# ON THE SHARP DIMENSION ESTIMATE OF CR HOLOMORPHIC FUNCTIONS IN SASAKIAN MANIFOLDS

*SHU-CHENG CHANG, †YINGBO HAN, AND **CHIEN LIN

Abstract. This is the very first paper to focus on the CR analogue of Yau's uniformization conjecture in a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion (i.e. Sasakian manifold) which is an odd dimensional counterpart of Kähler geometry. In this paper, we mainly deal with the problem of the sharp dimension estimate of CR holomorphic functions in a complete noncompact pseudohermitian manifold of vanishing torsion with nonnegative pseudohermitian bisectional curvature.

## 1. Introduction

S.-Y. Cheng and S.-T. Yau( [Y1], [CY]) derived the well-known gradient estimate for positive harmonic functions and obtained the classical Liouville theorem, which states that any bounded harmonic function is constant in complete noncompact $m$-dimensional Riemannian manifolds with nonnegative Ricci curvature. Yau conjectured that the dimension $h^d(M)$ of the space $\mathcal{H}^d(M)$ consisting of harmonic functions of polynomial growth of degree at most $d$, is finite for each positive integer $d$ and satisfies the estimate:

$$h^d(M) \leq h^d(\mathbf{R}^m).$$

1991 *Mathematics Subject Classification.* Primary 32V05, 32V20; Secondary 53C56.

*Key words and phrases.* CR heat kernel, CR Yau's uniformization conjecture, CR holomorphic function, Sharp dimension estimate.

*Shu-Cheng Chang and **Chien Lin are partially supported in part by the MOST of Taiwan.

†(Corresponding author) Yingbo Han is partially supported by an NSFC grant No. 11201400, Nanhu Scholars Program for Young Scholars of Xinyang Normal University and the Universities Young Teachers Program of Henan Province (2016GGJS-096).

Colding and Minicozzi ([CM]) affirmatively answered the first question and proved that

$$h^d(M) \leq C_0 d^{m-1}$$

for manifolds of nonnegative Ricci curvature. Later, Li ([Li2]) produced a shorter proof requiring only the manifold to satisfy the volume doubling property and the mean value inequality. For the latter question, the sharp upper bound estimate is still missing except for the special cases $m = 2$ or $d = 1$ obtained by Li-Tam ([LT], [LT1]) and Kasue ([K]), and the rigidity part is only known for the special case $d = 1$ obtained by Li ([Li1]) and Cheeger-Colding-Minicozzi ([CCM]).

In [CKT], by modifying the arguments of [Y1], [CY], and [CKL], Chang, Kuo and Tie derived a sub-gradient estimate for positive pseudoharmonic functions in a complete noncompact pseudohermitian manifold $(M, J, \theta)$. This sub-gradient estimate can serve as the CR version of Yau's gradient estimate. As an application of the sub-gradient estimate, the CR analogue of Liouville-type theorem holds for positive pseudoharmonic functions. In the recent paper ([CCHT]), we study the CR analogue of Yau's conjecture on the space $\mathcal{H}^d(M)$ consisting of all pseudoharmonic functions of polynomial growth of degree at most $d$ in a complete noncompact pseudohermitian manifold. We showed that the first part of CR Yau's conjecture holds for pseudoharmonic functions of polynomial growth.

In Kähler geometry, Yau ([ScY]) proposed a variety of uniformization-type problems on complete Kähler manifolds with nonnegative holomorphic bisectional curvature. The first conjecture is

**Conjecture 1.** *If $M$ is a complete noncompact $n$-dimensional Kähler manifold with nonnegative holomorphic bisectional curvature, then*

$$\dim_{\mathbb{C}} \left( \mathcal{O}_d \left( M^n \right) \right) \leq \dim_{\mathbb{C}} \left( \mathcal{O}_d \left( \mathbb{C}^n \right) \right).$$

*The equality holds if and only if $M$ is isometrically biholomorphic to $\mathbb{C}^n$. Here $\mathcal{O}_d(M^n)$ denotes the family of all holomorphic functions on a complete $n$-dimensional Kähler manifold $M$ of polynomial growth of degree at most $d$.*

In [N], Ni established the validity of this conjecture by deriving the monotonicity formula for the heat equation under the extra assumption that $M$ has maximal volume growth

$$\lim_{r\to+\infty}\frac{Vol\left(B_p\left(r\right)\right)}{r^{2n}}\geq c$$

for a fixed point $p$ and a positive constant $c$. Later, in [CFYZ], Chen, Fu, Yin, and Zhu improved Ni's result without the assumption of maximal volume growth. One should refer to [Liu1] for more general results recently.

The second conjecture is

**Conjecture 2.** *If $M$ is a complete noncompact $n$-dimensional Kähler manifold with nonnegative holomorphic bisectional curvature, then the ring $O_P(M)$ of all holomorphic functions of polynomial growth is finitely generated.*

This one was solved completely by G. Liu ([Liu2]) quite recently. He mainly deployed four techniques to attack this conjecture via Cheeger-Colding-Tian's theory ([ChCo1], [ChCo2], [CCT]), methods of heat flow developed by Ni and Tam ([N], [NT1], [NT4]), Hormander $L^2$-estimate of $\overline{\partial}$ ([De]) and three circle theorem ([Liu1]).

The third uniformization conjecture is

**Conjecture 3.** *If $M$ is a complete noncompact $n$-dimensional Kähler manifold with positive holomorphic bisectional curvature, then $M$ is biholomorphic to the standard $n$-dimensional complex space $\mathbb{C}^n$.*

The first giant progress pertaining to the third conjecture could be attributed to Mok, Siu and Yau. In their papers ([MSY] and [M1]), they showed that, under the assumptions of the maximal volume growth and the scalar curvature $R(x)$ decays as

$$0 \leq r(x) \leq \frac{C}{(1+d(x,x_0))^{2+\epsilon}}$$

for some positive constant $C$ and any arbitrarily small positive number $\epsilon$, a complete noncompact $n$-dimensional Kähler manifold $M$ with nonnegative holomorphic bisectional curvature is isometrically biholomorphic to $\mathbb{C}^n$. A Riemannian version was proved in [GW2] shortly afterwards. Since then there are several further works aiming to prove the optimal result and reader is referred to [M2], [CTZ], [CZ], [N2], [NT1], [NT2] and [NST]. For example, A. Chau and L. F. Tam ([CT]) proved that a complete noncompact Kähler manifold with bounded nonnegative bisectional curvature and maximal volume growth is biholomorphic to $\mathbb{C}^n$. Recently, G. Liu ([Liu3]) confirmed Yau's uniformization conjecture when $M$ has maximal volume growth.

This is the very first paper to focus on the CR analogue of Yau's uniformization conjectures in a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion (i.e. Sasakian manifold) which is an odd dimensional counterpart of Kähler geometry. We refer to the next section for the detailed notations. The following is the so-called CR Yau's uniformization conjecture.

**Conjecture 4.** *(CR Yau's Uniformization Conjecture) Let $M$ be a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with positive pseudohermitian bisectional curvature. Then $M$ is CR biholomorphic to the standard Heisenberg group $\mathbf{H}_n = \mathbb{C}^n \times \mathbb{R}$.*

In this paper, it is very natural to concerned the CR analogue of Yau's first conjecture in a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian bisectional curvature. A smooth complex-valued function on a pseudohermitian $(2n+1)$-manifold $(M,J,\theta)$ is called CR-holomorphic if

$$\overline{\partial}_b f = 0.$$

For any fixed point $x \in M$, a CR-holomorphic function $f$ is called to be of polynomial growth if there are a nonnegative number $d$ and a positive constant $C = C(x, d, f)$, depending on $x$, $d$ and $f$, such that

$$|f(y)| \leq C(1 + d_{cc}(x, y))^d$$

for all $y \in M$, where $d_{cc}(x, y)$ denotes the Carnot-Caratheodory distance between $x$ and $y$. In the following sections, we sometimes would use the notation $r(x, y)$ for the Carnot-Caratheodory distance. Furthermore, we could define the degree of a CR-holomorphic function $f$ of polynomial growth by

$$\deg(f) = \inf\left\{ d \geq 0 \left| \begin{array}{c} |f(y)| \leq C(1 + d_{cc}(x, y))^d \quad \forall\, y \in M, \\ for\ some\ d \geq 0\ and\ C = C(x, d, f) \end{array} \right. \right\}$$

as well as the aforementioned holomorphic case. In fact, the definition above is independent of the choice of the point $x \in M$. Finally we denote $\mathcal{O}_d^{CR}(M)$ the family of all CR-holomorphic functions $f$ of polynomial growth of degree at most $d$ with $Tf(x) = f_0(x) = 0$ :

$$\mathcal{O}_d^{CR}(M) = \{ f(x) \,|\overline{\partial}_b f(x) = 0,\ f_0(x) = 0 \text{ and } |f(x)| \leq C(1 + d_{cc}(x, y))^d \text{ for some constant } C \}.$$

Now we explain the extra condition $Tf = 0$ in the definition of $\mathcal{O}_d^{CR}(M)$ in a Sasakian manifold. Follow the notion as in [FOW] : Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of the Sasakian manifold $(M^{2n+1}, g)$ and $\pi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n$ submersion such that $\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)$ is biholomorphic. On each $V_\alpha$, there is a canonical isomorphism

$$d\pi_\alpha : D_p \to T_{\pi_\alpha(p)} V_\alpha$$

for any $p \in U_\alpha$, where $D = \ker \theta \subset TM$. Since $T$ generates isometries, the restriction of the Sasakian metric $g$ to $D$ gives a well-defined Hermitian metric $g_\alpha^T$ on $V_\alpha$. This Hermitian metric in fact is Kähler. More precisely, let $z^1, z^2, \cdots, z^n$ be the local holomorphic coordinates on $V_\alpha$. We pull back these to $U_\alpha$ and still write the same. Let $x$ be the coordinate along the leaves with $T = \frac{\partial}{\partial x}$. Then we have the local coordinate $\{x, z^1, z^2, \cdots, z^n\}$ on $U_\alpha$ and $(D \otimes \mathbb{C})$ is spanned by the form

$$Z_\alpha = \left( \frac{\partial}{\partial z^\alpha} - \theta \left( \frac{\partial}{\partial z^\alpha} \right) T \right), \quad \alpha = 1, 2, \cdots, n.$$

Since $i(T) d\theta = 0$,

$$d\theta(Z_\alpha, \overline{Z_\beta}) = d\theta(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\overline{\partial z^\beta}}).$$

Then the Kähler 2-form $\omega_\alpha^T$ of the Hermitian metric $g_\alpha^T$ on $V_\alpha$, which is the same as the restriction of the Levi form $\frac{1}{2} d\theta$ to $\widetilde{D_\alpha^n}$, the slice $\{x = \text{constant}\}$ in $U_\alpha$, is closed. The collection of Kähler metrics $\{g_\alpha^T\}$ on $\{V_\alpha\}$ is so-called a transverse Kähler metric. We often refer to $\frac{1}{2} d\theta$ as the Kähler form of the transverse Kähler metric $g^T$in the leaf space $\widetilde{D^n}$. As an example, $\left\{ Z_\alpha = \frac{\partial}{\partial z^\alpha} + i \overline{z}^\alpha \frac{\partial}{\partial t} \right\}_{\alpha=1}^n$ is exactly a local frame on in the $(2n+1)$-dimensional Heisenberg group $\mathbf{H}_n = \mathbb{C}^n \times \mathbb{R}$. Here

$$\theta = dt + i \sum_{\alpha \in I_n} \left( z^\alpha d\overline{z}^\alpha - \overline{z}^\alpha dz^\alpha \right)$$

is a pseudohermitian contact structure on $\mathbf{H}_n$ and $T = \frac{\partial}{\partial t}$. In this case, $\widetilde{D^n} = \mathbb{C}^n$ and then

$$\dim_{\mathbb{C}} \mathcal{O}_d(\mathbb{C}^n) = \dim_{\mathbb{C}} \left( \mathcal{O}_d^{CR} (\mathbf{H}_n) \right). \tag{1.1}$$

Now, for a nontrivial function $f \in \mathcal{O}_d^{CR}(M)$ with $f(x) = 0$, we would define the vanishing order of $f$ at $x \in M$ by

$$ord_x(f) = \max \{ m \in \mathbb{N} \mid D^\alpha f = 0, \quad \forall \ |\alpha| < m \},$$

where $D^\alpha = \prod_{j \in I_n} Z_j^{\alpha_j}$ with $\alpha = (\alpha_1, \alpha_{2,\ldots,} \alpha_n)$.

Now we state our main theorem in this paper.

**Theorem 1.1.** *If $(M, J, \theta)$ is a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian bisectional curvature, then*

$$\dim_{\mathbb{C}} \left( \mathcal{O}_d^{CR}(M) \right) \leq \dim_{\mathbb{C}} \left( \mathcal{O}_d^{CR}(\mathbf{H}_n) \right) \tag{1.2}$$

*for each* $d \in \mathbb{N}$ *and* $\mathbf{H}_n = \mathbb{C}^n \times \mathbb{R}$ *is the* $(2n+1)$*-dimensional Heisenberg group.*

*Remark* 1.1. 1. In order to prove Theorem 1.1, it follows from (1.1) that it suffices to show that

$$\dim_{\mathbb{C}} \left( \mathcal{O}_d^{CR}(M) \right) \leq \dim_{\mathbb{C}} \mathcal{O}_d(\mathbb{C}^n) \tag{1.3}$$

in a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian bisectional curvature.

2. It is interesting to know whether $M$ is CR equivalent to the Heisenberg group $\mathbf{H}_n$ if the equality holds

$$\dim_{\mathbb{C}} \left( \mathcal{O}_d^{CR}(M) \right) = \dim_{\mathbb{C}} \left( \mathcal{O}_d^{CR}(\mathbf{H}_n) \right).$$

The method here is inspired mainly from [N] and [CFYZ] which is organized as follows. In Section 2, we introduce some basic notions about pseudohermitian manifolds and the necessary results for this paper. In Section 3, we show the existence of solutions to the CR heat equation with the initial condition under some appropriate assumptions. In Section 4, we prove the CR analogue of the rough dimension estimate. In Section 5, we show the CR sharp dimension estimate. In Appendix, we would give the complete proof of the $L^p$-submean value inequality which is a key step in showing the existence of solutions to the CR heat equation in Section 3.

## 2. Preliminaries

We introduce some basic materials about a pseudohermitian manifold (see [L] and [DT] for more details). Let $(M, \xi) = (M, J, \theta)$ be a $(2n+1)$-dimensional, orientable, contact manifold with the contact structure $\xi$. A CR structure compatible with $\xi$ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that $J$ satisfies the integrability condition: If $X$ and $Y$ are in $\xi$, then so are $[JX, Y]+[X, JY]$ and $J([JX, Y]+[X, JY]) = [JX, JY]-[X, Y]$.

Let $\{T, Z_\alpha, Z_{\bar\alpha}\}_{\alpha\in I_n}$ be a frame of $TM\otimes\mathbb{C}$, where $Z_\alpha$ is any local frame of $T_{1,0}M$, $Z_{\bar\alpha} = \overline{Z_\alpha} \in T_{0,1}M$ ,and $T$ is the characteristic vector field and $I_n = \{1, 2, ..., n\}$.Then $\{\theta, \theta^\alpha, \theta^{\bar\alpha}\}$, the coframe dual to $\{T, Z_\alpha, Z_{\bar\alpha}\}$, satisfies

$$d\theta = ih_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar\beta})$. If we have this contact structure, we call such $M$ a pseudohermitian $(2n+1)$-manifold or strictly pseudoconvex CR $(2n+1)$-manifold as well.

The Levi form $\langle\ ,\ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}M$ defined by

$$\langle Z, W\rangle_{L_\theta} = -i\left\langle d\theta, Z\wedge\overline{W}\right\rangle .$$

We can extend $\langle\ ,\ \rangle_{L_\theta}$ to $T_{0,1}M$ by defining $\left\langle\overline{Z}, \overline{W}\right\rangle_{L_\theta} = \overline{\langle Z, W\rangle}_{L_\theta}$ for all $Z, W \in T_{1,0}M$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}M$, denoted by $\langle\ ,\ \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over $M$ with respect to the volume form $d\mu = \theta\wedge(d\theta)^n$, we get an inner product on the space of sections of each tensor bundle.

The pseudohermitian connection of $(J, \theta)$ is the connection $\nabla$ on $TM\otimes\mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha \in T_{1,0}M$ by

$$\nabla Z_\alpha = {\omega_\alpha}^\beta \otimes Z_\beta, \quad \nabla Z_{\bar\alpha} = {\omega_{\bar\alpha}}^{\bar\beta}\otimes Z_{\bar\beta}, \quad \nabla T = 0,$$

where ${\omega_\alpha}^\beta$ are the 1-forms uniquely determined by the following equations :

$$\begin{cases} d\theta^\alpha + \omega^\alpha_\beta \wedge \theta^\beta = \theta\wedge\tau^\alpha \\ \tau_\alpha\wedge\theta^\alpha = 0 \\ \omega^\alpha_\beta + \omega^{\overline{\beta}}_{\overline{\alpha}} = 0 \end{cases} .$$

We can write (by Cartan lemma) $\tau_\alpha = A_{\alpha\gamma}\theta^\gamma$ with $A_{\alpha\gamma} = A_{\gamma\alpha}$. The curvature of Tanaka-Webster connection, expressed in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \theta^{\bar\alpha}\}$, is

$$\begin{cases} \Pi_\beta^\alpha = \overline{\Pi_{\overline{\beta}}^{\overline{\alpha}}} = d\omega_\beta^\alpha + \omega_\gamma^\alpha \wedge \omega_\beta^\gamma \\ \Pi_0^\alpha = \Pi_\alpha^0 = \Pi_0^{\overline{\alpha}} = \Pi_{\overline{\alpha}}^0 = \Pi_0^0 = 0 \end{cases}.$$

Webster showed that ${\Pi_\beta}^\alpha$ can be written

$$\Omega_\beta^\alpha = \Pi_\beta^\alpha + i\tau^\alpha \wedge \theta_\beta - i\theta^\alpha \wedge \tau_\beta = {R_\beta}^\alpha{}_{\gamma\overline{\delta}}\theta^\gamma \wedge \theta^{\overline{\delta}} + W_{\beta\gamma}^\alpha \theta^\gamma \wedge \theta - W_{\beta\overline{\gamma}}^\alpha \theta^{\overline{\gamma}} \wedge \theta$$

where the coefficients satisfy

$$R_{\beta\bar\alpha\rho\bar\sigma} = \overline{R_{\alpha\bar\beta\sigma\bar\rho}} = R_{\bar\alpha\beta\bar\sigma\rho} = R_{\rho\bar\alpha\beta\bar\sigma}, \quad W_{\beta\bar\alpha\gamma} = W_{\gamma\bar\alpha\beta}.$$

Here ${R_\beta}^\alpha{}_{\gamma\overline{\delta}}$ is the pseudohermitian curvature tensor, $R_{\alpha\bar\beta} = {R_\gamma}^\gamma{}_{\alpha\bar\beta}$ is the pseudohermitian Ricci curvature tensor and $A_{\alpha\beta}$ is the pseudohermitian torsion. $R = h^{\alpha\overline{\beta}}R_{\alpha\bar\beta}$ denotes the pseudohermitian scalar curvature. Besides, we define the pseudohermitian bisectional curvature

$$R_{\alpha\bar\alpha\beta\overline{\beta}}(X,Y) = R_{\alpha\bar\alpha\beta\overline{\beta}}X_\alpha X_{\overline{\alpha}}Y_\beta Y_{\bar\beta},$$

the bitorsion tensor

$$T_{\alpha\overline{\beta}}(X,Y) = \frac{1}{i}(A_{\alpha\gamma}X^\gamma Y_{\overline{\beta}} - A_{\overline{\beta}\overline{\gamma}}X^{\overline{\gamma}}Y_\alpha),$$

the torsion tensor

$$Tor\,(X,Y) = tr\left(T_{\alpha\overline{\beta}}\right) = \frac{1}{i}\left(A_{\alpha\beta}X^\beta Y^\alpha - A_{\overline{\alpha}\overline{\beta}}X^{\overline{\beta}}Y^{\overline{\alpha}}\right),$$

and the tensor

$$(\operatorname{div} A)^2(X,Y) = A_{\alpha\gamma,\overline{\gamma}}A_{\overline{\beta}\overline{\delta},\delta}X^\alpha Y^{\overline{\beta}}$$

where $X = X^\alpha Z_\alpha, Y = Y^\alpha Z_\alpha$ in $T_{1,0}M$.

We will denote the components of the covariant derivatives with indices preceded by comma. The indices $\{0, \alpha, \bar\alpha\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_{\bar\alpha}\}$. For derivatives of a real-valued/complex-valued function, we will often omit the comma, for instance, $u_\alpha =$

$Z_\alpha u,\ u_{\alpha\bar{\beta}} = Z_{\bar{\beta}} Z_\alpha u - \omega_\alpha{}^\gamma (Z_{\bar{\beta}}) Z_\gamma u$. The subgradient $\nabla_b \varphi$ of a smooth real-valued function $\varphi$ is defined by

$$\langle \nabla_b \varphi, Z \rangle_{L_\theta} = Z\varphi$$

for $Z \in \Gamma(\xi)$ where $\Gamma(\xi)$ denotes the family of all smooth vector fields tangent to the conact plane $\xi$. We could locally write the subgradient $\nabla_b \varphi$ as

$$\nabla_b u = u_{\bar{\alpha}} Z_\alpha + u_\alpha Z_{\bar{\alpha}}.$$

Accordingly, we could define the subhessian $Hess_b$ as the complex linear map

$$Hess_b : T_{1,0}M \oplus T_{0,1}M \longrightarrow T_{1,0}M \oplus T_{0,1}M$$

by

$$(Hess_b \varphi)\, Z = \nabla_Z \nabla_b \varphi$$

for $Z \in \Gamma(\xi)$ and a smooth real-valued function $\varphi$.

Next we recall the following commutation relations. (see [L]) Let $\varphi$ be a smooth real-valued function, $\sigma = \sigma_\alpha \theta^\alpha$ be a $(1,0)$-form and $\varphi_0 = T\varphi$, then we have

$$\left\{ \begin{array}{ccl} \varphi_{\alpha\beta} & = & \varphi_{\beta\alpha}, \\ \varphi_{\alpha\bar{\beta}} - \varphi_{\bar{\beta}\alpha} & = & ih_{\alpha\overline{\beta}} \varphi_0 \\ \varphi_{0\alpha} - \varphi_{\alpha 0} & = & A_{\alpha\beta} \varphi^\beta \\ \sigma_{\alpha,\beta\gamma} - \sigma_{\alpha,\gamma\beta} & = & i\left(A_{\alpha\gamma}\sigma_\beta - A_{\alpha\beta}\sigma_\gamma\right) \\ \sigma_{\alpha,\overline{\beta}\overline{\gamma}} - \sigma_{\alpha,\overline{\gamma}\overline{\beta}} & = & -i\left(h_{\alpha\overline{\gamma}} A^{\delta}_{\overline{\beta}} \sigma_\delta - h_{\alpha\overline{\beta}} A^{\delta}_{\overline{\gamma}} \sigma_\delta\right) \\ \sigma_{\alpha,\beta\overline{\gamma}} - \sigma_{\alpha,\overline{\gamma}\beta} & = & ih_{\beta\overline{\gamma}} \sigma_{\alpha,0} + R_\alpha{}^\delta{}_{\beta\overline{\gamma}} \sigma_\delta \\ \sigma_{\alpha,0\beta} - \sigma_{\alpha,\beta 0} & = & A^{\overline{\gamma}}{}_\beta \sigma_{\alpha,\bar{\gamma}} - A_{\alpha\beta,\bar{\gamma}} \sigma^{\overline{\gamma}} \\ \sigma_{\alpha,0\bar{\beta}} - \sigma_{\alpha,\bar{\beta}0} & = & A^\gamma{}_{\bar{\beta}} \sigma_{\alpha,\gamma} + A_{\bar{\gamma}\bar{\beta},\alpha} \sigma^{\overline{\gamma}} \end{array} \right. .$$

Last, we would introduce the concept of the adapted metric. A family of Webster adapted metrics $h_\epsilon$ of a strictly pseudoconvex CR $(2n+1)$-manifold $(M, J, \theta)$ are the Riemannian metrics

$$h_{\epsilon} = h + \frac{1}{\epsilon^2}\theta^2 \tag{2.1}$$

where $h = \langle\ ,\ \rangle_{L_\theta}$ is the Levi metric. In [CC], they show that

$$\Delta_\epsilon g = 2\Delta_b g + \epsilon^2 T^2 g$$

for any real-valued smooth function $g$ and

$$d\mu_\epsilon = \frac{1}{\epsilon 2^n n!} d\mu$$

where $d\mu\,(y) = dy = \theta \wedge (d\theta)^n$ is a volume form on $(M, J, \theta)$. Here $\Delta_\epsilon$ and $d\mu_\epsilon$ are the Riemannian Laplace operator and the Riemannian volume element with respective to the adapted Riemannian metric $h_\epsilon$, respectively. In this paper, we choose $\epsilon = \frac{1}{2^{n-1}n!}$such that $d\mu_\epsilon = \frac{1}{2}d\mu$.

## 3. The CR heat equation

In this section, we will derive the essential fact about the existence of solutions to the CR heat equation on a complete noncompact pseudohermitian $(2n+1)$-manifold. First of all, we give the following lemmas so that one can establish the existence of solutions to the CR heat equation. It is also very useful to study the CR Poincaré-Lelong equation ([CCHL]).

We will use the semigroup method in [BBGM]. It is known that the heat semigroup $(P_t)_{t\geq 0}$ is given by

$$P_t = \int_0^\infty e^{-\lambda t} dE_\lambda$$

for the spectral decomposition of $\Delta_b = -\int_0^\infty \lambda dE_\lambda$ in $L^2\,(M)$. It is a one-parameter family of bounded operators on $L^2\,(M)$ . We denote

$$P_t f\,(x) = \int_M H\,(x, y, t)\, f\,(y)\, d\mu\,(y)\,.$$

Here $H(x,y,t) > 0$ is the so-called symmetric heat kernel associated to $P_t$. Due to hypoellipticity of $\Delta_b$, the function $(x,t) \to P_t f(x)$ is smooth on $M \times (0,\infty)$, $f \in C_0^\infty(M)$. Moreover

$$u(x,t) = P_t f(x)$$

is a solution of CR heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_b u \\ u(x,0) = f(x) \end{cases}.$$

Under these settings, we're going to prove some results derived by estimating the CR heat kernel $H(x,y,t)$. In the following, $V_x(r)$ denotes the volume of the geodesic ball $B_x(r)$ with respect to the Carnot-Carathéodory distance $r(x,y)$ between $x$ and $y$, and $r(x) = r(x,o)$ where $o \in M$ is a fixed point.

**Theorem 3.1.** *Let $(M,J,\theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold with nonnegative pseudohermitian Ricci curvature tensors and vanishing pseudohermitian torsion. If $f$ is a continuous function on $M$ such that*

$$\frac{1}{V(B_o(r))} \int_{B_o(r)} |f|(x)dx \leq exp(ar^2+b)$$

*for some positive constants $a > 0$ and $b > 0$, then the following initial value problem*

$$\begin{cases} (\frac{\partial}{\partial t} - \triangle_b)v(x,t) = 0 \\ v(x,0) = f(x) \end{cases} \tag{3.1}$$

*has a solution on $M \times (0, \frac{C}{16a}]$; moreover,*

$$v(x,t) = \int_M H(x,y,t)f(y)dy,$$

*where $H(x,y,t)$ is the Heat kernel on $(M,J,\theta)$.*

The proof is based on the effect estimate on the CR heat kernel, CR volume doubling property and $L^p$ submean value inequality which do not need the stronger assumption on vanishing pseudohermitian torsion as in Proposition 3.1. However, for the later purpose as in the following sections, we do need the condition of vanishing torsion, in which the heat kernel estimate is obtained also in [BBGM]. Then here we focus on the case of a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion only.

**Proposition 3.1.** *(*[CCHT], [BBGM]*) Let $(M, J, \theta)$ be a complete pseudohermitian $(2n+1)$-manifold with the pseudohermitian Ricci curvature*

$$Ric\,(X,X) \geq k_0 \left\langle X, X\right\rangle_{L_\theta}$$

*and*

$$\sup_{i,j\in I_n} |A_{ij}| \leq k_1 < \infty \text{ and } \sup_{i,j\in I_n} |A_{ij,\bar{i}}|^2 \leq k_2 < \infty,$$

*for $X = X^\alpha Z_\alpha \in T_{1,0}M$ and $k_0,\ k_1,\ k_2$ are constants with $k_1,\ k_2 \geq 0$. Then*

*(i) There exist positive constants $C_3, C_4, C_5$ such that for $x,\ y \in M,\ t > 0$*

$$H\,(x,y,t) \leq \frac{C_3}{V_x\left(\sqrt{t}\right)^{\frac{1}{2}} V_y\left(\sqrt{t}\right)^{\frac{1}{2}}} \exp\Big(-C_5 \frac{d_{cc}^2\,(x,y)}{t} + C_4\kappa t\Big). \tag{3.2}$$

*(ii) There exist positive constants $C_6, C_7, C_8$ such that for $x,\ y \in M,\ t > 0$*

$$H(x,y,t) \geq \frac{C_6}{V_x\left(\sqrt{t}\right)} \exp\Big[-C_7 \frac{{d_{cc}}^2\,(x,y)}{t} - C_8\kappa(t + d_{cc}^2(x,y))\Big]. \tag{3.3}$$

*(iii) There exist positive constants $C_9, C_{10}$ such that for $0 < s < t$*

$$\frac{H(x,x,s)}{H(x,x,t)} \leq \left(\frac{t}{s}\right)^{C_9} e^{C_{10}\kappa(t-s)}. \tag{3.4}$$

*(iv) (CR Volume Doubling Property) For any $\sigma > 1$, then there exist a positive constant $C_1$ such that*

$$V_x\left(\sigma\rho\right) \leq C_1 \sigma^{2C_9} e^{(C_1\sigma^2 + C_8)\kappa\rho^2} V_x\left(\rho\right).$$

*Here $\kappa = \kappa(k_0, k_1, k_2) \geq 0$.*

**Lemma 3.1.** *Let $(M, J, \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian Ricci curvature tensors. Assume that $u$ is defined by*

$$u(x,t) = \int_M H(x,y,t) f(y) dy,$$

*on $M \times [0,T]$ for some $T > 0$. Here $f$ is a nonnegative function with*

$$\lim_{r \to +\infty} exp(-\frac{C(n)\, r^2(x)}{4t}) \int_{B_o(r)} f(x) dx = 0. \tag{3.5}$$

*Then for any $0 < t \leq r^2$, and $p \geq 1$,*

$$\begin{aligned} \tfrac{1}{V(B_o(r))} \textstyle\int_{B_o(r)} u^p(x,t) dx \;&\leq\; C(n)[\tfrac{1}{V(B_o(4r))} \textstyle\int_{B_o(4r)} f^p(x) dx \\ &\quad + (\tfrac{1}{t} \textstyle\int_{4r}^{\infty} exp(-\tfrac{C(n)s^2}{8t}) s (\tfrac{1}{V(B_o(s))} \textstyle\int_{B_o(s)} f(x) dx) ds)^p]. \end{aligned} \tag{3.6}$$

*Proof.* For any $p \geq 1$ and $r \geq \sqrt{t}$,

$$\begin{aligned} \textstyle\int_{B_o(r)} u^p(x,t) dx &= \textstyle\int_{B_o(r)} (\int_M H(x,y,t) f(y) dy)^p dx \\ &\leq 2^{p-1} [\textstyle\int_{B_o(r)} (\int_{B_o(4r)} H(x,y,t) f(y) dy)^p dx \\ &\quad + \textstyle\int_{B_o(r)} (\int_{M \backslash B_o(4r)} H(x,y,t) f(y) dy)^p dx]. \end{aligned} \tag{3.7}$$

We first will estimate the second term on the right hand side in (3.7). Now for $x \in B_o(r)$ and $y \in M \backslash B_o(4r)$, we have $r(x,y) \geq \frac{3}{4} r(y)$. By the estimates of the CR heat kernel as in Proposition 3.1, we have

$$\begin{aligned} & \textstyle\int_{M \backslash B_o(4r)} H(x,y,t) f(y) dy \\ \leq\; & C_1 \textstyle\int_{M \backslash B_o(4r)} \frac{1}{V(B_x(\sqrt{t}))} exp(-\frac{C_2 r^2(x,y)}{2t}) f(y) dy \\ \leq\; & C_1 \textstyle\int_{M \backslash B_o(4r)} \frac{1}{V(B_x(r+\sqrt{t}))} (\frac{r+\sqrt{t}}{\sqrt{t}})^{2C_9} exp(-\frac{C_2 r^2(x,y)}{2t}) f(y) dy \\ \leq\; & \tfrac{C_1}{V(B_o(\sqrt{t}))} \textstyle\int_{M \backslash B_o(4r)} (\frac{r+\sqrt{t}}{\sqrt{t}})^{2C_9} exp(-\frac{C_2 r^2(x,y)}{2t}) f(y) dy \\ \leq\; & \tfrac{C_1}{V(B_o(\sqrt{t}))} (\tfrac{r+\sqrt{t}}{\sqrt{t}})^{2C_9} \textstyle\int_{4r}^{\infty} [exp(-\frac{C_2 s^2}{4t}) (\int_{\partial B_o(s)} f)] ds \\ \leq\; & \tfrac{C_1}{V(B_o(\sqrt{t}))} (\tfrac{r+\sqrt{t}}{\sqrt{t}})^{2C_9} \textstyle\int_{4r}^{\infty} [exp(-\frac{C_2 s^2}{4t}) (\int_{B_o(s)} f)] d(\frac{s^2}{t}) \\ \leq\; & C_1 (\tfrac{r+\sqrt{t}}{\sqrt{t}})^{2C_9} \textstyle\int_{4r}^{\infty} [\frac{V(B_o(s))}{V(B_o(\sqrt{t}))} exp(-\frac{C_2 s^2}{4t}) (\frac{1}{V(B_o(s))} \int_{B_o(s)} f)] d(\frac{s^2}{t}) \\ \leq\; & C_1 \textstyle\int_{4r}^{\infty} (\frac{s}{\sqrt{t}})^{4C_9} exp(-\frac{C_2 s^2}{4t}) (\frac{1}{V(B_o(s))} \int_{B_o(s)} f) d(\frac{s^2}{t}) \\ \leq\; & C \tfrac{1}{t} \textstyle\int_{4r}^{\infty} exp(-\frac{C_2 s^2}{8t}) s (\frac{1}{V(B_o(s))} \int_{B_o(s)} f) ds. \end{aligned} \tag{3.8}$$

where $C_9$ is the constant as in Proposition 3.1. Here, besides utilizing the inequality $r(x,y) \geq \frac{3}{4}r(y)$, we have used the volume doubling property and the assumption (3.5) when performing integration by parts in the fifth inequality. Note that when separating $M \backslash B_o(4r)$ into the shells, the concept of the Legendrian normal has come into our case (see [CCW]). From now on, once we deal with such process of the integration, we keep the idea in mind.

As for the first term on the right hand side in (3.7), by Hölder's inequality and the fact ([BBGM]) that

$$\int_M H(x,y,t)dy = 1,$$

we have

$$\left(\int_{B_o(4r)} H(x,y,t)f(y)dy\right)^p \leq \int_{B_o(4r)} H(x,y,t)f^p(y)dy.$$

Hence

$$\begin{aligned} \int_{B_o(r)}\left(\int_{B_o(4r)} H(x,y,t))f(y)dy\right)^p dx &\leq \int_{B_o(r)}\int_{B_o(4r)} H(x,y,t)f^p(y)dydx \\ &\leq \int_{B_o(4r)} f^p(y)\left(\int_{B_o(r)} H(x,y,t)dx\right)dy \\ &\leq \int_{B_o(4r)} f^p(y)dy. \end{aligned} \tag{3.9}$$

From (3.7)-(3.9), we know that the Lemma holds. □

From the preceding lemma, we could control the upper bound of subsolutions to the power of $p$ of the CR heat equation by its $L^p$-norm under the same hypotheses. On the other hand, on the course of proving the existence of solutions to the CR heat equation, we also need the $L^p$ submean value inequality for $p \in (0,\infty)$, which will be shown in the Appendix.

**Proposition 3.2.** *(CR $L^p$ submean value inequality ) Let $(M,J,\theta)$ be a complete pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian Ricci curvature tensors. Let $Q(\tau,x,r,s) = (s-\tau r^2, s) \times B_x(r)$ and $Q_\delta(\tau,x,r,s) = (s-\delta\tau r^2, s) \times B_x(\delta r)$ for $\tau > 0$, $x \in M$, $r > 0$, $s \in R$, $\delta \in (0,1)$. If $\tau > 0$ and $p \in (0,+\infty)$ are given, then there exists a constant $A(\tau,\nu,p) > 0$ such that if $u$ is a positive subsolution to the CR heat* *equation* $(\frac{\partial}{\partial t}-\triangle_b)u\leq 0$ *in* $Q(\tau,x,r,s)$*, then, for* $0<\delta<\delta'\leq 1$,

$$\sup_{Q_\delta}\{u^p\}\leq\frac{A(\tau,\nu,p)}{(\delta'-\delta)^{2+\nu}r^2V\left(B_x(r)\right)}\int_{Q_{\delta'}}u^p d\overline{\mu},$$

*where* $d\overline{\mu}=d\mu dt$. *Here the constant* $\nu$ *is the exponential constant in CR Sobolev inequality (*[CCHT]*)*

$$(3.10)\qquad (\int_{B_x(\rho)}|\varphi|^{\frac{2\nu}{\nu-2}}d\mu)^{\frac{\nu-2}{\nu}}\leq C_s\rho^2V\left(B_x(r)\right)^{-\frac{2}{\nu}}[\int_{B_x(\rho)}|\nabla_b\varphi|^2d\mu+\rho^{-2}\int_{B_x(\rho)}\varphi^2d\mu]$$

*for any* $\varphi\in C_c^\infty(B_x(\rho))$*,* $x\in M$*.*

Now we are ready to prove **Theorem 3.1 :**

*Proof.* For all $j\geq 1$, let $0\leq\varphi_j\leq 1$ be a smooth cut-off function such that $\varphi_j=1$ on $B_o(j)$ and $\varphi_j=0$ on $M\backslash B_o(2j)$. Let $f_j=\varphi_j f$ be continuous with compact support. Hence one can solve (3.1) with the initial value $f_j$ for all time. The solution $v_j$ is given by

$$v_j(x,t)=\int_M H(x,y,t)f_j(y)dy$$

for $(x,t)\in M\times(0,\infty)$. The existence and uniqueness of such $v_j$ of the form could be found in [Li]. By Lemma 3.1, for $0<t\leq\min\left\{r^2,\frac{C}{16a}\right\}$, we have

$$\begin{array}{rcl}\frac{1}{V(B_o(r))}\int_{B_o(r)}|v_j|dx & \leq & \frac{1}{V(B_o(r))}\int_{B_o(r)}\int_M H(x,y,t)|f_j(y)|dydx\\ & \leq & C[\frac{1}{V(B_o(4r))}\int_{B_o(4r)}|f_j(y)|dy\\ & & +\frac{1}{t}\int_{4r}^\infty exp(-\frac{Cs^2}{8t})s(\frac{1}{V(B_o(s))}\int_{B_o(s)}f_j)ds]\\ & \leq & Ce^b[exp(16ar^2)+\int_{4r}^\infty exp(-\frac{Cs^2}{16t})d(\frac{s^2}{t})]\\ & \leq & Ce^b\left[exp(16ar^2)+1\right].\end{array}$$

Assume $\frac{C}{16a} \leq R^2$. Let $(\tau, x, r, s, p, \delta, \delta') = (2, o, R, \frac{C}{16a}, 1, \frac{1}{2}, 1)$ in the CR $L^p$ submean value inequality, and we know that $|v_j|$ is a subsolution to the CR heat equation, then we have

$$\begin{aligned}
\sup_{B_o(\frac{R}{2})\times(0,\frac{C}{16a})} |v_j| &\leq \frac{A(2,\nu,1)}{\left(\frac{1}{2}\right)^{2+\nu} R^2 V(B_o(R))} \int_{Q_1} |v_j|\, d\overline{\mu} \\
&\leq 2^{3+\nu} A\left(2,\nu,1\right) \frac{1}{V(B_o(R))} \int_{B_o(R)} |v_j| dx \\
&\leq Cexp(16aR^2 + b).
\end{aligned}$$

From this, it's easy to see that, after passing to a subsequence, $\{v_j\}_{j\in\mathbb{N}}$ together with their derivatives uniformly converge on compact sets on $M \times (0, \frac{C}{16a})$ to a solution $v$ of the CR heat equation. Moreover, for any $(x,t) \in M \times (0, \frac{C}{16a})$ as in (3.8), we have

$$\begin{aligned}
\left|\int_M H(x,y,t)dy - v_j(x,t)\right| &\leq \left|\int_M H(x,y,t)(f(y) - f_j(y))dy\right| \\
&\leq \int_{M\setminus B_o(j)} H(x,y,t)|f\left(y\right)|dy \\
&\leq C\int_j^\infty exp(-\tfrac{Cs^2}{16t})d(\tfrac{s^2}{t}) \\
&\leq C\int_{\frac{j^2}{t}}^\infty exp(-\tfrac{C\tau}{16})d\tau.
\end{aligned}$$

Let $j$ tend to $+\infty$, we have

$$v(x,t) = \int_M H(x,y,t)f(y)dy.$$

This completes the proof. $\square$

## 4. Rough Dimension Estimates

After settling the existence of solutions to the CR heat equation, we will give the asymptotic behavior of the solutions. It's a crucial step on the course of proving the sharp dimension estimate. And it's worth to note that because of the following lemma, we could drop the assumption that $M$ is of maximum volume growth in the hypotheses of the sharp dimension estimate. That's why the statement about the sharp dimension estimate in [CFYZ] could hold without the assumption of the maximum volume growth as in [N].

**Lemma 4.1.** *Let $(M, J, \theta)$ be a complete pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian Ricci curvature tensors. Assume that $v(x,t)$ is a solution to the CR heat equation*

$$(\frac{\partial}{\partial t} - \Delta_b) v(x,t) = 0$$

*on $M \times (0, +\infty)$ with the initial condition*

$$v(x,0) = 2 \log |f(x)|,$$

*where $f \in \mathcal{O}_d^{CR}(M)$. Then*

$$\limsup_{t \to +\infty} \frac{v(x,t)}{\log t} \leq d. \tag{4.1}$$

*Proof.* From the fact $f \in \mathcal{O}_d^{CR}(M)$, we see that for all $\epsilon > 0$, there's a constant $C' = C'(f, x, d, \epsilon) > 0$ such that

$$|f(y)| \leq C'(1 + r(x,y)^{d+\epsilon}).$$

Hereafter, in this proof, we would denote the positive constants by $C' = C'(f, x, d, \epsilon)$ and $C = C(n, d, \epsilon)$. These constants may be different line by line.

Because

$$\frac{1}{V(B_o(r))} \int\limits_{B_o(r)} \log |f(x)| \, dx \leq exp(ar^2 + b)$$

for any $a > 0$, then, from Theorem 3.1, we know the solution $v(x,t)$ has the form

$$v(x,t) = 2 \int\limits_M H(x,y,t) \log |f(y)| \, dy.$$

Hence

$$\begin{aligned}
v(x,t) &= 2\int_M H(x,y,t)\log|f(y)|\,dy \\
&= 2\Big[\int_{\left[r(x,y)\le\sqrt{t}\right]} H(x,y,t)\log|f(y)|\,dy + \int_{\left[r(x,y)>\sqrt{t}\right]} H(x,y,t)\log|f(y)|\,dy\Big] \\
&\le \int_{\left[r(x,y)\le\sqrt{t}\right]} H(x,y,t)\left((d+\epsilon)\log t + C'\right)dy \\
&\quad + \int_{\left[r(x,y)>\sqrt{t}\right]} H(x,y,t)\left(2(d+\epsilon)\log r(x,y) + C'\right)dy
\end{aligned}$$

for $t>1$. By the CR heat kernel estimate and the CR volume doubling property in [CCHT] and [BBGM], we obtain

$$\begin{aligned}
& v(x,t) \\
\le\;& (d+\epsilon)\log t + C' + \int_{\left[r(x,y)>\sqrt{t}\right]} (d+\epsilon)\,H(x,y,t)\log\left(\tfrac{r(x,y)^2}{t}\right)dy \\
\le\;& C\int_{\left[r(x,y)>\sqrt{t}\right]} \tfrac{d+\epsilon}{V\left(B_x\left(\sqrt{t}\right)\right)}\exp\left(-C\tfrac{r(x,y)^2}{2t}\right)\log\left(\tfrac{r(x,y)^2}{t}\right)dy + (d+\epsilon)\log t + C' \\
\le\;& C\sum_{k\ge 0}\int_{\left[2^k\sqrt{t}<r(x,y)\le 2^{k+1}\sqrt{t}\right]} \tfrac{d+\epsilon}{V\left(B_x\left(\sqrt{t}\right)\right)}\exp\left(-C\tfrac{r(x,y)^2}{2t}\right)\log\left(\tfrac{r(x,y)^2}{t}\right)dy + (d+\epsilon)\log t + C' \\
\le\;& C(d+\epsilon)\sum_{k\ge 0}\left[2^{2(k+1)C_9}\exp\left(-2^{2k-1}C\right)2(k+1)\right] + (d+\epsilon)\log t + C' \\
\le\;& (d+\epsilon)\log t + C' + C.
\end{aligned}$$

Here the CR volume doubling property is used in the fourth inequality. Then we are done. $\square$

Now we give another lemma which is also essential for the proof of the rough dimension estimate. In the proof of the following lemma, we would find that it's closely related to the CR moment-type estimate as in [CF].

**Lemma 4.2.** *Let $(M,J,\theta)$ be a complete pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian Ricci curvature tensors. Assume that $v(x,t)$ is a* *solution to the CR heat equation*

$$\left(\frac{\partial}{\partial t}-\Delta_b\right)v\left(x,t\right)=0 \tag{4.2}$$

*on* $M\times(0,+\infty)$ *with the initial condition*

$$v\left(x,0\right)=2\log\left|f\left(x\right)\right| \tag{4.3}$$

*for* $f\in\mathcal{O}_d^{CR}\left(M\right)$*, and*

$$w\left(x,t\right)=\frac{\partial}{\partial t}v\left(x,t\right).$$

*Then there exists a positive constant* $C$ *such that*

$$C(n)ord_xf\leq\lim_{t\to0^+}tw\left(x,t\right). \tag{4.4}$$

*Proof.* On account of the equation (4.2) and the initial condition (4.3), we have

$$w\left(x,t\right)=2\int_M H(x,y,t)\Delta_b\log\left|f\left(y\right)\right|dy.$$

by the uniqueness theorem in [Do]. So by the CR heat kernel estimate in [CCHT], we know

$$\begin{aligned} w\left(x,t\right) &= 2\int_M H(x,y,t)\Delta_b\log\left|f\left(y\right)\right|dy \\ &\geq \frac{C}{V\left(B_x\left(\sqrt{t}\right)\right)}\int_M \exp(-C\tfrac{d(x,y)^2}{t})\Delta_b\log\left|f\left(y\right)\right|dy \\ &\geq \frac{C}{V\left(B_x\left(\sqrt{t}\right)\right)}\int_{\left[0,\sqrt{t}\right]}\exp\left(-C\tfrac{r^2}{t}\right)\left(\int_{\partial B_x(r)}\Delta_b\log\left|f\left(s\right)\right|d\sigma\left(s\right)\right)dr \\ &\geq \frac{C}{V\left(B_x\left(\sqrt{t}\right)\right)}\int_{B_x\left(\sqrt{t}\right)}\Delta_b\log\left|f\left(y\right)\right|dy. \end{aligned} \tag{4.5}$$

Furthermore, from the local property of the Sasakian manifolds and $Tf = 0$, we have, as $t << 1$,

$$\begin{aligned} & \frac{C}{V\left(B_x\left(\sqrt{t}\right)\right)} \int\limits_{B_x\left(\sqrt{t}\right)} \Delta_b \log\left|f\left(y\right)\right| dy \\ = & \frac{C}{V\left(B_x\left(\sqrt{t}\right)\right)} \int_{-\sqrt{t}}^{\sqrt{t}} \int_{B_{\widetilde{x}}\left(\sqrt{t-l^2}\right)} \Delta_{\widetilde{y}} \log\left|f\left(\left(\widetilde{y}, t\right)\right)\right| d\widetilde{y} dl \\ \geq & \frac{C}{t^{\frac{2n+1}{2}}} ord_{\widetilde{x}}\left(f\right) \int_{-\sqrt{t}}^{\sqrt{t}} \frac{V\left(B_{\widetilde{x}}\left(\sqrt{t-l^2}\right)\right)}{\left(t-l^2\right)} dl \quad \text{by } (4.7) \\ = & \frac{C}{t^{\frac{2n+1}{2}}} ord_x\left(f\right) \int_{-\sqrt{t}}^{\sqrt{t}} \frac{V\left(B_{\widetilde{x}}\left(\sqrt{t-l^2}\right)\right)}{\left(t-l^2\right)} dl \\ \geq & \frac{C}{t^{\frac{2n+1}{2}}} ord_x\left(f\right) \int_{-\sqrt{t}}^{\sqrt{t}} \left(t-l^2\right)^{n-1} dl \\ = & \frac{C}{t^{\frac{2n+1}{2}}} t^{n-\frac{1}{2}} ord_x\left(f\right) \\ = & \frac{C}{t} ord_x\left(f\right). \end{aligned} \tag{4.6}$$

Here we have used the identity

$$ord_{\widetilde{x}}\left(f\right) = \frac{1}{2n} \lim_{r \to 0^+} \frac{r^2}{V\left(B_{\widetilde{x}}\left(r\right)\right)} \int\limits_{B_{\widetilde{x}}(r)} \widetilde{\Delta} \log\left|f(\widetilde{y})\right| d\widetilde{y} \tag{4.7}$$

where $\widetilde{\Delta}$ denotes the Laplace operator on the slice $\widetilde{D^n}$ with the trasversal Kähler structure. Also we adopt the adapted metric and

$$ord_x\left(f\right) = ord_{\widetilde{x}}\left(f\right),$$

which comes from the fact $Z_\alpha f = \frac{\partial f}{\partial z^\alpha}$, $f_0 = 0$ and utilize the equality

$$\int_{-s}^{s} \left(s^2 - l^2\right)^{n-1} dl = s^{2n-1} \sqrt{\pi} \frac{\Gamma\left(n\right)}{\Gamma(n + \frac{1}{2})} \tag{4.8}$$

in the third equality of (4.6) with $s=\sqrt{t}$ which is derived as follows :

$$\begin{aligned}
\int_{-s}^{s}\left(s^2-l^2\right)^{n-1}dl &= s^{2n-1}\int_{-1}^{1}\left(1-x^2\right)^{n-1}dx \\
&= 2s^{2n-1}\int_{0}^{1}\left(1-x^2\right)^{n-1}dx \\
&= s^{2n-1}\int_{0}^{1}\left(1-z\right)^{n-1}\tfrac{1}{\sqrt{z}}dz \\
&= s^{2n-1}B(\tfrac{1}{2},n) \\
&= s^{2n-1}\tfrac{\Gamma\left(\frac{1}{2}\right)\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)} \\
&= s^{2n-1}\sqrt{\pi}\tfrac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)},
\end{aligned}$$

where $B\left(z,w\right)$ and $\Gamma\left(z\right)$ denote the beta function and the gamma function respectively.

This completes the proof of Lemma 4.2. □

Prior to showing the sharp dimension estimate, we would obtain the dimension estimates of the rough version as follows :

**Theorem 4.1.** *Let $(M,J,\theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian Ricci curvature tensors. Then there exists a positive constant $C\left(n\right)$ such that*

$$\dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}\left(M\right)\right)\leq C\left(n\right)d^n. \tag{4.9}$$

*Proof.* We adopt the notations in the proof of Lemma 4.2. Set $w\left(x,t\right)=\frac{\partial}{\partial t}v\left(x,t\right)$. By the similar deductions in [CF, Lemma 4.2.], replaced the moment-type estimate in [CF, (3.10) and (5.12)] by (4.1), we came out with [CF, (5.14)] and then

$$\frac{\partial}{\partial t}\left(tw\left(x,t\right)\right)\geq 0. \tag{4.10}$$

In fact, we could check the hypotheses of Theorem 4.1 in [CF] from the upper bound of $v_{\alpha\overline{\beta}}\left(x,t\right)$ by $Cd\log t$ as in Lemma 4.1.

We first show that

$$ord_x f\leq Cd. \tag{4.11}$$

Basically, if we could prove

$$tw(x,t) \leq Cd \tag{4.12}$$

for $t > 1$, then we have (4.11) by Lemma 4.2. We observe that on account of the definition of $\mathcal{O}_d^{CR}(M)$ for all $x \in M$, there exists a constant $\widetilde{C} = \widetilde{C}(f,x,d,\epsilon) > 0$ such that

$$\log^+ |f(y)| \leq \widetilde{C} + d(\log(1 + d(x,y))).$$

Accordingly we have

$$\begin{aligned}
& v(x,t) \\
\leq\ & \frac{C}{V(B_x(\sqrt{t}))} \int_0^{+\infty} [\exp\left(-C\frac{r^2}{2t}\right) (\int\limits_{\partial B_x(r)} \log^+ |f(s)|\, d\sigma(s))] dr \\
\leq\ & \frac{C}{V(B_x(\sqrt{t}))} \int_{\sqrt{t}}^{+\infty} [\exp\left(-C\frac{r^2}{2t}\right) (\int\limits_{\partial B_x(r)} \log^+ |f(s)|\, d\sigma(s))] dr + Cd\log(1+t) \\
\leq\ & \frac{C}{V(B_x(\sqrt{t}))} \int_{\sqrt{t}}^{+\infty} [\exp\left(-C\frac{r^2}{2t}\right) \frac{Cr}{t} (\int\limits_{B_x(r)} \log^+ |f(y)|\, dy)] dr + Cd\log(1+t) \\
\leq\ & C\int_{\sqrt{t}}^{+\infty} [\left(\frac{r}{\sqrt{t}}\right)^{2C} \exp\left(-C\frac{r^2}{2t}\right) \frac{Cr}{t} d\log(1+r)] dr + Cd\log(1+t) \\
\leq\ & C\int_{\sqrt{t}}^{+\infty} [\frac{\tau^C}{t^{C+1}} \exp\left(-C\frac{\tau}{2t}\right) d\log(1+\tau)] d\tau + Cd\log(1+t) \\
\leq\ & C\int_0^{+\infty} \left[s^C \exp(-s)\, sd\log(1+t)\right] ds + Cd\log(1+t) \\
\leq\ & Cd\log(1+t) + Cd\log(1+t)\,\Gamma(C+2) \\
\leq\ & Cd\log t
\end{aligned} \tag{4.13}$$

for $t > 1$. From (4.13), we claim that the inequality (4.12) holds; for if there's some small positive constant $\epsilon$ and $t_0 > 1$ such that

$$tw(x,t) > (C+\epsilon)d$$

for $t > t_0$. Here we have utilized the monotonicity of $tw(x,t)$. Therefore, by integrating both sides, we have

$$v(x,t) \geq (C+\epsilon)d\log t - A$$

where the constant $A$ is independent of $t$. But this contradicts the inequality (4.13). So we obtain (4.12). As a result, we deduce (4.11).

Now we fix the constant $C$ chosen in (4.11) and may assume such constant is a positive integer. Next we want to settle the dimension estimate (4.9). In spite of the proof of (4.9) is the same as the one in [M1], we would write it down for completeness. Let $k(n)$ be a constant satisfying

$$q(m) = \left( \begin{array}{c} n+m \\ n \end{array} \right) < k(n)\, m^n.$$

Consider the map

$$\begin{aligned} \Phi : \mathcal{O}_d^{CR}(M) &\longrightarrow \mathbb{C}^{q(C)} \\ f &\longmapsto (D^\alpha f)_{|\alpha| \le Cd} \, . \end{aligned}$$

From (4.11), we see that $\Phi$ is injective. Suppose

$$\dim_{\mathbb{C}} \left( \mathcal{O}_d^{CR}(M) \right) > C' d^n$$

where $C'$ is chosen with $C' > k(n)\, C^n$, this implies that

$$\dim_{\mathbb{C}} \left( \mathcal{O}_d^{CR}(M) \right) > k(n)\,(Cd)^n > q(C)\, .$$

However, this contradicts with the fact that $\Phi$ is injective. We complete the proof. $\square$

## 5. Sharp Dimension Estimates

Subsequently, we give another lemma also substantial to the proof of the sharp dimension estimate. In contrast with Lemma 4.1, one could find the following lemma is the stronger version of that. In fact, that is why we could derive the more advanced result-the sharp dimension estimate than the one in the last section.

**Lemma 5.1.** *let $(M, J, \theta)$ be a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian bisectional curvature. Assume that*

*$v(x,t)$ is a solution to the CR heat equation*

$$\left(\frac{\partial}{\partial t}-\Delta_b\right)v(x,t)=0$$

*on $M\times(0,+\infty)$ with the initial condition*

$$v(x,0)=2\log|f(x)|$$

*for $f\in\mathcal{O}_d^{CR}(M)$, and*

$$w(x,t)=\frac{\partial}{\partial t}v(x,t).$$

*Then*

$$\lim_{t\to0^+}tw(x,t)=ord_xf. \tag{5.1}$$

*Proof.* From the definition of $w(x,t)$, we have

$$w(x,t)=2\int_M H(x,y,t)\Delta_b\log|f(y)|\,dy. \tag{5.2}$$

By the CR heat kernel estimate, the same computations of (4.5) in Lemma 4.1 give us the inequality

$$\frac{1}{V(B_x(r))}\int_{B_x(r)}\Delta_b\log|f(y)|\,dy\le Cw\left(x,r^2\right). \tag{5.3}$$

Combing (5.3) and (4.12), we obtain

$$\frac{1}{V(B_x(r))}\int_{B_x(r)}\Delta_b\log|f(y)|\,dy\le C\frac{d}{r^2} \tag{5.4}$$

for any $r>0$ by (4.10). By the equality

$$ord_{\widetilde{x}}(f)=\frac{1}{2n}\lim_{r\to0^+}\frac{r^2}{V(B_{\widetilde{x}}(r))}\int_{B_{\widetilde{x}}(r)}\widetilde{\Delta}\log|f|,$$

we know that for any $\epsilon>0$, there is $\delta>0$ such that

$$\left|2nord_{\widetilde{x}}(f)-\frac{r^2}{V(B_{\widetilde{x}}(r))}\int_{B_{\widetilde{x}}(r)}\widetilde{\Delta}\log|f|\right|<\frac{\epsilon}{6} \tag{5.5}$$

for $0<r<\delta<1$. Now we separate the integration in (5.2) into two parts:

$$\begin{aligned} & tw(x,t) \\ = \; & 2t\int_M H(x,y,t)\Delta_b \log|f(y)|\,dy \\ = \; & 2t[\int_{B_x(\delta)^c} H(x,y,t)\Delta_b \log|f(y)|\,dy + \int_{B_x(\delta)} H(x,y,t)\Delta_b \log|f(y)|\,dy]. \end{aligned} \tag{5.6}$$

So if we could show that (I) the first integration is close to zero and (II) the second one is close to $ord_x(f)$ as $t\to 0^+$ in the last line of (5.6), then (5.1) holds.

(I). Now we're going to show the first integration in (5.6) goes to zero for sufficiently small $t$ as follows: By CR heat kernel estimate, we have, for $t\leq\delta^2$,

$$\begin{aligned} & 2t\int_{B_x(\delta)^c} H(x,y,t)\Delta_b \log|f(y)|\,dy \\ \leq \; & 2t\tfrac{C}{V\left(B_x\left(\sqrt{t}\right)\right)}\int_\delta^{+\infty}\exp\left(-C\tfrac{r^2}{2t}\right)\left(\int_{\partial B_x(r)}\Delta_b\log|f(s)|\,d\sigma(s)\right)dr \\ \leq \; & -\tfrac{C}{V\left(B_x\left(\sqrt{t}\right)\right)}\exp\left(-C\tfrac{\delta^2}{2t}\right)\int_{B_x(\delta)}\Delta_b\log|f(y)|\,dy \\ & +Ct\int_\delta^{+\infty}[\exp\left(-C\tfrac{r^2}{2t}\right)\tfrac{r}{t}\left(\tfrac{r}{\sqrt{t}}\right)^{2C}\left(\tfrac{1}{V(B_x(r))}\int_{B_x(r)}\Delta_b\log|f(y)|\,dy\right)]dr. \end{aligned} \tag{5.7}$$

Here we use integration by part, (5.4) and the CR volume doubling property in the second inequality. From the CR volume doubling property and the inequality (5.4), we obtain

$$\lim_{t\to 0^+}\frac{C}{V\left(B_x\left(\sqrt{t}\right)\right)}\exp\left(-C\frac{\delta^2}{2t}\right)\int_{B_x(\delta)}\Delta_b\log|f(y)|\,dy=0. \tag{5.8}$$

Set $\tau = \frac{r^2}{2t}$ in the second integration in the right hand side in (5.7), we have

$$\begin{aligned} & C\int_{\delta}^{+\infty}[\exp(-C\tfrac{r^2}{2t})r(\tfrac{r}{\sqrt{t}})^{2C_9}(\tfrac{1}{V(B_x(r))}\int\limits_{B_x(r)}\Delta_b\log|f(y)|\,dy)]dr \\ \leq\ & C\int_{\frac{\delta^2}{2t}}^{+\infty}[\exp(-C\tau)(\tau t)(2\tau)^{C_9-1}(\tfrac{1}{V\left(B_x\left(\sqrt{2\tau t}\right)\right)}\int\limits_{B_x\left(\sqrt{2\tau t}\right)}\Delta_b\log|f(y)|\,dy)]d\tau \\ \leq\ & Cd\int_{\frac{\delta^2}{2t}}^{+\infty}\exp(-C\tau)(2\tau)^{C_9-1}\,d\tau \\ \to\ & 0^+ \quad as\ t\to 0^+ \end{aligned} \tag{5.9}$$

by (5.4). From (5.7), (5.8) and (5.9), we see that the first integration in the last line of (5.6) is close to zero, as $t$ goes to zero.

(II). Next we show the following two inequalities from which the fact that the second integration in the last line of (5.6) approximates to $ord_x(f)$ as $t$ small enough, could be deduced :

$$ord_x(f) - \epsilon < \liminf_{t\to 0^+}(2t\int\limits_{B_x(\delta)}H(x,y,t)\Delta_b\log|f(y)|\,dy) \tag{5.10}$$

and

$$\limsup_{t\to 0^+}(2t\int\limits_{B_x(\delta)}H(x,y,t)\Delta_b\log|f(y)|\,dy) < ord_x(f)+\epsilon. \tag{5.11}$$

Now we prove the inequality (5.10). As for (5.11), the deduction is almost the same as well as (5.10).

In the rest of the proof, we adopt the adapted metric as in (2.1). From the small time asymptotic expansion for the heat kernel $H_\epsilon(x,y,t)$ with respect to the adapted metric,

$$H_\epsilon(x,y,t) \sim \frac{1}{(4\pi t)^{\frac{2n+1}{2}}}\exp(-\frac{d^2(x,y)}{4t}) + L.O.T.$$

if $t$ is sufficiently small, we have, for $t << 1$,

$$
\begin{aligned}
& 2t \int_{B_x(\delta)} H(x,y,t)\Delta_b \log|f(y)|\, dy \\
= {} & 2t \int_{B_x(\delta)} H_\epsilon(x,y,t)\Delta_\epsilon \log|f(y)|\, d\mu_\epsilon(y) \\
\geq {} & 2t \int_0^\delta [\tfrac{1}{(4\pi t)^{\frac{2n+1}{2}}} \exp\left(-\tfrac{s^2}{4t}\right) ( \int_{\partial B_x(s)} \Delta_\epsilon \log|f(y)|\, d\sigma_\epsilon(y))]ds - \tfrac{\epsilon}{2} \\
= {} & 2t \tfrac{1}{(4\pi t)^{\frac{2n+1}{2}}} \exp\left(-\tfrac{\delta^2}{4t}\right) ( \int_{B_x(\delta)} \Delta_\epsilon \log|f(y)|\, d\mu_\epsilon(y)) \\
& +2t \int_0^\delta [\tfrac{1}{(4\pi t)^{\frac{2n+1}{2}}} \exp\left(-\tfrac{s^2}{4t}\right) \tfrac{s}{2t} ( \int_{B_x(s)} \Delta_\epsilon \log|f(y)|\, d\mu_\epsilon(y))]ds - \tfrac{\epsilon}{2} \\
= {} & 2t \tfrac{1}{(4\pi t)^{\frac{2n+1}{2}}} \exp\left(-\tfrac{\delta^2}{4t}\right) ( \int_{B_x(\delta)} \Delta_b \log|f(y)|\, d\mu(y)) \\
& +2t \int_0^\delta [\tfrac{1}{(4\pi t)^{\frac{2n+1}{2}}} \exp\left(-\tfrac{s^2}{4t}\right) \tfrac{s}{2t} ( \int_{B_x(s)} \Delta_b \log|f(y)|\, d\mu(y))]ds - \tfrac{\epsilon}{2}.
\end{aligned} \tag{5.12}
$$

Here we use the fact that $f_0 = 0$ and use integration by part, (5.4) and

$$
\begin{cases}
\Delta_\epsilon \log|f(y)| = 2\Delta_b \log|f(y)|, \\
d\mu_\epsilon(y) = \frac{1}{2} d\mu.
\end{cases}.
$$

More precisely, we explain why the first equality in (5.12) holds as follows : On $B_x(\delta)$, the equation

$$
\begin{cases}
\left(\frac{\partial}{\partial t} - \Delta_b\right) w(x,t) = 0 \\
w(x,0) = 2\Delta_b \log|f(x)|
\end{cases} \tag{5.13}
$$

admits the solution

$$
w(x,t) = 2 \int_{B_x(\delta)} H(x,y,t)\Delta_b \log|f(y)|\, dy.
$$

Differentiating the equation (5.13) along the direction $T$, we have, by $f_0 = 0$,

$$
\begin{cases}
\left(\frac{\partial}{\partial t} - \Delta_b\right) w_0(x,t) = 0, \\
w_0(x,0) = 0.
\end{cases}
$$

By Theorem 3.1, we know that

$$w_0(x,t)=0.$$

On the other hand, it follows from the equality $\Delta_\epsilon = 2\Delta_b + \epsilon^2 T^2$ that the equation (5.13) becomes

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta_\epsilon\right) w(x,t) = 0, \\ w(x,0) = 2\Delta_b \log|f(x)|. \end{cases}.$$

Substituting the new variable $\widetilde{t} = \frac{1}{2}t$ into the equation, it becomes

$$\begin{cases} \left(\frac{\partial}{\partial \widetilde{t}} - \Delta_\epsilon\right) w\left(x,\widetilde{t}\right) = 0, \\ w(x,0) = 2\Delta_b \log|f(x)|. \end{cases}.$$

From the result in [Do], we obtain that such equation admits a unique solution

$$w\left(x,\widetilde{t}\right) = 2\int\limits_{B_x(\delta)} H_\epsilon(x,y,\widetilde{t})\Delta_b \log|f(y)|\, d\mu_\epsilon(y),$$

that is,

$$w(x,t) = 2\int\limits_{B_x(\delta)} H_\epsilon(x,y,t)\Delta_b \log|f(y)|\, d\mu_\epsilon(y).$$

By the uniqueness, we have

$$2\int\limits_{B_x(\delta)} H(x,y,t)\Delta_b \log|f(y)|\, dy = 2\int\limits_{B_x(\delta)} H_\epsilon(x,y,t)\Delta_b \log|f(y)|\, d\mu_\epsilon(y).$$

By the inequality (5.4) again, we also have

$$\begin{aligned} & \lim_{t\to 0^+} t\frac{1}{(4\pi t)^{\frac{2n+1}{2}}}\exp(-\tfrac{\delta^2}{4t})(\int\limits_{B_x(\delta)} \Delta_b \log|f(y)|\, d\mu(y)) \\ \leq\ & \lim_{t\to 0^+} t\frac{1}{(4\pi t)^{\frac{2n+1}{2}}}\exp(-\tfrac{\delta^2}{4t}) V(B_x(\delta))\, C\frac{d}{\delta^2} \\ =\ & 0. \end{aligned}$$

From the result above, we simplify the inequality (5.12) into

$$\begin{aligned} & 2t \int_{B_x(\delta)} H(x,y,t) \Delta_b \log |f(y)| \, dy \\ \geq \ & 2t \int_0^\delta \tfrac{1}{(4\pi t)^{\frac{2n+1}{2}}} \exp(-\tfrac{s^2}{4t}) \tfrac{s}{2t} ( \int_{B_x(s)} \Delta_b \log |f(y)| \, dy) ds - \tfrac{3\epsilon}{4} \end{aligned}$$

for $t << 1$. Hence we just need to claim that

$$2t \int_0^\delta [\frac{1}{(4\pi t)^{\frac{2n+1}{2}}} \exp(-\frac{s^2}{4t}) \frac{s}{2t} ( \int_{B_x(s)} \Delta_b \log |f(y)| \, dy)] ds > ord_x f - \frac{\epsilon}{4} \tag{5.14}$$

for $t << 1$. By the hypotheses, we see that

$$V(B_{\widetilde{x}}(r)) \sim \omega_{2n} r^{2n} \tag{5.15}$$

as $r \to 0^+$, where $\omega_k$ is the volume of the unit ball in $\mathbb{R}^k$ for all $k \in \mathbb{N}$. Utilizing this volume approximation and (5.5), we obtain, for $t << 1$,

$$\begin{aligned} & 2t \int_0^\delta [\tfrac{1}{(4\pi t)^{\frac{2n+1}{2}}} \exp(-\tfrac{s^2}{4t}) \tfrac{s}{2t} ( \int_{B_x(s)} \Delta_b \log |f(y)| \, dy)] ds \\ = \ & 2t \int_0^\delta \tfrac{1}{(4\pi t)^{\frac{2n+1}{2}}} \exp(-\tfrac{s^2}{4t}) \tfrac{s}{2t} [\int_{-s}^{s} ( \int_{B_{\widetilde{x}}\left(\sqrt{s^2-l^2}\right)} \Delta_{\widetilde{y}} \log |f((\widetilde{y},t))| \, d\widetilde{y}) dl] ds \\ > \ & \int_0^\delta [\tfrac{1}{(4\pi t)^{\frac{2n+1}{2}}} \exp(-\tfrac{s^2}{4t}) 2ns \left(ord_x(f)\right) \omega_{2n} (\int_{-s}^{s} (s^2-l^2)^{n-1} dl)] ds - \tfrac{\epsilon}{6} \\ = \ & \int_0^\delta [\tfrac{1}{(4\pi t)^{\frac{2n+1}{2}}} \exp(-\tfrac{s^2}{4t}) 2ns \left(ord_x(f)\right) \omega_{2n} (s^{2n-1} \sqrt{\pi} \tfrac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)})] ds - \tfrac{\epsilon}{6} \\ = \ & \tfrac{4nt}{(4\pi t)^{\frac{2n+1}{2}}} ord_x(f) \, \omega_{2n} \sqrt{\pi} \tfrac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)} (\int_0^{\frac{\delta^2}{4t}} \exp(-\tau) (4\tau t)^{\frac{2n-1}{2}} d\tau) - \tfrac{\epsilon}{6} \\ = \ & \tfrac{n\omega_{2n} ord_x(f)}{\pi^n} \tfrac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)} \int_0^{\frac{\delta^2}{4t}} \exp(-\tau) \, \tau^{\frac{2n-1}{2}} d\tau - \tfrac{\epsilon}{6} \\ > \ & ord_x(f) - \tfrac{\epsilon}{4} \end{aligned} \tag{5.16}$$

where we use the error term $\frac{\epsilon}{12}$ twice in the third line, that are (5.5) and (5.15), and the fact that

$$\int_0^{\frac{\delta^2}{4t}} \exp(-\tau) \, \tau^{\frac{2n-1}{2}} d\tau \to \Gamma(n + \frac{1}{2})$$

as $t$ goes to zero, in the last inequality. For the second equality of (5.16), we use the equality (4.8). This completes the proof. □

Finally, we would show the main theorem-the sharp dimension estimate in this paper as follows.

*Proof.* of **Theorem 1.1**

Here we follow the ideas and arguments in [N] and [CFYZ] to derive this result and use the notations of the hypotheses in the preceding lemma.

From

$$\frac{\partial}{\partial t}\left(tw\left(x,t\right)\right)\geq 0,$$

we have, for the positive integer $k\geq 2$,

$$\begin{aligned} v\left(x,t\right) &= \Big(\int\limits_{(1,t)} w\left(x,s\right)ds\Big)+u\left(x,1\right) \\ &\geq \Big(\int\limits_{\left(t^{\frac{1}{k}},t\right)} \tfrac{t^{\frac{1}{k}}}{s} w\left(x,t^{\frac{1}{k}}\right)ds\Big)+u\left(x,1\right) \\ &= t^{\frac{1}{k}}w(x,t^{\frac{1}{k}})(1-\tfrac{1}{k})\log t+u\left(x,1\right). \end{aligned}$$

Then, by Lemma 4.1, we obtain

$$\limsup_{t\to+\infty}[t^{\frac{1}{k}}w(x,t^{\frac{1}{k}})(1-\frac{1}{k})]\leq d.$$

On account of the arbitrariness of $k$, we have

$$\limsup_{t\to+\infty} tw\left(x,t\right)\leq d.$$

By the monotonicity of $tw\left(x,t\right)$ and Lemma 5.1, we get

$$ord_x f\leq d \tag{5.17}$$

for all nontrivial $f\in\mathcal{O}_d^{CR}\left(M\right)$. Now we perform the Poincaré-Siegel argument and note that the argument only depends on the local property of the CR holomorphic functions of polynomial growth. The Poincaré-Siegel map

$$\begin{aligned} \Phi:\mathcal{O}_d^{CR}\left(M\right) &\longrightarrow \mathbb{C}^{q(d)} \\ f &\longmapsto \left(D^{\alpha}f\right)_{|\alpha|\leq d}, \end{aligned}$$

where $q(m) = \begin{pmatrix} n+m \\ n \end{pmatrix}$ is the complex dimension of $\mathcal{O}_d(\mathbb{C}^n)$. So if $f \neq 0$ but $D^\alpha f = 0$ for all $\alpha$ with $|\alpha| \leq d$, then it contradicts with (5.17). Hence, we know that the Poincaré-Siegel map $\Phi$ is injective and this implies that

$$\dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}(M)\right) \leq \dim_{\mathbb{C}} \mathcal{O}_d(\mathbb{C}^n) = \dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}(H^n)\right).$$

Therefore we complete the proof. □

## Appendix A.

In this appendix, we would show the CR $L^p$ submean value inequality. Although the details are almost the same as in [SC2], we give its proof here for completeness.

*Proof.* of **Proposition 3.2** : Initially we prove the case $p = 2$ and may assume $\tau = 1 = \delta'$. On account of

$$\int_{B_x(r)} \langle (\frac{\partial}{\partial t} - \triangle_b)u, \phi\rangle d\mu \leq 0$$

for any $\phi \in C_c^\infty(B_x(r))$, choosing $\phi = \psi^2 u$ for $\psi \in C_c^\infty(B_x(r))$, we have

$$\begin{aligned}\int_{B_x(r)} [\psi^2 u \tfrac{\partial u}{\partial t} + \psi^2 |\nabla_b u|^2] d\mu &\leq 2\left|\int_{B_x(r)} u\psi\langle\nabla_b u, \nabla_b\psi\rangle d\mu\right| \\ &\leq 2\int_{B_x(r)} |\nabla_b\psi|^2 u^2 d\mu + \tfrac{1}{2}\int_{B_x(r)} \psi^2|\nabla_b u|^2 d\mu.\end{aligned}$$

That is,

$$\int_{B_x(r)} \left(2\psi^2 u\frac{\partial u}{\partial t} + \psi^2|\nabla_b u|^2\right) d\mu \leq 4\int_{B_x(r)} |\nabla_b\psi|^2 u^2 d\mu.$$

So there exists a positive constant $C > 0$ such that,

$$\int_{B_x(r)} \left(2\psi^2 u\frac{\partial u}{\partial t} + \psi^2|\nabla_b u|^2\right) d\mu \leq C||\nabla_b\psi||^2_{L^\infty(B_x(r))} \int_{supp(\psi)} u^2 d\mu.$$

Hereafter we adopt the notations $C$ and $A$ as constants which may be different in each line.

Let $\chi(t) \in C^{\infty}(R)$ be a smooth function on $R$. We have

$$\begin{array}{lrl} \text{(A.1)} & & \frac{\partial}{\partial t}\int\limits_{B_x(r)}(\chi\psi u)^2 d\mu + \chi^2 \int\limits_{B_x(r)} |\nabla_b(\psi u)|^2 d\mu \\ & \leq & C\chi\left(\chi||\nabla_b\psi||^2_{L^\infty(B_x(r))} + \|\psi\|^2_{L^\infty(B_x(r))}||\chi'||^2_{L^\infty(R)}\right)\int_{supp\psi} u^2 d\mu. \end{array}$$

Now we choose the appropriate the smooth functions $\psi$ and $\chi$ as follows : Let $0 < \sigma' < \sigma < 1$ and $\omega = (\sigma - \sigma')$, we define

$$\psi(y) = \begin{cases} 1, & on\ B_x(\sigma' r), \\ 0, & on\ M\backslash B_x(\sigma r), \end{cases}$$

with $0 \leq \psi \leq 1$ and $|\nabla_b\psi| \leq \frac{2}{\omega r}$. Define

$$\chi(t) = \begin{cases} 0, & on\ (-\infty, s - \sigma r^2), \\ 1, & on\ (s - \sigma' r^2, \infty), \end{cases}$$

with $0 \leq \chi \leq 1$ and $|\chi'| \leq \frac{2}{\omega r^2}$. Let $I_\sigma = (s - \sigma r^2, s)$. Integrating (A.1) over $(s - r^2, t)$ for $t \in I_{\sigma'}$, we have

$$\text{(A.2)} \qquad \sup_{I_{\sigma'}}\{\int_{B_x(r)}(\psi u)^2 d\mu\} + \int_{B_x(r)\times I_{\sigma'}} |\nabla_b(\psi u)|^2 d\overline{\mu} \leq \frac{C}{(\omega r)^2}\int_{Q_\sigma} u^2 d\overline{\mu}.$$

Let $k(r,\nu) = \frac{Cr^2}{V(B_x(r))^{\frac{2}{\nu}}}$. By Hölder's inequality and CR Sobolev inequality (3.10), we have

$$\begin{array}{rcl} \int_{B_x(\sigma' r)}(\psi u)^{2(1+\frac{2}{\nu})}d\mu & \leq & \int_{B_x(r)}(\psi u)^{2(1+\frac{2}{\nu})}d\mu \\ & = & \int_{B_x(r)}(\psi u)^2(\psi u)^{\frac{4}{\nu}}d\mu \\ & \leq & (\int_{B_x(r)}(\psi u)^{\frac{2\nu}{\nu-2}}d\mu)^{\frac{\nu-2}{\nu}}(\int_{B_x(r)}(\psi u)^2 d\mu)^{\frac{2}{\nu}} \\ & \leq & [k\int_{B_x(r)}(|\nabla_b(\psi u))|^2 + |\frac{\psi u}{r}|^2)d\mu]||\psi u||^{\frac{4}{\nu}}_{L^2(B_x(r))}. \end{array}$$

Then, integrating over $I_{\sigma'}$,

$$\int_{Q_{\sigma'}} u^{2(1+\frac{2}{\nu})}d\overline{\mu} \leq k(\frac{A}{(r\omega)^2}\int_{Q_\sigma} u^2 d\overline{\mu})^{1+\frac{2}{\nu}}$$

by utilizing $(A.2)$ twice. Let $\theta = \left(1 + \frac{2}{\nu}\right)$ and we note that $u^p$ is also a positive smooth subsolution to the CR heat equation if $u$ is a positive smooth subsolution to the CR heat equation for $p \geq 1$. So we have the following inequality

$$\int_{Q_{\sigma'}} u^{2p\theta} d\overline{\mu} \leq k(\frac{A}{(r\omega)^2} \int_{Q_\sigma} u^{2p} d\overline{\mu})^\theta. \tag{A.3}$$

Let $\omega_j = \frac{(1-\delta)}{2^j}$ and $\sigma_0 = 1$ and $\sigma_{k+1} = 1 - \sum_{j\in I_k} \omega_j$ for $k \geq 0$. Apply $(A.3)$ with $p = \theta^i$, $\sigma = \sigma_i$ and $\sigma' = \sigma_{i+1}$, we have

$$\int_{Q_{\sigma_{i+1}}} u^{2\theta^{i+1}} d\overline{\mu} \leq k(\frac{A}{(r\omega_i)^2} \int_{Q_{\sigma_i}} u^{2\theta^i} d\overline{\mu})^\theta. \tag{A.4}$$

By iteration, we obtain

$$(\int_{Q_{\sigma_{i+1}}} u^{2\theta^{i+1}})^{\theta^{-1-i}} \leq A(\nu) k^{(\sum_{l\in I_i'} \theta^{l-1-i})} ((1-\delta)r)^{(-2\sum_{l\in I_i'} \theta^{l-i})} (\int_{Q_0} u^2 d\overline{\mu}).$$

Letting $i$ tend to $\infty$, we have

$$\sup_{Q_\delta}\{u^2\} \leq A(\nu) k^{\frac{\nu}{2}} ((1-\delta)r)^{-2-\nu} \int_Q u^2 d\overline{\mu}. \tag{A.5}$$

This completes the case $p = 2$. When $p > 2$, then the transformation $u \mapsto u^{\frac{p}{2}}$ still satisfies the hypothesis for $p = 2$. Hence, it is easy to validate this case. As for $p \in (0, 2)$, the case $p = 2$ yields that

$$\sup_{\widetilde{Q}_\sigma} \{u\} \leq \frac{\widetilde{k}(\tau, \nu, r)}{(\rho - \sigma)^{1+\frac{\nu}{2}}} (\int_{\widetilde{Q}_\rho} u^2 d\overline{\nu})^{\frac{1}{2}},$$

where $\widetilde{k}(\tau, \nu, r) = \frac{A(\tau,\nu,2) C^{\frac{\nu}{2}}}{r^2}$, $d\overline{\nu} = d\nu dt =$ the volume-normalized measure $\frac{d\overline{\mu}}{V(\widetilde{Q})}$, and $\widetilde{Q} = Q_{\delta'}$, $\widetilde{Q}_\sigma = \left(\widetilde{Q}\right)_\sigma$, $\widetilde{Q}_\rho = \left(\widetilde{Q}\right)_\rho$, $\sigma \in \left(\frac{1}{2}, 1\right)$, $(1 - \rho) = \frac{3}{4}(1 - \sigma)$. Due to the inequality

$$\|u\|_{L^2(\widetilde{Q}_\rho, \overline{\nu})} \leq \|u\|^{1-\frac{p}{2}}_{L^\infty(\widetilde{Q}_\rho, \overline{\nu})} \|u\|^{\frac{p}{2}}_{L^p(\widetilde{Q}, \overline{\nu})},$$

we have

$$\|u\|_{L^\infty(\widetilde{Q}_\sigma, \overline{\nu})} \leq (\widetilde{k}(\tau, \nu, r)\, 4^{\left(1+\frac{\nu}{2}\right)} \|u\|^{\frac{p}{2}}_{L^p(\widetilde{Q}, \overline{\nu})}) \frac{1}{(1-\sigma)^{1+\frac{\nu}{2}}} \|u\|^{1-\frac{p}{2}}_{L^\infty(\widetilde{Q}_\rho, \overline{\nu})}. \tag{A.6}$$

Set $\sigma_0 = \delta_1 = \frac{\delta}{\delta'} \in (0,1)$, $(1-\sigma_i) = \left(\frac{3}{4}\right)^i (1-\delta_1)$. Taking $\sigma = \sigma_i$, $\rho = \sigma_{i+1}$ in $(A.6)$, we have

$$\|u\|_{L^\infty\left(\widetilde{Q}_{\sigma_i},\overline{\nu}\right)} \leq \left(\frac{4}{3}\right)^{i\left(1+\frac{\nu}{2}\right)} w \, \|u\|^{1-\frac{p}{2}}_{L^\infty\left(\widetilde{Q}_{\sigma_{i+1}},\overline{\nu}\right)} \tag{A.7}$$

for $w = [\widetilde{k}(\tau,\nu,r)\, 4^{\left(1+\frac{\nu}{2}\right)} \|u\|^{\frac{p}{2}}_{L^p\left(\widetilde{Q},\overline{\nu}\right)} (1-\delta_1)^{-1-\frac{\nu}{2}}]$.

By iterating the inequality $(A.7)$, we obtain

$$\|u\|_{L^\infty\left(\widetilde{Q}_{\delta_1},\overline{\nu}\right)} \leq \left(\frac{4}{3}\right)^{\left(1+\frac{\nu}{2}\right)\left(\sum_{l\in I_{i-1}} l\left(1-\frac{p}{2}\right)^l\right)} w^{\sum_{l\in I'_{i-1}}\left(1-\frac{p}{2}\right)^l} \|u\|^{1-\frac{p}{2}}_{L^\infty\left(\widetilde{Q}_{\sigma_i},\overline{\nu}\right)},$$

for $i \in \mathbb{N}$. As before, letting $i$ tend to $\infty$, we have

$$\|u\|_{L^\infty\left(\widetilde{Q}_{\delta_1},\overline{\nu}\right)} \leq \left(\frac{4}{3}\right)^{\frac{(2+\nu)(2-p)}{p^2}} \frac{A(\tau,\nu,2)^{\frac{2}{p}} C^{\frac{\nu}{p}}}{r^{\frac{4}{p}}} 4^{\frac{(2+\nu)}{p}} (1-\delta_1)^{-\frac{2+\nu}{p}} \|u\|_{L^p\left(\widetilde{Q},\overline{\nu}\right)}.$$

This completes the proof. □

## References


[BBGM] F. Baudoin, M. Bonnefont, N. Garofalo and I. Munive, Volume and distance comparison theorems for sub-Riemannian manifolds, J. Funct. Anal. 267(2014) 2005-2027.

[C] Laurent Saloff-Coste, Aspects of Sobolev-Type Inequalities, Cam. Uni. Press (2002).

[CC] S.-C. Chang, H.-L. Chiu, Nonnegativity of CR Paneitz operator and its applications to the CR Obata's theorem, J. Geom. Anal. 19(2009), 261-287

[CCHL] D.-C. Chang, S.-C. Chang, Y.-B. Han and C. Lin, On the CR Poincaré-Lelong equation, Yamabe steady solitons and structures of complete noncompact Sasakian manifolds, 2017, submitted.

[CCHT] D.-C. Chang, S.-C. Chang, Y.-B. Han and J.-Z. Tie, A CR Analogue of Yau's Conjecture On Pseudoharmonic Functions of Polynomial Growth, 2017, submmited.

[CCM] J. Cheeger, T. H. Colding and W. P. Minicozzi, Lirlear growth harmonic functions on completemanifoldswithnonnegativeRiccicurvature, Geom. Funct. Anal. 5 (1995), no. 6, 948-954.

[CCL] D.-C. Chang, S.-C. Chang and C. Lin, On Li-Yau gradient estimate for sum of squares of vector fields up to higher step, arXiv:1602.01531.

[CCT] J. Cheeger, T. H. Colding and G. Tian, On the singularities of spaces with bounded Ricci curvature, Geom. Funct. Anal. 12(2002) 873-914.

[CCW] S.-C. Chang, C.-W. Chen and C.-T. Wu, On the CR analogue of Reilly formula and Yau eigenvalue conjecture, arXiv:1503.07639, to appear in Asian J. Math.

[CF] S.-C Chang, Y.-W. Fan, An optimal gap theorem in a complete strictly pseudoconvex CR $(2n+1)$-Manifold, J. Geom. Anal. 26(2016): 2425-2449.

[CFYZ] B.-L. Chen, X.-Y. Fu, L. Yin and X.-P. Zhu, Sharp dimension estimates of holomorphic functions and rigidity, Trans. of AMS, Volume 358, Number 4(2005) 1435-1454.

[ChCo1] J. Cheeger and T. H. Colding, Lower bounds on Ricci curvature and the almost rigidity of wraped products, Annals of Math. 144(1996) 189-237.

[ChCo2] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below I, II, III, J. Differential Geometry 46(1997) 406-480, 54(2000) 13-35, 54(2000) 37-74.

[CKL] S.-C. Chang, T.-L. Kuo, and S.-H. Lai, Li-Yau Gradient Estimate and Entropy Formulae for the CR Heat Equation in a Closed Pseudohermitian 3-manifold, *J. Diff. Geom.* **89** (2011), 185-216.

[CKT] S.-C. Chang, T.-J. Kuo and J.Z. Tie, Yau's gradient estimate and Liouville theorem for positive pseduoharmonic functions in a complete pseudohermitian manifold, arXiv:1506.03270.

[CM] T. H. Colding and W. P. Minicozzi, Harmonic functions on manifolds, *Ann. of Math.* **146** (1997), no. 3, 725-747.

[CT] A. Chau and L.-F. Tam, On the complex structure of Kaehler manifolds with nonnegative curvature, J. Diff. Geom, 73(2006), 491-530.

[CTW] S.-C. Chang, J. Tie and C.-T. Wu, Subgradient Estimate and Liouville-type Theorems for the CR Heat Equation on Heisenberg groups $\mathbf{H}^n$, *Asian J. Math.* **14**, No. 1 (2010), 041–072.

[CTZ] B.-L. Chen, S.-H. Tamg and X.-P. Zhu, A uniformization theorem for complete noncompact Kahler surfaces with positive bisectional curvature, J. Differential Geometry 67(2004) 519-570.

[CY] S.-Y. Cheng and S.-T. Yau, Differential equations on Riemannian manifolds and their geometric applications, *Comm. Pure Appl. Math.* **28** (1975), 333-354.

[CZ] B.-L. Chen and X.-P. Zhu, On complete noncompact Kaehler manifolds with positive bisectional curvature, Math. Ann. 327(2003) 1-23.

[De] J.P. Demailly, Complex analytic and differential geometry, Institut Fourier (2012).

[Do] H. Donnelly, Uniqueness of positive solutions of the heat equation.AMS 99(2)(1987).

[DT] S. Dragomir and G. Tomassini, Differential Geometry and Analysis on CR manifolds, Progress in Mathematics, Volume 246, Birkhauser 2006.

[FOW] A. Futaki, H. Ono and G.-F. Wang, Transverse Kahler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds, J. Differential Geometry 83(2009) 585-635.

[GH] P.A. Griffiths and J. Harris, Principles of algebraic geometry, Wylie (1978).

[GW1] R.E. Green, H. Wu, Integrals of subharmonic functions on manifolds on negative curvature, Invent. Math. 27(1974) 265-298.

[GW2] R.E. Green, H. Wu, Gap theorems for noncompact Riemannian manifolds, Duke Math. Journal 49(3) (1982) 731-756.

[K] A. Kasue, Harmonic functions with growth conditions on a manifold of asymptotically non- negative curvature II, Recent Topics in Differential and Analytic Geometry, Adv. Stud. Pure Math., vol. 18, North-Holland, 1989.

[KS] A. Koranyi and N. Stanton, Liouville Type Theorems for Some Complex Hypoelliptic Operators, *J. of Funct. Anal.* **60** (1985), 370-377.

[KS1] S. Kusuoka and D. Stroock, Application of Malliavin Calculus (Part lII), *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **34** (1987), 391-442.

[KS2] S. Kusuoka and D. Stroock, Long time estimates for the heat kernel associated with a uniformly sbellitic symmetric second order operator. *Ann. Math.* **127** (1988), 165-189.

[L] J. M. Lee, Pseudo-Einstein Structure on $CR$ Manifolds, *Amer. J. Math.* **110** (1988), 157-178.

[Li] P. Li, "Geometric Analysis ", *Cambridge Studies in Advanced Mathematics*, **134**, Cambridge University Press, 2012.

[Li1] P. Li, Linear growth harmonic functions on Kahler manifolds with nonnegative Ricci curva- ture, Math. Res. Lett. 2 (1995), 79-94.

[Li2] P. Li, Harmonic sections of polynomial growth, *Math. Res. Lett.* **4** (1997), 35-44.

[Liu1] Gang Liu, Three circles theorem on Kahler manifolds and applications, Duke Math J.

[Liu2] Gang Liu, Gromov-Hausdorff limits of Kahler manifolds and the finite generation conjecture, Annals of Math. 184(2016) 775-815

[Liu3] Gang Liu, On Yau's Uniformization Conjecture, arXiv: 1606.08958v2.

[LT] P. Li and L.-F. Tam, Complete surfaces with finite total curvature, J. Differential Geom. 33 (1991), 139-168.

[LT1] P. Li and L.-F. Tam, Linear growth harmonic functions on a complete manifold, J. Differential Geom. 29 (1989), 421-425.

[LY1] P. Li and S. -T. Yau, Estimates of Eigenvalues of a Compact Riemannian Manifold, *AMS Proc. Symp. in Pure Math.* **36** (1980), 205-239.

[LY2] P. Li and S. -T. Yau, On the Parabolic Kernel of the Schrődinger Operator, *Acta Math.* **156** (1985), 153-201.

[LY] P. Li and S.-T. Yau, On the parabolic kernel of the Schrödinger operator, Acta Math. 156(1986) 139-168.

[M1] N. Mok, An embedding theorem of complete Kahler manifolds of positive bisectional curvature onto affine algebraic varieties, Bull. Soc. Math. France 112(1984) 197-250.

[M2] N. Mok, The uniformization theorem for compact Kahler manifolds of nonnegative holomorphic bisectional curvature, J. Differential Geometry 27(1988) 179-214.

[MSY] N. Mok, Y.-T. Siu and S.-T. Yau, The Poincare-Lelong equation on complete Kahler manifolds Compos. Math. 44(1981) 183-218.

[N] Lei Ni, A monotonicity formula on complete Kahler manifolds with nonnegative bisectional curvature, J. of AMS, Volume17, Number 4(2004) 909-946.

[N2] Lei Ni, Vanishing theorems on complete Kaehler manifolds and their applications, J. Differential Geometry 50(1998) 89-122.

[NST] L. Ni, Y. Shi and L.-F. Tam, Poisson equation, Poincare-Lelong equation and curvature decay on complete Kähler manifolds, J. Differential Geometry 57(2001) 339-388.

[NT1] L. Ni, L.-F. Tam, Plurisubharmonic functions and the structure of complete Kähler manifolds with nonnegative curvature, J. Differential Geometry 64(2003) 457-524.

[NT2] L. Ni, L.-F. Tam, Kähler-Ricci flow and the Poincaré-Lelong equation, Comm. Anal. Geom. Math. 12(2004) 111-141.

[NT3] L. Ni, L.-F. Tam, Plurisubharmonic functions and the Kahler-Ricci flow, Amer. Jour. of Math. 125(2003) 623-645.

[NT4] L. Ni, L.-F. Tam, Poincare-Lelong equation via the Hodge-Laplace heat equation, Compos. Math. 149 1856-1870.

[SC] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, *J. Diff. Geom.* **36** (1992), no. 2, 417-450.

[SC2] L. Saloff-Coste,“Aspect of Sobolev-type inequalities”, London Math. Soc. Lecture Notes Series **289**. Cambridge University Press, 2002.

[ScY] Richard Schoen and S.-T. Yau, Lectures on differential gemeotry, International Press(1994).

[SY] Y.-T. Siu and S.-T. Yau, Complete Kähler manifolds with non-positive curvature of faster than quadratic decay, Ann. Math. 105 (1977), 225-264.

[Y1] S. -T. Yau, Harmonic functions on complete Riemannian manifolds, *Comm. Pure Appl. Math.* **28** (1975), 201-228.

[Y2] S. -T. Yau (ed.), "Seminar on Differential Geometry ", *Annals of Math. Studies* **102**, Princeton, New Jersey, 1982.

[Y3] S. -T. Yau, Open prohlems in geometry, Lectures on Differential Geometry, by Schoen and Yau, International Press (1994), 365-404.

Department of Mathematics and Taida Institute for Mathematical Sciences (TIMS), National Taiwan University, Taipei 10617, Taiwan, R.O.C., Current Address: Yau Mathematical Sciences Center, Tsinghua University, Beijing, China

*E-mail address*: scchang@math.ntu.edu.tw

School of Mathematics and Statistics, Xinyang Normal University, Xinyang,464000, Henan, P.R. China

*E-mail address*: yingbohan@163.com

Department of Mathematics, National Tsing Hua University, Hsinchu 30013, Taiwan, R.O.C

*E-mail address*: r97221009@ntu.edu.tw