve up to second derivatives of the boundary curve as a function of  $\psi$  and the first partial derivatives of the integrand with respect to both  $\psi$  and  $w$ . The interior term involves the second  $\psi$ -partial derivatives of the integrand. This gives an overall estimate for any second partial derivative of the function  $N_0^\varepsilon$  at  $\phi$  of the form

$$O(AB_{2,0}^{\text{int}} + LB_{1,1}^{\text{bdy}} C_2),$$

where

- $A=\pi\varepsilon^2$  is the area of the region of integration at  $\psi=\phi$ ,
- $B_{2,0}^{\text{int}}$  is an interior upper bound (i.e. on  $|w|<\varepsilon$ ) for the integrand and its  $\psi$ -partial derivatives of order at most 2,
- $B_{1,1}^{\text{bdy}}$  is a boundary upper bound (i.e. on  $|w|=\varepsilon$ ) for the integrand and its first partial derivatives with respect to  $\psi$  and  $w$ ,
- $L=2\pi\varepsilon$  is the length of the boundary curve, and
- $C_2$  is a bound on the  $\psi$ -derivatives of the boundary.

Since  $h$  and  $u$  are holomorphic, their derivatives of any fixed order are uniformly bounded once we take  $U$  and  $\varepsilon$  small enough. Examining the integrand of (5.2) we then find  $B_{2,0}^{\text{int}}=O(\varepsilon^k)$ ,  $B_{1,1}^{\text{bdy}}=O(\varepsilon^{k-1})$  (with the  $w$ -derivative of the integrand on  $|w|=\varepsilon$  being the dominant term), and  $C_2=O(1)$ . Hence  $\partial\bar{\partial}N_0^\varepsilon$  is  $O(\varepsilon^2\varepsilon^k + \varepsilon\varepsilon^{k-1}\cdot 1)=O(\varepsilon^k)$  as  $\varepsilon\rightarrow 0$ . As  $k>0$ , this establishes (III).  $\square$

Before proceeding to relate the Kähler structure of Theorem 5.3 to the symplectic structure of  $\mathcal{MF}(S)$ , we will need to describe convenient local coordinates for both spaces. After discussing suitable period and train-track coordinates, we return to the matter of relating these spaces in §5.6.

### 5.3. Double covers and periods

For any  $\phi \in Q(X)$  let  $\tilde{X}_\phi$  denote the Riemann surface of the locally defined one-form  $\sqrt{\phi}$  on  $X$ , i.e.  $\tilde{X}_\phi$  is a branched double cover of  $X$  in which  $\phi$  is canonically expressed as the square of a 1-form (also denoted by  $\sqrt{\phi}$ ). By construction,  $\tilde{X}_\phi$  has a holomorphic involution  $\sigma$  such that  $\sigma^*\sqrt{\phi} = -\sqrt{\phi}$  and  $X = \tilde{X}_\phi/\sigma$ .

The 1-form  $\sqrt{\phi}$  on  $\tilde{X}_\phi$  has *absolute periods* obtained by integration along cycles in  $H_1(\tilde{X}_\phi)$  and *relative periods* obtained by integration along cycles in  $H_1(\tilde{X}_\phi, \tilde{Z}_\phi)$ , where  $\tilde{Z}_\phi$  is the set of zeros of  $\sqrt{\phi}$ . Since  $\sigma^*\sqrt{\phi} = -\sqrt{\phi}$ , these integrals vanish for cycles invariant under  $\sigma$  and non-trivial periods are only obtained from cycles in the  $(-1)$ -eigenspace, which we denote by

$$H_1^-(\phi) := \{c \in H_1(\tilde{X}_\phi, \tilde{Z}_\phi; \mathbb{C}) : \sigma_*c = -c\}.$$

Collectively, the periods of  $\sqrt{\phi}$  determine its cohomology class as an element of

$$H_1^-(\phi) := \{\theta \in H^1(\tilde{X}_\phi, \tilde{Z}_\phi; \mathbb{C}) : \sigma^*\theta = -\theta\}.$$

Note that a saddle connection  $I$  of  $\phi$  determines an element  $[I] \in H_1^-(\phi)$  by taking the difference of its two lifts to  $\tilde{X}_\phi$ . The result is well defined up to sign. We can therefore consider relative periods of  $\sqrt{\phi}$  along such saddle connections.

Since integration of  $\sqrt{\phi}$  gives a local natural coordinate for  $\phi$ , the relative period of a saddle connection is simply its displacement vector in such a coordinate system. In particular, the height of a saddle connection  $I$  is given by

$$\phi\text{-height}(I) = \left| \text{Im} \int_{[I]} \sqrt{\phi} \right|. \quad (5.3)$$

### 5.4. Period coordinates for strata

Let  $\mathcal{Q}(S, \pi)$  be a stratum in  $\mathcal{Q}(S)$ . The topological type of the double cover  $\tilde{X}_\phi \rightarrow X$  is determined by the symbol of  $\phi$ , so in a small neighborhood  $U$  of  $\phi$  in  $\mathcal{Q}(S, \pi)$  we can trivialize the family of double covers and (co)homology groups. Thus we can regard each space  $H_1^-(\psi)$ , where  $\psi \in U$ , as an instance of a single cohomology space  $H_1^-(\pi)$  that is determined by topological information contained in  $(X, \phi)$ ; we let  $H_1^-(\pi)$  denote the corresponding trivialization of the family of homology groups. When considering a class  $a \in H_1^-(\pi)$  we write  $a_\psi$  for a representing cycle in  $H_1^-(\psi)$ .

Using this local trivialization, the cohomology class of  $\sqrt{\phi}$  determines the *relative period map*

$$\text{Per}: U \longrightarrow H_1^-(\pi).$$

Explicitly, as a linear function on cycles  $a \in H_1^-(\pi)$  the map is given by

$$\text{Per}(\phi)(a) = \int_{a_\phi} \sqrt{\phi}.$$

This map provides local coordinates for strata [V2], [V1, §28], [MaS].

**THEOREM 5.4.** *The relative period construction gives local biholomorphic coordinates for  $\mathcal{Q}(S, \pi)$ , i.e. for any sufficiently small open set  $U \subset \mathcal{Q}(S, \pi)$  the period map*

$$\text{Per}: U \longrightarrow H_-^1(\pi)$$

*is a diffeomorphism onto an open set. In particular, we have*

$$\dim_{\mathbb{C}} H_-^1(\pi) = \dim_{\mathbb{C}} \mathcal{Q}(S, \pi).$$

While the result above applies to strata in  $\mathcal{Q}(S)$ , each stratum  $Q_j(X)$  of  $Q(X)$  is a complex submanifold of  $\mathcal{Q}(S, \pi)$  for some  $\pi \in \mathcal{S}$ , so we have the following.

**COROLLARY 5.5.** *Let  $\phi \in Q_j(X)$  be a quadratic differential with symbol  $\pi$ . Then there is an open neighborhood of  $\phi$  in  $Q_j(X)$  in which the relative period map to  $H_-^1(\pi)$  is biholomorphic onto its image.*

Later we will need the following formula for the derivative of the relative period coordinates.

**LEMMA 5.6.** (Douady–Hubbard) *Let  $\phi \in Q_j(X)$  and  $\psi \in T_\phi Q_j(X)$ . Then for any  $a \in H_1^-(\pi)$  we have*

$$d\text{Per}_\phi(\psi)(a) = \int_{a_\phi} \frac{\psi}{2\sqrt{\phi}}.$$

The proof in [DH] is for differentials with simple zeros, however, the argument only uses the fact that the period of a saddle connection for a family  $\phi_t$  (where  $\phi_0 = \phi$  and  $(\partial\phi/\partial t)|_{t=0} = \psi$ ) can be expressed in the form

$$\int_{a_t}^{b_t} \sqrt{\phi_t(z)} dz,$$

where  $a_t$  and  $b_t$  are smooth paths traced out by the zeros of  $\phi_t$  as  $t$  varies near 0. The assumptions on  $\phi$  and  $\psi$  in the lemma above imply that  $\psi$  is tangent to a family of differentials whose zeros have constant multiplicity, and so the same argument applies.

### 5.5. Adapted train tracks

We now consider train-track coordinates for  $\mathcal{MF}(S)$  compatible with the relative period construction described above. The following refinement of Lemma 4.3 ensures that we can always choose these coordinates so that the foliation in question lies in the interior of the train-track chart.

LEMMA 5.7. *For each non-zero  $\phi \in Q(X)$ , there exist a triangulation  $\Delta$  of  $X$  by saddle connections and a dual maximal train track  $\tau$  satisfying conditions (i)–(iii) of Lemma 4.3 and such that none of the saddle connections in  $\Delta$  are horizontal. In particular, the point  $[\mathcal{F}(\phi)]$  lies in the interior of the train-track chart  $\mathcal{MF}(\tau)$ .*

*Proof.* Each face of the train-track chart  $\mathcal{MF}(\tau)$  is defined by the weight of some branch of the track being zero. In this case the weights are heights of saddle connections, so excluding horizontal edges will result in  $\mathcal{F}(\phi)$  being in the interior of the chart.

If the Delaunay triangulation of Lemma 4.3 has no horizontal edges, or if  $\phi$  itself has no horizontal saddle connections, then we are done. Otherwise we must alter the construction of the triangulation to eliminate the horizontal edges. Note that  $\phi$  has only finitely many horizontal saddle connections.

Consider the Teichmüller geodesic  $(X_t, \phi_t)$  determined by  $e^{i\theta}\phi$ . The Riemann surfaces and quadratic differentials in this family are identified by locally affine maps, so saddle connections of  $\phi$  are also saddle connections of  $\phi_t$  and vice versa. Furthermore, if  $\theta \neq \pi$  then horizontal saddle connections of  $\phi$  have  $\phi_t$ -length growing exponentially in  $t$ . If we choose  $\theta \neq \pi$  so that the geodesic is recurrent in moduli space (a dense set of directions have this property [KW]), then by choosing  $t$  large enough we may assume that  $X_t$  has bounded  $|\phi_t|$ -diameter while the  $\phi$ -horizontal saddle connections are arbitrarily long with respect to  $|\phi_t|$ .

Since the length of an edge of the Delaunay triangulation is bounded by the diameter of the surface [MaS, Theorem 4.4], this shows that for large  $t$  the  $\phi$ -horizontal saddle connections are not edges of the Delaunay triangulation of  $\phi_t$ . Thus the Delaunay triangulation of  $\phi_t$  gives the desired triangulation by non-horizontal saddle connections of  $\phi$ .  $\square$

### 5.6. The symplectomorphism

Hubbard and Masur showed that for any  $X \in \mathcal{J}(S)$ , the foliation map  $\mathcal{F}: Q(X) \rightarrow \mathcal{MF}(S)$  is a homeomorphism [HM]. We now show that this map relates the Kähler structure on  $Q(X)$  introduced above to the Thurston symplectic structure of  $\mathcal{MF}(S)$ .



**THEOREM 5.8.** *For any  $X \in \mathcal{T}(S)$ , the map  $\mathcal{F}: Q(X) \rightarrow \mathcal{MF}(S)$  is a real-analytic stratified symplectomorphism. That is,*

(i) *for any  $\phi \in Q_j(X)$  there exist an open neighborhood  $U \subset Q_j(X)$  of  $\phi$  in its stratum and a train-track coordinate chart  $\mathcal{MF}(\tau) \subset \mathcal{MF}(S)$  covering  $\mathcal{F}(U)$  so that the restriction*

$$\mathcal{F}: U \longrightarrow \mathcal{MF}(\tau)$$

*is a real-analytic diffeomorphism onto its image, and*

(ii) *the derivative  $d\mathcal{F}_\phi$  defines a symplectic linear map from  $T_\phi Q_j(X)$  into  $W(\tau) \simeq T_{\mathcal{F}(\phi)} \mathcal{MF}(\tau)$ , where  $T_\phi Q_j(X)$  is equipped with the symplectic form  $\omega_\phi$  and  $W(\tau)$  is given the Thurston symplectic form.*

*Proof.* (i) Let  $\phi \in Q_j(X)$ . Applying Lemma 5.7 we obtain a neighborhood  $U \subset Q_j(X)$  of  $\phi$  and a train track  $\tau$  that carries the horizontal foliation of each  $\psi \in U$  by assigning to each branch the height of an associated edge of the  $\psi$ -geodesic triangulation.

Lift  $\tau$  and the dual  $\phi$ -geodesic triangulation  $\Delta$  to the cover  $\tilde{X}_\phi$ , obtaining a triangulation  $\hat{\Delta}$  and double covering of train tracks  $\hat{\tau} \rightarrow \tau$ . Orient the edges of  $\hat{\Delta}$  so that the integral of  $\text{Im} \sqrt{\phi}$  over any edge is positive. (This integral is non-zero because the original triangulation did not have any  $\phi$ -horizontal edges.) Then the integral of  $\text{Im} \sqrt{\phi}$  over an edge  $\hat{e}$  is the  $\phi$ -height of the corresponding edge  $e$  of  $\Delta$ .

The covering train tracks and oriented triangulations obtained in this way for other  $\psi \in U$  are naturally isotopic to  $\hat{\tau}$  and  $\hat{\Delta}$ , so this construction extends throughout  $U$ . Thus for all  $\psi \in U$  we have realized the weights on  $\tau$  defining  $[\mathcal{F}(\psi)]$  as the imaginary parts of periods of  $\sqrt{\psi}$ , which by Corollary 5.5 are real-analytic functions.

It remains to show that the derivative of the map to  $\mathcal{ML}(\tau)$  is an isomorphism, so that after shrinking  $U$  appropriately we have a diffeomorphism onto an open set. However, this is a consequence of the proof of (ii) below since the Thurston symplectic form is non-degenerate.

(ii) We need to show that any  $\psi_1, \psi_2 \in T_\phi Q_j(X)$  satisfy

$$\omega_\phi(\psi_1, \psi_2) = \omega_{\text{Th}}(d\mathcal{F}_\phi(\psi_1), d\mathcal{F}_\phi(\psi_2)). \quad (5.4)$$

We begin by analyzing the left-hand side. Let  $\hat{\psi}_j$  denote the lift of  $\psi_j$  to the double cover  $\tilde{X}_\phi$  and define

$$\theta_j = \frac{\hat{\psi}_j}{2\sqrt{\phi}} \in \Omega(\tilde{X}_\phi).$$

These 1-forms are holomorphic because all poles of  $\psi_j/\phi$  are simple and occur at branch points of the covering  $\tilde{X}_\phi \rightarrow X$ . Since the 2-form  $\frac{1}{2}i\theta_1 \wedge \bar{\theta}_2$  is the lift of the integrand of  $\langle \psi_1, \psi_2 \rangle_\phi$  to the degree-2 cover  $\tilde{X}_\phi$ , we have

$$\omega_\phi(\psi_1, \psi_2) = \frac{1}{2} \text{Im} \int_{\tilde{X}_\phi} \frac{i}{2} \theta_1 \wedge \bar{\theta}_2 = \frac{1}{4} \text{Re} \int_{\tilde{X}_\phi} \theta_1 \wedge \bar{\theta}_2.$$

For any holomorphic 1-forms  $\theta_j$  we have

$$\operatorname{Re}(\theta_1 \wedge \bar{\theta}_2) = 2(\operatorname{Re} \theta_1) \wedge (\operatorname{Re} \theta_2) = 2(\operatorname{Im} \theta_1) \wedge (\operatorname{Im} \theta_2),$$

so we can express the integral above as

$$\omega_\phi(\psi_1, \psi_2) = \frac{1}{2} \int_{\tilde{X}_\phi} (\operatorname{Im} \theta_1) \wedge (\operatorname{Im} \theta_2) = \frac{1}{2} [\operatorname{Im} \theta_1] \cdot [\operatorname{Im} \theta_2], \quad (5.5)$$

where in the last expression  $[\alpha]$  denotes the de Rham cohomology class of a closed 1-form  $\alpha$  and  $[\alpha] \cdot [\beta]$  is the cup product.

Now consider the pairing  $\omega_{\text{Th}}(d\mathcal{F}_\phi(\psi_1), d\mathcal{F}_\phi(\psi_2))$ . The tangent vector  $d\mathcal{F}_\phi(\psi_j) \in W(\tau)$  is a weight function whose value on a branch  $e$  is the derivative of the height of the associated edge  $e'$  of  $\Delta$ . The height of an edge is the imaginary part of the period of  $\sqrt{\phi}$ , so Lemma 5.6 gives a formula for the derivatives of these periods. Namely, after lifting to the covering train track  $\hat{\tau}$  we find that  $d\mathcal{F}_\phi(\psi_j)$  corresponds to the weight function  $\hat{w}_j \in W(\hat{\tau})$  defined by

$$\hat{w}_j(e) = \int_{\hat{e}} \operatorname{Im} \theta_j,$$

where  $e$  is a branch of  $\tau$  (identified with its dual edge of  $\Delta$ ) and  $\hat{e}$  is an associated oriented edge of  $\hat{\Delta}$ . The orientation of  $\hat{\Delta}$  induces a consistent orientation of  $\hat{\tau}$  so that all intersections of  $\hat{\tau}$  with  $\hat{\Delta}$  become positively oriented. In terms of this orientation, the expression above shows that the de Rham cohomology class  $[\operatorname{Im} \theta_j]$  is Poincaré dual to the cycle

$$\hat{c}_j = \sum_{e \in \hat{\tau}} w_j(e) \vec{e}.$$

Using the formula (2.2) for the Thurston form as a homological intersection of such cycles and the duality of intersection and cup product, we have

$$\omega_{\text{Th}}(d\mathcal{F}_\phi(\psi_1), d\mathcal{F}_\phi(\psi_2)) = \frac{1}{2} (\hat{c}_1 \cdot \hat{c}_2) = \frac{1}{2} [\operatorname{Im} \theta_1] \cdot [\operatorname{Im} \theta_2].$$

With (5.5) this gives the desired equality between symplectic pairings.  $\square$

*Remark.* The smoothness of the foliation map when restricted to a set of quadratic differentials with constant symbol is implicit in [HM]. Because we consider only tangent vectors to strata in  $Q(X)$ , the subtle issues that arise from breaking up high-order zeros (and which underlie the failure of differentiability for the full map  $Q(X) \rightarrow \mathcal{ML}(S)$ ) do not arise here.

### 5.7. Application: The Hubbard–Masur constant

Here we mention an application of Theorem 5.8 that is not used in the sequel. It is immediate from the definition (5.1) that the hermitian form  $\langle \psi_1, \psi_2 \rangle_\phi$  is invariant under the action of  $S^1 \simeq \{e^{i\theta}\}$  on  $Q(X)$  by scalar multiplication:

$$\langle c\psi_1, c\psi_2 \rangle_{c\phi} = \langle \psi_1, \psi_2 \rangle_\phi \quad \text{if } |c| = 1.$$

It follows that this  $S^1$ -action preserves the volume form associated with the stratified Kähler structure on  $Q(X)$ , and thus the symplectomorphism with  $\mathcal{MF}(S)$  gives the following result.

COROLLARY 5.9. *The action of  $S^1$  on  $\mathcal{MF}(S)$  induced by the foliation map*

$$\mathcal{F}: Q(X) \longrightarrow \mathcal{MF}(S)$$

*preserves the volume form associated with the Thurston symplectic structure.*

In particular this corollary applies to the *antipodal involution*  $i_X: \mathcal{MF}(S) \rightarrow \mathcal{MF}(S)$  which corresponds to multiplication by  $-1$  in  $Q(X)$ . This map exchanges the vertical and horizontal measured foliations of any quadratic differential on  $X$ .

Let  $b(X) \subset \mathcal{MF}(S)$  denote the unit ball of the extremal length function on  $X$ :

$$b(X) = \{[\nu] \in \mathcal{MF}(S) : \text{Ext}_{[\nu]}(X) \leq 1\}.$$

Equivalently,  $b(X)$  is the image of the  $L^1$ -norm unit ball in  $Q(X)$  under the foliation map, so it is invariant under  $i_X$ .

Let  $\Lambda(X)$  denote the volume of this set with respect to the Thurston symplectic form on  $\mathcal{MF}(S)$ ; this defines the *Hubbard–Masur function*  $\Lambda: \mathcal{T}(S) \rightarrow \mathbb{R}^+$ . This function appears as a coefficient in various counting problems related to the action of the mapping class group  $\text{Mod}(S)$  on  $\mathcal{T}(S)$  studied in [ABEM].

Using Corollary 5.9, Mirzakhani has shown the following (personal communication).

THEOREM 5.10. (Mirzakhani) *The Hubbard–Masur function is constant. That is, the volume of  $b(X)$  depends only on the topological type of  $S$  and is independent of the point  $X \in \mathcal{T}(S)$ .*

The following argument is based on the above-cited communication with Mirzakhani. An analogous statement in a different dynamical context is established in [Y].

*Proof.* We will use the antipodal map  $i_X$  to show that the derivative of  $\Lambda$  vanishes identically. Since  $\mathcal{T}(S)$  is connected it will then follow that  $\Lambda$  is constant.

Let  $S(X)=\partial b(X)$  denote the extremal-length unit sphere in  $\mathcal{MF}(S)$ . We think of this as a family of hypersurfaces in  $\mathcal{MF}(S)$  parameterized by  $X \in \mathcal{T}(S)$ .

Fix a point  $X_0 \in \mathcal{T}(S)$ . For any other point  $X \in \mathcal{T}(S)$ , both  $S(X_0)$  and  $S(X)$  intersect each ray in  $\mathcal{MF}(S)$  in a single point, so we can consider  $S(X)$  as obtained from  $S(X_0)$  by scaling each point  $[\nu] \in S(X_0)$  by a positive real number

$$(\text{Ext}_{[\nu]}(X_0)/\text{Ext}_{[\nu]}(X))^{1/2}. \quad (5.6)$$

Regarding this expression as a function of  $[\nu]$ , we have described the spheres  $S(X)$  as a family of “radial graphs” over  $S(X_0)$ . Using this description, the derivative of this family at  $X=X_0$  (if it exists) is a vector field along  $S(X_0)$  which is *radial*, i.e. it is a pointwise multiple of the vector field  $\partial/\partial t$  generating the  $\mathbb{R}^+$  action. Since the scaling function relating  $S(X_0)$  to  $S(X)$  is a quotient of powers of the extremal length functions, differentiability of this family at  $X=X_0$  is a consequence of Gardiner’s formula [Gar], which states that the derivative of extremal length is given by

$$\left. \frac{d}{dt} \text{Ext}_{[\nu]}(X_t) \right|_{t=0} = 2 \text{Re} \int_{X_0} \mu \mathcal{F}^{-1}([\nu]),$$

where  $X_t$  is smooth a path in  $\mathcal{T}(S)$  and  $\mu$  is a Beltrami coefficient on  $X_0$  representing

$$\left. \frac{d}{dt} X_t \right|_{t=0}.$$

Differentiating (5.6) using this formula, we find that the derivative of  $S(X)$  at  $X=X_0$  is the continuous vector field

$$V_\mu([\nu]) = - \left( \text{Re} \int_{X_0} \mu \mathcal{F}^{-1}([\nu]) \right) \frac{\partial}{\partial t}.$$

As above  $\partial/\partial t$  is the vector field on  $\mathcal{MF}(S)$  generating the  $\mathbb{R}^+$ -action. The derivative of the volume enclosed by  $S(X)$  at  $X=X_0$  is therefore the integral over  $S(X_0)$  of the interior product of this vector field with the Thurston volume form,

$$d\Lambda_{X_0}(\mu) = \int_{S(X_0)} V_\mu \lrcorner \omega_{\text{Th}}^n,$$

where  $n = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{MF}(S)$ . Since it corresponds to the  $L^1$ -norm unit sphere in  $Q(X)$ , the sphere  $S(X_0)$  is invariant under the antipodal involution. By Corollary 5.9, the volume form  $\omega_{\text{Th}}^n$  is also  $i_X$ -invariant. But, as  $\mathcal{F}^{-1}(i_X([\nu])) = -\mathcal{F}^{-1}([\nu])$ , the vector field  $V_\mu$  is odd under this involution (i.e.  $i_X^*(V_\mu) = -V_\mu$ ) as is the integrand  $V_\mu \lrcorner \omega_{\text{Th}}^n$ . Since the integral of an odd form over  $S(X_0)$  vanishes, we have  $d\Lambda_{X_0}(\mu) = 0$ .  $\square$

## 6. Character varieties and holonomy

### 6.1. Character varieties

Let  $G$  be one of the complex algebraic groups  $\mathrm{SL}_2\mathbb{C}$  or  $\mathrm{PSL}_2\mathbb{C}$ , and let  $\Gamma$  be a finitely generated group. We denote by  $\mathcal{R}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)$  the  $G$ -representation variety of  $\Gamma$ , which carries an action of  $G$  by conjugation. The categorical quotient

$$\mathcal{X}(\Gamma, G) := \mathcal{R}(\Gamma, G) // G$$

is the *character variety*, or more precisely, the variety of characters of representations of  $\Gamma$  in  $G$ . See [CS], [MoS1, §II.4] and [HP] for detailed discussions of these spaces. Both  $\mathcal{R}(\Gamma, G)$  and  $\mathcal{X}(\Gamma, G)$  are affine algebraic varieties defined over  $\mathbb{Q}$ . The ring  $\mathbb{Q}[\mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C})]$  is generated by the *trace functions*  $\{t_\gamma\}_{\gamma \in \Gamma}$  which are induced by the conjugation-invariant functions on  $\mathcal{R}(\Gamma, G)$  defined by

$$t_\gamma(\varrho) = \mathrm{tr}(\varrho(\gamma)).$$

Similarly, the ring  $\mathbb{Q}[\mathcal{X}(\Gamma, \mathrm{PSL}_2\mathbb{C})]$  is generated by the squares of trace functions.

There are two types of natural maps between character varieties that we will use in the sequel. First, the covering map  $\mathrm{SL}_2\mathbb{C} \rightarrow \mathrm{PSL}_2\mathbb{C}$  induces a map of character varieties

$$r: \mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C}) \longrightarrow \mathcal{X}(\Gamma, \mathrm{PSL}_2\mathbb{C}),$$

which is finite-to-one, proper, and whose image is a union of irreducible components; in fact, the group  $H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$  acts on  $\mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C})$  by biregular maps, and  $r$  is the quotient mapping for this action [MoS3, §V.1]. Secondly, if  $\phi: \Gamma \rightarrow \Gamma'$  is a group homomorphism, then composing representations with  $\phi$  induces a map of character varieties

$$\phi^*: \mathcal{X}(\Gamma', G) \longrightarrow \mathcal{X}(\Gamma, G),$$

which is a regular map.

These constructions are functorial in the sense that the maps  $r$  and  $\phi^*$  fit into a commutative diagram

$$\begin{array}{ccc} \mathcal{X}(\Gamma', \mathrm{SL}_2\mathbb{C}) & \xrightarrow{\phi^*} & \mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C}) \\ r \downarrow & & \downarrow r \\ \mathcal{X}(\Gamma', \mathrm{PSL}_2\mathbb{C}) & \xrightarrow{\phi^*} & \mathcal{X}(\Gamma, \mathrm{PSL}_2\mathbb{C}). \end{array} \quad (6.1)$$

Since the character varieties that we consider are for groups of the form  $\Gamma = \pi_1 N$ , where  $N$  is a compact 2- or 3-manifold, we often use the abbreviated notation

$$\mathcal{X}(N, G) := \mathcal{X}(\pi_1 N, G).$$

Note that this algebraic variety does not depend on a choice of orientation for  $N$ .

## 6.2. The Morgan–Shalen compactification

In [MoS1] a compactification of  $\mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C})$  is defined by mapping  $\mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C})$  into the infinite-dimensional projective space  $\mathbb{P}(\mathbb{R}^\Gamma) := (\mathbb{R}^\Gamma \setminus \{0\})/\mathbb{R}^+$  by

$$[\varrho] \mapsto (\log(|t_\gamma(\varrho)| + 2))_{\gamma \in \Gamma}. \quad (6.2)$$

The image of this map is precompact, and taking the closure gives the *Morgan–Shalen compactification* of  $\mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C})$ . A boundary point  $[\ell]$  of this compactification is a projective equivalence class of functions  $\ell: \Gamma \rightarrow \mathbb{R}$ , and any function arising this way is the translation-length function of an action of  $\Gamma$  on an  $\mathbb{R}$ -tree by isometries. These  $\mathbb{R}$ -trees are constructed algebraically, using valuations on the function fields of subvarieties of  $\mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C})$ . The intermediate stages of this algebraic construction also involve  $\Lambda$ -trees of higher rank. For later use we will now recall some key steps in their construction.

## 6.3. Valuation constructions

In what follows we consider irreducible subvarieties  $V \subset \mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C})$ , and  $k$  will denote a countable subfield of  $\mathbb{C}$  over which  $V$  is defined. (For example, if  $\mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C})$  is irreducible, we can take  $V = \mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C})$  and  $k = \mathbb{Q}$ .) The function field  $k(V)$  of such a variety is a finitely generated extension of  $k$ , and we consider  $k$ -valuations  $v: k(V)^* \rightarrow \Lambda$ , where  $\Lambda$  is an ordered abelian group. Without loss of generality, we may assume that  $\Lambda$  has finite rank [ZS, §6.10]. A valuation is *supported at infinity* if there exists a regular function  $f \in k[V]$  with  $v(f) < 0$ .

Boundary points of the Morgan–Shalen compactification correspond to valuations as follows.

**THEOREM 6.1.** ([MoS1, Theorem I.3.6]) *If  $V \subset \mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C})$  is an irreducible subvariety defined over  $k$  and  $[\ell]$  is a boundary point of  $V$  in the Morgan–Shalen compactification, then there exists a valuation  $v: k(V)^* \rightarrow \Lambda$  such that*

- (i)  $v$  is supported at infinity,
- (ii) if  $\Lambda_1 \subset \Lambda$  is the minimal non-trivial convex subgroup, then for each  $\gamma \in \Gamma$  either  $v(t_\gamma) > 0$  or  $v(t_\gamma) \in \Lambda_1$ , and
- (iii) there is an order-preserving embedding  $p: \Lambda_1 \rightarrow \mathbb{R}$  such that

$$\ell(\gamma) = p(\max\{-v(t_\gamma), 0\}).$$

Note that the embedding  $p$  is unique up to multiplication by a positive constant (by Theorem 2.2) and that condition (iii) above shows that  $\ell$  can be recovered from the valuation  $v$ .

The link between valuations and  $\Lambda$ -trees is given by the following result.

**THEOREM 6.2.** ([MoS1, Theorem II.4.3 and Lemma II.4.5]) *If  $V \subset \mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C})$  is an irreducible subvariety defined over  $k$  and  $v: k(V)^* \rightarrow \Lambda$  is a valuation supported at infinity, then there is an isometric action of  $\Gamma$  on a  $\Lambda$ -tree whose translation-length function  $\ell: \Gamma \rightarrow \Lambda$  satisfies*

$$\ell(\gamma) = \max\{-v(t_\gamma), 0\}.$$

*Remark.* The statement of [MoS1, Lemma II.4.5] involves only  $\mathbb{R}$ -trees, however, a  $\Lambda$ -tree satisfying the conditions above is constructed as part of its proof. The lemma also involves an additional condition on the valuation (equivalent to Theorem 6.1 (ii) above), but this condition is only used at the final step to produce an  $\mathbb{R}$ -tree from the  $\Lambda$ -tree. Additional discussion of the  $\Lambda$ -tree construction underlying Theorem 6.2 can be found in [Mor2, Theorem 16] and [MoS2, pp. 232–233].

#### 6.4. The extension variety

Let  $M$  be a compact 3-manifold with connected boundary  $S$ , and let  $i_*: \pi_1 S \rightarrow \pi_1 M$  be the map induced by inclusion of the boundary. As discussed above, such a homomorphism induces a map of character varieties

$$i^*: \mathcal{X}(M, G) \longrightarrow \mathcal{X}(S, G).$$

We call this the *restriction map*. Since it is a regular map of algebraic varieties, the image of  $i^*$  is a constructible set which contains a Zariski open subset of its closure.

Considering the case  $G = \mathrm{SL}_2\mathbb{C}$ , we denote the closure of the image by

$$\mathcal{E}_M := \overline{i^*(\mathcal{X}(M, \mathrm{SL}_2\mathbb{C}))}^{\mathrm{Zariski}},$$

which we call the *extension variety*, since its generic points are conjugacy classes of representations of  $\pi_1 S$  that are trivial on  $\ker(i_*)$  and which admit an extension from  $i_*(\pi_1 S)$  to its supergroup  $\pi_1 M$ . Note that  $\mathcal{E}_M$  is an algebraic subvariety of  $\mathcal{X}(S, \mathrm{SL}_2\mathbb{C})$ .

Since points in the Morgan–Shalen boundary of  $\mathcal{X}(S, \mathrm{SL}_2\mathbb{C})$  correspond to length functions of actions of  $\pi_1 S$  on  $\mathbb{R}$ -trees, it is natural to expect that the length functions that arise as boundary points of  $\mathcal{E}_M$  would have a similar extension property. We now show that this is true if we allow the extended length function to take values in a higher-rank group,  $\mathbb{R}^n$  with the lexicographical order.

**THEOREM 6.3.** *Let  $[\ell]$  be a boundary point of  $\mathcal{E}_M$  in the Morgan–Shalen compactification of  $\mathcal{X}(S, \mathrm{SL}_2\mathbb{C})$ . Then  $\ell: \pi_1 S \rightarrow \mathbb{R}$  extends to a length function of an action of  $\pi_1 M$  on a  $\mathbb{R}^n$ -tree, i.e. there exists a function  $\hat{\ell}: \pi_1 M \rightarrow \mathbb{R}^n$  such that*

- (i) *the group  $\pi_1 M$  acts isometrically on a  $\mathbb{R}^n$ -tree with translation-length function  $\hat{\ell}$ ,*

(ii) for each  $\gamma \in \pi_1 S$  we have

$$\hat{\ell}(i_*(\gamma)) = i_n(\ell(\gamma)),$$

where  $i_*: \pi_1 S \rightarrow \pi_1 M$  is the map induced by the inclusion of  $S$  as the boundary of  $M$  and  $i_n: \mathbb{R} \rightarrow \mathbb{R}^n$  is the order-preserving inclusion as the last (least significant) factor.

*Proof.* As  $[\ell]$  is a boundary point of  $\mathcal{E}_M$ , it is a boundary point of one of its irreducible components. Let  $\mathcal{E}_M^0$  be such a component, and let  $\mathcal{X}_M^0$  be a corresponding irreducible component of  $\mathcal{X}(M, \mathrm{SL}_2\mathbb{C})$  so that  $i^*(\mathcal{X}_M^0)$  contains a Zariski open subset of  $\mathcal{E}_M^0$ .

Since  $i^*: \mathcal{X}_M^0 \rightarrow \mathcal{E}_M^0$  is dominant, it induces an extension of function fields  $k(\mathcal{E}_M^0) \hookrightarrow k(\mathcal{X}_M^0)$ , where  $k$  is a finite extension of  $\mathbb{Q}$  over which  $\mathcal{E}_M^0$  and  $\mathcal{X}_M^0$  are defined. Note that when considering  $k(\mathcal{E}_M^0)$  as a subfield of  $k(\mathcal{X}_M^0)$ , the element of  $k(\mathcal{E}_M^0)$  represented by the trace function  $t_\gamma$ ,  $\gamma \in \pi_1 S$ , is identified with the element of  $k(\mathcal{X}_M^0)$  represented by the trace function  $t_{i_*(\gamma)}$ .

Let  $v: k(\mathcal{E}_M^0)^* \rightarrow \Lambda$  and  $p: \Lambda_1 \rightarrow \mathbb{R}$  be the valuation and embedding associated with  $\ell$  by Theorem 6.1. As  $k(\mathcal{X}_M^0)$  is finitely generated over  $k(\mathcal{E}_M^0)$ , the standard extension theorem for valuations (see [ZS, p. 13, Theorem 5'] and [MoS1, Lemma II.4.4]) gives an ordered abelian group  $\Lambda'$  such that  $\Lambda \subset \Lambda'$  and  $\Lambda' \subset m\Lambda$  for some  $m \in \mathbb{N}$ , and a valuation

$$v': k(\mathcal{X}_M^0)^* \longrightarrow \Lambda'$$

such that  $v'(f) = v(f)$  for any  $f \in k(\mathcal{E}_M)$ . Since  $\Lambda' \subset m\Lambda$ , it follows that the minimal convex subgroups satisfy  $\Lambda_1 \subset \Lambda'_1$ .

By Lemma 2.4, we have a commutative diagram of order-preserving embeddings

$$\begin{array}{ccc} \Lambda_1 & \longrightarrow & \Lambda' \\ p \downarrow & & \downarrow F \\ \mathbb{R} & \xrightarrow{i_n} & \mathbb{R}^n. \end{array} \tag{6.3}$$

We can arrange that the left vertical map in this diagram agrees with the embedding  $p: \Lambda_1 \rightarrow \mathbb{R}$  considered above; this is possible since there is a unique such embedding up to scale (by Theorem 2.2), and both vertical maps in the diagram can be multiplied by an arbitrary positive constant while preserving commutativity and order.

Applying Theorem 6.2 to  $v'$ , we obtain a  $\Lambda'$ -tree  $T'$  on which  $\pi_1 M$  acts by isometries with length function  $\ell'$ . Let  $T = T' \otimes_{\Lambda'} \mathbb{R}^n$  be the  $\mathbb{R}^n$ -tree associated with  $T'$  by the embedding  $F$ , and let  $\hat{\ell}: \pi_1 M \rightarrow \mathbb{R}^n$  be its length function. Condition (i) is satisfied by definition.



It remains to verify condition (ii). For any  $\gamma \in \pi_1 S$  we have  $t_{i_*(\gamma)} \in k(\mathcal{X}_M^0)$  and the length function  $\hat{\ell}$  satisfies

$$\begin{aligned}
\hat{\ell}(i_*(\gamma)) &= F(\ell'(i_*(\gamma))) && \text{by definition of } T, \\
&= F(\max\{-v'(t_{i_*(\gamma)}), 0\}) && \text{by Theorem 6.2,} \\
&= F(\max\{-v(t_\gamma), 0\}) && \text{since } v' \text{ extends } v, \\
&= i_n(p(\max\{-v(t_\gamma), 0\})) && \text{by commutativity of (6.3),} \\
&= i_n(\ell(\gamma)) && \text{by Theorem 6.1.} \quad \square
\end{aligned} \tag{6.4}$$

*Remark.* It is natural to ask whether the extended length function  $\hat{\ell}$  of Theorem 6.3 can always be taken to be  $\mathbb{R}$ -valued, thus avoiding the introduction of  $\mathbb{R}^n$ -trees ( $n > 1$ ). The proof above shows a potential obstruction. For any  $\gamma \in \pi_1 S$  the valuation  $v(t_\gamma)$  is either positive or it lies in the rank-1 convex subgroup  $\Lambda_1$ , but it is not clear whether this holds for the extended valuation  $v'$  applied to a trace function of an element in  $\pi_1 M$ . A rank-1 subgroup containing the negative valuations of all trace functions is needed in order to apply the construction of [MoS1, §II.4] to produce an  $\mathbb{R}$ -tree from the  $\Lambda$ -tree while preserving the action of  $\pi_1 M$  and the length function.

Since these valuations are associated with boundary points of the Morgan–Shalen compactification, this is effectively a question about comparing the rate of growth of traces in a sequence of  $\pi_1 S$ -representations in  $\mathcal{E}_M$  to that of an associated sequence of  $\pi_1 M$ -representations. Alternatively, in the terminology of [D, §6], we ask whether the *local scales* of  $\mathrm{PSL}_2\mathbb{C}$ -representations of  $\pi_1 M$  are comparable (within a uniform multiplicative constant) to those of the restrictions to  $i_*(\pi_1 S)$ , or if such a uniform comparison is possible for *some* sequence representing any given boundary point.

### 6.5. The holonomy variety

Let  $X$  be a marked Riemann surface structure on  $S$ . Here we allow that the complex structure of  $X$  induces an orientation opposite that of  $S$ , so either  $X \in \mathcal{T}(S)$  or  $X \in \mathcal{T}(\bar{S})$ .

The vector space  $Q(X)$  of holomorphic quadratic differentials can be identified with the set of complex projective structures ( $\mathbb{CP}^1$ -structures) on  $X$ . Here  $0 \in Q(X)$  corresponds to the projective structure induced by the uniformization of  $X$ .

Each  $\mathbb{CP}^1$  structure on  $X$  has an associated holonomy representation  $\pi_1 S \rightarrow \mathrm{PSL}_2\mathbb{C}$ , which is well defined up to conjugacy. Considering the conjugacy class of the holonomy representation as a function of the projective structure gives the *holonomy map*

$$\mathrm{hol}: Q(X) \longrightarrow \mathcal{X}(S, \mathrm{PSL}_2\mathbb{C}).$$

This map can be lifted through  $r: \mathcal{X}(S, \mathrm{SL}_2\mathbb{C}) \rightarrow \mathcal{X}(S, \mathrm{SL}_2\mathbb{C})$  in several ways; the set of such lifts is naturally in bijection with the set  $\mathrm{Spin}(X)$  of spin structures on  $X$ , which is a finite set acted upon, simply transitively, by  $H_1(S, \mathbb{Z}/2\mathbb{Z})$ . For each  $\varepsilon \in \mathrm{Spin}(X)$  we denote the corresponding lifted holonomy map by

$$\mathrm{hol}_\varepsilon: Q(X) \longrightarrow \mathcal{X}(S, \mathrm{SL}_2\mathbb{C}),$$

so  $\mathrm{hol} = r \circ \mathrm{hol}_\varepsilon$ . The maps  $\mathrm{hol}$  and  $\mathrm{hol}_\varepsilon$  are proper holomorphic embeddings [GKM, Theorem 11.4.1].

Define

$$\mathcal{H}_{X,\varepsilon} := \mathrm{hol}_\varepsilon(Q(X)),$$

which is therefore a complex-analytic subvariety of  $\mathcal{X}(S, \mathrm{SL}_2\mathbb{C})$ . Taking the union of these subvarieties we obtain the *holonomy variety*

$$\mathcal{H}_X := \bigcup_{\varepsilon \in \mathrm{Spin}(X)} \mathcal{H}_{X,\varepsilon} \subset \mathcal{X}(S, \mathrm{SL}_2\mathbb{C}),$$

an analytic variety with irreducible components  $\mathcal{H}_{X,\varepsilon}$ . Equivalently, we have

$$\mathcal{H}_X = r^{-1}(\mathrm{hol}(Q(X))).$$

We will be interested in the limiting behavior of  $\mathcal{H}_X$  in the Morgan–Shalen compactification and how this relates to the parametrizations of its components by  $Q(X)$ . Consider a divergent sequence  $\phi_n \in Q(X)$ . Since the unit sphere in  $Q(X)$  is compact, by passing to a subsequence we may assume that  $c_n \phi_n \rightarrow \phi$  as  $n \rightarrow \infty$ , where  $c_n \in \mathbb{R}^+$  is a suitable sequence of scale factors with  $c_n \rightarrow 0$ . We call a limit point  $\phi$  obtained this way a *projective limit* of  $\{\phi_n\}_{n=1}^\infty$ . Projective limits in  $Q(X)$  are related to limits of holonomy representations in  $\mathcal{X}(S, \mathrm{SL}_2\mathbb{C})$  as follows.

**THEOREM 6.4.** ([D, Theorem A]) *If  $\phi_n \in Q(X)$  is a divergent sequence with projective limit  $\phi$ , then any accumulation point of  $\mathrm{hol}_\varepsilon(\phi_n)$  in the Morgan–Shalen boundary is represented by an  $\mathbb{R}$ -tree  $T$  that admits an equivariant, surjective straight map  $T_\phi \rightarrow T$ .*

## 6.6. Intersections and the isotropic cone

We now consider the intersection of the holonomy variety and the extension variety. Since  $\mathcal{H}_{X,\varepsilon}$  is parameterized by  $Q(X)$ , the intersection  $\mathcal{H}_{X,\varepsilon} \cap \mathcal{E}_M$  is parameterized by a subset of  $Q(X)$  which we denote by

$$\mathcal{V}_{M,\varepsilon} = \mathrm{hol}_\varepsilon^{-1}(\mathcal{E}_M) \subset Q(X),$$

and  $\mathcal{H}_X \cap \mathcal{E}_M$  is the union of the images of these sets under the respective holonomy maps.

Combining the main results of §4 with Theorems 6.3 and 6.4, we have the following characterization of the limit points of  $\mathcal{V}_{M,\varepsilon}$ .

**THEOREM 6.5.** *Let  $\{\phi_n\}_{n=1}^\infty \subset \mathcal{V}_{M,\varepsilon}$  be a divergent sequence with projective limit  $\phi$ . Then  $[\mathcal{F}(\phi)] \in \mathcal{L}_{M,X}$ , where  $\mathcal{L}_{M,X}$  is the isotropic cone of Theorem 4.1.*

Note that in this theorem we regard  $\mathcal{L}_{M,X}$  as a subset of  $\mathcal{MF}(S)$  regardless of whether  $X \in \mathcal{T}(S)$  or  $X \in \mathcal{T}(\bar{S})$ . This is possible since the natural identification  $\mathcal{MF}(S) \simeq \mathcal{MF}(\bar{S})$  preserves the property of being an isotropic cone (while changing the sign of the symplectic form).

*Proof.* Let  $[\ell]$  be an accumulation point of  $\text{hol}_\varepsilon(\phi_n)$  in the Morgan–Shalen compactification. By Theorem 6.4, there is a surjective, equivariant straight map  $T_\phi \rightarrow T$ , where  $T$  is an  $\mathbb{R}$ -tree on which  $\pi_1 S$  acts with length function  $\ell$ .

By Theorem 6.3, the length function  $\ell$  extends to a length function  $\hat{\ell}: \pi_1 M \rightarrow \mathbb{R}^n$  of an action of  $\pi_1 M$  on an  $\mathbb{R}^n$ -tree  $\hat{T}$ . By Lemma 2.7, there is an  $\mathbb{R}$ -tree  $T' \subset \hat{T}$  on which  $\pi_1 S$  acts with length function  $\ell = \hat{\ell}|_{\pi_1 S}$  (where we use the embedding  $i_n: \mathbb{R} \rightarrow \mathbb{R}^n$  as the least significant factor to identify  $\mathbb{R}$  with the minimal convex subgroup of  $\mathbb{R}^n$ ). Note that  $T$  and  $T'$  are then isospectral  $\mathbb{R}$ -trees in the terminology of §4.3.

Finally we apply Theorem 4.4 to the isospectral  $\mathbb{R}$ -trees  $T$  and  $T'$ , the straight map  $T_\phi \rightarrow T$  and the inclusion of  $T'$  as a subtree of  $\hat{T}$  to conclude that  $[\mathcal{F}(\phi)] \in \mathcal{L}_{M,X}$ .  $\square$

## 7. Analytic geometry in $Q(X)$

Theorem 6.5 shows that the large-scale behavior of the set  $\mathcal{V}_{M,\varepsilon} \subset Q(X)$  is constrained by an isotropic cone in  $\mathcal{MF}(S)$ . The goal of this section is to complete the proof of Theorem B by showing that only a discrete set can satisfy this constraint.

We begin with some generalities on real and complex limit points of sets in a complex vector space.

### 7.1. Real and complex boundaries

The vector space  $\mathbb{C}^n$  can be compactified to  $\mathbb{C}\mathbb{P}^n$  by adjoining a hyperplane at infinity  $\mathbb{C}\mathbb{P}^{n-1} = (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$ . When considering this compactification we regard  $\mathbb{C}\mathbb{P}^{n-1}$  as the *complex boundary* of  $\mathbb{C}^n$ , and write  $\partial_{\mathbb{C}}\mathbb{C}^n = \mathbb{C}\mathbb{P}^{n-1}$ .

Given a set  $R \subset \mathbb{C}^n$ , let  $\partial_{\mathbb{C}}R \subset \mathbb{C}\mathbb{P}^{n-1}$  denote its set of accumulation points in the complex boundary  $\partial_{\mathbb{C}}R = \bar{R} \cap \mathbb{C}\mathbb{P}^{n-1}$ , where  $\bar{R}$  is the closure of  $R$  in  $\mathbb{C}\mathbb{P}^n$ .

In the real analogue of these constructions we identify the sphere  $S^{2n-1}$  with the set of rays from the origin in  $\mathbb{C}^n$ ; for any  $q \in S^{2n-1}$  let  $r_q \subset \mathbb{C}^n$  denote the corresponding open ray. There is a corresponding compactification  $\bar{B}^{2n} = \mathbb{C}^n \sqcup S^{2n-1}$ , where  $\bar{B}^{2n}$  is the closed ball of dimension  $2n$ ; here a sequence  $z_k$  converges to  $q \in S^{2n-1}$  if it can be rescaled by positive real constants so as to converge to a point in  $r_q$ . In this sense  $S^{2n-1}$  is the *real boundary* of  $\mathbb{C}^n$  and we write  $\partial_{\mathbb{R}}\mathbb{C}^n = S^{2n-1}$ .

Given a set  $R \subset \mathbb{C}^n$ , let  $\partial_{\mathbb{R}}R \subset S^{2n-1}$  denote the accumulation points of  $R$  in the boundary of the real compactification  $\bar{B}^{2n}$ . This real boundary of  $R$  corresponds to a set of open rays in  $\mathbb{C}^n$ , and we denote by  $\text{Cone}_{\mathbb{R}}(R)$  the union of these rays, i.e.

$$\text{Cone}_{\mathbb{R}}(R) = \bigcup_{q \in \partial_{\mathbb{R}}R} r_q.$$

Equivalently  $\text{Cone}_{\mathbb{R}}(R)$  is the set of projective limits of the set  $R$  in the sense of §6.5.

Mapping a ray in  $\mathbb{C}^n$  to the complex line it spans induces the *Hopf fibration*  $\Pi: S^{2n-1} \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ , which gives  $S^{2n-1}$  the structure of a principal  $S^1$ -bundle. Identifying  $S^{2n-1}$  with the unit sphere in  $\mathbb{C}^n$ , the map  $\Pi$  is the restriction of the quotient map  $\hat{\Pi}: \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ , which is holomorphic. It is immediate from the definitions that for any set  $R \subset \mathbb{C}^n$  we have  $\partial_{\mathbb{C}}R = \Pi(\partial_{\mathbb{R}}R)$ .

We call a fiber of  $\Pi$  a *Hopf circle*. For any Hopf circle  $C = \Pi^{-1}(p) \subset S^{2n-1}$ , the set  $\bigcup_{q \in C} r_q = \hat{\Pi}^{-1}(p) \subset \mathbb{C}^n$  is a punctured complex line, and more generally if  $I \subset S^{2n-1}$  is an open arc of a Hopf circle, then  $\bigcup_{q \in I} r_q$  is an open sector in a complex line.

The following elementary lemma allows us to recognize totally real submanifolds of  $\mathbb{C}\mathbb{P}^{n-1}$  arising as boundaries of cones in  $\mathbb{C}^n$ .

**LEMMA 7.1.** *Let  $L \subset \mathbb{C}^n \setminus \{0\}$  be an  $\mathbb{R}^+$ -invariant and totally real submanifold of dimension  $m$ . Then  $\partial_{\mathbb{C}}L$  (i.e. the projection of  $L$  to  $\mathbb{C}\mathbb{P}^{n-1}$ ) is an immersed totally real submanifold of  $\mathbb{C}\mathbb{P}^{n-1}$  of dimension  $m-1$ .*

*Proof.* By  $\mathbb{R}^+$ -invariance, intersecting  $L$  with the unit sphere gives a manifold  $L_1 \subset S^{2n-1}$  of dimension  $m-1$  naturally identified with  $\partial_{\mathbb{R}}L$ , and  $\partial_{\mathbb{C}}L = \Pi(L_1)$ . Let  $x \in L_1$ . Because  $L$  is totally real, the tangent space to  $T_x L_1$  is transverse to  $i\mathbb{R} \cdot x = \ker d\Pi_x$ , thus  $\Pi|_{L_1}$  is an immersion. Since the  $\mathbb{C}$ -span of  $T_x L_1$  has complex dimension  $m-1$  and is transverse to  $\mathbb{C} \cdot x$ , the differential  $d\hat{\Pi}$  maps it injectively and complex-linearly to  $T_{\Pi(x)}\mathbb{C}\mathbb{P}^{n-1}$ . The image of the totally real subspace  $T_x L_1$  under such a map is totally real, and the lemma follows.  $\square$

While the real and complex boundary constructions have been described for  $\mathbb{C}^n$ , they apply naturally to any finite-dimensional complex vector space; we will apply them to  $Q(X) \simeq \mathbb{C}^{3g-3}$ .

## 7.2. Real and complex boundaries of $\mathcal{V}_{M,\varepsilon}$

Using the symplectic properties of the foliation map (Theorem 5.8), we can now translate the properties of  $\mathcal{V}_{M,\varepsilon}$  established in Theorem 6.5 into analytic conditions satisfied by its real and complex boundaries.

**THEOREM 7.2.** *Let  $\mathcal{V}_{M,\varepsilon} = \text{hol}_\varepsilon^{-1}(\mathcal{E}_M) \subset Q(X)$  be the set of holomorphic quadratic differentials corresponding to projective structures with holonomy in the extension variety of the 3-manifold  $M$ . Then, we have the following:*

(i)  $\partial_{\mathbb{C}}\mathcal{V}_{M,\varepsilon}$  is locally contained in a totally real manifold; that is, either  $\partial_{\mathbb{C}}\mathcal{V}_{M,\varepsilon} = \emptyset$  or there exist  $p \in \partial_{\mathbb{C}}\mathcal{V}_{M,\varepsilon}$ , a neighborhood  $U$  of  $p$  in  $\partial_{\mathbb{C}}Q(X)$ , and a totally real, real-analytic submanifold  $N \subset U$  such that  $\partial_{\mathbb{C}}\mathcal{V}_{M,\varepsilon} \cap U \subset N$ ;

(ii)  $\partial_{\mathbb{R}}\mathcal{V}_{M,\varepsilon}$  does not contain an open arc of any Hopf circle; that is, for any  $p \in \partial_{\mathbb{C}}\mathcal{V}_{M,\varepsilon}$  the intersection  $\Pi^{-1}(p) \cap \partial_{\mathbb{R}}\mathcal{V}_{M,\varepsilon}$  has empty interior in the relative topology of  $\Pi^{-1}(p) \simeq S^1$ .

*Proof.* (i) Define

$$\mathcal{C}_{M,\varepsilon} := \text{Cone}_{\mathbb{R}}(\mathcal{V}_{M,\varepsilon}),$$

and suppose that  $\partial_{\mathbb{C}}\mathcal{V}_{M,\varepsilon} \neq \emptyset$  so that  $\mathcal{C} \neq \emptyset$ . Note that

$$\partial_{\mathbb{R}}\mathcal{V}_{M,\varepsilon} = \partial_{\mathbb{R}}\mathcal{C}_{M,\varepsilon} \quad \text{and} \quad \partial_{\mathbb{C}}\mathcal{V}_{M,\varepsilon} = \partial_{\mathbb{C}}\mathcal{C}_{M,\varepsilon}.$$

By Theorem 6.5, we have  $\mathcal{C}_{M,\varepsilon} \subset \mathcal{F}^{-1}(\mathcal{L}_{M,X})$ , and so

$$\partial_{\mathbb{C}}\mathcal{V}_{M,\varepsilon} \subset \partial_{\mathbb{C}}\mathcal{F}^{-1}(\mathcal{L}_{M,X}).$$

Thus it will suffice to locally cover  $\partial_{\mathbb{C}}\mathcal{F}^{-1}(\mathcal{L}_{M,X})$  by a totally real, real-analytic manifold and to ensure that this set contains a limit point of  $\mathcal{C}_{M,\varepsilon}$ .

Among the strata  $Q_j(X)$  intersected by  $\mathcal{C}_{M,\varepsilon}$ , let  $Q_k(X)$  be a maximal element (one not contained in the boundary of another stratum intersecting  $\mathcal{C}_{M,\varepsilon}$ ). Thus  $Q_k(X)$  has an open tubular neighborhood  $U_0 \subset Q(X)$  disjoint from all other strata intersecting  $\mathcal{C}_{M,\varepsilon}$ , so that  $\emptyset \neq \mathcal{C}_{M,\varepsilon} \cap U_0 \subset Q_k(X)$ . Furthermore since  $Q_k(X)$  is  $\mathbb{C}^*$ -invariant, we can choose  $U_0$  to be  $\mathbb{C}^*$ -invariant as well. In particular, the set of complex lines in  $U_0$  is an open set  $\partial_{\mathbb{C}}U_0 \subset \partial_{\mathbb{C}}Q(X)$ .

By Theorem 5.8, the intersection  $\mathcal{F}^{-1}(\mathcal{L}_{M,X}) \cap Q_k(X)$  is a semianalytic set, i.e. it is locally defined by finitely many equations and inequalities of real-analytic functions. Indeed, for each  $p \in \mathcal{C}_{M,\varepsilon} \cap U_0$  the theorem gives an open neighborhood  $V$  of  $p$  in  $Q_k(X)$  such that  $\mathcal{F}: V \rightarrow \mathcal{ML}(\tau)$  is a real-analytic map into a train-track chart, and  $\mathcal{ML}(\tau) \cap \mathcal{L}_{M,X}$  is a union of convex cones in linear subspaces (thus semianalytic).

As  $\mathcal{F}^{-1}(\mathcal{L}_{M,X}) \cap Q_k(X)$  is a union of rays in  $Q(X)$ , its real boundary is the same as its intersection with the unit sphere in  $Q(X)$  and it is also semianalytic. Thus the set

$$\partial_{\mathbb{C}}(\mathcal{F}^{-1}(\mathcal{L}_{M,X}) \cap Q_k(X)) = \Pi(\partial_{\mathbb{R}}(\mathcal{F}^{-1}(\mathcal{L}_{M,X}) \cap Q_k(X)))$$

is the image of a semianalytic set under a proper real-analytic mapping, i.e. a subanalytic set. Such sets can be stratified by connected, real-analytic, subanalytic manifolds (see [Hi] and [Har]), and furthermore stratifications of  $\partial_{\mathbb{R}}(\mathcal{F}^{-1}(\mathcal{L}_{M,X}) \cap Q_k(X))$  and  $\partial_{\mathbb{C}}(\mathcal{F}^{-1}(\mathcal{L}_{M,X}) \cap Q_k(X))$  can be chosen so that  $\Pi$  maps strata to strata and so that the differential of  $\Pi$  has constant rank on each stratum [Har, Corollary 4.4].

Let  $N \subset \partial_{\mathbb{C}}(\mathcal{F}^{-1}(\mathcal{L}_{M,X}) \cap Q_k(X))$  be a stratum maximal among those intersecting  $\partial_{\mathbb{C}}\mathcal{V}_{M,\varepsilon}$ . Taking a tubular neighborhood  $U$  of  $N$  in  $\partial_{\mathbb{C}}U_0$  gives an open set in  $\partial_{\mathbb{C}}Q(X)$  in which  $\emptyset \neq \partial_{\mathbb{C}}\mathcal{V}_{M,\varepsilon} \cap U \subset N \cap U$ .

It remains to show that the real-analytic manifold  $N$  is totally real. Since  $\Pi$  is surjective, for each  $p \in N$  there is a stratum

$$N' \subset \partial_{\mathbb{R}}(\mathcal{F}^{-1}(\mathcal{L}_{M,X}) \cap Q_k(X))$$

such that  $\Pi(N')$  contains an open neighborhood of  $p$  in  $N$ . Thus  $N'' := \text{Cone}_{\mathbb{R}}(N')$  is a real-analytic submanifold of  $Q(X)$  and  $N'' \subset \mathcal{F}^{-1}(\mathcal{L}_{M,X}) \cap Q_k(X)$ .

Theorem 5.8 then implies that  $N''$  locally maps, by  $\mathcal{F}$ , into a finite union of isotropic subspaces of a train-track chart  $\mathcal{ML}(\tau)$ , and that in these local charts  $\mathcal{F}$  is a real-analytic, symplectic map. Thus  $N''$  is isotropic with respect to the symplectic structure of  $Q_k(X)$  given by Theorem 5.3. As the symplectic form on  $Q_k(X)$  is induced by a Kähler structure, an isotropic manifold is totally real. We have therefore described an open neighborhood of an arbitrary point  $p \in N$  as  $\Pi(\partial_{\mathbb{R}}N'') = \partial_{\mathbb{C}}N''$ , where  $N'' \subset Q(X)$  is  $\mathbb{R}^+$ -invariant and totally real. It follows by Lemma 7.1 that  $N$  itself is totally real.

(ii) Suppose on the contrary that  $\partial_{\mathbb{R}}\mathcal{V}_{M,\varepsilon}$  contains an open arc of a Hopf circle, or equivalently that the set  $\mathcal{C}_{M,\varepsilon}$  contains an open sector  $D$  in a complex line in  $Q(X)$ . Then  $D$  lies in a stratum  $Q_j(X) \subset Q(X)$ , so after possibly shrinking  $D$  we can apply Theorem 5.8 as in the previous paragraph to conclude that  $D$  is totally real, a contradiction.  $\square$

In order to show that  $\mathcal{V}_{M,\varepsilon}$  is discrete, we will derive a contradiction from the conditions (i) and (ii) of Theorem 7.2 and the assumption that  $\mathcal{V}_{M,\varepsilon}$  contains an analytic curve. The next few paragraphs develop necessary machinery for analyzing the real and complex boundaries of such curves, after which we return to the proof of Theorem B in §7.5 and §7.6.

### 7.3. Tangent cones and analytic curves

A general reference for the following material is [Ch2]. If  $E$  is a subset of  $\mathbb{C}^n$  and  $p \in \mathbb{C}^n$ , the *tangent cone* of  $E$  at  $p$  is the set  $C(E, p) \subset \mathbb{C}^n$  of points of the form

$$\lim_{k \rightarrow \infty} t_k(z_k - p),$$

where  $z_k \in E$ ,  $z_k \rightarrow p$ , and  $t_k \rightarrow 0^+$  as  $k \rightarrow \infty$ . The following basic properties of the tangent cone follow immediately from the definition:

- (1)  $C(E, p)$  is a closed  $\mathbb{R}^+$ -invariant set;
- (2)  $C(E, p) \neq 0$  if and only if  $p$  lies in the closure  $\bar{E}$ ;
- (3) for any  $E_1, E_2 \subset \mathbb{C}^n$  we have  $C(E_1 \cap E_2, p) \subset C(E_1, p) \cap C(E_2, p)$ ;
- (4) if  $E$  is a submanifold in a neighborhood of  $p \in E$ , then  $C(E, p)$  is the tangent space  $T_p E$ .

It follows from (4) that if  $E$  is an analytic curve (i.e. a complex-analytic set of dimension 1) and  $p \in E$  is a smooth point, then  $C(E, p)$  is a complex line. More generally, the tangent cone of an analytic curve at any point is a finite union of complex lines.

Furthermore, just as a complex submanifold is locally a graph over its tangent space, in a neighborhood of a point an analytic curve can be parameterized as follows (see [Ch2, §1.6.1]).

LEMMA 7.3. *Let  $U$  be an open neighborhood of  $0 \in \mathbb{C}^n$  and  $E \subset U$  be an analytic curve containing 0 and irreducible at that point. Suppose that  $C(E, 0) = \{(w_1, 0, \dots, 0) : w_1 \in \mathbb{C}\}$ . Then there exist a natural number  $m$  and a holomorphic map  $f: (\Delta, 0) \rightarrow (E, 0)$  such that  $f(\zeta) = (f_1(\zeta), \dots, f_n(\zeta))$ , where*

$$f_1(\zeta) = \zeta^m$$

and for each  $k > 1$  we have

$$f_k(\zeta) = \zeta^{m_k} h_k(\zeta),$$

where  $m_k > m$  and either  $h_k(\zeta) \equiv 0$  or  $h_k(\zeta)$  is holomorphic with  $h_k(0) \neq 0$ . The image  $f(\Delta)$  contains  $E \cap U'$  for some neighborhood  $U'$  of 0.  $\square$

Of course one can permute coordinates to obtain a similar parametrization when  $C(E, 0)$  is any of the coordinate axes in  $\mathbb{C}^n$ . Also note that in this lemma the number  $m$  is the *multiplicity* of  $E$  at  $p$ , and  $m > 1$  if and only if  $p$  is a singular point.

### 7.4. Analytic curves near a totally real manifold

Two results characterizing the behavior of an analytic curve near a totally real submanifold of  $\mathbb{C}^n$  will be essential in the sequel. The first describes the tangent cone of an

analytic curve in the complement of a totally real manifold at a boundary point. A (closed) *complex half-line* is a set of the form  $\{L(x+iy):y\geq 0\}$ , where  $L:\mathbb{C}\rightarrow\mathbb{C}^n$  is an injective complex linear map.

**THEOREM 7.4.** (Chirka [Ch1, Proposition 19]) *Let  $E\subset U\setminus M$  be an analytic curve where  $U\subset\mathbb{C}^n$  is open and  $M\subset U$  is a closed, totally real submanifold of class  $C^k$  for some  $k>1$ . Then for any  $p\in\overline{E}\cap M$  the tangent cone  $C(E,p)$  is a non-empty finite union of complex lines and half-lines.*

With additional regularity for the submanifold  $M$ , one has the following extension result.

**THEOREM 7.5.** (Chirka [Ch1, §1], Alexander [A]) *Let  $E\subset U\setminus M$  be an analytic curve, where  $U\subset\mathbb{C}^n$  is open and  $M\subset U$  is a closed, totally real, real-analytic submanifold. Then for any  $p\in\overline{E}\cap M$ , the set  $E$  admits an analytic continuation near  $p$ , i.e. there exist a neighborhood  $U'_p$  of  $p$  and an analytic curve  $E'_p\subset U'_p$  such that  $E\cap U'_p\subset E'_p$ .*

Further discussion of this result can be found in [Ch2, §20.5].

## 7.5. Hopf circles

If  $E\subset\mathbb{C}^n$  is an algebraic curve, then  $\partial_{\mathbb{C}}E$  is a finite set and  $\partial_{\mathbb{R}}E$  is the union of Hopf circles lying over  $\partial_{\mathbb{C}}E$ . The next theorem establishes a similar property of  $\partial_{\mathbb{R}}E$  when the algebraic assumption is replaced by the condition that  $\partial_{\mathbb{C}}E$  locally lie in a totally real manifold.

**THEOREM 7.6.** *Let  $E$  be an analytic curve in  $\mathbb{C}^n$  and suppose that for some  $p\in\partial_{\mathbb{C}}E$  there is a neighborhood  $U$  of  $p$  in  $\partial_{\mathbb{C}}\mathbb{C}^n$  and a totally real, real-analytic submanifold  $N$  of  $U$  such that  $\partial_{\mathbb{C}}E\cap U\subset N$ . Then  $\partial_{\mathbb{R}}E$  contains an open arc of a Hopf circle.*

In the proof we will consider  $\mathbb{C}^n$  as an affine chart of its compactification  $\mathbb{C}\mathbb{P}^n=\mathbb{C}^n\cup\partial_{\mathbb{C}}\mathbb{C}^n$ , but other affine charts of  $\mathbb{C}\mathbb{P}^n$  will also be used. In order to distinguish among them, we use the notation  $\mathbb{C}_z^n$  for the original affine chart (in which  $E$  is an analytic curve) with coordinates  $z_1, \dots, z_n$ , and  $\mathbb{C}_w^n$  will denote another affine chart with coordinates  $w_1, \dots, w_n$ .

*Proof.* Let  $V$  be an open neighborhood of  $p$  in  $\mathbb{C}\mathbb{P}^n$  such that  $V\cap\partial_{\mathbb{C}}\mathbb{C}_z^n=U$ . After possibly shrinking  $V$  and  $U$ , we may assume that  $V$  lies in an affine chart  $\mathbb{C}_w^n$  of  $\mathbb{C}\mathbb{P}^n$ .

By Theorem 7.5 we may assume, after further shrinking  $V$ , that  $E\cap V$  is a subset of an analytic curve  $E'\subset V$ . One of the irreducible components of  $E'$  at  $p$  must intersect  $E$ , so let  $\widehat{E}'$  denote such a component. Then  $\widehat{E}=\widehat{E}'\cap E$  is an analytic curve in  $U\cap\mathbb{C}_z^n$  such that  $\partial_{\mathbb{R}}\widehat{E}\subset N$ , and  $p$  is an accumulation point of  $\widehat{E}$ . By Theorem 7.4, the set  $C(\widehat{E},p)$



contains a complex half-line  $H$ . Since  $\widehat{E} \subset \widehat{E}'$ , the complex line  $L$  containing  $H$  is one of the finite set of lines comprising  $C(\widehat{E}', p)$ .

There are now two cases to consider, based on the relative position of  $L$  and the hyperplane  $\partial_{\mathbb{C}}\mathbb{C}_z^n$ :

- (1)  $L$  is transverse to  $\partial_{\mathbb{C}}\mathbb{C}_z^n$  (equivalently, it intersects  $\mathbb{C}_z^n$ ),
- (2)  $L$  is contained in  $\partial_{\mathbb{C}}\mathbb{C}_z^n$ .

Intuitively, these two cases correspond to whether the analytic curve  $E$  meets the hyperplane at infinity  $\partial_{\mathbb{C}}\mathbb{C}_z^n$  transversely or tangentially at  $p$ .

*Case 1.* By changing the affine chart  $\mathbb{C}_w^n$  and making an affine change of coordinates in  $\mathbb{C}_z^n$  and  $\mathbb{C}_w^n$ , we put  $p$ ,  $L$ , and  $H$  in a standard position; specifically, we suppose that in a homogeneous coordinate system for  $\mathbb{C}\mathbb{P}^n$  the inclusions of  $\mathbb{C}_z^n$  and  $\mathbb{C}_w^n$  are given by

$$\begin{aligned} (z_1, \dots, z_n) &\longmapsto [z_1 : \dots : z_n : 1], \\ (w_1, \dots, w_n) &\longmapsto [w_1 : \dots : w_{n-1} : 1 : w_n], \end{aligned}$$

that  $L$  is the  $w_n$ -axis in  $\mathbb{C}_w^n$ , and that  $H \subset L$  is defined by  $\text{Im}(w_n) \geq 0$ .

Let  $f: \Delta \rightarrow \mathbb{C}_w^n$  be a local parametrization of  $\widehat{E}'$  at 0 as in Lemma 7.3; note that here the tangent cone of  $\widehat{E}'$  is the  $w_n$ -axis. Then we have for any  $\zeta \neq 0$  that

$$\begin{aligned} f(\zeta) &= (\zeta^{m_1} h_1(\zeta), \dots, \zeta^{m_{n-1}} h_{n-1}(\zeta), \zeta^m) \in \mathbb{C}_w^n, \\ f(\zeta) &= (\zeta^{m_1-m} h_1(\zeta), \dots, \zeta^{m_{n-1}-m} h_{n-1}(\zeta), \zeta^{-m}) \in \mathbb{C}_z^n. \end{aligned}$$

As  $H$  is the tangent cone of  $\widehat{E} \subset \widehat{E}'$  as a subset of  $\mathbb{C}_w^n$ , for each  $\theta \in [0, \pi]$  we have a sequence  $\zeta_k \rightarrow 0$  such that  $\lim_{k \rightarrow \infty} \arg(\zeta_k^m) = \theta$  and  $f(\zeta_k) \in \widehat{E}$ . As a point in  $\mathbb{C}_z^n$ ,  $f(\zeta_k)$  lies on the same ray as  $|\zeta_k|^m f(\zeta_k)$  which has coordinates

$$(|\zeta_k|^m \zeta_k^{m_1-m} h_1(\zeta_k), \dots, |\zeta_k|^m \zeta_k^{m_{n-1}-m} h_{n-1}(\zeta_k), |\zeta_k|^m \zeta_k^{-m}) \in \mathbb{C}_z^n.$$

Since  $m_j > 0$  and  $h_j(\zeta_k)$  is bounded as  $\zeta_k \rightarrow 0$  for each  $1 \leq j \leq m-1$ , we have that the sequence  $|\zeta_k|^m f(\zeta_k) \in \mathbb{C}_z^n$  converges to  $(0, \dots, 0, e^{-i\theta}) \in \partial_{\mathbb{R}} E$ . Since  $\theta \in [0, \pi]$  was arbitrary, we find that  $\partial_{\mathbb{R}} E$  contains half of the Hopf circle containing  $(0, \dots, 0, 1)$ , completing this case.

*Case 2.* We begin as before, altering the argument as necessary.

Choose coordinates so that  $z_j$  and  $w_j$  are related to one another as above but now we take  $L$  to be the  $w_1$ -axis and  $H$  the subset with  $\text{Im}(w_1) \geq 0$ . Parameterizing  $\widehat{E}'$  and calculating as above we find that

$$\begin{aligned} f(\zeta) &= (\zeta^m, \zeta^{m_2} h_2(\zeta), \dots, \zeta^{m_n} h_n(\zeta)) \in \mathbb{C}_w^n, \\ f(\zeta) &= (\zeta^{m-m_n} h_n(\zeta)^{-1}, \zeta^{m_2-m_n} h_2(\zeta) h_n(\zeta)^{-1}, \\ &\quad \dots, \zeta^{m_{n-1}-m_n} h_{n-1}(\zeta) h_n(\zeta)^{-1}, \zeta^{-m_n} h_n(\zeta)^{-1}) \in \mathbb{C}_z^n. \end{aligned}$$

The coordinate expression in  $\mathbb{C}_z^n$  is well defined since  $h_n(\zeta) \neq 0$  for  $\zeta$  in a small punctured neighborhood of zero. Indeed, while Lemma 7.3 includes the possibility that  $h_n(\zeta) \equiv 0$ , this would mean that  $\widehat{E}' \cap \mathbb{C}_z^n = \emptyset$ , contradicting the assumption that  $\widehat{E}'$  intersects  $E$ . By a further linear change of coordinates, we may also assume without loss of generality that  $h_n(0) = 1$ . As before, the condition that  $H$  is the tangent cone of  $\widehat{E}$  gives for each  $\theta \in [0, \pi]$  a sequence  $\zeta_k \rightarrow 0$  such that  $\lim_{k \rightarrow \infty} \arg(\zeta_k^m) = \theta$  and  $f(\zeta_k) \in \widehat{E}$ . As a point in  $\mathbb{C}_z^n$ ,  $f(\zeta_k)$  lies on the same ray as  $|\zeta_k|^{m_n} f(\zeta_k)$  which has coordinates

$$\begin{aligned} & (|\zeta_k|^{m_n} \zeta_k^{m-m_n} h_n(\zeta_k)^{-1}, |\zeta_k|^{m_n} \zeta_k^{m_2-m_n} h_2(\zeta_k) h_n(\zeta_k)^{-1}, \\ & \dots, |\zeta_k|^{m_n} \zeta_k^{m_{n-1}-m_n} h_{n-1}(\zeta_k) h_n(\zeta_k)^{-1}, |\zeta_k|^{m_n} \zeta_k^{-m_n} h_n(\zeta_k)^{-1}) \in \mathbb{C}_z^n. \end{aligned}$$

As  $k \rightarrow \infty$  we find  $|\zeta_k|^{m_n} f(\zeta_k) \rightarrow (0, \dots, 0, e^{-(m_n/m)i\theta'}) \in \partial_{\mathbb{R}} E$  for some  $\theta' \equiv \theta \pmod{2\pi}$ . Allowing  $\theta$  to vary over  $[0, \pi]$ , we find that  $\partial_{\mathbb{R}} E$  contains an arc of a Hopf circle.  $\square$

## 7.6. Discreteness

Using the above results on analytic curves near a totally real manifold, the discreteness of  $\mathcal{H}_X \cap \mathcal{E}_M$  now follows easily.

*Proof of Theorem B (connected boundary).* Suppose on the contrary that the intersection  $\mathcal{H}_X \cap \mathcal{E}_M$  is not discrete. Since this set is a finite union of analytic subvarieties  $\{\mathcal{H}_{X,\varepsilon} \cap \mathcal{E}_M : \varepsilon \in \text{Spin}(X)\}$ , at least one of these subvarieties is not discrete. Thus there exists some  $\varepsilon \in \text{Spin}(X)$  such that  $\mathcal{H}_{X,\varepsilon} \cap \mathcal{E}_M$  contains an analytic curve, as does its preimage  $\mathcal{V}_{M,\varepsilon} = \text{hol}^{-1}(\mathcal{E}_M)$ .

By the maximum principle,  $\mathcal{V}_{M,\varepsilon}$  is non-compact and  $\partial_{\mathbb{C}} \mathcal{V}_{M,\varepsilon} \neq \emptyset$ , so part (i) of Theorem 7.2 implies that  $\partial_{\mathbb{C}} \mathcal{V}_{M,\varepsilon}$  is locally contained in a real-analytic, totally real manifold. But then Theorem 7.6 gives an open arc of a Hopf circle contained in  $\partial_{\mathbb{R}} \mathcal{V}_{M,\varepsilon}$ , contradicting part (ii) of Theorem 7.2.  $\square$

## 8. Skinning maps: The connected boundary case

### 8.1. Hyperbolic structures

Let  $M$  be a compact, irreducible, atoroidal 3-manifold with connected incompressible boundary  $S = \partial M$  of genus  $g \geq 2$ . Then the interior  $M^\circ$  admits a complete hyperbolic structure by Thurston's geometrization theorem for Haken manifolds. Let  $\text{AH}(M) \subset \mathcal{X}(M, \text{PSL}_2\mathbb{C})$  denote the set of isometry classes of marked hyperbolic structures on  $M^\circ$ . The closed set  $\text{AH}(M)$  lies in the smooth locus of  $\mathcal{X}(M, \text{PSL}_2\mathbb{C})$  (see [Kap, §8.8]), and

its interior  $\text{GF}(M)$  consists of the convex cocompact hyperbolic structures. The quasi-conformal deformation theory of Kleinian groups gives a holomorphic parametrization of  $\text{GF}(M)$  by the Teichmüller space  $\mathcal{T}(S)$  (see e.g. [Be2] or [Kr]). We denote this *Ahlfors–Bers parametrization* by

$$\begin{aligned} \mathcal{T}(S) &\longrightarrow \text{GF}(M) \subset \mathcal{X}(M, \text{PSL}_2\mathbb{C}), \\ X &\longmapsto \varrho_X^M. \end{aligned}$$

## 8.2. Quasi-Fuchsian groups

The set  $\text{QF}(S) \subset \mathcal{X}(S, \text{PSL}_2\mathbb{C})$  of characters of quasi-Fuchsian representations is an open subset of the smooth locus in  $\mathcal{X}(S, \text{PSL}_2\mathbb{C})$  that has a natural parametrization by the product of Teichmüller spaces  $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ . As a set  $\text{QF}(S)$  does not depend on the orientation of  $S$ , but the orientation *is* used to distinguish the factors in this parametrization. This coordinate system for  $\text{QF}(S)$  is a particular case of the Ahlfors–Bers coordinates, since  $\text{QF}(S) = \text{GF}(S \times I) \subset \mathcal{X}(S \times I, \text{PSL}_2\mathbb{C}) = \mathcal{X}(S, \text{PSL}_2\mathbb{C})$ . We write  $Q(X, Y)$  for the point in  $\text{QF}(S)$  corresponding to the pair  $(X, Y) \in \mathcal{T}(S) \times \mathcal{T}(\bar{S})$ .

Given  $Y \in \mathcal{T}(\bar{S})$ , the *Bers slice* is the subset

$$B_Y = \{Q(X, Y) : X \in \mathcal{T}(S)\} \subset \text{QF}(S),$$

that is,  $B_Y$  is a “horizontal slice” of the product structure of the quasi-Fuchsian space. Similarly, we can define the vertical Bers slices  $B_X = \{Q(X, Y) : Y \in \mathcal{T}(\bar{S})\}$  for  $X \in \mathcal{T}(S)$ .

Note  $B_Y$  is naturally in one-to-one correspondence with  $\mathcal{T}(S)$ . An inequality of Bers shows that the lengths of geodesics in the hyperbolic 3-manifold corresponding to a quasi-Fuchsian group  $Q(X, Y)$  have an upper bound in terms of lengths in the uniformization of either  $X$  or  $Y$  [Be1, Theorem 3]. Since one of these is fixed in a Bers slice, there are uniform bounds on the traces of any finite set of elements in  $\pi_1 S$  over  $B_Y$ , and thus the following result holds.

LEMMA 8.1. (Bers) *For each  $Y \in \mathcal{T}(\bar{S})$ , the Bers slice  $B_Y$  is precompact.*

Each point in the Bers slice  $B_Y$  induces a  $\mathbb{C}\mathbb{P}^1$ -structure on  $Y$  by taking the quotient of one of the domains of discontinuity of the associated quasi-Fuchsian group. This construction is a local inverse of the holonomy map, i.e. it gives an open subset of  $Q(Y)$  (the *Bers embedding*) that maps biholomorphically onto  $B_Y$  by *hol*. In particular, we have  $B_Y \subset r(\mathcal{H}_Y)$ , where  $r: \mathcal{X}(S, \text{SL}_2\mathbb{C}) \rightarrow \mathcal{X}(S, \text{PSL}_2\mathbb{C})$  is the map induced by the covering  $\text{SL}_2\mathbb{C} \rightarrow \text{PSL}_2\mathbb{C}$ . This applies equally to the vertical slices, i.e.  $B_X \subset r(\mathcal{H}_X)$  for  $X \in \mathcal{T}(S)$ .

### 8.3. The skinning map

The restriction map  $i^*: \mathcal{X}(M, \mathrm{PSL}_2\mathbb{C}) \rightarrow \mathcal{X}(S, \mathrm{PSL}_2\mathbb{C})$  sends  $\mathrm{GF}(M)$  into  $\mathrm{QF}(S)$ , and in terms of the Ahlfors–Bers parametrization it is the identity on one Teichmüller space factor, i.e.

$$i^*(\varrho_X^M) = Q(X, \sigma_M(X)),$$

which defines a map

$$\sigma_M: \mathcal{T}(S) \longrightarrow \mathcal{T}(\bar{S}),$$

called the *skinning map* of  $M$ .

Fibers of the skinning map are related to sets  $r(\mathcal{H}_Y \cap \mathcal{E}_M)$  as follows.

LEMMA 8.2. *The preimage  $\sigma_M^{-1}(Y)$  is in bijection with a precompact set*

$$F_Y \subset r(\mathcal{H}_Y \cap \mathcal{E}_M).$$

*Proof.* Given  $Y \in \mathcal{T}(\bar{S})$ , we consider the set of quasi-Fuchsian groups

$$F_Y = i^*(\mathrm{GF}(M)) \cap B_Y,$$

i.e.

$$F_Y = \{Q(X, Y) : \text{there exists } X \in \mathcal{T}(S) \text{ such that } i^*(\varrho_X^M) = Q(X, Y)\}.$$

From the definition of the skinning map it is immediate that  $F_Y$  is in bijection with the preimage  $\sigma_M^{-1}(Y)$  by

$$Q(X, Y) \in F_Y \quad \text{if and only if} \quad X \in \sigma_M^{-1}(Y).$$

Furthermore  $F_Y \subset B_Y \subset r(\mathcal{H}_Y)$ , and precompactness of  $F_Y$  follows from that of  $B_Y$ , so it remains only to show that  $F_Y \subset r(\mathcal{E}_M)$ .

By definition  $\mathcal{E}_M$  contains a Zariski dense subset  $i^*(\mathcal{X}(M, \mathrm{SL}_2\mathbb{C}))$ , using the commutative diagram (6.1) for  $r$  and  $i^*$  we have

$$i^*(r(\mathcal{X}(M, \mathrm{SL}_2\mathbb{C}))) = r(i^*(\mathcal{X}(M, \mathrm{SL}_2\mathbb{C}))) \subset r(\mathcal{E}_M).$$

Since  $F_Y \subset i^*(\mathrm{GF}(M))$ , it is enough to know that  $\mathrm{GF}(M) \subset r(\mathcal{X}(M, \mathrm{SL}_2\mathbb{C}))$ , i.e. that the  $\mathrm{PSL}_2\mathbb{C}$ -representations arising from hyperbolic structures on  $M$  can be lifted to  $\mathrm{SL}_2\mathbb{C}$ . This is a well-known consequence of the parallelizability of 3-manifolds (see [Cu] or [CS, Theorem 3.1.1] for details).  $\square$

Finally, using this lemma we have the following result.

*Proof of Theorem A (connected boundary).* Suppose on the contrary that  $\sigma_M^{-1}(Y)$  is infinite. Then, by the previous lemma, the set  $r(\mathcal{H}_Y \cap \mathcal{E}_M)$  has an infinite and precompact subset, and therefore an accumulation point. Since  $\mathcal{H}_Y \cap \mathcal{E}_M$  is discrete by Theorem B and the map  $r: \mathcal{X}(S, \mathrm{SL}_2\mathbb{C}) \rightarrow \mathcal{X}(S, \mathrm{PSL}_2\mathbb{C})$  is proper, this yields a contradiction.  $\square$

## 9. Disconnected boundary and tori

The previous sections established the main theorems for a 3-manifold with connected boundary. We now adapt the statements and proofs to the more general case of a compact oriented 3-manifold  $M$  whose boundary has at least one connected component that is not a torus.

As in the introduction, we denote by  $\partial_0 M$  the union of the non-torus boundary components of  $M$ . Let  $S_1, \dots, S_m$  denote the connected components of  $\partial_0 M$ , each equipped with the boundary orientation. Let  $\partial_1 M = \partial M \setminus \partial_0 M$  denote the union of the torus boundary components of  $M$ .

### 9.1. Measured foliations and Teichmüller spaces

In several cases we first need to adapt definitions of spaces associated with a surface to the disconnected case. We define the measured foliation space  $\mathcal{MF}(\partial_0 M)$  and Teichmüller space  $\mathcal{T}(\partial_0 M)$  to be the cartesian products of the spaces corresponding to the connected components  $S_j$ , e.g.

$$\mathcal{MF}(\partial_0 M) = \prod_{j=1}^m \mathcal{MF}(S_j).$$

The sum of the symplectic forms of the factors (using the boundary orientation) gives the Thurston symplectic form on  $\mathcal{MF}(\partial_0 M)$ .

For a point  $X = (X_1, \dots, X_m) \in \mathcal{T}(\partial_0 M)$ , we denote by  $Q(X)$  the direct sum of quadratic differential spaces,

$$Q(X) = \bigoplus_{j=1}^m Q(X_j).$$

The product of foliation maps of the factors gives the homeomorphism

$$\mathcal{F}: Q(X) \longrightarrow \mathcal{MF}(\partial_0 M).$$

For the 3-manifold  $M$  and its fundamental group, in some cases we must treat torus boundary components differently. Instead of considering arbitrary isometric actions of  $\pi_1 M$  on  $\mathbb{R}$ -trees, we restrict attention to actions in which each subgroup of  $\pi_1 M$  represented by a component of  $\partial_1 M$ —that is, each *boundary torus subgroup*—has a fixed point.

### 9.2. Isotropic cones

The isotropic cone construction (Theorem 4.4, the main result of §§3–4) generalizes to the disconnected case as follows.

**THEOREM 9.1.** *For each  $X=(X_1, \dots, X_M) \in \mathcal{J}(\partial_0 M)$  there exists an isotropic piecewise linear cone  $\mathcal{L}_{M,X} \subset \mathcal{MF}(\partial_0 M)$  with the following property: Let  $T$  be a  $\Lambda$ -tree on which  $\pi_1 M$  acts so that each boundary torus subgroup has a fixed point. Let  $\mathfrak{o}: \mathbb{R} \rightarrow \Lambda$  be an order-preserving embedding. Suppose that, for each  $1 \leq j \leq m$ , we have*

- a pair  $T_{\ell_j}, T'_{\ell_j}$  of  $\mathbb{R}$ -trees on which  $\pi_1 S_j$  acts minimally with length function  $\ell_j$ ;
- a holomorphic quadratic differential  $\phi_j \in Q(X_j)$ ;
- a  $\pi_1 S_j$ -equivariant straight map  $T_{\phi_j} \rightarrow T_{\ell_j}$  with respect to  $\mathfrak{o}$ ;
- a  $\pi_1 S_j$ -equivariant isometric embedding  $k: T'_{\ell_j} \rightarrow T$ .

Then  $[\mathcal{F}(\phi)] \in \mathcal{L}_{M,X}$ , where  $\phi = (\phi_1, \dots, \phi_m)$ .

*Proof.* First we adapt the definition of the cone  $\mathcal{L}_{M,X}$  from the connected case. We choose a finite set of triangulations of  $M$  that extend the triangulations of  $S_j$  given by Lemma 4.3. For each such triangulation  $\Delta_M$  we have a space  $W_4(\Delta_M, \Lambda)$  of weights satisfying the four-point condition in each 3-simplex, but we now also require these weights to be identically zero on the edges of each boundary torus.

As in Lemmas 3.5 and 3.7, we find that the restriction of  $W_4(\Delta_M, \mathbb{R})$  to the edges of the boundary triangulation gives a finite union of isotropic subspaces for a symplectic form that is the sum of the Thurston forms for the triangulated surfaces  $S_j$  and a similar alternating 2-form for the weight space of each boundary torus. Since the weights in  $W_4(\Delta_M)$  are identically zero in the torus components, these subspaces are still isotropic when projected to the product of non-torus factors. Thus  $W_4(\Delta_M)$  gives an isotropic cone in a product of train-track charts  $\mathcal{MF}(\tau_j)$  for the surfaces  $S_j$ , and taking the union of these over the finite set of triangulations of  $M$  gives the cone  $\mathcal{L}_{M,X} \subset \mathcal{MF}(\partial_0 M)$ .

Now we show that  $[\mathcal{F}(\phi)] \in \mathcal{L}_{M,X}$ , or equivalently that the train-track coordinates of  $[\mathcal{F}(\phi)]$  are obtained by restricting an element of  $W_4(\Delta_M)$  to the boundary. Here  $\Delta_M$  is the triangulation from our finite set in which the edges on  $S_j$  can be realized  $\phi_j$ -geodesically.

Using the straight maps  $T_{\phi_j} \rightarrow T_{\ell_j}$  and either the isometry  $T_{\ell_j} \rightarrow T'_{\ell_j}$  of Theorem 4.5 (in the case of a non-abelian length function) or the partially defined map  $T_{\ell_j} \dashrightarrow T'_{\ell_j}$  of Theorem 4.7 (in the abelian case), we obtain a map from the non-torus boundary vertices,  $\tilde{\Delta}_M^{(0)} \cap \partial_0 M$ , to the tree  $T$ . We extend this over the vertices on the torus boundary components by mapping all vertices in a given boundary torus to a fixed point of the associated subgroup of  $\pi_1 M$ .

Extending over the remaining (interior) vertices of  $\Delta_M$  as in Propositions 3.2 and 4.2, we obtain a map  $\tilde{\Delta}_M^{(0)} \rightarrow T$  whose associated weight function  $w$  lies in  $W_4(\Delta_M, \Lambda)$ ; note that since all vertices of a boundary torus are mapped to a single point of  $T$ , the associated weight vanishes on edges of the torus boundary components as required. We push forward by a left inverse of  $\mathfrak{o}$  to obtain an element of  $W_4(\Delta_M)$  whose values on the non-torus

boundary edges give the train-track coordinates of  $[\mathcal{F}(\phi)]$ . Thus  $[\mathcal{F}(\phi)] \in \mathcal{L}_{M,X}$ .  $\square$

### 9.3. Kähler structure and symplectomorphism

The results of §5 generalize easily to disconnected surfaces by taking products of the spaces, maps, and stratifications considered there.

Specifically, for any  $X \in \mathcal{T}(\partial_0 M)$ , Lemma 5.1 provides a stratification of  $Q(X_j)$ . There is an induced *product stratification* of the product space

$$Q(X) = \bigoplus_{j=1}^m Q(X_j)$$

consisting of products of strata in the factors. Note that the origin  $\{0\} \in Q(X_j)$  is the minimal stratum in each factor, so  $Q(X)$  now has non-trivial (positive-dimensional) strata consisting of quadratic differentials that are zero on one or more of the boundary components.

Similarly, we take the product of the stratified Kähler structures on the factors  $Q(X_j)$  to obtain a Kähler structure on  $Q(X)$ , smooth relative to the product stratification. Applying Theorem 5.8 to each factor of the map  $\mathcal{F}: Q(X) \rightarrow \mathcal{MF}(\partial_0 M)$ , we obtain the following result.

**THEOREM 9.2.** *For any  $X \in \mathcal{T}(\partial_0 M)$  the map  $\mathcal{F}: Q(X) \rightarrow \mathcal{MF}(\partial_0 M)$  is a real-analytic stratified symplectomorphism. That is, if  $Q_k(X)$  is a stratum of  $Q(X)$ , then*

(i) *for any  $\phi = (\phi_1, \dots, \phi_m) \in Q_k(X)$  there exist an open neighborhood  $U \subset Q_k(X)$  of  $\phi$  and a product of train-track coordinate charts  $\prod_{j=1}^m \mathcal{MF}(\tau_j) \subset \mathcal{MF}(\partial_0 M)$  covering  $\mathcal{F}(U)$  so that the restriction*

$$\mathcal{F}: U \longrightarrow \prod_{j=1}^m \mathcal{MF}(\tau_j)$$

*is a real-analytic diffeomorphism onto its image, and*

(ii) *the derivative  $d\mathcal{F}_\phi$  defines a symplectic linear map from  $T_\phi Q_k(X)$  into*

$$\bigoplus_{j=1}^m W(\tau_j),$$

*where  $T_\phi Q_k(X)$  is equipped with the symplectic form  $\sum_j \omega_{\phi_j}$  and  $\bigoplus_{j=1}^m W(\tau_j)$  is given the Thurston symplectic form.*

To make sense of this statement in case the differentials in the stratum  $Q_k(X)$  are identically zero in some factor, say  $Q(X_j)$ , we adopt the convention that  $\tau_j$  is the empty train track and that  $\mathcal{MF}(\tau_j) = W(\tau_j) = \{0\}$  is a point representing the empty foliation on  $X_j$ , which is the image of 0 under the map  $\mathcal{F}: Q(X_j) \rightarrow \mathcal{MF}(S_j)$ .

#### 9.4. Character varieties and extension varieties

We generalize the character variety of a connected surface to  $\partial_0 M$  by taking the product of character varieties of components

$$\mathcal{X}(\partial_0 M, G) := \prod_{j=1}^m \mathcal{X}(S_j, G),$$

and similarly for the representation variety  $\mathcal{R}(\partial_0 M, G)$ . Note that while  $\mathcal{R}(\partial_0 M, G)$  can also be described as the representation variety of the free product  $\pi_1 S_1 * \dots * \pi_1 S_m$ , the character variety of this free product does *not* agree with our definition of  $\mathcal{X}(\partial_0 M, G)$ . To obtain  $\mathcal{X}(\partial_0 M, G)$  from  $\mathcal{R}(\partial_0 M, G)$  one must take the quotient by the action of  $G^m$ .

The 3-manifold character variety also requires modification to account for the presence of boundary tori. As is standard when considering complete hyperbolic structures, rather than working with the full character variety of  $\pi_1 M$ , we consider the subvariety

$$\mathcal{X}(M, \partial_1 M, G) \subset \mathcal{X}(M, G)$$

consisting of characters of representations that map each boundary torus subgroup (that is, the fundamental group of each connected component of  $\partial_1 M$ ) to parabolic elements of  $G$ .

The inclusion of each boundary component  $S_j \hookrightarrow M$  induces a restriction map

$$\mathcal{X}(M, \partial_1 M, \mathrm{SL}_2\mathbb{C}) \longrightarrow \mathcal{X}(S_j, \mathrm{SL}_2\mathbb{C}),$$

and taking the product of these we obtain a regular map

$$i^*: \mathcal{X}(M, \partial_1 M, \mathrm{SL}_2\mathbb{C}) \longrightarrow \mathcal{X}(\partial_0 M, \mathrm{SL}_2\mathbb{C}).$$

We define the extension variety  $\mathcal{E}_M \subset \mathcal{X}(\partial_0 M, \mathrm{SL}_2\mathbb{C})$  as the Zariski closure of the image  $i^*(\mathcal{X}(M, \partial_1 M, \mathrm{SL}_2\mathbb{C}))$ .

The Morgan–Shalen compactification of  $\mathcal{X}(\Gamma, \mathrm{SL}_2\mathbb{C})$  was defined in §6.2 using trace functions of elements of  $\Gamma$ . On the product  $\mathcal{X}(\partial_0 M, \mathrm{SL}_2\mathbb{C})$  we have trace functions for the elements of each component  $\pi_1 S_j$ , so the family of all such functions is indexed by the disjoint union

$$H := \pi_1 S_1 \sqcup \dots \sqcup \pi_1 S_m.$$

To adapt the Morgan–Shalen compactification to this case, we map the character variety of  $\partial_0 M$  to the projective space  $\mathbb{P}(\mathbb{R}^H)$  using formula (6.2) and take its closure. Indeed, a compactification in this generality was already discussed in [MoS1], where the map to projective space arising from an arbitrary collection of regular functions on an algebraic variety is considered.



A boundary point of the resulting compactification of  $\mathcal{X}(\partial_0 M, \mathrm{SL}_2\mathbb{C})$  is therefore an  $\mathbb{R}^+$ -equivalence class  $[\ell]$ , where  $\ell=(\ell_1, \dots, \ell_m)$  is a tuple of functions,  $\ell_j: \pi_1 S_j \rightarrow \mathbb{R}$ . The factor  $\ell_j$  is either a non-trivial length function of an action of  $\pi_1 S_j$  on an  $\mathbb{R}$ -tree or is identically zero (which is the length function of the action of  $\pi_1 S_j$  on a point).

Having adapted the definitions of its objects suitably, Theorem 6.3 on extensions of length functions arising from the boundary of  $\mathcal{E}_M$  generalizes to the following.

**THEOREM 9.3.** *Let  $[\ell]$  be a boundary point of  $\mathcal{E}_M$  in the Morgan–Shalen compactification of  $\mathcal{X}(\partial_0 M, \mathrm{SL}_2\mathbb{C})$ , where  $\ell=(\ell_1, \dots, \ell_m)$ . Then there exists a function  $\hat{\ell}: \pi_1 M \rightarrow \mathbb{R}^n$  such that the following conditions hold:*

- (i) *the group  $\pi_1 M$  acts isometrically on a  $\mathbb{R}^n$ -tree with length function  $\hat{\ell}$  such that each boundary torus subgroup of  $\pi_1 M$  has a fixed point;*
- (ii) *the function  $\hat{\ell}$  is a simultaneous extension of the functions  $\ell_j$ , i.e. for each  $\gamma \in \pi_1 S_j$  we have  $\hat{\ell}(i_*(\gamma))=i_n(\ell(\gamma))$ , where  $i_*: \pi_1 S_j \rightarrow \pi_1 M$  is induced by the inclusion of  $S_j$  as a boundary component of  $M$  and  $i_n(x)=(0, \dots, 0, x)$ .*

*Proof.* As in the proof of Theorem 6.3 we have an extension of fields  $k(\mathcal{E}_M^0) \rightarrow k(\mathcal{X}_M^0)$ , where now  $\mathcal{E}_M^0 \subset \mathcal{X}(\partial_0 M, \mathrm{SL}_2\mathbb{C})$  is an irreducible component of  $\mathcal{E}_M$  which has  $[\ell]$  in its boundary and  $\mathcal{X}_M^0 \subset \mathcal{X}(M, \partial_1 M, \mathrm{SL}_2\mathbb{C})$ .

The boundary point  $[\ell]$  determines a valuation  $v: k(\mathcal{E}_M^0) \rightarrow \Lambda$  by the analogue of Theorem 6.1 for subvarieties of the product of character varieties  $\mathcal{X}(\partial_0 M, \mathrm{SL}_2\mathbb{C})$ ; like Theorem 6.1, this case is covered by the more general comparison of valuation- and projectivization-based compactifications of [MoS1, Theorem I.3.6].

Extending this valuation to the superfield  $k(\mathcal{X}_M^0)$  and proceeding as in the proof of Theorem 6.3 then gives a function  $\hat{\ell}: \pi_1 M \rightarrow \mathbb{R}^n$  satisfying condition (ii) above; note that the argument of (6.4) applies to each function  $\ell_j: \pi_1 S_j \rightarrow \mathbb{R}$ ,  $1 \leq j \leq m$ , giving that  $\hat{\ell}$  is a simultaneous extension.

The resulting length function  $\hat{\ell}$  arises from an action of  $\pi_1 M$  on an  $\mathbb{R}^n$ -tree  $T$ , but we must show that each boundary torus subgroup has a fixed point. As  $\mathcal{X}_M^0$  is an irreducible subvariety of  $\mathcal{X}(M, \partial_1 M, \mathrm{SL}_2\mathbb{C})$ , the trace function  $t_\gamma$  of an element  $\gamma$  of a boundary torus subgroup restricts to a constant  $\pm 2$  on  $\mathcal{X}_M^0$  (the possible traces of parabolics). Therefore we have  $v'(t_\gamma)=0$ , where  $v'$  is the valuation of  $k(\mathcal{X}_M^0)$  from which  $T$  is constructed. By Lemma 2.7, each boundary torus subgroup leaves invariant a subtree of  $T$  on which it acts with zero length function, and by [MoS1, Proposition II.2.15] there is a fixed point.  $\square$

### 9.5. Holonomy, the isotropic cone, and discreteness

The construction of the holonomy variety in §6.5 extends to  $\partial_0 M$  by taking products; that is, for  $X \in \mathcal{T}(\partial_0 M)$  or  $X \in \mathcal{T}(\overline{\partial_0 M})$ , where  $X = (X_1, \dots, X_m)$ , we define

$$\mathcal{H}_X := \bigcup_{\varepsilon \in \text{Spin}(X)} \mathcal{H}_{X,\varepsilon}, \quad \text{where } \mathcal{H}_{X,\varepsilon} := \prod_{j=1}^m \mathcal{H}_{X_j,\varepsilon_j}.$$

Here  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$  is a tuple of spin structures on the components.

The subvariety of the quadratic differential space corresponding to the intersection  $\mathcal{H}_{X,\varepsilon} \cap \mathcal{E}_M$  is

$$\mathcal{V}_{M,\varepsilon} := \text{hol}_\varepsilon^{-1}(\mathcal{E}_M) \subset Q(\partial_0 M).$$

Generalizing Theorem 6.5, we have the following.

**THEOREM 9.4.** *Let  $\phi = (\phi_1, \dots, \phi_m)$  be the projective limit of a divergent sequence in  $\mathcal{V}_{M,\varepsilon} \subset Q(\partial_0 M)$ . Then  $[\mathcal{F}(\phi)] \in \mathcal{L}_{M,X}$ , where  $\mathcal{L}_{M,X}$  is the isotropic cone of Theorem 9.1.*

*Proof.* Theorems 9.1 and 9.3 provide the necessary generalizations to adapt the proof of Theorem 6.5 to this situation, except for the existence of the straight maps  $T_{\phi_j} \rightarrow T_{\ell_j}$ .

In the connected boundary case, a straight map  $T_\phi \rightarrow T_\ell$  is given by Theorem 6.4 (i.e. [D, Theorem A]). Both the projective limit  $\phi$  and the representative  $\ell$  of the Morgan–Shalen boundary point are only well defined up to multiplication by a positive constant in this case, but it suffices to have such a straight map for some pair of representatives  $\ell$  and  $\phi$ .

In the disconnected case, both tuples  $\phi = (\phi_1, \dots, \phi_m)$  and  $\ell = (\ell_1, \dots, \ell_m)$  represent equivalence classes up to a single multiplicative constant, whereas a factorwise application of Theorem 6.4 would seem to require a separate multiplicative factor for each connected component of the boundary.

To remedy this, we choose representatives  $\ell$  and  $\phi$  as in [D], where it is shown that for a connected surface  $S$ ,  $X \in \mathcal{T}(S)$ , and a divergent sequence  $\{\phi_n\}_{n=1}^\infty \subset Q(X)$ , we can extract a representative  $\ell$  of the Morgan–Shalen limit of  $\text{hol}(\phi_n)$  by scaling the functions  $\gamma \mapsto \log(|t_\gamma(\text{hol}(\phi_n))| + 2)$  by the factors  $\|\phi_n\|^{-1/2}$  and taking the limit  $\ell \in \mathbb{R}^{\pi_1 S}$ . Furthermore, the straight map  $T_\phi \rightarrow T_\ell$  of Theorem 6.4 is defined for this function  $\ell$  and for the projective limit  $\phi \in Q(X)$  satisfying  $\|\phi\| = C$ , for a universal constant  $C$ .

Correspondingly, for the disconnected case we consider  $X \in \mathcal{T}(\partial_0 M)$  and a sequence of tuples  $\{(\phi_{1,n}, \dots, \phi_{m,n})\}_{n=1}^\infty$  which converges projectively. Equivalently, the sequence

$$\{c_n(\phi_{1,n}, \dots, \phi_{m,n})\}_{n=1}^\infty$$

converges in  $Q(X)$  as  $n \rightarrow \infty$ , where

$$c_n = \left( \sum_{j=1}^m \|\phi_{j,n}\| \right)^{-1}.$$

Applying Theorem 6.4 to each factor and using the representative length functions and projective limits discussed above, after passing to a subsequence we obtain straight maps

$$T_{\phi_j^{(1)}} \longrightarrow T_{\ell_j^{(1)}}, \quad (9.1)$$

where

$$\phi_j^{(1)} = C \lim_{n \rightarrow \infty} \frac{\phi_{j,n}}{\|\phi_{j,n}\|}$$

and  $\ell_j^{(1)}: \pi_1 S_j \rightarrow \mathbb{R}$  is the limit as  $n \rightarrow \infty$  of the functions

$$\gamma \mapsto \frac{1}{\|\phi_{j,n}\|^{1/2}} \log(|t_\gamma(\text{hol}(\phi_{j,n}))| + 2).$$

Since  $0 \leq \|\phi_{j,n}\| \leq c_n^{-1}$  we can take a further subsequence so that for each  $j$  the limit

$$r_j = \lim_{n \rightarrow \infty} c_n \|\phi_{j,n}\| \in [0, 1]$$

exists. Then

$$\phi := (r_1 \phi_1^{(1)}, \dots, r_n \phi_n^{(1)}) = C \lim_{n \rightarrow \infty} c_n (\phi_{1,n}, \dots, \phi_{m,n}) \in Q(X)$$

is a projective limit of the sequence of quadratic differentials and

$$\ell := (r_1^{1/2} \ell_1^{(1)}, \dots, r_n^{1/2} \ell_n^{(1)}) = \lim_{n \rightarrow \infty} (\gamma \mapsto c_n^{1/2} \log(|t_\gamma(\text{hol}(\phi_{j,n}))| + 2))_{\gamma \in \pi_1 S_j, 1 \leq j \leq n} \in \mathbb{R}^H$$

represents the limit of the holonomy representations in the Morgan–Shalen compactification of  $\mathcal{X}(\partial_0 M, \text{SL}_2 \mathbb{C})$ . The desired straight maps  $T_{\phi_j} \rightarrow T_{\ell_j}$  are therefore given by (9.1) for each  $j$  with  $r_j \neq 0$ , since the trees  $T_{\phi_j}$  and  $T_{\ell_j}$  are obtained from  $T_{\phi_j^{(n)}}$ ,  $T_{\ell_j^{(n)}}$  by multiplying the metrics by  $r_j^{1/2}$ . In each factor where  $r_j = 0$  we have  $\phi_j = 0$  and  $\ell_j = 0$ , so each of  $T_{\phi_j}$  and  $T_{\ell_j}$  is a point, and there is nothing to prove.  $\square$

Using Theorems 9.4 and 9.2 in place of their counterparts for the connected boundary case (Theorems 6.5 and 5.8, respectively), the properties of the real and complex boundaries of  $\mathcal{V}_{M,\varepsilon}$  from Theorem 7.2 follow for the general case as well. The argument of §7.6 then gives the discreteness of these varieties, completing the proof of Theorem B in the general case.

## 9.6. Skinning maps

Now suppose that in addition to the hypotheses of Theorem B that the 3-manifold  $M$  has incompressible boundary and that its interior admits a complete hyperbolic structure with no accidental parabolics. The space  $\text{GF}(M)$  of geometrically finite hyperbolic

structures on  $M$  is naturally a subset of  $\mathcal{X}(M, \partial_1 M, \mathrm{PSL}_2\mathbb{C})$  which is parameterized by the Teichmüller space  $\mathcal{T}(\partial_0 M)$ ; as before we denote this parametrization by  $X \mapsto \varrho_M^X$ , where  $X = (X_1, \dots, X_m)$ .

The restriction map  $i^*: \mathcal{X}(M, \partial_1 M, \mathrm{PSL}_2\mathbb{C}) \rightarrow \mathcal{X}(\partial_0 M, \mathrm{PSL}_2\mathbb{C})$  sends  $\varrho_M^X$  to a tuple of quasi-Fuchsian groups

$$i^*(\varrho_M^X) = (Q(X_1, Y_1), \dots, Q(X_m, Y_m)) \in \prod_{j=1}^m \mathrm{QF}(S_j),$$

and this defines the *skinning map*

$$\begin{aligned} \sigma_M: \mathcal{T}(\partial_0 M) &\longrightarrow \mathcal{T}(\overline{\partial_0 M}), \\ (X_1, \dots, X_m) &\longmapsto (Y_1, \dots, Y_m). \end{aligned}$$

With this definition in hand, the generalization of the proof of the main theorem from the connected case is straightforward.

*Proof of Theorem A (general case).* Suppose on the contrary that  $\sigma_M^{-1}(Y)$  is infinite. As in Lemma 8.2, it follows from the definition of  $\sigma_M$  that the preimage  $\sigma_M^{-1}(Y)$  is in bijection with a subset of  $B_Y \cap r(\mathcal{E}_M)$ , where  $B_Y = \prod_{j=1}^m B_{Y_j}$  is the product of Bers slices. Here  $r: \mathcal{X}(\partial_0 M, \mathrm{SL}_2\mathbb{C}) \rightarrow \mathcal{X}(\partial_0 M, \mathrm{PSL}_2\mathbb{C})$  is the finite-sheeted, proper map induced by  $\mathrm{SL}_2\mathbb{C} \rightarrow \mathrm{PSL}_2\mathbb{C}$ . Using Theorem B we find that  $r(\mathcal{H}_Y \cap \mathcal{E}_M)$  is a discrete set. But since  $B_Y$  is a precompact subset of  $r(\mathcal{H}_Y)$ , the infinite subset we have identified with  $\sigma_M^{-1}(Y)$  has an accumulation point, which is a contradiction.  $\square$

## 10. Extended skinning maps of acylindrical manifolds

As mentioned in the introduction, the proof of Theorem A can be adapted to show finiteness of fibers of Thurston's continuous extension of the skinning map

$$\hat{\sigma}_M: \mathrm{AH}(M) \longrightarrow \mathcal{T}(\overline{\partial_0 M})$$

for an acylindrical manifold  $M$ . A proof of the existence of this continuous extension by Brock, Kent, and Minsky can be found in [Ke, §9].

The only property of the extension we will use is the following, which can be taken as the definition of  $\hat{\sigma}_M$ : If the image of  $\varrho \in \mathrm{AH}(M)$  under the extended skinning map is

$$\hat{\sigma}_M(\varrho) = (Y_1, \dots, Y_m) \in \mathcal{T}(\overline{\partial_0 M}),$$

and if the image of  $\varrho$  under the restriction map is

$$i^*(\varrho) = (\eta_1, \dots, \eta_m) \in \mathcal{X}(\partial_0 M, \mathrm{PSL}_2\mathbb{C}),$$

then  $\eta_j$  is the character of a discrete, faithful  $\pi_1 S_j$ -representation which has an invariant disk in its domain of discontinuity for which the quotient Riemann surface is  $Y_j$ .

The analogue of Theorem A for  $\hat{\sigma}_M$  is the following.

**THEOREM 10.1.** *Let  $M$  be a compact, oriented, irreducible, acylindrical 3-manifold with incompressible boundary that is not empty and not a union of tori. Let*

$$\hat{\sigma}_M: \text{AH}(M) \longrightarrow \mathcal{T}(\overline{\partial_0 M})$$

be the extension of the skinning map of  $M$ . Then, for each  $Y \in \mathcal{T}(\overline{\partial_0 M})$ , the set  $\hat{\sigma}_M^{-1}(Y)$  is finite.

*Proof.* First we show that  $i^*: \mathcal{X}(M, \partial_1 M, \text{PSL}_2\mathbb{C}) \rightarrow \mathcal{X}(\partial_0 M, \text{PSL}_2\mathbb{C})$  is injective when restricted to  $\text{AH}(M)$ . Suppose  $i^*(\varrho) = i^*(\varrho')$  with  $\varrho, \varrho' \in \text{AH}(M)$ . We claim the hyperbolic structures on  $M$  associated with  $\varrho$  and  $\varrho'$  are bilipschitz. As  $i^*(\varrho) = i^*(\varrho')$ , these hyperbolic structures are actually *isometric* in a neighborhood of each non-torus end. Smooth bilipschitz maps of the torus ends can be obtained by choosing affine maps between the cusp tori and extending normally. We have therefore defined a smooth bilipschitz equivalence on all of the ends, and any diffeomorphic extension to the remaining compact part is globally bilipschitz. By Sullivan's rigidity theorem [S], we conclude that  $\varrho = \varrho'$ .

Now suppose, by contradiction, that  $\varrho^{(n)} \in \text{AH}(M)$  is an infinite sequence of pairwise distinct points satisfying  $\hat{\sigma}_M(\varrho^{(n)}) = Y = (Y_1, \dots, Y_m)$  for all  $n$ . As in the proof of Theorem A, it is enough to show that this leads to an accumulation point in  $r(\mathcal{H}_Y \cap \mathcal{E}_M)$ .

Consider the sequence

$$i^*(\varrho^{(n)}) = (\eta_1^{(n)}, \dots, \eta_m^{(n)}) \in \prod_{j=1}^m \mathcal{X}(S_j, \text{PSL}_2\mathbb{C}),$$

which is also pairwise distinct, since  $i^*$  is injective on  $\text{AH}(M)$ . By the definition of  $\hat{\sigma}_M$ , the character  $\eta_j^{(n)}$  corresponds to a discrete  $\pi_1 S_j$ -representation with an invariant disk in its domain of discontinuity having quotient Riemann surface  $Y_j$ . Since this quotient describes a projective structure on  $Y_j$  with holonomy  $\eta_j^{(n)}$ , we find that  $i^*(\varrho^{(n)}) \in r(\mathcal{H}_Y)$ . As  $r(\mathcal{E}_M)$  contains the range of  $i^*$ , we in fact have  $i^*(\varrho^{(n)}) \in r(\mathcal{H}_Y \cap \mathcal{E}_M)$ .

While the representations  $\eta_j^{(n)}$  are not necessarily quasi-Fuchsian, a generalization of Bers' inequality (e.g. [Oh, Proposition 2.1]) shows that the fixed quotient Riemann surface  $Y_j$  constrains  $\eta_j^{(n)}$  to lie in a compact subset of  $\mathcal{X}(S_j, \text{PSL}_2\mathbb{C})$ . Thus  $i^*(\varrho^{(n)})$  lies in the product of these compact sets for all  $n$ , giving an accumulation point of  $r(\mathcal{H}_Y \cap \mathcal{E}_M)$  and the desired contradiction.  $\square$

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