ON THE THREE-CIRCLE THEOREM AND ITS APPLICATIONS IN SASAKIAN MANIFOLDS

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Abstract. This paper mainly focuses on the CR analogue of the three-circle theorem in a complete noncompact pseudohermitian manifold of vanishing torsion being odd dimensional counterpart of Kähler geometry. In this paper, we show that the CR three-circle theorem holds if its pseudohermitian sectional curvature is nonnegative. As an application, we confirm the first CR Yau's uniformization conjecture and obtain the CR analogue of the sharp dimension estimate for CR holomorphic functions of polynomial growth and its rigidity when the pseudohermitian sectional curvature is nonnegative. This is also the first step toward second and third CR Yau's uniformization conjecture. Moreover, in the course of the proof of the CR three-circle theorem, we derive CR sub-Laplacian comparison theorem. Then Liouville theorem holds for positive pseudoharmonic functions in a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion and nonnegative pseudohermitian Ricci curvature.

1. Introduction

In 1896, J. Hadamard ([Ha]) published the so-called classical three-circle theorem which says that, on the annulus A with inner radius r_1 and outer radius r_2, the logarithm for the modulus of a holomorphic function on the closure $\overline{A}$ of the annulus is convex with respect to $\log r$ for r lying between r_1 and r_2. Recently, G. Liu ([Liu1]) generalized the three-circle

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theorem to complete Kähler manifolds and characterized a necessary and sufficient condition for the three-circle theorem. Here a complete Kähler manifold M is said to satisfy the three-circle theorem if, for any point $p \in M$, $R > 0$, any holomorphic function f on the geodesic ball $B(p, R)$, $\log M_f(r)$ is convex with respect to $\log r$, namely, for $0 < r_1 \leq r_2 \leq r_3 < R$,

$$\log\left(\frac{r_3}{r_1}\right)\log M_f(r_2) \leq \log\left(\frac{r_3}{r_2}\right)\log M_f(r_1) + \log\left(\frac{r_2}{r_1}\right)\log M_f(r_3) \tag{1.1}$$

where $M_f(r) = \sup\limits_{x \in B(p,r)} |f(x)|$. More precisely, he showed that a complete Kähler manifold satisfies the three-circle theorem if and only if its holomorphic sectional curvature is nonnegative. The proof employed the Hessian comparison and the maximum principle. There are many substantial applications pertaining to the three-circle theorem, especially to the uniformization-type problems proposed by Yau ([ScY]) on complete Kähler manifolds with nonnegative holomorphic bisectional curvature. It could be summarized as follows. The first Yau's uniformization conjecture is that if M is a complete noncompact n-dimensional Kähler manifold with nonnegative holomorphic bisectional curvature, then

$$\dim_{\mathbb{C}}(\mathcal{O}_d(M^n)) \leq \dim_{\mathbb{C}}(\mathcal{O}_d(\mathbb{C}^n)).$$

The equality holds if and only if M is isometrically biholomorphic to $\mathbb{C}^n$. Here $\mathcal{O}_d(M^n)$ denotes the family of all holomorphic functions on a complete n-dimensional Kähler manifold M of polynomial growth of degree at most d. In [N], Ni established the validity of this conjecture by deriving the monotonicity formula for the heat equation under the assumption that M has maximal volume growth

$$\lim_{r \to +\infty} \frac{Vol(B_p(r))}{r^{2n}} \geq c$$

for a fixed point p and a positive constant c. Later, in [CFYZ], the authors improved Ni's result without the assumption of maximal volume growth. In recent years, G. Liu ([Liu1]) generalized the sharp dimension estimate by only assuming that M admits nonnegative holomorphic sectional curvature. Note that there are noncompact complex manifolds admitting complete Kähler metrics with positive holomorphic sectional curvature but not admitting complete Kähler metrics with nonnegative Ricci curvature (see [Hi]).

The second Yau's uniformization conjecture is that if M is a complete noncompact n-dimensional Kähler manifold with nonnegative holomorphic bisectional curvature, then the ring $\mathcal{O}_P(M)$ of all holomorphic functions of polynomial growth is finitely generated. This one was solved completely by G. Liu ([Liu2]) quite recently. He mainly deployed four techniques to attack this conjecture via Cheeger-Colding-Tian's theory ([ChCo1], [ChCo2], [CCT]), methods of heat flow developed by Ni and Tam ([N], [NT1], [NT4]), Hörmander L^2-estimate of $\overline{\partial}$ ([De]) and three-circle theorem ([Liu1]) as well.

The third Yau's uniformization conjecture is that if M is a complete noncompact n-dimensional Kähler manifold with positive holomorphic bisectional curvature, then M is biholomorphic to the standard n-dimensional complex Euclidean space $\mathbb{C}^n$. The first giant progress relating to the third conjecture could be attributed to Mok, Siu and Yau. In their papers ([MSY] and [M1]), they showed that, under the assumptions of the maximal volume growth and the scalar curvature $R(x)$ decays as

$$0 \leq R(x) \leq \frac{C}{(1 + d(x, x_0))^{2+\epsilon}}$$

for some positive constant C and any arbitrarily small positive number ϵ, a complete noncompact n-dimensional Kähler manifold M with nonnegative holomorphic bisectional curvature is isometrically biholomorphic to $\mathbb{C}^n$. A Riemannian version was solved in [GW2] shortly afterwards. Since then there are several further works aiming to prove the optimal result and reader is referred to [M2], [CTZ], [CZ], [N2], [NT1], [NT2] and [NST]. For example, A. Chau and L. F. Tam ([CT]) proved that a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature and maximal volume growth is biholomorphic to $\mathbb{C}^n$. Recently, G. Liu ([Liu3]) confirmed Yau's uniformization conjecture when M has maximal volume growth. Later, M.-C. Lee and L.-F. Tam ([LT]) also confirmed Yau's uniformization conjecture with the maximal volume growth condition.

For the corresponding first uniformization conjectures in a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion (i.e. Sasakian manifold) which is an odd dimensional counterpart of Kähler geometry (see the next section for its definition and some properties), it was settled that the CR sharp dimension estimate for CR holomorphic functions of polynomial growth with nonnegative pseudohermitian bisectional curvature in [CHL] of which proof is inspired primarily from [N] and [CFYZ]. So it's natural to concern whether the second and third CR Yau's uniformization conjectures hold as well. However, as inspired by recent works of G. Liu ([Liu2], [Liu3]), it is crucial to have the CR analogue of the three-circle theorem which is a step towards establishing the validity of such CR Yau's uniformization conjectures.

In this paper, we mainly focus on the CR three-circle theorem in a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian sectional curvature which is weak than nonnegative pseudohermitian bisectional curvature.

A smooth complex-valued function on a pseudohermitian $(2n+1)$-manifold (M, J, θ) is called CR-holomorphic if

$$\overline{\partial}_b f = 0.$$

We recall $\mathcal{O}^{CR}(M)$ the family of all CR-holomorphic functions f with $Tf(x) = f_0(x) = 0$ ([CHL])

$$\mathcal{O}^{CR}(M) = \{f(x) \in C^{\infty}_{\mathbb{C}}(M) \ |\overline{\partial}_b f(x) = 0 \text{ and } \ f_0(x) = 0 \ \},$$

where the extra condition $Tf(x) = 0$ is included, the interested readers could refer to [CHL] or [FOW].

Next we give the definition of the CR three-circle theorem generalizing the classical Hadamard's three-circle theorem to CR manifolds :

Definition 1.1. *Let (M, J, θ) be a complete pseudohermitian $(2n+1)$-manifold. (M, J, θ) is said to satisfy the CR three-circle theorem if, for any point $p \in M$, any positive number $R > 0$, and any function $f \in \mathcal{O}^{CR}(B_{cc}(p, R))$ on the ball $B_{cc}(p, R)$, $\log M_f(r)$ is convex* *with respect to* $\log r$ *for* $0 < r < R$. *Here* $M_f(r) = \sup_{x \in B_{cc}(p,r)} |f(x)|$ *and* $B_{cc}(p,R)$ *is the Carnot-Carathéodory ball centered at* p *with radius* R.

Now we state our main theorem in this paper as follows:

Theorem 1.1. *If* (M, J, θ) *is a complete noncompact pseudohermitian* $(2n+1)$*-manifold with vanishing torsion, then the CR three-circle theorem holds on* M *if the pseudohermitian sectional curvature is nonnegative; moreover, we have that, for any* $f \in \mathcal{O}^{CR}(M)$,

$$\frac{M_f(kr)}{M_f(r)} \tag{1.2}$$

is increasing with respect to r *for any positive number* $k \geq 1$.

Remark 1.1. 1. In the course of the proof of the CR three-circle theorem, we derive the following CR sub-Laplacian comparison

$$\Delta_b r \leq \frac{(2n-1)}{r}$$

if (M, J, θ) is a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian bisectional curvature. It is sharp in case of $n = 1$. See Corollary 3.1 for details.

2. As a consequence of the sub-Laplacian comparison, Liouville theorem holds for positive pseduoharmonic functions in a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion and nonnegative pseudohermitian Ricci curvature ([CKLT]).

As an application of the preceding theorem, we have the enhanced version of the sharp dimension estimate for CR holomorphic functions of polynomial growth and its rigidity which is served a generalization of authors previous results ([CHL]).

Theorem 1.2. *If* (M, J, θ) *is a complete noncompact pseudohermitian* $(2n+1)$*-manifold of vanishing torsion with nonnegative pseudohermitian sectional curvature, then*

$$\dim_{\mathbb{C}} \left(\mathcal{O}_d^{CR}(M) \right) \leq \dim_{\mathbb{C}} \left(\mathcal{O}_d^{CR}(\mathbb{H}^n) \right)$$

for any positive integer $d \in \mathbb{N}$; morever, if M is simply connected, then the equality holds if only if (M, J, θ) is CR-isomorphic to $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$.

Remark 1.2. 1. In the forthcoming paper, one of the most important application for CR three-circle theorem, we expect that there exists a nonconstant CR-holomorphic function of polynomial growth on a complete noncompact pseudohermitian $(2n+1)$-manifold (M, J, θ) of vanishing torsion, nonnegative pseudohermitian bisectional curvature and maximal volume growth. This is the first step toward the second CR Yau's uniformization conjecture ([Liu2]).

2. As in the paper of G. Liu ([Liu3]), by applying Cheeger-Colding theory for the Webster metric on CR manifolds and Hörmander L^2-technique of $\overline{\partial}_b$ on the space of basic forms and this CR three-circle theorem, we shall work on the third CR Yau's uniformization conjecture as well.

Besides, the CR three-circle theorem could be extended to the case when M admits the pseudohermitian sectional curvature bounded below:

Theorem 1.3. *Let (M, J, θ) be a complete noncompact pseudohermitian $(2n+1)$-manifold, $r(x) = d_{cc}(p, x)$ and $Z_1 = \frac{1}{\sqrt{2}}(\nabla_b r - iJ\nabla_b r)$ for some fixed point $p \in M$. If the pseudohermitian sectional curvature $R_{1\bar{1}1\bar{1}}(x)$ has the inequality*

$$R_{1\bar{1}1\bar{1}}(x) \geq g(r(x)) \tag{1.3}$$

for $g \in C^0([0, +\infty))$ and the pseudohermitian torsion vanishes, a function $u(r) \in C^1(\mathbb{R}^+)$ satisfies

$$2u^2 + u' + \frac{g}{2} \geq 0 \tag{1.4}$$

with $u(r) \sim \frac{1}{2r}$ as $r \to 0^+$ and a function $h(r) \in C^1(\mathbb{R}^+)$ satisfies

$$h'(r) > 0 \tag{1.5}$$

and

$$\frac{1}{2}h''(r)+h'(r)\,u(r)\leq 0 \tag{1.6}$$

with $h(r)\sim\log r$ *as* $r\to 0^{+}$, *then* $\log M_f(r)$ *is convex with respect to the function* $h(r)$ *for* $f\in\mathcal{O}^{CR}(M)$*; moreover, if the vanishing order* $ord_p(f)$ *of* $f\in\mathcal{O}^{CR}(M)$ *at* p *is equal to* d, *then* $\frac{M_f(r)}{\exp(dh(r))}$ *is increasing with respect to* r.

As precedes, there is also a sharp dimension estimate when the pseudohermitian sectional curvature is asymptotically nonnegative.

Theorem 1.4. *If* (M,J,θ) *is a complete noncompact pseudohermitian* $(2n+1)$*-manifold,* $r(x)=d_{cc}(p,x)$ *for some fixed point* $p\in M$, *and there are two positive constants* ϵ, A *such that*

$$R_{j\bar{j}j\bar{j}}(x)\geq-\frac{A}{(1+r(x))^{2+\epsilon}}$$

for any $Z_j\in T_x^{1,0}M$ *with* $|Z_j|=1$, *then there is a constant* $C(\epsilon,A)>0$ *such that, for any* $d\in\mathbb{N}$,

$$\dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}(M)\right)\leq C(\epsilon,A)\,d^{n}.$$

Furthermore, if $d\leq e^{-\frac{3A}{\epsilon}}$, *then*

$$\dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}(M)\right)=1.$$

Finally, if

$$\frac{A}{\epsilon}\leq\frac{1}{4d}$$

for $d\in\mathbb{N}$, *then we have*

$$\dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}(M)\right)\leq\dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}(\mathbb{H}^n)\right).$$

The method we adopt here is inspired from [Liu1]. This paper is organized as follows. In Section 2, we introduce some basic notions about pseudohermitian manifolds and the necessary results for this paper. In Section 3, we show that the CR analogue of the three-circle theorem and some of its applications; specially, we confirm the first CR Yau's uniformization conjecture on the sharp dimension estimate for CR holomorphic functions of polynomial growth and its rigidity. As a by-product, we obtain the CR sub-Laplacian comparison theorem. In Section 4, we generalize the CR three-circle theorem to the case when the pseudohermitian sectional curvature is bounded below. It enables us to derive the dimension estimate when the pseudohermitian sectional curvature is asymptotically nonnegative.

2. Preliminaries

We introduce some basic materials about a pseudohermitian manifold (see [L] and [DT] for more details). Let $(M,\xi) = (M,J,\theta)$ be a $(2n+1)$-dimensional, orientable, contact manifold with the contact structure ξ. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -Id$. We also assume that J satisfies the integrability condition: If X and Y are in ξ, then so are $[JX,Y]+[X,JY]$ and $J([JX,Y]+[X,JY]) = [JX,JY]-[X,Y]$. A contact manifold $(M,\xi) = (M,J,\theta)$ with a CR structure J compatible with ξ together with a contact form θ is called a pseudohermitian manifold or a strictly pseudoconvex CR manifold as well. Such a choice induces a unique vector field $T \in \Gamma(TM)$ transverse to the contact structure ξ, which is called the Reeb vector field or the characteristic vector field of θ such that $\iota_T\theta = 0 = \iota_T d\theta$. A CR structure J could be extended to the complexified space $\xi^{\mathbb{C}} = \mathbb{C} \otimes \xi$ of the contact structure $\xi$and decompose it into the direct sum of $T_{1,0}M$ and $T_{0,1}M$ which are eigenspaces of J corresponding to the eigenvalues 1 and -1, respectively.

Let $\{T, Z_\alpha, Z_{\bar{\alpha}}\}_{\alpha\in I_n}$ be a frame of $TM\otimes\mathbb{C}$, where Z_α is any local frame of $T_{1,0}M$, $Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T_{0,1}M$, and T is the Reeb vector field (or the characteristic direction) and $I_n = \{1,2,...,n\}$. Then $\{\theta^\alpha, \theta^{\bar{\alpha}}, \theta\}$, the coframe dual to $\{Z_\alpha, Z_{\bar{\alpha}}, T\}$, satisfies

$$d\theta = ih_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}$$

for some positive definite matrix of functions $(h_{\alpha\bar{\beta}})$. Usually, we assume such matrix $(h_{\alpha\bar{\beta}})$ is the identity matrix A pseudohermitian manifold (M, J, θ) is called a Sasakian manifold if the pseudohermitian torsion $\tau = \iota_T T_D = 0$

The Levi form $\langle\ ,\ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}M$ defined by

$$\langle Z, W\rangle_{L_\theta} = -i\left\langle d\theta, Z \wedge \overline{W}\right\rangle.$$

We can extend $\langle\ ,\ \rangle_{L_\theta}$ to $T_{0,1}M$ by defining $\left\langle \overline{Z}, \overline{W}\right\rangle_{L_\theta} = \overline{\langle Z, W\rangle}_{L_\theta}$ for all $Z, W \in T_{1,0}M$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}M$, denoted by $\langle\ ,\ \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over M with respect to the volume form $d\mu = \theta \wedge (d\theta)^n$, we get an inner product on the space of sections of each tensor bundle.

The Tanaka-Webster connection of (M, J, θ) is the connection ∇ on $TM \otimes \mathbb{C}$ (and extended to tensors) given, in terms of a local frame $Z_\alpha \in T_{1,0}M$, by

$$\nabla Z_\alpha = {\omega_\alpha}^\beta \otimes Z_\beta, \quad \nabla Z_{\bar{\alpha}} = {\omega_{\bar{\alpha}}}^{\bar{\beta}} \otimes Z_{\bar{\beta}}, \quad \nabla T = 0,$$

where ${\omega_\alpha}^\beta$ are the 1-forms uniquely determined by the following equations:

$$\begin{cases} d\theta^\alpha + \omega_\beta^\alpha \wedge \theta^\beta = \theta \wedge \tau^\alpha \\ \tau_\alpha \wedge \theta^\alpha = 0 \\ \omega_\beta^\alpha + \omega_{\overline{\alpha}}^{\overline{\beta}} = 0 \end{cases}.$$

We can write (by Cartan lemma) $\tau_\alpha = A_{\alpha\gamma}\theta^\gamma$ with $A_{\alpha\gamma} = A_{\gamma\alpha}$. The curvature of Tanaka-Webster connection ∇, expressed in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \theta^{\bar{\alpha}}\}$, is

$$\begin{cases} \Pi_\beta^\alpha = \overline{\Pi_{\overline{\beta}}^{\overline{\alpha}}} = d\omega_\beta^\alpha + \omega_\gamma^\alpha \wedge \omega_\beta^\gamma \\ \Pi_0^\alpha = \Pi_\alpha^0 = \Pi_0^{\overline{\alpha}} = \Pi_{\overline{\alpha}}^0 = \Pi_0^0 = 0 \end{cases}.$$

Webster showed that ${\Pi_\beta}^\alpha$ can be written

$$\Omega_\beta^\alpha = \Pi_\beta^\alpha + i\tau^\alpha \wedge \theta_\beta - i\theta^\alpha \wedge \tau_\beta = {R_\beta}^\alpha{}_{\gamma\bar{\delta}}\theta^\gamma \wedge \theta^{\overline{\delta}} + W_{\beta\gamma}^\alpha \theta^\gamma \wedge \theta - W_{\beta\overline{\gamma}}^\alpha \theta^{\overline{\gamma}} \wedge \theta$$

where the coefficients satisfy

$$R_{\beta\bar{\alpha}\rho\bar{\sigma}} = \overline{R_{\alpha\bar{\beta}\sigma\bar{\rho}}} = R_{\bar{\alpha}\beta\bar{\sigma}\rho} = R_{\rho\bar{\alpha}\beta\bar{\sigma}};\ W^{\alpha}_{\beta\gamma} = A_{\beta\gamma},{}^{\alpha};\ W^{\alpha}_{\beta\bar{\gamma}} = A_{\bar{\gamma}}{}^{\alpha}{}_{,\beta}\,.$$

Besides $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$, the other part of the curvature of Tanaka-Webster connection are clear:

$$\tag{2.1} \begin{cases} R_{\alpha\overline{\beta}\gamma\overline{\mu}} = -2i\left(A_{\alpha\mu}\delta_{\beta\gamma} - A_{\alpha\gamma}\delta_{\beta\mu}\right) \\ R_{\alpha\overline{\beta}\overline{\gamma}\overline{\mu}} = -2i\left(A_{\overline{\beta}\overline{\mu}}\delta_{\alpha\gamma} - A_{\overline{\beta}\overline{\gamma}}\delta_{\alpha\mu}\right) \\ R_{\alpha\overline{\beta}0\gamma} = A_{\gamma\alpha,\overline{\beta}} \\ R_{\alpha\overline{\beta}0\overline{\gamma}} = -A_{\overline{\beta}\overline{\gamma},\alpha} \end{cases} .$$

Here $R_{\beta}{}^{\alpha}{}_{\gamma\bar{\delta}}$ is the pseudohermitian curvature tensor field, $R_{\alpha\bar{\beta}} = R_{\gamma}{}^{\gamma}{}_{\alpha\bar{\beta}}$ is the pseudohermitian Ricci curvature tensor field and $A_{\alpha\beta}$ is the pseudohermitian torsion. $R = h^{\alpha\overline{\beta}}R_{\alpha\bar{\beta}}$ denotes the pseudohermitian scalar curvature. Moreover, we define the pseudohermitian bisectional curvature tensor field

$$R_{\alpha\bar{\alpha}\beta\overline{\beta}}(X,Y) = R_{\alpha\bar{\alpha}\beta\overline{\beta}}X_{\alpha}X_{\overline{\alpha}}Y_{\beta}Y_{\overline{\beta}},$$

the bitorsion tensor field

$$T_{\alpha\overline{\beta}}(X,Y) = \frac{1}{i}(A_{\alpha\gamma}X^{\gamma}Y_{\overline{\beta}} - A_{\overline{\beta}\overline{\gamma}}X^{\overline{\gamma}}Y_{\alpha}),$$

and the torsion tensor field

$$Tor\,(X,Y) = tr\left(T_{\alpha\overline{\beta}}\right) = \frac{1}{i}(A_{\alpha\beta}X^{\beta}Y^{\alpha} - A_{\overline{\alpha}\overline{\beta}}X^{\overline{\beta}}Y^{\overline{\alpha}}),$$

where $X = X^{\alpha}Z_{\alpha}, Y = Y^{\alpha}Z_{\alpha}$ in $T_{1,0}M$.

We will denote the components of the covariant derivatives with indices preceded by comma. The indices $\{0,\alpha,\bar{\alpha}\}$ indicate the covariant derivatives with respect to $\{T, Z_{\alpha}, Z_{\bar{\alpha}}\}$. For the covariant derivatives of a real-valued function, we will often omit the comma, for instance, $u_{\alpha} = Z_{\alpha}u,\ u_{\alpha\bar{\beta}} = Z_{\bar{\beta}}Z_{\alpha}u - \omega_{\alpha}{}^{\gamma}(Z_{\bar{\beta}})Z_{\gamma}u$. The subgradient $\nabla_b\varphi$ of a smooth real-valued function φ is defined by

$$\langle\nabla_b\varphi, Z\rangle_{L_\theta} = Z\varphi$$

for $Z \in \Gamma(\xi)$ where $\Gamma(\xi)$ denotes the family of all smooth vector fields tangent to the conact plane ξ. We could locally write the subgradient $\nabla_b \varphi$ as

$$\nabla_b u = u^{\alpha} Z_{\alpha} + u^{\overline{\alpha}} Z_{\bar{\alpha}}.$$

Accordingly, we could define the subhessian $Hess_b$ as the complex linear map

$$Hess_b : T_{1,0}M \oplus T_{0,1}M \longrightarrow T_{1,0}M \oplus T_{0,1}M$$

by

$$(Hess_b \varphi)\, Z = \nabla_Z \nabla_b \varphi$$

for $Z \in \Gamma(\xi)$ and a smooth real-valued function φ.

Also, the sub-Laplacian is defined by

$$\Delta_b u = tr\,(Hess_b u) = u_{\alpha}{}^{\alpha} + u^{\alpha}{}_{\alpha}.$$

Now we recall the following commutation relations (see [L]). Let φ be a smooth real-valued function, $\sigma = \sigma_{\alpha}\theta^{\alpha}$ be a $(1,0)$-form and $\varphi_0 = T\varphi$, then we have

$$\left\{ \begin{array}{rcl} \varphi_{\alpha\beta} & = & \varphi_{\beta\alpha}, \\ \varphi_{\alpha\bar{\beta}} - \varphi_{\bar{\beta}\alpha} & = & ih_{\alpha\overline{\beta}}\varphi_0 \\ \varphi_{0\alpha} - \varphi_{\alpha 0} & = & A_{\alpha\beta}\varphi^{\beta} \\ \sigma_{\alpha,\beta\gamma} - \sigma_{\alpha,\gamma\beta} & = & i\left(A_{\alpha\gamma}\sigma_{\beta} - A_{\alpha\beta}\sigma_{\gamma}\right) \\ \sigma_{\alpha,\overline{\beta}\overline{\gamma}} - \sigma_{\alpha,\overline{\gamma}\overline{\beta}} & = & -i(h_{\alpha\overline{\gamma}}A^{\delta}_{\overline{\beta}}\sigma_{\delta} - h_{\alpha\overline{\beta}}A^{\delta}_{\overline{\gamma}}\sigma_{\delta}) \\ \sigma_{\alpha,\beta\overline{\gamma}} - \sigma_{\alpha,\overline{\gamma}\beta} & = & ih_{\beta\overline{\gamma}}\sigma_{\alpha,0} + R_{\alpha}{}^{\delta}{}_{\beta\overline{\gamma}}\sigma_{\delta} \\ \sigma_{\alpha,0\beta} - \sigma_{\alpha,\beta 0} & = & A^{\overline{\gamma}}{}_{\beta}\sigma_{\alpha,\bar{\gamma}} - A_{\alpha\beta,\bar{\gamma}}\sigma^{\overline{\gamma}} \\ \sigma_{\alpha,0\bar{\beta}} - \sigma_{\alpha,\bar{\beta}0} & = & A^{\gamma}{}_{\bar{\beta}}\sigma_{\alpha,\gamma} + A_{\bar{\gamma}\bar{\beta},\alpha}\sigma^{\overline{\gamma}} \end{array} \right. . \tag{2.2}$$

Subsequently, we introduce the notion about the Carnot-Carathéodory distance.

Definition 2.1. *A piecewise smooth curve* $\gamma : [0,1] \to (M, \xi)$ *is said to be horizontal if* $\gamma'(t) \in \xi$ *whenever* $\gamma'(t)$ *exists. The length of* γ *is then defined by*

$$L(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle_{L_\theta}^{\frac{1}{2}} \, dt.$$

The Carnot-Carathéodory distance between two points $p,\ q \in M$ *is*

$$d_{cc}(p,q) = \inf \{ L(\gamma) |\ \gamma \in C_{p,q} \},$$

where $C_{p,q}$ *is the set of all horizontal curves joining* p *and* q*. We say* (M, ξ) *is complete if it's complete as a metric space. By Chow's connectivity theorem, there always exists a horizontal curve joining* p *and* q*, so the distance is finite. The diameter* d_c *is defined by*

$$d_c(M) = \sup \{ d_{cc}(p,q) |\ p, q \in M \}.$$

Note that there is a minimizing geodesic joining p *and* q *so that its length is equal to the distance* $d_{cc}(p,q)$*.*

For any fixed point $x \in M$, a CR-holomorphic function f is called to be of polynomial growth if there are a nonnegative number d and a positive constant $C = C(x, d, f)$, depending on x, d and f, such that

$$|f(y)| \leq C \left(1 + d_{cc}(x,y)\right)^d$$

for all $y \in M$, where $d_{cc}(x,y)$ denotes the Carnot-Carathéodory distance between x and y. Furthermore, we could define the degree of a CR-holomorphic function f of polynomial growth by

$$\deg(f) = \inf \left\{ d \geq 0 \left| \begin{array}{l} |f(y)| \leq C \left(1 + d_{cc}(x,y)\right)^d \quad \forall\ y \in M, \\ \text{for some } \ d \geq 0 \ \text{ and } \ C = C(x,d,f) \end{array} \right. \right\}. \tag{2.3}$$

With these notions, we could define the family $\mathcal{O}_d^{CR}(M)$ of all CR-holomorphic functions f of polynomial growth of degree at most d with $Tf(x) = f_0(x) = 0$:

$$\mathcal{O}_d^{CR}(M) = \{ f \in \mathcal{O}^{CR}(M) \mid \ \deg(f) \leq d \ \}. \tag{2.4}$$

Finally, we denote by $ord_p(f) = \max\{m \in \mathbb{N} \mid D^\alpha f(p) = 0, \forall\ |\alpha| < m\}$ the vanishing order of CR-holomorphic function f at p where $D^\alpha = \prod_{j\in I_n} Z_j^{\alpha_j}$ with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$.

3. CR Three-Circle Theorem

In this section, we will derive the CR analogue of three-circle theorem on a complete noncompact pseudohermitian $(2n+1)$-manifold. Before that, we need a lemma which is essential in the course of the proof of the CR three-circle theorem as follows:

Lemma 3.1. *If (M, J, θ) is a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian sectional curvature, the Carnot-Carathéodory distance function $r(x) = d_{cc}(p, x)$ from a fixed point p to a point x in M is smooth at $q \in M$ and $Z_1 = \frac{1}{\sqrt{2}}(\nabla_b r - iJ\nabla_b r)$, then*

$$r_{1\bar{1}} \leq \frac{1}{2r}. \tag{3.1}$$

In particular, we have

$$(\log r)_{1\bar{1}} \leq 0.$$

Proof. Let $\left\{e_j, e_{\tilde{j}}, T\right\}_{j\in I_n}$ be an orthonormal frame at q where $e_{\tilde{j}} = Je_j$ and $e_1 = \nabla_b r$. By Corollary 2.3 in [DZ] and vanishing pseudohermitian torsion, we could parallel transport such frame at q to obtain the orthonormal frame along the radial ∇-geodesic γ from p to q. Hence we have an orthonormal frame $\{Z_j, Z_{\bar{j}}, T\}_{j\in I_n}$ along γ where $Z_j = \frac{1}{\sqrt{2}}\left(e_j - ie_{\tilde{j}}\right)$ and $Z_{\bar{j}} = \overline{Z_j}$. By the fact that γ is the ∇-geodesic, we have

$$\begin{aligned} r_{11} &= -\tfrac{1}{2}(ie_2e_1 + e_2e_2)r - (\nabla_{Z_1}Z_1)r \\ &= -\tfrac{1}{2}(ie_2e_1 + e_2e_2)r + \tfrac{1}{2}\left[i\nabla_{(J\nabla_b r)}\nabla_b r + J\left(\nabla_{(J\nabla_b r)}\nabla_b r\right)\right] \end{aligned}$$

and

$$\begin{aligned} r_{1\bar{1}} &= \tfrac{1}{2}(ie_2e_1 + e_2e_2)r - \left(\nabla_{Z_{\bar{1}}}Z_1\right)r \\ &= \tfrac{1}{2}(ie_2e_1 + e_2e_2)r - \tfrac{1}{2}\left[i\nabla_{(J\nabla_b r)}\nabla_b r + J\left(\nabla_{(J\nabla_b r)}\nabla_b r\right)\right]. \end{aligned}$$

These imply that

$$r_{11} = -r_{1\bar{1}} \tag{3.2}$$

along γ. From the construction of the curve γ, we derive that

$$r_0 = 0 \tag{3.3}$$

on the ∇-geodesic γ. Besides, we could see that $r_{1\bar{1}}$ is real by the commutation formula. At the point q,

$$\begin{aligned} 0 &= \tfrac{1}{2}\left(|\nabla_b r|^2\right)_{1\bar{1}} \\ &= \sum_\alpha \left(|r_{\alpha 1}|^2 + |r_{\alpha\bar{1}}|^2 + r_{\alpha 1\bar{1}} r_{\bar{\alpha}} + r_{\bar{\alpha}1\bar{1}} r_\alpha\right) \\ &\geq |r_{11}|^2 + |r_{1\bar{1}}|^2 + r_{11\bar{1}} r_{\bar{1}} + r_{\bar{1}1\bar{1}} r_1 \\ &= 2r_{1\bar{1}}^2 + \left(r_{1\bar{1}1} + ir_{10} + R^1_{11\bar{1}} r_1\right) r_{\bar{1}} + \left(r_{1\bar{1}} - ir_0\right)_{\bar{1}} r_1 \\ &= 2r_{1\bar{1}}^2 + \langle \nabla_b r_{1\bar{1}}, \nabla_b r\rangle_{L_\theta} + \tfrac{1}{2} R_{1\bar{1}1\bar{1}} \\ &\geq 2r_{1\bar{1}}^2 + \left(\nabla_b r\right) r_{1\bar{1}} \\ &= 2r_{1\bar{1}}^2 + \left(\nabla r\right) r_{1\bar{1}} \\ &= 2r_{1\bar{1}}^2 + \tfrac{\partial r_{1\bar{1}}}{\partial r}. \end{aligned} \tag{3.4}$$

Here we use the facts that $r_1 = \frac{1}{\sqrt{2}}$ and $r_{1\bar{1}}$ is real, the equality (3.2), (3.3), and the commutation formulas (2.2). Together with the initial condition of $r_{1\bar{1}}$ as r goes to zero, we have

$$r_{1\bar{1}} \leq \frac{1}{2r}.$$

In particular, (3.1) indicates that

$$(\log r)_{1\bar{1}} = \frac{r_{1\bar{1}}}{r} - \frac{|r_1|^2}{r^2} \leq 0.$$

This completes the proof. □

Actually, the similar deductions enable us to derive the substantial CR sub-Laplacian comparisons below.

Corollary 3.1. *If (M, J, θ) is a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian Ricci curvature, then we have the CR sub-Laplacian comparison, for $n \geq 2$,*

$$\Delta_b r \leq \frac{2^n}{r}; \tag{3.5}$$

furthermore, if the pseudohermitian bisectional curvature is nonnegative, then

$$\Delta_b r \leq \frac{2n-1}{r}. \tag{3.6}$$

Finally, the equality holds in (3.6) only if, for any $j \in I_n$,

$$R_{1\bar{1}j\bar{j}} = 0. \tag{3.7}$$

Remark 3.1. When (M, J, θ) is a complete noncompact pseudohermitian 3-manifold of vanishing torsion with nonnegative pseudohermitian Ricci curvature, it's also easy to derive

$$\Delta_b r \leq \frac{1}{r}$$

from the estimate (3.9). The estimate is sharp in the sense of the equality holds only if M is flat as in the proof of Theorem 1.2 .

Proof. By the similar computation as precedes, for any $j \neq 1$,

$$\begin{array}{rcl} 0 & = & \frac{1}{2}\left(|\nabla_b r|^2\right)_{j\bar{j}} \\ & = & \sum\limits_{\alpha}\left(|r_{\alpha j}|^2 + |r_{\alpha\bar{j}}|^2 + r_{\alpha j\bar{j}}r_{\bar{\alpha}} + r_{\bar{\alpha}j\bar{j}}r_{\alpha}\right) \\ & \geq & |r_{j\bar{j}}|^2 + r_{1j\bar{j}}r_{\bar{1}} + r_{\bar{1}j\bar{j}}r_1 \\ & = & r_{j\bar{j}}^2 + \left(r_{1\bar{j}j} + ir_{10} + R^1_{1j\bar{j}}r_1\right)r_{\bar{1}} + r_{\bar{1}j\bar{j}}r_1 \\ & \geq & r_{j\bar{j}}^2 + \left\langle \nabla_b r_{j\bar{j}}, \nabla_b r\right\rangle_{L_\theta} \\ & = & r_{j\bar{j}}^2 + (\nabla_b r)\, r_{j\bar{j}} \\ & = & r_{j\bar{j}}^2 + \frac{\partial}{\partial r}r_{j\bar{j}}, \end{array} \tag{3.8}$$

and the inequality (3.1), it's easy to derive

$$\Delta_b r \leq \frac{2n-1}{r}.$$

From the inequalities (3.4) and (3.8), it follows that

$$\begin{aligned} 0 &\geq \left(2r_{1\bar{1}}^2 + \tfrac{\partial r_{1\bar{1}}}{\partial r} + \tfrac{1}{2}R_{1\bar{1}1\bar{1}}\right) + \sum_{j\neq 1}\left(r_{j\bar{j}}^2 + \tfrac{\partial}{\partial r} r_{j\bar{j}} + \tfrac{1}{2}R_{1\bar{1}j\bar{j}}\right) \\ &\geq 2^{1-n}\left(\sum_j r_{j\bar{j}}\right)^2 + \tfrac{\partial}{\partial r}\sum_j r_{j\bar{j}} + \tfrac{1}{2}R_{1\bar{1}} \\ &\geq 2^{1-n}\left(\sum_j r_{j\bar{j}}\right)^2 + \tfrac{\partial}{\partial r}\sum_j r_{j\bar{j}} \end{aligned} \tag{3.9}$$

and then

$$\Delta_b r = 2\sum_j r_{j\bar{j}} \leq \frac{2^n}{r}.$$

□

Now, we could proceed with the proof of the CR three-circle theorem below:

Proof. (of the Theorem 1.1) First of all, we prove that if (M, J, θ) admits nonnegative pseudohermitian sectional curvature, then the CR three-circle theorem holds. On the closure $\overline{A}\,(p; r_1, r_3)$ of the annulus

$$A\,(p; r_1, r_3) = \{x \in M \mid r_1 < r\,(x) = d\,(p, x) < r_3\}$$

for $0 < r_1 < r_3$, we define

$$F\,(x) = (\log r_3 - \log r\,(x)) \log M_f\,(r_1) + (\log r\,(x) - \log r_1) \log M_f\,(r_3)$$

and

$$G\,(x) = (\log r_3 - \log r_1) \log |f\,(x)|\,.$$

May assume that $M_f\,(r_1) < M_f\,(r_3)$. Let f be a CR-holomorphic function on M with

$$f_0 = 0. \tag{3.10}$$

It suffices to claim

$$G \leq F$$

on $\overline{A}(p;r_1,r_3)$. It's clear that $G \leq F$ on the boundary $\partial A(p;r_1,r_3)$ of the annulus $A(p;r_1,r_3)$. Suppose that $G(x) > F(x)$ for some interior point x in $A(p;r_1,r_3)$, then we could choose a point $q \in A(p;r_1,r_3)$ such that the function $(G-F)$ attains the maximum value at q.

If $q \notin Cut(p)$, then

$$i\partial_b\overline{\partial}_b(G-F)(q) \leq 0$$

by observing that the inequality $i\partial_b\overline{\partial}_b(G-F)(q) > 0$, which says that $(G-F)_{\alpha\overline{\beta}}$ is positive definite, implies the positivity of the sub-Lplacian $\Delta_b(G-F)(q) > 0$ (this contradicts that q is a maximum point of $(G-F)$). In particular,

$$(G-F)_{1\overline{1}}(q) \leq 0 \tag{3.11}$$

where $Z_1 = \frac{1}{\sqrt{2}}(\nabla_b r - iJ\nabla_b r)$. Note that $(G-F)_0(q) = 0$ due to (3.10) and (3.3).

On the other hand, it follows from [FOW] or [CHL] that there is a transverse Kähler structure at the point q and we denoted such local coordinates in some open neighborhood U of the point q by $\{z_\alpha, x\}_{\alpha\in I_n}$ with $T = \frac{\partial}{\partial x}$ and

$$Z_1(q) = \left(\frac{\partial}{\partial z_1} - \theta\left(\frac{\partial}{\partial z_1}\right)T\right)|_q.$$

Restrict to the leaf space $\widetilde{D} = [x=0]$ and write the point y in U as $(\widetilde{y}, x)$. It's clear that $q = (\widetilde{q}, 0)$. Hereafter the quantity with the tilde means such one lies in the slice $\widetilde{D}$. This enables us to transfer the local property of the Kähler manifolds to the CR manifolds. Let $\widetilde{G}$ and $\widetilde{F}$ denote the restrictions of G and F to the leaf space $\widetilde{D}$. So $\widetilde{q}$ is a maximum point of $\left(\widetilde{G}-\widetilde{F}\right)$ and $\widetilde{f} = f|_{\widetilde{D}}$ is a holomorphic function on $U \cap \widetilde{D}$. Because $\left(\widetilde{G}-\widetilde{F}\right)$ attains the maximum value at $\widetilde{q}$ (this implies that $\left|\widetilde{f}(\widetilde{q})\right| \neq 0$), the Poincaré-Lelong equation

$$\frac{i}{2\pi}\partial\overline{\partial}\log\left|\widetilde{f}\right|^2 = \left[D\left(\widetilde{f}\right)\right]$$

gives

$$\widetilde{G}_{1\bar{1}}\left(\widetilde{q}\right)=0$$

where $D\left(\widetilde{f}\right)$ is the divisor of $\widetilde{f}$. However, the condition (3.10) implies that G is independent of the characteristic direction T. So we have the equality

$$G_{1\bar{1}}\left(q\right)=0.$$

This and Lemma 3.1 indicate that

$$\left(G-F\right)_{1\bar{1}}\left(q\right)\geq 0;$$

however, it's not enough to obtain the contradiction from (3.11). So we take a modified function F_{ϵ} for replacing the original function F so that it enables us to get the contradiction. Set

$$F_{\epsilon}\left(x\right)=a_{\epsilon}\log\left(r\left(x\right)-\epsilon\right)+b_{\epsilon}$$

for any sufficiently small positive number ϵ. Here the two constants a_{ϵ} and b_{ϵ} are determined by the following equations:

$$F_{\epsilon}\left(r_j\right)=F_{\epsilon}\left(\partial B\left(p,r_j\right)\right)=\log\left(\frac{r_3}{r_1}\right)\log M_f\left(r_j\right)$$

for $j=1,3$. It's apparent to see $F_{\epsilon}\longrightarrow F$ on the annulus $A\left(p;r_1,r_3\right)$ as $\epsilon\longrightarrow 0^{+}$ and $a_{\epsilon}>0$. Let q_{ϵ} be a maximum point in $A\left(p;r_1,r_3\right)$ of the function $\left(G-F_{\epsilon}\right)$. With the same deduction, we have

$$\left(G-F_{\epsilon}\right)_{1\bar{1}}\left(q_{\epsilon}\right)\leq 0 \tag{3.12}$$

and

$$G_{1\bar{1}}\left(q_{\epsilon}\right)=0.$$

From Lemma 3.1 again, we obtain

$$\left(\log\left(r-\epsilon\right)\right)_{1\bar{1}}\left(q_{\epsilon}\right)=\frac{r_{1\bar{1}}}{\left(r-\epsilon\right)}-\frac{1}{2\left(r-\epsilon\right)^{2}}<0.$$

Accordingly, we have

$$(G - F_\epsilon)_{1\bar{1}}(q_\epsilon) > 0.$$

It contradicts with the inequality (3.12). This indicates that

$$(G - F_\epsilon) \leq 0$$

in the annulus $A(p; r_1, r_3)$.

If $q_\epsilon \in Cut(p)$, then we adopt the trick of Calabi as follows. Choose a number $\epsilon_1 \in (0, \epsilon)$ and the point p_1 lying on the minimal D-geodesic from p to q_ϵ with $d(p, p_1) = \epsilon_1$. Set

$$\widehat{r}(x) = d(p_1, x)$$

and consider the slight modification of the function $F_\epsilon(x)$

$$F_{\epsilon,\epsilon_1}(x) = a_\epsilon \log(\widehat{r} + \epsilon_1 - \epsilon) + b_\epsilon.$$

It's not hard to observe that $F_\epsilon(q_\epsilon) = F_{\epsilon,\epsilon_1}(q_\epsilon)$ and $F_\epsilon \leq F_{\epsilon,\epsilon_1}$; hence, we know $(G - F_{\epsilon,\epsilon_1})$ also attains the maximum value at q_ϵ. Then, applying the similar argument as precedes, we still have

$$G - F_{\epsilon,\epsilon_1} \leq 0$$

in the annulus $A(p; r_1, r_3)$. Letting $\epsilon_1 \to 0^+$, then $\epsilon \to 0^+$, the validity of the CR three circle theorem is settled.

As for the monotonicity of (1.2), it's easily derived by taking the 3-tuples (r_1, r_2, kr_2) and (r_1, kr_1, kr_2) into the convexity of the CR three circle theorem for $0 < r_1 \leq r_2 < +\infty$. $\square$

It's clear that we have the following sharp monotonicity:

Proposition 3.1. *If (M, J, θ) is a complete noncompact pseudohermitian $(2n+1)$-manifold on which CR three-circle theorem holds and $ord_p f \geq k \geq 1$ for some $f \in \mathcal{O}^{CR}(M)$ and $p \in M$, then*

$$\frac{M_f(r)}{r^k}$$

is increasing in terms of r.

Proof. Let $0 < r_2 \leq r_3 < +\infty$. Since the vanishing order of f at p is at least k, for any $\epsilon > 0$, there is a sufficiently small number $0 < r_1 < r_2$, such that

$$\log M_f(r_1) \leq \log M_f(r_3) + (k-\epsilon)\log \frac{r_1}{r_3}. \tag{3.13}$$

Substituting (3.13) into the inequality (1.1), we get

$$\frac{M_f(r_2)}{r_2^{k-\epsilon}} \leq \frac{M_f(r_3)}{r_3^{k-\epsilon}}.$$

The proposition is accomplished by letting $\epsilon \to 0^+$. □

Therefore, Theorem 1.1 and Proposition 3.1 imply

Corollary 3.2. *If* (M, J, θ) *is a complete noncompact pseudohermitian* $(2n+1)$*-manifold of vanishing torsion with nonnegative pseudohermitian sectional curvature, and* $ord_p f \geq k \geq 1$ *for some* $f \in \mathcal{O}^{CR}(M)$ *and* $p \in M$, *then*

$$\frac{M_f(r)}{r^k}$$

is increasing in terms of r.

Before giving a variety of applications of the CR three-circle theorem, we need some notations about the CR-holomorphic functions of polynomial growth.

Definition 3.1. *Let* (M, J, θ) *be a complete noncompact pseudohermitian* $(2n+1)$*-manifold. We denote the collections*

$$\left\{ f \in \mathcal{O}^{CR}(M) \,\middle|\, \limsup_{r\to+\infty} \frac{M_f(r)}{r^d} < +\infty \right\}$$

and

$$\left\{ f \in \mathcal{O}^{CR}(M) \,\middle|\, \liminf_{r\to+\infty} \frac{M_f(r)}{r^d} < +\infty \right\}$$

by $\widetilde{\mathcal{O}}_d^{CR}(M)$ *and* $\widehat{\mathcal{O}}_d^{CR}(M)$, *respectively.*

It's clear that $\widetilde{\mathcal{O}}_d^{CR}(M) \subseteq \mathcal{O}_d^{CR}(M) \cap \widehat{\mathcal{O}}_d^{CR}(M)$.

Proposition 3.2. *If (M, J, θ) is a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian sectional curvature, then $f \in \widehat{\mathcal{O}}_d^{CR}(M)$ if and only if $\frac{M_f(r)}{r^d}$ is decreasing with respect to r for any $f \in \mathcal{O}^{CR}(M)$.*

Proof. It's straightforward to see that the sufficient part holds. On the other hand, let $0 < r_1 \leq r_2 < +\infty$, by the assumption that

$$f \in \widehat{\mathcal{O}}_d^{CR}(M),$$

for any positive number ϵ, there's a sequence $\{\lambda_j\} \nearrow +\infty$ such that

$$\log M_f(\lambda_j) \leq \log M_f(r_1) + (d+\epsilon)\log \lambda_j.$$

From Theorem 1.1, by taking $r_3 = \lambda_j$, it follows that

$$\log M_f(r_2) \leq \log M_f(r_1) + (d+\epsilon)\log \frac{r_2}{r_1}.$$

Let ϵ go to zero, the necessary part follows. $\square$

From the last proposition, it's easy to deduce the conclusion below.

Corollary 3.3. *If (M, J, θ) is a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian sectional curvature, then*

$$\widetilde{\mathcal{O}}_d^{CR}(M) = \widehat{\mathcal{O}}_d^{CR}(M).$$

In addition, we have the asymptotic property for the degree of CR-holomorphic functions of polynomial growth as follows:

Corollary 3.4. *Let (M, J, θ) be a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian sectional curvature. If*

$$f \in \widetilde{\mathcal{O}}_{d+\epsilon}^{CR}(M)$$

for any number $\epsilon > 0$, then

$$f \in \widetilde{\mathcal{O}}_d^{CR}(M).$$

Proof. From Proposition 3.2 and Corollary 3.3, we know

$$\frac{M_f(r)}{r^{d+\epsilon}}$$

is decreasing with respect to r for any positive number ϵ. By acontradiction, it's easy to validate the monotonicity of

$$\frac{M_f(r)}{r^d}.$$

Then, by the last two corollaries, we have

$$f \in \widetilde{\mathcal{O}}_d^{CR}(M).$$

□

From Corollary 3.4, we know

Corollary 3.5. *If (M, J, θ) is a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with nonnegative pseudohermitian sectional curvature, then*

$$\mathcal{O}_d^{CR}(M) = \widetilde{\mathcal{O}}_d^{CR}(M).$$

As an application, we could recover and generalize the CR sharp dimension estimate in [CHL] under the assumption of nonnegative pseudohermitian sectional curvature instead of nonnegative pseudohermitian bisectional curvature.

Proof. (of Theorem 1.2) Suppose on the contrary, i.e.

$$\dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}(M)\right) > \dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}(\mathbb{H}^n)\right)$$

for some positive integer $d \in \mathbb{N}$. Then for any point $p \in M$, there's a nonzero CR-holomorphic function f of polynomial growth of degree at most d with

$$ord_p f \geq d+1$$

Concerning the existence of such function f, one could refer to the proof of Theorem 1.1 in [CHL, (5.17)] which is just a method of the Poincaré-Siegel argument via linear algebra ([M1]). Therefore, we have

$$\lim_{r \to 0^+} \frac{M_f(r)}{r^d} = 0.$$

However, this contradicts with the monotonicity of the function $\frac{M_f(r)}{r^d}$ as in Proposition 3.2. Hence, the sharp dimension estimate holds. As for the rigidity part, we just claim that if

$$\dim_{\mathbb{C}} \left(\mathcal{O}_d^{CR}(M)\right) = \dim_{\mathbb{C}} \left(\mathcal{O}_d^{CR}(\mathbb{H}^n)\right), \tag{3.14}$$

then (M, J, θ) is CR-isomorphic to $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$. From Proposition 4.1 in [T] (or Theorem 7.15 in [B]), it suffices to show that M has constant J-holomorphic sectional curvature -3. From the equation, for any $p \in M$, $Z \in (T_{1,0}M)_p$ with $|Z| = 1$,

$$\begin{aligned} R^\theta\left(Z, \overline{Z}, Z, \overline{Z}\right) &= R\left(Z, \overline{Z}, Z, \overline{Z}\right) + g_\theta\left(\left(Z \wedge \overline{Z}\right) Z, \overline{Z}\right) + 2d\theta\left(Z, \overline{Z}\right) g_\theta\left(JZ, \overline{Z}\right) \\ &= R\left(Z, \overline{Z}, Z, \overline{Z}\right) - 3, \end{aligned}$$

the proof of the rigidity part is completed if we justify that the pseudohermitian sectional curvature vanishes. Adopting the notations as in Theorem 1.1, we just claim that, for simplification,

$$R\left(Z_1, \overline{Z}_1, Z_1, \overline{Z}_1\right)(p) = 0 \tag{3.15}$$

where

$$Z_1 = \frac{\partial}{\partial z_1} - \theta\left(\frac{\partial}{\partial z_1}\right) T.$$

The equality (3.14) enables us to see that there is a function $f \in \mathcal{O}_d^{CR}(M)$ such that

$$f(z_1, ..., z_n, x) = z_1^d + O\left(r^{d+1}\right)$$

locally. This impies that

$$ord_p f = d.$$

Therefore, from Corollary 3.2 and Proposition 3.2, we obtain

$$\frac{M_f(r)}{r^d}$$

is constant. In the proofs of Theorem 1.1 and Lemma 3.1, it's not difficult to find that $G - F$ attains the maximum value 0 on $\partial B(p, r)$ at the point $q(r)$ for any positive number r and then

$$R(\nabla_b r, J\nabla_b r, \nabla_b r, J\nabla_b r)(q(r)) = 0.$$

From the definition of the chosen function f, we could take a subsequence $\{(\nabla_b r)(q(r_j))\}_{j\in\mathbb{N}}$ such that its limit, as $r \to 0^+$, lies in the tangent space at p spanned by $\frac{\partial}{\partial z_1}|_p$ and $T|_p$. Then the equality (3.15) holds by the formula (2.1). Accordingly, this theorem is accomplished. □

4. An extension of CR Three-Circle Theorem

Subsequently, we will give the proof of the CR three-circle theorem when the pseudohermitian sectional curvature is bounded from below by a function.

Proof. (of Theorem 1.3) Although this proof is similar to the one of the CR three-circle theorem, we give its proof for completeness. Here we adopt the same notations as in the proof of Theorem 1.1. Define

$$F(x) = (h(r_3) - h(r(x))) \log M_f(r_1) + (h(r(x)) - h(r_1)) \log M_f(r_3)$$

and

$$G(x) = (h(r_3) - h(r_1)) \log |f(x)|$$

on the annulus $A(p; r_1, r_3)$ for $0 < r_1 < r_3 < +\infty$. We still assume that

$$M_f(r_1) < M_f(r_3). \tag{4.1}$$

It's clear that $G \leq F$ on the boundary $\partial A(p; r_1, r_3)$ by (1.5). It suffices to show that

$$G \leq F$$

on the annulus $A(p; r_1, r_3)$ to reach our first conclusion by the maximum principle. Suppose that $G(x) > F(x)$ for some interior point x in $A(p; r_1, r_3)$, then we could choose a point $q \in A(p; r_1, r_3)$ such that the function $(G - F)$ attains the maximum value at q.

If $q \notin Cut(p)$, then

$$(G - F)_{1\bar{1}}(q) \leq 0. \tag{4.2}$$

With the same deduction in Theorem 1.1, we have

$$G_{1\bar{1}}(q) = 0$$

from the Poincaré-Lelong equation and the fact that $f(q) \neq 0$ with the help of the transverse Kähler structure. Due to the fact that $h(r) \sim \log r$ as $r \to 0^+$, we could define

$$F_\epsilon(x) = a_\epsilon \log\left(e^{h(r)} - \epsilon\right) + b_\epsilon$$

for any sufficiently small number $\epsilon > 0$ and the two constants a_ϵ and b_ϵ are restricted by the following equations

$$F_\epsilon(r_j) = (h(r_3) - h(r_1)) \log M_f(r_j)$$

for $j = 1, 3$. It's obvious that $F_\epsilon \to F$ on the annulus $A(p; r_1, r_3)$ as $\epsilon \to 0^+$. Due to the inequality (1.5) and the assumption (4.1), we see that $a_\epsilon > 0$. Denote by q_ϵ a maximum point in $A(p; r_1, r_3)$ of the function $(G - F_\epsilon)$ and modify the point q into the point q_ϵ. From (1.4), (3.4), (1.3), and the initial condition $u(r) \sim \frac{1}{2r}$ as $r \to 0^+$ imply that

$$r_{1\bar{1}} \leq u(r). \tag{4.3}$$

By the hypotheses (1.5), (1.6) and the inequality (4.3), we get

$$\begin{aligned}&\left(\log\left(e^{h(r)}-\epsilon\right)\right)_{1\bar{1}}(q_\epsilon)\\&=\frac{-\epsilon e^h(h')^2|r_1|^2+\left(e^{h(r)}-\epsilon\right)\left(e^h h''|r_1|^2+e^h h' r_{1\bar{1}}\right)}{\left(e^{h(r)}-\epsilon\right)^2}\\&\leq-\frac{\epsilon e^h(h')^2}{2\left(e^{h(r)}-\epsilon\right)^2}\\&<0\end{aligned}$$

for sufficiently small positive number ϵ. It yields that

$$(G-F_\epsilon)_{1\bar{1}}(q_\epsilon)>0.$$

This contradicts with (4.2). Therefore we obtain

$$G\leq F_\epsilon$$

for sufficeintly small number $\epsilon>0$. If $q\in Cut(p)$, then, by the trick of Calabi again, let $\epsilon_1\in(0,\epsilon)$ and the point p_1 lying on the minimal D-geodesic from p to q_ϵ with $d(p,p_1)=\epsilon_1$. Set

$$\widehat{r}(x)=d(p_1,x)$$

and consider the modified function $F_{\epsilon,\epsilon_1}(x)$ of the function $F_\epsilon(x)$

$$F_{\epsilon,\epsilon_1}(x)=a_\epsilon\log\left(e^{h(\widehat{r}+\epsilon_1)}-\epsilon\right)+b_\epsilon.$$

Due to the monotonicity of the function h, we see that $F_\epsilon\leq F_{\epsilon,\epsilon_1}$. It's clear that $F_\epsilon(q_\epsilon)=F_{\epsilon,\epsilon_1}(q_\epsilon)$. So the point q_ϵ is still a maximum point of $(G-F_{\epsilon,\epsilon_1})$. Set

$$\widehat{Z}_1=\frac{1}{\sqrt{2}}\left(\nabla_b\widehat{r}-iJ\nabla_b\widehat{r}\right).$$

By observing the expansion of $\left(\log\left(e^{h(\widehat{r}+\epsilon_1)}-\epsilon\right)\right)_{\widehat{1}\widehat{\bar{1}}}(q_\epsilon)$, (1.5), (1.6), and the continuity of the pseudohermitian sectional curvature imply that

$$(F_{\epsilon,\epsilon_1})_{\widehat{1}\widehat{\bar{1}}}(q_\epsilon)<0$$

for sufficiently small $\epsilon_1 > 0$ for fixed ϵ. Here the property that the pseudohermitian sectional curvature is continuous is utilized to obtain the estimate

$$\widehat{r}_{\widehat{1}\widehat{\bar{1}}} \leq u + \epsilon'$$

for small positive error $\epsilon' = \epsilon'(\epsilon_1)$. Then $(G - F_{\epsilon,\epsilon_1})_{\widehat{1}\widehat{\bar{1}}}(q_\epsilon) > 0$. However, it contradicts with the fact that $(G - F_{\epsilon,\epsilon_1})$ attains a maximum point at q_ϵ. Accordingly, the inequality

$$G \leq F_{\epsilon,\epsilon_1}$$

holds. Letting $\epsilon_1 \to 0^+$, then $\epsilon \to 0^+$, we have

$$G \leq F$$

on the annulus $A(p; r_1, r_3)$. Because $ord_p(f) = d$ and $h(r) \sim \log r$ as $r \to 0^+$, then we have, for any $\epsilon > 0$,

$$\log M_f(r_1) \leq \log M_f(r_2) + (d - \epsilon)(h(r_1) - h(r_2))$$

for sufficiently small positive number r_1 and $r_1 < r_2$. By the convexity of $\log M_f(r)$ with respect to the function $h(r)$

$$\log M_f(r) \leq \frac{h(r_2) - h(r)}{h(r_2) - h(r_1)} \log M_f(r_1) + \frac{h(r) - h(r_1)}{h(r_2) - h(r_1)} \log M_f(r_2)$$

for $r_1 \leq r \leq r_2$, we obtain the monotonicity of $\frac{M_f(r)}{\exp(dh(r))}$. This completes the proof. □

Choosing the functions $g(r) = -1$, $u(r) = \frac{(e^{2r}+1)}{2(e^{2r}-1)}$, and $h(r) = \log \frac{e^r-1}{e^r+1}$ in Theorem 1.3, we have the following consequence:

Corollary 4.1. *If (M, J, θ) is a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with the pseudohermitian sectional curvature bounded from below by -1, $f \in O^{CR}(M)$, then $\log M_f(r)$ is convex with respect to the function $\log \frac{e^r-1}{e^r+1}$. In particular, $\frac{M_f(r)}{\left(\frac{e^r-1}{e^r+1}\right)^d}$ is increasing for $ord_p(f) = d$.*

Similarly, choosing the functions $g(r) = 1$, $u(r) = \frac{1}{2}\cot r$, and $h(r) = \log\tan\frac{r}{2}$ in Theorem 1.3, we obtain

Corollary 4.2. *If (M, J, θ) is a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion with the pseudohermitian sectional curvature bounded from below by 1, $f \in O^{CR}(B(p,R))$, then $\log M_f(r)$ is convex with respect to the function $\log\tan\frac{r}{2}$. In particular, $\frac{M_f(r)}{\left(\tan\frac{r}{2}\right)^d}$ is increasing for $ord_p(f) = d$.*

With the help of Theorem 1.3, we have the dimension estimate when the pseudohermitian sectional curvature is asymptotically nonnegative.

Proof. (of Theorem 1.4) Although the proof is almost the same as in [Liu1], we give its proof for completeness. May assume $\epsilon < \frac{1}{2}$. Choose

$$u(r) = \frac{1}{2r} + \frac{A}{(1+r)^{1+\epsilon}};$$

hence, the inequality holds

$$2u^2 + u' - \frac{1}{2}\frac{A}{(1+r)^{2+\epsilon}} \geq 0.$$

Suppose $h(r)$ is the solution to the equation

$$\begin{cases} \frac{1}{2}h'' + h'u = 0, \\ \lim\limits_{r\to 0^+} \frac{\exp(h(r))}{r} = 1, \end{cases}$$

then

$$h'(r) = \frac{\exp\left(\frac{2A}{\epsilon(1+r)^\epsilon}\right)}{r}\exp\left(-\frac{2A}{\epsilon}\right).$$

Accordingly,

$$h(r) \geq \exp\left(-\frac{2A}{\epsilon}\right)\log r + C$$

for any number $r \geq 1$. Here $C = C(A,\epsilon)$. Theorem 1.3 implies that if the vanishing order $ord_p(f)$ of $f \in \mathcal{O}^{CR}(M)$ at p is equal to d, then $\frac{M_f(r)}{\exp(dh(r))}$ is increasing with respect to r. So

$$M_f(r) \geq \exp(dh(r)) \lim_{s\to 0^+} \frac{M_f(s)}{\exp(dh(s))} \geq C_1 r^{d\exp\left(-\frac{2A}{\epsilon}\right)} \tag{4.4}$$

where

$$C_1 = \exp(Cd) \lim_{s \to 0^+} \frac{M_f(s)}{\exp(dh(s))}.$$

Consider the Poincaré-Siegel map

$$\Phi : \mathcal{O}_d^{CR}(M) \longrightarrow \mathbb{C}^{q\left(\left[d \exp\left(\frac{2A}{\epsilon}\right)\right]\right)}$$
$$f \longmapsto (D^\alpha f)_{|\alpha| \leq \left[d \exp\left(\frac{2A}{\epsilon}\right)\right]}$$

where $q(m) = \binom{n+m}{n}$ for any $m \in \mathbb{N}$ and $[a]$ denotes the greatest integer less than or equal to a. We would claim that Φ is injective; for if $0 \neq f \in \mathcal{O}_d^{CR}(M)$ and $D^\alpha f = 0$ for any $|\alpha| \leq d' = \left[d \exp\left(\frac{2A}{\epsilon}\right)\right]$, then

$$ord_p(f) \geq d' + 1.$$

Hence, by (4.4), we obtain

$$M_f(r) \geq C_1 r^{(1+d') \exp\left(-\frac{2A}{\epsilon}\right)};$$

however, this contradicts with the fact

$$f \in \mathcal{O}_d^{CR}(M).$$

Therefore we have the dimension estimate

$$\dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}(M)\right) \leq \dim_{\mathbb{C}}\left(\mathcal{O}^{CR}_{\left[d \exp\left(\frac{2A}{\epsilon}\right)\right]}(H^n)\right) = C(\epsilon, A) d^n$$

for any $d \in \mathbb{N}$. If $d \leq e^{-\frac{3A}{\epsilon}}$, then

$$\dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}(M)\right) \leq \dim_{\mathbb{C}}\left(\mathcal{O}_{d \exp\left(\frac{2A}{\epsilon}\right)}(\mathbb{C}^n)\right) \leq \dim_{\mathbb{C}}\left(\mathcal{O}_{\exp\left(-\frac{A}{\epsilon}\right)}(\mathbb{C}^n)\right) = 1.$$

Last, if $\frac{A}{\epsilon} \leq \frac{1}{4d}$, then

$$d \exp\left(\frac{2A}{\epsilon}\right) < d + 1$$

and

$$\dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}(M)\right) \leq \dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}(\mathbb{H}^n)\right).$$

This theorem is accomplished. $\square$

It's not difficult to observe that Theorem 1.4 includes the case when the pseudohermitian sectional curvature is nonnegative outside a compact set as follows:

Corollary 4.3. *Let (M, J, θ) is a complete noncompact pseudohermitian $(2n+1)$-manifold of vanishing torsion of which the pseudohermitian sectional curvature is nonnegative outside a compact subset K and is bounded from below by $-a$ for some $a > 0$ on M. If $\lambda = a\,(d_c\,(K))^2$ where $d_c\,(K)$ denotes the diameter of K, then there is a positive constant $C\,(\lambda, n)$ such that*

$$\dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}\,(M)\right) \leq C\,(\lambda, n)\,d^n$$

for any positive integer d. For any $d \in \mathbb{N}$, there is a positive number $\epsilon\,(d)$ such that if $\lambda \leq \epsilon\,(d)$, then we have

$$\dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}\,(M)\right) \leq \dim_{\mathbb{C}}\left(\mathcal{O}_d^{CR}\,(\mathbb{H}^n)\right).$$

Furthermore, there exists a number $\delta\,(\lambda) > 0$ such that

$$\dim_{\mathbb{C}}\left(\mathcal{O}_{\delta(\lambda)}^{CR}\,(M)\right) = 1.$$

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