An SPD stabilized finite element method for Stokes equations

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Abstract A new residual-based stabilized finite element method is analyzed for solving the Stokes equations in terms of velocity and pressure, where the H^{-1} norm is introduced in the measurement of the residuals to obtain a symmetric positive definite (SPD) method. The H^{-1} norm is computable and can be always easily realized offline by the continuous *linear* finite element solution or the preconditioner counterpart of the Poisson Dirichlet problem. Although the H^{-1} norm is computed in the linear element space, no matter what the finite element spaces for the velocity and the pressure are, optimal error bounds can be established when using continuous finite element pairs $R_l - R_m$ for velocity and pressure for any $l, m \geq 1$. Numerical experiments are performed to confirm the theoretical results obtained.

Keywords: Stokes equations; stabilized finite element method; H^{-1} norm; symmetric positive definiteness; linear finite element solution of Poisson Dirichlet problem.

1 Introduction

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, a bounded and connected open domain with a Lipschitz-continuous boundary Γ, we are interested in the finite element method for the Stokes problem [36, 23]:

$$-\Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma, \tag{1.1}$$

where u, p, f are the velocity, the pressure and the given source, respectively. Introduce standard Hilbert spaces [1]: $L^2(\Omega) = \{v : ||v||_0^2 := \int_\Omega |v|^2 < \infty\}$, $H^1(\Omega) = \{v \in L^2(\Omega) : \nabla v \in (L^2(\Omega))^d\}$, $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_\Gamma = 0\}$, $H^1(\Omega)/\mathbb{R} = \{v \in H^1(\Omega) : \int_\Omega v = 0\}$, where we use the same notation $||v||_1$ and $|\cdot|_1$ to respectively denote the norm and the semi-norm for these H^1 spaces, with $||v||_1^2 := ||v||_0^2 + ||\nabla v||_0^2$ and with $|v|_1 := ||\nabla v||_0$. The dual of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$, with the duality pairing $\langle \cdot, \cdot \rangle$, where the norm for $H^{-1}(\Omega)$ is denoted by $||\cdot||_{-1}$ and is defined by [1, 40]

$$||\chi||_{-1} := \sup_{0 \neq v \in H_0^1(\Omega)} \frac{\langle \chi, v \rangle}{||v||_1}, \tag{1.2}$$

where if $\chi \in L^2(\Omega)$, the duality pairing $\langle \chi, v \rangle$ is the L^2 inner product $(\chi, v) = \int_\Omega \chi v$. Set

$$X = (H_0^1(\Omega))^d, \quad M = L^2(\Omega)/\mathbb{R}, \tag{1.3}$$

where X is equipped with the norm $||v||_1$ and M with the norm $||q||_{0/\mathbb{R}} := \inf_{c \in \mathbb{R}} ||q + c||_0$ (hereafter, $||q||_{0/\mathbb{R}}$ is still denoted by $||q||_0$ for convenience). We define the bounded bilinear forms as follows:

$$a(u, v) = (\nabla u, \nabla v) : X \times X \to \mathbb{R}, \quad b(v, q) = -(\operatorname{div} v, q) : X \times M \to \mathbb{R}. \tag{1.4}$$

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Let $f \in (H^{-1}(\Omega))^d$. The standard Galerkin variational problem for the Stokes problem (1.1) is to find $(u,p) \in X \times M$ such that

$$\mathscr{B}(u,p;v,q) := a(u,v) + b(v,p) + b(u,q) = \langle f, v \rangle \quad \forall (v,q) \in X \times M. \tag{1.5}$$

Problem (1.5) is well-posed, since it is well known that [12, 23, 36, 27, 8]:

$$\sup_{0 \neq (v,q) \in X \times M} \frac{\mathscr{B}(u,p;v,q)}{||v||_1 + ||q||_0} \geq C(||u||_1 + ||p||_0) \quad \forall (u,p) \in X \times M. \tag{1.6}$$

As usual, let h denote the mesh size of the finite element triangulation $\mathcal{T}_h$ of Ω. Introduce finite element spaces $X_h \subset X$ and $M_h \subset M \cap H^1(\Omega)$. Unfortunately, the finite element problem of (1.5) over $X_h \times M_h$ is not always well-posed, since, in general, (1.6) cannot hold over $X_h \times M_h$ uniformly in h. Readers may refer to [8, 23, 38] for more details. Another difficulty is the need to solve an indefinite saddle-point system resulting from the finite element problem. There have been much progress in iterative methods for solving saddle-point systems in the past decade [5, 18]. But, a symmetric positive definite (SPD) system is still the most desirable in large-scale computations, since there are many readily available preconditioning techniques and algorithms for iterative solutions of SPD system [29]. The stabilization methods as in [19, 20, 21, 13] are not SPD. For the purpose of practical applications, on the other hand, there has been increasingly interest in employing equal-order continuous elements for the velocity and the pressure of the Stokes problem. Least-squares (LS) methods are thus suitable for these considerations [2]. The idea for the LS method is quite simple [2]. Measuring the residuals of the underlying partial differential equations in some suitable Hilbert norms, to find the minimizer which belongs to suitable Hilbert spaces such that the residuals is minimized in those Hilbert norms. Trivially, the original solution is usually exactly the minimizer of the LS minimization problem. The obvious advantages of LS method are that the minimization problem is at least semi-positive definite and symmetric and that the Inf-Sup constraint (1.6) between X_h and M_h is not required and any conforming finite elements can be employed. The LS method may be divided into two main subclasses. One subclass is the LS method for the second-order system of partial differential equations like (1.1), see [25]. The other subclass is the LS for the first-order system of partial differential equations which is usually formulated by the introduction of additional unknowns other than the velocity and the pressure from a second-order system [16, 2, 14, 25].

In the context of second-order system like (1.1), the H^{-1} LS method is a desirable method that allows the velocity and the pressure to still belong to $X \times M$ and behaves just like the standard finite element method for elliptic problems in [10], for instance, the condition number is $\mathcal{O}(h^{-2})$. The H^{-1} LS method is to use the H^{-1} norm to measure some of the residuals of the underlying partial differential equations [25]. For the Stokes problem (1.1), the H^{-1} LS method is to find the minimizer $(u,p) \in X \times M$ such that $\mathscr{J}(u,p;f) = \min_{(v,q)\in X\times M} \mathscr{J}(v,q;f)$, where $\mathscr{J}(v,q;f) = || -\Delta v + \nabla q - f||_{-1}^2 + ||\mathrm{div}\, v||_0^2$. It is obvious that if (u,p) solves (1.1), it indeed minimizes $\mathscr{J}(v,q;f)$ at the zero minimum, i.e., $\mathscr{J}(u,p;f) = 0$. Also, it can be shown that the solution of (1.5) is the minimizer and vice versa. The Galerkin problem of the H^{-1} LS minimization is coercive over $X \times M$, inheriting from (1.6). From Lax-Milgram lemma, one can infer that the H^{-1} LS Galerkin problem admits a unique solution $(u,p) \in X \times M$. It then follows from the standard finite element theory that the solution (u,p) can be numerically solved in any $X \times M$ conforming finite element space and the convergence optimal relative to the order of approximation and the required regularity can be obtained. Now, *the only question is how to compute the H^{-1} norm.* As in [25, 24], the H^{-1} norm is approximately computed from the finite element solution operator or its preconditioner counterpart of the Poisson equation of Laplace operator with homogeneous Dirichlet boundary condition. The idea comes from the Riesz representation theorem [40], since the H^{-1} norm of any $\chi \in H^{-1}(\Omega)$ is equal to the H^1 norm of the Riesz representation in $H_0^1(\Omega)$ of χ, i.e., the solution of the Poisson Dirichlet problem of Laplace operator with right-hand side χ. In implementation, the H^{-1} LS finite element method involves three stages where three symmetric and positive definite algebraic systems are solved. The first stage is to solve the L^2 projections of $-\Delta u + \nabla p$ onto X_h. The second stage is to solve the Poisson Dirichlet problem in X_h with the finite element solution operator $T_h : (H^{-1}(\Omega))^d \to X_h$ or to construct the symmetric positive definite preconditioner $B_h : X_h \to X_h$, which satisfies the spectral equivalence to T_h. The third stage is to numerically solve (u, p) from the H^{-1} LS finite element problem in $X_h \times M_h$. All these stages are realized in X_h, the finite element space of the velocity.

In this paper, we shall propose a new H^{-1} LS finite element method. The key feature is that all the three stages mentioned as above are implemented only in the continuous linear element space, denoted by $V_h \subset X$, no matter what the finite element spaces $X_h \times M_h$ for the velocity and the pressure are. In other words, here the H^{-1}-norm is computed in the linear element space V_h, unlike the method elsewhere where the H^{-1}-norm is computed in the finite element space X_h of velocity. More importantly, employing this newly proposed H^{-1} LS method with finite element spaces $X_h \times M_h$ of velocity and pressure, although the H^{-1}-norm is only computed in V_h (instead of X_h), we can still obtain optimal error bounds in both order of approximations of $X_h \times M_h$ as well as required regularity of velocity and pressure. Besides, other advantages of H^{-1}-LS methods are all inherited, for example, the "norm equivalence" property in H^1 norm of velocity and in L^2 norm of pressure still holds. In the new H^{-1} LS method, the L^2 projections of $-\Delta u + \nabla p$ and the finite element solution operator of the Poisson equation of Laplace operator with homogeneous Dirichlet boundary condition are defined only in the linear element space V_h, which are respectively denoted as $\mathcal{A}_h$ and $\mathcal{S}_h$, i.e., $\mathcal{A}_h : (H^{-1}(\Omega))^d \to V_h$ and $\mathcal{S}_h : V_h \to V_h$, see (2.2) and (2.3) in section 2. Both $\mathcal{A}_h$ and $\mathcal{S}_h$ are used for computing the H^{-1}-norm, see (2.9) and (2.11) in section 2. They can be easily obtained from simple numerical quadrature methods, such as 'mass-lumping' method with a diagonal resultant mass matrix for the former and one point quadrature scheme for the latter, see [10]. The preconditioner of the finite element solution operator $\mathcal{S}_h$ which is defined in the linear element space V_h is denoted as $\mathcal{B}_h$: $V_h \to V_h$, which is spectrally equivalent to $\mathcal{S}_h$, see (2.13) in section 2. This linear element preconditioner $\mathcal{B}_h$ can also be easily obtained from a *one* step smoothing in *one* time V-cycle multigrid algorithm on nested meshes. Nowadays, the multigrid algorithm for linear element method, which is built-in most existing softwares, is readily available. Many other algorithms are also available for easily generating the preconditioner $\mathcal{B}_h$ in the linear element space V_h for the finite element solution operator $\mathcal{S}_h$ which is defined in the linear element space V_h, such as domain decomposition method.

The new H^{-1} LS method is then highly attractive for the case where X_h of the velocity is involved with higher-order elements, nonnested meshes, nonnested elements, and three-dimensional problems. Among others, no matter what finite element spaces $X_h \times M_h$ for solving the velocity and pressure are, we always compute the H^{-1} norm in the continuous linear element space V_h. As such, whenever the degrees of piecewise polynomials change globally or locally in $X_h \times M_h$ by adding more nodes in all or part of the elements of $\mathcal{T}_h$, the L^2 projections $\mathcal{A}_h$ and the finite element solution operator $\mathcal{S}_h$ or its preconditioner counterpart $\mathcal{B}_h$ which are all defined in the linear element space V_h always live on only the vertices of the elements of $\mathcal{T}_h$. Therefore, the new H^{-1} LS method would be potentially very useful in several circumstances, say *hp*-version and *p*-version methods [33, 35], discontinuous Galerkin methods [11], adaptive methods [31, 32], etc. All these methods may involve locally or globally higher-order approximations in $X_h \times M_h$ so that the velocity and the pressure can have higher accuracy locally or globally.

Here we should note a fact. The preconditioner $\mathcal{B}_h$ we shall define in the linear element space V_h is not a preconditioning of the one in [25]. In other words, the preconditioner $\mathcal{B}_h$ (or the finite element solution operator $\mathcal{S}_h$) in the linear element space V_h is not involved with the preconditioning of the finite element space X_h of the velocity. In any case, the linear element preconditioner $\mathcal{B}_h$ is only the preconditioner of the linear element solution operator $\mathcal{S}_h$ in the linear element space V_h, no matter what the finite element space X_h of the velocity is. The only role of the linear element preconditioner $\mathcal{B}_h$ (or the linear element solution operator $\mathcal{S}_h$) which is an approximation of the Riesz representation operator associated with the Poisson equations of Laplace operator with homogeneous Dirichlet boundary condition is for computing the H^{-1}-norm.

In this paper, we shall also prove the optimal L^2-norm error bound for the velocity with one order higher than the H^1-norm error bound if the H^2 regularity of the solutions of the Stokes problem and the linear elasticity problem hold (e.g., the H^2 regularity holds for convex domain). This type of error estimate has not appeared elsewhere in the literature for the H^{-1} LS method of the Stokes problem, to the best of the authors' knowledge. We elaborate an ad hoc duality argument to achieve this.

The rest of this paper is outlined as follows. In section 2, we define the L^2 projection and the finite element solution of the Poisson Dirichlet problem (including the preconditioner) and the discrete H^{-1} norm in the linear element space. In section 3, the new H^{-1} LS finite element method is defined and the consistency property is shown. In section 4, the coercivity/ellipticity or the norm equivalence is established and the estimate of the condition number of the resulting algebraic system is given, and the optimal error bounds in H^1 norm for the velocity and in L^2 norm for the pressure are obtained. In section 5, the argument for deriving the optimal error bound in L^2 norm for the velocity is developed. In section 6, numerical experiments are presented to confirm the theoretical results obtained, and a conclusion is given in the last section.

2 L^2 projection, finite element solution of Poisson Dirichlet problem, discrete H^{-1} norm and preconditioner in linear element space

Let Ω be a simply-connected polygon or polyhedron, with connected polygonal boundary Γ. For any $h > 0$, let $\mathcal{T}_h$ denote a family of shape-regular conforming triangulations of Ω into elements [10, 6], such as triangles or tetrahedra or quadrilaterals or hexahedra. As usual, $h = \max_{K\in\mathcal{T}_h} h_K$, where h_K denotes the diameter of K. Denote by $\mathcal{F}_h^I$ the set of all interior element boundaries in $\mathcal{T}_h$, and we denote by $\mathcal{F}_h$ the set of all element boundaries in $\mathcal{T}_h$. The diameter of $F \in \mathcal{F}_h$ is denoted by h_F. For any given interior edge or face $F \in \mathcal{F}_h^I$ which is the intersection of two elements $K_1, K_2 \in \mathcal{T}_h$, of v across F we define the jump $[v]|_F = (v|_{K_1} - v|_{K_2})|_F$. Let $R_l(K)$, $l \geq 1$ being an integer, either denote the space of polynomials over $K \in \mathcal{T}_h$ of total degree in all coordinates variables not greater than l for simplexes, or denote the space of isoparametric functions over K from the polynomials over the fixed reference element $\hat{K}$ of the respective degree in each coordinates variable not greater than l.

We define the linear element space:

$$V_h = \{v \in X : v|_K \in (R_1(K))^d, \forall K \in \mathcal{T}_h\}. \tag{2.1}$$

Associated with V_h, we introduce a discrete L^2 inner product, denoted by $(\cdot,\cdot)_{0,h}$, which is an approximation of the L^2 inner product $(\cdot,\cdot)$. We also introduce a discrete H^1 inner product $((\cdot,\cdot))_{1,h}$, which is an approximation of the H^1 inner product $((\cdot,\cdot))_1 := (\nabla\cdot,\nabla\cdot) + (\cdot,\cdot)$ or $(\nabla\cdot,\nabla\cdot)$. Firstly, we define a generalized linear element L^2 projection operator $\mathcal{A}_h : (H^{-1}(\Omega))^d \to V_h$: given any $\chi \in (H^{-1}(\Omega))^d$, $\mathcal{A}_h\chi \in V_h$ satisfies

$$(\mathcal{A}_h\chi, z_h)_{0,h} = \langle \chi, z_h\rangle \quad \forall z_h \in V_h. \tag{2.2}$$

Note that the above is not a genuine L^2 projection. Nevertheless, the left-hand side $(\cdot,\cdot)_{0,h}$ is the approximation of the L^2 inner product and $\langle \chi, z\rangle = (\chi, z)$ for $\chi, z \in (L^2(\Omega))^d$, so $\mathcal{A}_h\chi$ is essentially the L^2 projection of χ when $\chi \in (L^2(\Omega))^d$, and we will simply call $\mathcal{A}_h$ the L^2 projection operator. Next, we define a linear finite element solution operator $\mathcal{S}_h : V_h \to V_h$: given any $\chi_h \in V_h$, $\mathcal{S}_h\chi_h \in V_h$ satisfies

$$((\mathcal{S}_h\chi_h, z_h))_{1,h} = (\chi_h, z_h)_{0,h} \quad \forall z_h \in V_h. \tag{2.3}$$

It is obvious that $\mathcal{S}_h\mathcal{A}_h : (H^{-1}(\Omega))^d \to V_h$ gives the relation: given any $\chi \in (H^{-1}(\Omega))^d$, $\mathcal{S}_h\mathcal{A}_h\chi \in V_h$ satisfies

$$((\mathcal{S}_h\mathcal{A}_h\chi, z_h))_{1,h} = \langle \chi, z_h\rangle \quad \forall z_h \in V_h. \tag{2.4}$$

In other words, according to which one $((\cdot,\cdot))_{1,h}$ is taken as the approximation of either $((\cdot,\cdot))_1 = (\nabla\cdot,\nabla\cdot) + (\cdot,\cdot)$ or $((\cdot,\cdot))_1 = (\nabla\cdot,\nabla\cdot)$, $\mathcal{S}_h\mathcal{A}_h$ (or $\mathcal{S}_h$) is the linear finite element solution operator, with respect to $((\cdot,\cdot))_{1,h}$, for the Poisson Dirichlet problem as follows:

$$-\Delta\omega + \omega = \chi \quad \text{or} \quad -\Delta\omega = \chi \quad \text{in } \Omega, \quad \omega = 0 \quad \text{on } \Gamma. \tag{2.5}$$

Below we formulate a better understanding of $\mathcal{A}_h$ and $\mathcal{S}_h$ for Stokes problem. For any given $(u, p) \in X \times M$, letting $\chi := -\Delta u + \nabla p \in (H^{-1}(\Omega))^d$, in terms of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ in (1.4), since $\langle \chi, v\rangle = a(u, v) + b(v, p)$, we have from (2.3) and (2.4)

$$(\mathcal{A}_h(-\Delta u + \nabla p), z_h)_{0,h} = ((\mathcal{S}_h\mathcal{A}_h(-\Delta u + \nabla p), z_h))_{1,h} = a(u, z_h) + b(z_h, p) \quad \forall z_h \in V_h. \tag{2.6}$$

Assumption A1) Let $||z_h||_{0,h}^2 := (z_h, z_h)_{0,h}$ and $||z_h||_{1,h}^2 := ((z_h, z_h))_{1,h}$ for all $z_h \in V_h$. We require that the discrete L^2 inner product $(\cdot,\cdot)_{0,h}$ and the discrete H^1 inner product $((\cdot,\cdot))_{1,h}$ respectively satisfies

$$C_1||z_h||_0 \le ||z_h||_{0,h} \le C_2||z_h||_0 \quad \forall z_h \in V_h, \quad C_3||z_h||_1 \le ||z_h||_{1,h} \le C_4||z_h||_1 \quad \forall z_h \in V_h, \tag{2.7}$$

From the definition of $\mathcal{S}_h$ we can easily show the following Proposition 2.1.

Proposition 2.1 *Assume that Assumption A1) holds. The finite element solution solver $\mathcal{S}_h$ is symmetric, positive definite with respect to both $(\cdot,\cdot)_{0,h}$ and $((\cdot,\cdot))_{1,h}$:*

$$((\mathcal{S}_h\chi_h, \chi_h))_{1,h} = ||\chi_h||_{0,h}^2 \ge C||\chi_h||_0^2 > 0 \quad \forall 0 \ne \chi_h \in V_h,$$

$$(\mathcal{S}_h\chi_h, \chi_h)_{0,h} = ((\mathcal{S}_h\chi_h, \mathcal{S}_h\chi_h))_{1,h} = ||\mathcal{S}_h\chi_h||_{1,h}^2 > 0 \quad \forall 0 \ne \chi_h \in V_h.$$

We can now define a discrete version of the H^{-1} norm (1.2), according to the linear element space V_h, as follows:

$$||\chi_h||_{-1,h} := \sup_{0\ne z_h\in V_h} \frac{(\chi_h, z_h)_{0,h}}{||z_h||_{1,h}} \quad \forall \chi_h \in V_h. \tag{2.8}$$

From the definition of $\mathcal{S}_h$ in (2.3), we have

$$||\chi_h||_{-1,h} = ||\mathcal{S}_h\chi_h||_{1,h} \quad \forall \chi_h \in V_h, \tag{2.9}$$

and we have

$$||\chi_h||_{-1,h}^2 = (\chi_h, \mathcal{S}_h\chi_h)_{0,h} \quad \forall \chi_h \in V_h. \tag{2.10}$$

From the definition of $\mathcal{A}_h$ in (2.2) we have, for all $\chi \in (H^{-1}(\Omega))^d$,

$$||\mathcal{A}_h\chi||_{-1,h} = \sup_{0\ne z_h\in V_h} \frac{\langle \chi, z_h\rangle}{||z_h||_{1,h}}. \tag{2.11}$$

Proposition 2.2 *Assuming Assumption A1), we have, for all $\chi \in (H^{-1}(\Omega))^d$,*

$$||\mathcal{S}_h\mathcal{A}_h\chi||_{1,h} = ||\mathcal{A}_h\chi||_{-1,h} \le C||\chi||_{-1}. \tag{2.12}$$

Proof. From (2.3) and (2.10) it follows that

$$||\mathcal{S}_h\mathcal{A}_h\chi||_{1,h}^2 = ((\mathcal{S}_h\mathcal{A}_h\chi, \mathcal{S}_h\mathcal{A}_h\chi))_{1,h} = ((\mathcal{A}_h\chi, \mathcal{S}_h\mathcal{A}_h\chi))_{0,h} = ||\mathcal{A}_h\chi||_{-1,h}^2.$$

From (2.11), (1.2) and Assumption A1) we have

$$||\mathcal{A}_h\chi||_{-1,h} = \sup_{0\ne z_h\in V_h} \frac{\langle \chi, z_h\rangle}{||z_h||_{1,h}} \le \sup_{0\ne z_h\in V_h} \frac{||\chi||_{-1}||z_h||_1}{||z_h||_{1,h}} \le \frac{1}{C_3}||\chi||_{-1} \sup_{0\ne z_h\in V_h} \frac{||z_h||_{1,h}}{||z_h||_{1,h}} = \frac{1}{C_3}||\chi||_{-1}.$$

□

In practice, to compute $\mathcal{S}_h$, we may choose a spectral equivalent preconditioner $\mathcal{B}_h$ so that $\mathcal{B}_h\mathcal{S}_h^{-1}$ is well-conditioned, and so that the computation of $\mathcal{S}_h$ can be efficiently implemented by the preconditioned conjugate gradient algorithm with a convergence rate uniform in the mesh size h. On the other hand, we may directly replace $\mathcal{S}_h$ by its preconditioner $\mathcal{B}_h$ in computing the H^{-1} norm for developing the H^{-1} LS finite element method, so we do not need to properly or exactly solve the Poisson Dirichlet problem (2.5). Namely, it is unnecessary to solve (2.3) to a full extent.

To achieve this, we make an assumption on the preconditioner $\mathcal{B}_h$ of $\mathcal{S}_h^{-1}$.

Assumption A2) We require that there exists a preconditioner $\mathcal{B}_h : V_h \to V_h$, which is symmetric, positive definite with respect to $(\cdot,\cdot)_{0,h}$, such that the spectral equivalence to $\mathcal{S}_h$ holds:

$$C_5(\mathcal{S}_hz_h, z_h)_{0,h} \le (\mathcal{B}_hz_h, z_h)_{0,h} \le C_6(\mathcal{S}_hz_h, z_h)_{0,h} \quad \forall z_h \in V_h. \tag{2.13}$$

From (2.9) and (2.10) we can see that (2.13) implies the following

$$C_5||\mathcal{S}_h z_h||_{1,h}^2 = C_5||z_h||_{-1,h}^2 \leq (\mathcal{B}_h z_h, z_h)_{0,h} \leq C_6||z_h||_{-1,h}^2 = C_6||\mathcal{S}_h z_h||_{1,h}^2 \quad \forall z_h \in V_h. \tag{2.14}$$

Lemma 2.1 *Let $\mathcal{B}_h$ be stated as in Assumption A2). Assuming that Assumption A1) holds, we have*

$$C_5||\mathcal{S}_h z_h||_{1,h} \leq ||\mathcal{B}_h z_h||_{1,h} \leq C_6||\mathcal{S}_h z_h||_{1,h} \quad \forall z_h \in V_h. \tag{2.15}$$

Proof. Firstly, from the definition of the discrete H^{-1} minus norm $||\cdot||_{-1,h}$ in (2.8), we find that for all $0 \neq z_h \in V_h$ and for all $\chi_h \in V_h$,

$$\frac{|(\chi_h, z_h)_{0,h}|}{||z_h||_{1,h}} \leq \sup_{0 \neq w_h \in V_h} \frac{(\chi_h, w_h)_{0,h}}{||w_h||_{1,h}} = ||\chi_h||_{-1,h},$$

that is,

$$|(\chi_h, z_h)_{0,h}| \leq ||\chi_h||_{-1,h}||z_h||_{1,h}.$$

Clearly, for $z_h = 0$, the above still holds. Hence,

$$|(\chi_h, z_h)_{0,h}| \leq ||\chi_h||_{-1,h}||z_h||_{1,h} \quad \forall z_h, \chi_h \in V_h. \tag{2.16}$$

Secondly, from Proposition 2.1 we know that $\mathcal{S}_h$ is symmetric, positive definite with respect to the discrete L^2 inner product $(\cdot,\cdot)_{0,h}$, and we have the following generalized Cauchy-Schwarz inequality

$$|(\mathcal{S}_h \chi_h, z_h)_{0,h}| \leq \Big((\mathcal{S}_h \chi_h, \chi_h)_{0,h}\Big)^{\frac{1}{2}} \Big((\mathcal{S}_h z_h, z_h)_{0,h}\Big)^{\frac{1}{2}} \quad \forall \chi_h, z_h \in V_h, \tag{2.17}$$

and from Assumption A2) we know that the preconditioner $\mathcal{B}_h$, which is symmetric, positive definite with respect to the discrete L^2 inner product $(\cdot,\cdot)_{0,h}$, also satisfies the generalized Cauchy-Schwarz inequality

$$|(\mathcal{B}_h \chi_h, z_h)_{0,h}| \leq \Big((\mathcal{B}_h \chi_h, \chi_h)_{0,h}\Big)^{\frac{1}{2}} \Big((\mathcal{B}_h z_h, z_h)_{0,h}\Big)^{\frac{1}{2}} \quad \forall \chi_h, z_h \in V_h. \tag{2.18}$$

Thus, from (2.14), (2.16) and (2.9), we have

$$||\mathcal{S}_h z_h||_{1,h}^2 \leq \frac{1}{C_5}(\mathcal{B}_h z_h, z_h)_{0,h} \leq \frac{1}{C_5}||\mathcal{B}_h z_h||_{1,h}||z_h||_{-1,h} = \frac{1}{C_5}||\mathcal{B}_h z_h||_{1,h}||\mathcal{S}_h z_h||_{1,h},$$

and it follows that the left-hand side of (2.15) holds. On the other hand,

$$||\mathcal{B}_h z_h||_{1,h} = \sup_{0 \neq w_h \in V_h} \frac{((\mathcal{B}_h z_h, w_h))_{1,h}}{||w_h||_{1,h}}, \tag{2.19}$$

where, from Proposition 2.1 we know that $\mathcal{S}_h^{-1}$ exists, since the coercivity holds with respect to the discrete H^1 inner product $((\cdot,\cdot))_{1,h}$, and from (2.3), (2.18) and (2.14), we have

$$\begin{aligned} &((\mathcal{B}_h z_h, w_h))_{1,h} = ((\mathcal{B}_h z_h, \mathcal{S}_h \mathcal{S}_h^{-1} w_h))_{1,h} = (\mathcal{B}_h z_h, \mathcal{S}_h^{-1} w_h)_{0,h} \\ &\leq \Big((\mathcal{B}_h z_h, z_h)_{0,h}\Big)^{\frac{1}{2}} \Big((\mathcal{B}_h \mathcal{S}_h^{-1} w_h, \mathcal{S}_h^{-1} w_h)_{0,h}\Big)^{\frac{1}{2}} \leq C_6||\mathcal{S}_h z_h||_{1,h}||w_h||_{1,h}. \end{aligned} \tag{2.20}$$

It follows from (2.19) and (2.20) that the right-hand side of (2.15) holds. □

So far, we have completed the definitions of the linear element L^2 projections, the linear finite element solution, the linear element discrete H^{-1} norm, and the linear element spectral equivalent preconditioner $\mathcal{B}_h$. All these will be used to define the linear-element-based H^{-1} LS finite element method in the next section.

In what follows, we shall give some remarks.

Remark 2.1 Due to the linear element space V_h, for simplex meshes, we may use the 'mass-lumping' L^2 inner product [37, 6]: $(u, v)_{0,h} := \sum_{K \in \mathcal{T}_h} \frac{|K|}{d+1} \sum_{i=1}^{d+1} u(a_i)v(a_i)$, where $|K|$ is the area or volume of $K \in \mathcal{T}_h$, and a_i, $1 \le i \le d+1$, are the vertices of K. Note that the resulting matrix is diagonal. For quadrilateral or hexahedral meshes, we may use the four-node or eight-node quadrature formula. Regarding defining $((\cdot,\cdot))_{1,h}$, since it is the approximation of the H^1 inner product $((\cdot,\cdot))_1$ in the linear element space V_h, we may use the same quadrature schemes as defining $(\cdot,\cdot)_{0,h}$. With these choices, Assumption A1) can be easily verified to hold true. We refer to [10] for a complete theory on numerical quadrature schemes. Anyway, both $\mathcal{A}_h$ and $\mathcal{S}_h$ can be easily implemented in the linear element space V_h.

Remark 2.2 To generate a preconditioner $\mathcal{B}_h$ to satisfy Assumption A2), one may use the multigrid algorithm one time on the h level [26, 6, 7, 15]. For the linear element V_h on nested meshes, applying a one time V-cycle multigrid algorithm on the h level with one smoothing, we can obtain a preconditioner $\mathcal{B}_h$ which satisfies the spectral equivalence (2.13) or equivalently the norm equivalence (2.15). See, e.g., Section 7.4 in [26] for V-cycle multigrid preconditioner and Section 4 in [26] for multilevel additive preconditioner. The preconditioner $\mathcal{B}_h$ satisfying Assumption A2) may be also generated from domain decomposition methods, see [9, 34, 39]. Indeed, as a preconditioner in the linear element space V_h of the Poisson Dirichlet problem, many existing algorithms are readily available for generating such a $\mathcal{B}_h$.

Remark 2.3 Since $\mathcal{A}_h$ and $\mathcal{S}_h$ are not involved with the finite element spaces $X_h \times M_h$, they can be obtained in advance, preceding the solution of the Stokes problem in $X_h \times M_h$. Note that the T_h, as mentioned earlier in the Introduction section, is defined by $((T_h\chi, v_h))_1 = \langle \chi, v_h\rangle$ for all $v_h \in X_h$ for given $\chi \in (H^{-1}(\Omega))^d$. Only when $X_h = V_h$, may the combination $\mathcal{S}_h\mathcal{A}_h$ in (2.4) be viewed as T_h. However, more importantly, here the combination $\mathcal{S}_h\mathcal{A}_h$ is defined only onto the linear element space V_h, whatever the finite element space X_h of velocity is. This is a significant difference from T_h which is defined onto and varies with X_h. When X_h is high-order elements, a 'high-order' T_h need to solve and must be realized at the realtime when X_h is specified (i.e., the realization of T_h cannot be performed before the specification of X_h).

3 Stabilization, finite element method and consistency

For solving the velocity and the pressure of the Stokes problem, we introduce two finite element spaces as follows:

$$X_h = \{v \in X : v|_K \in (R_l(K))^d, \forall K \in \mathcal{T}_h\}, \quad M_h = \{q \in M \cap H^1(\Omega) : q|_K \in R_m(K), \forall K \in \mathcal{T}_h\}. \tag{3.1}$$

We first define two types of mesh-dependent residual-based stabilization terms.

For all $(u,p),(v,q) \in \prod_{K\in\mathcal{T}_h}(H^2(K))^d \times \prod_{K\in\mathcal{T}_h} H^1(K)$, we define the stabilizing bilinear form

$$\mathscr{C}_h(u,p;v,q) := \sum_{K\in\mathcal{T}_h} h_K^2(-\Delta u + \nabla p, -\Delta v + \nabla q)_{0,K} + \sum_{F\in\mathcal{F}_h^I} h_F \int_F [\frac{\partial u}{\partial n}][\frac{\partial v}{\partial n}], \tag{3.2}$$

and the corresponding right-hand side linear form for $f \in (L^2(\Omega))^d$

$$\mathscr{G}_h(f;v,q) := \sum_{K\in\mathcal{T}_h} h_K^2(f, -\Delta v + \nabla q)_{0,K}. \tag{3.3}$$

When $(u,p) \in (H^2(\Omega))^d \times H^1(\Omega)/\mathbb{R}$ is the exact solution solving the Stokes problem (1.1), we have the consistency for all $(v,q) \in \prod_{K\in\mathcal{T}_h}(H^2(K))^d \times \prod_{K\in\mathcal{T}_h} H^1(K)$:

$$\mathscr{C}_h(u,p;v,q) = \mathscr{G}_h(f;v,q). \tag{3.4}$$

We also have the following boundedness:

$$|\mathscr{C}_h(u_h,p_h;v_h,q_h)| \le C(||u_h||_1 + ||p_h||_0)(||v_h||_1 + ||q_h||_0) \quad \forall (u_h,p_h),(v_h,q_h) \in X_h \times M_h, \tag{3.5}$$

$$|\mathscr{G}_h(f;v_h,q_h)| \le C||f||_0(||v_h||_1 + ||q_h||_0) \quad \forall (v_h,q_h) \in X_h \times M_h. \tag{3.6}$$

If the exact solution (u,p) of the Stokes problem is not so smooth that they belong to $(H^2(\Omega))^d \times H^1(\Omega)/\mathbb{R}$ and f may be in $(H^{-1}(\Omega))^d$, we have to adopt the embedded-bubble technique in [17] to formulate the stabilizing bilinear form and its right-hand side in the following.

For each interior edge/face $F \in \mathcal{F}_h^I$ which is shared by two elements K_1 and K_2, let $M_F = K_1 \cup K_2$ denote the macro-element, and let $\mathcal{M}_h$ be the set of all these macro-elements M_F when collecting all the interior edges/faces $F \in \mathcal{F}_h^I$, i.e., $\mathcal{M}_h = \{M_F, \forall F \in \mathcal{F}_h^I\}$. Let $b_{M_F} \in H_0^1(M_F)$ denote the macro-element bubble over $M_F = K_1 \cup K_2$, whose restriction to F is an edge/face bubble, i.e., $b_{M_F}|_F \in H_0^1(F)$. It may be generated by the local basis functions of $R_1(K_1)$ and $R_1(K_2)$ associated with the vertices of $F \in \partial K_1 \cap \partial K_2$. For example, in the case of simplexes, letting λ_j^K, $1 \le j \le d$, denote the d local basis of the linear element $P_1(K)$ associated with the d vertices of F, we define $b_{M_F}|_{K_i} = \lambda_1^{K_i} \cdots \lambda_d^{K_i}$, $i = 1, 2$. Clearly, this $b_{M_F} \in H_0^1(M_F)$ and $b_{M_F}|_F \in H_0^1(F)$. Meanwhile, let $b_K \in H_0^1(K)$ denote the element bubble, for example, $b_K = \lambda_1^K \cdots \lambda_d^K \lambda_{d+1}^K$ for simplexes, where λ_{d+1}^K denotes the local basis of $P_1(K)$ associated with the vertex opposite F. For each $F \in \mathcal{F}_h^I$ which corresponds to the macro-element $M_F \in \mathcal{M}_h$ and for each element $K \in \mathcal{T}_h$, we introduce two local function spaces

$$\Theta(M_F) = \mathrm{span}\{\theta_j \in (H^1(M_F))^d, 1 \le j \le J_F\}, \quad W(K) = \mathrm{span}\{\psi_j \in (H^1(K))^d, 1 \le j \le J_K\}, \tag{3.7}$$

where those local functions $\theta_j, 1 \le j \le J_F$, and $\psi_j, 1 \le j \le J_K$, with two integers J_F and J_K, are chosen so that the following local inclusions hold:

$$(-\Delta v_h + \nabla q_h)|_K \in W(K), \qquad [\![\frac{\partial v_h}{\partial n}]\!]|_F \in \Theta(M_F)|_F \quad \forall (v_h, q_h) \in X_h \times M_h. \tag{3.8}$$

These local inclusions can be easily done. In fact, for example, considering the case where $\mathcal{T}_h$ is composed of simplexes, we may choose $\Theta(M_F)$ and $W(K)$ as the spaces of polynomials as follows:

$$\Theta(M_F) = (R_{l-1}(M_F))^d, \qquad W(K) = (R_{\max(l-2,m-1)}(K))^d.$$

With this choice we can easily verify the above local inclusions (3.8). Let $\gamma > 0$ be a stabilization constant independent of h. For all $(u,p), (v,q) \in X \times M$, we define the stabilizing bilinear form and linear form:

$$\begin{aligned}\mathscr{C}_h(u,p;v,q) := \sum_{K \in \mathcal{T}_h} \frac{\sum_{j=1}^{J_K} \Big((\nabla u, \nabla(\psi_j b_K))_{0,K} - (\mathrm{div}\,(\psi_j b_K), p)_{0,K}\Big)\Big((\nabla v, \nabla(\psi_j b_K))_{0,K} - (\mathrm{div}\,(\psi_j b_K), q)_{0,K}\Big)}{\sum_{j=1}^{J_K} ||\nabla(\psi_j b_K)||_{0,K}^2} \\ +\gamma \sum_{F \in \mathcal{F}_h^I} \frac{\sum_{j=1}^{J_F} \Big((\nabla u, \nabla(\theta_j b_{M_F}))_{0,M_F} - (\mathrm{div}\,(\theta_j b_{M_F}), p)_{0,M_F}\Big)\Big((\nabla v, \nabla(\theta_j b_{M_F}))_{0,M_F} - (\mathrm{div}\,(\theta_j b_{M_F}), q)_{0,M_F}\Big)}{\sum_{j=1}^{J_F} ||\nabla(\theta_j b_{M_F})||_{0,M_F}^2}\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mathscr{G}_h(f;v,q) := \sum_{K \in \mathcal{T}_h} \frac{\sum_{j=1}^{J_K} \Big((f, \psi_j b_K)_{0,K}\Big)\Big((\nabla v, \nabla(\psi_j b_K))_{0,K} - (\mathrm{div}\,(\psi_j b_K), q)_{0,K}\Big)}{\sum_{j=1}^{J_K} ||\nabla(\psi_j b_K)||_{0,K}^2} \\ +\gamma \sum_{F \in \mathcal{F}_h^I} \frac{\sum_{j=1}^{J_F} \Big((f, \theta_j b_{M_F})_{0,M_F}\Big)\Big((\nabla v, \nabla(\theta_j b_{M_F}))_{0,M_F} - (\mathrm{div}\,(\theta_j b_{M_F}), q)_{0,M_F}\Big)}{\sum_{j=1}^{J_F} ||\nabla(\theta_j b_{M_F})||_{0,M_F}^2}.\end{aligned} \tag{3.10}$$

Clearly, when $(u,p) \in X \times M$ is the solution of the Galerkin problem (1.5), we have the consistency:

$$\mathscr{C}_h(u,p;v,q) = \mathscr{G}_h(f;v,q) \quad \forall (v,q) \in X \times M. \tag{3.11}$$

We also have the boundedness:

$$|\mathscr{C}_h(u,p;v,q)| \le C(||u||_1 + ||p||_0)(||v||_1 + ||q||_0) \quad \forall (u,p), (v,q) \in X \times M, \tag{3.12}$$

$$|\mathscr{G}_h(f;v,q)| \le C||f||_{-1}(||v||_1 + ||q||_1) \quad \forall (v,q) \in X \times M. \tag{3.13}$$

On the other hand, for all $(u,p),(v,q) \in \prod_{K\in\mathcal{T}_h}(H^2(K))^d \times \prod_{K\in\mathcal{T}_h} H^1(K)$, we have

$$\begin{aligned}\mathscr{C}_h(u,p;v,q) &:= \sum_{K\in\mathcal{T}_h} \frac{\sum_{j=1}^{J_K}\Big((-\Delta u + \nabla p, \psi_j b_K)_{0,K}\Big)\Big((-\Delta v + \nabla q, \psi_j b_K)_{0,K}\Big)}{\sum_{j=1}^{J_K} ||\nabla(\psi_j b_K)||^2_{0,K}} \\ &+\gamma \sum_{F\in\mathcal{F}_h^I} \frac{\sum_{j=1}^{J_F}\Big((-\Delta u + \nabla p, \theta_j b_{M_F})_{0,M_F} + \int_F \theta_j b_{M_F}[\frac{\partial u}{\partial n}]\Big)\Big((-\Delta v + \nabla q, \theta_j b_{M_F})_{0,M_F} + \int_F \theta_j b_{M_F}[\frac{\partial v}{\partial n}]\Big)}{\sum_{j=1}^{J_F} ||\nabla(\theta_j b_{M_F})||^2_{0,M_F}}.\end{aligned} \tag{3.14}$$

Under the local inclusions in (3.8), following the argument in [17], for a suitable $\gamma > 0$, we can show the following equivalence:

$$C_7||(v_h,q_h)||_h^2 \le \mathscr{C}_h(v_h,q_h;v_h,q_h) \le C_8||(v_h,q_h)||_h^2 \quad \forall (v_h,q_h) \in X_h \times M_h, \tag{3.15}$$

where

$$||(v_h,q_h)||_h^2 := \left(\sum_{K\in\mathcal{T}_h} h_K^2||-\Delta v_h + \nabla q_h||^2_{0,K} + \sum_{F\in\mathcal{F}_h^I} h_F \int_F |[\frac{\partial v_h}{\partial n}]|^2\right). \tag{3.16}$$

Remark 3.1 The stabilization terms in (3.9)-(3.10) are advantageous over the ones (3.2)-(3.3), since they are meaningful for the weak solution only in $X \times M$ of the Galerkin Stokes problem (1.5). With this choice, we can establish the following quasi-optimal error estimates between the exact solution $(u,p) \in X\times M$ and the finite element solution $(u_h,p_h) \in X_h \times M_h$: $||u-u_h||_1 + ||p-p_h||_0 \le C \inf_{(v_h,q_h)\in X_h\times M_h}(||u-v_h||_1 + ||p-q_h||_0)$. If (3.2) and (3.3) are instead used, a weaker result holds. See Theorem 4.3 in the next section.

We are now in a position to state the finite element problem for solving the Stokes problem (1.1): Find $(u_h,p_h) \in X_h \times M_h$ such that for all $(v_h,q_h) \in X_h \times M_h$,

$$\begin{aligned}&\mathscr{B}_h(u_h,p_h;v_h,q_h) := (\mathcal{A}_h(-\Delta u_h + \nabla p_h), \mathcal{S}_h\mathcal{A}_h(-\Delta v_h + \nabla q_h))_{0,h} + \mathscr{C}_h(u_h,p_h;v_h,q_h) + (\operatorname{div} u_h, \operatorname{div} v_h) \\ &= \mathscr{R}_h(f;v_h,q_h) := \langle f, \mathcal{S}_h\mathcal{A}_h(-\Delta v_h + \nabla q_h)\rangle + \mathscr{G}_h(f;v_h,q_h).\end{aligned} \tag{3.17}$$

With the preconditioner $\mathcal{B}_h$ replacing $\mathcal{S}_h$ in the above, the alternative finite element problem for solving the Stokes problem (1.1) reads: Find $(u_h,p_h) \in X_h \times M_h$ such that for all $(v_h,q_h) \in X_h \times M_h$,

$$\begin{aligned}&\widetilde{\mathscr{B}}_h(u_h,p_h;v_h,q_h) := (\mathcal{A}_h(-\Delta u_h + \nabla p_h), \mathcal{B}_h\mathcal{A}_h(-\Delta v_h + \nabla q_h))_{0,h} + \mathscr{C}_h(u_h,p_h;v_h,q_h) + (\operatorname{div} u_h, \operatorname{div} v_h) \\ &= \widetilde{\mathscr{R}}_h(f;v_h,q_h) := \langle f, \mathcal{B}_h\mathcal{A}_h(-\Delta v_h + \nabla q_h)\rangle + \mathscr{G}_h(f;v_h,q_h).\end{aligned} \tag{3.18}$$

In both (3.17) and (3.18), if $(u,p) \in (H^2(\Omega))^d \times H^1(\Omega)/\mathbb{R}$ solve the Stokes problem (1.1) and $f \in (L^2(\Omega))^d$, we employ the stabilizations (3.2) and (3.3). If $(u,p) \in X \times M$ solve the Galerkin problem (1.5) and $f \in (H^{-1}(\Omega))^d$, then the stabilizations (3.9) and (3.10) are chosen.

Remark 3.2 Both (3.17) and (3.18) are least-squares methods based on discrete H^{-1}-norm. Note that, from (2.9) and (2.10), we have $(\mathcal{A}_h(-\Delta v_h + \nabla q_h), \mathcal{S}_h\mathcal{A}_h(-\Delta v_h + \nabla q_h))_{0,h} = ||\mathcal{A}_h(-\Delta v_h + \nabla q_h)||^2_{-1,h} = ||\mathcal{S}_h\mathcal{A}_h(-\Delta v_h + \nabla q_h)||^2_{1,h}$ for all $(v_h,q_h) \in X_h \times M_h$. From (2.14), $||\mathcal{S}_h\mathcal{A}_h(-\Delta v_h + \nabla q_h)||^2_{1,h}$ is equivalent to $(\mathcal{A}_h(-\Delta v_h + \nabla q_h), \mathcal{B}_h\mathcal{A}_h(-\Delta v_h + \nabla q_h))_{0,h}$. Note that $||\mathcal{A}_h(-\Delta v_h + \nabla q_h)||^2_{-1,h} + ||\operatorname{div} v_h||_0^2$ is not yet equivalent to $||v_h||_1^2 + ||q_h||_0^2$, although $||-\Delta v_h + \nabla q_h||^2_{-1} + ||\operatorname{div} v_h||_0^2$ is. To obtain this equivalence for $||\mathcal{A}_h(-\Delta v_h + \nabla q_h)||^2_{-1,h} + ||\operatorname{div} v_h||_0^2$, the stabilization $\mathscr{C}_h(u_h,p_h;v_h,q_h)$ must be introduced (see Lemma 4.1 in the next section), while the right-hand side $\mathscr{G}_h(f;v_h,q_h)$ is only for consistency.

We next show how to use the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ in (1.4) to express the above finite element problems so that we can easily see that the finite element problems satisfy the consistency property. For any given $(v_h,q_h)\in X_h\times M_h$, we can associate with a $\widetilde{v}\in V_h\subset X$ as follows:

$$\widetilde{v}=\mathcal{S}_h\mathcal{A}_h(-\Delta v_h+\nabla q_h)\quad\text{or}\quad \widetilde{v}=\mathcal{B}_h\mathcal{A}_h(-\Delta v_h+\nabla q_h), \tag{3.19}$$

and from (2.6) we have

$$(\mathcal{A}_h(-\Delta u_h+\nabla p_h),\widetilde{v})_{0,h}=a(u_h,\widetilde{v})+b(\widetilde{v},p_h). \tag{3.20}$$

From (3.19), (3.20), (3.4) or (3.11) and (1.5) we can show the consistency property of the finite element methods (3.17) and (3.18).

Theorem 3.1 *Let $(u,p)\in(H^2(\Omega))^d\times H^1(\Omega)/\mathbb{R}$ solve the Stokes problem (1.1) or $(u,p)\in X\times M$ solve the Galerkin problem (1.5), where the stabilizations $\mathscr{C}_h(\cdot,\cdot;\cdot,\cdot)$ and $\mathscr{G}_h(f;\cdot,\cdot)$ are correspondingly chosen as (3.2)-(3.3) or (3.9)-(3.10). Let (u_h,p_h) solve the finite element problem (3.17) or (3.18). Then, we have the following consistency property:*

$$\widetilde{\mathscr{B}}_h(u-u_h,p-p_h;v_h,q_h)=\mathscr{B}_h(u-u_h,p-p_h;v_h,q_h)=0\quad\forall(v_h,q_h)\in X_h\times M_h. \tag{3.21}$$

$\Box$

4 Coercivity, condition number and error bound

In this section we shall establish the coercivity/ellipticity over $X_h\times M_h$ and the quasi-optimal error bound.

Lemma 4.1 *Assume that Assumption A1) holds. We have*

$$C(||u||_1^2+||p||_0^2)\le(\mathcal{A}_h(-\Delta u+\nabla p),\mathcal{S}_h\mathcal{A}_h(-\Delta u+\nabla p))_{0,h}+||(u,p)||_h^2+||\mathrm{div}\,u||_0^2, \tag{4.1}$$

for all $(u,p)\in X\times M\cap(\prod_{K\in\mathcal{T}_h}(H^2(K))^d\times\prod_{K\in\mathcal{T}_h}H^1(K))$, where $||(u,p)||_h$ is defined by (3.16).

Proof. For any given $(u,p)\in X\times M\cap(\prod_{K\in\mathcal{T}_h}(H^2(K))^d\times\prod_{K\in\mathcal{T}_h}H^1(K))$, from (1.6) we have

$$||u||_1+||p||_0\le C\sup_{0\neq(v,q)\in X\times M}\frac{\mathscr{B}(u,p;v,q)}{||v||_1+||q||_0}. \tag{4.2}$$

Letting $\Pi_h:X\to V_h$ denote the linear finite element interpolator [10, 6], we have

$$\left(\sum_{K\in\mathcal{T}_h}h_K^{-2}||v-\Pi_hv||_{0,K}^2+\sum_{F\in\mathcal{F}_h}h_F^{-1}||v-\Pi_hv||_{0,F}^2\right)^{\frac{1}{2}}+||\Pi_hv||_1\le C||v||_1.$$

From (1.5) we have

$$\mathscr{B}(u,p;v,q)=a(u,v-\Pi_hv)+b(v-\Pi_hv,p)+b(u,q)+a(u,\Pi_hv)+b(\Pi_hv,p), \tag{4.3}$$

where

$$\begin{aligned}a(u,v-\Pi_hv)+b(v-\Pi_hv,p)&=\sum_{K\in\mathcal{T}_h}(-\Delta u+\nabla p,v-\Pi_hv)_{0,K}+\sum_{F\in\mathcal{F}_h^I}\int_F[\frac{\partial u}{\partial n}](v-\Pi_hv)\\ &\le C||(u,p)||_h\left(\sum_{K\in\mathcal{T}_h}h_K^{-2}||v-\Pi_hv||_{0,K}^2+\sum_{F\in\mathcal{F}_h}h_F^{-1}||v-\Pi_hv||_{0,F}^2\right)^{\frac{1}{2}}\le C||(u,p)||_h||v||_1,\end{aligned} \tag{4.4}$$

$$b(u,q)\le||\mathrm{div}\,u||_0||q||_0. \tag{4.5}$$

Thus, we have

$$\sup_{0\neq(v,q)\in X\times M}\frac{\mathscr{B}(u,p;v,q)}{||v||_1+||q||_0}\le C(||(u,p)||_h+||\mathrm{div}\,u||_0)+\sup_{0\neq(v,q)\in X\times M}\frac{a(u,\Pi_hv)+b(\Pi_hv,p)}{||v||_1+||q||_0}, \tag{4.6}$$

where

$$
\begin{aligned}
\sup_{0\neq(v,q)\in X\times M}\frac{a(u,\Pi_h v)+b(\Pi_h v,p)}{||v||_1+||q||_0} &\leq \sup_{0\neq v\in X}\frac{a(u,\Pi_h v)+b(\Pi_h v,p)}{||v||_1},\\
&\leq C\sup_{0\neq v\in X}\frac{a(u,\Pi_h v)+b(\Pi_h v,p)}{||\Pi_h v||_1}\\
&\leq C\sup_{0\neq v\in X}\sup_{0\neq z_h\in V_h}\frac{a(u,z_h)+b(z_h,p)}{||z_h||_1}\\
&= C\sup_{0\neq z_h\in V_h}\frac{a(u,z_h)+b(z_h,p)}{||z_h||_1}\\
&= C\sup_{0\neq z_h\in V_h}\frac{(\mathcal{A}_h(-\Delta u+\nabla p),z_h)_{0,h}}{||z_h||_1}\\
&\leq C\sup_{0\neq z_h\in V_h}\frac{(\mathcal{A}_h(-\Delta u+\nabla p),z_h)_{0,h}}{||z_h||_{1,h}}\\
&= C||\mathcal{A}_h(-\Delta u+\nabla p)||_{-1,h}\\
&= C\Big((\mathcal{A}_h(-\Delta u+\nabla p),\mathcal{S}_h\mathcal{A}_h(-\Delta u+\nabla p))_{0,h}\Big)^{\frac{1}{2}}.
\end{aligned}
\tag{4.7}
$$

Therefore, summarizing (4.2)-(4.7), we have

$$
C(||u||_1+||p||_0)\leq\Big((\mathcal{A}_h(-\Delta u+\nabla p),\mathcal{S}_h\mathcal{A}_h(-\Delta u+\nabla p))_{0,h}\Big)^{\frac{1}{2}}+||(u,p)||_h+||\mathrm{div}\,u||_0,
$$

which completes the proof. □

Theorem 4.1 *Assume that Assumptions A1) and A2) hold. We have the following coercvity/ellipticity:*

$$
\widetilde{\mathscr{B}}_h(u,p;u,p),\qquad \mathscr{B}_h(u,p;u,p)\geq C(||u||_1^2+||p||_0^2)\quad\forall(u,p)\in X_h\times M_h. \tag{4.8}
$$

Proof. From Lemma 4.1 and Assumption A2) it follows that (4.8) holds. □

Lemma 4.2 *Assume that Assumptions A1) and A2) hold. We have*

$$
|\widetilde{\mathscr{B}}_h(u,p;v,q)|,\qquad |\mathscr{B}_h(u,p;v,q)|\leq C(||u||_1+||p||_0)(||v||_1+||q||_0). \tag{4.9}
$$

$$
|\widetilde{\mathscr{R}}_h(f;v,q)|,\qquad |\mathscr{R}_h(f;v,q)|\leq C||f||_*(||v||_1+||q||_0) \tag{4.10}
$$

for all $(u,p),(v,q)\in X_h\times M_h$ if the stabilizations are (3.2)-(3.3) and $||f||_*=||f||_0$, or for all $(u,p),(v,q)\in X\times M$ if the stabilizations are (3.9)-(3.10) and $||f||_*=||f||_{-1}$.

Proof. Below we only show (4.9), while (4.10) can be similarly shown. From (2.16) and (2.9) we have

$$
\begin{aligned}
|(\mathcal{A}_h(-\Delta u+\nabla p),\mathcal{S}_h\mathcal{A}_h(-\Delta v+\nabla q))_{0,h}| &\leq ||\mathcal{A}_h(-\Delta u+\nabla p)||_{-1,h}||\mathcal{S}_h\mathcal{A}_h(-\Delta v+\nabla q)||_{1,h}\\
&= ||\mathcal{A}_h(-\Delta u+\nabla p)||_{-1,h}||\mathcal{A}_h(-\Delta v+\nabla q)||_{-1,h},
\end{aligned}
$$

but, for all $(u,p)\in X\times M$, we have from (2.11), (2.6) and Assumption A1)

$$
\begin{aligned}
||\mathcal{A}_h(-\Delta u+\nabla p)||_{-1,h} &= \sup_{0\neq z_h\in V_h}\frac{(\mathcal{A}_h(-\Delta u+\nabla p),z_h)_{0,h}}{||z_h||_{1,h}}=\sup_{0\neq z_h\in V_h}\frac{a(u,z_h)+b(z_h,p)}{||z_h||_{1,h}}\\
&\leq C\sup_{0\neq z_h\in V_h}\frac{a(u,z_h)+b(z_h,p)}{||z_h||_1}\leq C(||u||_1+||p||_0),
\end{aligned}
\tag{4.11}
$$

and we have for all $(u,p)\in X\times M$ and for all $(v,q)\in X\times M$,

$$
|(\mathcal{A}_h(-\Delta u+\nabla p),\mathcal{S}_h\mathcal{A}_h(-\Delta v+\nabla q))_{0,h}|\leq C(||u||_1+||p||_0)(||v||_1+||q||_0). \tag{4.12}
$$

Thanks to (2.16), Lemma 2.1, Proposition 2.2 and (4.11), for $\mathcal{B}_h$, we also have

$$
\begin{aligned}
|(\mathcal{A}_h(-\Delta u+\nabla p),\mathcal{B}_h\mathcal{A}_h(-\Delta v+\nabla q))_{0,h}| &\leq ||\mathcal{A}_h(-\Delta u+\nabla p)||_{-1,h}||\mathcal{B}_h\mathcal{A}_h(-\Delta v+\nabla q)||_{1,h}\\
&\leq C(||u||_1+||p||_0)||\mathcal{S}_h\mathcal{A}_h(-\Delta v+\nabla q)||_{1,h}\\
&= C(||u||_1+||p||_0)||\mathcal{A}_h(-\Delta v+\nabla q)||_{-1,h}\\
&\leq C(||u||_1+||p||_0)(||v||_1+||q||_0).
\end{aligned}
\tag{4.13}
$$

for all $(u,p) \in X \times M$ and for all $(v,q) \in X \times M$. Finally, from (3.5)-(3.6) if the stabilizations are (3.2)-(3.3), or from (3.12)-(3.13) if the stabilizations are (3.9)-(3.10), from (4.12) and (3.17) or from (4.13) and (3.18) we obtain (4.9). $\square$

Theorem 4.2 *Assume that Assumptions A1) and A2) hold. We have the following norm-equivalence over $X_h \times M_h$:*

$$C_9(||u||_1^2 + ||p||_0^2) \leq \widetilde{\mathscr{B}}_h(u,p;u,p), \qquad \mathscr{B}_h(u,p;u,p) \leq C_{10}(||u||_1^2 + ||p||_0^2) \quad \forall (u,p) \in X_h \times M_h. \tag{4.14}$$

Proof. Consequently, Lemma 4.2 and Theorem 4.1 lead to this result. $\square$

Corollary 4.1 *Assuming the same assumptions as in Theorem 4.2, the finite element problem (3.17) or (3.18) admits a unique solution $(u_h, p_h) \in X_h \times M_h$, satisfying*

$$||u_h||_1 + ||p_h||_0 \leq C||f||_0 \quad or \quad C||f||_{-1}. \tag{4.15}$$

Proof. From Theorem 4.1 and Lax-Milgram lemma, we have the existence and uniqueness of the solution of the finite element problems over $X_h \times M_h$. Following the same argument as proving Lemma 4.2, from (3.5)-(3.6) or (3.12)-(3.13), Theorem 4.2, (3.17) (or (3.18)) and (4.10), we can have the continuous dependence on f or the stability of the finite element solution $(u_h, p_h) \in X_h \times M_h$. In other words, (4.15) holds. $\square$

Meanwhile, from Theorem 4.2, a standard argument (see page 261-265 in [6]) and the inverse estimate in Theorem 17.2, page 135 in [10], we can have the estimation of the condition number for the resulting system of the finite element problem.

Corollary 4.2 *Assuming the same assumptions as in Theorem 4.2, and additionally assuming quasi-uniform meshes, we have the condition number $\mathcal{O}(h^{-2})$ of the resulting algebraic system from the finite element problem (3.17) or (3.18).* $\square$

We shall next analyze the error bounds.

Theorem 4.3 *Assume that Assumptions A1) and A2) hold. Let $(u,p) \in (H^2(\Omega))^d \times H^1(\Omega)/\mathbb{R}$ solve the Stokes problem (1.1) or let $(u,p) \in X \times M$ solve the Galerkin Stokes problem (1.5). Let (u_h, p_h) denote the finite element solution of problem (3.17) or (3.18). If the stabilizations are chosen as (3.9)-(3.10), then for $(u,p) \in X \times M$*

$$||u - u_h||_1 + ||p - p_h||_0 \leq C \inf_{(v_h,q_h)\in X_h\times M_h} (||u - v_h||_1 + ||p - q_h||_0). \tag{4.16}$$

If the stabilization terms are chosen as (3.2)-(3.3), then for $(u,p) \in (H^2(\Omega))^d \times H^1(\Omega)/\mathbb{R}$,

$$\begin{aligned} ||u - u_h||_1 + ||p - p_h||_0 \; &\leq C \inf_{(v_h,q_h)\in X_h\times M_h} (||u - v_h||_1 + ||p - q_h||_0) \\ &+ C \inf_{(v_h,q_h)\in X_h\times M_h} \left(\sum_{K\in\mathcal{T}_h} h_K^2(||u - v_h||_{2,K}^2 + ||p - q_h||_{1,K}^2) \right)^{\frac{1}{2}}. \end{aligned} \tag{4.17}$$

Proof. Take any $(v_h, q_h) \in X_h \times M_h$ and put

$$\widetilde{v} := u_h - v_h, \quad \widetilde{q} := p_h - q_h, \quad e_u = u - v_h, \quad e_p := p - q_h.$$

From Theorem 4.1 and Theorem 3.1, we have

$$\begin{aligned} ||\widetilde{v}||_1^2 + ||\widetilde{q}||_0^2 \; &\leq C\mathscr{B}_h(\widetilde{v},\widetilde{q};\widetilde{v},\widetilde{q}), \qquad C\widetilde{\mathscr{B}}_h(\widetilde{v},\widetilde{q};\widetilde{v},\widetilde{q}) \\ &= C\mathscr{B}_h(e_u,e_p;\widetilde{v},\widetilde{q}), \qquad C\widetilde{\mathscr{B}}_h(e_u,e_p;\widetilde{v},\widetilde{q}), \end{aligned}$$

where

$$\mathscr{B}_h(e_u,e_p;\widetilde{v},\widetilde{q}) = (\mathcal{A}_h(-\Delta e_u + \nabla e_p), \mathcal{S}_h\mathcal{A}_h(-\Delta\widetilde{v} + \nabla\widetilde{q}))_{0,h} + \mathscr{C}_h(e_u,e_p;\widetilde{v},\widetilde{q}) + (\operatorname{div} e_u, \operatorname{div}\widetilde{v}),$$

$$\widetilde{\mathscr{B}}_h(e_u,e_p;\widetilde{v},\widetilde{q}) = (\mathcal{A}_h(-\Delta e_u + \nabla e_p), \mathcal{B}_h\mathcal{A}_h(-\Delta\widetilde{v} + \nabla\widetilde{q}))_{0,h} + \mathscr{C}_h(e_u,e_p;\widetilde{v},\widetilde{q}) + (\operatorname{div} e_u, \operatorname{div}\widetilde{v}).$$

Similar to (4.12) and (4.13), we can have

$$\begin{aligned}&|(\mathcal{A}_h(-\Delta e_u+\nabla e_p),\mathcal{S}_h\mathcal{A}_h(-\Delta\widetilde{v}+\nabla\widetilde{q}))_{0,h}|, \qquad |(\mathcal{A}_h(-\Delta e_u+\nabla e_p),\mathcal{B}_h\mathcal{A}_h(-\Delta\widetilde{v}+\nabla\widetilde{q}))_{0,h}|\\ &\le C(||e_u||_1+||e_p||_0)(||\widetilde{v}||_1+||\widetilde{q}||_0).\end{aligned}$$

If the stabilizations are chosen as (3.9)-(3.10), then for $(u,p)\in X\times M$

$$|\mathscr{C}_h(e_u,e_p;\widetilde{v},\widetilde{q})|\le C(||e_u||_1+||e_p||_0)(||\widetilde{v}||_1+||\widetilde{q}||_0).$$

If the stabilizations are chosen as (3.2)-(3.3), then for $(u,p)\in(H^2(\Omega))^d\times H^1(\Omega)/\mathbb{R}$

$$\begin{aligned}\mathscr{C}_h(e_u,e_p;\widetilde{v},\widetilde{q}) &= \sum_{K\in\mathcal{T}_h}h_K^2(-\Delta e_u+\nabla e_p,-\Delta\widetilde{v}+\nabla\widetilde{q})_{0,K}+\sum_{F\in\mathcal{F}_h^I}h_F\int_F[\frac{\partial e_u}{\partial n}][\frac{\partial\widetilde{v}}{\partial n}]\\ &\le C(||\widetilde{v}||_1+||\widetilde{q}||_0)\left(||e_u||_1+\left(\sum_{K\in\mathcal{T}_h}h_K^2(||e_u||_{2,K}^2+||e_p||_{1,K}^2)\right)^{\frac{1}{2}}\right).\end{aligned}$$

$$(\operatorname{div}e_u,\operatorname{div}\widetilde{v})\le C||e_u||_1||\widetilde{v}||_1.$$

The proof is finished by putting the above together and by applying the triangle-inequality. □

Corollary 4.3 *Under the same assumptions as in Theorem 4.3, the estimation (4.16) implies the convergence if the exact solution is only in $X\times M$. Namely,*

$$\lim_{h\to 0}(||u-u_h||_1+||p-p_h||_0)=0.$$

Proof. The argument for proving the above convergence from (4.16) is quite standard, see page 139 in [10]. □

Corollary 4.4 *Under the same assumptions as in Theorem 4.3, for $(u,p)\in(H^{l+1}(\Omega))^d\times H^m(\Omega)$ for $l,m\ge 1$, then for X_h and M_h, which are defined as in (3.1),*

$$||u-u_h||_1+||p-p_h||_0\le Ch^{\min(l,m)}(||u||_{l+1}+||p||_m). \tag{4.18}$$

Proof. Let $(\pi_h u,\rho_h p)\in X_h\times M_h$ denote the finite element interpolations of $(u,p)\in X\times M$, where π_h and ρ_h represent the finite element interpolators. From the classical finite element interpolation theory [10, 6], e.g., see Theorem 16.2 on page 128 in [10], we have for $(u,p)\in(H^{l+1}(\Omega))^d\times H^m(\Omega)$ for $l,m\ge 1$,

$$||u-\pi_h u||_1+||p-\rho_h p||_0+\left(\sum_{K\in\mathcal{T}_h}h_K^2(||u-\pi_h u||_{2,K}^2+||p-\rho_h p||_{1,K}^2)\right)^{\frac{1}{2}}\le Ch^{\min(l,m)}(||u||_{l+1}+||p||_m). \tag{4.19}$$

It then follows from Theorem 4.3 and (4.19) that (4.18) holds. □

5 L^2 error bounds

In this section, we shall establish the L^2 error bound for the velocity. This error bound says that the error between the exact solution and the finite element solution in L^2 norm would have one order higher than that in H^1 norm. We shall elaborate an ad hoc duality argument to achieve this.

For that goal, we shall make a series of assumptions of the H^2 regularity of the solutions of the Stokes problem and the elasticity problem in the following.

Assumption A3) We require that for any given $f\in(L^2(\Omega))^d$, the solution (u,p) of the Stokes problem (1.1) satisfies

$$||u||_2+||p||_1\le C||f||_0.$$

Assumption A4) We require that for any given $f\in(L^2(\Omega))^d$ and for all $\lambda\ge 0$, the following elasticity problem

$$-\Delta u-\lambda\nabla\operatorname{div}u=f\quad\text{in }\Omega,\quad u=0\quad\text{on }\Gamma,$$

has a solution $u \in (H^2(\Omega))^d$, satisfying

$$||u||_2 + \lambda||\text{div}\, u||_1 \leq C||f||_0,$$

where C is independent of λ.

Assumption A5) We require that for any given $f \in (L^2(\Omega))^d$ and for any given $v \in (H^2(\Omega) \cap H_0^1(\Omega))^d$ the following generalized Stokes problem

$$-\Delta u + \nabla p = f, \qquad \text{div}\, u = \text{div}\, v \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma,$$

has a solution pair $(u,p) \in (H^2(\Omega))^d \times H^1(\Omega)$, satisfying

$$||u||_2 + ||p||_1 \leq C(||f||_0 + ||\text{div}\, v||_1).$$

Remark 5.1 On convex polygon or on smooth domains the above H^2 regularity assumptions A3)-A4) hold [3, 6, 23]. Concerning assumption A5), it essentially results from A3). For example, let us consider a convex polygon. From (11.4.21), (11.4.22), page 326 in [6] (or, Lemma 2.1, page 323 in [30]), for the given $v \in (H^2(\Omega) \cap H_0^1(\Omega))^2$, there exists a $v^* \in (H^2(\Omega) \cap H_0^1(\Omega))^2$ such that $\text{div}\, v^* = \text{div}\, v$, satisfying $||v^*||_2 \leq C||\text{div}\, v||_1$. Thus, with the right-hand side $f + \Delta v^*$ in assumption A3), we find $(w,\pi) \in (H^2(\Omega) \cap H_0^1(\Omega))^2 \times H^1(\Omega)/\mathbb{R}$, which solves $-\Delta w + \nabla \pi = f + \Delta v^*$ in Ω, $\text{div}\, w = 0$ in Ω and $w = 0$ on $\partial\Omega$ and satisfies $||w||_2 + ||\pi||_1 \leq C||f + \Delta v^*||_0 \leq C(||f||_0 + ||v^*||_2) \leq C(||f||_0 + ||\text{div}\, v||_1)$. Put $u := v^* + w$ and $p := \pi$. Such (u,p) is the desired in assumption A5) with the given f and v.

Assumption A6) We require that the numerical quadrature $((\cdot,\cdot))_{1,h}$ is chosen so that

$$|((w_h, z_h))_{1,h} - ((w_h, z_h))_1| \leq Ch||w_h||_1||z_h||_1 \quad \forall w_h, z_h \in V_h,$$

where $((\cdot,\cdot))_1 = (\nabla\cdot, \nabla\cdot) + (\cdot,\cdot)$ or $(\nabla\cdot, \nabla\cdot)$.

Remark 5.2 See for a complete theory in [10] from which the above holds, e.g., see Theorem 28.2, page 199 in [10].

Lemma 5.1 *Assume that $((\cdot,\cdot))_{1,h}$ is the approximation of $((\cdot,\cdot))_1 = (\nabla\cdot,\nabla\cdot)$. For any given $w_h \in V_h$, letting $\widetilde{w} := \mathcal{S}_h\mathcal{A}_h(-\Delta w_h) \in V_h$ denote the linear finite element solution to the Poisson Dirichlet problem for the right-hand side $-\Delta w_h \in (H^{-1}(\Omega))^d$: $z \in H_0^1(\Omega)$ solves*

$$-\Delta z = -\Delta w_h \quad \textit{in } \Omega, \quad z = 0 \quad \textit{on } \Gamma. \tag{5.1}$$

Assuming Assumption A6) and Assumption A1), we have

$$||w_h - \widetilde{w}||_1, \qquad ||w_h - \widetilde{w}||_{1,h} \leq Ch||w_h||_1. \tag{5.2}$$

Proof. From (2.4), we have

$$((\widetilde{w}, z_h))_{1,h} = ((\mathcal{S}_h\mathcal{A}_h(-\Delta w_h), z_h))_{1,h} = \langle -\Delta w_h, z_h\rangle = (\nabla w_h, \nabla z_h) = ((w_h, z_h))_1 \quad \forall z_h \in V_h.$$

Choosing

$$z_h := \widetilde{w} - w_h \in V_h,$$

from Assumption A6) and Assumption A1), we have

$$\begin{aligned}&||z_h||_{1,h}^2 = ((z_h, z_h))_{1,h} = ((\widetilde{w} - w_h, z_h))_{1,h} = ((\widetilde{w}, z_h))_{1,h} - ((w_h, z_h))_{1,h} = ((w_h, z_h))_1 - ((w_h, z_h))_{1,h}\\ &\leq Ch||w_h||_1||z_h||_1 \leq Ch||w_h||_1||z_h||_{1,h},\end{aligned}$$

and we obtain

$$||\widetilde{w} - w_h||_{1,h} = ||z_h||_{1,h} \leq Ch||w_h||_1.$$

From Assumption A1), we further have $||\widetilde{w} - w_h||_1 \leq Ch||w_h||_1$. $\square$

For the choice of $((\cdot,\cdot))_{1,h}$ being the approximation of $((\cdot,\cdot))_1 = (\nabla\cdot,\nabla\cdot)+(\cdot,\cdot)$, we can similarly prove the following lemma.

Lemma 5.2 *Assume that $((\cdot,\cdot))_{1,h}$ is the approximation of $((\cdot,\cdot))_1 = (\nabla\cdot,\nabla\cdot)+(\cdot,\cdot)$. For any given $w_h \in V_h$, letting $\widetilde{w} := \mathcal{S}_h\mathcal{A}_h(-\Delta w_h + w_h) \in V_h$ denote the linear finite element solution to the Poisson Dirichlet problem for the right-hand side $-\Delta w_h + w_h \in (H^{-1}(\Omega))^d$: $z \in H_0^1(\Omega)$ solves*

$$-\Delta z + z = -\Delta w_h + w_h \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \Gamma. \tag{5.3}$$

Assuming Assumption A6) and Assumption A1), we have

$$||w_h - \widetilde{w}||_1, \qquad ||w_h - \widetilde{w}||_{1,h} \le Ch||w_h||_1. \tag{5.4}$$

□

Remark 5.3 Note that $w_h \in V_h$ is in fact the linear finite element solution to the Poisson Dirichlet problem which is solved using the bilinear form $((\cdot,\cdot))_1$, while $\widetilde{w} = \mathcal{S}_h\mathcal{A}_h(-\Delta w_h)$ or $\widetilde{w} = \mathcal{S}_h\mathcal{A}_h(-\Delta w_h + w_h)$ is also the linear finite element solution to the same problem but it is solved using the numerical quadrature $((\cdot,\cdot))_{1,h}$. Also, note that if we choose $((\cdot,\cdot))_{1,h} := ((\cdot,\cdot))_1$, i.e., if no numerical quadrature is used for defining $\mathcal{S}_h$, then $\widetilde{w} = w_h$. In that case, Assumption A6) is unnecessary.

We shall now prove the L^2 error bound of the velocity variable for the case where $\mathcal{S}_h$ is employed in the finite element problem (3.17).

Theorem 5.1 *Assume that Assumptions A1)-A6) hold. Let $(u,p) \in (H^2(\Omega))^d \times H^1(\Omega)/\mathbb{R}$ denote the solution pair of the Stokes problem (1.1) and let (u_h,p_h) denote the finite element solution of (3.17). Then*

$$||u-u_h||_0 \le Ch\left(||u-u_h||_1 + ||p-p_h||_0 + \left(\sum_{K\in\mathcal{T}_h} h_K^2(||u-u_h||_{2,K}^2 + ||p-p_h||_{1,K}^2)\right)^{\frac{1}{2}}\right). \tag{5.5}$$

Proof. From Assumption A3) we first know that the solution pair (u,p) of Stokes problem (1.1) belongs to $H^2(\Omega)\times H^1(\Omega)$. Putting

$$E_u := u - u_h \in X, \quad E_p = p - p_h \in M \cap H^1(\Omega), \quad \lambda := \frac{1}{h},$$

from Assumption A4) we have a solution $w \in (H_0^1(\Omega)\cap H^2(\Omega))^d$ which solves

$$-\Delta w - \lambda\,\nabla \mathrm{div}\, w = E_u \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \Gamma,$$

and which satisfies

$$||w||_2 + \lambda\,||\mathrm{div}\, w||_1 \le C||E_u||_0. \tag{5.6}$$

Put

$$\theta := \lambda\,\mathrm{div}\, w, \quad \text{where } ||\theta||_1 \le C||E_u||_0,$$

and take $\bar{w} \in V_h$ such that

$$||w-\bar{w}||_1 \le Ch||w||_2. \tag{5.7}$$

In terms of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ in (1.4), we have

$$\begin{aligned}&||E_u||_0^2 = (-\Delta w - \nabla\theta, E_u) = a(E_u,w) - b(E_u,\theta)\\ &= a(E_u, w-\bar{w}) + b(w-\bar{w},E_p) - b(w,E_p) - b(E_u,\theta) + a(E_u,\bar{w}) + b(\bar{w},E_p),\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}a(E_u,w-\bar{w}) + b(w-\bar{w},E_p) \le C(||E_u||_1 + ||E_p||_0)||w-\bar{w}||_1 &\le Ch(||E_u||_1 + ||E_p||_0)||w||_2\\ &\le Ch(||E_u||_1 + ||E_p||_0)||E_u||_0,\end{aligned} \tag{5.9}$$

$$-b(w,E_p) \le ||\mathrm{div}\, w||_0||E_p||_0 = \lambda^{-1}||\theta||_0||E_p||_0 = h||\theta||_0||E_p||_0 \le h||\theta||_1||E_p||_0 \le Ch||E_p||_0||E_u||_0, \tag{5.10}$$

and if $((\cdot,\cdot))_{1,h}$ is chosen as the approximation of $((\cdot,\cdot))_1 = (\nabla\cdot,\nabla\cdot)$, then from (2.6)

$$
\begin{aligned}
a(E_u,\bar w)+b(\bar w,E_p)=(\mathcal{A}_h(-\Delta E_u+\nabla E_p),\bar w)_{0,h} \;&= (\mathcal{A}_h(-\Delta E_u+\nabla E_p),\bar w-\mathcal{S}_h\mathcal{A}_h(-\Delta\bar w))_{0,h}\\
&+(\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{S}_h\mathcal{A}_h(-\Delta(\bar w-w)))_{0,h}\\
&+(\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{S}_h\mathcal{A}_h(-\Delta w))_{0,h},
\end{aligned}\tag{5.11}
$$

alternatively, if $((\cdot,\cdot))_{1,h}$ is chosen as the approximation of $((\cdot,\cdot))_1 = (\nabla\cdot,\nabla\cdot)+(\cdot,\cdot)$, then

$$
\begin{aligned}
a(E_u,\bar w)+b(\bar w,E_p) \;&= (\mathcal{A}_h(-\Delta E_u+\nabla E_p),\bar w)_{0,h}\\
&= (\mathcal{A}_h(-\Delta E_u+\nabla E_p),\bar w-\mathcal{S}_h\mathcal{A}_h(-\Delta\bar w+\bar w))_{0,h}\\
&+(\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{S}_h\mathcal{A}_h(-\Delta(\bar w-w)+\bar w-w))_{0,h}\\
&+(\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{S}_h\mathcal{A}_h(-\Delta w+w))_{0,h}.
\end{aligned}\tag{5.12}
$$

Below we only consider (5.11), while (5.12) can be similarly dealt with. From (2.6), Lemma 5.1, (5.7), (5.6), (2.12) and the fact that from (1.2) we have

$$
\|-\Delta(\bar w-w)\|_{-1}=\sup_{0\neq v\in X}\frac{\langle-\Delta(\bar w-w),v\rangle}{\|v\|_1}=\sup_{0\neq v\in X}\frac{a(\bar w-w,v)}{\|v\|_1}\le C\|\bar w-w\|_1,
$$

it follows that

$$
\begin{aligned}
&(\mathcal{A}_h(-\Delta E_u+\nabla E_p),\bar w-\mathcal{S}_h\mathcal{A}_h(-\Delta\bar w))_{0,h}=a(E_u,\bar w-\mathcal{S}_h\mathcal{A}_h(-\Delta\bar w))+b(\bar w-\mathcal{S}_h\mathcal{A}_h(-\Delta\bar w),E_p)\\
&\le C(\|E_u\|_1+\|E_p\|_0)\|\bar w-\mathcal{S}_h\mathcal{A}_h(-\Delta\bar w)\|_1\le Ch(\|E_u\|_1+\|E_p\|_0)\|\bar w\|_1\\
&\le Ch(\|E_u\|_1+\|E_p\|_0)\|w\|_2\le Ch(\|E_u\|_1+\|E_p\|_0)\|E_u\|_0,
\end{aligned}\tag{5.13}
$$

$$
\begin{aligned}
&(\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{S}_h\mathcal{A}_h(-\Delta(\bar w-w)))_{0,h}=a(E_u,\mathcal{S}_h\mathcal{A}_h(-\Delta(\bar w-w)))+b(\mathcal{S}_h\mathcal{A}_h(-\Delta(\bar w-w)),E_p)\\
&\le C(\|E_u\|_1+\|E_p\|_0)\|\mathcal{S}_h\mathcal{A}_h(-\Delta(\bar w-w))\|_1\le C(\|E_u\|_1+\|E_p\|_0)\|\mathcal{S}_h\mathcal{A}_h(-\Delta(\bar w-w))\|_{1,h}\\
&\le C(\|E_u\|_1+\|E_p\|_0)\|-\Delta(\bar w-w)\|_{-1}\le C(\|E_u\|_1+\|E_p\|_0)\|\bar w-w\|_1\\
&\le Ch(\|E_u\|_1+\|E_p\|_0)\|w\|_2\le Ch(\|E_u\|_1+\|E_p\|_0)\|E_u\|_0.
\end{aligned}\tag{5.14}
$$

Thus, from (5.8)-(5.11) and (5.13)-(5.14), we find that

$$
\|E_u\|_0^2\le Ch(\|E_u\|_1+\|E_p\|_0)\|E_u\|_0+|\mathcal{E}|,\tag{5.15}
$$

where

$$
\mathcal{E}:=-b(E_u,\theta)+(\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{S}_h\mathcal{A}_h(-\Delta w))_{0,h}.
$$

In what follows, we estimate $\mathcal{E}$. For that goal, we let $(u^*,p^*)\in(H_0^1(\Omega)\cap H^2(\Omega))^d\times H^1(\Omega)/\mathbb{R}$, and solve the following generalized Stokes problem

$$
-\Delta u^*+\nabla p^*=-\Delta w,\quad \operatorname{div}u^*=\theta\quad\text{in }\Omega,\quad u^*=0\quad\text{on }\Gamma,
$$

where $-\Delta w\in(L^2(\Omega))^d$ and $\theta=\lambda\operatorname{div}w$ with $w\in(H_0^1(\Omega)\cap H^2(\Omega))^d$, and from Assumption A5), we have

$$
\|u^*\|_2+\|p^*\|_1\le C(\|-\Delta w\|_0+\|\theta\|_1)\le C\|E_u\|_0.
$$

Take $(\bar u^*,\bar p^*)\in V_h\times Q_h\subset X_h\times M_h$, where $Q_h=\{q\in H^1(\Omega)/\mathbb{R}: q|_K\in R_1(K),\forall K\in\mathcal{T}_h\}$ is the linear element subspace of M_h, such that, e.g., see Theorem 16.2 on page 128 in [10],

$$
\|u^*-\bar u^*\|_1+\|p^*-\bar p^*\|_0+\left(\sum_{K\in\mathcal{T}_h}h_K^2(\|u^*-\bar u^*\|_{2,K}^2+\|p^*-\bar p^*\|_{1,K}^2)\right)^{\frac12}\le Ch(\|u^*\|_2+\|p^*\|_1).\tag{5.16}
$$

We then have

$$
\begin{aligned}
\mathcal{E}\;&=(\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{S}_h\mathcal{A}_h(-\Delta w))_{0,h}+(\operatorname{div}E_u,\theta)\\
&=(\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{S}_h\mathcal{A}_h(-\Delta u^*+\nabla p^*))_{0,h}+(\operatorname{div}E_u,\operatorname{div}u^*)\\
&=(\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{S}_h\mathcal{A}_h(-\Delta(u^*-\bar u^*)+\nabla(p^*-\bar p^*)))_{0,h}+(\operatorname{div}E_u,\operatorname{div}(u^*-\bar u^*))\\
&+(\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{S}_h\mathcal{A}_h(-\Delta\bar u^*+\nabla\bar p^*))_{0,h}+(\operatorname{div}E_u,\operatorname{div}\bar u^*)\\
&:=\mathcal{I}_1+\mathcal{I}_2,
\end{aligned}\tag{5.17}
$$

where

$$\mathcal{I}_1 := (\mathcal{A}_h(-\Delta E_u + \nabla E_p), \mathcal{S}_h\mathcal{A}_h(-\Delta(u^* - \bar{u}^*) + \nabla(p^* - \bar{p}^*)))_{0,h} + (\operatorname{div} E_u, \operatorname{div}(u^* - \bar{u}^*)),$$

$$\mathcal{I}_2 := (\mathcal{A}_h(-\Delta E_u + \nabla E_p), \mathcal{S}_h\mathcal{A}_h(-\Delta\bar{u}^* + \nabla\bar{p}^*))_{0,h} + (\operatorname{div} E_u, \operatorname{div} \bar{u}^*).$$

Noting that

$$(\operatorname{div} E_u, \operatorname{div}(u^* - \bar{u}^*)) \le C||E_u||_1||u^* - \bar{u}^*||_1,$$

from (4.12), we have

$$\begin{aligned}
\mathcal{I}_1 &= (\mathcal{A}_h(-\Delta E_u + \nabla E_p), \mathcal{S}_h\mathcal{A}_h(-\Delta(u^* - \bar{u}^*) + \nabla(p^* - \bar{p}^*)))_{0,h} + (\operatorname{div} E_u, \operatorname{div}(u^* - \bar{u}^*)) \\
&\le C(||E_u||_1 + ||E_p||_0)(||u^* - \bar{u}^*||_1 + ||p^* - \bar{p}^*||_0) \\
&\le Ch(||E_u||_1 + ||E_p||_0)(||u^*||_2 + ||p^*||_1) \le Ch(||E_u||_1 + ||E_p||_0)||E_u||_0.
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
\mathcal{I}_2 &= (\mathcal{A}_h(-\Delta E_u + \nabla E_p), \mathcal{S}_h\mathcal{A}_h(-\Delta\bar{u}^* + \nabla\bar{p}^*))_{0,h} + (\operatorname{div} E_u, \operatorname{div} \bar{u}^*) \\
&= (\mathcal{A}_h(-\Delta E_u + \nabla E_p), \mathcal{S}_h\mathcal{A}_h(-\Delta\bar{u}^* + \nabla\bar{p}^*))_{0,h} + (\operatorname{div} E_u, \operatorname{div} \bar{u}^*) + \mathscr{C}_h(E_u, E_p; \bar{u}^*, \bar{p}^*) \\
&\quad -\mathscr{C}_h(E_u, E_p; \bar{u}^*, \bar{p}^*),
\end{aligned} \tag{5.19}$$

where from Theorem 3.1, we have

$$\begin{aligned}
&(\mathcal{A}_h(-\Delta E_u + \nabla E_p), \mathcal{S}_h\mathcal{A}_h(-\Delta\bar{u}^* + \nabla\bar{p}^*))_{0,h} + (\operatorname{div} E_u, \operatorname{div} \bar{u}^*) + \mathscr{C}_h(E_u, E_p; \bar{u}^*, \bar{p}^*) \\
&= \mathscr{B}_h(E_u, E_p; \bar{u}^*, \bar{p}^*) = 0,
\end{aligned}$$

and

$$-\mathscr{C}_h(E_u, E_p; \bar{u}^*, \bar{p}^*) = \mathscr{C}_h(E_u, E_p; u^* - \bar{u}^*, p^* - \bar{p}^*) - \mathscr{C}_h(E_u, E_p; u^*, p^*),$$

where if choosing the stabilizations (3.9)-(3.10) then

$$\begin{aligned}
\mathscr{C}_h(E_u, E_p; u^* - \bar{u}^*, p^* - \bar{p}^*) &\le C(||E_u||_1 + ||E_p||_0)(||u^* - \bar{u}^*||_1 + ||p^* - \bar{p}^*||_0) \\
&\le Ch(||E_u||_1 + ||E_p||_0)(||u^*||_2 + ||p^*||_1) \\
&\le Ch(||E_u||_1 + ||E_p||_0)||E_u||_0,
\end{aligned}$$

or if choosing (3.2)-(3.3) then

$$\begin{aligned}
&\mathscr{C}_h(E_u, E_p; u^* - \bar{u}^*, p^* - \bar{p}^*) = \sum_{K\in\mathcal{T}_h} h_K^2(-\Delta E_u + \nabla E_p, -\Delta(u^* - \bar{u}^*) + \nabla(p^* - \bar{p}^*))_{0,K} \\
&+ \sum_{F\in\mathcal{F}_h^I} h_F \int_F [\frac{\partial E_u}{\partial n}][\frac{\partial(u^* - \bar{u}^*)}{\partial n}] \\
&\le C\left(\sum_{K\in\mathcal{T}_h} h_K^2(||E_u||_{2,K}^2 + ||E_p||_{1,K}^2)\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h} h_K^2(||u^* - \bar{u}^*||_{2,K}^2 + ||p^* - \bar{p}^*||_{1,K}^2)\right)^{\frac{1}{2}} \\
&+ \left(\sum_{K\in\mathcal{T}_h} ||E_u||_{1,K}^2 + h_K^2||E_u||_{2,K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h} ||u^* - \bar{u}^*||_{1,K}^2 + h_K^2||u^* - \bar{u}^*||_{2,K}^2\right)^{\frac{1}{2}} \\
&\le Ch\left(||E_u||_1 + \left(\sum_{K\in\mathcal{T}_h} h_K^2(||E_u||_{2,K}^2 + ||E_p||_{1,K}^2)\right)^{\frac{1}{2}}\right)(||u^*||_2 + ||p^*||_1) \\
&\le Ch\left(||E_u||_1 + \left(\sum_{K\in\mathcal{T}_h} h_K^2(||E_u||_{2,K}^2 + ||E_p||_{1,K}^2)\right)^{\frac{1}{2}}\right)||E_u||_0.
\end{aligned} \tag{5.20}$$

Since $u^* \in (H^2(\Omega))^d$, we have $[\frac{\partial u^*}{\partial n}] = 0$ across each $F \in \mathcal{F}_h^I$, and from either (3.2) or (3.14), we can follow the routine in proving (5.20) to obtain

$$-\mathscr{C}_h(E_u, E_p; u^*, p^*) \le Ch\left(||E_u||_1 + \left(\sum_{K\in\mathcal{T}_h} h_K^2(||E_u||_{2,K}^2 + ||E_p||_{1,K}^2)\right)^{\frac{1}{2}}\right)||E_u||_0.$$

Hence, we obtain

$$\mathcal{I}_2 \le Ch \left(||E_u||_1 + ||E_p||_0 + \left(\sum_{K \in \mathcal{T}_h} h_K^2 (||E_u||_{2,K}^2 + ||E_p||_{1,K}^2) \right)^{\frac{1}{2}} \right) ||E_u||_0. \tag{5.21}$$

From (5.18) and (5.21) we have

$$|\mathcal{E}| \le Ch \left(||E_u||_1 + ||E_p||_0 + \left(\sum_{K \in \mathcal{T}_h} h_K^2 (||E_u||_{2,K}^2 + ||E_p||_{1,K}^2) \right)^{\frac{1}{2}} \right) ||E_u||_0. \tag{5.22}$$

Therefore, (5.5) follows from (5.15) and (5.22). $\square$

In what follows, we shall prove (5.5) for the finite element problem (3.18), where the preconditioner $\mathcal{B}_h$ is employed. For that goal, we shall make an additional assumption on the preconditioner $\mathcal{B}_h$ as well as the spectral equivalence in Assumption A2).

Assumption A7) In addition to the spectral equivalence (2.13), we require that the preconditioner $\mathcal{B}_h$ satisfies

$$||\mathcal{S}_h z_h||_{0,h} \le C||\mathcal{B}_h z_h||_{0,h}.$$

Remark 5.4 If $\mathcal{B}_h$ is obtained from the multigrid algorithm on the h level, then we can show that $\mathcal{B}_h$ satisfies Assumption A7) from the L^2 norm convergence in [4].

Lemma 5.3 *Assumption A7) implies*

$$||\mathcal{B}_h^{-1} \mathcal{S}_h z_h||_{0,h} \le C||z_h||_{0,h} \quad \forall z_h \in V_h.$$

Proof. In fact,

$$||\mathcal{B}_h^{-1} \mathcal{S}_h z_h||_{0,h} = \sup_{0 \ne w_h \in V_h} \frac{(\mathcal{B}_h^{-1} \mathcal{S}_h z_h, w_h)_{0,h}}{||w_h||_{0,h}} = \sup_{0 \ne w_h \in V_h} \frac{(\mathcal{S}_h z_h, \mathcal{B}_h^{-1} w_h)_{0,h}}{||w_h||_{0,h}},$$

where, putting $\chi_h := \mathcal{B}_h^{-1} w_h$, we have from Assumption A7)

$$\frac{(\mathcal{S}_h z_h, \mathcal{B}_h^{-1} w_h)_{0,h}}{||w_h||_{0,h}} = \frac{(\mathcal{S}_h z_h, \chi_h)_{0,h}}{||\mathcal{B}_h \chi_h||_{0,h}} = \frac{(z_h, \mathcal{S}_h \chi_h)_{0,h}}{||\mathcal{B}_h \chi_h||_{0,h}} \le \frac{||z_h||_{0,h} ||\mathcal{S}_h \chi_h||_{0,h}}{||\mathcal{B}_h \chi_h||_{0,h}} \le C||z_h||_{0,h}.$$

The proof is then finished. $\square$

Assumption A8) We require that the discrete L^2 inner product $(\cdot,\cdot)_{0,h}$ satisfies the following property

$$|(w_h, z_h)_{0,h} - (w_h, z_h)| \le Ch||w||_1 ||z_h||_0 \quad \forall w_h, z_h \in V_h.$$

Remark 5.5 Assumption A8) is true for the numerical quadrature for the L^2 inner product, see [10] for more details. For example, if we choose the mass-lumping L^2 inner product in Remark 2.1, then Assumption A8) holds as well [37].

Under Assumptions A1)-A8), we prove the L^2 error bound for the finite element problem (3.18).

Theorem 5.2 *Let Assumptions A1)-A8) hold. Let $(u, p) \in (H^2(\Omega))^d \times H^1(\Omega)/\mathbb{R}$ denote the solution pair of the Stokes problem (1.1) and let (u_h, p_h) denote the finite element solution of (3.18) Then*

$$||u - u_h||_0 \le Ch \left(||u - u_h||_1 + ||p - p_h||_0 + \left(\sum_{K \in \mathcal{T}_h} h_K^2 (||u - u_h||_{2,K}^2 + ||p - p_h||_{1,K}^2) \right)^{\frac{1}{2}} \right). \tag{5.23}$$

Proof. The argument is quite similar to the one for proving Theorem 5.1 for problem (3.17). Following the same argument until (5.11) or (5.12), we shall only deal with $(\mathcal{A}_h(-\Delta E_u + \nabla E_p), \mathcal{S}_h \mathcal{A}_h(-\Delta w))_{0,h}$ in (5.11) in a different way as follows:

$$(\mathcal{A}_h(-\Delta E_u + \nabla E_p), \mathcal{S}_h \mathcal{A}_h(-\Delta w))_{0,h} = (\mathcal{B}_h \mathcal{A}_h(-\Delta E_u + \nabla E_p), \mathcal{B}_h^{-1} \mathcal{S}_h \mathcal{A}_h(-\Delta w))_{0,h},$$

where

$$\begin{aligned}(\mathcal{B}_h\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w))_{0,h} &= (\mathcal{B}_h\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w))_{0,h}\\ &-(\mathcal{B}_h\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w))\\ &+(\mathcal{B}_h\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w)),\end{aligned}$$

and from Assumption A8) and Assumption A1), we have

$$\begin{aligned}&(\mathcal{B}_h\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w))_{0,h}-(\mathcal{B}_h\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w))\\ &\leq Ch||\mathcal{B}_h\mathcal{A}_h(-\Delta E_u+\nabla E_p)||_1||\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w)||_0\\ &\leq Ch||\mathcal{B}_h\mathcal{A}_h(-\Delta E_u+\nabla E_p)||_{1,h}||\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w)||_{0,h},\end{aligned}$$

where from Lemma 2.1, (2.12) and (4.11), we have

$$||\mathcal{B}_h\mathcal{A}_h(-\Delta E_u+\nabla E_p)||_{1,h}\leq C||\mathcal{S}_h\mathcal{A}_h(-\Delta E_u+\nabla E_p)||_{1,h}\leq C||\mathcal{A}_h(-\Delta E_u+\nabla E_p)||_{-1,h}\leq C(||E_u||_1+||E_p||_0),$$

and from Lemma 5.3 and the definition of $\mathcal{A}_h$ in (2.2), where from Assumption A1), we know that $||\mathcal{A}_h\chi||_{0,h}\leq C||\chi||_0$ for any given $\chi\in(L^2(\Omega))^d$, we have

$$||\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w)||_{0,h}\leq C||\mathcal{A}_h(-\Delta w)||_{0,h}\leq C||\Delta w||_0\leq C||w||_2\leq C||E_u||_0, \tag{5.24}$$

and from the definition of $\mathcal{A}_h$ in (2.2), we have

$$\begin{aligned}(\mathcal{B}_h\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w)) &= (\mathcal{B}_h\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{A}_h\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w))_{0,h}\\ &= (\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{B}_h\mathcal{A}_h\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w))_{0,h}.\end{aligned}$$

Now, we can obtain (5.15), and we need only to estimate

$$\mathcal{E}:=-b(E_u,\theta)+(\mathcal{A}_h(-\Delta E_u+\nabla E_p),\mathcal{B}_h\mathcal{A}_h\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w))_{0,h}.$$

For that goal, we let $(u^*,p^*)\in(H_0^1(\Omega)\cap H^2(\Omega))^d\times H^1(\Omega)/\mathbb{R}$ solve the following generalized Stokes problem:

$$-\Delta u^*+\nabla p^*=\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w),\quad \mathrm{div}\,u^*=\theta\quad\text{in }\Omega,\quad u^*=0\quad\text{on }\Gamma,$$

where $\theta=\lambda\mathrm{div}\,w$ satisfies $||\theta||_1\leq C||E_u||_0$ (see (5.6)), and we have from Assumption A5), Assumption A1) and (5.24)

$$||u^*||_2+||p^*||_1\leq C(||\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w)||_0+||\theta||_1)\leq C(||\mathcal{B}_h^{-1}\mathcal{S}_h\mathcal{A}_h(-\Delta w)||_{0,h}+||\theta||_1)\leq C||E_u||_0.$$

Hereafter, just following the same argument from (5.16) to (5.22), with $\mathcal{B}_h$ replacing $\mathcal{S}_h$ and with $\widetilde{\mathscr{B}}_h$ replacing $\mathscr{B}_h$, we can obtain (5.23). □

Corollary 5.1 *Under the same assumptions as in Theorem 5.1 or in Theorem 5.2, for* $(u,p)\in(H^{l+1}(\Omega))^d\times H^m(\Omega)$ *for* $l,m\geq 1$*, then*

$$||u-u_h||_0\leq Ch^{\min(l,m)+1}(||u||_{l+1}+||p||_m). \tag{5.25}$$

Proof. Let $(\pi_h u,\rho_h p)\in X_h\times M_h$ be the finite element interpolant to (u,p) as in (4.19). Observe that from the local inverse estimates [23, 10, 6], we have

$$h_K||\pi_h u-u_h||_{2,K}\leq C||\pi_h u-u_h||_{1,K}\leq C(||\pi_h u-u||_{1,K}+||u-u_h||_{1,K}),$$
$$h_K||\rho_h p-p_h||_{1,K}\leq C||\rho_h p-p_h||_{0,K}\leq C(||\rho_h p-p||_{0,K}+||p-p_h||_{0,K}).$$

Hence, using the triangle inequality, from (4.18) and (4.19), we have

$$\begin{aligned}C\left(\sum_{K\in\mathcal{T}_h}h_K^2(||u-u_h||_{2,K}^2+||p-p_h||_{1,K}^2)\right)^{\frac{1}{2}} &\leq ||u-u_h||_1+||p-p_h||_0+||u-\pi_h u||_1+||p-\rho_h p||_0\\ &+\left(\sum_{K\in\mathcal{T}_h}h_K^2(||u-\pi_h u||_{2,K}^2+||p-\rho_h p||_{1,K}^2)\right)^{\frac{1}{2}}\\ &\leq Ch^{\min(l,m)}(||u||_{l+1}+||p||_m).\end{aligned}$$

In conclusion, combining Theorem 5.1 or Theorem 5.2 and Corollary 4.4, we obtain the desired (5.25). □

6 Numerical experiments

In this section we report some numerical results to support the proposed method and the theory developed in the earlier sections. Given $\Omega = (0,1)^2 \subset \mathbb{R}^2$, and $f \in (L^2(\Omega))^2$ and $g \in L^2(\Omega)$, with $\int_\Omega g = 0$. Consider the following Stokes problem:

$$-\Delta u + \nabla p = f, \quad \operatorname{div} u = g \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma.$$

We choose f and g so that the exact solution $(u, p) = (u_1, u_2, p)$ is

$$u_1(x,y) = \sin(\pi x)\sin(\pi y), \quad u_2(x,y) = \sin(\pi x)\sin(\pi y), \quad p(x,y) = \cos(\pi x)\exp(\pi y).$$

We partition Ω into uniform square elements, and we perform the finite element method stated in (3.17), with an additional term $(g, \operatorname{div} v_h)$ in the right-hand side (3.17) to account for the consistency in the above model with non-zero divergence. For the finite element spaces $X_h \subset (H_0^1(\Omega))^2$ and $M_h \subset H^1(\Omega)/\mathbb{R}$ of velocity and pressure, we consider three cases:

Case 1. Equal-order bilinear elements (denoted by Q_1 elements)

Case 2. Equal-order biquadratic elements (denoted by Q_2 elements)

Case 3. Unequal-order biquadratic-bilinear elements (denoted by $Q_2 - Q_1$ elements)

In three cases, we employ the bilinear element for $V_h \subset (H_0^1(\Omega))^2$ to define the involved bilinear finite element solution solver $\mathcal{S}_h\mathcal{A}_h$ from (2.6), and we use (3.20) to compute the first term of (3.17). The mesh-dependent bilinear and linear forms $\mathscr{C}_h$ and $\mathscr{G}_h$ are chosen as (3.2) and (3.3). The theoretical results of the error bounds are as follows:

$$||u - u_h||_1 + ||p - p_h||_0 \le Ch^l(||u||_{l+1} + ||p||_l), \quad ||u - u_h||_0 \le Ch^{l+1}(||u||_{l+1} + ||p||_l),$$

where $l = 1$ and $l = 2$ are theoretical convergence order, respectively corresponding to equal-order Q_1 elements and equal-order Q_2 elements or unequal-order $Q_2 - Q_1$ elements of velocity and pressure. Due to the limitations of our Laptop computer's power, we provide numerical results on a sequence of coarser meshes. The computed results in L^2 norm and H^1 norm are listed in Tables 1-6. We find that the computed and the predicted are consistent. Equal-order and unequal-order elements work for velocity and pressure, and, in particular, even if the H^{-1} norm is computed only in the linear element space V_h, when using higher-order elements Q_2 for velocity and pressure, we obtain the predicted convergence.

Table 1: Equal-order Q_1 elements
Errors and convergence order in L^2 norms for velocity and pressure

$1/h$	$\|u_1 - u_{1h}\|_0$	order	$\|u_2 - u_{2h}\|_0$	order	$\|p - p_h\|_0$	order
4	3.477980E-01		3.325297E-01		3.799489E+00	
8	2.423516E-01	0.52	2.385534E-01	0.48	2.995609E+00	0.34
12	1.719798E-01	0.85	1.706970E-01	0.83	2.208155E+00	0.75
16	1.241632E-01	1.13	1.236360E-01	1.12	1.626164E+00	1.06
20	9.212606E-02	1.34	9.187738E-02	1.33	1.223679E+00	1.27
24	7.035846E-02	1.48	7.022841E-02	1.47	9.452716E-01	1.42
28	5.517351E-02	1.58	5.509985E-02	1.57	7.485025E-01	1.51
32	4.427169E-02	1.65	4.422725E-02	1.65	6.056930E-01	1.59
36	3.622929E-02	1.70	3.620107E-02	1.70	4.993316E-01	1.64
40	3.014983E-02	1.74	3.013114E-02	1.74	4.182407E-01	1.68
44	2.545444E-02	1.78	2.544162E-02	1.78	3.551265E-01	1.72
48	2.175914E-02	1.80	2.175007E-02	1.80	3.051079E-01	1.75
52	1.880254E-02	1.83	1.879596E-02	1.82	2.648349E-01	1.77
56	1.640237E-02	1.84	1.639750E-02	1.84	2.319540E-01	1.79
60	1.442874E-02	1.86	1.442506E-02	1.86	2.047758E-01	1.81
64	1.278720E-02	1.87	1.278437E-02	1.87	1.820639E-01	1.82

Table 2: Equal-order Q_1 elements
Errors and convergence order in H^1 norms for velocity and pressure

1/h	$\|u_1 - u_{1h}\|_1$	order	$\|u_2 - u_{2h}\|_1$	order	$\|p - p_h\|_1$	order
4	1.591977E+00		1.524381E+00		1.403737E+01	
8	1.113779E+00	0.52	1.096775E+00	0.48	1.042898E+01	0.43
12	7.950230E-01	0.83	7.892634E-01	0.81	7.602035E+00	0.78
16	5.782950E-01	1.11	5.759270E-01	1.10	5.594013E+00	1.07
20	4.328143E-01	1.30	4.316983E-01	1.29	4.230709E+00	1.25
24	3.337075E-01	1.43	3.331248E-01	1.42	3.298303E+00	1.37
28	2.643488E-01	1.51	2.640195E-01	1.51	2.643626E+00	1.44
32	2.143727E-01	1.57	2.141746E-01	1.57	2.169811E+00	1.48
36	1.773616E-01	1.61	1.772361E-01	1.61	1.816890E+00	1.51
40	1.492722E-01	1.64	1.491894E-01	1.64	1.547216E+00	1.53
44	1.274908E-01	1.66	1.274342E-01	1.65	1.336536E+00	1.54
48	1.102809E-01	1.67	1.102410E-01	1.67	1.168765E+00	1.54
52	9.645765E-02	1.67	9.642885E-02	1.67	1.032933E+00	1.54
56	8.519314E-02	1.68	8.517189E-02	1.68	9.213590E-01	1.54
60	7.589567E-02	1.68	7.587969E-02	1.67	8.285424E-01	1.54
64	6.813399E-02	1.67	6.812178E-02	1.67	7.504583E-01	1.53

Table 3: Equal-order Q_2 elements
Errors and convergence order in L^2 norms for velocity and pressure

1/h	$\|u_1 - u_{1h}\|_0$	order	$\|u_2 - u_{2h}\|_0$	order	$\|p - p_h\|_0$	order
4	1.816323E-02	–	1.615629E-02	–	2.298031E-01	–
8	1.559541E-03	3.54	1.536019E-03	3.39	4.340678E-02	2.40
12	4.050852E-04	3.32	4.055466E-04	3.28	1.511455E-02	2.60
16	1.501849E-04	3.45	1.508731E-04	3.44	7.028019E-03	2.66
20	6.863550E-05	3.51	6.900508E-05	3.51	3.935371E-03	2.60
24	3.603071E-05	3.53	3.622434E-05	3.53	2.484405E-03	2.52
28	2.086180E-05	3.54	2.096791E-05	3.55	1.701130E-03	2.46
32	1.299041E-05	3.55	1.305165E-05	3.55	1.234097E-03	2.40

Table 4: Equal-order Q_2 elements
Errors and convergence order in H^1 norms for velocity and pressure

$1/h$	$\|u_1 - u_{1h}\|_1$	order	$\|u_2 - u_{2h}\|_1$	order	$\|p - p_h\|_1$	order
4	1.557982E-01	–	1.523290E-01	–	2.798491E+00	–
8	2.865841E-02	2.44	2.862063E-02	2.41	1.109327E+00	1.33
12	1.059827E-02	2.45	1.060124E-02	2.45	6.606824E-01	1.28
16	5.287807E-03	2.42	5.290716E-03	2.42	4.676084E-01	1.20
20	3.123207E-03	2.36	3.124905E-03	2.36	3.615504E-01	1.15
24	2.048378E-03	2.31	2.049358E-03	2.31	2.946316E-01	1.12
28	1.441736E-03	2.28	1.442326E-03	2.28	2.485808E-01	1.10
32	1.067474E-03	2.25	1.067845E-03	2.25	2.149591E-01	1.09

Table 5: Unequal-order $Q_2 - Q_1$ elements
Errors and convergence order in L^2 norms for velocity and pressure

1/h	$\|u_1 - u_{1h}\|_0$	order	$\|u_2 - u_{2h}\|_0$	order	$\|p - p_h\|_0$	order
4	8.080257E-02	–	1.381396E-02	–	7.752562E-01	–
8	7.244889E-03	3.48	1.780281E-03	2.96	9.695358E-02	3.00
12	1.476571E-03	3.92	5.229798E-04	3.02	3.146412E-02	2.78
16	4.630260E-04	4.03	2.156166E-04	3.08	1.582979E-02	2.39
20	1.868489E-04	4.07	1.057930E-04	3.19	9.630225E-03	2.23
24	8.897097E-05	4.07	5.810773E-05	3.29	6.494196E-03	2.16
28	4.760877E-05	4.06	3.461162E-05	3.36	4.679310E-03	2.13
32	2.777557E-05	4.04	2.192604E-05	3.42	3.533364E-03	2.10

Table 6: Unequal-order $Q_2 - Q_1$ elements
Errors and convergence order in H^1 norms for velocity and pressure

$1/h$	$\|u_1 - u_{1h}\|_1$	order	$\|u_2 - u_{2h}\|_1$	order	$\|p - p_h\|_1$	order
4	4.227891E-01	–	1.584266E-01	–	7.060317E+00	–
8	4.883858E-02	3.11	3.030656E-02	2.39	3.365978E+00	1.07
12	1.389177E-02	3.10	1.158630E-02	2.37	2.224405E+00	1.02
16	6.182536E-03	2.81	5.808297E-03	2.40	1.661126E+00	1.01
20	3.461147E-03	2.60	3.399543E-03	2.40	1.325575E+00	1.01
24	2.206119E-03	2.47	2.202112E-03	2.38	1.102865E+00	1.01
28	1.526170E-03	2.39	1.531721E-03	2.36	9.442559E-01	1.01
32	1.117043E-03	2.34	1.122661E-03	2.33	8.255476E-01	1.01

7 Conclusion

In this paper we have proposed a new least-squares finite element method for Stokes equations, where the discrete H^{-1}-norm is used for measuring the residuals from the momentum equation of Stokes equations. The main novelty and advantage is the computation of the H^{-1} norm is always only in the linear element space, whatever the finite element spaces of velocity and pressure are. Since the discrete H^{-1}-norm involves only the linear element solution of Poisson Dirichlet problem, it can be realized cheaply and 'offline' or in advance, before the solution procedure of Stokes equations. We have presented a theoretical analysis to give a rigorous justification for this novelty and advantage, and optimal error bounds are shown to hold true, even if the discrete H^{-1}-norm always lives in the linear element space. In addition, an ad hoc argument is developed for the derivation of the optimal L^2-norm error bounds for velocity. We have performed some numerical experiments to illustrate the performance of the proposed method and to confirm the theoretical results obtained.

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