A LOCAL-GLOBAL EQUALITY ON EVERY AFFINE VARIETY ADMITTING POINTS IN AN ARBITRARY RANK-ONE SUBGROUP OF A GLOBAL FUNCTION FIELD

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ABSTRACT. For every affine variety over a global function field, we show that the set of its points with coordinates in an arbitrary rank-one multiplicative subgroup of this function field is topologically dense in the set of its points with coordinates in the topological closure of this subgroup in the product of the multiplicative group of those local completions of this function field over all but finitely many places.

1. INTRODUCTION

Let K be a global function field over a finite field k of positive characteristic p. We denote by k^{alg} the algebraic closure of k inside a fixed algebraic closure K^{alg} of K. Let Σ_K be the set of all places of K. For each $v \in \Sigma_K$, denote by K_v the completion of K at v; by O_v, $\mathfrak{m}_v$, and $\mathbb{F}_v$ respectively the valuation ring, the maximal ideal, and the residue field associated to v. For each finite subset $S \subset \Sigma_K$, we denote by O_S the ring of S-integers in K. For any commutative ring R with unity, denote by R^* the group of its units. We fix a subgroup $\Gamma \subset O_S^*$ for some finite $S \subset \Sigma_K$. Let M be a natural number, and $\mathbb{A}^M$ be the affine M-space, whose coordinate is denoted by $\mathbf{X} = (X_1, \ldots, X_M)$. For each polynomial $f \in K[X_1, \ldots, X_M]$, we denote by H_f the hypersurface in $\mathbb{A}^M$ defined by f. If the total degree of f is one, we say that H_f is a hyperplane. By a linear K-variety in $\mathbb{A}^M$, we mean an intersection of K-hyperplanes. We say that a closed K-variety W in $\mathbb{A}^M$ is homogeneous if W can be defined by homogeneous polynomials.

For any closed K-variety W in $\mathbb{A}^M$ and any subset Θ of some ring containing K, let $W(\Theta)$ denote the set of points on W with each coordinate in Θ. For each subset

$\widetilde{S} \subset \Sigma_K$, we endow $\prod_{v\in\widetilde{S}} K_v^*$ with the natural product topology; via the diagonal embedding, we identify $W(\Gamma)$ with its image $W(\Gamma)_{\widetilde{S}}$ in $W\left(\prod_{v\in\widetilde{S}} K_v^*\right)$ and denote by $\overline{W(\Gamma)_{\widetilde{S}}}$ its topological closure. We naturally identify Γ with $\mathbb{A}^1(\Gamma)$, and write $\overline{\Gamma_{\widetilde{S}}}$ for $\overline{\mathbb{A}^1(\Gamma)_{\widetilde{S}}}$. For each place $v \in \Sigma_K$, we write $\overline{\Gamma_v}$ for $\overline{\Gamma_{\{v\}}}$. Note that $\overline{\Gamma_{\widetilde{S}}} \subset \prod_{v\in\widetilde{S}} \overline{\Gamma_v}$. We fix a cofinite subset $\Sigma \subset \Sigma_K$, and drop the lower subscript Σ in the notation of topological closure; for example, we simply write $\overline{\Gamma}$ for $\overline{\Gamma_\Sigma}$.

For any closed K-variety W in $\mathbb{A}^M$, we consider the following conjecture.

Conjecture 1. $W(\overline{\Gamma}) = \overline{W(\Gamma)}$.

First formulated in this form by the author [Sun14], Conjecture 1 is an analog for split algebraic tori to Conjecture C in [PV10] for Abelian varieties. One of the deepest aspects of Conjecture 1 is explained as follows. If $W = \bigcup_{i\in I} W_i$ is a finite union of closed K-varieties in $\mathbb{A}^M$, then it is easy to see that

$$W(\overline{\Gamma}) \supset \bigcup_{i\in I} W_i(\overline{\Gamma}) \supset \bigcup_{i\in I} \overline{W_i(\Gamma)} = \overline{W(\Gamma)};$$

thus if Conjecture 1 holds for W, then we must have $W(\overline{\Gamma}) = \bigcup_{i\in I} W_i(\overline{\Gamma})$. In fact, following the idea proposed by Stoll (Question 3.12 in [Sto07], so-called "Adelic Mordell-Lang Conjecture") and first realized by Poonen and Voloch [PV10], all previous results [Sun13, Sun14] dealing with Conjecture 1 for reducible W are established by reducing it to the assertion that $Z(\overline{\Gamma}) = \bigcup_{i\in I} Z_i(\overline{\Gamma})$ for every finite union $Z = \bigcup_{i\in I} Z_i$ of irreducible zero-dimensional K-varieties in $\mathbb{A}^M$, and proving this assertion via an argument invented by Poonen and Voloch [PV10], who managed to bypass the difficulty encountered when one tries to develop the function-field analog of the proof by Stoll [Sto07] of the number-field counterpart of this assertion. In the present setting, the Mordell-Lang Conjecture is treated by Derksen and Masser [DM12] in full generalities. The author [Sun14] establishes some "adelic analog" (restated as Proposition 3 below) of their result in certain case, and completes the aforementioned reduction under the artificial hypothesis induced from this analog; this hypothesis is then put in the main result in [Sun14] on Conjecture 1. In the remaining case, however, no adelic analog exists (see Remark 4); thus it is hopeless to completely solve Conjecture 1 via this "Mordell-Lang approach".

In this paper, we tackle Conjecture 1 in a different approach; although we still need Proposition 3, its present usage is to reduce Conjecture 1 to the situation where W is defined over a hopefully smaller subfield of K without putting on any assumption on W. In the case where Γ has rank one, we directly prove Conjecture 1 in this situation by generalizing of Proposition 24 in the author's recent work [Sunar] via introducing Lemma 10, which is an elementary linear-algebra argument. Our approach leads to the following main result, in which no hypothesis is put on W.

Theorem 2. *Suppose that $\Gamma \cap O_S^*$ has rank at most one, where $S = \Sigma_K \setminus \Sigma$. Then for every closed K-variety W in $\mathbb{A}^M$, we have that $W(\overline{\Gamma}) = \overline{W(\Gamma)}$.*

2. The proof of Theorem 2

For any subgroup $\Delta \subset K^*$, we denote by $k(\Delta)$ the smallest subfield of K containing k and Δ, by $\rho(\Delta)$ the subgroup $\bigcap_{m\geq 0}(K^{p^m})^*\Delta$ of K^*, and by $\sqrt[K]{\Delta}$ the subgroup $\{x \in K^* : x^n \in \Delta \text{ for some } n \in \mathbb{N}\}$ of K^*. The following result is a special case of Proposition 6 in [Sun14].

Proposition 3. *Let d be the dimension of W. Suppose that W is a union of homogeneous linear K-varieties, and that each d-dimensional irreducible component of W is not $\rho(\Gamma)$-isotrivial. Then there exists a finite union V of homogeneous linear K-subvarieties of W with dimension smaller than d such that $W(\overline{\Gamma_v}) = V(\overline{\Gamma_v})$ for every $v \in \Sigma_K$; in particular, we have $W(\overline{\Gamma}) = V(\overline{\Gamma})$.*

Remark 4. Proposition 3 is an analog for $W(\overline{\Gamma})$ to the qualitative part of conclusion (a) in Main Estimate of Section 9 in [DM12]. However, there is no analog for $W(\overline{\Gamma})$ to the qualitative part of conclusion (b). To be precise, we consider the example where $W = H_f \subset \mathbb{A}^2$ with $f(X_1, X_2) = X_1 + X_2 - 1 \in k[X_1, X_2]$. Note that W is irreducible of dimension one. By the qualitative part of conclusion (b), there is a finite set $\mathcal{V}$ of irreducible proper K-subvarieties of W such that

$W\left(\sqrt[K]{\Gamma}\right) = \bigcup_{V\in\mathcal{V}}\bigcup_{e\in\mathbb{N}\cup\{0\}}\left(V\left(\sqrt[K]{\Gamma}\right)\right)^{p^e}$. Considering the proper K-subvariety $Z_0 = \bigcup_{V\in\mathcal{V}} V$ of W, we see that $\bigcup_{V\in\mathcal{V}}\left(V\left(\sqrt[K]{\Gamma}\right)\right)^{p^e} = \left(Z\left(\sqrt[K]{\Gamma}\right)\right)^{p^e}$ for each $e\in\mathbb{N}\cup\{0\}$; it follows that

$$W\left(\sqrt[K]{\Gamma}\right) = \bigcup_{e\in\mathbb{N}\cup\{0\}}\left(Z\left(\sqrt[K]{\Gamma}\right)\right)^{p^e}.$$

Nevertheless, in the case where $K=\mathbb{F}_p(T)$ and $\Gamma=\{cT^n(1-T)^m : c\in\mathbb{F}_p^*,\ (n,m)\in\mathbb{Z}^2\} = \sqrt[K]{\Gamma}$ and Σ is the maximal subset of Σ_K such that $\Gamma\subset O_v^*$ for each $v\in\Sigma$, we claim that

$$W\left(\overline{\Gamma}\right) \not\subset \bigcup_{e\in\mathbb{N}\cup\{0\}}\left(Z\left(\overline{\Gamma}\right)\right)^{p^e}$$

for every proper K-subvariety Z of W. To see this, first note that such Z must have dimension zero. Moreover, the sequence $(T^{p^{n!}})_{n\geq 0}$ converges to some element $\alpha = (\alpha_v)_{v\in\Sigma}\in\overline{\Gamma_T}\subset\overline{\Gamma}$, where $\Gamma_T\subset\Gamma$ is the cyclic subgroup generated by T. (Example 1 in [Sun13]) Similarly, we see that $1-\alpha\in\overline{\Gamma_{1-T}}\subset\overline{\Gamma}$, where $\Gamma_{1-T}\subset\Gamma$ is the cyclic subgroup generated by T. Thus we have that $(\alpha,1-\alpha)\in W(\overline{\Gamma})$. Based on these facts, our claim can be proved by either of the two following arguments.

(1) Example 1 in [Sun13] also shows that $\alpha\notin K^*$, thus $(\alpha,1-\alpha)\notin W(K^*)$. However, since Z has dimension zero, we have that $Z\left(\overline{\Gamma}\right) = Z(\Gamma)$. (Proposition 2 in [Sun14]) Because $Z(\Gamma)\subset Z(K^*)\subset W(K^*)$, this proves our claim.
(2) For each $v\in\Sigma$, note that $\alpha_v\in O_v^*$ and let $P_v(T)\in\mathbb{F}_p[T]$ be the unique irreducible polynomial such that $P_v(T)\in\mathfrak{m}_v$; thus we have that $P_v(T^{p^{n!}}) = P_v(T)^{p^{n!}}\in\mathfrak{m}_v^{p^{n!}}$ for each $n\geq 0$, which implies that $P_v(\alpha_v)\in\cap_{n\geq 0}\mathfrak{m}_v^{p^{n!}} = \{0\}$; it follows that P_v is the minimal polynomial for α_v over $\mathbb{F}_p$. On the other hand, for any $\alpha\in k^{\mathrm{alg}}$ and any $e\in\mathbb{N}\cup\{0\}$, the element α^{p^e} is a zero of the minimal point for α over $\mathbb{F}_P$. As Z is a zero-dimensional K-variety, it follows that the degrees of minimal polynomials of torsion points in $\left(Z\left(\overline{\Gamma_v}\right)\right)^{p^e}$ are uniformly bounded over all $(v,e)\in\Sigma\times(\mathbb{N}\cup\{0\})$. Because the degree of P_v can be arbitrarily large as v ranges over Σ, there

must be some $v_0 \in \Sigma$ such that $\alpha_{v_0} \notin \bigcup_{e\in\mathbb{N}\cup\{0\}} \left(Z\left(\overline{\Gamma_{v_0}}\right)\right)^{p^e}$; since $Z\left(\overline{\Gamma}\right) \subset \prod_{v\in\Sigma} Z\left(\overline{\Gamma_v}\right)$, this proves our claim.

Proposition 5. *For any closed K-variety $W \subset \mathbb{A}^M$, there exists some closed $k(\rho(\Gamma))$-subvariety V of W such that $W(\overline{\Gamma_v}) = V(\overline{\Gamma_v})$ for every $v \in \Sigma_K$; in particular, we have $W(\overline{\Gamma}) = V(\overline{\Gamma})$.*

Proof. Let $\{f_j : 1 \leq j \leq J\} \subset K[X_1, \ldots, X_M]$ be a set of polynomials defining W. Choose $D \in \mathbb{N}$ such that for each $j \in \{1, \ldots, J\}$, we may write

$$f_j(X_1, \ldots, X_M) = \sum_{(d_1,\cdots,d_M)\in\{0,1,\cdots,D\}^M} c_{(j,d_1,\cdots,d_M)} X_1^{d_1} \cdots X_M^{d_M}$$

with each $c_{(j,d_1,\cdots,d_M)} \in K$. Consider the tuple $\mathbf{Y} = (Y_{(d_1,\cdots,d_M)})_{(d_1,\cdots,d_M)\in\{0,1,\cdots,D\}^M}$ of new variables, in which we define linear forms

$$\ell_j(\mathbf{Y}) = \sum_{(d_1,\cdots,d_M)\in\{0,1,\cdots,D\}^M} c_{(j,d_1,\cdots,d_M)} Y_{(d_1,\cdots,d_M)}$$

for each $j \in \{1, \ldots, J\}$. Let $N = (D+1)^M$ and $W' \subset \mathbb{A}^N$ be the homogeneous linear variety defined by $\{\ell_j : 1 \leq j \leq J\}$. By Proposition 3, there exists a finite union V' of homogeneous linear K-subvarieties of W' such that each irreducible component of V' is $\rho(\Gamma)$-isotrivial and that $W'(\overline{\Gamma_v}) = V'(\overline{\Gamma_v})$ for every $v \in \Sigma_K$. In particular, each irreducible component of V' is defined over $k(\rho(\Gamma))$, thus so is V'. Let $\{g'_j : 1 \leq j \leq J'\} \subset k(\rho(\Gamma))[\mathbf{Y}]$ be a set of polynomials defining V'. For each $j \in \{1, \ldots, J'\}$, we construct $f'_j(X_1, \ldots, X_M)$ by substituting each variable $Y_{(d_1,\cdots,d_M)}$ in $g'_j(\mathbf{Y})$ by the monomial $X_1^{d_1} \cdots X_M^{d_M}$, thus we have that $f'_j(X_1, \ldots, X_M) \in k(\rho(\Gamma))[X_1, \ldots, X_M]$. Let $V \subset \mathbb{A}^M$ be the $k(\rho(\Gamma))$-variety whose vanishing ideal is generated defined by $\{f'_j : 1 \leq j \leq J'\}$. For every $j \in \{1, \ldots, J'\}$ and every $(x_1, \ldots, x_M) \in V(K^{\mathrm{alg}})$, we have $f'_j(x_1, \ldots, x_M) = 0$, thus the point $(x_1^{d_1} \cdots x_M^{d_M})_{(d_1,\cdots,d_M)\in\{0,1,\cdots,D\}^M} \in \mathbb{A}^N(K^{\mathrm{alg}})$ is a zero of $g'_j(\mathbf{Y})$ by construction; this shows that $(x_1^{d_1} \cdots x_M^{d_M})_{(d_1,\cdots,d_M)\in\{0,1,\cdots,D\}^M} \in V'(K^{\mathrm{alg}}) \subset W'(K^{\mathrm{alg}})$ and thus the construction yields $(x_1, \ldots, x_M) \in W(K^{\mathrm{alg}})$. Hence we see that $V \subset W$.

Similar reasonings gives the other desired conclusion that $W(\overline{\Gamma_v}) = V(\overline{\Gamma_v})$ for every $v \in \Sigma_K$. $\square$

For any finitely generated subgroup $\Delta \subset K^*$, Lemma 3 of [Vol98] shows that $\rho(\Delta) \subset \sqrt[K]{\Delta}$, and thus that Δ and $\rho(\Delta)$ have the same rank; we also note that $\Delta \subset \rho(\Delta) = \rho(\rho(\Delta))$ by definition.

Proposition 6. *Letting $S = \Sigma_K \setminus \Sigma$, there exists a free subgroup $\Phi \subset O_S^*$ which has the same rank as $\Gamma \cap O_S^*$ and satisfies the following property: if $V(\overline{\Phi}) = \overline{V(\Phi)}$ for every closed $k(\Phi)$-variety $V \subset \mathbb{A}^M$, then $W(\overline{\Gamma}) = \overline{W(\Gamma)}$ for every closed K-variety $W \subset \mathbb{A}^M$.*

Proof. Let Φ be a maximal free subgroup of the finitely generated abelian group $\rho(\Gamma \cap O_S^*)$. Since $\Phi \subset \rho(\Phi) \subset \rho(\rho(\Gamma \cap O_S^*)) = \rho(\Gamma \cap O_S^*)$, it follows that Φ is a maximal free subgroup of $\rho(\Phi)$, and this implies that $\rho(\Phi) = \mathrm{Tor}(\rho(\Phi))\Phi = k^*\Phi$. Letting $S_0 \subset \Sigma_K$ be a finite subset such that $\Gamma \subset O_{S_0}^*$, we see that the image of Γ in $\prod_{v\in\Sigma} K_v^*$ is contained in $\left(\prod_{v\in\Sigma\cap S_0} K_v^*\right) \times \left(\prod_{v\in\Sigma\setminus S_0} O_v^*\right)$; since $S = \Sigma_K \setminus \Sigma$, this shows that the image of $\Gamma \cap O_S^*$ in $\prod_{v\in\Sigma} K_v^*$ is exactly the intersection of the image of Γ in $\prod_{v\in\Sigma} K_v^*$ with the open subgroup $\left(\prod_{v\in\Sigma\cap S_0} O_v^*\right) \times \left(\prod_{v\in\Sigma\setminus S_0} K_v^*\right)$ of $\prod_{v\in\Sigma} K_v^*$. It follows that $\Gamma \cap O_S^*$ is open in Γ. Since the index of $\Phi \cap \Gamma \cap O_S^*$ in $\Gamma \cap O_S^*$ is finite, Corollary 2 of [Sun14] shows that $\Phi \cap \Gamma \cap O_S^*$ is open in $\Gamma \cap O_S^*$, and thus is open in Γ. We note that $\Phi \cap \Gamma \cap O_S^* = \Phi \cap \Gamma$ since $\Phi \subset O_S^*$.

Fix a closed K-variety $W \subset \mathbb{A}^M$. Consider an arbitrary $\mathbf{x} \in W(\overline{\Gamma})$, which is the limit of a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $\mathbb{A}^M(\Gamma)$. Since $\Phi \cap \Gamma$ is open in Γ, we may assume that $\mathbf{x}_n = \mathbf{r}\mathbf{y}_n$ with some $\mathbf{r} \in \mathbb{A}^M(\Gamma)$ and a sequence $(\mathbf{y}_n)_{n\in\mathbb{N}}$ in $\mathbb{A}^M(\Phi \cap \Gamma)$. Note that the sequence $(\mathbf{y}_n)_{n\in\mathbb{N}}$ converges to $\mathbf{r}^{-1}\mathbf{x} \in (\mathbf{r}^{-1}W)(\overline{\Phi})$. Recalling that $\rho(\Phi) = k^*\Phi$, Proposition 5 says that there exist some closed $k(\Phi)$-subvariety V of $\mathbf{r}^{-1}W$ such that $(\mathbf{r}^{-1}W)(\overline{\Phi}) = V(\overline{\Phi})$. Assuming $V(\overline{\Phi}) = \overline{V(\Phi)}$, we see that $\mathbf{r}^{-1}\mathbf{x} \in (\mathbf{r}^{-1}W)(\overline{\Phi}) = V(\overline{\Phi}) = \overline{V(\Phi)} \subset \overline{(\mathbf{r}^{-1}W)(\Phi)} \subset \mathbf{r}^{-1}\left(\overline{W(\Phi)}\right)$, i.e. $\mathbf{x} \in \overline{W(\Phi)}$ is the limit of some sequence $(\mathbf{x}'_n)_{n\in\mathbb{N}}$ in $W(\Phi)$. Letting $(\mathbf{x}''_n)_{n\in\mathbb{N}} \subset \mathbb{A}^M(\Phi\Gamma)$ be the sequence defined by $\mathbf{x}''_{2n-1} = \mathbf{x}_n$ and $\mathbf{x}''_{2n} = \mathbf{x}'_n$, we see that the sequence

$(\mathbf{x}''_n)_{n\in\mathbb{N}} \subset \mathbb{A}^M(\Phi\Gamma)$ is Cauchy. As abelian groups, $\Phi\Gamma/\Gamma$ is isomorphic to $\Phi/(\Gamma\cap\Phi)$, which is finite by the construction of Φ; thus Corollary 2 of [Sun14] shows that Γ is open in $\Phi\Gamma$. It follows that $\Phi \cap \Gamma$ is open in $\Phi\Gamma$. Hence, for sufficiently large $n \in \mathbb{N}$, we have that $(\mathbf{r}^{-1}\mathbf{x}_n)^{-1}(\mathbf{r}^{-1}\mathbf{x}'_n) = (\mathbf{x}''_{2n-1})^{-1}\mathbf{x}''_{2n} \in \mathbb{A}^M(\Phi \cap \Gamma)$; since $\mathbf{r}^{-1}\mathbf{x}_n = \mathbf{y}_n \in \mathbb{A}^M(\Phi \cap \Gamma)$, we conclude that $\mathbf{r}^{-1}\mathbf{x}'_n \in \mathbb{A}^M(\Phi \cap \Gamma)$ and thus $\mathbf{x}'_n \in \mathbb{A}^M(\Gamma) \cap W(\Phi) \subset W(\Gamma)$, i.e. $\mathbf{x} \in \overline{W(\Gamma)}$. This finishes the proof.

□

For any $a \in \mathbb{N}$ and $b \in \mathbb{N} \setminus p\mathbb{N}$, consider the polynomial

$$g_{a,b}(T) = \frac{T^{ab} - 1}{T^a - 1} \in k[T].$$

We make the following convention. For a polynomial $Q(T) \in k[T]$ and a rational function $P(T) \in k(T)$, we say that $Q(T)$ divides $P(T)$ if any zero of $Q(T)$ in k^{alg} is not a pole of $\frac{P(T)}{Q(T)}$. The long proof of the following proposition, which is the core in the proof of Theorem 2, is postponed to Section 3.

Proposition 7. *Let $\mathcal{S}$ be a finite set of irreducible polynomials in $k[T]$. Let J be a natural number. For each $j \in \{1, \ldots, J\}$, let $f_j(X_1, \ldots, X_M) \in k[T][X_1, \ldots, X_M]$.*

Assume that there exists a sequence $\{(e_{1,n}, \ldots, e_{M,n})\}_{n\geq 1}$ in $\mathbb{A}^M(\mathbb{Z})$ satisfying the following conditions:

For every $Q(T) \in k[T]$ not divisible by any element in $\mathcal{S}$, there is an $N_Q \in \mathbb{N}$ such that for any $n \geq N_Q$ we have that $Q(T)$ divides $f_j(T^{e_{1,n}}, \ldots, T^{e_{M,n}})$ for all $j \in \{1, \ldots, J\}$.

Then there exists a sequence $\{(e'_{1,n}, \ldots, e'_{M,n})\}_{n\in\mathcal{N}}$ in $\mathbb{A}^M(\mathbb{Z})$ indexed by an infinite subset $\mathcal{N} \subset \mathbb{N}$ with the following properties:

(1) *For each $n \in \mathcal{N}$ we have $f_j(T^{e'_{1,n}}, \ldots, T^{e'_{M,n}}) = 0$ for all $j \in \{1, \ldots, J\}$.*
(2) *For every $\widetilde{Q}(T) \in k[T]$ not divisible by T, there is an $\widetilde{N}_{\widetilde{Q}} \in \mathbb{N}$ such that for any $n \in \mathcal{N}$ with $n \geq \widetilde{N}_{\widetilde{Q}}$ we have that $\widetilde{Q}(T)$ divides $T^{e_{i,n}} - T^{e'_{i,n}}$ for all i.*

The next theorem, which follows formally from Proposition 7, is proved in the same way that in [Sunar] Theorem 25 is formally deduced from Proposition 24.

Theorem 8. *Let W be a closed $k(\Gamma)$-variety in $\mathbb{A}^M$. Suppose that Γ is free with rank one, and that is contained in O_S^*, where $S = \Sigma_K \setminus \Sigma$. Then we have that $W(\overline{\Gamma}) = \overline{W(\Gamma)}$.*

Proof. Let γ be a generator of Γ. Let $\Sigma|_{k(\gamma)} \subset \Sigma$ be the subset satisfying the following property that for each $v \in \Sigma$ there exists a unique $w \in \Sigma|_{k(\gamma)}$ such that both v and w restrict to the same place of $k(\gamma)$. Consider the k-isomorphism between fields

$$k(T) \to k(\gamma),\ T \mapsto \gamma. \tag{2.1}$$

Through the isomorphism (2.1), the set $\Sigma|_{k(\gamma)}$ is injectively mapped onto a subset of the set of places of $k(T)$. For each $v \in \Sigma|_{k(\gamma)}$, we have that $\gamma \in O_v^*$; let $P_v(T) \in k[T]$ be the irreducible polynomial corresponding to the image of v under this map. Let $\mathcal{S}$ be the complement of the subset $\{P_v(T) : v \in \Sigma|_{k(\gamma)}\}$ of the set of all irreducible polynomials in $k[T]$. Note that $\mathcal{S}$ is a finite set containing the polynomial T, and that $k[\Gamma] \subset \prod_{v\in\Sigma} O_v$, where $k[\Gamma]$ is the smallest subring of K containing both k and Γ.

Write $W = \bigcap_{j=1}^{J} H_{f_j}$, where $f_j(X_1, \ldots, X_M) \in k[\gamma][X_1, \ldots, X_M]$ for each j. Let $\{(\gamma^{e_{1,n}}, \ldots, \gamma^{e_{M,n}})\}_{n\geq 1}$ be a sequence in $\mathbb{A}^M(\Gamma)$ which converges to a point $(x_1, \ldots, x_M) \in W(\overline{\Gamma}) \subset \mathbb{A}^M\left(\prod_{v\in\Sigma} K_v^*\right)$, where $e_{i,n} \in \mathbb{Z}$. In fact, this sequence lies in the image of $\mathbb{A}^M\left(\prod_{v\in\Sigma|_{k(\gamma)}} k(\gamma)_v^*\right)$ in $\mathbb{A}^M\left(\prod_{v\in\Sigma} K_v^*\right)$ under the natural map, where $k(\gamma)_v$ denotes the topological closure of the subfield $k(\gamma)$ in K_v. Note that this image is a closed subset. The topology on $\overline{\Gamma}$ is induced from the usual product topology on $\prod_{v\in\Sigma} k(\gamma)_v^*$, and the latter topology is the same as the subspace topology restricted from the usual product topology on $\prod_{v\in\Sigma} k(\gamma)_v$. Thus for each $i \in \{1, \ldots, M\}$ the sequence $(\gamma^{e_{i,n}})_{n\geq 1}$ converges to x_i in $\prod_{v\in\Sigma} k(\gamma)_v$ Therefore, from the continuity of each f_j at $(x_1, \ldots, x_M) \in \mathbb{A}^M\left(\prod_{v\in\Sigma} k(\gamma)_v\right)$, we see that each sequence $(f_j(\gamma^{e_{1,n}}, \ldots, \gamma^{e_{M,n}}))_{n\geq 1}$ converges to $f_j(x_1, \ldots, x_M) = 0$ in $\prod_{v\in\Sigma} k(\gamma)_v$. Consider the sequence $\{(e_{1,n}, \ldots, e_{M,n})\}_{n\geq 1}$ in $\mathbb{A}^M(\mathbb{Z})$. Fix an arbitrary $Q(T) \in k[T]$ not divisible by any element in $\mathcal{S}$. Thus we have the prime decomposition $Q(T) = \prod_{v\in\Sigma|_{k(\gamma)}} P_v(T)^{n_v}$ in $k[T]$, where there are only finitely many $v \in \Sigma|_{k(\gamma)}$ with $n_v > 0$. In particular,

$$U_Q = \prod_{\substack{v \in \Sigma|_{k(\gamma)} \\ n_v = 0}} k(\gamma)_v \times \prod_{\substack{v \in \Sigma|_{k(\gamma)} \\ n_v > 0}} (\mathfrak{m}_v \cap k(\gamma)_v)^{n_v}$$

is an open subset in $\prod_{v\in\Sigma|_{k(\gamma)}} k(\gamma)_v$ endowed with the the product topology. Note that $f_j(\gamma^{e_{1,n}}, \ldots, \gamma^{e_{M,n}}) \in k[\gamma, \gamma^{-1}]$ for each $j \in \{1, \ldots, J\}$ and $n \in \mathbb{N}$. The intersection of U_Q with the image of $k[\gamma, \gamma^{-1}]$ in $\prod_{v\in\Sigma|_{k(\gamma)}} k(\gamma)_v$ is the image of $Q[\gamma]k[\gamma, \gamma^{-1}]$, which is thus an open subset of $k[\gamma, \gamma^{-1}]$ containing zero with respect to the subspace topology restricted from $\prod_{v\in\Sigma|_{k(\gamma)}} k(\gamma)_v$. Therefore, from the fact each sequence $(f_j(\gamma^{e_{1,n}}, \ldots, \gamma^{e_{M,n}}))_{n\geq 1}$ converges to zero in $\prod_{v\in\Sigma} k(\gamma)_v$, it follows that there is an $N_Q \in \mathbb{N}$ such that for any $n \geq N_Q$ we have that $f_j(\gamma^{e_{1,n}}, \ldots, \gamma^{e_{M,n}}) \in Q[\gamma]k[\gamma, \gamma^{-1}]$ for each $j \in \{1, \ldots, J\}$; thus by the isomorphism (2.1) we have that $Q(T)$ divides $f_j(T^{e_{1,n}}, \ldots, T^{e_{M,n}})$, because 0 is not a zero of $Q(T)$. Therefore the assumption of Proposition 7 is verified. Applying the isomorphism (2.1) to the conclusion of Proposition 7, we see that there exists a sequence $\{(e'_{1,n}, \ldots, e'_{M,n})\}_{n\in\mathcal{N}}$ in $\mathbb{A}^M(\mathbb{Z})$ indexed by an infinite subset $\mathcal{N} \subset \mathbb{N}$ satisfying the following properties: for each $n \in \mathcal{N}$ we have $f_j(\gamma^{e'_{1,n}}, \ldots, \gamma^{e'_{M,n}}) = 0$ for all $j \in \{1, \ldots, J\}$, and for every $\widetilde{Q}(T) \in k[T]$ not divisible by T, there is an $\widetilde{N}_{\widetilde{Q}} \in \mathbb{N}$ such that for any $n \in \mathcal{N}$ with $n \geq \widetilde{N}_{\widetilde{Q}}$ we have that $\gamma^{e_{i,n}} - \gamma^{e'_{i,n}} \in \widetilde{Q}(\gamma)k[\gamma, \gamma^{-1}]$ for all $i \in \{1, \ldots, M\}$. The first property says that $(\gamma^{e'_{1,n}}, \ldots, \gamma^{e'_{M,n}}) \in W(\Gamma)$ for each $n \in \mathcal{N}$. On the other hand, because the image of $k[\gamma, \gamma^{-1}]$ in $\prod_{v\in\Sigma|_{k(\gamma)}} k(\gamma)_v$ lies in $\prod_{v\in\Sigma|_{k(\gamma)}}(O_v \cap k(\gamma)_v)$, one may argue similarly as above that the topology on $k[\gamma, \gamma^{-1}]$, which is induced from the usual product topology on $\prod_{v\in\Sigma} k(\gamma)_v$, is generated by those subset $\widetilde{Q}(\gamma)k[\gamma, \gamma^{-1}]$ with $\widetilde{Q}(T) \in k[T]$ not divisible by any element in the set $\mathcal{S}$. Since $\mathcal{S}$ contains the polynomial T, the second property implies that for each $i \in \{1, \ldots, M\}$ the sequence $(\gamma^{e_{i,n}} - \gamma^{e'_{i,n}})_{n\in\mathcal{N}}$ converges to zero in $\prod_{v\in\Sigma} k(\gamma)_v$; this shows that the two sequences $(\gamma^{e_{i,n}})_{n\in\mathcal{N}}$ and $(\gamma^{e'_{i,n}})_{n\in\mathcal{N}}$ converge to the same element in $\prod_{v\in\Sigma} k(\gamma)_v$. Hence, for each $i \in \{1,\ldots,M\}$, the sequence $(\gamma^{e'_{i,n}})_{n\in\mathcal{N}}$ converges to x_i in $\prod_{v\in\Sigma} k(\gamma)_v$; since $x_i \in \prod_{v\in\Sigma} k(\gamma)^*_v$, it follows from what is explained above that the same convergence also happens in $\prod_{v\in\Sigma} k(\gamma)^*_v$. This shows that $(x_1,\ldots,x_M) \in \overline{\{(\gamma^{e'_{1,n}},\ldots,\gamma^{e'_{M,n}})\}_{n\in\mathcal{N}}} \subset \overline{W(\Gamma)}$, which completes the proof. □

Proof of Theorem 2. Combine Proposition 6 and Theorem 8. □

3. The Proof of Proposition 7

The following result is proved in the author's recent work [Sunar].

Lemma 9. *Let $f(T) = \sum_{i\in I} c_i T^{e_i} \in k(T)$ with each $c_i \in k$ and $e_i \in \mathbb{Z}$, where I is a finite index set. Let $a \in \mathbb{N}$, $b \in \mathbb{N}\setminus p\mathbb{N}$ with b greater than the cardinality of I. Denote by $\mathscr{C}$ the collection of those partitions $\mathscr{P}$ of the set I such that for each set $\Omega \in \mathscr{P}$ we have $\sum_{i\in\Omega} c_i = 0$ and for each nonempty proper subset $\Omega' \subset \Omega$ we have $\sum_{i\in\Omega'} c_i \neq 0$. Suppose that $g_{a,b}(T)$ divides $f(T)$. Then there is some $\mathscr{P} \in \mathscr{C}$ such that for each set $\Omega \in \mathscr{P}$ and each $i_1, i_2 \in \Omega$ we have that ab divides $e_{i_1} - e_{i_2}$.*

Proved by an elementary linear-algebra argument, the following result plays a crucial role so that Proposition 24 in the author's recent work [Sunar] can be generalized to Proposition 7, which is the core in the proof of Theorem 2.

Lemma 10. *Let $\mathcal{N} \subset \mathbb{N}$ be a subset such that for each $m \in \mathbb{N}$ there is some $n \in \mathcal{N}$ divisible by m. Let $a_{j,i} \in \mathbb{Z}$ and $b_j \in \mathbb{Z}$, $(j,i) \in \{1,\ldots,J\} \times \{1,\ldots,M\}$ be fixed integers. Suppose that for each $n \in \mathcal{N}$ there are some $e_{i,n}$, $i \in \{1,\ldots,M\}$ such that n divides $b_j - \sum_{i=1}^{M} a_{j,i} e_{i,n}$ for each j. Then there is some $n_0 \in \mathcal{N}$ with the following property: for each $n \in \mathcal{N}$ divisible by n_0, there are some $e'_{i,n}$, $i \in \{1,\ldots,M\}$, such that $\frac{n}{n_0}$ divides $e_{i,n} - e'_{i,n}$ and that $b_j = \sum_{i=1}^{M} a_{j,i} e'_{i,n}$ for each j.*

Proof. Consider the J-by-$(M+1)$ matrix $(a_{j,i} \,|\, b_j)$, where j indices rows and i indices the first M columns. Applying a sequence of the following operations: interchanging any two rows or any two of the first M columns, multiplying some row by an integer, adding some row to another one, we can transform this matrix to $(a'_{j,i} \,|\, b'_j)$ such that for some $R \le \min\{J, M\}$ we have that $a'_{j,i} = 0$ for any $(j,i) \in (\{1,\dots,J\} \times \{1,\dots,R\}) \cup (\{R+1,\dots,J\} \times \{1,\dots,M\})$ with $i \neq j$, and that $a'_{i,i} \neq 0$ if and only if $i \in \{1,\dots,R\}$. Then there is some permutation σ on $\{1,\dots,M\}$ such that n divides $b'_j - \sum_{i=1}^M a'_{j,i} e_{\sigma(i),n}$ for each $n \in \mathcal{N}$ and each $j \in \{1,\dots,J\}$. By the properties of $\mathcal{N}$, there is some $n_0 \in \mathcal{N}$ divisible by $\prod_{i=1}^R a'_{i,i}$. For any $j \in \{1,\dots,R\}$ and any $n \in \mathcal{N}$ divisible by n_0, from the fact that

$$b'_j - \sum_{i=1}^M a'_{j,i} e_{\sigma(i),n} = b'_j - a'_{j,j} e_{\sigma(j),n} - \sum_{i=R+1}^M a'_{j,i} e_{\sigma(i),n}$$

is divisible by $n \in a'_{j,j}\mathbb{Z}$, we see that $a'_{j,j}$ divides $b'_j - \sum_{i=R+1}^M a'_{j,i} e_{\sigma(i),n}$, and thus there exists a unique $e'_{\sigma(j),n} \in \mathbb{Z}$ satisfying $b'_j - a'_{j,j} e'_{\sigma(j),n} - \sum_{i=R+1}^M a'_{j,i} e_{\sigma(i),n} = 0$; hence n divides $a'_{j,j}(e'_{\sigma(j),n} - e_{\sigma(j),n})$. For any $j \in \{1,\dots,R\}$, since n_0 divisible by $a'_{j,j}$, we conclude that $e_{\sigma(j),n} - e'{}_{\sigma(j),n}$ is divisible by $\frac{n}{a'_{j,j}}$ and thus by $\frac{n}{n_0}$ as desired. For any $j \in \{R+1,\dots,J\}$ and any $n \in \mathcal{N}$ divisible by n_0, we simply define $e'_{\sigma(j),n} = e_{\sigma(j),n}$; thus $\frac{n}{n_0}$ divides $e_{\sigma(j),n} - e'_{\sigma(j),n}$ trivially. For every pair $(j,i) \in \{R+1,\dots,J\} \times \{1,\dots,M\}$, we have $a'_{j,i} = 0$, hence b'_j is divisible by every integer, and therefore $b'_j = 0$. Combined with the construction of $e'_{\sigma(j),n}$ for any $j \in \{1,\dots,R\}$, we see that for any $j \in \{1,\dots,J\}$ and any $n \in \mathcal{N}$ divisible by n_0, we always have $b'_j - \sum_{i=1}^M a'_{j,i} e'_{\sigma(i),n} = 0$. Transforming the matrix $(a'_{j,i} \,|\, b'_j)$ back to $(a_{j,i} \,|\, b_j)$, we obtain that $b_j - \sum_{i=1}^M a_{j,i} e'_{i,n} = 0$ as desired. □

We are ready to present the

Proof of Proposition 7. Choose $D \in \mathbb{N}$ such that for each $j \in \{1,\dots,J\}$, we may write

$$f_j(X_1,\dots,X_M) \quad = \sum_{(d_0,d_1,\cdots,d_M)\in\{0,1,\cdots,D\}^{M+1}} c_{(j,d_0,d_1,\cdots,d_M)} T^{d_0} X_1^{d_1} \cdots X_M^{d_M}$$

with each $c_{(j,d_0,d_1,\cdots,d_M)} \in k$. For each $j \in \{1,\dots,J\}$, denote by $\mathscr{C}_j$ the collection of those partitions $\mathscr{P}$ of the set $\{0,1,\cdots,D\}^{M+1}$ such that for each set $\Omega \in \mathscr{P}$ we have $\sum_{(d_0,d_1,\cdots,d_M)\in\Omega} c_{(j,d_0,d_1,\cdots,d_M)} = 0$ and for each nonempty proper subset

$\Omega' \subset \Omega$ we have $\sum_{(d_0,d_1,\cdots,d_M)\in\Omega'} c_{(d_0,d_1,\cdots,d_M)} \neq 0$. By Remark 14 in [Sunar], we may choose some $a_0 \in \mathbb{N}\setminus p\mathbb{N}$ and $b_0 \in \mathbb{N}\setminus p\mathbb{N}$ with $b_0 > (D+1)^{M+1}$ such that for any $a \in a_0\mathbb{N}$ the polynomial $g_{a,b_0}(T)$ is not divisible by any element in $\mathcal{S}$. By our assumption, there is a strictly increasing sequence $\{N_a\}_{a\in a_0\mathbb{N}}$ such that $g_{a,b_0}(T)$ divides

$$f_j(T^{e_{1,N_a}},\ldots,T^{e_{M,N_a}}) = \sum_{(d_0,d_1,\cdots,d_M)\in\{0,1,\cdots,D\}^{M+1}} c_{(j,d_0,d_1,\cdots,d_M)} T^{d_0+d_1e_{1,N_a}+\cdots+d_Me_{M,N_a}}$$

for any $j \in \{1,\ldots,J\}$. Thus, by Lemma 9, for any $a \in a_0\mathbb{N}$ and any $j \in \{1,\ldots,J\}$, there is some $\mathscr{P}_{j,a} \in \mathscr{C}_j$ such that for each set $\Omega \in \mathscr{P}_{j,a}$, each $(d_0,d_1,\cdots,d_M)$ and $(d'_0,d'_1,\cdots,d'_M)$ in Ω, any $i \in \{1,\ldots,M\}$ and any $n \geq N_a$, we have that ab_0 divides both $d_0-d'_0+(d_1-d'_1)e_{1,N_a}+\cdots+(d_M-d'_M)e_{M,N_a}$. Consider the subset $\{\prod_{i=1}^{a_0+n} i :\ n \in \mathbb{N}\} \subset a_0\mathbb{N}$. For each $j \in \{1,\ldots,J\}$ the collection $\mathscr{C}_j$ is finite while $\{\prod_{i=1}^{a_0+n} i :\ n \in \mathbb{N}\}$ is infinite; thus there is an infinite subset $\mathcal{A}$ of the set $\{\prod_{i=1}^{a_0+n} i :\ n \in \mathbb{N}\}$, which is contained in $a_0\mathbb{N}$, such that for each $j \in \{1,\ldots,J\}$ the collection $\{\mathscr{P}_{j,a} : a \in \mathcal{A}\}$ consists of only one partition, denoted by $\mathscr{P}_j$. Since $\mathcal{A} \subset \{\prod_{i=1}^{a_0+n} i :\ n \in \mathbb{N}\}$ is an infinite subset, it has the property that for each $m \in \mathbb{N}$ there is some $a \in \mathcal{A}$ divisible by m.

For any $a \in \mathcal{A}$, any $j \in \{1,\ldots,J\}$, we observe that $(e_{1,N_a},\cdots,e_{M,N_a})$ satisfies the condition that for each set $\Omega \in \mathscr{P}_j$, each $(d_0,d_1,\cdots,d_M)$ and $(d'_0,d'_1,\cdots,d'_M)$ in Ω, we have that a divides $d_0-d'_0+(d_1-d'_1)e_{1,N_a}+\cdots+(d_M-d'_M)e_{M,N_a}$. Applying Lemma 10, we obtain some $n_0 \in \mathcal{A}$ with the following property: for each $a \in \mathcal{A}$ divisible by n_0, there are some e'_{i,N_a}, $i \in \{1,\ldots,M\}$, such that $\frac{a}{n_0}$ divides $e_{i,N_a}-e'_{i,N_a}$ and that for each $j \in \{1,\ldots,J\}$, each set $\Omega \in \mathscr{P}_j$, each $(d_0,d_1,\cdots,d_M)$ and $(d'_0,d'_1,\cdots,d'_M)$ in Ω, we have

$$d_0-d'_0+(d_1-d'_1){e'}_{1,N_a}+\cdots+(d_M-d'_M){e'}_{M,N_a}=0;$$

thus we may let $m_{a,j,\Omega} = d_0+d_1{e'}_{1,N_a}+\cdots+d_M{e'}_{M,N_a}$ for any $(d_0,d_1,\cdots,d_M) \in \Omega$. Letting $\mathcal{N} = \{N_a :\ a \in \mathcal{A}\cap n_0\mathbb{N}\}$, which is an infinite subset of $\mathbb{N}$ since $\mathcal{A} \subset a_0\mathbb{N}$ and the sequence $\{N_a\}_{a\in a_0\mathbb{N}}$ is strictly increasing, we now show that the constructed sequence $\{(e'_{1,n}, \ldots, e'_{M,n})\}_{n\in\mathcal{N}}$ satisfies the desired properties. To verify Property (1), we fix some $j \in \{1, \ldots, J\}$ and $n = N_a \in \mathcal{N}$ with $a \in \mathcal{A} \cap n_0\mathbb{N}$. From construction, we have

$$\begin{aligned} & f_j(T^{e'_{1,n}}, \ldots, T^{e'_{M,n}}) \\ = \; & \textstyle\sum_{(d_0,d_1,\cdots,d_M)\in\{0,1,\cdots,D\}^{M+1}} c_{(j,d_0,d_1,\cdots,d_M)} T^{d_0+d_1 e'_{1,N_a}+\cdots+d_M e'_{M,N_a}} \\ = \; & \textstyle\sum_{\Omega\in\mathscr{P}_j} T^{m_{a,j,\Omega}} \sum_{(d_0,d_1,\cdots,d_M)\in\Omega} c_{(j,d_0,d_1,\cdots,d_M)} \\ = \; & 0 \end{aligned}$$

as desired. To verify Property (2), we fix some $\widetilde{Q}(T) \in k[T]$, not divisible by T. Since each zero of $\widetilde{Q}(T)$ is in $(k^{\mathrm{alg}})^*$ and thus has a finite order, we can use the property that that for each $m \in \mathbb{N}$ there is some element of $\mathcal{A} \cap n_0\mathbb{N}$ divisible by m and get some $a \in \mathcal{A} \cap n_0\mathbb{N}$ such that $\widetilde{Q}(T)$ divides $T^a - 1$. Using this property again yields some $a_{\widetilde{Q}} \in \mathcal{A} \cap n_0\mathbb{N}$ divisible by an_0. Let $\widetilde{N}_{\widetilde{Q}} = N_{a_{\widetilde{Q}}}$ and fix some $n \in \mathcal{N}$ with $n \geq \widetilde{N}_{\widetilde{Q}} = N_{a_{\widetilde{Q}}}$. Then $n = N_{a'} \in \mathcal{N}$ with some $a' \in \mathcal{A} \cap n_0\mathbb{N}$; the latter condition implies, by construction, that $\frac{a'}{n_0}$ divides $e_{i,N_{a'}} - e'_{i,N_{a'}} = e_{i,n} - e'_{i,n}$ for each $i \in \{1, \ldots, M\}$. Since the sequence $\{N_a\}_{a\in a_0\mathbb{N}}$ is strictly increasing, this implies that $a' \geq a_{\widetilde{Q}}$; by construction, both a' and $a_{\widetilde{Q}}$ are in the set $\{\prod_{i=1}^{a_0+n} i : \; n \in \mathbb{N}\}$, thus we get that a' is divisible by $a_{\widetilde{Q}}$. Because $\frac{a_{\widetilde{Q}}}{n_0}$ is divisible by a, we conclude that a divides $\frac{a'}{n_0}$ and thus divides $e_{i,n} - e'_{i,n}$ for each $i \in \{1, \ldots, M\}$; equivalently, we have shown that $T^{e_{i,n}} - T^{e'_{i,n}}$ is divisible by $T^a - 1$ and thus by $\widetilde{Q}(T)$ as desired. This completes the proof. □

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