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Enumeration of points, lines, planes, etc.

by

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1. Introduction

One of the earliest results in enumerative combinatorial geometry is the following theorem of de Bruijn and Erdős [dBE]:

Every finite set of points E in a projective plane determines at least $|E|$ lines, unless E is contained in a line.

In other words, if E is not contained in a line, then the number of lines in the plane containing at least two points in E is at least $|E|$. See [dW1] and [dW2] for an interesting account of its history and a survey of known proofs.

The following more general statement, conjectured by Motzkin in [Mo1], was subsequently proved by many in various settings:

Every finite set of points E in a projective space determines at least $|E|$ hyperplanes, unless E is contained in a hyperplane.

Motzkin proved the above for E in real projective spaces [Mo2]. Basterfield and Kelly [BK] showed the statement in general, and Greene [G] strengthened the result by showing that there is a *matching* from E to the set of hyperplanes determined by E, unless E is contained in a hyperplane:

For every point in E one can choose a hyperplane containing the point in such a way that no hyperplane is chosen twice.

Mason [Ma] and Heron [He] obtained similar results by different methods.

Let $\mathbb{P}$ be the projectivization of an r-dimensional vector space over a field, $E \subseteq \mathbb{P}$ be a finite subset not contained in any hyperplane, and $\mathscr{L}$ be the poset of subspaces of $\mathbb{P}$

spanned by the subsets of E. The poset $\mathscr{L}$ is a graded lattice, and its rank function satisfies the submodular inequality

$$\operatorname{rank}(F_1)+\operatorname{rank}(F_2) \geqslant \operatorname{rank}(F_1 \vee F_2)+\operatorname{rank}(F_1 \wedge F_2) \quad \text{for all } F_1, F_2 \in \mathscr{L}.$$

For a non-negative integer p, we write $\mathscr{L}^p$ for the set of rank-p elements in the lattice $\mathscr{L}$. Thus $\mathscr{L}^1$ is the set of points in E, $\mathscr{L}^2$ is the set of lines joining points in E, and $\mathscr{L}^r$ is the set with one element, $\mathbb{P}$. Graded posets obtained in this way are standard examples of *geometric lattices* [W]. These include the lattice of all subsets of a finite set (boolean lattices), the lattice of all partitions of a finite set (partition lattices), and the lattice of all subspaces of a finite vector space (projective geometries). In [DW2], Dowling and Wilson further generalized the above results for geometric lattices:

For every non-negative integer p less than $\frac{1}{2}r$, there is a matching from the set of rank-at-most-p elements of $\mathscr{L}$ to the set of corank-at-most-p elements of $\mathscr{L}$.

The matching can be chosen to match the minimum of $\mathscr{L}$ to the maximum of $\mathscr{L}$, and hence the above statement covers all the results introduced above. Kung gave another proof of the same result from the point of view of Radon transformations in [Ku1] and [Ku2].

In [DW1] and [DW2], Dowling and Wilson stated the following "top-heavy" conjecture.

Conjecture 1. Let $\mathscr{L}$ be a geometric lattice of rank r.

(1) For every non-negative integer p less than $\frac{1}{2}r$,

$$|\mathscr{L}^p| \leqslant |\mathscr{L}^{r-p}|.$$

In fact, there is an injective map $\iota\colon \mathscr{L}^p \to \mathscr{L}^{r-p}$ satisfying $x \leqslant \iota(x)$ for all x.

(2) For every non-negative integer p less than $\frac{1}{2}r$,

$$|\mathscr{L}^p| \leqslant |\mathscr{L}^{p+1}|.$$

In fact, there is an injective map $\iota\colon \mathscr{L}^p \to \mathscr{L}^{p+1}$ satisfying $x \leqslant \iota(x)$ for all x.

The conjecture was reproduced in [Stl2, Exercise 3.37] and [KRY, Exercise 3.5.7]. For an overview and related results, see [A]. When $\mathscr{L}$ is a boolean lattice or a projective geometry, the validity of Conjecture 1 is a classical result. We refer to [HM$^+$] and [Stl3] for recent expositions. In these cases, Conjecture 1 implies that $\mathscr{L}$ has the *Sperner property*:

The maximal number of incomparable elements in $\mathscr{L}$ is the maximum of $|\mathscr{L}^p|$ over p.

Kung proved the second part of Conjecture 1 for partition lattices in [Ku3]. Later he showed the second part of Conjecture 1 for $p \leqslant 2$ when every line contains the same number of points [Ku4].

We now state our main result. As before, we write $\mathbb{P}$ for the projectivization of an r-dimensional vector space over a field.

THEOREM 2. *Let $E \subseteq \mathbb{P}$ be a finite subset not contained in any hyperplane, and $\mathscr{L}$ be the poset of subspaces of $\mathbb{P}$ spanned by subsets of E.*

(1) *For all non-negative integers $p \leqslant q$ satisfying $p+q \leqslant r$,*

$$|\mathscr{L}^p| \leqslant |\mathscr{L}^{r-q}|.$$

In fact, there is an injective map $\iota: \mathscr{L}^p \to \mathscr{L}^{r-q}$ satisfying $x \leqslant \iota(x)$ for all x.

(2) *For every positive integer p less than $\frac{1}{2}r$,*

$$0 \leqslant |\mathscr{L}^{p+1}| - |\mathscr{L}^p| \leqslant (|\mathscr{L}^p| - |\mathscr{L}^{p-1}|)^{\langle p \rangle}.$$

Equivalently, $(|\mathscr{L}^0|, |\mathscr{L}^1| - |\mathscr{L}^0|, \ldots, |\mathscr{L}^{p+1}| - |\mathscr{L}^p|)$ is the h-vector of a shellable simplicial complex.

The first part of Theorem 2 settles Conjecture 1 for all $\mathscr{L}$ realizable over some field. We believe this to be a good demonstration of the power of the main ingredient in the proof, the decomposition theorem package for intersection complexes [BBD].

The expression $n^{\langle p \rangle}$ used in the second part of Theorems 2 is defined as follows: For any positive integer p and non-negative integer n, write $n_1, \ldots, n_p$ for the unique increasing sequence of non-negative integers satisfying

$$n = \sum_{k=1}^{p} \binom{n_k}{k}.$$

Then, by definition,

$$n^{\langle p \rangle} = \sum_{k=1}^{p} \binom{n_k+1}{k+1}.$$

For h-vectors of shellable simplicial complexes, we refer to [Stl1, Chapter II].

Example 3. Modular geometric lattices, such as boolean lattices or finite projective geometries, satisfies a stronger matching property:

For every p, there is an injective or surjective map $\iota: \mathscr{L}^p \to \mathscr{L}^{p+1}$ satisfying $x \leqslant \iota(x)$.

As noted before, this implies that modular geometric lattices have the Sperner property.

Dilworth and Greene constructed in [DG] a configuration of 21 points in any 10-dimensional projective space over a field with the property that there is neither injective nor surjective map

$$\iota\colon \mathscr{L}^6 \longrightarrow \mathscr{L}^7, \quad x \leqslant \iota(x).$$

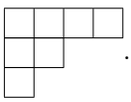
Canfield [C] found such "no-matching" successive rank-level sets as above in partition lattices with sufficiently many elements (exceeding $10^{10^{20}}$). These geometric lattices satisfy Conjecture 1 but do not have the Sperner property.

Rota conjectured [R], [RH] that the sizes of the rank-level sets of a geometric lattice form a unimodal sequence

$$|\mathscr{L}^0| \leqslant \ldots \leqslant |\mathscr{L}^{p-1}| \leqslant |\mathscr{L}^p| \geqslant |\mathscr{L}^{p+1}| \geqslant \ldots \geqslant |\mathscr{L}^r| \quad \text{for some } p.$$

Stronger versions of this conjecture were proposed by Mason [Ma]. The unimodality for the "upper half" remains as an outstanding open problem.

Example 4. Let λ be a partition of a positive integer, which we view as a Young diagram [F2]. For example, the partition $(4, 2, 1)$ of 7 corresponds to the Young diagram



The *Young's lattice* associated with λ is the graded poset $\mathscr{L}_\lambda$ of all partitions whose Young diagram fit inside λ. The poset $\mathscr{L}_\lambda$ is usually not a geometric lattice, but Björner and Ekedahl [BE] showed that $\mathscr{L}_\lambda$ satisfies both conclusions of Conjecture 1 when r is the number of boxes in λ:

(1) For p less than $\frac{1}{2}r$, there is an injective map $\iota\colon \mathscr{L}^p \to \mathscr{L}^{r-p}$ satisfying $x \leqslant \iota(x)$ for all x.

(2) For p less than $\frac{1}{2}r$, there is an injective map $\iota\colon \mathscr{L}^p \to \mathscr{L}^{p+1}$ satisfying $x \leqslant \iota(x)$ for all x.

However, according to Stanton [Stt], Young's lattice for the partition $(8, 8, 4, 4)$ defines a non-unimodal sequence

$$\begin{aligned}&(|\mathscr{L}_\lambda^0|, |\mathscr{L}_\lambda^1|, |\mathscr{L}_\lambda^2|, \ldots, |\mathscr{L}_\lambda^{24}|)\\ &\quad = (1, 1, 2, 3, 5, 6, 9, 11, 15, 17, 21, 23, 27, 28, 31, 30, 31, 27, 24, 18, 14, 8, 5, 2, 1).\end{aligned}$$

Face lattices of simplicial polytopes behave similarly, starting from dimension 20 [BL], [B]. See [Z, Chapter 8] for a discussion of unimodality in the case of polytopes.

2. The graded Möbius algebra

We use the language of matroids, and use [W] and [O] as basic references. Let r and n be positive integers, and let M be a rank-r simple matroid on the ground set

$$E = \{1, ..., n\}.$$

Write $\mathscr{L}$ for the lattice of flats of M. We define a graded analogue of the Möbius algebra for $\mathscr{L}$.

Definition 5. Introduce symbols y_F, one for each flat F of M, and construct vector spaces

$$B^p(\mathrm{M}) = \bigoplus_{F \in \mathscr{L}^p} \mathbb{Q} y_F \quad \text{and} \quad B^*(\mathrm{M}) = \bigoplus_{F \in \mathscr{L}} \mathbb{Q} y_F.$$

We equip $B^*(\mathrm{M})$ with the structure of a commutative graded algebra over $\mathbb{Q}$ by setting

$$y_{F_1} y_{F_2} = \begin{cases} y_{F_1 \vee F_2}, & \text{if } \operatorname{rank}(F_1) + \operatorname{rank}(F_2) = \operatorname{rank}(F_1 \vee F_2), \\ 0, & \text{if } \operatorname{rank}(F_1) + \operatorname{rank}(F_2) > \operatorname{rank}(F_1 \vee F_2). \end{cases}$$

For simplicity, we write $y_1, ..., y_n$ instead of $y_{\{1\}}, ..., y_{\{n\}}$.

Maeno and Numata introduced this algebra in a slightly different form in [MN], and used it to show that modular geometric lattices have the Sperner property. Note that $B^*(\mathrm{M})$ is generated by $B^1(\mathrm{M})$ as an algebra: If I_F is any basis of a flat F of M, then

$$y_F = \prod_{i \in I_F} y_i.$$

Unlike its ungraded counterpart, which is isomorphic to the product of $\mathbb{Q}$'s as a $\mathbb{Q}$-algebra [So], the graded Möbius algebra $B^*(\mathrm{M})$ has a non-trivial algebra structure. Define

$$L = \sum_{i \in E} y_i.$$

We deduce Theorem 2 from the following algebraic statement. Similar injectivity properties have appeared in the context of Kac–Moody Schubert varieties [BE] and toric hyperkähler varieties [Ha].

THEOREM 6. *For non-negative integer p less than $\frac{1}{2}r$, the multiplication map*

$$\begin{aligned} B^p(\mathrm{M}) &\longrightarrow B^{r-p}(\mathrm{M}), \\ \xi &\longmapsto L^{r-2p}\xi, \end{aligned}$$

is injective, when M *is realizable over some field.*

It follows that, for non-negative integers $p \leqslant q$ satisfying $p+q \leqslant r$, the multiplication map

$$\begin{aligned} B^p(\mathrm{M}) &\longrightarrow B^{r-q}(\mathrm{M}), \\ \xi &\longmapsto L^{r-p-q}\xi, \end{aligned}$$

is injective, when M is realizable over some field. To deduce the first part of Theorem 2 from this, consider the matrix of the multiplication map with respect to the standard bases of the source and the target. Entries of this matrix are labeled by pairs of elements of $\mathscr{L}$, and all the entries corresponding to incomparable pairs are zero. The matrix has full rank, so there is a maximal square submatrix with non-zero determinant. In the standard expansion of this determinant, there must be a non-zero term, and the permutation corresponding to this term produces the injective map ι. The second part of Theorem 2 also follows from Theorem 6. To see this, note that the algebra $B^*(\mathrm{M})$ is generated by its degree-1 elements, and apply Macaulay's theorem to the quotient of $B^*(\mathrm{M})$ by the ideal generated by L [Stl1, Chapter II, Corollary 2.4].

Conjecture 7. Theorem 6 holds without the assumption of realizability.

Let M be as before, and let $\overline{\mathrm{M}}$ be a simple matroid on the ground set

$$\overline{E} = \{0, 1, ..., n\}.$$

Let $\bar{\mathscr{L}}$ be the lattice of flats of $\overline{\mathrm{M}}$. We suppose that $\mathrm{M} = \overline{\mathrm{M}}/0$, that is, M is obtained from $\overline{\mathrm{M}}$ by contracting the element 0.

Definition 8. Introduce variables $x_{\bar{F}}$, one for each non-empty proper flat $\overline{F}$ of $\overline{\mathrm{M}}$, and set

$$S_{\overline{\mathrm{M}}} = \mathbb{Q}[x_{\bar{F}}]_{\bar{F} \neq \varnothing, \bar{F} \neq \overline{E}, \bar{F} \in \bar{\mathscr{L}}}.$$

The *Chow ring* $A^*(\overline{\mathrm{M}})$ is the quotient of $S_{\overline{\mathrm{M}}}$ by the ideal generated by the linear forms

$$\sum_{i_1 \in \bar{F}} x_{\bar{F}} - \sum_{i_2 \in \bar{F}} x_{\bar{F}},$$

one for each pair of distinct elements i_1 and i_2 of $\overline{E}$, and the quadratic monomials

$$x_{\bar{F}_1} x_{\bar{F}_2},$$

one for each pair of incomparable non-empty proper flats of $\overline{\mathrm{M}}$.

The algebra $A^*(\overline{\mathrm{M}})$ and its generalizations were studied by Feichtner and Yuzvinsky in [FY]. For every i in E, we define an element of $A^1(\overline{\mathrm{M}})$ by setting

$$\beta_i = \sum_{\overline{F}} x_{\overline{F}},$$

where the sum is over all flats $\overline{F}$ of $\overline{\mathrm{M}}$ that contain 0 and do not contain i. The linear relations show that we may equivalently define

$$\beta_i = \sum_{\overline{F}} x_{\overline{F}},$$

where the sum is over all flats $\overline{F}$ of $\overline{\mathrm{M}}$ that contain i and do not contain 0. We record here three basic implications of the defining relations of $A^*(\overline{\mathrm{M}})$:

(R1) When $\overline{F}$ is a non-empty proper flat of $\overline{\mathrm{M}}$ containing exactly one of i and 0,

$$\beta_i \cdot x_{\overline{F}} = 0.$$

This follows from the quadratic monomial relations.

(R2) For every element i in E,

$$\beta_i \cdot \beta_i = 0.$$

This follows from the previous statement.

(R3) For any two maximal chains of non-empty proper flats of $\overline{\mathrm{M}}$, say $\{\overline{F}_k\}_{1 \leqslant k}$ and $\{\overline{G}_k\}_{1 \leqslant k}$,

$$\prod_{k=1}^{r} x_{\overline{F}_k} = \prod_{k=1}^{r} x_{\overline{G}_k} \neq 0.$$

The proofs of (R1) and (R2) are straightforward. The proof of (R3) can be found in [AHK, §5].

PROPOSITION 9. *There is a unique injective graded $\mathbb{Q}$-algebra homomorphism*

$$\begin{aligned} \varphi \colon B^*(\mathrm{M}) &\longrightarrow A^*(\overline{\mathrm{M}}), \\ y_i &\longmapsto \beta_i. \end{aligned}$$

Proof. First, we show that there is a well-defined $\mathbb{Q}$-linear map

$$\begin{aligned} \varphi \colon B^*(\mathrm{M}) &\longrightarrow A^*(\overline{\mathrm{M}}), \\ y_F &\longmapsto \prod_{i \in I_F} \beta_i, \end{aligned}$$

where I_F is any basis of a flat F of M. In other words, if J_F is any other basis of F, then

$$\prod_{i\in I_F}\beta_i = \prod_{i\in J_F}\beta_i.$$

Since any basis of F can be obtained from any other basis of F by a sequence of elementary exchanges, it is enough to check the equality in the special case when $I_F\setminus J_F=\{1\}$ and $J_F\setminus I_F=\{2\}$. Assuming that this is the case, we write the left-hand side of the claimed equality by

$$\Big(\prod_{i\in I_F\cap J_F}\beta_i\Big)\Big(\sum_{\overline{G}} x_{\overline{G}}\Big),$$

where the sum is over all non-empty proper flats $\overline{G}$ of $\overline{\mathrm{M}}$ that contain 0 and do not contain 1. The relation (R1) shows that we may take the sum only over those $\overline{G}$ satisfying

$$0\in\overline{G}, \quad 1\notin\overline{G} \quad \text{and} \quad I_F\cap J_F\subseteq\overline{G}.$$

Since $I_F\cup\{0\}$ and $J_F\cup\{0\}$ are bases of the same flat of $\overline{\mathrm{M}}$, the above condition is equivalent to

$$0\in\overline{G}, \quad 2\notin\overline{G} \quad \text{and} \quad I_F\cap J_F\subseteq\overline{G}.$$

This proves the claimed equality, which shows that φ is a well-defined linear map.

Second, we show that φ is a ring homomorphism. Given flats F_1 and F_2 of M, we show that

$$\Big(\prod_{i\in I_{F_1}}\beta_i\Big)\Big(\prod_{i\in I_{F_2}}\beta_i\Big)=0 \quad \text{when the rank of } F_1\vee F_2 \text{ is less than } |I_{F_1}|+|I_{F_2}|.$$

If the independent sets I_{F_1} and I_{F_2} intersect, this follows from the relation (R2). If otherwise, the condition on the rank of $F_1\vee F_2$ implies that there are two distinct bases of $F_1\vee F_2$ contained in $I_{F_1}\cup I_{F_2}$, say

$$I_{F_1\vee F_2}\subseteq I_{F_1}\cup I_{F_2} \quad \text{and} \quad J_{F_1\vee F_2}\subseteq I_{F_1}\cup I_{F_2}.$$

Using the first part of the proof, once again from the relation (R2), we deduce that

$$\begin{aligned}\Big(\prod_{i\in I_{F_1}}\beta_i\Big)\Big(\prod_{i\in I_{F_2}}\beta_i\Big) &= \Big(\prod_{i\in I_{F_1\vee F_2}}\beta_i\Big)\Big(\prod_{i\in I_{F_1}\cup I_{F_2}\setminus I_{F_1\vee F_2}}\beta_i\Big)\\ &= \Big(\prod_{i\in J_{F_1\vee F_2}}\beta_i\Big)\Big(\prod_{i\in I_{F_1}\cup I_{F_2}\setminus I_{F_1\vee F_2}}\beta_i\Big)=0.\end{aligned}$$

This completes the proof that φ is a ring homomorphism.

Third, we show that φ is injective in degree r. Choose any ordered basis $\{i_1, ..., i_r\}$ of M. For each $q{=}1, ..., r$, we let $\overline{G}_q$ be the closure of $\{0, i_1, ..., i_{q-1}\}$ in $\overline{\mathrm{M}}$. We deduce from the relation (R1) that

$$(\beta_{i_1} \dots \beta_{i_{r-1}})\beta_{i_r} = (\beta_{i_1} \dots \beta_{i_{r-1}})x_{\overline{G}_r}.$$

Similarly, for any positive integer $q \leqslant r$, we have

$$(\beta_{i_1} \dots \beta_{i_{q-2}}\beta_{i_{q-1}})x_{\overline{G}_q} = (\beta_{i_1} \dots \beta_{i_{q-2}})x_{\overline{G}_{q-1}}x_{\overline{G}_q} = x_{\overline{G}_1} \dots x_{\overline{G}_{q-1}}x_{\overline{G}_q},$$

since $\overline{G}_{q-1}$ is the only flat of $\overline{\mathrm{M}}$ containing $\overline{G}_{q-1}$, comparable to $\overline{G}_q$, and not containing i_{q-1}. Combining the above formulas, we deduce from the relation (R3) that

$$\beta_{i_1} \dots \beta_{i_r} = x_{\overline{G}_1} \dots x_{\overline{G}_r} \neq 0.$$

This proves that φ is injective in degree r.

Last, we show that φ is injective in any degree q less than r. For this we analyze the bilinear map given by the multiplication

$$\varphi(B^q(\mathrm{M})) \times \bigoplus_{\overline{G}} \mathbb{Q}x_{\overline{G}} \longrightarrow A^{q+1}(\overline{\mathrm{M}}),$$

where the sum is over all rank-$(q{+}1)$ flats $\overline{G}$ of $\overline{\mathrm{M}}$ containing 0. For any independent set $\{i_1, ..., i_q\}$ of M, we claim that, for any $\overline{G}$ as in the previous sentence,

$$(\beta_{i_1} \dots \beta_{i_q})x_{\overline{G}} \neq 0 \quad \text{if and only if} \quad \overline{G} \text{ is the closure of } \{0, i_1, ..., i_q\} \text{ in } \overline{\mathrm{M}}.$$

The "if" statement follows from the analysis made above. For the "only if" statement, suppose that the product is non-zero. Since $\overline{G}$ contains 0, it must contain $i_1, ..., i_q$ by the relation (R1). Since $\overline{G}$ and the closure both have the same rank, we have that

$$\overline{G} \text{ is the closure of } \{0, i_1, ..., i_q\} \text{ in } \overline{\mathrm{M}}.$$

This proves the claimed equivalence, and it follows that the image of the basis $\{y_F\}$ of $B^q(\mathrm{M})$ under φ is linearly independent in $A^q(\overline{\mathrm{M}})$. □

3. The simplex, the cube, and the permutohedron

In this section, we give a toric preparation for the proof of our main result, Theorem 6. For undefined terms in toric geometry and intersection theory, we refer to [F1] and [F3]. All the Chow groups and rings will have rational coefficients.

As in the previous section, we fix a positive integer n and work with the sets

$$E = \{1, ..., n\} \quad \text{and} \quad \overline{E} = \{0, 1, ..., n\}.$$

Let $\mathbb{Z}^{\overline{E}}$ be the abelian group generated by the basis vectors $\mathbf{e}_i$ corresponding to $i \in \overline{E}$. For an arbitrary subset $\bar{I} \subseteq \overline{E}$, we define

$$\mathbf{e}_{\bar{I}} = \sum_{i \in \bar{I}} \mathbf{e}_i.$$

We associate with $\overline{E}$ the abelian group $N_{\overline{E}} = \mathbb{Z}^{\overline{E}}/\mathbb{Z}\mathbf{e}_{\overline{E}}$ and the vector space

$$N_{\overline{E},\mathbb{R}} = \mathbb{R}^{\overline{E}}/\mathbb{R}\mathbf{e}_{\overline{E}}.$$

(1) Let $\Sigma(\mathrm{S}_n) \subseteq N_{\overline{E},\mathbb{R}}$ be the image of the normal fan of the standard n-dimensional simplex

$$\mathrm{S}_n = \mathrm{conv}\{\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_n\} \subseteq \mathbb{R}^{\overline{E}}.$$

There are $n+1$ maximal cones in $\Sigma(\mathrm{S}_n)$, one for each maximal proper subset $\bar{I}$ of $\overline{E}$:

$$\sigma_{\bar{I}} = \mathrm{cone}\{\mathbf{e}_i : i \in \bar{I}\} \subseteq N_{\overline{E},\mathbb{R}}.$$

This fan defines the n-dimensional projective space $\mathbb{P}^n$, whose homogeneous coordinates are labeled by $i \in \overline{E}$.

(2) Let $\Sigma(\mathrm{C}_n) \subseteq \mathbb{R}^E$ be the normal fan of the standard n-dimensional cube

$$\mathrm{C}_n = \mathrm{conv}\{\pm\mathbf{e}_1, ..., \pm\mathbf{e}_n\} \subseteq \mathbb{R}^E.$$

There are 2^n maximal cones in $\Sigma(\mathrm{C}_n)$, one for each subset I of E:

$$\sigma_I = \mathrm{cone}\{\mathbf{e}_i : i \in I\} - \mathrm{cone}\{\mathbf{e}_i : i \notin I\} \subseteq \mathbb{R}^E.$$

This fan defines the product of n projective lines $(\mathbb{P}^1)^n$, whose multi-homogeneous coordinates are labeled by $i \in E$.

(3) Let $\Sigma(\mathrm{P}_n) \subseteq N_{\overline{E},\mathbb{R}}$ be the image of the normal fan of the n-dimensional permutohedron

$$\mathrm{P}_n = \mathrm{conv}\{(x_0, ..., x_n) : x_0, ..., x_n \text{ is a permutation of } 0, ..., n\} \subseteq \mathbb{R}^{\overline{E}}.$$

There are $(n+1)!$ maximal cones in $\Sigma(\mathrm{P}_n)$, one for each maximal chain $\mathscr{I}$ in $2^{\overline{E}}$:

$$\sigma_{\mathscr{I}} = \mathrm{cone}\{\mathbf{e}_{\bar{I}} : \bar{I} \in \mathscr{I}\} \subseteq N_{\overline{E},\mathbb{R}}.$$

This fan defines the n-dimensional permutohedral space, denoted X_{A_n}. See [BB] for a detailed study of X_{A_n} and its analogues for other root systems.

The inclusion $\mathbb{Z}^E \subseteq \mathbb{Z}^{\overline{E}}$ induces an isomorphism

$$\psi^{-1}\colon \mathbb{R}^E \longrightarrow N_{\overline{E},\mathbb{R}}.$$

This identifies the underlying vector spaces of the normal fans $\Sigma(\mathrm{S}_n)$, $\Sigma(\mathrm{P}_n)$ and $\Sigma(\mathrm{C}_n)$:

$$\begin{array}{ccc} & N_{\overline{E},\mathbb{R}} & \\ {\scriptstyle \mathrm{id}}\swarrow & & \searrow{\scriptstyle \psi} \\ N_{\overline{E},\mathbb{R}} & & \mathbb{R}^E. \end{array}$$

We observe that id and ψ induce morphisms between the fans and their toric varieties:

$$\begin{array}{ccc} & \Sigma(\mathrm{P}_n) & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ \Sigma(\mathrm{S}_n) & & \Sigma(\mathrm{C}_n) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & X_{A_n} & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ \mathbb{P}^n & & (\mathbb{P}^1)^n. \end{array}$$

The morphism p_1 is the standard barycentric subdivision. We check that p_2 is a subdivision.

PROPOSITION 10. *The isomorphism ψ induces a morphism p_2.*

In other words, the image of a cone in $\Sigma(\mathrm{P}_n)$ under ψ is contained in a cone in $\Sigma(\mathrm{C}_n)$.

Proof. For each $i \in E$, define ψ_i as the composition of ψ with the ith projection

$$\psi_i = \mathrm{proj}_i \circ \psi, \quad \mathrm{proj}_i\colon \mathbb{R}^E \longrightarrow \mathbb{R}^{\{i\}} \simeq \mathbb{R}.$$

For any subset $\bar{I} \subseteq \overline{E}$, we have

$$\psi_i(\mathbf{e}_{\bar{I}}) = \begin{cases} \mathbf{e}_i, & \text{if } \bar{I} \text{ contains } i \text{ and does not contain } 0, \\ -\mathbf{e}_i, & \text{if } \bar{I} \text{ contains } 0 \text{ and does not contain } i, \\ 0, & \text{otherwise.} \end{cases}$$

It is enough to check that ψ_i induces a morphism $\Sigma(\mathrm{P}_n) \to \Sigma(\mathrm{C}_1)$.

Recall that any non-zero cone in the normal fan of P_n is of the form

$$\sigma_{\mathscr{I}} = \mathrm{cone}\{\mathbf{e}_{\bar{I}} : \bar{I} \in \mathscr{I}\},$$

where $\mathscr{I}$ is a non-empty chain in $2^{\overline{E}}$. Viewing $\mathscr{I}$ as an ordered collection of sets, we see that $\psi_i(\sigma_{\mathscr{I}})$ is contained in the cone generated by

$$\begin{cases} \mathbf{e}_i, & \text{if } i \text{ appears before 0 in } \mathscr{I}, \\ -\mathbf{e}_i, & \text{if } i \text{ appears after 0 in } \mathscr{I}. \end{cases}$$

Thus, the image of a cone in $\Sigma(\mathrm{P}_n)$ under ψ_i is contained in a cone in $\Sigma(\mathrm{C}_1)$, for each $i \in E$. □

Geometrically, π_1 is the blow-up of all the torus-invariant points in $\mathbb{P}^n$, all the strict transforms of torus-invariant $\mathbb{P}^1$'s in $\mathbb{P}^n$, all the strict transforms of torus-invariant $\mathbb{P}^2$'s in $\mathbb{P}^n$, and so on. The map π_2 is the blow-up of points 0^n and ∞^n, all the strict transforms of torus-invariant $\mathbb{P}^1$'s in $(\mathbb{P}^1)^n$ containing 0^n or ∞^n, all the strict transforms of torus-invariant $(\mathbb{P}^1)^2$'s in $(\mathbb{P}^1)^n$ containing 0^n or ∞^n, and so on.

Remark 11. For later use, we record here a combinatorial description of the pull-back of piecewise linear functions under the linear map $\psi_i = \mathrm{proj}_i \circ \psi$:

Let α be the piecewise linear function on $\Sigma(\mathrm{C}_1)$ determined by its values

$$\alpha(\mathbf{e}_i) = 1 \quad \textit{and} \quad \alpha(-\mathbf{e}_i) = 0.$$

Then $\psi_i^*(\alpha)$ is the piecewise linear function on $\Sigma(\mathrm{P}_n)$ determined by its values

$$\psi_i^*(\alpha)(\mathbf{e}_{\bar{I}}) = \begin{cases} 1, & \textit{if } \bar{I} \textit{ contains } i \textit{ and does not contain } 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Using the correspondence between piecewise linear functions on fans and torus-invariant divisors on toric varieties [F1, Chapter 3], the above can be used to describe the pull-back homomorphism between the Chow rings

$$\pi_2^* \colon A^*((\mathbb{P}^1)^n) \longrightarrow A^*(X_{A_n}).$$

Explicitly, writing y_i for the divisor of $\mathbf{e}_i$ in $(\mathbb{P}^1)^n$ and $x_{\bar{I}}$ for the divisor of $\mathbf{e}_{\bar{I}}$ in X_{A_n},

$$\pi_2^*(y_i) = \sum_{\bar{I}} x_{\bar{I}},$$

where the sum is over all subsets $\bar{I} \subseteq \overline{E}$ that contain i and do not contain 0.

4. Proof of Theorem 6

Let M be a simple matroid on E, and let $\overline{\mathrm{M}}$ be a simple matroid on $\overline{E}$ with $\mathrm{M} = \overline{\mathrm{M}}/0$. For simplicity, we take $\overline{\mathrm{M}}$ to be the direct sum of M and the rank-1 matroid on $\{0\}$, so that M and $\overline{\mathrm{M}}$ share the same set of circuits.

Suppose that M is realizable over some field. Then M is realizable over some finite field [O, Corollary 6.8.13], and hence over the algebraically closed field $\overline{\mathbb{F}}_p$ for some prime number p. The matroid $\overline{\mathrm{M}}$ is realizable over the same field, say by a spanning set of vectors

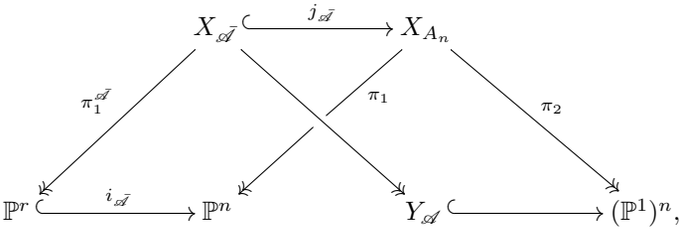
$$\bar{\mathscr{A}} = \{f_0, f_1, ..., f_n\} \subseteq \overline{\mathbb{F}}_p^{\,r+1}.$$

Dually, the realization $\mathscr{A}^{\bar{}}$ of $\overline{\mathrm{M}}$ corresponds to an injective linear map between projective spaces

$$i_{\mathscr{A}^{\bar{}}} : \mathbb{P}^r \longrightarrow \mathbb{P}^n, \quad i_{\mathscr{A}^{\bar{}}} = [f_0 : f_1 : \ldots : f_n].$$

The collection $\mathscr{A} = \{f_1, \ldots, f_n\}$ is a realization of the matroid M.

The restriction of the torus-invariant hyperplanes of $\mathbb{P}^n$ to $\mathbb{P}^r$ defines an arrangement of hyperplanes in $\mathbb{P}^r$, which we denote by the same symbol $\mathscr{A}^{\bar{}}$. We use $i_{\mathscr{A}^{\bar{}}}$ to construct the commutative diagram



where $X_{\mathscr{A}^{\bar{}}}$ is the strict transform of $\mathbb{P}^r$ under π_1 and $Y_{\mathscr{A}}$ is the image of $X_{\mathscr{A}^{\bar{}}}$ under π_2.

The induced map $\pi_1^{\mathscr{A}^{\bar{}}}$ is the blow-up of all the zero-dimensional flats of $\mathscr{A}^{\bar{}}$, all the strict transforms of 1-dimensional flats of $\mathscr{A}^{\bar{}}$, all the strict transforms of 2-dimensional flats of $\mathscr{A}^{\bar{}}$, and so on. The variety $X_{\mathscr{A}^{\bar{}}}$ is the *wonderful model* of $\mathscr{A}^{\bar{}}$ corresponding to the maximal building set [dCP1]. The variety $Y_{\mathscr{A}}$ is studied in [AB], and its affine part centered at ∞^n is the *reciprocal plane* in [EPW], [PS]. To apply the decomposition theorem of [BBD], we notice that all varieties, maps and sheaves under consideration may be defined over some finite extension of $\mathbb{F}_p$.

We know that the Chow ring of $X_{\mathscr{A}^{\bar{}}}$ is determined by the matroid $\overline{\mathrm{M}}$: There is an isomorphism of graded algebras

$$A^*(\overline{\mathrm{M}}) \simeq A^*(X_{\mathscr{A}^{\bar{}}}),$$

where $x_{\bar{F}}$ is identified with the class of the strict transform of the exceptional divisor produced when blowing up the flat of $\mathscr{A}^{\bar{}}$ corresponding to $\overline{F}$. See [dCP2, §1.1], and also [dCP1] and [FY]. When $\overline{\mathrm{M}}$ is the boolean matroid $\overline{\mathrm{B}}$ on $\overline{E}$, this describes the Chow ring of the permutohedral space $A^*(X_{A_n})$. In general, the pull-back homomorphism

$$A^*(\overline{\mathrm{B}}) \simeq A^*(X_{A_n}) \xrightarrow{j^*_{\mathscr{A}^{\bar{}}}} A^*(X_{\mathscr{A}^{\bar{}}}) \simeq A^*(\overline{\mathrm{M}})$$

is determined by the assignment, for non-empty proper subsets $\bar{I}$ of $\overline{E}$,

$$x_{\bar{I}} \longmapsto \begin{cases} x_{\bar{I}}, & \text{if } \bar{I} \text{ is a flat of } \overline{\mathrm{M}}, \\ 0, & \text{if } \bar{I} \text{ is a not flat of } \overline{\mathrm{M}}. \end{cases}$$

Fix a prime number ℓ different from p, and consider the ℓ-adic étale cohomology rings and the ℓ-adic étale intersection cohomology groups of the varieties in the diagram above. These are $\overline{\mathbb{Q}}_\ell$-vector spaces of the form

$$\mathrm{H}^*(X, \overline{\mathbb{Q}}_\ell) := \mathrm{H}^*(X, \overline{\mathbb{Q}}_{\ell,X}) \quad \text{and} \quad \mathrm{IH}^*(X, \overline{\mathbb{Q}}_\ell) := \mathrm{H}^*(X, \mathrm{IC}_X),$$

where $\overline{\mathbb{Q}}_{\ell,X}$ and IC_X are constructible complexes of $\overline{\mathbb{Q}}_\ell$-sheaves on X as in [BBD]. The blow-up construction of $X_{\mathscr{A}}$ shows that the cycle class map induces an isomorphism of commutative graded $\overline{\mathbb{Q}}_\ell$-algebras

$$A^*(X_{\mathscr{A}}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \simeq \mathrm{H}^{2*}(X_{\mathscr{A}}, \overline{\mathbb{Q}}_\ell);$$

see [Ke, Appendix]. For the variety $Y_{\mathscr{A}}$, which may be singular, we show in Theorem 14 that there is an isomorphism of commutative graded $\overline{\mathbb{Q}}_\ell$-algebras

$$B^*(\mathrm{M}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \simeq \mathrm{H}^{2*}(Y_{\mathscr{A}}, \overline{\mathbb{Q}}_\ell).$$

In general, the intersection cohomology $\mathrm{IH}^*(X, \overline{\mathbb{Q}}_\ell)$ is a module over the cohomology $\mathrm{H}^*(X, \overline{\mathbb{Q}}_\ell)$, satisfying the Poincaré duality and the hard Lefschetz theorems. See [dCM] for an introduction and precise statements.

We obtain Theorem 6 from the following general observation. Let f be a proper map from an r-dimensional smooth projective variety

$$f\colon X_1 \longrightarrow X_2,$$

and let L be a fixed ample line bundle on X_2. Consider the pull-back homomorphism of cohomology in even degrees

$$\mathrm{H}^{2*}(X_2, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathrm{H}^{2*}(X_1, \overline{\mathbb{Q}}_\ell).$$

The image of the pull-back is a commutative graded algebra over $\overline{\mathbb{Q}}_\ell$, denoted $B^*(f)_{\overline{\mathbb{Q}}_\ell}$:

$$B^*(f)_{\overline{\mathbb{Q}}_\ell} = \mathrm{im}(\mathrm{H}^{2*}(X_2, \overline{\mathbb{Q}}_\ell) \to \mathrm{H}^{2*}(X_1, \overline{\mathbb{Q}}_\ell)).$$

$B^*(f)_{\overline{\mathbb{Q}}_\ell}$ is the cyclic $\mathrm{H}^{2*}(X_2, \overline{\mathbb{Q}}_\ell)$-submodule of $\mathrm{H}^{2*}(X_1, \overline{\mathbb{Q}}_\ell)$ generated by the element 1.

Proposition 12. *If f is birational onto its image, then the multiplication map*

$$\begin{aligned} B^p(f)_{\overline{\mathbb{Q}}_\ell} &\longrightarrow B^{r-p}(f)_{\overline{\mathbb{Q}}_\ell}, \\ \xi &\longmapsto L^{r-2p}\ \xi, \end{aligned}$$

is injective for every non-negative integer p less than $\frac{1}{2}r$.

Proof. We reduce to the case when f is surjective. For this consider the factorization

$$X_1 \xrightarrow{\ g\ } f(X_1) \xhookrightarrow{\ h\ } X_2, \quad f = h \circ g.$$

Then $B^*(f)_{\overline{\mathbb{Q}}_\ell}$ is a subalgebra of $B^*(g)_{\overline{\mathbb{Q}}_\ell}$, and hence the statement (f, L) follows from (g, h^*L).

Suppose that f is surjective. The decomposition theorem [BBD, §4.3] says that the intersection complex of X_2 appears as a direct summand of the direct image of the constant sheaf $\overline{\mathbb{Q}}_\ell$ on X_1:

$$Rf_*\overline{\mathbb{Q}}_{\ell,X_1} \simeq \mathrm{IC}_{X_2} \oplus \mathscr{C}.$$

Taking cohomology of both sides, we obtain a splitting injection of $\mathrm{H}^*(X_2, \overline{\mathbb{Q}}_\ell)$-modules

$$\Phi\colon \mathrm{IH}^*(X_2, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathrm{H}^*(X_1, \overline{\mathbb{Q}}_\ell).$$

Since Φ is an isomorphism in degree zero, it restricts to an isomorphism of commutative algebras

$$\mathrm{im}(\mathrm{H}^{2*}(X_2, \overline{\mathbb{Q}}_\ell) \to \mathrm{IH}^{2*}(X_2, \overline{\mathbb{Q}}_\ell)) \simeq B^*(f)_{\overline{\mathbb{Q}}_\ell}.$$

The conclusion follows from the hard Lefschetz theorem for L on $\mathrm{IH}^{2*}(X_2, \overline{\mathbb{Q}}_\ell)$. □

Theorem 6 will be deduced from the case when f is the map $X_{\mathscr{A}^\perp} \to (\mathbb{P}^1)^n$. For each $i \in E$, let f_i be the composition of f with the ith projection:

$$f_i = \mathrm{proj}_i \circ f, \quad \mathrm{proj}_i\colon (\mathbb{P}^1)^n \longrightarrow \mathbb{P}^1.$$

As in Proposition 9, for each $i \in E$ we write β_i for the sum of $x_{\overline{F}}$ over all flats $\overline{F}$ of $\overline{\mathrm{M}}$ that contain i and do not contain 0. As mentioned before, the blow-up construction of $X_{\mathscr{A}^\perp}$ shows that the cycle class map induces an isomorphism of commutative graded $\overline{\mathbb{Q}}_\ell$-algebras

$$A^*(X_{\mathscr{A}^\perp}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \simeq \mathrm{H}^{2*}(X_{\mathscr{A}^\perp}, \overline{\mathbb{Q}}_\ell),$$

Let Ψ be the composition of isomorphisms

$$\Psi\colon A^*(\overline{\mathrm{M}}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \simeq A^*(X_{\mathscr{A}^\perp}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \simeq \mathrm{H}^{2*}(X_{\mathscr{A}^\perp}, \overline{\mathbb{Q}}_\ell),$$

which maps $x_{\overline{F}}$ to the class of the strict transform in $X_{\mathscr{A}^\perp}$ of the exceptional divisor produced when blowing up the flat of $\mathscr{A}^\perp$ in $\mathbb{P}^r$ corresponding to $\overline{F}$.

PROPOSITION 13. *The element $\Psi(\beta_i)$ is the pull-back of the class of a point in $\mathbb{P}^1$ under f_i.*

Proof. We factor f into the composition

$$X_{\mathscr{A}_{\bar{I}}} \overset{j_{\mathscr{A}_{\bar{I}}}}{\hookrightarrow} X_{A_n} \overset{\pi_2}{\twoheadrightarrow} (\mathbb{P}^1)^n.$$

As noted before, the pull-back map associated with the inclusion $j_{\mathscr{A}_{\bar{I}}}$ satisfies

$$x_{\bar{I}} \longmapsto \begin{cases} x_{\bar{I}}, & \text{if } \bar{I} \text{ is a flat of } \overline{\mathrm{M}}, \\ 0, & \text{if } \bar{I} \text{ is a not flat of } \overline{\mathrm{M}}. \end{cases}$$

Thus, it is enough to prove the claim when $X_{\mathscr{A}_{\bar{I}}} = X_{A_n}$. This is the case when $\overline{\mathrm{M}}$ is the boolean matroid on $\overline{E}$, and the claim in this case was proved in Remark 11 at the level of Chow rings. □

Since the cohomology ring of $(\mathbb{P}^1)^n$ is generated by the pull-backs under f_i, Propositions 9 and 13 together show that Ψ induces an isomorphism between $B^*(\mathrm{M}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell$ and $B^*(f)_{\overline{\mathbb{Q}}_\ell}$, which we denote by Ψ'. More precisely, there is a commutative diagram

$$\begin{array}{ccc} A^*(\overline{\mathrm{M}}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell & \xrightarrow{\Psi} & \mathrm{H}^{2*}(X_{\mathscr{A}_{\bar{I}}}, \overline{\mathbb{Q}}_\ell) \\ {\scriptstyle \varphi \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell} \uparrow & & \uparrow \\ B^*(\mathrm{M}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell & \xrightarrow{\Psi'} & B^*(f)_{\overline{\mathbb{Q}}_\ell}, \end{array}$$

where φ is the injective ring homomorphism of Proposition 9.

Proof of Theorem 6. It is enough to show that the multiplication map

$$\begin{aligned} B^p(\mathrm{M}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell &\longrightarrow B^{r-p}(\mathrm{M}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell, \\ \xi &\longmapsto L^{r-2p}\xi, \end{aligned}$$

is injective. Under the isomorphism Ψ', the statement to be proved translates to the conclusion of Proposition 12 when f is the map $X_{\mathscr{A}_{\bar{I}}} \to (\mathbb{P}^1)^n$. □

With more work, we can show that the graded Möbius algebra of the matroid M is isomorphic to the cohomology ring of the variety $Y_{\mathscr{A}}$. Write L_i for the first Chern class of the pull-back of $\mathcal{O}(1)$ under the composition

$$Y_{\mathscr{A}} \hookrightarrow (\mathbb{P}^1)^n \overset{\mathrm{proj}_i}{\twoheadrightarrow} \mathbb{P}^1.$$

THEOREM 14. *There is an isomorphism of commutative graded $\overline{\mathbb{Q}}_\ell$-algebras*

$$\begin{aligned} B^*(\mathrm{M}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell &\xrightarrow{\simeq} H^{2*}(Y_{\mathscr{A}}, \overline{\mathbb{Q}}_\ell), \\ y_i &\longmapsto L_i. \end{aligned}$$

In what follows, we write $z_0, z_1, ..., z_n$ for the homogeneous coordinates of $\mathbb{P}^n$, and write $(z_1, w_1), ..., (z_n, w_n)$ for the multi-homogeneous coordinates of $(\mathbb{P}^1)^n$.

Proof. Recall that M and $\overline{\mathrm{M}}$ share the same set of circuits. For every circuit C of M, there are non-zero constants $a_c \in \overline{\mathbb{F}}_p$, one for each element $c \in C$, such that

$$\sum_{c \in C} a_c z_c = 0 \quad \text{on the image of } i_{\mathscr{A}^{\bar{}}} \colon \mathbb{P}^r \longrightarrow \mathbb{P}^n.$$

The collection $(a_c)_{c \in C}$ is uniquely determined by the circuit C, up to a common multiple.

A defining set of multi-homogeneous equations of $Y_{\mathscr{A}}$ is explicitly described by Ardila and Boocher in [AB, Theorem 1.3]:

$$Y_{\mathscr{A}} = \left\{ ((z_1, w_1), ..., (z_n, w_n)) \in (\mathbb{P}^1)^n : \sum_{c \in C} a_c z_c \left(\prod_{d \in C \setminus \{c\}} w_d \right) = 0 \text{ for all circuits } C \text{ of M} \right\}.$$

This shows that $Y_{\mathscr{A}}$ has an *algebraic cell decomposition*, in the sense of [BE, §3],

$$Y_{\mathscr{A}} = \coprod_F \mathbb{A}^{\mathrm{rank}(F)},$$

where the disjoint union is over all flats F of M, and $\mathbb{A}^{\mathrm{rank}(F)}$ is the intersection of $Y_{\mathscr{A}}$ with the affine space

$$\mathbb{A}^{|F|} = \{ ((z_1, w_1), ..., (z_n, w_n)) \in (\mathbb{P}^1)^n : w_i = 0 \text{ if and only if } i \text{ is not in } F \} \subseteq (\mathbb{P}^1)^n.$$

The existence of the cell decomposition has the following implications [BE, Theorem 3.1]:

(CD1) The natural map $\mathrm{H}^{2*}(Y_{\mathscr{A}}, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathrm{IH}^{2*}(Y_{\mathscr{A}}, \overline{\mathbb{Q}}_\ell)$ is injective.

(CD2) The dimension of $\mathrm{H}^{2k}(Y_{\mathscr{A}}, \overline{\mathbb{Q}}_\ell)$ is the number of k-dimensional cells in $Y_{\mathscr{A}}$ for all k.

All the odd-cohomology groups of $Y_{\mathscr{A}}$ are zero. When combined with the decomposition theorem for $X_{\mathscr{A}^{\bar{}}} \to Y_{\mathscr{A}}$, the statement (CD1) shows that the pull-back homomorphism in cohomology

$$\mathrm{H}^{2*}(Y_{\mathscr{A}}, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathrm{H}^{2*}(X_{\mathscr{A}^{\bar{}}}, \overline{\mathbb{Q}}_\ell)$$

is injective. According to Proposition 13, the pull-back of L_i in the cohomology of $X_{\mathscr{A}^{\bar{}}}$ is $\Psi(\beta_i)$, and hence the previous sentence implies that there is an injective graded ring homomorphism

$$\begin{aligned} B^*(f)_{\overline{\mathbb{Q}}_\ell} &\longrightarrow \mathrm{H}^{2*}(Y_{\mathscr{A}}, \overline{\mathbb{Q}}_\ell), \\ \Psi(\beta_i) &\longmapsto L_i. \end{aligned}$$

Composing with the isomorphism Ψ', we get the injective graded ring homomorphism

$$B^*(\mathrm{M})\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}_\ell \longrightarrow \mathrm{H}^{2*}(Y_{\mathscr{A}}, \overline{\mathbb{Q}}_\ell),$$
$$y_i \longmapsto L_i.$$

The statement (CD2) shows that the source and the target are $\overline{\mathbb{Q}}_\ell$-vector spaces of the same dimension, which is the number of flats of M. Therefore, the map must be an isomorphism. □

Remark 15. Let M be a simple matroid on $E=\{1,...,n\}$ with rank $r\geqslant 2$. We write "deg" for the isomorphism

$$\deg\colon B^r(\mathrm{M}) \longrightarrow \mathbb{Q},$$
$$y_E \longmapsto 1.$$

Let HR(M) be the symmetric $n\times n$ matrix with entries

$$\mathrm{HR}(\mathrm{M})_{ij} = \begin{cases} 0, & \text{if } i=j, \\ b_{ij}(\mathrm{M}), & \text{if } i\neq j, \end{cases}$$

where $b_{ij}(\mathrm{M})$ is the number of bases of M containing i and j. The matrix HR(M) represents the Hodge–Riemann form

$$B^1(\mathrm{M})\times B^1(\mathrm{M}) \longrightarrow \mathbb{Q},$$
$$(\xi_1,\xi_2) \longmapsto \deg(L^{r-2}\xi_1\xi_2),$$

with respect to the standard basis $\{y_1,...,y_n\}$. It can be shown that the matrix HR(M) has exactly one positive eigenvalue [HW].

Consider the restriction of HR(M) to the 3-dimensional subspace of $B^1(\mathrm{M})$ spanned by y_i, y_j and L. The one-positive-eigenvalue condition says that the determinant of the resulting symmetric 3×3 matrix is non-negative, and this implies

$$2 > \frac{b(\mathrm{M})b_{ij}(\mathrm{M})}{b_i(\mathrm{M})b_j(\mathrm{M})},$$

where $b(\mathrm{M})$ is the number of bases of M and $b_i(\mathrm{M})$ is the number of bases of M containing i. More detailed arguments will be given in [HW].

Question. How large can the ratio $b(\mathrm{M})b_{ij}(\mathrm{M})/b_i(\mathrm{M})b_j(\mathrm{M})$ be?

For graphic matroids, the work of Kirchhoff on electric circuits shows that the ratio is bounded above by 1, see [FM]. In other words, for a randomly chosen spanning tree of a graph, the presence of an edge can only make any other edge less likely. It was once conjectured that this is the case for all matroids, but Seymour and Welsh found an example with the ratio $\simeq 1.02$ [SW].

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