CR SUB-LAPLACIAN COMPARISON AND LIOUVILLE-TYPE THEOREM IN A COMPLETE NONCOMPACT SASAKIAN MANIFOLD

*SHU-CHENG CHANG[1], *TING-JUNG KUO[2], *CHIEN LIN[3], AND JINGZHI TIE[4]

Abstract. In this paper, we first obtain the sub-Laplacian comparison theorem in a complete noncompact pseudohermitian manifold of vanishing torsion (i.e. Sasakian manifold). Secondly, we derive the sub-gradient estimate for positive pseudoharmonic functions in a complete noncompact pseudohermitian manifold which satisfies the CR sub-Laplacian comparison property. It is served as the CR analogue of Yau's gradient estimate. As a consequence, we have the natural CR analogue of Liouville-type theorems in a complete noncompact Sasakian manifold of nonnegative pseudohermitian Ricci curvature tensors.

1. Introduction

In [Y1] and [CY], S.-Y. Cheng and S.-T. Yau derived a well known gradient estimate for positive harmonic functions in a complete noncompact Riemannian manifold.

Proposition 1.1. *([Y1], [CY]) Let M be a complete noncompact Riemannian m-manifold with Ricci curvature bounded from below by $-K$ $(K \geq 0)$. If $u(x)$ is a positive harmonic function on M, then there exists a positive constant $C = C(m)$ such that*

$$|\nabla f(x)|^2 \leq C(\sqrt{K} + \frac{1}{R}) \tag{1.1}$$

on the ball $B(R)$ with $f(x) = \ln u(x)$. As a consequence, the Liouville theorem holds for complete noncompact Riemannian m-manifolds of nonnegative Ricci curvature.

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In this paper, by modifying the arguments of [Y1], [CY] and [CKL], we derive a subgradient estimate for positive pseudoharmonic functions in a complete noncompact pseudohermitian $(2n+1)$-manifold (M, J, θ) of vanishing pseudohermitian torsion (i.e. Sasakian manifold) which is an odd dimensional counterpart of Kähler geometry. It is served as the CR version of Yau's gradient estimate. As a consequence, we prove that the CR analogue of Liouville-type theorem holds for complete noncompact Sasakian manifolds of nonnegative pseudohermitian Ricci curvature.

We first define Ric and Tor on $T_{1,0}$ by

$$Ric(X,Y) = R_{\alpha\bar{\beta}} X^{\alpha} Y^{\bar{\beta}} \tag{1.2}$$

and

$$Tor(X,Y) = i \sum_{\alpha,\beta} (A_{\bar{\alpha}\bar{\beta}} X^{\bar{\alpha}} Y^{\bar{\beta}} - A_{\alpha\beta} X^{\alpha} Y^{\beta}). \tag{1.3}$$

Here $X = X^{\alpha} Z_{\alpha}$, $Y = Y^{\beta} Z_{\beta}$ for a frame $\{T, Z_{\alpha}, Z_{\bar{\alpha}}\}$ of $TM \otimes \mathbb{C} = T_{1,0} \oplus T_{0,1}$ with $Z_{\alpha} \in T_{1,0}$ and $Z_{\bar{\alpha}} = \overline{Z_{\alpha}} \in T_{0,1}$. $R_{\gamma}{}^{\gamma}{}_{\alpha\bar{\beta}}$ is the pseudohermitian curvature tensor, $R_{\alpha\bar{\beta}} = R_{\gamma}{}^{\delta}{}_{\alpha\bar{\beta}}$ is the pseudohermitian Ricci curvature tensor and $A_{\alpha\beta}$ is the torsion tensor. We refer to section 2 for more details about the notions of pseudohermitian geometry.

In Yau's method for the proof of gradient estimates, one can estimate $\Delta(\eta(x)\,|\nabla f(x)|^2)$ for a nonegative cut-off function $\eta(x)$ on $B(2R)$ via Bochner formula and Laplacian comparison. At the end, one has gradient estimate (1.1) by applying the maximum principle to $\eta(x)\,|\nabla f(x)|^2$.

However in order to derive the CR subgradient estimate, one of difficulties is to deal with the following CR Bochner formula (Lemma 2.1) which involving a term $\langle J\nabla_b\varphi,\ \nabla_b\varphi_0\rangle$ that has no analogue in the Riemannian case.

$$\begin{aligned} \Delta_b |\nabla_b\varphi|^2 &= 2\left|\left(\nabla^H\right)^2\varphi\right|^2 + 2\langle \nabla_b\varphi,\ \nabla_b\Delta_b\varphi\rangle \\ &+ \ (4Ric - 2(n-2)Tor)\left((\nabla_b\varphi)_C,\ (\nabla_b\varphi)_C\right) + 4\langle J\nabla_b\varphi,\ \nabla_b\varphi_0\rangle . \end{aligned}$$

Here $\left(\nabla^{H}\right)^{2}$, Δ_b, ∇_b are the subhessian, sub-Laplacian and sub-gradient respectively. We also denote $\varphi_0 = T\varphi$. In order to overcome this difficulty, we introduce a real-valued function $F(x,\ t,\ R,\ b) : M \times [0,\ 1] \times (0,\ \infty) \times (0,\ \infty) \to \mathbb{R}$ by adding an extra term $t\eta(x) f_0^2(x)$ to $|\nabla_b f(x)|^2$ as following

$$F(x,\ t,\ R,\ b) = t\left(|\nabla_b f(x)|^2 + bt\eta(x) f_0^2(x)\right)$$

on the Carnot-Carathéodory ball $B(2R)$ with a constant b to be determined. In section 4, we derive the CR subgradient estimate (1.11) and (1.7) by applying the maximum principle to $\eta(x) F(x,t)$ for each fixed $t \in (0,1]$ if the CR sub-Laplacian comparison property (1.1) holds on (M, J, θ) which is the case when M is Sasakian (Theorem 1.2).

We recall that a piecewise smooth curve $\gamma : [0,1] \to M$ is said to be horizontal if $\gamma'(t) \in \xi$ whenever $\gamma'(t)$ exists. The length of γ is then defined by

$$l(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle_{L_\theta}^{\frac{1}{2}} dt.$$

Here $\langle\ ,\ \rangle_{L_\theta}$ is the Levi form as in (2.2). The Carnot-Carathéodory distance between two points p, $q \in M$ is

$$d_c(p,q) = \inf\{l(\gamma) \mid \gamma \in C_{p,q}\}$$

where $C_{p,q}$ is the set of all horizontal curves joining p and q. We say M is complete if it is complete as a metric space. We refer to [S] for some details. By Chow connectivity theorem [Cho], there always exists a horizontal curve joining p and q, so the distance is finite. Furthermore, there is a minimizing geodesic joining p and q so that its length is equal to the distance $d_c(p,q)$.

Firstly, by applying the Ricatti inequality for sub-Laplacian of Carnot-Caratheodory distance as in Lemma 3.1 and Theorem 3.2, we have the following Bishop-type sub-Laplacian comparison property in a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing pseudohermitian torsion tensors.

Theorem 1.2. *Let* $(M,\ J,\ \theta)$ *be a complete noncompact pseudohermitian* $(2n+1)$*-manifold of vanishing pseudohermitian torsion tensors with*

$$Ric\,(Z,\ Z) \geq -k\,|Z|^2$$

for all $Z \in T_{1,0}$ *and* k *is an nonnegative constant. Then*

(i) $n = 1$

$$\Delta_b r \leq \frac{1}{r} + \sqrt{k}.$$

(ii) $n \geq 2$

$$\Delta_b r \leq \frac{2^n}{r} + \sqrt{2^n}\sqrt{k}.$$

in the sense of distributions.

In the case of nonvanishing torsion, we make the following assumption:

Definition 1.1. *Let* $(M,\ J,\ \theta)$ *be a complete noncompact pseudohermitian* $(2n+1)$*-manifold with*

$$(2Ric - (n-2)Tor)\,(Z,\ Z) \geq -2k\,|Z|^2 \tag{1.4}$$

for all $Z \in T_{1,0}$*, and* k *is an nonnegative constant. We say that* $(M,\ J,\ \theta)$ *satisfies the CR sub-Laplacian comparison property if there exists a positive constant* $C_0 = C_0(k,n)$ *such that*

$$\Delta_b r \leq C_0(\frac{1}{r} + \sqrt{k}) \tag{1.5}$$

We now state the following general sub-gradient estimate for positive pseudoharmonic functions u.

Theorem 1.3. *Let* $(M,\ J,\ \theta)$ *be a complete noncompact pseudohermitian* $(2n+1)$*-manifold with*

$$(2Ric - (n-2)Tor)\,(Z,\ Z) \geq -2k\,|Z|^2$$

for all $Z \in T_{1,0}$, *and* $k \geq 0$. *Furthermore, we assume that* $(M,\ J,\ \theta)$ *satisfies the CR sub-Laplacian comparison property (1.5). If* $u\,(x)$ *is a positive pseudoharmonic function (i.e.* $\Delta_b u = 0$*) with*

$$[\Delta_b, T]u = 0 \tag{1.6}$$

on M. *Then for each constant* $b > 0$, *there exists a positive constant* $C_2 = C_2(k)$ *such that*

$$\frac{|\nabla_b u|^2}{u^2} + b\frac{u_0^2}{u^2} < \frac{(n+5+2bk)^2}{(5+2bk)}\left(k + \frac{2}{b} + \frac{C_2}{R}\right) \tag{1.7}$$

on the ball $B\,(R)$ *of a large enough radius* R *which depends only on* b, k.

Remark 1.1. *It is shown that (Lemma 2.3)*

$$[\Delta_b,\ T]\,u = 4\,\mathrm{Im}[i\sum_{\alpha,\beta=1}^{n}(A_{\bar{\alpha}\bar{\beta}}u_{\beta})_{,\alpha}]. \tag{1.8}$$

If $(M,\ J,\ \theta)$ *is a complete noncompact pseudohermitian* $(2n+1)$*-manifold of vanishing torsion. Then*

$$[\Delta_b,\ T]\,u = 0.$$

It follows easily from the Theorem 1.2 and Theorem 1.3 that we have our main results on the CR Yau's gradient estimate (1.9) and Liouville-type theorem on a complete noncompact Sasakian $(2n+1)$-manifold in this paper.

Theorem 1.4. *Let* $(M,\ J,\ \theta)$ *be a complete noncompact pseudohermitian* $(2n+1)$*-manifold of vanishing pseudohermitian torsion and*

$$Ric\,(Z,\ Z) \geq -k\,|Z|^2$$

for all $Z \in T_{1,0}$, *and* $k \geq 0$. *Let* $u\,(x)$ *be a positive pseudoharmonic function. Then for each constant* $b > 0$, *there exists a positive constant* $\overline{C}_2 = \overline{C}_2(k)$ *such that*

$$\frac{|\nabla_b u|^2}{u^2} + b\frac{u_0^2}{u^2} < \frac{(n+5+2bk)^2}{(5+2bk)}\left(k + \frac{2}{b} + \frac{\overline{C}_2}{R}\right) \tag{1.9}$$

on the ball $B\,(R)$ *of a large enough radius* R *which depends only on* b, k.

As a consequence, let $R \to \infty$ and then $b \to \infty$ with $k = 0$ in (1.9),we have the following CR Liouville-type theorem.

Corollary 1.5. *Let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold of nonnegative pseudohermitian Ricci curvature tensors and vanishing torsion. Then any positive pseudoharmonic function is constant.*

Corollary 1.6. *There does not exist any positive nonconstant pseudoharmonic function in a standard Heisenberg $(2n+1)$-manifold $(\mathbf{H}^n,\ \mathbf{J},\ \boldsymbol{\theta})$.*

Remark 1.2. *Koranyi and Stanton (*[KS]*) proved the Liouville theorem in $(\mathbf{H}^n,\ \mathbf{J},\ \boldsymbol{\theta})$ by a different method.*

In general if the positive pseudoharmonic function u does not satisfy the condition $[\Delta_b,\ T]\, u = 0$, we have the following weak sub-gradient estimate.

Theorem 1.7. *Let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold with*

$$(2Ric - (n-2)Tor)\,(Z,\ Z) \geq -2k\,|Z|^2$$

and

$$\max\left\{|A_{\alpha\beta}|\,,\ |A_{\alpha\beta,\bar{\alpha}}|\right\} \leq k_1 \tag{1.10}$$

for all $Z \in T_{1,0}$ and $k \geq 0,\ k_1 > 0$. Furthermore, we assume that $(M,\ J,\ \theta)$ satisfies the CR sub-Laplacian comparison property. If $u\,(x)$ is a positive pseudoharmonic function on M. Then there exists a small constant $b_0 = b_0(n,k,k_1) > 0$ and $C_3 = C_4(k,\ k_1,\ k_2)$ such that for any $0 < b \leq b_0$,

$$\frac{|\nabla_b u|^2}{u^2} + b\frac{u_0^2}{u^2} < \frac{(n+5)^2}{5}\left(k + n\,(1+b)\,k_1 + \frac{2}{b} + \frac{C_3}{R}\right) \tag{1.11}$$

on the ball $B\,(R)$ of a large enough radius R which depends only on b.

Remark 1.3. *By comparing the Yau's gradient estimate (1.1), we need an extra assumption (1.10) to obtain the CR subgradient estimate (1.11) due to the natural of sub-Laplacian in pseudohermitian geometry. However, we do obtain an extra estimate on the derivative of pseudoharmonic functions $u(x)$ along the missing direction of characteristic vector field T.*

We briefly describe the methods used in our proofs. In section 2, we first introduce some basic materials in a pseudohermitian $(2n+1)$-manifold. Then we are able to get the CR Bochner-type estimate and derive some key Lemmas. In section 3, we give a proof of sub-Laplacian comparison theorem in a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing pseudohermitian torsion tensors. In section 4, let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold with the CR sub-Laplacian comparison property, we obtain subgradient estimates for positive pseudoharmonic functions. As a consequence, the natural analogue of Liouville-type theorem for the sub-Laplacian holds in a complete noncompact pseudohermitian $(2n+1)$-manifold of nonnegative pseudohermitian Ricci curvature tensor and vanishing torsion.

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2. CR Bochner-Type Estimate

In this section, we derive some key lemmas. In particular, we obtain the CR Bochner-type estimate as in Lemma 2.2. We first introduce some basic materials in a pseudohermitian $(2n+1)$-manifold (see [L1], [L2] for more details).

Let (M,ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ. A CR structure compatible with ξ is an endomorphism $J:\xi\to\xi$ such that $J^2=-1$. We also assume that J satisfies the following integrability condition: If X and Y are in ξ, then so are $[JX,Y]+[X,JY]$ and $J([JX,Y]+[X,JY])=[JX,JY]-[X,Y]$.

Let $\{T,Z_\alpha,Z_{\bar\alpha}\}$ be a frame of $TM\otimes\mathbb{C}$, where Z_α is any local frame of $T_{1,0},\ Z_{\bar\alpha}=\overline{Z_\alpha}\in T_{0,1}$ and T is the characteristic vector field. Then $\left\{\theta,\theta^\alpha,\theta^{\bar\alpha}\right\}$, which is the coframe dual to $\{T,Z_\alpha,Z_{\bar\alpha}\}$, satisfies

$$d\theta=ih_{\alpha\overline{\beta}}\theta^\alpha\wedge\theta^{\overline{\beta}} \tag{2.1}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar\beta})$. Actually we can always choose Z_α such that $h_{\alpha\bar\beta}=\delta_{\alpha\beta}$; hence, throughout this note, we assume $h_{\alpha\bar\beta}=\delta_{\alpha\beta}$.

The Levi form $\left\langle\ ,\ \right\rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by

$$\left\langle Z,W\right\rangle_{L_\theta}=-i\left\langle d\theta,Z\wedge\overline{W}\right\rangle. \tag{2.2}$$

We can extend $\left\langle\ ,\ \right\rangle_{L_\theta}$ to $T_{0,1}$ by defining $\left\langle\overline{Z},\overline{W}\right\rangle_{L_\theta}=\overline{\left\langle Z,W\right\rangle}_{L_\theta}$ for all $Z,W\in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\left\langle\ ,\ \right\rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over M with respect to the volume form $d\mu=\theta\wedge(d\theta)^n$, we get an inner product on the space of sections of each tensor bundle. We denote the inner product by the notation $\left\langle\ ,\ \right\rangle$. For example

$$\langle u,v\rangle=\int_M u\overline{v}\ d\mu,$$

for functions u and v.

The pseudohermitian connection of (J,θ) is the connection ∇ on $TM\otimes\mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha\in T_{1,0}$ by

$$\nabla Z_\alpha={\theta_\alpha}^\beta\otimes Z_\beta,\quad \nabla Z_{\bar\alpha}={\theta_{\bar\alpha}}^{\bar\beta}\otimes Z_{\bar\beta},\quad \nabla T=0,$$

where ${\theta_\alpha}^\beta$ are the 1-forms uniquely determined by the following equations:

$$
\begin{aligned}
d\theta^{\beta} &= \theta^{\alpha} \wedge \theta_{\alpha}{}^{\beta} + \theta \wedge \tau^{\beta}, \\
0 &= \tau_{\alpha} \wedge \theta^{\alpha}, \\
0 &= \theta_{\alpha}{}^{\beta} + \theta_{\bar{\beta}}{}^{\bar{\alpha}},
\end{aligned}
\tag{2.3}
$$

We can write (by Cartan lemma) $\tau_{\alpha} = A_{\alpha\gamma}\theta^{\gamma}$ with $A_{\alpha\gamma} = A_{\gamma\alpha}$. The curvature of Tanaka-Webster connection, expressed in terms of the coframe $\{\theta = \theta^0, \theta^{\alpha}, \theta^{\bar{\alpha}}\}$, is

$$
\begin{aligned}
&\Pi_{\beta}{}^{\alpha} = \overline{\Pi_{\bar{\beta}}{}^{\bar{\alpha}}} = d\omega_{\beta}{}^{\alpha} - \omega_{\beta}{}^{\gamma} \wedge \omega_{\gamma}{}^{\alpha}, \\
&\Pi_{0}{}^{\alpha} = \Pi_{\alpha}{}^{0} = \Pi_{0}{}^{\bar{\beta}} = \Pi_{\bar{\beta}}{}^{0} = \Pi_{0}{}^{0} = 0.
\end{aligned}
$$

Webster showed that $\Pi_{\beta}{}^{\alpha}$ can be written

$$
\Pi_{\beta}{}^{\alpha} = R_{\beta}{}^{\alpha}{}_{\rho\bar{\sigma}}\theta^{\rho} \wedge \theta^{\bar{\sigma}} + W_{\beta}{}^{\alpha}{}_{\rho}\theta^{\rho} \wedge \theta - W^{\alpha}{}_{\beta\bar{\rho}}\theta^{\bar{\rho}} \wedge \theta + i\theta_{\beta} \wedge \tau^{\alpha} - i\tau_{\beta} \wedge \theta^{\alpha}
\tag{2.4}
$$

where the coefficients satisfy

$$
R_{\beta\bar{\alpha}\rho\bar{\sigma}} = \overline{R_{\alpha\bar{\beta}\sigma\bar{\rho}}} = R_{\bar{\alpha}\beta\bar{\sigma}\rho} = R_{\rho\bar{\alpha}\beta\bar{\sigma}}, \quad W_{\beta\bar{\alpha}\gamma} = W_{\gamma\bar{\alpha}\beta}.
$$

We will denote components of covariant derivatives with indices preceded by a comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0, \alpha, \bar{\alpha}\}$ indicate derivatives with respect to $\{T, Z_{\alpha}, Z_{\bar{\alpha}}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $u_{\alpha} = Z_{\alpha}u$, $u_{\alpha\bar{\beta}} = Z_{\bar{\beta}}Z_{\alpha}u - \omega_{\alpha}{}^{\gamma}(Z_{\bar{\beta}})Z_{\gamma}u$.

For a real function u, the subgradient ∇_b is defined by $\nabla_b u \in \xi$ and $\langle Z, \nabla_b u\rangle_{L_\theta} = du(Z)$ for all vector fields Z tangent to contact plane. Locally $\nabla_b u = \sum_{\alpha} u_{\bar{\alpha}} Z_{\alpha} + u_{\alpha} Z_{\bar{\alpha}}$. We can use the connection to define the subhessian as the complex linear map

$$
(\nabla^H)^2 u : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1}
$$

by

$$
(\nabla^H)^2 u(Z) = \nabla_Z \nabla_b u.
$$

In particular,

$$|\nabla_b u|^2 = 2u_\alpha u_{\overline{\alpha}}, \quad |\nabla_b^2 u|^2 = 2(u_{\alpha\beta}u_{\overline{\alpha}\overline{\beta}} + u_{\alpha\overline{\beta}}u_{\overline{\alpha}\beta}).$$

Also

$$\Delta_b u = Tr\left((\nabla^H)^2 u\right) = \textstyle\sum_\alpha (u_{\alpha\bar{\alpha}} + u_{\bar{\alpha}\alpha}).$$

Next we recall the following commutation relations ([L1]). Let φ be a scalar function and $\sigma = \sigma_\alpha \theta^\alpha$ be a $(1,0)$ form, then we have

$$\begin{aligned}
\varphi_{\alpha\beta} &= \varphi_{\beta\alpha}, \\
\varphi_{\alpha\bar{\beta}} - \varphi_{\bar{\beta}\alpha} &= ih_{\alpha\overline{\beta}}\varphi_0, \\
\varphi_{0\alpha} - \varphi_{\alpha 0} &= A_{\alpha\beta}\varphi_{\bar{\beta}}, \\
\sigma_{\alpha,0\beta} - \sigma_{\alpha,\beta 0} &= \sigma_{\alpha,\bar{\gamma}}A_{\gamma\beta} - \sigma_\gamma A_{\alpha\beta,\bar{\gamma}}, \\
\sigma_{\alpha,0\bar{\beta}} - \sigma_{\alpha,\bar{\beta}0} &= \sigma_{\alpha,\gamma}A_{\bar{\gamma}\bar{\beta}} + \sigma_\gamma A_{\bar{\gamma}\bar{\beta},\alpha},
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
\sigma_{\alpha,\beta\gamma} - \sigma_{\alpha,\gamma\beta} &= iA_{\alpha\gamma}\sigma_\beta - iA_{\alpha\beta}\sigma_\gamma, \\
\sigma_{\alpha,\bar{\beta}\bar{\gamma}} - \sigma_{\alpha,\bar{\gamma}\bar{\beta}} &= ih_{\alpha\overline{\beta}}A_{\bar{\gamma}\bar{\rho}}\sigma_\rho - ih_{\alpha\overline{\gamma}}A_{\bar{\beta}\bar{\rho}}\sigma_\rho, \\
\sigma_{\alpha,\beta\bar{\gamma}} - \sigma_{\alpha,\bar{\gamma}\beta} &= ih_{\beta\overline{\gamma}}\sigma_{\alpha,0} + R_{\alpha\bar{\rho}\beta\bar{\gamma}}\sigma_\rho.
\end{aligned} \tag{2.6}$$

Now we recall a lemma from A. Greenleaf ([Gr]) and also ([CC2]).

Lemma 2.1. *For a real function φ,*

$$\begin{aligned}
\Delta_b |\nabla_b\varphi|^2 &= 2\left|\left(\nabla^H\right)^2\varphi\right|^2 + 2\left\langle \nabla_b\varphi,\ \nabla_b\Delta_b\varphi\right\rangle \\
&+ \left(4Ric - 2\left(n-2\right)Tor\right)\left(\left(\nabla_b\varphi\right)_C,\ \left(\nabla_b\varphi\right)_C\right) + 4\left\langle J\nabla_b\varphi,\ \nabla_b\varphi_0\right\rangle,
\end{aligned} \tag{2.7}$$

where $\left(\nabla_b\varphi\right)_C = \varphi_{\bar{\alpha}}Z_\alpha$ is the corresponding complex $(1,\ 0)$-vector of $\nabla_b\varphi$.

Lemma 2.2. *For a smooth real-valued function φ and any $\nu > 0$, we have*

$$\begin{aligned}
\Delta_b |\nabla_b\varphi|^2 &\geq 4\left(\sum_{\alpha,\beta=1}^{n}\left|\varphi_{a\beta}\right|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|\varphi_{a\bar{\beta}}\right|^2\right) + \tfrac{1}{n}\left(\Delta_b\varphi\right)^2 + n\varphi_0^2 + 2\left\langle\nabla_b\varphi,\ \nabla_b\Delta_b\varphi\right\rangle \\
&+ \left(4Ric - 2\left(n-2\right)Tor - \tfrac{4}{\nu}\right)\left(\left(\nabla_b\varphi\right)_C,\ \left(\nabla_b\varphi\right)_C\right) - 2\nu\left|\nabla_b\varphi_0\right|^2.
\end{aligned}$$

Proof. Since

$$\begin{aligned}
|(\nabla^H)^2\varphi|^2 &= 2\textstyle\sum_{\alpha,\beta=1}^{n}(\varphi_{\alpha\beta}\varphi_{\overline{\alpha}\overline{\beta}}+\varphi_{\alpha\overline{\beta}}\varphi_{\overline{\alpha}\beta})\\
&= 2\textstyle\sum_{\alpha,\beta=1}^{n}(|\varphi_{\alpha\beta}|^2+|\varphi_{\alpha\overline{\beta}}|^2)\\
&= 2(\textstyle\sum_{\alpha,\beta=1}^{n}|\varphi_{\alpha\beta}|^2+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{n}|\varphi_{\alpha\overline{\beta}}|^2+\sum_{\alpha=1}^{n}|\varphi_{\alpha\overline{\alpha}}|^2)
\end{aligned}$$

and from the commutation relation (2.5)

$$\begin{aligned}
\textstyle\sum_{\alpha=1}^{n}|\varphi_{\alpha\overline{\alpha}}|^2 &= \tfrac{1}{4}\textstyle\sum_{\alpha=1}^{n}\left(|\varphi_{\alpha\overline{\alpha}}+\varphi_{\overline{\alpha}\alpha}|^2+\varphi_0^2\right)\\
&= \tfrac{1}{4}\textstyle\sum_{\alpha=1}^{n}|\varphi_{\alpha\overline{\alpha}}+\varphi_{\overline{\alpha}\alpha}|^2+\tfrac{n}{4}\varphi_0^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
|(\nabla^H)^2\varphi|^2 &= 2(\textstyle\sum_{\alpha,\beta=1}^{n}|\varphi_{\alpha\beta}|^2+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{n}|\varphi_{\alpha\overline{\beta}}|^2)+\tfrac{1}{2}\sum_{\alpha=1}^{n}|\varphi_{\alpha\overline{\alpha}}+\varphi_{\overline{\alpha}\alpha}|^2+\tfrac{n}{2}\varphi_0^2\\
&\leq 2(\textstyle\sum_{\alpha,\beta=1}^{n}|\varphi_{\alpha\beta}|^2+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{n}|\varphi_{\alpha\overline{\beta}}|^2)+\tfrac{1}{2n}\left(\Delta_b\varphi\right)^2+\tfrac{n}{2}\varphi_0^2.
\end{aligned}$$

On the other hand, for all $\nu>0$

$$\begin{aligned}
4\left\langle J\nabla_b\varphi,\ \nabla_b\varphi_0\right\rangle &\geq -4\left|\nabla_b\varphi\right|\left|\nabla_b\varphi_0\right|\\
&\geq -\tfrac{2}{\nu}\left|\nabla_b\varphi\right|^2-2\nu\left|\nabla_b\varphi_0\right|^2.
\end{aligned}$$

Then the result follows easily from Lemma 2.1. □

Definition 2.1. *([GL]) Let (M,J,θ) be a pseudohermitian $(2n+1)$-manifold. We define the purely holomorphic second-order operator Q by*

$$Q\varphi=2i\sum_{\alpha,\beta=1}^{n}(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta})_{,\alpha}\,.$$

By apply the commutation relations (2.5), one obtains

Lemma 2.3. *([GL], [CKL]) Let $\varphi\left(x\right)$ be a smooth function defined on M. Then*

$$\Delta_b\varphi_0=\left(\Delta_b\varphi\right)_0+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}\varphi_{\bar{\beta}}\right)_{\bar{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\right)_{\alpha}\right].$$

That is

$$2\,\mathrm{Im}\,Q\varphi=\left[\Delta_b,\ T\right]\varphi.$$

Proof. By direct computation and the commutation relation (2.5), we have

$$\begin{aligned}
\Delta_b\varphi_0 &= \varphi_{0\alpha\overline{\alpha}}+\varphi_{0\overline{\alpha}\alpha}\\
&= \left(\varphi_{\alpha0}+A_{\alpha\beta}\varphi_{\overline{\beta}}\right)_{\overline{\alpha}}+\text{conjugate}\\
&= \varphi_{\alpha0\overline{\alpha}}+\left(A_{\alpha\beta}\varphi_{\overline{\beta}}\right)_{\overline{\alpha}}+\text{conjugate}\\
&= \varphi_{\alpha\overline{\alpha}0}+\varphi_{\overline{\alpha}\alpha0}+2\left[\left(A_{\alpha\beta}\varphi_{\overline{\beta}}\right)_{\overline{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\right)_{\alpha}\right]\\
&= \left(\Delta_b\varphi\right)_0+2\left[\left(A_{\alpha\beta}\varphi_{\overline{\beta}}\right)_{\overline{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\right)_{\alpha}\right].
\end{aligned}$$

This completes the proof. □

Let u be a positive pseudoharmonic function and $f\left(x\right)=\ln u\left(x\right)$. Then

$$\Delta_b f=-\left|\nabla_b f\right|^2.$$

We first define

$$V\left(\varphi\right)=\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}\varphi_{\bar{\beta}}\right)_{\bar{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\right)_{\alpha}+A_{\alpha\beta}\varphi_{\bar{\beta}}\varphi_{\bar{\alpha}}+A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\varphi_{\alpha}\right].$$

Lemma 2.4. *Let u be a positive pseudoharmonic function with $f=\ln u$. Then*

$$\Delta_b f_0=-2\left\langle\nabla_b f,\ \nabla_b f_0\right\rangle+2V\left(f\right).$$

Proof. From Lemma 2.3

$$\Delta_b f_0=\left(\Delta_b f\right)_0+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}\varphi_{\bar{\beta}}\right)_{\bar{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\right)_{\alpha}\right].$$

Since

$$\Delta_b f=-\left|\nabla_b f\right|^2,$$

it follows from the commutation relation (2.5) that

$$\begin{aligned}
\Delta_b f_0 &= \left(\Delta_b f\right)_0+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar{\beta}}\right)_{\bar{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}f_{\beta}\right)_{\alpha}\right]\\
&= \left(-\left|\nabla_b f\right|^2\right)_0+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar{\beta}}\right)_{\bar{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}f_{\beta}\right)_{\alpha}\right]\\
&= -2\left\langle\nabla_b f_0,\ \nabla_b f\right\rangle+2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar{\beta}}\right)_{\bar{\alpha}}+\left(A_{\bar{\alpha}\bar{\beta}}f_{\beta}\right)_{\alpha}+A_{\alpha\beta}f_{\bar{\alpha}}f_{\bar{\beta}}+A_{\bar{\alpha}\bar{\beta}}f_{\alpha}f_{\beta}\right].
\end{aligned}$$

□

Lemma 2.5. *Let (M, J, θ) be a pseudohermitian $(2n+1)$-manifold and u be a positive function with $f = \ln u$. Suppose that*

$$2\operatorname{Im} Qu = [\Delta_b,\ T]\, u = 0.$$

Then

$$V(f) = 0. \tag{2.8}$$

Proof. We compute

$$\begin{aligned}
V(f) &= \sum_{\alpha,\beta=1}^{n} \left[\left(A_{\alpha\beta} f_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} f_{\beta}\right)_{\alpha} + A_{\alpha\beta} f_{\bar\alpha} f_{\bar\beta} + A_{\bar\alpha\bar\beta} f_{\alpha} f_{\beta} \right] \\
&= \sum_{\alpha,\beta=1}^{n} \left[A_{\alpha\beta} f_{\bar\beta\bar\alpha} + A_{\alpha\beta,\bar\alpha} f_{\bar\beta} + A_{\bar\alpha\bar\beta} f_{\beta\alpha} + A_{\bar\alpha\bar\beta,\alpha} f_{\beta} + A_{\alpha\beta} f_{\bar\alpha} f_{\bar\beta} + A_{\bar\alpha\bar\beta} f_{\alpha} f_{\beta} \right] \\
&= \sum_{\alpha,\beta=1}^{n} \left\{ A_{\bar\alpha\bar\beta} \left(\tfrac{u_{\beta\alpha}}{u} - \tfrac{u_\alpha u_\beta}{u^2} \right) + A_{\alpha\beta} \left(\tfrac{u_{\bar\beta\bar\alpha}}{u} - \tfrac{u_{\bar\alpha} u_{\bar\beta}}{u^2} \right) \right. \\
&\quad \left. + A_{\bar\alpha\bar\beta,\alpha} \tfrac{u_\beta}{u} + A_{\alpha\beta,\bar\alpha} \tfrac{u_{\bar\beta}}{u} + A_{\bar\alpha\bar\beta} \tfrac{u_\alpha u_\beta}{u^2} + A_{\alpha\beta} \tfrac{u_{\bar\alpha} u_{\bar\beta}}{u^2} \right\} \\
&= \sum_{\alpha,\beta=1}^{n} \tfrac{1}{u} \left[\left(A_{\alpha\beta} u_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} u_{\beta}\right)_{\alpha} \right] \\
&= \tfrac{1}{2u} [\Delta_b,\ T]\, u.
\end{aligned} \tag{2.9}$$

This completes the proof. □

3. CR Sub-Laplacian Comparison Theorem

In this section, we give the proof of sub-Laplacian comparison theorems in a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing pseudohermitian torsion tensors. In order to prove Theorem 1.2, we first derive the Ricatti inequality for sub-Laplacian of Carnot-Carathéodory distance. We refer to [CHL, Corollary 3.1.] for some of computations.

Lemma 3.1. *Let (M, J, θ) be a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing pseudohermitian torsion with*

$$R_{\alpha\bar\beta} \geq k_2 h_{\alpha\bar\beta}$$

for some constant k_2. Then, for any $x \in M$ where $r(x)$ is smooth, we have *(i) For $n = 1$,*

$$\partial_r(\Delta_b r) + (\Delta_b r)^2 + k_2 \leq 0.$$

(ii) For $n \geq 2$,

$$\partial_r(\Delta_b r) + 2^{-n}(\Delta_b r)^2 + k_2 \leq 0.$$

Proof. We choose $\left\{e_j, e_{\tilde{j}}, T\right\}_{j\in I_n}$ to be an orthonormal frame at q where $e_{\tilde{j}} = Je_j$ and $e_1 = \nabla_b r$. Since the pseudohermitian torsion is vanishing, by a result in [DZ, Corollary 2.3], we could parallel transport such frame at q to obtain the orthonormal frame along the radial ∇-geodesic γ from p to q. Then there is an orthonormal frame $\{Z_j, Z_{\bar{j}}, T\}_{j\in I_n}$ along γ with

$$Z_j = \frac{1}{\sqrt{2}}\left(e_j - ie_{\tilde{j}}\right).$$

By the fact that γ is the ∇-geodesic, one can compute the following in Z_1-direction as

$$\begin{array}{rcl} r_{11} & = & -\frac{1}{2}\left(ie_2e_1 + e_2e_2\right)r - \left(\nabla_{Z_1}Z_1\right)r \\ & = & -\frac{1}{2}\left(ie_2e_1 + e_2e_2\right)r + \frac{1}{2}\left[i\nabla_{(J\nabla_b r)}\nabla_b r + J\left(\nabla_{(J\nabla_b r)}\nabla_b r\right)\right] \end{array}$$

and

$$\begin{array}{rcl} r_{1\bar{1}} & = & \frac{1}{2}\left(ie_2e_1 + e_2e_2\right)r - \left(\nabla_{Z_{\bar{1}}}Z_1\right)r \\ & = & \frac{1}{2}\left(ie_2e_1 + e_2e_2\right)r - \frac{1}{2}\left[i\nabla_{(J\nabla_b r)}\nabla_b r + J\left(\nabla_{(J\nabla_b r)}\nabla_b r\right)\right]. \end{array}$$

Therefore along γ

$$r_{11} = -r_{1\bar{1}}. \tag{3.1}$$

Moreover, by computing

$$\begin{array}{rcl} r_1 & = & Z_1 r \\ & = & \frac{1}{\sqrt{2}}\left(\nabla_b r - iJ\nabla_b r\right)r \\ & = & \frac{1}{\sqrt{2}}\left(\left|\nabla_b r\right|^2 - i\left\langle\nabla_b r, J\nabla_b r\right\rangle\right) \\ & = & \frac{1}{\sqrt{2}} \end{array}$$

and

$$\begin{aligned} r_{1\bar{1}} &= Z_{\bar{1}}Z_1 r - \Gamma^1_{\bar{1}1} r_1 \\ &= Z_{\bar{1}}Z_1 r - g_\theta\left(\left[Z_{\bar{1}}, Z_1\right], Z_1\right) r_1 \\ &= Z_{\bar{1}}Z_1 r - \tfrac{1}{\sqrt{2}} g_\theta\left(\left[Z_{\bar{1}}, Z_1\right], Z_1\right), \end{aligned}$$

we derive that $r_{1\bar{1}}$ is real by the commutation formula. Therefore, we have

$$r_0 = 0 \tag{3.2}$$

along the ∇-geodesic γ.

Now at the point q, by the facts that $r_1 = \frac{1}{\sqrt{2}}$ and $r_{1\bar{1}}$ is real, the equalities (3.1), (3.2) and the commutation formulas (2.5), (2.6), we have the following computation as well.

$$\begin{aligned} 0 &= \tfrac{1}{2}\left(|\nabla_b r|^2\right)_{1\bar{1}} \\ &= \sum_\alpha \left(|r_{\alpha 1}|^2 + |r_{\alpha\bar{1}}|^2 + r_{\alpha 1\bar{1}} r_{\bar{\alpha}} + r_{\bar{\alpha}1\bar{1}} r_\alpha\right) \\ &\geq |r_{11}|^2 + |r_{1\bar{1}}|^2 + r_{11\bar{1}} r_{\bar{1}} + r_{\bar{1}1\bar{1}} r_1 \\ &= 2r_{1\bar{1}}^2 + \left(r_{1\bar{1}1} + i r_{10} + R^1_{11\bar{1}} r_1\right) r_{\bar{1}} + \left(r_{1\bar{1}} - i r_0\right)_{\bar{1}} r_1 \\ &= 2r_{1\bar{1}}^2 + \langle \nabla_b r_{1\bar{1}}, \nabla_b r\rangle_{L_\theta} + \tfrac{1}{2} R_{1\bar{1}1\bar{1}} \\ &\geq 2r_{1\bar{1}}^2 + \left(\nabla_b r\right) r_{1\bar{1}} + \tfrac{1}{2} R_{1\bar{1}1\bar{1}} \\ &= 2r_{1\bar{1}}^2 + \left(\nabla r\right) r_{1\bar{1}} + \tfrac{1}{2} R_{1\bar{1}1\bar{1}} \\ &= 2r_{1\bar{1}}^2 + \tfrac{\partial r_{1\bar{1}}}{\partial r} + \tfrac{1}{2} R_{1\bar{1}1\bar{1}}. \end{aligned} \tag{3.3}$$

(i) For $n = 1$: Since

$$\Delta_b r = r_{1\bar{1}} + r_{\bar{1}1} = 2r_{1\bar{1}},$$

it follows from (3.3) that

$$\partial_r(\Delta_b r) + (\Delta_b r)^2 + R_{1\bar{1}} \leq 0$$

and then

$$\partial_r(\Delta_b r) + (\Delta_b r)^2 + k_2 \leq 0$$

as well.

(ii) For $n \geq 2:$ The similar computation as before, for any $j \neq 1$,

$$\begin{aligned}
0 &= \tfrac{1}{2}\left(|\nabla_b r|^2\right)_{j\bar{j}} \\
&= \sum_{\alpha}\left(\left|r_{\alpha j}\right|^2 + \left|r_{\alpha\bar{j}}\right|^2 + r_{\alpha j\bar{j}}r_{\overline{\alpha}} + r_{\overline{\alpha}j\bar{j}}r_{\alpha}\right) \\
&\geq \left|r_{j\bar{j}}\right|^2 + r_{1j\bar{j}}r_{\bar{1}} + r_{\bar{1}j\bar{j}}r_1 \\
&= r_{j\bar{j}}^2 + \left(r_{1\bar{j}j} + ir_{10} + R^1_{1j\bar{j}}r_1\right)r_{\bar{1}} + r_{\bar{1}j\bar{j}}r_1 \qquad (3.4)\\
&\geq r_{j\bar{j}}^2 + \left\langle \nabla_b r_{j\bar{j}}, \nabla_b r\right\rangle_{L_\theta} + \tfrac{1}{2}R_{1\bar{1}j\bar{j}} \\
&= r_{j\bar{j}}^2 + \left(\nabla_b r\right) r_{j\bar{j}} + \tfrac{1}{2}R_{1\bar{1}j\bar{j}} \\
&= r_{j\bar{j}}^2 + \tfrac{\partial}{\partial r} r_{j\bar{j}} + \tfrac{1}{2}R_{1\bar{1}j\bar{j}}.
\end{aligned}$$

It follows from the inequalities (3.3) and (3.4) that

$$\begin{aligned}
0 &\geq \left(2r_{1\bar{1}}^2 + \tfrac{\partial r_{1\bar{1}}}{\partial r} + \tfrac{1}{2}R_{1\bar{1}1\bar{1}}\right) + \sum_{j\neq 1}\left(r_{j\bar{j}}^2 + \tfrac{\partial}{\partial r}r_{j\bar{j}} + \tfrac{1}{2}R_{1\bar{1}j\bar{j}}\right) \qquad (3.5)\\
&\geq 2^{1-n}\left(\sum_{j=1}^{n} r_{j\bar{j}}\right)^2 + \tfrac{\partial}{\partial r}\sum_{j=1}^{n} r_{j\bar{j}} + \tfrac{1}{2}R_{1\bar{1}}.
\end{aligned}$$

Hence

$$\partial_r(\Delta_b r) + 2^{-n}(\Delta_b r)^2 + R_{1\bar{1}} \leq 0$$

and then

$$\partial_r(\Delta_b r) + 2^{-n}(\Delta_b r)^2 + k_2 \leq 0$$

as well. $\square$

Now Theorem 3.2 will follows from the Lemma 3.1 easily ([Li], [W]).

Theorem 3.2. *Let (M, J, θ) be a complete pseudohermitian $(2n+1)$-manifold of vanishing pseudohermitian torsion with*

$$R_{\alpha\bar{\beta}} \geq k_2 h_{\alpha\bar{\beta}}$$

for some constant k_2. Then, for any $x \in M$ where $r(x)$ is smooth, we have *(i)* $n = 1$

$$\Delta_b r \leq \begin{cases} \sqrt{k_2}\cot(\sqrt{k_2}r), & k_2 > 0, \\ \frac{1}{r}, & k_2 = 0, \\ \sqrt{|k_2|}\coth(\sqrt{|k_2|}r), & k_2 < 0. \end{cases} \tag{3.6}$$

(ii) $n \geq 2$:

$$\Delta_b r \leq \begin{cases} \sqrt{2^n k_2}\cot(\sqrt{2^{-n}k_2}), & k_2 > 0, \\ \frac{2^n}{r}, & k_2 = 0, \\ \sqrt{2^n|k_2|}\coth(\sqrt{2^{-n}|k_2|}r), & k_2 < 0. \end{cases} \tag{3.7}$$

Moreover, it holds on the whole manifold in the sense of distribution.

4. CR Analogue of Yau's Gradient Estimate

In this section, we will prove main Theorem 1.7 and Theorem 1.3. We first recall a real-valued function

$$F(x,\ t,\ R,\ b) : M \times [0,\ 1] \times (0,\ \infty) \times (0,\ \infty) \to \mathbb{R}$$

defined by

$$F(x,\ t,\ R,\ b) = t\left(|\nabla_b f|^2(x) + bt\eta(x) f_0^2(x)\right), \tag{4.1}$$

where $\eta(x) : M \to [0,\ 1]$ is a smooth cut-off function defined by

$$\eta(x) = \eta(r(x)) = \begin{cases} 1, & x \in\ B(R) \\ 0, & x \in M \backslash B(2R) \end{cases}$$

such that

$$-\frac{C}{R}\eta^{\frac{1}{2}} \leq \eta' \leq 0 \tag{4.2}$$

and

$$\left|\eta''\right| \leq \frac{C}{R^2}, \tag{4.3}$$

where we denote $\frac{\partial}{\partial r}\eta$ by $\eta^{'}$ and $r(x)$ is the Carnot-Carathéodory distance to a fixed point x_0.

In the following calculation, the universal constant C might be changed from lines to lines.

Proposition 4.1. *Let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold with*

$$\left(2Ric-(n-2)\,Tor\right)(Z,\ Z)\geq -2k\left|Z\right|^{2} \tag{4.4}$$

for all $Z\in T_{1,0}$, where k is an nonnegative constant. Suppose that $(M,\ J,\ \theta)$ satisfies the CR sub-Laplacian comparison property. Then

$$\begin{array}{rcl}\Delta_b F & \geq & -2\left\langle \nabla_b f,\ \nabla_b F\right\rangle + t\left[4\sum\limits_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2 + 4\sum\limits_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2+\frac{1}{n}\left(\Delta_b f\right)^2\right.\\ & & \left.+\left(n-\frac{bCt}{R}\right)f_0^2-\left(2k+\frac{4}{bt\eta}\right)\left|\nabla_b f\right|^2-\frac{bCt}{R}\eta\left|\nabla_b f\right|^2 f_0^2+4bt\eta f_0 V(f)\right].\end{array}$$

Proof. By CR sub-Laplacian comparison property,

$$\begin{array}{rcl}\Delta_b\eta & = & \eta^{''}+\eta^{'}\Delta_b r\\ & \geq & -\frac{C}{R^2}-\frac{C}{R}\left(\frac{C_1}{R}+C_2\right)\\ & \geq & -\frac{C}{R}.\end{array}$$

First we compute

$$\begin{array}{rcl}\Delta_b\left(bt\eta f_0^2\right) & = & bt\left(f_0^2\Delta_b\eta+\eta\Delta_b f_0^2+2\left\langle\nabla_b\eta,\ \nabla_b f_0^2\right\rangle\right)\\ & \geq & bt\left(-\frac{C}{R}f_0^2+2\eta f_0\Delta_b f_0+2\eta\left|\nabla_b f_0\right|^2+4f_0\left\langle\nabla_b\eta,\ \nabla_b f_0\right\rangle\right)\\ & \geq & bt\left(-\frac{C}{R}f_0^2+2\eta f_0\Delta_b f_0+2\eta\left|\nabla_b f_0\right|^2-4\left|f_0\right|\left|\nabla_b\eta\right|\left|\nabla_b f_0\right|\right)\\ & \geq & bt[-\frac{C}{R}f_0^2+2\eta f_0\Delta_b f_0+\left(2\eta-2\cdot\frac{1}{2}\eta\right)\left|\nabla_b f_0\right|^2-2\cdot 2\eta^{-1}\left|\nabla_b\eta\right|^2 f_0^2]\\ & \geq & bt[-\frac{C}{R}f_0^2+2\eta f_0\Delta_b f_0+\left(2\eta-2\cdot\frac{1}{2}\eta\right)\left|\nabla_b f_0\right|^2],\end{array} \tag{4.5}$$

where we use the Young's inequality and the inequality (4.2) which implies that

$$\eta^{-1}\left|\nabla_b\eta\right|^2\leq\frac{C}{R^2}.$$

Second, it follows from assumption (4.4), Lemma 2.2 and (4.5) that

$$\begin{aligned}
\Delta_b F &= t\left(\Delta_b\left|\nabla_b f\right|^2+\Delta_b\left(bt\eta f_0^2\right)\right)\\
&\geq t\Bigg(4\sum_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2+4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar\beta}\right|^2+\tfrac{1}{n}\left(\Delta_b f\right)^2+nf_0^2+2\left\langle\nabla_b f,\ \nabla_b\Delta_b f\right\rangle\\
&\quad -2\left(k+\tfrac{1}{\nu}\right)\left|\nabla_b f\right|^2-2\nu\left|\nabla_b f_0\right|^2-\tfrac{bCt}{R}f_0^2+2bt\eta f_0\Delta_b f_0+2\cdot\tfrac{bt}{2}\eta\left|\nabla_b f_0\right|^2\Bigg)\\
&\geq t\Bigg[4\sum_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2+4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar\beta}\right|^2+\tfrac{1}{n}\left(\Delta_b f\right)^2+\left(n-\tfrac{bCt}{R}\right)f_0^2+2\left\langle\nabla_b f,\ \nabla_b\Delta_b f\right\rangle\\
&\quad -2\left(k+\tfrac{1}{\nu}\right)\left|\nabla_b f\right|^2+2\left(\tfrac{bt}{2}\eta-\nu\right)\left|\nabla_b f_0\right|^2+2bt\eta f_0\Delta_b f_0\Bigg].
\end{aligned}$$

Then taking $\nu=\frac{bt\eta}{2}$,

$$\begin{aligned}
\Delta_b F &\geq t\Bigg[4\sum_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2+4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar\beta}\right|^2+\tfrac{1}{n}\left(\Delta_b f\right)^2+\left(n-\tfrac{bCt}{R}\right)f_0^2\\
&\quad -2\left(k+\frac{2}{bt\eta}\right)\left|\nabla_b f\right|^2+2\left\langle\nabla_b f,\ \nabla_b\Delta_b f\right\rangle+2bt\eta f_0\Delta_b f_0\Bigg].
\end{aligned}\tag{4.6}$$

Finally, by Lemma 2.4

$$\begin{aligned}
&2\left\langle\nabla_b f,\ \nabla_b\Delta_b f\right\rangle+2bt\eta f_0\Delta_b f_0\\
=\ &2\left\langle\nabla_b f,\ \nabla_b\left(-\left|\nabla_b f\right|^2\right)\right\rangle+2bt\eta f_0\left[-2\left\langle\nabla_b f,\ \nabla_b f_0\right\rangle+2V(f)\right]\\
=\ &-2\left\langle\nabla_b f,\ \nabla_b\left(\tfrac{F}{t}-bt\eta f_0^2\right)\right\rangle-4bt\eta f_0\left\langle\nabla_b f,\ \nabla_b f_0\right\rangle+4bt\eta f_0V(f)\\
=\ &-\tfrac{2}{t}\left\langle\nabla_b f,\ \nabla_b F\right\rangle+2bt\left\langle\nabla_b f,\ \nabla_b\left(\eta f_0^2\right)\right\rangle-4bt\eta f_0\left\langle\nabla_b f,\ \nabla_b f_0\right\rangle+4bt\eta f_0V(f)\\
=\ &-\tfrac{2}{t}\left\langle\nabla_b f,\ \nabla_b F\right\rangle+2btf_0^2\left\langle\nabla_b f,\ \nabla_b\eta\right\rangle+4bt\eta f_0V(f)
\end{aligned}$$

Now by Young's inequality, we have

$$\begin{aligned}
&2\left\langle\nabla_b f,\ \nabla_b\Delta_b f\right\rangle+2bt\eta f_0\Delta_b f_0\\
=\ &-\tfrac{2}{t}\left\langle\nabla_b f,\ \nabla_b F\right\rangle+2btf_0^2\left\langle\nabla_b f,\ \nabla_b\eta\right\rangle+4bt\eta f_0V(f)\\
\geq\ &-\tfrac{2}{t}\left\langle\nabla_b f,\ \nabla F\right\rangle-2btf_0^2\left|\nabla_b f\right|\left|\nabla_b\eta\right|+4bt\eta f_0V(f)\\
\geq\ &-\tfrac{2}{t}\left\langle\nabla_b f,\ \nabla F\right\rangle-\tfrac{2Cbt}{R}f_0^2\left|\nabla_b f\right|\eta^{\frac{1}{2}}+4bt\eta f_0V(f)\\
\geq\ &-\tfrac{2}{t}\left\langle\nabla_b f,\ \nabla F\right\rangle-\tfrac{Cbt}{R}f_0^2-\tfrac{Cbt}{R}\eta f_0^2\left|\nabla_b f\right|^2+4bt\eta f_0V(f).
\end{aligned}\tag{4.7}$$

Substituting (4.7) into (4.6),

$$\begin{aligned}\Delta_b F \ \ \geq\ \ & -2\left\langle \nabla_b f,\ \nabla F\right\rangle + t\left[4\sum_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2 + 4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2 + \tfrac{1}{n}\left(\Delta_b f\right)^2\right.\\ &\left. + \left(n - \tfrac{bCt}{R}\right) f_0^2 - 2\left(k + \tfrac{2}{bt\eta}\right)\left|\nabla_b f\right|^2 - \tfrac{Cbt}{R}\eta f_0^2\left|\nabla_b f\right|^2 + 4bt\eta f_0 V(f)\right].\end{aligned}$$

□

Proposition 4.2. *Let $(M,\ J,\ \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold with*

$$\left(2Ric - (n-2)\,Tor\right)(Z,\ Z) \geq -2k\left|Z\right|^2$$

for all $Z \in T_{1,0}$, where k is an nonnegative constant. Suppose that $(M,\ J,\ \theta)$ satisfies the CR sub-Laplacian comparison property. Then for all $a \neq 0$

$$\begin{aligned}t\eta\Delta_b(\eta F) \ \ \geq\ \ & \tfrac{1}{na^2}(\eta F)^2 - \tfrac{C}{R}(\eta F) + 2t\eta\left\langle \nabla_b\eta,\ \nabla_b F\right\rangle - 2t\eta^2\left\langle \nabla_b f,\ \nabla_b F\right\rangle\\ & + 4t^2\eta^2\left(\sum_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2\right)\\ & + \left[n - \tfrac{bC}{R} - \left(\tfrac{2b}{na^2} + \tfrac{bC}{R}\right)(\eta F)\right] t^2\eta^2 f_0^2\\ & + \left[-\tfrac{2(1+a)}{na^2}(\eta F) - 2k - \tfrac{4}{b}\right] t\eta\left|\nabla_b f\right|^2 + 4bt^3\eta^3 f_0 V(f).\end{aligned} \tag{4.8}$$

Proof. By using Proposition 4.1, we first compute

$$\begin{aligned}\Delta_b(\eta F) \ \ =\ \ & (\Delta_b\eta) F + 2\left\langle \nabla_b\eta,\ \nabla_b F\right\rangle + \eta\Delta_b F\\ \geq\ \ & -\tfrac{C}{R}F + 2\left\langle \nabla_b\eta,\ \nabla_b F\right\rangle - 2\eta\left\langle \nabla_b f,\ \nabla F\right\rangle\\ & + t\eta\left[4\left(\sum_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2\right) + \tfrac{1}{n}\left(\Delta_b f\right)^2 + \left(n - \tfrac{bCt}{R}\right) f_0^2\right.\\ & \left. - 2\left(k + \tfrac{2}{bt\eta}\right)\left|\nabla_b f\right|^2 - \tfrac{Cbt}{R}\eta f_0^2\left|\nabla_b f\right|^2 + 4bt\eta f_0 V(f)\right]\end{aligned}$$

and for each $a \neq 0$

$$\begin{aligned}
(\Delta_b f)^2 &= \left(-|\nabla_b f|^2\right)^2 \\
&= \left(\tfrac{1}{at}F - \tfrac{1}{a}|\nabla_b f|^2 - \tfrac{1}{a}bt\eta f_0^2 - |\nabla_b f|^2\right)^2 \\
&= \left(\tfrac{1}{at}F - \tfrac{a+1}{a}|\nabla_b f|^2 - \tfrac{1}{a}bt\eta f_0^2\right)^2 \\
&= \tfrac{1}{a^2t^2}F^2 + \left(\tfrac{a+1}{a}\right)^2|\nabla_b f|^4 + \tfrac{1}{a^2}b^2t^2\eta^2 f_0^4 \\
&\quad -\tfrac{2(a+1)}{a^2t}F|\nabla_b f|^2 - \tfrac{2b}{a^2}\eta F f_0^2 + \tfrac{2(a+1)bt}{a^2}\eta|\nabla_b f|^2 f_0^2 \\
&\geq \tfrac{1}{a^2t^2}F^2 - \tfrac{2(a+1)}{a^2t}F|\nabla_b f|^2 - \tfrac{2b}{a^2}\eta F f_0^2.
\end{aligned}$$

Then

$$\begin{aligned}
\Delta_b(\eta F) \geq\ & \tfrac{1}{na^2t}\eta F^2 - \tfrac{C}{R}F + 2\langle\nabla_b\eta,\ \nabla_b F\rangle - 2\eta\langle\nabla_b f,\ \nabla F\rangle \\
&+4t\eta\left(\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2\right) \\
&+\left(n - \tfrac{bCt}{R} - \tfrac{2b}{na^2}\eta F\right)t\eta f_0^2 + \left(-\tfrac{2(1+a)}{na^2}\eta F - 2kt\eta - \tfrac{4}{b}\right)|\nabla_b f|^2 \\
&-\tfrac{Cb}{R}\left(t\eta|\nabla_b f|^2\right)\left(t\eta f_0^2\right) + 4bt^2\eta^2 f_0 V(f).
\end{aligned}$$

Hence

$$\begin{aligned}
\Delta_b(\eta F) \geq\ & \tfrac{1}{na^2t}\eta F^2 - \tfrac{C}{R}F + 2\langle\nabla_b\eta,\ \nabla_b F\rangle - 2\eta\langle\nabla_b f,\ \nabla F\rangle \\
&+4t\eta\left(\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2\right) \\
&+\left[n - \tfrac{bCt}{R} - \left(\tfrac{2b}{na^2} + \tfrac{bC}{R}\right)\eta F\right]t\eta f_0^2 \\
&+\left(-\tfrac{2(1+a)}{na^2}\eta F - 2kt\eta - \tfrac{4}{b}\right)|\nabla_b f|^2 + 4bt^2\eta^2 f_0 V(f).
\end{aligned} \tag{4.9}$$

Finally, multiply $t\eta$ on the both sides of (4.9) and note that $t \leq 1$, $\eta \leq 1$

$$\begin{aligned}
t\eta\Delta_b(\eta F) \geq\ & \tfrac{1}{na^2}(\eta F)^2 - \tfrac{C}{R}\eta F + 2t\eta\langle\nabla_b\eta,\ \nabla_b F\rangle - 2t\eta^2\langle\nabla_b f,\ \nabla_b F\rangle \\
&+4t^2\eta^2\left(\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2\right) \\
&+\left[n - \tfrac{bC}{R} - \left(\tfrac{2b}{na^2} + \tfrac{bC}{R}\right)(\eta F)\right]t^2\eta^2 f_0^2 \\
&+\left[-\tfrac{2(1+a)}{na^2}(\eta F) - 2k - \tfrac{4}{b}\right]t\eta|\nabla_b f|^2 + 4bt^3\eta^3 f_0 V(f).
\end{aligned}$$

$\square$

Proposition 4.3. *Let* $(M,\ J,\ \theta)$ *be a complete noncompact pseudohermitian* $(2n+1)$*-manifold with*

$$\left(2Ric-(n-2)\,Tor\right)(Z,\ Z)\geq -2k\left|Z\right|^2$$

for all $Z\in T_{1,0}$*, where* k *is an nonnegative constant. Suppose that* $(M,\ J,\ \theta)$ *satisfies the CR sub-Laplacian comparison property. Let* $b,\ R$ *be fixed, and* $p(t)\in B(2R)$ *be the maximal point of* ηF *for each* $t\in(0,1]$*. Then at* $(\ p(t)\,,t)$ *we have*

$$\begin{array}{rcl}0 & \geq & \left(\frac{1}{na^2}-\frac{C}{R}\right)(\eta F)^2-\frac{3C}{R}(\eta F)+4t^2\eta^2\left(\sum\limits_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2+\sum\limits_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2\right)\\ & & +\left[n-\frac{bC}{R}-\left(\frac{2b}{na^2}+\frac{bC}{R}\right)(\eta F)\right]t^2\eta^2f_0^2\\ & & +\left[-\frac{2(1+a)}{na^2}(\eta F)-2k-\frac{4}{b}-\frac{C}{R}\right]t\eta\left|\nabla_bf\right|^2+4bt^3\eta^3f_0V(f)\,.\end{array}\tag{4.10}$$

Proof. Since $(\eta F)(p(t)\,,\ t,\ R,\ b)=\max\limits_{x\in B(2R)}(\eta F)(x,\ t,\ R,\ b)$, at a critical point $(p(t)\,,\ t)$ of $(\eta F)(x,\ t,\ R,\ b)$, we have

$$\nabla_b(\eta F)(p(t)\,,\ t,\ R,\ b)=0.$$

This implies that

$$F\nabla_b\eta+\eta\nabla_bF=0\tag{4.11}$$

at $(p(t)\,,\ t)$. On the other hand,

$$\Delta_b(\eta F)(p(t)\,,\ t,\ R,\ b)\leq 0\tag{4.12}$$

at $(p(t)\,,\ t)$.

Now we apply (4.11) to $2t\eta\left\langle\nabla_b\eta,\ \nabla_bF\right\rangle$ and $-2t\eta^2\left\langle\nabla_bf,\ \nabla_bF\right\rangle$ in (4.8), we can derive the following estimates.

$$\begin{array}{rcl}2t\eta\left\langle\nabla_b\eta,\ \nabla_bF\right\rangle & = & -2tF\left|\nabla_b\eta\right|^2\\ & \geq & -\frac{2tC}{R^2}\eta F\\ & \geq & -\frac{2C}{R}\eta F\end{array}\tag{4.13}$$

and

$$
\begin{aligned}
-2t\eta^2 \left\langle \nabla_b f,\ \nabla_b F \right\rangle &= 2t\eta F \left\langle \nabla_b f,\ \nabla_b \eta \right\rangle \\
&\geq -2t\left(\eta F\right)\left|\nabla_b f\right| \left|\ \nabla_b \eta\right| \\
&\geq -\tfrac{2tC}{R}\left(\eta F\right)\eta^{\frac{1}{2}}\left|\nabla_b f\right| \\
&\geq -\tfrac{Ct}{R}\left(\eta F\right)^2 - \tfrac{C}{R}t\eta\left|\nabla_b f\right|^2 .
\end{aligned}
\tag{4.14}
$$

Here we have applied the Young's inequality for (4.14).

Finally, substituting (4.12), (4.13) and (4.14) into (4.8) in Proposition 4.2, and noting that $t \leq 1$,

$$
\begin{aligned}
0 \ \geq\ & \left(\tfrac{1}{na^2} - \tfrac{C}{R}\right)\left(\eta F\right)^2 - \tfrac{3C}{R}\left(\eta F\right) + 4t^2\eta^2 \left(\sum_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2\right) \\
& + \left[n - \tfrac{bC}{R} - \left(\tfrac{2b}{na^2} + \tfrac{bC}{R}\right)\left(\eta F\right)\right] t^2\eta^2 f_0^2 \\
& + \left[-\tfrac{2(1+a)}{na^2}\left(\eta F\right) - 2k - \tfrac{4}{b} - \tfrac{C}{R}\right] t\eta\left|\nabla_b f\right|^2 + 4bt^3\eta^3 f_0 V\left(f\right).
\end{aligned}
$$

This completes the proof. □

Now, we are ready to prove our main theorems.

Proof of Theorem 1.3 :

Proof. We observe that

$$
V\left(f\right) = 0 \tag{4.15}
$$

by assumption (1.6) and Lemma 2.5.

We begin by substituting (4.15) into (4.10) in Proposition 4.3 at the maximum point $(p(t), t)$. Hence

$$
\begin{aligned}
0 \ \geq\ & \left(\tfrac{1}{na^2} - \tfrac{C}{R}\right)\left[\left(\eta F\right)\right]^2 - \tfrac{3C}{R}\left[\left(\eta F\right)\right] \\
& + \left[n - \tfrac{bC}{R} - \left(\tfrac{2b}{na^2} + \tfrac{bC}{R}\right)\left(\eta F\right)\right] t^2\eta^2 f_0^2 \\
& + \left[-\tfrac{2(1+a)}{na^2}\left(\eta F\right) - 2k - \tfrac{4}{b} - \tfrac{C}{R}\right] t\eta\left|\nabla_b f\right|^2 \\
& + 4t_0^2\eta^2 \left(\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2\right).
\end{aligned}
\tag{4.16}
$$

We claim at $t=1$

$$(\eta F)(p(1),\ 1,\ R,\ b) < \frac{na^2}{-2(1+a)}\left(2k+\frac{4}{b}+\frac{C}{R}\right) \tag{4.17}$$

for a large enough R which to be determined later. Here $(1+a)<0$ for some a to be chosen later (say $1+a=-\frac{5+2bk}{n}$).

We prove it by contradiction. Suppose not, that is

$$(\eta F)(p(1),\ 1,\ R,\ b) \geq \frac{na^2}{-2(1+a)}\left(2k+\frac{4}{b}+\frac{C}{R}\right).$$

Since $(\eta F)(p(t),\ t,\ R,\ b)$ is continuous in the variable t and $(\eta F)(p(0),\ 0,\ R,\ b)=0$, by Intermediate-value theorem there exists a $t_0\in(0,\ 1]$ such that

$$(\eta F)(p(t_0),\ t_0,\ R,\ b) = \frac{na^2}{-2(1+a)}\left(2k+\frac{4}{b}+\frac{C}{R}\right). \tag{4.18}$$

Now we apply (4.16) at the point $(p(t_0),t_0)$, denoted by (p_0,t_0). We have by using (4.18)

$$\begin{aligned} 0 \ \geq\ & \left(\tfrac{1}{na^2}-\tfrac{C}{R}\right)\left[(\eta F)(p_0,t_0)\right]^2-\tfrac{3C}{R}\left[(\eta F)(p_0,t_0)\right] \\ & +\left[n-\tfrac{bC}{R}-\left(\tfrac{2b}{na^2}+\tfrac{bC}{R}\right)(\eta F)(p_0,t_0)\right]t^2\eta^2 f_0^2 \\ & +4t_0^2\eta^2\left(\sum_{\alpha,\beta=1}^{n}\left|f_{\alpha\beta}\right|^2+\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2\right). \end{aligned} \tag{4.19}$$

Moreover, we compute

$$\begin{aligned} & \left[\left(\tfrac{1}{na^2}-\tfrac{C}{R}\right)(\eta F)(p_0,t_0)-\tfrac{3C}{R}\right] \\ & =\left[\left(\tfrac{1}{na^2}-\tfrac{C}{R}\right)\left(\tfrac{na^2}{-2(1+a)}\right)\left(2k+\tfrac{4}{b}+\tfrac{C}{R}\right)-\tfrac{3C}{R}\right] \\ & =\left\{\tfrac{-1}{2(1+a)}\left(2k+\tfrac{4}{b}\right)-\tfrac{C}{R}\left[\tfrac{na^2}{-2(1+a)}\left(2k+\tfrac{4}{b}+\tfrac{C}{R}\right)+\tfrac{1}{2(1+a)}+3\right]\right\} \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} & \left[n-\tfrac{bC}{R}-\left(\tfrac{2b}{na^2}+\tfrac{bC}{R}\right)(\eta F)(p_0,t_0)\right] \\ & =n-\tfrac{bC}{R}-\left(\tfrac{2b}{na^2}+\tfrac{bC}{R}\right)\left(\tfrac{na^2}{-2(1+a)}\right)\left(2k+\tfrac{4}{b}+\tfrac{C}{R}\right) \\ & =n-\tfrac{bC}{R}+\tfrac{b}{(1+a)}\left(2k+\tfrac{4}{b}+\tfrac{C}{R}\right)+\tfrac{bC}{R}\left(\tfrac{na^2}{2(1+a)}\right)\left(2k+\tfrac{4}{b}+\tfrac{C}{R}\right) \\ & =\left(n+\tfrac{4}{1+a}+\tfrac{2bk}{1+a}\right)+\tfrac{C}{R}\left[-\tfrac{ab}{1+a}+\tfrac{na^2b}{2(1+a)}\left(2k+\tfrac{4}{b}+\tfrac{C}{R}\right)\right]. \end{aligned} \tag{4.21}$$

Now we choose a such that

$$(1+a) < -\frac{4+2bk}{n}$$

and then

$$\left(n+\frac{4}{1+a}+\frac{2bk}{1+a}\right) > 0.$$

In particular, we let

$$1+a = -\frac{5+2bk}{n}. \tag{4.22}$$

Then for $R = R(b,k)$ large enough, one obtains

$$\left[\left(\frac{1}{na^2}-\frac{C}{R}\right)(\eta F)(p_0,t_0)-\frac{3C}{R}\right] > 0$$

and

$$\left[n-\frac{bC}{R}-\left(\frac{2b}{na^2}+\frac{bC}{R}\right)(\eta F)(p_0,t_0)\right] > 0.$$

This leads to a contradiction with (4.19). Hence from (4.17) and (4.22)

$$(\eta F)(1,\ p(1),\ R,\ b) < \frac{(n+5+2bk)^2}{2(5+2bk)}\left(2k+\frac{4}{b}+\frac{C}{R}\right).$$

This implies

$$\max_{x\in B(2R)}\left(|\nabla_b f|^2+b\eta f_0^2\right)(x) < \frac{(n+5+2bk)^2}{2(5+2bk)}\left(2k+\frac{4}{b}+\frac{C}{R}\right).$$

When we fix on the set $x \in B(R)$, we obtain

$$|\nabla_b f|^2+bf_0^2 < \frac{(n+5+2bk)^2}{2(5+2bk)}\left(2k+\frac{4}{b}+\frac{C}{R}\right)$$

on $B(R)$.

This completes the proof.

Next we prove Theorem 1.7. The proof is similar to Theorem 1.3.

Proof of Theorem 1.7 :

Proof. Firstly, we recall (Proposition 4.2) that

$$
\begin{aligned}
& t\eta\Delta_b(\eta F) \\
\geq\ & \tfrac{1}{na^2}(\eta F)^2 - \tfrac{C}{R}(\eta F) + 2t\eta\langle\nabla_b\eta,\ \nabla_b F\rangle - 2t\eta^2\langle\nabla_b f,\ \nabla_b F\rangle \\
& +4t^2\eta^2\left(\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2\right) \\
& +\left[n - \tfrac{bC}{R} - \left(\tfrac{2b}{na^2} + \tfrac{bC}{R}\right)(\eta F)\right]t^2\eta^2 f_0^2 \\
& +\left[-\tfrac{2(1+a)}{na^2}(\eta F) - 2k - \tfrac{4}{b}\right]t\eta|\nabla_b f|^2 + 4bt^3\eta^3 f_0 V(f).
\end{aligned}
\tag{4.23}
$$

Now we need to deal with the term $4bt^3\eta^3 f_0 V(f)$ in (4.23).

$$
\begin{aligned}
& 4bt^3\eta^3 f_0 V(f) \\
&= 4bt^3\eta^3 f_0\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar{\beta}}\right)_{\bar{\alpha}} + \left(A_{\bar{\alpha}\bar{\beta}}f_{\beta}\right)_{\alpha} + A_{\alpha\beta}f_{\bar{\alpha}}f_{\bar{\beta}} + A_{\bar{\alpha}\bar{\beta}}f_{\alpha}f_{\beta}\right] \\
&= 4bt^3\eta^3 f_0\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar{\beta}\bar{\alpha}} + A_{\bar{\alpha}\bar{\beta}}f_{\beta\alpha}\right) + \left(A_{\alpha\beta,\bar{\alpha}}f_{\bar{\beta}} + A_{\bar{\alpha}\bar{\beta},\alpha}f_{\beta}\right) + \left(A_{\alpha\beta}f_{\bar{\alpha}}f_{\bar{\beta}} + A_{\bar{\alpha}\bar{\beta}}f_{\alpha}f_{\beta}\right)\right] \\
&\geq -8bt^3\eta^3|f_0|\sum_{\alpha,\beta=1}^{n}\left(\left|A_{\bar{\alpha}\bar{\beta}}\right||f_{\beta\alpha}| + |A_{\alpha\beta,\bar{\alpha}}|\left|f_{\bar{\beta}}\right| + \left|A_{\bar{\alpha}\bar{\beta}}\right||f_{\alpha}||f_{\beta}|\right)
\end{aligned}
\tag{4.24}
$$

In (4.24), by Young's inequality and noting that $t \leq 1$, $\eta \leq 1$, we have following estimates:

$$
\begin{aligned}
-8bt^3\eta^3|f_0|\sum_{\alpha,\beta=1}^{n}\left|A_{\bar{\alpha}\bar{\beta}}\right||f_{\beta\alpha}| &\geq \sum_{\alpha,\beta=1}^{n} - 8k_1bt^3\eta^3|f_0||f_{\beta\alpha}| \\
&\geq \sum_{\alpha,\beta=1}^{n}\left(-4k_1bt^3\eta^3|f_{\beta\alpha}|^2 - 4k_1bt^3\eta^3 f_0^2\right) \\
&\geq -4k_1bn^2\left(t^2\eta^2 f_0^2\right) - 4k_1bt^2\eta^2\sum_{\alpha,\beta=1}^{n}|f_{\beta\alpha}|^2
\end{aligned}
\tag{4.25}
$$

and

$$
\begin{aligned}
-8bt^3\eta^3|f_0|\sum_{\alpha,\beta=1}^{n}|A_{\alpha\beta,\bar{\alpha}}|\left|f_{\bar{\beta}}\right| &\geq -8k_1bt^3\eta^3\sum_{\alpha,\beta=1}^{n}|f_0|\left|f_{\bar{\beta}}\right| \\
&\geq -8k_1bt^3\eta^3\sum_{\alpha,\beta=1}^{n}\left(\tfrac{1}{2}f_0^2 + \tfrac{1}{2}\left|f_{\bar{\beta}}\right|^2\right) \\
&\geq -4k_1bn^2t^3\eta^3 f_0^2 - 4k_1bnt^3\eta^3\sum_{\beta=1}^{n}\left|f_{\bar{\beta}}\right|^2 \\
&\geq -4k_1bn^2\left(t^2\eta^2 f_0^2\right) - 2k_1bn\left(t\eta|\nabla_b f|^2\right)
\end{aligned}
\tag{4.26}
$$

and

$$
\begin{aligned}
-8bt^3\eta^3\left|f_0\right|\sum_{\alpha,\beta=1}^{n}\left|A_{\bar{\alpha}\bar{\beta}}\right|\left|f_\alpha\right|\left|f_\beta\right| &\geq -8k_1bt^3\eta^3\left|f_0\right|\sum_{\alpha,\beta=1}^{n}\left(\tfrac{1}{2}\left|f_\alpha\right|^2+\tfrac{1}{2}\left|f_\beta\right|^2\right)\\
&\geq -4k_1bt^3\eta^3\left|f_0\right|\left(n\sum_{\alpha=1}^{n}\left|f_\alpha\right|^2+n\sum_{\beta=1}^{n}\left|f_\beta\right|^2\right)\\
&\geq -4k_1bnt^3\eta^3\left|f_0\right|\left|\nabla_bf\right|^2\\
&\geq -2k_1b^2nt^3\eta^3f_0^2\left|\nabla_bf\right|^2-2k_1nt^3\eta^3\left|\nabla_bf\right|^2\\
&= -2k_1b^2n\left(t\eta\left|\nabla_bf\right|^2\right)\left(t^2\eta^2f_0^2\right)-2k_1nt^3\eta^3\left|\nabla_bf\right|^2.
\end{aligned}\tag{4.27}
$$

Substitute estimates (4.25), (4.26), and (4.27) into (4.23), one obtains

$$
\begin{aligned}
t\eta\Delta_b\left(\eta F\right) \geq\ & \tfrac{1}{na^2}\left(\eta F\right)^2-\tfrac{C}{R}\left(\eta F\right)+2t\eta\left\langle\nabla_b\eta,\ \nabla_bF\right\rangle-2t\eta^2\left\langle\nabla_bf,\ \nabla_bF\right\rangle\\
&+4t^2\eta^2\left[\left(1-bk_1n\right)\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2+\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2\right]\\
&+\left[n-8bk_1n^2-\tfrac{bC}{R}-\left(\tfrac{2b}{na^2}+2b^2k_1n+\tfrac{bC}{R}\right)\left(\eta F\right)\right]t^2\eta^2f_0^2\\
&+\left[-\tfrac{2(1+a)}{na^2}\left(\eta F\right)-2k-2n\left(1+b\right)k_1-\tfrac{4}{b}\right]t\eta\left|\nabla_bf\right|^2.
\end{aligned}
$$

Next as shown in the same computation as in Proposition 4.3, at the maximal point $(p(t),\ t)$

$$
\begin{aligned}
0 \geq\ & \left(\tfrac{1}{na^2}-\tfrac{C}{R}\right)\left(\eta F\right)^2-\tfrac{3C}{R}\left(\eta F\right)\\
&+4t^2\eta^2\left[\left(1-bk_1n\right)\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2+\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2\right]\\
&+\left[n-8bk_1n^2-\tfrac{bC}{R}-\left(\tfrac{2b}{na^2}+2b^2k_1n+\tfrac{bC}{R}\right)\left(\eta F\right)\right]t^2\eta^2f_0^2\\
&+\left[-\tfrac{2(1+a)}{na^2}\left(\eta F\right)-2k-2n\left(1+b\right)k_1-\tfrac{4}{b}-\tfrac{C}{R}\right]t\eta\left|\nabla_bf\right|^2.
\end{aligned}\tag{4.28}
$$

We claim at $t=1$, there exists a small constant $b_0=b_0(n,k,k_1)>0$ such that for any $0<b\leq b_0$

$$
\left(\eta F\right)\left(p\left(1\right),\ 1,\ R,\ b\right)<\frac{na^2}{-2\left(1+a\right)}\left(2k+2n\left(1+b\right)k_1+\frac{4}{b}+\frac{C}{R}\right)
$$

if R is large enough which to be determined later. Here $(1+a)<0$ for some a to be chosen later (say $1+a=-\frac{5}{n}$).

We prove it by contradiction. Suppose not, that is

$$(\eta F)\left(p\left(1\right),\ 1,\ R,\ b\right) \geq \frac{na^2}{-2\left(1+a\right)}\left(2k+2n\left(1+b\right)k_1+\frac{4}{b}+\frac{C}{R}\right).$$

Since $(\eta F)\left(p\left(t\right),\ t,\ R,\ b\right)$ is continuous in the variable t and $(\eta F)\left(p\left(0\right),\ 0,\ R,\ b\right)=0$, by Intermediate-value theorem there exists a $t_0 \in (0,\ 1]$ such that

$$(\eta F)\left(p\left(t_0\right),\ t_0,\ R,\ b\right) = \frac{na^2}{-2\left(1+a\right)}\left(2k+2n\left(1+b\right)k_1+\frac{4}{b}+\frac{C}{R}\right).$$

Now we apply (4.28) at the point $\left(p\left(t_0\right), t_0\right)$, denoted by $\left(p_0, t_0\right)$. We have

(4.29)
$$\begin{aligned}
&\left[\left(\tfrac{1}{na^2}-\tfrac{C}{R}\right)(\eta F)\left(p_0,t_0\right)-\tfrac{3C}{R}\right] \\
&= \left[\left(\tfrac{1}{na^2}-\tfrac{C}{R}\right)\left(\tfrac{na^2}{-2(1+a)}\right)\left(2k+2n\left(1+b\right)k_1+\tfrac{4}{b}+\tfrac{C}{R}\right)-\tfrac{3C}{R}\right] \\
&= \left\{\tfrac{-1}{2(1+a)}\left(2k+2n\left(1+b\right)k_1+\tfrac{4}{b}\right)-\tfrac{C}{R}\left[\tfrac{na^2}{-2(1+a)}\left(2k+2n\left(1+b\right)k_1+\tfrac{4}{b}+\tfrac{C}{R}\right)+\tfrac{1}{2(1+a)}+3\right]\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\left[n-8bk_1n^2-\tfrac{bC}{R}-\left(\tfrac{2b}{na^2}+2b^2k_1n+\tfrac{bC}{R}\right)(\eta F)\left(p_0,t_0\right)\right] \\
&= n-8bk_1n^2-\tfrac{bC}{R}-\left(\tfrac{2b}{na^2}+2b^2k_1n+\tfrac{bC}{R}\right)\left(\tfrac{na^2}{-2(1+a)}\right)\left(2k+2n\left(1+b\right)k_1+\tfrac{4}{b}+\tfrac{C}{R}\right) \\
&= n-8bk_1n^2-\tfrac{bC}{R}+\left(\tfrac{na^2}{2(1+a)}\right)\left(\tfrac{2b}{na^2}+2b^2k_1n\right)\left(2k+2n\left(1+b\right)k_1+\tfrac{4}{b}+\tfrac{C}{R}\right) \\
&+\tfrac{bC}{R}\left(\tfrac{na^2}{2(1+a)}\right)\left(2k+2n\left(1+b\right)k_1+\tfrac{4}{b}+\tfrac{C}{R}\right) \\
&= \left\{n-8bk_1n^2+\left(\tfrac{b+a^2b^2n^2k_1}{(1+a)}\right)\left[2k+2n\left(1+b\right)k_1+\tfrac{4}{b}\right]\right\} \\
&+\tfrac{C}{R}\left\{-b+\left(\tfrac{b+a^2b^2n^2k_1}{(1+a)}\right)+\tfrac{na^2b}{2(1+a)}\left[2k+2n\left(1+b\right)k_1+\tfrac{4}{b}+\tfrac{C}{R}\right]\right\}.
\end{aligned} \tag{4.30}$$

Now we choose a and b such that

$$\begin{aligned}
&n-8bk_1n^2+\left(\tfrac{b+a^2b^2n^2k_1}{(1+a)}\right)\left[2k+2n\left(1+b\right)k_1+\tfrac{4}{b}\right] \\
&= n-b\left\{8k_1n^2-\left(\tfrac{1+a^2bn^2k_1}{(1+a)}\right)\left[2k+2n\left(1+b\right)k_1\right]-\left(\tfrac{4a^2n^2k_1}{1+a}\right)\right\}+\tfrac{4}{1+a} \\
&> 0.
\end{aligned} \tag{4.31}$$

This can be done by choosing

$$(1+a) < -\frac{4}{n}$$

and then choose a small $b_0 = b_0(n, k, k_1) > 0$ such that for any $b \leq b_0$

$$n - b\{8k_1n^2 - (\frac{1 + a^2bn^2k_1}{(1+a)})[2k + 2n\,(1+b)\,k_1] - (\frac{4a^2n^2k_1}{1+a})\} + \frac{4}{1+a} > 0$$

and

$$(1 - bk_1n) > 0.$$

In particular, we let

$$1 + a = -\frac{5}{n}.$$

Then for any $0 < b \leq b_0$, one obtains

$$\left[\left(\frac{1}{na^2} - \frac{C}{R}\right)(\eta F)\,(p_0, t_0) - \frac{3C}{R}\right] > 0$$

and

$$\left[n - 8bk_1n^2 - \frac{bC}{R} - \left(\frac{2b}{na^2} + 2b^2k_1n + \frac{bC}{R}\right)(\eta F)\,(p_0, t_0)\right] > 0$$

for $R = R(b, k, k_1)$ large enough. This leads to a contradiction with (4.28). Hence

$$(\eta F)\,(1,\ p\,(1)\,,\ R,\ b) < \frac{na^2}{-\,(1+a)}\left(k + n\,(1+b)\,k_1 + \frac{2}{b} + \frac{C}{R}\right).$$

This implies for $1 + a = -\frac{5}{n}$

$$\max_{x \in B(2R)} \left(|\nabla_b f|^2 + b\eta f_0^2\right)(x) < \frac{(n+5)^2}{5}\left(k + n\,(1+b)\,k_1 + \frac{2}{b} + \frac{C}{R}\right).$$

When we fix on the set $x \in B\,(R)$, we obtain

$$|\nabla_b f|^2 + bf_0^2 < \frac{(n+5)^2}{5}\left(k + n\,(1+b)\,k_1 + \frac{2}{b} + \frac{C}{R}\right)$$

on $B\,(R)$. Note that the preceding computation is not valid if ηF is not smooth at x_0. In this case, we may use a trick due to E. Calabi (see [W] for details).

This completes the proof of Theorem 1.7.

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[1]Department of Mathematics and Taida Institute for Mathematical Sciences (TIMS), National Taiwan University, Taipei 10617, Taiwan, Current Address : Yau Mathematical Sciences Center, Tsinghua University, Beijing, China

E-mail address: scchang@math.ntu.edu.tw

[2]Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan

E-mail address: tjkuo1215@ntnu.edu.tw

[3]Department of Mathematics, National Tsing Hua University, Hsinchu 30013, Taiwan

E-mail address: r97221009@ntu.edu.tw

[4]Department of Mathematics, University of Georgia, Athens, GA 30602-7403, U.S.A.

E-mail address: jtie@math.uga.edu