A Novel Technique for Constructing Difference Schemes for Systems of Singularly Perturbed Equations

Po-Wen Hsieh[1], Yin-Tzer Shih[2], Suh-Yuh Yang[3]*, Cheng-Shu You[3]

[1] *Department of Applied Mathematics, Chung Yuan Christian University, Jhongli District, Taoyuan City 32023, Taiwan*
[2] *Department of Applied Mathematics, National Chung Hsing University, Taichung 40227, Taiwan*
[3] *Department of Mathematics, National Central University, Jhongli District, Taoyuan City 32001, Taiwan*

Manuscript submitted to CiCP

Abstract. In this paper, we propose a novel and simple technique to construct effective difference schemes for solving systems of singularly perturbed convection-diffusion-reaction equations, whose solutions may display boundary or interior layers. We illustrate the technique by taking the Il'in-Allen-Southwell scheme for 1-D scalar equations as a basis to derive a formally second-order scheme for 1-D coupled systems and then extend the scheme to 2-D case by employing an alternating direction approach. Numerical examples are given to demonstrate the high performance of the obtained scheme on uniform meshes as well as piecewise-uniform Shishkin meshes.

AMS subject classifications: 65N06, 65N12, 76M20

Key words: system of singularly perturbed equations, system of viscous Burgers' equations, boundary layer, interior layer, Il'in-Allen-Southwell scheme, formally second-order scheme.

1 Introduction

The aim of this paper is to demonstrate a novel and simple technique to construct effective difference schemes for solving systems of singularly perturbed convection-diffusion-reaction (CDR) equations, whose solutions may display boundary or interior layers. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and convex polygonal domain with boundary $\partial\Omega$. We consider the Dirichlet boundary value problem for a system of two linear CDR equations:

$$\left\{ \begin{array}{rcll} -E\Delta \boldsymbol{u} - A\boldsymbol{u}_x - B\boldsymbol{u}_y + C\boldsymbol{u} &=& \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} &=& \boldsymbol{g} & \text{on } \partial\Omega, \end{array} \right. \tag{1.1}$$

*Corresponding author. *Email addresses:* `pwhsieh0209@gmail.com` (P.-W. Hsieh), `yintzer.shih@gmail.com` (Y.-T. Shih), `syyang@math.ncu.edu.tw` (S.-Y. Yang), `csdyou@gmail.com` (C.-S. You)

where $\boldsymbol{u}=(u_1,u_2)^\top$ is the physical quantity of interest; $E=\mathrm{diag}\{\varepsilon_1,\varepsilon_2\}$ is a given constant matrix of diffusivities with $0<\varepsilon_i\leq 1$ and we are particularly interested in the case of $\varepsilon_1=\varepsilon_2\ll 1$; $A=(a_{ij})_{2\times 2}$ and $B=(b_{ij})_{2\times 2}$ are the given convection coefficient matrices and $C=(c_{ij})_{2\times 2}$ is the reaction coefficient matrix; $\boldsymbol{f}=(f_1,f_2)^\top$ is a given source term and $\boldsymbol{g}=(g_1,g_2)^\top$ is a prescribed boundary data. Throughout this paper, we always assume that the coefficient matrices A,B,C, with $a_{ii}\neq 0$ and $b_{ii}\neq 0$ in Ω for $i=1,2$, and $\boldsymbol{f},\boldsymbol{g}$ are sufficiently smooth such that problem (1.1) is well posed.

Compared with the scalar singularly perturbed CDR equation, system (1.1) can model more complicated physical phenomena, such as the turbulent interaction of waves and currents [15], the diffusion processes in the presence of chemical reactions [14], the optimal control and certain resistance-capacitor electrical circuits [6], and the magnetohydrodynamic duct flow problems [2, 3], etc. Similar to the scalar equation, as one of the diffusivities ε_i is small enough than the module of the corresponding convection or reaction coefficients, the solution component u_i of (1.1) may display boundary or interior layers. These layers are narrow regions where the solution component changes rapidly and it is often difficult to resolve numerically the high gradients near the layer regions. Therefore, the study of singularly perturbed problems has been the focus of intense research for quite some time. However, most numerical methods for such problems are lacking in either stability or accuracy (cf. [7, 12]). For example, the central difference scheme performs very poorly since large spurious oscillations appear.

Aiming to overcome the difficulties caused by high gradients of solution of system of singularly perturbed CDR equations, most difference schemes are essentially of the upwind type and thus have only first-order accuracy. In [8], O'Riordan *et al.* proposed a difference scheme for 1-D case which is consisting of simple upwinding with an appropriate piecewise-uniform Shishkin mesh. They showed the first-order convergence when A is a strictly diagonally dominant M-matrix and $C=\mathbf{0}$. In [9], with proper hypotheses placed on the coupling matrices A and C, a similar scheme was designed for more general 1-D systems. Also, they obtained the first-order approximations by using a Jacobi-type iteration [10]. For 2-D systems of singularly perturbed CDR equations, we proposed in [3] a compact difference scheme with accuracy of $O(\varepsilon^2(h+k)+\varepsilon(h^2+k^2)+(h^3+k^3))$ when A and B are symmetry matrices with zero diagonal entries and $C=\mathbf{0}$. To the best of our knowledge, it seems not easy to construct workable difference schemes, rather than the upwind-type schemes, for systems of singularly perturbed CDR equations.

In this paper, we will propose a novel and simple technique to construct effective difference schemes for solving 2-D systems of singularly perturbed CDR equations, whose solutions may display boundary or interior layers. We will illustrate the technique by taking the Il'in-Allen-Southwell scheme [11, 12] as a basis and combining with a novel treatment for the diffusion terms to derive a three-point difference scheme for the 1-D counterpart of system (1.1). We then extend the scheme to the 2-D coupled system (1.1) by employing an alternating direction approach [3, 16]. We remark that the Il'in-Allen-Southwell scheme is a formally second-order difference scheme for scalar CDR equations [12]. Consequently, the difference scheme developed in this paper for system (1.1) is of formally second-order accuracy. The proposed technique in this paper can be easily applied to other difference schemes, such as the El-Mistikawy-Werle scheme [12], for scalar CDR equations to generate the analogue for coupled systems. We will provide several numerical examples, including 1-D and 2-D nonlinearly coupled systems of viscous Burgers' equations, to illustrate the performance of the obtained scheme. From the numerical results, we observe that the scheme can achieve high accuracy and stability, even if the diffusivities are very small. Moreover, with appropriate piecewise-uniform Shishkin meshes, the numerical evidence shows that the computed solutions converge uniformly in the diffusivities in the discrete maximum norm.

Finally, we remark that the underlying idea developed in this paper can be directly applied to systems with more equations and in higher dimensional domains. The remainder of this paper is organized as follows. In Section 2, we first derive a difference scheme for the system (1.1) in 1-D case. We then extend the scheme to 2-D case in Section 3. Several numerical examples are provided in Section 4 to demonstrate the effectiveness of the scheme and the conclusions are made in Section 5.

2 The difference scheme for 1-D systems

In this section, we will derive a formally second-order difference scheme for the 1-D counterpart of (1.1). Consider the scalar CDR equation in, for simplicity, $I=(0,1)$:

$$-\varepsilon u''(x)-a(x)u'(x)+c(x)u(x)=f(x) \quad \text{for } x\in I, \tag{2.1}$$

where $0<\varepsilon\leq 1$; $-a(x)\neq 0$ is the convection direction at x, c is the reaction coefficient and f is a given source function. Let $\mathcal{P}=\{0=x_0<x_1<\cdots<x_N=1\}$ be a partition of I with $h_i=x_i-x_{i-1}$ for $1\leq i\leq N$ and $\overline{h}_i=(h_i+h_{i+1})/2$ for $1\leq i\leq N-1$. Define $a^i:=a(x_i)$, $c^i:=c(x_i)$ and $f^i:=f(x_i)$. The Il'in-Allen-Southwell difference scheme [11, 12] for approximating (2.1) at x_i can be expressed as

$$-\alpha^i\delta_x^2 u_i - a^i\delta_x u_i + c^i u_i = f^i, \tag{2.2}$$

where u_i denotes the approximation to $u(x_i)$ and the δ-operators on u_i are defined as

$$\delta_x^2 u_i := \frac{1}{\overline{h}_i}\left(\frac{u_{i+1}-u_i}{h_{i+1}} - \frac{u_i-u_{i-1}}{h_i}\right) \quad \text{and} \quad \delta_x u_i := \frac{u_{i+1}-u_{i-1}}{2\overline{h}_i}, \tag{2.3}$$

and the coefficient α^i is given by

$$\alpha^i = \frac{a^i h_i h_{i+1}\left(e^{a^i h_i/\varepsilon}-e^{-a^i h_{i+1}/\varepsilon}\right)}{2\left(h_i e^{-a^i h_{i+1}/\varepsilon} - 2\overline{h}_i + h_{i+1}e^{a^i h_i/\varepsilon}\right)}. \tag{2.4}$$

We note that if the partition $\mathcal{P}$ is uniform, i.e., $h_i=h$ for all i, then α^i becomes

$$\alpha^i = \frac{a^i h}{2}\coth\left(\frac{a^i h}{2\varepsilon}\right). \tag{2.5}$$

Reverting u_i in (2.2) to $u(x_i)$, we obtain

$$-\frac{a^i h}{2}\coth\left(\frac{a^i h}{2\varepsilon}\right)\delta_x^2 u(x_i) - a^i \delta_x u(x_i) + c^i u(x_i) = f^i. \tag{2.6}$$

By the Taylor expansion, the above equation is changed to

$$-\varepsilon u''(x_i) - a^i u'(x_i) + c^i u(x_i) = f^i + \left\{\frac{a^i h}{2}\coth\left(\frac{a^i h}{2\varepsilon}\right) - \varepsilon\right\} u''(x_i) + O(h^2). \tag{2.7}$$

Note that $\coth(x)$ has the series expansion,

$$\coth(x) = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + \cdots \quad \text{for } 0 < |x| < \pi. \tag{2.8}$$

Thus, if $0 < |a^i| h/(2\varepsilon) < \pi$, we have

$$\frac{a^i h}{2}\coth\left(\frac{a^i h}{2\varepsilon}\right) - \varepsilon = \frac{a^i h}{2}\left(\frac{2\varepsilon}{a^i h} - \frac{a^i h}{6\varepsilon} + \cdots\right) - \varepsilon = O(h^2/\varepsilon), \tag{2.9}$$

and this shows that (2.6) is a formally second-order scheme. On the other hand, when $|a^i| h/(2\varepsilon) \geq \pi$, since $\coth(x) \approx 1$ for $x \geq \pi$ and $\coth(x) \approx -1$ for $x \leq -\pi$, we have

$$-\frac{a^i h}{2}\coth\left(\frac{a^i h}{2\varepsilon}\right) \approx -\frac{|a^i| h}{2} \quad \text{for} \quad \frac{|a^i| h}{2\varepsilon} \geq \pi, \tag{2.10}$$

and therefore (2.6) is very close to the first-order upwind scheme when ε is very small. Moreover, one can verify that the Il'in-Allen-Southwell difference scheme is first-order uniformly convergent in the diffusivity ε in the discrete maximum norm, i.e., $\|u - u_h\|_\infty \leq Ch^1$, where C is independent of ε and h. For more details, we refer the reader to [12].

We next introduce a formally second-order difference scheme for the following 1-D coupled system of two linear CDR equations:

$$\begin{aligned} -\varepsilon_1 u_1'' - a_{11} u_1' - a_{12} u_2' + c_{11} u_1 + c_{12} u_2 &= f_1 \quad \text{in } I, \qquad (2.11)\\ -\varepsilon_2 u_2'' - a_{21} u_1' - a_{22} u_2' + c_{21} u_1 + c_{22} u_2 &= f_2 \quad \text{in } I, \qquad (2.12) \end{aligned}$$

where we assume $a_{11} \neq 0$ and $a_{22} \neq 0$ in I. In order to having the form $(-\varepsilon u'' - au' + cu)$ for both unknowns u_1 and u_2, we reformulate the coupled system (2.11)-(2.12) in each subinterval $I_i := (x_{i-1}, x_{i+1})$ centered at x_i as

$$\begin{aligned} -\varepsilon_1 u_1'' - a_{11} u_1' + (\mu_{12} u_2'' - \mu_{12} u_2'') - a_{12} u_2' + c_{11} u_1 + c_{12} u_2 &= f_1 \quad \text{in } I_i, \qquad (2.13)\\ (\mu_{21} u_1'' - \mu_{21} u_1'') - a_{21} u_1' - \varepsilon_2 u_2'' - a_{22} u_2' + c_{21} u_1 + c_{22} u_2 &= f_2 \quad \text{in } I_i, \qquad (2.14) \end{aligned}$$

where the coefficients μ_{12} and μ_{21} are defined by

$$\mu_{12} := \begin{cases} \mathrm{sign}(a_{12}^i a_{22}^i)\varepsilon_2, & \text{if } a_{12}^i \neq 0, \\ 0, & \text{if } a_{12}^i = 0, \end{cases} \qquad \mu_{21} := \begin{cases} \mathrm{sign}(a_{21}^i a_{11}^i)\varepsilon_1, & \text{if } a_{21}^i \neq 0, \\ 0, & \text{if } a_{21}^i = 0. \end{cases} \tag{2.15}$$

We remark that with the above definition of μ_{12}, the convection direction at x_i of $(-\mu_{12}u_2''-a_{12}u_2'+c_{12}u_2)$ in (2.13) is then consistent with that of $(-\varepsilon_2 u_2''-a_{22}u_2'+c_{22}u_2)$ in (2.14). Similarly, the definition of μ_{21} makes the convection directions at x_i of $(-\mu_{21}u_1''-a_{21}u_1'+c_{21}u_1)$ in (2.14) and $(-\varepsilon_1 u_1''-a_{11}u_1'+c_{11}u_1)$ in (2.13) are consistent as well.

Now, we have from (2.13) that

$$\begin{aligned} -\varepsilon_1 u_1''-a_{11}u_1'+c_{11}u_1 &= F_1 \quad \text{in } I_i, && (2.16)\\ -\mu_{12}u_2''-a_{12}u_2'+c_{12}u_2 &= F_2 \quad \text{in } I_i, && (2.17)\end{aligned}$$

where the functions F_1 and F_2 are respectively given by

$$\begin{aligned} F_1 &= f_1+a_{12}u_2'-c_{12}u_2, && (2.18)\\ F_2 &= f_1+\varepsilon_1 u_1''+a_{11}u_1'-\mu_{12}u_2''-c_{11}u_1. && (2.19)\end{aligned}$$

Applying the Il'in-Allen-Southwell scheme (2.2) to (2.16) and (2.17) at x_i, we obtain

$$\begin{aligned} -\alpha_{11}^i\delta_x^2 u_{1i}-a_{11}^i\delta_x u_{1i}+c_{11}^i u_{1i} &= F_1^i, && (2.20)\\ -\alpha_{12}^i\delta_x^2 u_{2i}-a_{12}^i\delta_x u_{2i}+c_{12}^i u_{2i} &= F_2^i, && (2.21)\end{aligned}$$

where u_{1i} and u_{2i} approximate $u_1(x_i)$ and $u_2(x_i)$, respectively, and

$$\alpha_{11}^i = \frac{a_{11}^i h_i h_{i+1}(e^{a_{11}^i h_i/\varepsilon_1}-e^{-a_{11}^i h_{i+1}/\varepsilon_1})}{2(h_i e^{-a_{11}^i h_{i+1}/\varepsilon_1}-2\overline{h}_i+h_{i+1}e^{a_{11}^i h_i/\varepsilon_1})}, \tag{2.22}$$

$$\alpha_{12}^i = \begin{cases} \dfrac{a_{12}^i h_i h_{i+1}(e^{a_{12}^i h_i/\mu_{12}}-e^{-a_{12}^i h_{i+1}/\mu_{12}})}{2(h_i e^{-a_{12}^i h_{i+1}/\mu_{12}}-2\overline{h}_i+h_{i+1}e^{a_{12}^i h_i/\mu_{12}})}, & \text{if } a_{12}^i\neq 0,\\ 0, & \text{if } a_{12}^i=0.\end{cases} \tag{2.23}$$

Notice that the sum of (2.16) and (2.17) gives (2.11). Therefore, adding (2.20) and (2.21) and using the fact $F_1+F_2=f_1-\mu_{12}u_2''$, and approximating $\mu_{12}u_2''$ at x_i by the central difference rule, we obtain a formally second-order scheme for (2.11) at x_i,

$$-\alpha_{11}^i\delta_x^2 u_{1i}-a_{11}^i\delta_x u_{1i}-(\alpha_{12}^i-\mu_{12})\delta_x^2 u_{2i}-a_{12}^i\delta_x u_{2i}+c_{11}^i u_{1i}+c_{12}^i u_{2i}=f_1^i. \tag{2.24}$$

With the same strategy, we have another formally second-order scheme for (2.12) at x_i,

$$-(\alpha_{21}^i-\mu_{21})\delta_x^2 u_{1i}-a_{21}^i\delta_x u_{1i}-\alpha_{22}^i\delta_x^2 u_{2i}-a_{22}^i\delta_x u_{2i}+c_{21}^i u_{1i}+c_{22}^i u_{2i}=f_2^i, \tag{2.25}$$

where α_{21}^i and α_{22}^i are given by

$$\alpha_{21}^i = \begin{cases} \dfrac{a_{21}^i h_i h_{i+1}(e^{a_{21}^i h_i/\mu_{21}}-e^{-a_{21}^i h_{i+1}/\mu_{21}})}{2(h_i e^{-a_{21}^i h_{i+1}/\mu_{21}}-2\overline{h}_i+h_{i+1}e^{a_{21}^i h_i/\mu_{21}})}, & \text{if } a_{21}^i\neq 0,\\ 0, & \text{if } a_{21}^i=0,\end{cases} \tag{2.26}$$

$$\alpha_{22}^i = \frac{a_{22}^i h_i h_{i+1}(e^{a_{22}^i h_i/\varepsilon_2}-e^{-a_{22}^i h_{i+1}/\varepsilon_2})}{2(h_i e^{-a_{22}^i h_{i+1}/\varepsilon_2}-2\overline{h}_i+h_{i+1}e^{a_{22}^i h_i/\varepsilon_2})}. \tag{2.27}$$

Finally, putting (2.24) and (2.25) together as a difference system, we reach a formally second-order difference scheme for the 1-D coupled system (2.11)-(2.12).

Remark 2.1. *In deriving (2.13), we add the term $(\mu_{12}u_2''-\mu_{12}u_2'')$ into (2.11) to integrating the form $(-\mu_{12}u_2''-a_{12}u_2'+c_{12}u_2)$ for the solution component u_2. A similar idea is applied to (2.12) by adding the term $(\mu_{21}u_1''-\mu_{21}u_1'')$ for u_1 to reach (2.14). This idea is very different from that in [2, 3], where we use a combination technique to decouple the MHD duct flow equations. With this novel idea here, we can easily maintain the convection directions at x_i for all solution components correctly in the difference scheme (2.24)-(2.25).*

3 The difference scheme for 2-D systems

In this section, we will extend the formally second-order difference scheme (2.24)-(2.25) to the 2-D coupled system (1.1). We assume that the 2-D domain is a unit square region $\Omega=(0,1)\times(0,1)$. Let $\{(x_i,y_j)\}$ be the collection of grid points of a rectangular mesh of Ω, and set $h_i=x_i-x_{i-1}$, $k_j=y_j-y_{j-1}$, $\overline{h}_i=(h_i+h_{i+1})/2$ and $\overline{k}_j=(k_j+k_{j+1})/2$. We then denote $v_{i,j}$ the finite difference approximation to some function v at the grid point (x_i,y_j), and introduce the following δ-operators:

$$\begin{aligned}
\delta_x^2 v_{i,j} &:= \frac{1}{\overline{h}_i}\left(\frac{v_{i+1,j}-v_{i,j}}{h_{i+1}}-\frac{v_{i,j}-v_{i-1,j}}{h_i}\right) \quad \text{and} \quad \delta_x v_{i,j}:=\frac{v_{i+1,j}-v_{i-1,j}}{2\overline{h}_i};\\
\delta_y^2 v_{i,j} &:= \frac{1}{\overline{k}_j}\left(\frac{v_{i,j+1}-v_{i,j}}{k_{j+1}}-\frac{v_{i,j}-v_{i,j-1}}{k_j}\right) \quad \text{and} \quad \delta_y v_{i,j}:=\frac{v_{i,j+1}-v_{i,j-1}}{2\overline{k}_j}.
\end{aligned} \tag{3.1}$$

We rewrite the 2-D coupled system (1.1) of singularly perturbed CDR equations as

$$\left\{\begin{array}{rcll} -\varepsilon_1\Delta u_1-a_{11}u_{1x}-a_{12}u_{2x}-b_{11}u_{1y}-b_{12}u_{2y}+c_{11}u_1+c_{12}u_2 &=& f_1 & \text{in } \Omega,\\ -\varepsilon_2\Delta u_2-a_{21}u_{1x}-a_{22}u_{2x}-b_{21}u_{1y}-b_{22}u_{2y}+c_{21}u_1+c_{22}u_2 &=& f_2 & \text{in } \Omega, \end{array}\right. \tag{3.2}$$

where we assume $a_{ii}\neq 0$ and $b_{ii}\neq 0$ in Ω for $i=1,2$. Using the alternating direction approach (cf. [3,16]), we have

$$\left\{\begin{array}{rcll} -\varepsilon_1 u_{1xx}-a_{11}u_{1x}-a_{12}u_{2x}+c_{11}u_1+c_{12}u_2 &=& F_1 & \text{in } \Omega,\\ -\varepsilon_2 u_{2xx}-a_{21}u_{1x}-a_{22}u_{2x}+c_{21}u_1+c_{22}u_2 &=& F_2 & \text{in } \Omega; \end{array}\right. \tag{3.3}$$

$$\left\{\begin{array}{rcll} -\varepsilon_1 u_{1yy}-b_{11}u_{1y}-b_{12}u_{2y}+c_{11}u_1+c_{12}u_2 &=& F_3 & \text{in } \Omega,\\ -\varepsilon_2 u_{2yy}-b_{21}u_{1y}-b_{22}u_{2y}+c_{21}u_1+c_{22}u_2 &=& F_4 & \text{in } \Omega, \end{array}\right. \tag{3.4}$$

where the functions F_i, $i=1,2,3,4$, are given by

$$F_1 = f_1-(-\varepsilon_1 u_{1yy}-b_{11}u_{1y}-b_{12}u_{2y}), \tag{3.5}$$
$$F_2 = f_2-(-\varepsilon_2 u_{2yy}-b_{21}u_{1y}-b_{22}u_{2y}), \tag{3.6}$$
$$F_3 = f_1-(-\varepsilon_1 u_{1xx}-a_{11}u_{1x}-a_{12}u_{2x}), \tag{3.7}$$
$$F_4 = f_2-(-\varepsilon_2 u_{2xx}-a_{21}u_{1x}-a_{22}u_{2x}). \tag{3.8}$$

Note that adding system (3.3) to system (3.4) gives system (3.2). Now applying the 1-D difference scheme (2.24)-(2.25) to systems (3.3) and (3.4) at (x_i,y_j), we obtain the schemes of formally second-order accuracy in the x- and y-directions, respectively:

$$\left\{\begin{array}{rcl} -\alpha_{11}^{i,j}\delta_x^2 u_{1i,j}-a_{11}^{i,j}\delta_x u_{1i,j}-(\alpha_{12}^{i,j}-\mu_{12})\delta_x^2 u_{2i,j}-a_{12}^{i,j}\delta_x u_{2i,j}+c_{11}^{i,j}u_{1i,j}+c_{12}^{i,j}u_{2i,j} &=& F_1^{i,j}, \\ -(\alpha_{21}^{i,j}-\mu_{21})\delta_x^2 u_{1i,j}-a_{21}^{i,j}\delta_x u_{1i,j}-\alpha_{22}^{i,j}\delta_x^2 u_{2i,j}-a_{22}^{i,j}\delta_x u_{2i,j}+c_{21}^{i,j}u_{1i,j}+c_{22}^{i,j}u_{2i,j} &=& F_2^{i,j}; \end{array}\right. \tag{3.9}$$

$$\left\{\begin{array}{rcl} -\beta_{11}^{i,j}\delta_y^2 u_{1i,j}-b_{11}^{i,j}\delta_y u_{1i,j}-(\beta_{12}^{i,j}-\nu_{12})\delta_y^2 u_{2i,j}-b_{12}^{i,j}\delta_y u_{2i,j}+c_{11}^{i,j}u_{1i,j}+c_{12}^{i,j}u_{2i,j} &=& F_3^{i,j}, \\ -(\beta_{21}^{i,j}-\nu_{21})\delta_y^2 u_{1i,j}-b_{21}^{i,j}\delta_y u_{1i,j}-\beta_{22}^{i,j}\delta_y^2 u_{2i,j}-b_{22}^{i,j}\delta_y u_{2i,j}+c_{21}^{i,j}u_{1i,j}+c_{22}^{i,j}u_{2i,j} &=& F_4^{i,j}, \end{array}\right. \tag{3.10}$$

where $g^{i,j}$ denotes the function value $g(x_i,y_j)$ for a given function g and for $p,q=1,2$,

$$\mu_{pq} = \left\{\begin{array}{ll} \mathrm{sign}(a_{pq}^{i,j}a_{qq}^{i,j})\varepsilon_q, & \text{if } a_{pq}^{i,j}\neq 0, \\ 0, & \text{if } a_{pq}^{i,j}=0; \end{array}\right. \qquad \eta_{pq}=\left\{\begin{array}{ll} \mu_{pq}, & \text{if } p\neq q, \\ \varepsilon_p, & \text{if } p=q; \end{array}\right. \tag{3.11}$$

$$\alpha_{pq}^{i,j} = \left\{\begin{array}{ll} \dfrac{a_{pq}^{i,j}h_i h_{i+1}(e^{a_{pq}^{i,j}h_i/\eta_{pq}}-e^{-a_{pq}^{i,j}h_{i+1}/\eta_{pq}})}{2(h_i e^{-a_{pq}^{i,j}h_{i+1}/\eta_{pq}}-2\bar{h}_i+h_{i+1}e^{a_{pq}^{i,j}h_i/\eta_{pq}})}, & \text{if } a_{pq}^{i,j}\neq 0, \\ \eta_{pq}, & \text{if } a_{pq}^{i,j}=0; \end{array}\right. \tag{3.12}$$

$$\nu_{pq} = \left\{\begin{array}{ll} \mathrm{sign}(b_{pq}^{i,j}b_{qq}^{i,j})\varepsilon_q, & \text{if } b_{pq}^{i,j}\neq 0, \\ 0, & \text{if } b_{pq}^{i,j}=0; \end{array}\right. \qquad \xi_{pq}=\left\{\begin{array}{ll} \nu_{pq}, & \text{if } p\neq q, \\ \varepsilon_p, & \text{if } p=q; \end{array}\right. \tag{3.13}$$

$$\beta_{pq}^{i,j} = \left\{\begin{array}{ll} \dfrac{b_{pq}^{i,j}k_j k_{j+1}(e^{b_{pq}^{i,j}k_j/\xi_{pq}}-e^{-b_{pq}^{i,j}k_{j+1}/\xi_{pq}})}{2(k_j e^{-b_{pq}^{i,j}k_{j+1}/\xi_{pq}}-2\bar{k}_j+k_{j+1}e^{b_{pq}^{i,j}k_j/\xi_{pq}})}, & \text{if } b_{pq}^{i,j}\neq 0, \\ \xi_{pq}, & \text{if } b_{pq}^{i,j}=0. \end{array}\right. \tag{3.14}$$

Adding system (3.9) to system (3.10) and using the fact that

$$F_1+F_3 = f_1+c_{11}u_1+c_{12}u_2, \tag{3.15}$$

$$F_2+F_4 = f_2+c_{21}u_1+c_{22}u_2, \tag{3.16}$$

we have the difference scheme for (3.2) at (x_i,y_j) with a formally second-order accuracy,

$$\left\{\begin{array}{l} -\alpha_{11}^{i,j}\delta_x^2 u_{1i,j}-a_{11}^{i,j}\delta_x u_{1i,j}-(\alpha_{12}^{i,j}-\mu_{12})\delta_x^2 u_{2i,j}-a_{12}^{i,j}\delta_x u_{2i,j} \\ \quad -\beta_{11}^{i,j}\delta_y^2 u_{1i,j}-b_{11}^{i,j}\delta_y u_{1i,j}-(\beta_{12}^{i,j}-\nu_{12})\delta_y^2 u_{2i,j}-b_{12}^{i,j}\delta_y u_{2i,j}+c_{11}^{i,j}u_{1i,j}+c_{12}^{i,j}u_{2i,j}=f_1^{i,j}, \\ -(\alpha_{21}^{i,j}-\mu_{21})\delta_x^2 u_{1i,j}-a_{21}^{i,j}\delta_x u_{1i,j}-\alpha_{22}^{i,j}\delta_x^2 u_{2i,j}-a_{22}^{i,j}\delta_x u_{2i,j} \\ \quad -(\beta_{21}^{i,j}-\nu_{21})\delta_y^2 u_{1i,j}-b_{21}^{i,j}\delta_y u_{1i,j}-\beta_{22}^{i,j}\delta_y^2 u_{2i,j}-b_{22}^{i,j}\delta_y u_{2i,j}+c_{21}^{i,j}u_{1i,j}+c_{22}^{i,j}u_{2i,j}=f_2^{i,j}. \end{array}\right. \tag{3.17}$$

4 Numerical experiments

In this section, we will present several numerical examples, including 1-D and 2-D nonlinearly coupled systems of viscous Burgers' equations, to illustrate the high performance of the obtained formally second-order difference scheme.

Example 4.1 *(1-D boundary layer problem with variable convection coefficients).* This example is taken from [8]. We consider the following 1-D coupled system in $I=(0,1)$:

$$\begin{cases} -\varepsilon u_1''-(4+xe^x)u_1'-(-1-2x)u_2'=-1-x-3x^2 & \text{in } I, \\ -\varepsilon u_2''-(-1-x)u_1'-(2+x^2)4u_2'=-2x-1 & \text{in } I, \end{cases} \tag{4.1}$$

with the Dirichlet boundary conditions $\boldsymbol{u}(0)=(2,1)^\top$ and $\boldsymbol{u}(1)=(2,2)^\top$. Since the exact solution is not available, we use an approximate solution as the exact solution which is produced by the second-order central difference scheme over a uniform mesh with a very small mesh size $h=2^{-16}\approx 1.5259\times 10^{-5}$. When $0<\varepsilon\ll 1$, a strong boundary layer appears near $x=0$ for both solution components; see Figure 1 for $\varepsilon=10^{-4}$. We test the 1-D scheme (2.24)-(2.25) for $\varepsilon=10^{-4}$ and compare the results with the O'Riordan-Stynes upwind scheme [8]. We adopt the piecewise-uniform Shishkin mesh [8] whose transition point is chosen as $\tau=\min\{1/2,(4\varepsilon\ln N)/3\}$, where N is the number of grid points, i.e., we partition the interval $[0,1]$ into two subintervals $[0,\tau]$ and $[\tau,1]$ and each subinterval is then subdivided into a equidistant mesh by $(N/2)+1$ grid points. The numerical results with $N=64$ are depicted in Figure 1, from which we find that the numerical solutions produced by the present scheme may capture the boundary layers accurately and the results seem more accurate than that of the O'Riordan-Stynes upwind scheme.

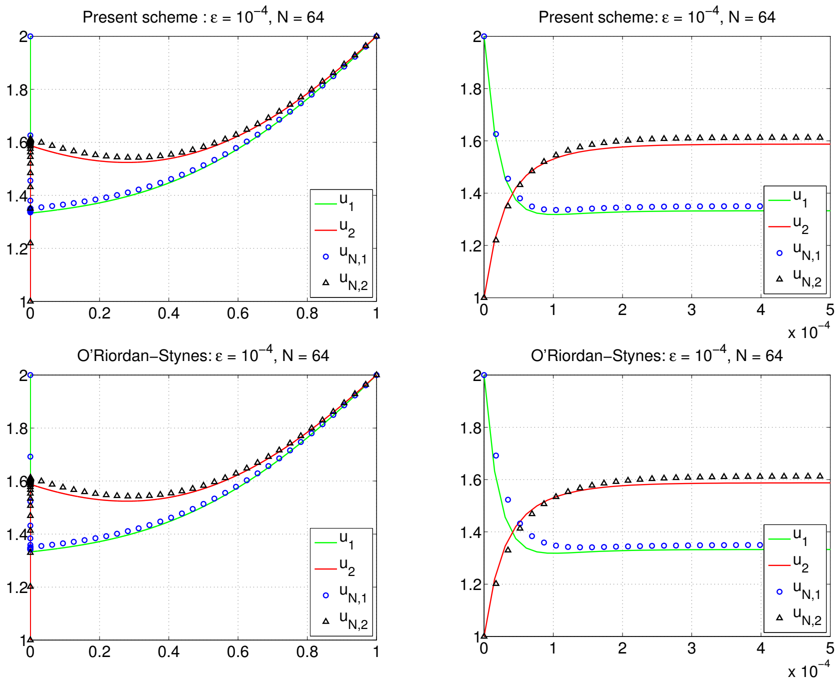

Figure 1. Plots of the exact solution $\boldsymbol{u}=(u_1,u_2)^\top$ and numerical solution $\boldsymbol{u}_N=(u_{N,1},u_{N,2})^\top$ of Example 4.1 with $\varepsilon=10^{-4}$ on a Shishkin mesh with $N=64$. (left) global region, $0\le x\le 1$; (right) a magnification of the boundary layer region, $0\le x\le 5\varepsilon$.

Example 4.2 *(1-D nonlinearly coupled system of viscous Burgers' equations).* The following system is quoted from [5] with a slight modification. Consider the nonlinearly coupled system of viscous Burgers' equations in 1-D domain $I=(-1,1)$:

$$\left\{\begin{array}{rcll} -\varepsilon u_1''+(u_1+u_2)u_1'+u_1u_2' &=& f_1 & \text{in } I, \\ -\varepsilon u_2''+u_2u_1'+(u_1+u_2)u_2' &=& f_2 & \text{in } I, \end{array}\right. \tag{4.2}$$

with the Dirichlet boundary conditions. We choose the source functions f_1 and f_2 such that the exact solution $\boldsymbol{u}=(u_1,u_2)^\top$ is given by

$$u_1(x)=-2\tanh\left(\frac{x}{2\varepsilon}\right) \quad \text{and} \quad u_2(x)=-\tanh\left(\frac{x}{2\varepsilon}\right).$$

To solve this nonlinear problem, a direct iterative procedure is associated with the considered scheme, where the stopping criterion of the iterative procedure is the maximum difference between successive approximations smaller than or equal to 10^{-5}. We consider the coupled system with $\varepsilon=10^{-\ell}$, $1\le\ell\le 4$, on a uniform mesh with $N=64$. The nonlinear terms u_iu_j' in the coupled system are linearized by the approximations $u_i^{(k)}u_j'$ in the $(k+1)$-th iteration. The numerical results are depicted in Figure 2, where the iteration numbers are 16,10,4,3, respectively. From Figure 2, we can find that the present scheme (2.24)-(2.25) may capture the interior layer structure very well.

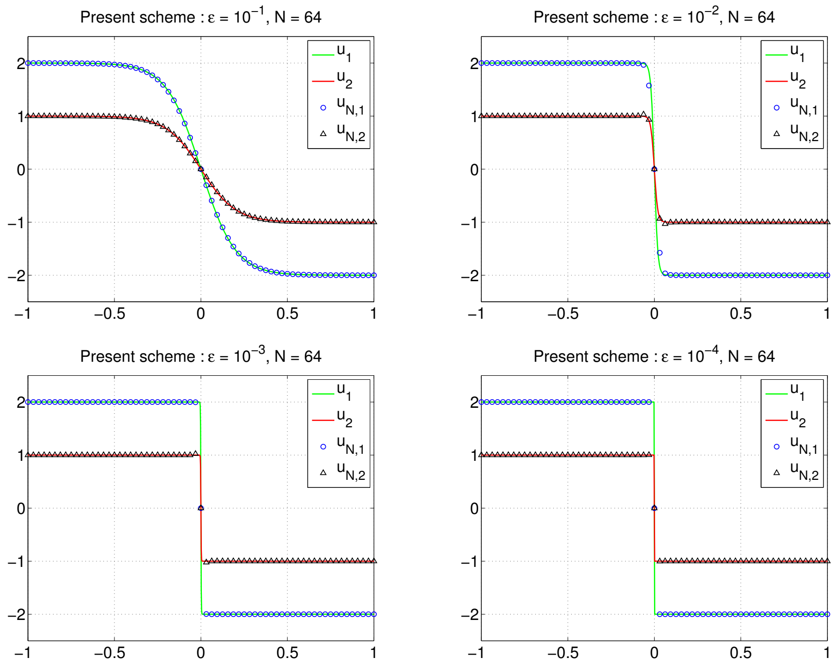

Figure 2. Plots of the exact solution $\boldsymbol{u}=(u_1,u_2)^\top$ and numerical solution $\boldsymbol{u}_N=(u_{N,1},u_{N,2})^\top$ of Example 4.2 with $\varepsilon=10^{-\ell}$, $\ell=1,2,3,4$, on a uniform mesh with $N=64$.

Example 4.3 *(2-D boundary layer problem with analytic solution).* We construct the following 2-D coupled system in domain $\Omega=(0,1)\times(0,1)$:

$$\left\{\begin{array}{rcll} -\varepsilon\Delta u_1-0.5u_{1x}-\frac{\sqrt{2}}{2}u_{1y}+0.1u_{2x}+0.3u_{2y}+2u_1+u_2 &=& f_1 & \text{in } \Omega, \\ -\varepsilon\Delta u_2+0.2u_{1x}+0.1u_{1y}-u_{2x}-\frac{\sqrt{3}}{2}u_{2y}+u_1+3u_2 &=& f_2 & \text{in } \Omega, \end{array}\right. \tag{4.3}$$

with the Dirichlet boundary conditions, where f_1 and f_2 are determined such that the exact solution $\boldsymbol{u}=(u_1,u_2)^\top$ is given by

$$\left\{\begin{array}{rcl} u_1(x,y) &=& \Big(-2x+\dfrac{2e^{-x/(2\varepsilon)}-2}{e^{-1/(2\varepsilon)}-1}\Big)\Big(-\sqrt{2}y+\dfrac{\sqrt{2}e^{-\sqrt{2}y/(2\varepsilon)}-\sqrt{2}}{e^{-\sqrt{2}/(2\varepsilon)}-1}\Big), \\ u_2(x,y) &=& \dfrac{(e^{-x/\varepsilon}-1)(e^{-\sqrt{3}y/(2\varepsilon)}-1)}{(e^{-1/\varepsilon}-1)(e^{-\sqrt{3}/(2\varepsilon)}-1)}. \end{array}\right.$$

When the perturbation parameter ε is very small, strong boundary layers appear near the x-axis and y-axis in both solution components u_1 and u_2; see Figure 3 for $\varepsilon=10^{-6}$.

We first consider the scheme (3.17) on uniform meshes with different mesh sizes h. Numerical results are reported in Table 1, from which we can find that when the diffusivity ε is not too small, the scheme displays a second-order accuracy. However, the convergence behavior is deteriorated when the diffusivity ε is getting smaller. In particular, as $\varepsilon\to 0^+$, the present scheme tends to completely lose its convergence order. With a closer inspection, we can find that the grid point at which the maximum error occurs is approaching to the boundary layers as halving the grid size. This behavior can also be observed in the numerical solutions of scalar singularly perturbed problems [4]. Indeed, since a finer mesh should allocate more grid points in the layer regions, it is not surprising to have a larger maximum error if the grid points are not appropriately distributed. Nevertheless, for a grid point being fixed, the errors in the max-norm approach to zero as $h=1/N\to 0^+$. For more details, we refer the reader to [4].

We next examine the scheme (3.17) on piecewise-uniform Shishkin meshes. Such a mesh is constructed by using the transition points $\tau^x:=\min\{1/2,(\varepsilon\ln N)/\eta^x\}$ and $\tau^y:=\min\{1/2,(\varepsilon\ln N)/\eta^y\}$, where $\eta^x=\min\{|a_{ij}(x,y)|:(x,y)\in(0,1)\times(0,1)\}=0.1$ and $\eta^y=\min\{|b_{ij}(x,y)|:(x,y)\in(0,1)\times(0,1)\}=0.1$. The numerical results are collected in Table 2, where the order of convergence is estimated by

$$\text{order}:=\frac{\log\|\boldsymbol{u}_{N_1}-\boldsymbol{u}\|_\infty-\log\|\boldsymbol{u}_{N_2}-\boldsymbol{u}\|_\infty}{\log N_2-\log N_1},$$

with N_1+1 and N_2+1 are the numbers of grid points of two partitions, and $\boldsymbol{u}_{N_1}$ and $\boldsymbol{u}_{N_2}$ are the corresponding difference solutions. From the numerical results reported in Table 2, we observe that the scheme with appropriate piecewise-uniform Shishkin meshes can achieve high accuracy and stability, even if the diffusivities are very small; see Figure 3 for 3-D plots of numerical solutions for a small perturbation parameter $\varepsilon=10^{-6}$. Moreover, numerical evidence also shows that the computed solutions converge uniformly in ε in the discrete maximum norm when ε is sufficiently small. A further theoretical analysis is needed to confirm this observation.

Table 1. Maximum errors of the numerical solution $\boldsymbol{u}_N$ of Example 4.3 produced by the present scheme (3.17) on uniform meshes.

	N					
ε	32	64	128	256	512	Order
10^{-1}	1.0192E-03	2.5495E-04	6.3749E-05	1.5938E-05	3.9845E-06	2.00
10^{-2}	1.2553E-01	3.9756E-02	1.0063E-02	2.5222E-03	6.3132E-04	1.91
10^{-3}	1.5969E-01	2.6389E-01	3.0389E-01	1.9297E-01	6.7970E-02	0.31
10^{-4}	1.8147E-02	3.9770E-02	8.2632E-02	1.5909E-01	2.5908E-01	-0.96

Table 2. Maximum errors of the numerical solution $\boldsymbol{u}_N$ of Example 4.3 produced by the present scheme (3.17) on piecewise-uniform Shishkin meshes.

	N					
ε	32	64	128	256	512	Order
10^{-1}	1.0192E-03	2.5495E-04	6.3749E-05	1.5938E-05	3.9845E-06	2.00
10^{-2}	7.1130E-02	2.7177E-02	9.4578E-03	2.5222E-03	6.3132E-04	1.71
10^{-3}	8.1056E-02	2.9941E-02	1.0492E-02	3.4474E-03	1.0918E-03	1.56
10^{-4}	8.2314E-02	3.0283E-02	1.0619E-02	3.4915E-03	1.1055E-03	1.55
10^{-5}	8.2448E-02	3.0318E-02	1.0632E-02	3.4960E-03	1.1069E-03	1.55
10^{-6}	8.2514E-02	3.0322E-02	1.0634E-02	3.4964E-03	1.1071E-03	1.55

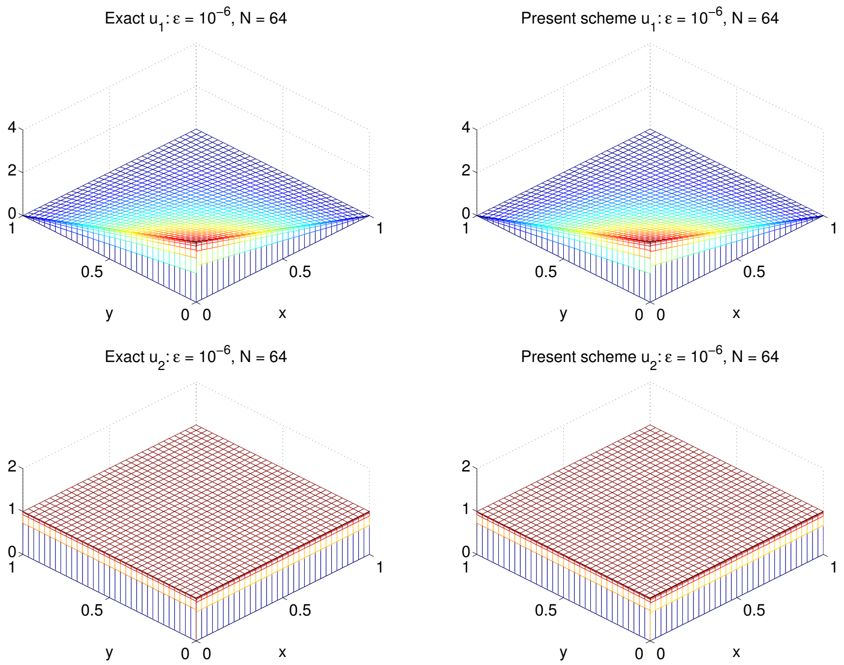

Figure 3. 3-D plots of the exact solution $\boldsymbol{u} = (u_1, u_2)^\top$ and numerical solution $\boldsymbol{u}_N = (u_{N,1}, u_{N,2})^\top$ of Example 4.3 with $\varepsilon = 10^{-6}$ on a piecewise-uniform Shishkin mesh.

Example 4.4 *(Time-dependent 2-D nonlinearly coupled system of viscous Burgers' equations).* This example is taken from [17]; see also [1,5,13]. Consider the following time-dependent, nonlinearly coupled system of viscous Burgers' equations in time interval $I=(0,T)$ and spatial domain $\Omega=(0,1)\times(0,1)$:

$$\begin{cases} \dfrac{\partial u_1}{\partial t}-\dfrac{1}{Re}\Delta u_1+u_1u_{1x}+u_2u_{1y} &= 0 \quad \text{in } I\times\Omega, \\ \dfrac{\partial u_2}{\partial t}-\dfrac{1}{Re}\Delta u_2+u_1u_{2x}+u_2u_{2y} &= 0 \quad \text{in } I\times\Omega, \end{cases} \tag{4.4}$$

with the initial and Dirichlet boundary conditions which are specified by the exact solution $\boldsymbol{u}(t,x,y)=(u_1(t,x,y),u_2(t,x,y))^\top$, where

$$u_1(t,x,y)=\frac{3}{4}-\frac{1}{4+4e^{Re(-4x+4y-t)/32}} \quad \text{and} \quad u_2(t,x,y)=\frac{3}{4}+\frac{1}{4+4e^{Re(-4x+4y-t)/32}}.$$

When the Reynolds number Re is large enough, strong interior layers appear in the solution components u_1 and u_2; see Figure 4 for $Re=10^3$.

We use the semi-implicit second-order time discretization to approximate time variable in system (4.4),

$$\begin{cases} \dfrac{u_1^{n+1}-u_1^n}{\Delta t}-\dfrac{1}{2Re}\Delta u_1^{n+1}+\dfrac{1}{2}(u_1^*u_{1x}^{n+1}+u_2^*u_{1y}^{n+1})=\dfrac{1}{2Re}\Delta u_1^n-\dfrac{1}{2}(u_1^nu_{1x}^n+u_2^nu_{1y}^n), \\ \dfrac{u_2^{n+1}-u_2^n}{\Delta t}-\dfrac{1}{2Re}\Delta u_2^{n+1}+\dfrac{1}{2}(u_1^*u_{2x}^{n+1}+u_2^*u_{2y}^{n+1})=\dfrac{1}{2Re}\Delta u_2^n-\dfrac{1}{2}(u_1^nu_{2x}^n+u_2^nu_{2y}^n), \end{cases} \tag{4.5}$$

where $u_1^*:=2u_1^n-u_1^{n-1}$ and $u_2^*:=2u_2^n-u_2^{n-1}$ and at each time step, we solve the resulting coupled system by using the present scheme (3.17). We remark that if we take $u_\ell^*:=u_\ell^{n+1}$, the above discretization is known as the Crank-Nicolson scheme. Hence, the difference approximation $2u_\ell^n-u_\ell^{n-1}$, which is an extrapolation of u_ℓ^{n+1} with the truncation error $O(\Delta t^2)$, can still retain the second-order accuracy in time.

We first verify the convergence of our scheme. The numerical results for different Reynolds numbers and a very small time step $\Delta t=10^{-4}$ are reported in Table 3, from which we can find that the present scheme with (4.5) achieves excellent approximations with the exact solutions for low Reynolds numbers. Next, we consider $Re=10^3$. The numerical computations are performed on a uniform mesh using $N=64$ and $\Delta t=1/64$. The results of $t=0.25,0.5,0.75,1$ are depicted in Figure 4, from which we can find that although our scheme gives a little bit over-diffused approximations near a small portion of the outflow boundary, it still shows a high stability for the larger Reynolds number.

Table 3. Maximum errors of the numerical solution $\boldsymbol{u}_N$ of Example 4.4 produced by the present scheme with (4.5) on uniform meshes at $T=0.5$ with time step $\Delta t=10^{-4}$.

	N					
Re	8	16	32	64	128	Order
1	3.4913E-09	8.9121E-10	2.2366E-10	5.5971E-11	1.3999E-11	1.99
10^1	7.1610E-05	1.8800E-05	4.7895E-06	1.2017E-06	3.0063E-07	1.98
10^2	3.2370E-02	1.7722E-02	6.8818E-03	2.0436E-03	5.3774E-04	1.48

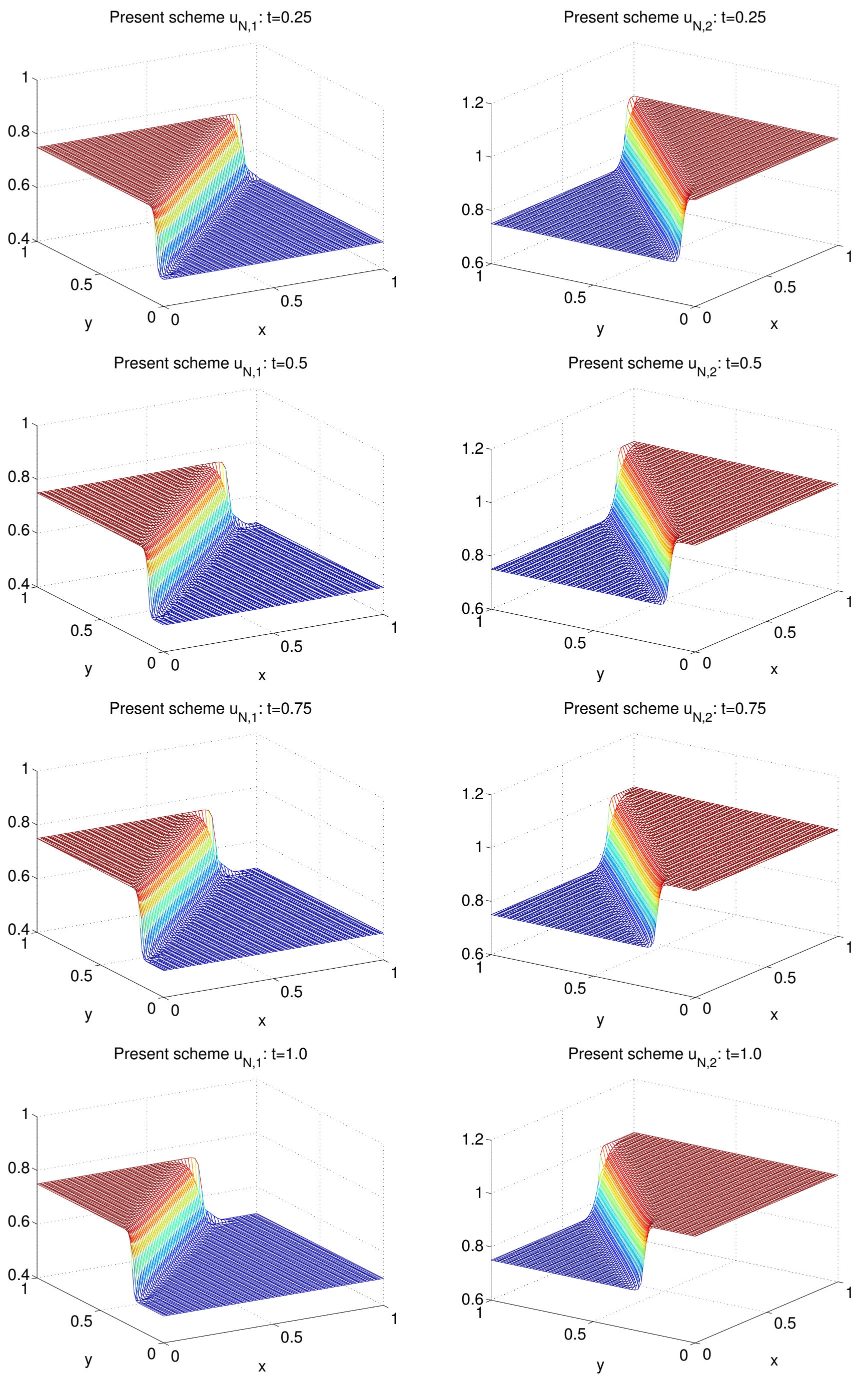

Figure 4. 3-D plots of the numerical solution $\boldsymbol{u}_N=(u_{N,1},u_{N,2})^\top$ at different times of Example 4.4 with $Re=10^3$ on a uniform mesh with $N=64$ and time step $\Delta t=1/64$.

5 Conclusions

In this paper, we have proposed a novel and simple technique to construct effective difference schemes for solving systems of singularly perturbed CDR equations. First, we have demonstrated the technique by deriving a formally second-order scheme for 1-D coupled systems. This scheme was formed by carefully adding some dummy terms in the form $(\mu u_i'' - \mu u_i'')$ into each equation of the 1-D coupled system and then applying the Il'in-Allen-Southwell scheme to each scalar CDR equation in the system. Then we have extended the scheme to 2-D case by employing the alternating direction approach. We have presented several numerical examples, including 1-D and 2-D nonlinearly coupled systems of viscous Burgers' equations, to illustrate the performance of the obtained scheme. We have found that the scheme can achieve high accuracy with high stability for coupled systems with small diffusivities. We have also observed that the solutions produced by the obtained scheme with appropriate piecewise-uniform Shishkin meshes converge uniformly in the perturbation parameters in the discrete maximum norm. A further theoretical analysis is needed to confirm this observation.

Acknowledgments

This work was supported by the National Science Council of Taiwan under the grant NSC 102-2115-M-033-007-MY2 (Po-Wen Hsieh) and the Ministry of Science and Technology of Taiwan under the grants MOST 103-2115-M-005-004-MY2 (Yin-Tzer Shih) and MOST 103-2115-M-008-009-MY3 (Suh-Yuh Yang). The work of Suh-Yuh Yang was also supported by the National Center for Theoretical Sciences, Taiwan.

References

[1] C. A. J. Fletcher, Generating exact solutions of the two-dimensional Burgers' equation, Int. J. Numer. Methods Fluids, 3 (1983), pp. 213-216.

[2] P.-W. Hsieh, Y. Shih, and S.-Y. Yang, A tailored finite point method for solving steady MHD duct flow problems with boundary layers, Commun. Comput. Phys., 10 (2011), pp. 161-182.

[3] P.-W. Hsieh and S.-Y. Yang, Two new upwind difference schemes for a coupled system of convection-diffusion equations arising from the steady MHD duct flow problems, J. Comput. Phys., 229 (2010), pp. 9216-9234.

[4] P.-W. Hsieh, S.-Y. Yang, and C.-S. You, A high-accuracy finite difference scheme for solving reaction-convection-diffusion problems with a small diffusivity, Adv. Appl. Math. Mech., 6 (2014), pp. 637-662.

[5] A. Kelleci and A. Yıldırım, An efficient numerical method for solving coupled Burgers' equation by combining homotopy perturbation and Pade techniques, Numer. Methods Partial Differential Eq., 27 (2011), pp. 982-995.

[6] P. V. Kokotović, Applications of singular perturbation techniques to control problems, SIAM Rev., 26 (1984), pp. 501-550.

[7] K. W. Morton, Numerical Solution of Convection-Diffusion Problems, Chapman & Hall, London, UK, 1996.

[8] E. O'Riordan and M. Stynes, Numerical analysis of a strongly coupled system of two singularly perturbed convection-diffusion problems, Adv. Comput. Math., 30 (2009), pp. 101-121.

[9] E. O'Riordan, J. Stynes, and M. Stynes, A parameter-uniform finite difference method for a coupled system of convection-diffusion two-point boundary value problems, Numer. Math. Theor. Meth. Appl., 1 (2008), pp. 176-197.

[10] E. O'Riordan, J. Stynes, and M. Stynes, An iterative numerical algorithm for a strongly coupled system of singularly perturbed convection-diffusion problems, in NAA 2008, LNCS 5434, Springer-Verlag, Berlin Heidelberg, 2009, pp. 104-115.

[11] H.-G. Roos, Ten ways to generate the Il'in and related schemes, J. Comput. Appl. Math., 53 (1994), pp. 43-59.

[12] H.-G. Roos, M. Stynes, and L. Tobiska, Robust Numerical Methods for Singularly Perturbed Differential Equations, Second Edition, Springer-Verlag, Berlin, 2008.

[13] Y. V. S. S. Sanyasiraju and N. Mishra, Exponential compact higher order scheme for nonlinear steady convection-diffusion equations, Commun. Comput. Phys., 9 (2011), 897-916.

[14] G. I. Shishkin, Mesh approximation of singularly perturbed boundary-value problems for systems of elliptic and parabolic equations, Comput. Maths. Math. Phys., 35 (1995), pp. 429-446.

[15] G. P. Thomas, Towards an improved turbulence model for wave-current interactions, in Second Annual Report to EU MAST-III Project "The Kinematics and Dynamics of Wave-Current Interactions," 1998.

[16] Z. F. Tian and S. Q. Dai, High-order compact exponential finite difference methods for convection-diffusion type problems, J. Comput. Phys., 220 (2007), pp. 952-974.

[17] H. Zhu, H. Shu, and M. Ding, Numerical solutions of two-dimensional Burgers' equations by discrete Adomian decomposition method, Comput. Math. Appl., 60 (2010), pp. 840-848.