TWO-SIDED SUB-GAUSSIAN ESTIMATES OF HEAT KERNELS ON INTERVALS FOR SELF-SIMILAR MEASURES WITH OVERLAPS

QINGSONG GU, JIAXIN HU, AND SZE-MAN NGAI

ABSTRACT. We obtain two-sided sub-Gaussian estimates of heat kernels for strongly local Dirichlet forms on intervals, equipped with self-similar measures generated by iterated function systems (IFS's) that do not satisfy the open set condition (OSC) and have overlaps. We first give a framework for heat kernel estimates on intervals, and then consider examples of self-similar measures to illustrate this phenomenon. These examples include the infinite Bernoulli convolution associated with the golden ratio, and a family of convolutions of Cantor-type measures. We make use of Strichartz second-order identities defined by auxiliary IFS's to compute measures of cells on different levels. These auxiliary IFS's do satisfy the OSC and are used to define new metrics. The walk dimensions obtained under these new metrics are strictly greater than 2 and are closely related to the spectral dimension of fractal Laplacians.

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1. INTRODUCTION

Heat kernel estimates for *local* Dirichlet forms on fractals are typically *sub-Gaussian*. This has been shown by plenty of examples: by Barlow and Perkins for the Sierpiński gasket [4], by Kumagai [24] for nested fractals, by Fitzsimmons, Hambly and Kumagai [7]

Date: November 9, 2016.

2010 *Mathematics Subject Classification.* Primary: 28A80, 35K08; Secondary: 35J05.

Key words and phrases. Heat kernel, sub-Gaussian, self-similar measure, second-order identities.

This research was supported in part by the National Natural Science Foundation of China, grants 11371217 and 11271122, and by SRFDP No. 20130002110003, and by a Hong Kong RGC grant. SN is also supported in part by Construct Program of the Key Discipline in Hunan Province, the Hunan Province Hundred Talents Program, and the Center of Mathematical Sciences and Applications of Harvard University.

for affine nested fractals, by Hambly and Kumagai [15] (see also [26]) for post-critically finite self-similar sets, by Kigami [19] and Kumagai [25] for resistance forms, and by Barlow and Bass [1, 2] for the Sierpiński carpets, and by Kigami [21, 22, 23] for doubling spaces, as well as many other authors. Equivalence conditions for two-sided estimates of heat kernels for local Dirichlet forms on metric measure spaces are given by Grigor'yan, Lau and the second author [10, 11], and Grigor'yan and Telcs [14], and references therein (see also [16] for a certain class of resistance forms).

Typical measures on fractals described above are s-dimensional Hausdorff measures for some number $s > 0$, and they are equivalent to self-similar measures generated by iterated function systems (IFS's) satisfying the open set condition (OSC). This paper studies self-similar measures generated by IFS's that do not satisfy the OSC. These measures are no longer regular but still possess doubling property after introducing suitable metrics. Although the self-similar sets themselves are intervals and hence Dirichlet forms can be defined easily, the associated self-similar measures exhibit complicated fractal behavior, and therefore heat kernel estimates become much more awkward.

Let $C^1(K)$ be the *space* of all functions, *continuous* on K and *first-order smooth* in open (a, b) with $a < b$. Consider the following form $(\mathcal{E}, C^1(K))$:

$$\mathcal{E}(u, v) = \int_a^b u'(x)v'(x)dx. \tag{1.1}$$

Note that for any $u \in C^1(K)$ and any $x, y \in K$,

$$|u(x) - u(y)|^2 \leq \mathcal{E}(u)|x - y|, \tag{1.2}$$

where $\mathcal{E}(u) := \mathcal{E}(u, u)$, since for any $x < y$ and any smooth u,

$$|u(x) - u(y)|^2 = \left\{\int_x^y u'(z)dz\right\}^2 \leq (y - x)\int_x^y [u'(z)]^2\, dz \leq (y - x)\mathcal{E}(u).$$

Let μ be a Radon measure with *full* $\mathrm{supp}(\mu) = K := [a, b]$ (that is, $\mu(I) > 0$ for any nonempty open interval in K). Clearly, the form $\left(\mathcal{E}, C^1(K)\right)$ is densely defined, non-negative definite, symmetric, bilinear and Markovian in $L^2(\mu) := L^2(K, \mu)$. In order to introduce a Dirichlet form for $\mathcal{E}$ in $L^2(\mu)$, we need to specify a *domain* $\mathcal{F}$ of $\mathcal{E}$ such that $\mathcal{F} \subset L^2(\mu)$. Let $H^1(K)$ be the usual *Sobolev space* on K against the *Lebesgue measure*. Using (1.2) and the Arzelà-Ascoli theorem, it is not hard to see that $H^1(K)$ is *complete* under *norm* $\sqrt{\mathcal{E}(u) + \|u\|^2_{L^2(\mu)}}$. Let

$$\mathcal{F} := H^1(K). \tag{1.3}$$

By (1.2), we have $\mathcal{F} \subset C(K)$ where $C(K)$ is the space of all *continuous* functions on K. The form $(\mathcal{E}, \mathcal{F})$ given by (1.1) and (1.3) is thus a Dirichlet form in $L^2(\mu)$ (cf. [8]). Moreover, $(\mathcal{E}, \mathcal{F})$ is *regular*, *conservative* , *strongly local* in $L^2(\mu)$ (since $1 \in \mathcal{F}$ and $\mathcal{E}(1) = 0$, the form $(\mathcal{E}, \mathcal{F})$ is conservative).

This paper studies two-sided estimates of the heat kernel of the form $(\mathcal{E}, \mathcal{F})$ for μ being certain self-similar measures with overlaps. Let $\{S_i\}_{i=0}^N$ be contractive similitudes on $\mathbb{R}$

such that

$$K = \bigcup_{i=0}^{N} S_i(K), \tag{1.4}$$

and let μ be a self-similar measure with weight $\{\rho_i\}_{i=0}^{N}$:

$$\mu = \bigcup_{i=0}^{N} \rho_i \left(\mu \circ S_i^{-1}\right), \tag{1.5}$$

where each $0 < \rho_i < 1$ and $\sum_{i=0}^{N} \rho_i = 1$, and

$$\mu(A) = 0 \tag{1.6}$$

if A is a singleton[1].

Let $\{T_j\}_{j=0}^{m}$ be an *auxiliary IFS* of contractive similitudes:

$$\left|T_j(x) - T_j(y)\right| = r_j|x - y| \quad \text{for any } x, y \in K, \tag{1.7}$$

where each $0 < r_j < 1$, such that $\{T_j(K) : j = 0, 1, \dots, m\}$ forms a partition of K:

$$K = \bigcup_{i=0}^{m} T_j(K), \tag{1.8}$$

and the intervals $T_i(K)$ and $T_j(K)$ can only intersect at their end-points if $i \neq j$. (Note that the *IFS* $\{T_j\}_{j=0}^{m}$ does not have overlaps but the IFS $\{S_i\}_{i=0}^{N}$ may have.)

For a word $\omega = \omega \cdots \omega_n$, we let $|\omega| = n$ denote the length of n and call K_ω an *n-cell*. Write $K_\omega \sim K_\tau$ if $K_\omega \cap K_\tau \neq \emptyset$. We say that two words ω, τ having the same length are neighbors if $K_\omega \sim K_\tau$. We use the notation

$$K_\omega := T_{\omega_1} \circ \cdots \circ T_{\omega_n}(K) \quad \text{and} \quad r_\omega := r_{\omega_1} \cdots r_{\omega_n}.$$

Note that $\{T_0, T_1, \dots, T_m\}$, not $\{S_0, S_1, \dots, S_N\}$, is used to define K_ω. For each $n \in \mathbb{N}$, let $\mathcal{J} = \{0, 1, \dots, m\}$ and let

$$\mathcal{J}^n := \{0, 1, \dots, m\}^n, \qquad \mathcal{J}^* := \bigcup_{k=0}^{\infty} \mathcal{J}^n$$

be respectively the sets of words with length n, with finite length. Here $\mathcal{J}^0$ is defined to be the empty word, and we use the convention that

$$\omega\emptyset = \emptyset\omega = \omega \;\text{ for any word } \omega.$$

For two finite words ω and τ, we say $\omega < \tau$ if there exists a non-empty word γ such that $\tau = \omega\gamma$. We write $\omega \leq \tau$ (and call ω a father of τ) if $\omega < \tau$ or $\omega = \tau$.

Let d_* be a metric on K, and let

$$V(x, r) := \mu(B_{d_*}(x, r)), \tag{1.9}$$

[1] In fact, if K is a self-similar set generated by a family of contractive similitudes in $\mathbb{R}^n$ and μ is an associated self-similar measure with positive weights supported on K, then $\mu\left(\partial B(x, r)\right) = 0$ for any Euclidean ball $B(x, r)$ in $\mathbb{R}^n$ provided that K is not a singleton. This fact can be shown by using Lemma 2.6 in the paper titled "Self-similar and self-affine sets: measure of the intersection of two copies". Ergodic Theory Dynam. Systems **30** (2010), 399–440 by M. Elekes, T. Keleti, and M. András − We learned this from De-Jun Feng.

where

$$B_{d_*}(x,r) = \{y \in K : d_*(y,x) < r\},$$

a ball with center x and radius r under the metric d_*.

Fix some number $\beta > 1$. We introduce the following conditions that may or may not be satisfied:

(1) *self-similarity* of $(\mathcal{E},\mathcal{F})$:

$$\mathcal{E}(u) = \sum_{i=0}^{m} \frac{1}{r_i}\mathcal{E}(u \circ T_i), \tag{1.10}$$

where r_j, $0 \le j \le m$, are given by (1.7);

(2) *comparability of neighboring cells*: if τ and σ are neighbors, then

$$\mu(K_\tau) \asymp \mu(K_\sigma); \tag{1.11}$$

(3) *generalized mid-point property*: for any points $x, y, z \in K$ with $x < y < z$,

$$d_*(x,z) = d_*(x,y) + d_*(y,z); \tag{1.12}$$

(4) *product* of Euclidean length and μ-measure of interval $[x, y]$:

$$|x-y|\mu\left([x,y]\right) \asymp d_*(x,y)^\beta; \tag{1.13}$$

(5) *volume doubling property (VD)*: there exists a constant $C > 0$ such that

$$V(x,2r) \le CV(x,r) \text{ for all } r > 0 \text{ and all } x \in K; \tag{1.14}$$

(6) *ratio* of volumes of two concentric balls $B_{d_*}(x,r)$ and $B_{d_*}(x,\eta r)$:

$$\frac{V(x,r)}{V(x,\eta r)} = o\left(\eta^{-\beta}\right) \text{ uniformly in } x, r \text{ as } \eta \to 0^+,$$

that is,

$$\sup_{x\in K, 0<r<1} \frac{\eta^\beta V(x,r)}{V(x,\eta r)} \to 0^+ \text{ as } \eta \to 0^+. \tag{1.15}$$

Here and below, the sign $f \asymp g$ means that $C^{-1}g \le f \le Cg$ for some universal constant $C > 0$ independent of the arguments f and g. Note that we can take $\beta = 2$ in above conditions if μ is the Lebesgue measure and d_* is the Euclidean metric. In this paper we are interested in the situation where β is strictly greater than 2 by introducing suitable μ and d_* on K.

Note that condition (1.12) implies the *mid-point property*, which in turn implies the *chain condition*, see for example [12, Definition 3.4]. (A distance d on a nonempty set X is said to have the *mid-point property* if for any $x, y \in X$, there exists some $z \in X$ such that, $d(x,z) = d(z,y) = d(x,y)/2$.)

Theorem 1.1. *Let μ be a Radon measure with full support $K = [a,b]$, and let $(\mathcal{E},\mathcal{F})$ be defined by (1.1), (1.3). Let $\{T_j\}_{j=0}^m$ be an auxiliary IFS defined by (1.7) such that (1.8)* *holds. Assume that conditions (1.10)–(1.15) are all satisfied for some metric d_* on K. Then the heat kernel $p_t(x, y)$ of $(\mathcal{E}, \mathcal{F})$ exists, and satisfies the upper estimate*

$$p_t(x, y) \leq \frac{C_1}{V(x, t^{1/\beta})} \exp\left(-c_1 \left(\frac{d_*(x, y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right) \tag{UE}$$

and the lower estimate

$$p_t(x, y) \geq \frac{C_2}{V(x, t^{1/\beta})} \exp\left(-c_2 \left(\frac{d_*(x, y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right) \tag{LE}$$

for all $t \in (0, 1)$ and all $x, y \in K$.

We will prove Theorem 1.1 in Section 2 .

We consider two specific Radon measures μ and introduce a new metric d_* accordingly. The introduction of this kind of new metric d_* (see (3.20) and (4.14) below) is partially motivated by that in [20, Section 5], but it is more involved because of the overlaps of the IFS. In this paper a lot of efforts go to verify conditions (1.10)–(1.15) for such (μ, d_*).

The first Radon measure we study is the infinite Bernoulli convolution associated with the golden ratio. Let

$$S_0(x) = \rho x, \qquad S_1(x) = \rho x + (1 - \rho), \qquad \rho = \frac{\sqrt{5} - 1}{2}, \tag{1.16}$$

and let μ be the self-similar measure with $\mathrm{supp}(\mu) = K$ satisfying:

$$\mu = \frac{1}{2}\mu \circ S_0^{-1} + \frac{1}{2}\mu \circ S_1^{-1}. \tag{1.17}$$

The metric d_* and the constant $\alpha \in \left(0, \frac{1}{2}\right)$ in the following theorem will be given in Section 3.

Theorem 1.2. *Let μ be defined by (1.17) and d_* by (3.20) below. Let $\alpha \in \left(0, \frac{1}{2}\right)$ be defined by (3.21) and $\beta := 1/\alpha > 2$. Then all the conditions (1.10)–(1.15) are satisfied. Consequently, the heat kernel $p_t(x, y)$ of $(\mathcal{E}, \mathcal{F})$ exists and satisfies the two-sided estimates (UE) and (LE) with such parameter β.*

Theorem 1.2 will be proved in Section 3.

The second measure we study is from a family of convolutions of Cantor-type measures. Let

$$S_0(x) = \frac{1}{m}x, \qquad S_1(x) = \frac{1}{m}x + \frac{m-1}{m}, \tag{1.18}$$

where $m \geq 3$ is an odd integer. The attractor of this IFS is a symmetric Cantor-type set. Let ν_m be the self-similar measure defined by the IFS (1.18) with probability weights $p_0 = p_1 = 1/2$. The m -fold convolution μ_m of ν_m^{*m} is the self-similar measure defined by the following IFS with overlaps (see [28]):

$$S_i(x) = \frac{1}{m}x + \frac{m-1}{m}i, \quad i = 0, 1, \ldots, m, \tag{1.19}$$

together with probability weights

$$w_i := \frac{1}{2^m}\binom{m}{i}, \quad i = 0, 1, \dots, m. \tag{1.20}$$

That is,

$$\mu_m = \sum_{i=0}^{m} \frac{1}{2^m}\binom{m}{i}\mu_m \circ S_i^{-1}, \tag{1.21}$$

with $\mathrm{supp}(\mu_m) = [0, m]$.

The metric d_* and the constant $\alpha \in \left(0, \frac{1}{2}\right)$ in the following theorem will be given in Section 4.

Theorem 1.3. *For any odd integer $m \geq 3$, let μ_m be the m -fold convolution of the Cantor measure defined as in (1.21). Let d_* be a metric defined by (4.14) below, and let $\alpha \in \left(0, \frac{1}{2}\right)$ be a constant defined by (4.15) and $\beta := 1/\alpha > 2$. Then the same conclusion of Theorem 1.2 holds with this value of β.*

We will prove Theorem 1.3 in Section 4 .

2. Two-sided heat kernel estimates

Let $K = [a, b]$ and let $(\mathcal{E}, \mathcal{F})$ be the regular, strongly local, and conservative Dirichlet form in $L^2(\mu)$ defined by (1.1) and (1.3). In this section, we will prove Theorem 1.1. Firstly, we derive the stated off-diagonal upper bound (UE) of the heat kernel by showing both of the following conditions (DUE) (see (2.27) below) and (E) (see (2.43) and (2.44) below) are satisfied, and then use the following equivalence

$$(UE) \Leftrightarrow (DUE) + (E) \tag{2.1}$$

that was obtained in [9, Theorem 2.2] or [14, Theorem 4.2] (although the metric space considered in both [9] and [14] is unbounded, the conclusion is also true for bounded metric space but with a finite range of time t in the heat kernel estimate). Secondly, we derive the lower bound (LE) of the heat kernel. To do this, first we have the on-diagonal lower bound from the off-diagonal upper bound, then we can use the Hölder continuity of the heat kernel to derive the near diagonal lower estimate (NLE) (see (2.48) below) and finally a chain argument yields the off-diagonal lower bound (see [12, Corollary 3.5]).

We use a Nash-type inequality to obtain the existence of the heat kernel, using ideas from [20, 21]. For a word ω, let

$$g(\omega) := \sqrt{r_\omega \mu(K_\omega)}. \tag{2.2}$$

For $s \in (0, 1]$, let

$$\Lambda_s := \{\omega = \omega_1 \cdots \omega_n : g(\omega) \leq s < g(\omega_1 \cdots \omega_{n-1}),\ \text{each } \omega_i \in \{0, 1, \dots, m\}\}.$$

For any $u \in \mathcal{F}$, let

$$\Lambda_s(u) := \{\omega \in \Lambda_s : K_\omega \cap \mathrm{supp}(u) \neq \emptyset\}.$$

The following lemma follows by modifying the proofs in [20, Theorem 5.3] and [21, Lemma 3.1.6]. We include a proof for completeness.

Lemma 2.1. *Assume that conditions (1.8) and (1.10) hold. Then there exist two positive universal constant C_1, C_2 such that for all $s \in (0,1]$ and all $u \in \mathcal{F}$,*

$$\mathcal{E}(u) + \frac{C_1}{s^2 \min\limits_{\omega\in\Lambda_s(u)} \mu(K_\omega)} \|u\|_1^2 \geq \frac{C_2}{s^2}\|u\|_2^2. \tag{2.3}$$

(If $\Lambda_s(u) = \emptyset$, then $\min\limits_{\omega\in\Lambda_s(u)} \mu(K_\omega) = 1$ since $K_\emptyset = K$ and $\mu(K) = 1$.)

Proof. Let $\mu^\omega(\cdot) := \mu(T_\omega(\cdot))/\mu(K_\omega)$. Clearly,

$$\mu^\omega(K) = 1 \quad \text{and} \quad \int_k f \circ T_\omega d\mu^\omega = \frac{1}{\mu(K_\omega)} \int_{K_\omega} f d\mu \tag{2.4}$$

for any $f \in L^1(\mu)$. Since for any $x \in K$,

$$|u(x) - \bar{u}|^2 = \left| \int_K (u(x) - u(y)) d\mu^\omega(y) \right|^2 \leq \int_K (u(x) - u(y))^2 d\mu^\omega(y),$$

where $\bar{u} := \int_K u\, d\mu^\omega$, we obtain from (1.2) that

$$\begin{aligned}\int_K |u - \bar{u}|^2 \, d\mu^\omega &\leq \int_{K\times K} (u(x) - u(y))^2 d\mu^\omega(y) d\mu^\omega(x) \\ &\leq \mathcal{E}(u) \int_{K\times K} |x - y| d\mu^\omega(y) d\mu^\omega(x) \leq C\mathcal{E}(u),\end{aligned} \tag{2.5}$$

where $C = b - a$, the Euclidean length of the interval K. Set $u_\omega := u \circ T_\omega$. Then using (1.10), (2.5) and (2.4), we have, for any $s \in (0,1]$,

$$\begin{aligned}\mathcal{E}(u) &= \sum_{\omega\in\Lambda_s(u)} \frac{1}{r_\omega} \mathcal{E}(u_\omega) \geq C \sum_{\omega\in\Lambda_s(u)} \frac{1}{r_\omega} \int_K |u \circ T_\omega - \overline{u \circ T_\omega}|^2 \, d\mu^\omega \\ &= C \sum_{\omega\in\Lambda_s(u)} \frac{1}{r_\omega \mu(K_\omega)} \left\{ \int_{K_\omega} u^2 \, d\mu - \frac{1}{\mu(K_\omega)} \left(\int_{K_\omega} u d\mu \right)^2 \right\}.\end{aligned} \tag{2.6}$$

For any $\omega \in \Lambda_s(u)$, we have $r_\omega \mu(K_\omega) \leq s^2 < r_{\omega_1} \cdots r_{\omega_{n-1}} \mu(K_{\omega_1\cdots\omega_{n-1}})$. Hence by (1.8)

$$C \sum_{\omega\in\Lambda_s(u)} \frac{1}{r_\omega \mu(K_\omega)} \int_{K_\omega} u^2 \, d\mu \geq C \sum_{\omega\in\Lambda_s(u)} \frac{1}{s^2} \int_{K_\omega} u^2 \, d\mu = \frac{C}{s^2} \|u\|_2^2. \tag{2.7}$$

On the other hand, by (2.2),

$$r_\omega \mu(K_\omega) \geq r_\omega \mu(K_{\omega_1\cdots\omega_{n-1}}) = r_{\omega_n} g^2(\omega_1 \cdots \omega_{n-1}) \geq \min_j \left\{ r_j \right\} s^2,$$

and hence, using (1.8) again,

$$\begin{aligned}C \sum_{\omega\in\Lambda_s(u)} \frac{1}{r_\omega \mu(K_\omega)} \frac{1}{\mu(K_\omega)} \left(\int_{K_\omega} u d\mu \right)^2 &\leq C' \sum_{\omega\in\Lambda_s(u)} \frac{1}{s^2} \frac{1}{\min\limits_{\omega\in\Lambda_s(u)} \mu(K_\omega)} \left(\int_{K_\omega} |u| d\mu \right)^2 \\ &\leq C' \frac{1}{s^2} \frac{1}{\min\limits_{\omega\in\Lambda_s(u)} \mu(K_\omega)} \|u\|_1^2 .\end{aligned} \tag{2.8}$$

The lemma now follows by combining (2.6), (2.7), and (2.8). □

Proposition 2.2. *Assume that condition (1.11) holds. Then there exists a constant* $\delta_1 \in (0,1)$ *such that for any word* ω,

$$\delta_1^{|\omega|} \le \mu(K_\omega) \quad \text{and} \quad r_\omega \le \delta_2^{|\omega|}, \tag{2.9}$$

where $\delta_2 = \max_j \left\{r_j\right\}$. *Consequently,*

$$\mu(K_\omega) \ge g(\omega)^\theta \tag{2.10}$$

for some constant $0 < \theta < 2$.

Proof. Let $\omega = \omega_1 \cdots \omega_n$. By (1.11), there exists a constant $\delta_1 \in (0,1)$ such that

$$\mu(K_\omega) \ge \delta_1 \mu(K_{\omega_1 \cdots \omega_{n-1}}) \ge \cdots \ge \delta_1^{|\omega|}\mu(K) = \delta_1^{|\omega|}. \tag{2.11}$$

Clearly,

$$r_\omega \le \left\{\max_j \{r_j : 1 \le j \le m\}\right\}^{|\omega|} = \delta_2^{|\omega|}, \tag{2.12}$$

and hence we get (2.9).

To prove (2.10), notice that by (2.2),

$$\mu(K_\omega) \ge g(\omega)^\theta \qquad \Leftrightarrow \qquad \mu(K_\omega)^{\frac{2-\theta}{\theta}} \ge r_\omega. \tag{2.13}$$

Hence it suffices to show that there exists some $0 < \theta < 2$ such that

$$\mu(K_\omega)^{\frac{2-\theta}{\theta}} \ge \left(\delta_1^{|w|}\right)^{\frac{2-\theta}{\theta}} \ge \delta_2^{|\omega|} \ge r_\omega, \tag{2.14}$$

where the first and third inequalities follow from (2.11) and (2.12) respectively. The second inequality also follows on taking θ sufficiently close to 2. The proof is complete. □

Lemma 2.3. *Assume that conditions (1.8), (1.10) , (1.11) hold. Then the following Nash inequality holds: there exists some* $\theta > 0$ *such that*

$$\|u\|_2^{2+\frac{4}{\theta}} \le C\left(\mathcal{E}(u) + \|u\|_2^2\right)\|u\|_1^{\frac{4}{\theta}} \quad \text{for all } u \in \mathcal{F}. \tag{2.15}$$

Consequently, the heat kernel of $(\mathcal{E}, \mathcal{F})$ *exists, and is jointly continuous by using (1.2).*

Proof. Substituting (2.10) into (2.3), and observing that $g(\omega) \asymp s$ for any $\omega \in \Lambda_s$, we obtain that for all $u \in \mathcal{F}$ and all $0 < s \le 1$,

$$\mathcal{E}(u) + \frac{C_3}{s^{2+\theta}}\|u\|_1^2 \ge \frac{C_4}{s^2}\|u\|_2^2.$$

We can rewrite it as

$$\|u\|_2^2 \le C\left(s^2\mathcal{E}(u) + s^{-\theta}\|u\|_1^2\right). \tag{2.16}$$

If $\mathcal{E}(u) \le \|u\|_1^2$, by letting $s = 1$ in (2.16), we have

$$\|u\|_2^2 \le C\|u\|_1^2, \tag{2.17}$$

which implies (2.15) but without the term $C\mathcal{E}(u)\|u\|_1^{\frac{4}{\theta}}$.

If $\mathcal{E}(u) > \|u\|_1^2$, by substituting $s = (\|u\|_1^2/\mathcal{E}(u))^{1/(2+\theta)}$ in (2.16), we have

$$\|u\|_2^2 \le C\|u\|_1^{\frac{4}{2+\theta}}\mathcal{E}(u)^{\frac{\theta}{2+\theta}}, \tag{2.18}$$

which implies (2.15) again but without the term $C\|u\|_2^2\|u\|_1^{\frac{4}{\theta}}$, after raising to the power $(2+\theta)/\theta$. In both cases, we get (2.15), as desired.

The ultra-contractivity of the heat semigroup now follows by (2.15) (see [6, Theorem 2.1]): $\|P_t\|_{1\to\infty} \le Ce^t t^{-\theta/2}$ and a unique heat kernel $p_t(x,y)$ exists. (We only need the existence; the exponents are not important here since our case is nonhomogeneous.) □

To obtain the on-diagonal upper bound, we will follow the argument in [3] on graphs (see also [16, p.184, Theorem 5.1] on metric spaces). Recall that the *effective resistance* $R(A,B)$ between two disjoint non-empty closed subsets A and B of K is defined by

$$R(A,B)^{-1} := \inf\{\mathcal{E}(u) : u \in \mathcal{F},\ u|_A = 1 \text{ and } u|_B = 0\}.$$

It follows from definition that for any fixed $A \subseteq K$, $R(A,B)$ is a non-increasing function of B. Denote

$$R(x,B) := R(\{x\},B) \text{ and } R(x,y) := R(\{x\},\{y\}).$$

Since $1 \in \mathcal{F} \subset C(K)$ and $\mathcal{E}(1) = 0$, we have the following equivalent definition:

$$R(x,y) = \sup\left\{\frac{|u(x)-u(y)|^2}{\mathcal{E}(u)} : u \in \mathcal{F},\ \mathcal{E}(u) > 0\right\}. \tag{2.19}$$

From this, we see that for any $t > 0$ and $x,y,z \in K$,

$$|p_t(x,y) - p_t(x,z)|^2 \le \mathcal{E}(p_t(x,\cdot))R(y,z). \tag{2.20}$$

The following proposition will be used.

Proposition 2.4. *Assume that conditions (1.12) and (VD) hold. Then there exists some universal constant $C > 0$ such that for all $x,y \in K$ with $x < y$,*

$$C^{-1}V(x,d_*(x,y)) \le \mu([x,y]) \le V(x,d_*(x,y)). \tag{2.21}$$

Consequently, if in addition (1.13) holds then

$$|x-y| \asymp R(x,y) \asymp \frac{d_*(x,y)^\beta}{V(x,d_*(x,y))}. \tag{2.22}$$

Proof. Let $x,y \in K$ with $x < y$. Then

$$[x,y] \subseteq B_{d_*}(x,d_*(x,y)), \tag{2.23}$$

since by (1.12), for any point $\xi \in [x,y]$,

$$d_*(x,y) = d_*(x,\xi) + d_*(\xi,y) \ge d_*(\xi,x).$$

Thus, the second inequality in (2.21) holds.

To show the first one, let $z \in [x,y]$ be the point such that

$$d_*(x,z) = d_*(z,y) = d_*(x,y)/2.$$

Then

$$[x,y] = B_{d_*}(z,d_*(x,y)/2),$$

and thus

$$\mu([x,y]) = V(z,d_*(x,y)/2). \tag{2.24}$$

On the other hand,

$$B_{d_*}(x, d_*(x,y)) \subseteq B_{d_*}(z, 3d_*(x,y)/2)\,, \tag{2.25}$$

since if $d_*(\xi, x) \leq d_*(x,y)$ then either $\xi < x$, which leads to

$$d_*(\xi, z) = d_*(\xi, x) + d_*(x,z) \leq d_*(x,y) + d_*(x,y)/2 = 3d_*(x,y)/2,$$

and (2.25) is true, or $\xi \in [x,y] \subset B_{d_*}(x, d_*(x,y))$ and (2.25) is also true. It follows that

$$V(x, d_*(x,y)) \leq V(z, 3d_*(x,y)/2)\,. \tag{2.26}$$

Therefore, we have from (2.24), (2.26), and condition (VD) that

$$\mu([x,y]) = V(z, d_*(x,y)/2) \geq C^{-1} V(z, 3d_*(x,y)/2) \geq C^{-1} V(x, d_*(x,y))\,,$$

proving the first inequality in (2.21).

Finally, note that there exists a constant $c > 0$ such that

$$c^{-1}|x-y| \leq R(x,y) \leq c|x-y| \quad \text{for all } x,y \in K$$

(see [18, formula (1.2)]). From this and using (1.13) and (2.21), we see that (2.22) follows. □

We introduce *condition (DUE)*.

(DUE) (on diagonal upper estimate): There exists a positive constant C such that

$$p_t(x,x) \leq \frac{C}{V(x, t^{1/\beta})} \tag{2.27}$$

for all $t \in (0,1)$ and all $x \in K$.

Lemma 2.5. *Assume that all conditions (1.12), (VD) and (1.15) hold. Then condition (DUE) is true.*

Proof. Fix a ball $B = B_{d_*}(x_0, r)$. Since $p_t(x_0, \cdot)$ is continuous, there is a point $y_0 \in \overline{B}$ such that $p_t(x_0, y_0) = \min_{y \in \overline{B}} p_t(x_0, y)$. Recall that $V(x_0, r) := \mu(B)$. Hence

$$p_t(x_0, y_0) V(x_0, r) \leq \int_{\overline{B}} p_t(x_0, y) d\mu(y) \leq 1,$$

and thus

$$p_t(x_0, y_0) \leq \frac{1}{V(x_0, r)}. \tag{2.28}$$

Observe by (1.15) and (VD) that there exists some constant $C > 0$ such that for all $x \in K$ and all $0 < s \leq r \leq 1$,

$$\frac{s^\beta}{V(x,s)} \leq \frac{Cr^\beta}{V(x,r)}. \tag{2.29}$$

In fact, let $s = \eta r$ for some $\eta \in (0,1]$. By (1.15), there exists some $\eta_0 \in (0,1)$ such that for all $0 < \eta \leq \eta_0$,

$$\frac{s^\beta}{V(x,s)} = \frac{(\eta r)^\beta}{V(x, \eta r)} \leq \frac{r^\beta}{V(x,r)},$$

whilst for all $\eta_0 < \eta \leq 1$, using (VD),

$$\frac{s^\beta}{V(x,s)} = \frac{(\eta r)^\beta}{V(x,\eta r)} \leq \frac{r^\beta}{V(x,\eta r)} \leq C\frac{r^\beta}{V(x,r)},$$

thus showing (2.29).

As $s := d_*(x_0,y_0) \leq r$, we have from (2.22) and (2.29) that

$$R(x_0,y_0) \asymp \frac{d_*(x_0,y_0)^\beta}{V(x_0,d_*(x_0,y_0))} \leq \frac{Cr^\beta}{V(x_0,r)} \tag{2.30}$$

for some universal constant $C > 0$. Using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ and

$$\begin{aligned}\mathcal{E}(p_t(x_0,\cdot)) &= -(\mathcal{L}p_t(x_0,\cdot), p_t(x_0,\cdot)) = -\left(\frac{\partial}{\partial t}p_t(x_0,\cdot), p_t(x_0,\cdot)\right)\\ &= -\frac{1}{2}\frac{\partial}{\partial t}\left(p_t(x_0,\cdot), p_t(x_0,\cdot)\right) = -\frac{1}{2}\frac{\partial}{\partial t}p_{2t}(x_0,x_0),\end{aligned}$$

we have

$$\begin{aligned}p_t^2(x_0,x_0) &\leq 2p_t^2(x_0,y_0) + 2\left|p_t(x_0,x_0) - p_t(x_0,y_0)\right|^2\\ &\leq \frac{2}{V(x_0,r)^2} + 2R(x_0,y_0)\mathcal{E}(p_t(x_0,\cdot)) \qquad \text{(by (2.28) and (2.20))}\\ &\leq \frac{2}{V(x_0,r)^2} + \frac{Cr^\beta}{V(x_0,r)}\left(-\frac{\partial}{\partial t}p_{2t}(x_0,x_0)\right) \qquad \text{(by (2.30)).}\end{aligned} \tag{2.31}$$

Letting $r = t^{1/\beta} \leq (\mathrm{diam}_{d_*}(K))/2 := c_1$, integrating both sides of (2.31) from $s/2$ to s, and using the monotonicity of $p_t(x_0,x_0)$ and $V(x_0,t^{1/\beta})$ on t, we have

$$\frac{s}{2}p_s^2(x_0,x_0) \leq \frac{s}{V\left(x_0,(s/2)^{1/\beta}\right)^2} + \frac{Cs}{V(x_0,(s/2)^{1/\beta})}p_s(x_0,x_0).$$

Solving this inequality gives

$$p_s(x_0,x_0) \leq \frac{C'}{V(x_0,(s/2)^{1/\beta})} \leq \frac{C}{V(x_0,s^{1/\beta})},$$

where $s \in (0,c_1^\beta)$, thus proving (2.27) by using the monotonicity of $p_t(x_0,x_0)$ again. □

We now show that condition (E) holds. Using a standard argument, we can derive this by first estimating $R(x,B^c)$, and then estimating the Green function on B.

For an open set Ω, let $\mathcal{F}(\Omega)$ be the closure of $\mathcal{F} \cap C_0(\Omega)$ under the $\mathcal{F}$-norm, where $C_0(\Omega)$ is the *space* of all continuous functions with supports contained in Ω. It is known that $(\mathcal{E},\mathcal{F}(\Omega))$ is a regular Dirichlet form in $L^2(\Omega,\mu)$ for any nonempty open $\Omega \subset K$.

Definition 2.6. *For any nonempty open set Ω, define the Green function $g_\Omega(x,y)$ as follows,*

$$g_\Omega(x,y) = \begin{cases} R(x,\Omega^c)\psi_\Omega(x,y), & \text{if } x,y \in \Omega,\\ 0, & \text{otherwise,}\end{cases} \tag{2.32}$$

where for any fixed $x \in \Omega$, the function ψ_Ω is a solution of the variational problem

$$\inf\left\{\mathcal{E}(u) : u \in \mathcal{F}(\Omega),\ u(x) = 1\right\};$$

that is, $\psi_\Omega(x,\cdot) \in \mathcal{F}(\Omega)$ *satisfies that* $\psi_\Omega(x,x) = 1$, $0 \le \psi_\Omega(x,\cdot) \le 1$ *on* K, *and*

$$R(x,\Omega^c)^{-1} = \mathcal{E}(\psi_\Omega(x,\cdot)). \tag{2.33}$$

Such a function $\psi_\Omega(x,\cdot)$ exists for any $x \in \Omega$ and any nonempty open Ω by using (1.2) (see for example [16, Proposition 4.2]).

For any ball B, let $\mathcal{L}_B$ be the *infinitesimal generator* of the form $(\mathcal{E},\mathcal{F}(B))$:

$$\mathcal{L}_B f := \lim_{t\to 0} \frac{P_t^B f - f}{t} \text{ in } L^2(\mu)\text{-norm},$$

where $\left\{P_t^B\right\}_{t>0}$ is the heat semigroup of $(\mathcal{E},\mathcal{F}(B))$. Then the *Poisson-type equation*

$$-\mathcal{L}_B u = f \text{ on } B \tag{2.34}$$

admits the unique weak solution $u \in \mathcal{F}(B)$ given by

$$u(x) = \int_B g_B(x,y) f(y) d\mu(y) \tag{2.35}$$

for any $f \in L^\infty(B)$ (see [16] or [13]).

Lemma 2.7. *Assume that conditions (1.12), (1.13), (VD), and (1.15) hold. Then there exists some constant* $C > 0$ *such that for any ball* $B = B_{d_*}(x_0, r)$,

$$\inf_{x\in\frac{1}{2}B} R(x,B^c) \ge C^{-1}\frac{r^\beta}{V(x,r)} \ge C'\frac{r^\beta}{V(x_0,r)}. \tag{2.36}$$

Proof. The proof given here is motivated by [3] on graphs (see also [16, Proposition 5.3] on metric spaces). For a point $x \in B \setminus \frac{1}{2}B$, let $\psi_x \in \mathcal{F}(B)$ be the optimal function satisfying that $\psi_x(x_0) = 1, \psi_x(x) = 0$, and $0 \le \psi_x \le 1$ on K, and

$$R(x_0,x)^{-1} = \mathcal{E}(\psi_x). \tag{2.37}$$

(Such a function ψ_x exists, see for example [16, Proposition 4.2].) For any $y \in B_{d_*}(x,\eta r)$ with $\eta \in (0,1/2)$, we have from (2.22) and (2.29) that

$$R(x,y) \le \frac{C' d_*(x,y)^\beta}{V(x,d_*(x,y))} \le \frac{C(\eta r)^\beta}{V(x,\eta r)}. \tag{2.38}$$

As $B_{d_*}(x_0,d_*(x_0,x)) \subseteq B_{d_*}(x,2r)$ and $d_*(x_0,x) \ge r/2$, we see from (2.22) that, using (VD),

$$\begin{aligned} R(x_0,x)^{-1} &\le C\frac{V(x_0,d_*(x_0,x))}{d_*(x_0,x)^\beta} \\ &\le C'\frac{V(x,2r)}{(r/2)^\beta} \le C\frac{V(x,r)}{r^\beta}. \end{aligned} \tag{2.39}$$

Combining (2.37), (2.38) and (2.39), we obtain

$$\begin{aligned} \psi_x(y)^2 = |\psi_x(y) - \psi_x(x)|^2 \le R(x,y)\mathcal{E}(\psi_x) = \frac{R(x,y)}{R(x_0,x)} \\ \le \frac{C(\eta r)^\beta}{V(x,\eta r)} \cdot \frac{V(x,r)}{r^\beta} = \frac{C\eta^\beta V(x,r)}{V(x,\eta r)}. \end{aligned}$$

In view of (1.15), we may choose $\eta > 0$ to be sufficiently small so that

$$C\frac{\eta^\beta V(x,r)}{V(x,\eta r)} \le \frac{1}{4}.$$

It follows that $\psi_x(y) \le \frac{1}{2}$, showing that

$$\psi_x \le \frac{1}{2} \text{ on } B_{d_*}(x,\eta r).$$

Since μ satisfies (VD), there exists a finite number of balls $\{B_{d_*}(x_i,\eta r)\}_{i=1}^N$, with $x_i \in B\backslash\frac{1}{2}B$, that covers the set $B\setminus\frac{1}{2}B$, where N does not depend on x_0 or r. Let

$$f = \psi_{x_1} \wedge \psi_{x_2} \wedge \cdots \wedge \psi_{x_N}.$$

Then $f(x_0) = 1$ and $f \le \frac{1}{2}$ on $B\setminus\frac{1}{2}B$, and $0 \le f \le 1$ on K. Let

$$g = 2\left(f - \frac{1}{2}\right)_+ .$$

Then $g(x_0) = 1$, $g = 0$ on $B\setminus\frac{1}{2}B$, and $0 \le g \le 1$ on K . Using the Markov property of $(\mathcal{E},\mathcal{F})$, we obtain

$$\mathcal{E}(g) \le 4\mathcal{E}(f) \le 4\sum_{i=1}^N \mathcal{E}(\psi_{x_i}) = 4\sum_{i=1}^N R(x_0,x_i)^{-1},$$

where the second inequality can be obtained as in [13, Proposition 6.9] and the third one follows from (2.37) with x replaced by x_i. Let ϕ be a cut-off function of the pair $\left(\frac{1}{2}B,B\right)$. Then $g1_{B^c} \in \mathcal{F}$ by using the facts that $g\phi \in \mathcal{F}$ and that $g1_{B^c} = g - g\phi$. Since the functions $g\phi$ and $g1_{B^c}$ have disjoint supports, we obtain, using the locality of $(\mathcal{E},\mathcal{F})$, that

$$\mathcal{E}(g\phi) = \mathcal{E}(g) - \mathcal{E}(g1_{B^c}) \le \mathcal{E}(g) \le 4\sum_{i=1}^N R(x_0,x_i)^{-1}.$$

Using (2.22) again, we see

$$R(x_0,x_i)^{-1} \le C\frac{V(x_0,d_*(x_0,x_i))}{d_*(x_0,x_i)^\beta} \le C'\frac{V(x_0,r)}{(r/2)^\beta} = C\frac{V(x_0,r)}{r^\beta},$$

which gives that

$$\mathcal{E}(g\phi) \le C'\frac{V(x_0,r)}{r^\beta}.$$

Noting that $g\phi$ satisfies the defining condition for $R(x_0,B^c)^{-1}$, we obtain

$$R(x_0,B^c) \ge \mathcal{E}(g\phi)^{-1} \ge C\frac{r^\beta}{V(x_0,r)},$$

Therefore, for any $x \in \frac{1}{2}B$,

$$R(x,B^c) \ge R(x,B_{d_*}(x,r/2)^c) \ge C\frac{(r/2)^\beta}{V(x,r/2)} \ge C'\frac{r^\beta}{V(x_0,r)},$$

thus proving (2.36). □

We estimate the Green function $g_B(x,y)$ when x,y are close to the center of B.

Lemma 2.8. *Assume the hypotheses of Lemma 2.7. Then there exist universal constants $C > 0$ and $\delta \in \left(0, \frac{1}{4}\right)$ such that*

$$g_B(x, y) \geq C^{-1} \frac{r^\beta}{V(x_0, r)} \quad \text{for all } x, y \in \delta B, \tag{2.40}$$

where $B = B_{d_*}(x_0, r)$ and g_B is the Green function on B.

Proof. Let $\delta \leq \frac{1}{4}$ be a positive number to be determined later on. For any $x \in \frac{1}{2}B$, let $\psi_B(x, \cdot)$ be the optimal function for $R(x, B^c)^{-1}$. For any $y \in B_{d_*}(x, 2\delta r)$, we obtain

$$\begin{aligned}
|\psi_B(x, y) - 1|^2 &= |\psi_B(x, y) - \psi_B(x, x)|^2 \leq R(x, y)\mathcal{E}(\psi_B(x, \cdot)) \\
&= R(x, y)R(x, B^c)^{-1} \qquad \text{(by (2.33) with } \Omega = B\text{)} \\
&\leq C \frac{d_*(x, y)^\beta}{V(x, d_*(x, y))} \frac{V(x, r)}{r^\beta} \qquad \text{(by (2.22) and (2.36))} \\
&\leq C' \frac{(2\delta r)^\beta}{V(x, 2\delta r)} \cdot \frac{V(x, r)}{r^\beta} \leq \frac{1}{4} \qquad \text{(by (2.29))}
\end{aligned}$$

for sufficiently small δ by using (1.15). Thus

$$\psi_B(x, y) \geq \frac{1}{2} \quad \text{for any } y \in B_{d_*}(x, 2\delta r). \tag{2.41}$$

From this and (2.36), we have

$$g_B(x, y) = R(x, B^c)\psi_B(x, y) \geq C^{-1} \frac{r^\beta}{V(x_0, r)} \psi_B(x, y) \geq (2C)^{-1} \frac{r^\beta}{V(x_0, r)}. \tag{2.42}$$

For any $x, y \in B_{d_*}(x_0, \delta r)$, note that $x \in \frac{1}{2}B$ as $\delta \leq \frac{1}{2}$, and $y \in B_{d_*}(x, 2\delta r)$ as

$$d_*(y, x) \leq d_*(y, x_0) + d_*(x_0, x) < \delta r + \delta r = 2\delta r.$$

Therefore, for any $x, y \in B_{d_*}(x_0, \delta r)$, it follows from (2.42) that

$$g_B(x, y) \geq (2C)^{-1} \frac{r^\beta}{V(x_0, r)},$$

proving (2.40). □

We say that *condition* $(E_\leq)$ holds if the solution u of (2.34) with $f = 1_B$ satisfies

$$\sup_B u \leq C r^\beta, \tag{2.43}$$

and *condition* $(E_\geq)$ holds if u satisfies

$$\inf_{\delta B} u \geq C^{-1} r^\beta \tag{2.44}$$

for some $\delta \in (0, 1)$ independent of u, B. We say that *Condition* (E) holds if both $(E_\leq)$ and $(E_\geq)$ are true.

Lemma 2.9. *Assume the hypotheses of Lemma 2.7. Then condition (E) holds.*

Proof. The condition $(E_{\leq})$ is straightforward by using the upper bound of $R(x, B^c)$. We now show condition $(E_{\geq})$. For any $B = B_{d_*}(x_0, r)$, let u be the solution of (2.34) with $f = 1_B$. Then for any $x \in B_{d_*}(x_0, \delta r)$, we have from (2.40) that

$$\begin{aligned} u(x) &= \int_B g_B(x, y) d\mu(y) \geq \int_{\delta B} g_B(x, y) f(y) d\mu(y) \\ &\geq C^{-1} \frac{r^\beta}{V(x_0, r)} \mu(\delta B) \geq C' r^\beta, \end{aligned}$$

thus showing that condition $(E_{\geq})$ is true. □

Proof of condition (UE) in Theorem 1.1. Since condition (DUE) is true by Lemma 2.5 and condition (E) is true by Lemma 2.9, condition (UE) follows by using (2.1). □

It remains to derive the lower estimate (LE) in Theorem 1.1. We first use (UE) to obtain (NLE), the method is standard, see [13, Theorem 6.17]. Since $(\mathcal{E}, \mathcal{F})$ is conservative, we have from (UE) that

$$\int_{B_{d_*}\left(x, (\delta^{-1} t)^{1/\beta}\right)} p_t(x, y) d\mu(y) = 1 - \int_{K \setminus B_{d_*}\left(x, (\delta^{-1} t)^{1/\beta}\right)} p_t(x, y) d\mu(y) \geq \frac{1}{2} \tag{2.45}$$

for sufficiently small $\delta > 0$, see [12, formula (3.8)]. By the semigroup property and (VD), we have, for any $x \in K$ and any $t \in (0, 1)$,

$$\begin{aligned} p_{2t}(x, x) &= \int_K p_t(x, y)^2 d\mu(y) \geq \int_{B_{d_*}\left(x, (\delta^{-1} t)^{1/\beta}\right)} p_t(x, y)^2 d\mu(y) \\ &\geq \frac{1}{\mu\left(B_{d_*}\left(x, \left(\delta^{-1} t\right)^{1/\beta}\right)\right)} \left[\int_{B_{d_*}\left(x, (\delta^{-1} t)^{1/\beta}\right)} p_t(x, y) d\mu(y)\right]^2 \\ &\geq \frac{1}{4V\left(x, \left(\delta^{-1} t\right)^{1/\beta}\right)} \geq \frac{C^{-1}}{V\left(x, t^{1/\beta}\right)}, \end{aligned}$$

which gives the on-diagonal lower bound: there exists $C > 0$ such that for all $x \in K$ and all $t \in (0, 1)$,

$$p_t(x, x) \geq \frac{C^{-1}}{V\left(x, (t/2)^{1/\beta}\right)} \geq \frac{C^{-1}}{V\left(x, t^{1/\beta}\right)}. \tag{2.46}$$

On the other hand, letting $f := p_{t/2}(x, \cdot)$, we have (see for example [16, (3.7)] or [13])

$$\begin{aligned} \mathcal{E}(p_t(x, \cdot)) = \mathcal{E}(P_{t/2} f) &= \int_0^\infty \lambda e^{-\lambda t} d(E_\lambda f, f) \\ &\leq \frac{1}{et} \int_0^\infty d(E_\lambda f, f) = \frac{1}{et} \|f\|_2^2 = \frac{1}{et} p_t(x, x) \\ &\leq \frac{C}{t V\left(x, t^{1/\beta}\right)} \qquad \text{(using condition (DUE))}. \end{aligned} \tag{2.47}$$

Then using (2.22) and (2.47), we have

$$
\begin{aligned}
|p_t(x,y)-p_t(x,x)|^2 &\le C\frac{d_*(x,y)^\beta}{V(x,d_*(x,y))}\mathcal{E}(p_t(x,\cdot)) \\
&\le C\frac{d_*(x,y)^\beta}{V(x,d_*(x,y))}\cdot\frac{1}{tV\left(x,t^{1/\beta}\right)}.
\end{aligned}
$$

Now we prove condition (*NLE*). Let $\eta\in(0,1)$ be some constant which will be determined later. For all $y\in B_{d_*}(x,(\eta t)^{1/\beta})$ and any $\varepsilon>0$,

$$
\begin{aligned}
p_t(x,y) &\ge p_t(x,x)-\sqrt{C\frac{d_*(x,y)^\beta}{V(x,d_*(x,y))}\cdot\frac{1}{tV\left(x,t^{1/\beta}\right)}} \\
&\ge \frac{C^{-1}}{V\left(x,t^{1/\beta}\right)}-\sqrt{CC'\frac{\eta t}{V(x,(\eta t)^{1/\beta})}\cdot\frac{1}{tV\left(x,t^{1/\beta}\right)}} \qquad \text{(by (2.29))} \\
&\ge \frac{C^{-1}}{V\left(x,t^{1/\beta}\right)}-\frac{\varepsilon}{V(x,t^{1/\beta})} \qquad \text{(by (1.15))} \\
&\ge \frac{C^{-1}}{2V\left(x,t^{1/\beta}\right)},
\end{aligned}
\tag{2.48}
$$

provided we choose η sufficiently small. This proves condition (*NLE*).

Proof of condition (LE) in Theorem 1.1. Condition (*NLE*) is true from above. Since the metric d_* satisfies the chain condition, we conclude that (LE) in Theorem 1.1 follows directly from (NLE) (see, for example, [12, Corollary 3.5]). We omit the detail. □

3. Infinite Bernoulli convolution associated with the golden ratio

Let $K=[0,1]$ and μ be given by (1.17) and (1.16). In this section we introduce a new metric d_* on K, and show that conditions (1.10)–(1.15) are all satisfied.

It is shown in [29] that by introducing the auxiliary IFS $\{T_0,T_1,T_2\}$:

$$
T_0(x)=\rho^2x,\qquad T_1(x)=\rho^3x+\rho^2,\qquad T_2(x)=\rho^2x+\rho, \tag{3.1}
$$

(see Figure 1), one can obtain the following second-order identities: For all Borel subsets $A\subset[0,1]$,

$$
\begin{bmatrix}\mu(T_0T_iA)\\ \mu(T_1T_iA)\\ \mu(T_2T_iA)\end{bmatrix}=M_i\begin{bmatrix}\mu(T_0A)\\ \mu(T_1A)\\ \mu(T_2A)\end{bmatrix},\qquad i=0,1,2, \tag{3.2}
$$

where

$$
M_0=\frac18\begin{bmatrix}2&0&0\\1&2&0\\0&4&0\end{bmatrix},\qquad M_1=\frac14\begin{bmatrix}0&1&0\\0&1&0\\0&1&0\end{bmatrix},\qquad M_2=\frac18\begin{bmatrix}0&4&0\\0&2&1\\0&0&2\end{bmatrix}. \tag{3.3}
$$

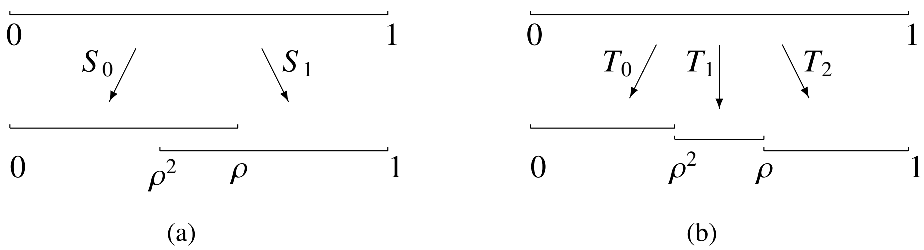

FIGURE 1. (a) The IFS $\{S_0, S_1\}$ has overlaps. (b) The auxiliary IFS $\{T_0, T_1, T_2\}$ does not have overlaps.

By (3.2) we have

$$\mu(T_0(K)) = \mu(T_1(K)) = \mu(T_2(K)) = \frac{1}{3} \tag{3.4}$$

(see [29, p.109]). Let

$$r_0 = r_2 = \rho^2 \text{ and } r_1 = \rho^3, \tag{3.5}$$

the contraction ratios of the auxiliary IFS $\{T_0, T_1, T_2\}$.

Proposition 3.1. *The Dirichlet form $(\mathcal{E}, \mathcal{F})$ defined by (1.1) and (1.3) in $L^2(\mu)$ satisfies the following self-similar identity: For any $u \in \mathcal{F}$,*

$$\mathcal{E}(u) = \sum_{i=0}^{2} \frac{1}{r_i} \mathcal{E}(u \circ T_i), \tag{3.6}$$

where r_0, r_1, r_2 are given by (3.5). Consequently, condition (1.10) holds.

Proof. Indeed, note that

$$\mathcal{E}(u \circ T_0) = \int_0^1 ((u \circ T_0)'(x))^2 \, dx = \int_0^1 \left(u'(x^2)\right)^2 \cdot \rho^4 dx = \rho^2 \int_0^{\rho^2} u'^2 dy. \tag{3.7}$$

Similarly,

$$\mathcal{E}(u \circ T_1) = \rho^3 \int_{\rho^2}^{\rho} u'^2 dy, \tag{3.8}$$

and

$$\mathcal{E}(u \circ T_2) = \rho^2 \int_{\rho}^{1} u'^2 dy. \tag{3.9}$$

Therefore, by summing up (3.7), (3.8) and (3.9), we have

$$\frac{1}{\rho^2}\mathcal{E}(u \circ T_0) + \frac{1}{\rho^3}\mathcal{E}(u \circ T_1) + \frac{1}{\rho^2}\mathcal{E}(u \circ T_2) = \int_0^1 u'^2 dy = \mathcal{E}(u),$$

thus showing (3.6). □

Note that the form $(\mathcal{E}, \mathcal{F})$ does not depend on any metric. We will introduce a new metric d_* on K below. Before this, we show condition (1.11). We use the following notation. For each $n \in \mathbb{N}$, let

$$\mathcal{J}^n := \{0, 1, 2\}^n, \qquad \mathcal{J}_0^n := \{0, 2\}^n, \qquad \mathcal{J}^* := \bigcup_{k=0}^{\infty} \mathcal{J}^n, \qquad \mathcal{J}_0^* := \bigcup_{k=0}^{\infty} \mathcal{J}_0^n,$$

where $\mathcal{J}^0$ and $\mathcal{J}_0^0$ are defined to be the empty word.

For a symbol $\omega \in \{0, 1, 2\}$, let $\mathbf{e}_\omega$ be the row matrix defined as

$$\mathbf{e}_\omega = \begin{cases} [1\ 0\ 0] & \text{if } \omega = 0, \\ [0\ 1\ 0] & \text{if } \omega = 1, \\ [0\ 0\ 1] & \text{if } \omega = 2. \end{cases}$$

By using (3.2), we obtain

$$\begin{aligned} \mu(K_\omega) = \mu(T_\omega(K)) &= \mathbf{e}_{\omega_1} \begin{bmatrix} \mu(T_0 T_{\omega_2} \cdots T_{\omega_n}(K)) \\ \mu(T_1 T_{\omega_2} \cdots T_{\omega_n}(K)) \\ \mu(T_2 T_{\omega_2} \cdots T_{\omega_n}(K)) \end{bmatrix} = \mathbf{e}_{\omega_1} M_{\omega_2} \begin{bmatrix} \mu(T_0 T_{\omega_3} \cdots T_{\omega_n}(K)) \\ \mu(T_1 T_{\omega_3} \cdots T_{\omega_n}(K)) \\ \mu(T_2 T_{\omega_3} \cdots T_{\omega_n}(K)) \end{bmatrix} \\ &= \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_n} \begin{bmatrix} \mu(T_0(K)) \\ \mu(T_1(K)) \\ \mu(T_2(K)) \end{bmatrix} = \frac{1}{3} \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_n} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{3.10}$$

Lemma 3.2. *For any two neighbouring words ω and τ, we have*

$$2^{-1}\mu(K_\tau) \le \mu(K_\omega) \le 2\mu(K_\tau). \tag{3.11}$$

Consequently, condition (1.11) is true.

Proof. Without loss of generality, we assume that K_ω is on the left of K_τ. Then either of the following relationships holds for such ω and τ:

$$\omega = \theta 0 \underbrace{2 \cdots 2}_{\ell} \quad \text{and} \quad \tau = \theta 1 \underbrace{0 \cdots 0}_{\ell}, \tag{3.12}$$

or

$$\omega = \theta 1 \underbrace{2 \cdots 2}_{\ell} \quad \text{and} \quad \tau = \theta 2 \underbrace{0 \cdots 0}_{\ell}, \tag{3.13}$$

where θ is some finite word (possibly empty word) and $\ell \ge 0$ is some integer.

We deal with the first case; the second one is similar.

Note that $\mu(K_\omega) = \mu(K_\tau) = \frac{1}{3}$ when $|\omega| = |\tau| = 1$ by using (3.4), and (3.11) holds trivially. We assume that $|\omega| = |\tau| = n \ge 2$ and ω_1, τ_1 are the first symbols of ω, τ respectively. Assume that $|\theta| = s \ge 1$ and write $\theta = \theta_1\theta_2 \cdots \theta_s$. As $\omega = \theta 0 \underbrace{2 \cdots 2}_{\ell}$, we have, using (3.10) that

$$\mu(K_\omega) = \frac{1}{3} \mathbf{e}_{\theta_1} M_{\theta_2} \cdots M_{\theta_s} \cdot M_0 \cdot M_2^\ell \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \cdot M_0 \cdot M_2^\ell \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{3.14}$$

where $\begin{bmatrix} a & b & c \end{bmatrix}$ is the row vector with $a, b, c \ge 0$. Similarly, we have

$$\mu(K_\tau) = \frac{1}{3} \mathbf{e}_{\theta_1} M_{\theta_2} \cdots M_{\theta_s} \cdot M_1 \cdot M_0^\ell \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \cdot M_1 \cdot M_0^\ell \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{3.15}$$

By comparing $\mu(K_\omega)$ and $\mu(K_\tau)$, it is easy to see that $\mu(K_\omega) \le 2\mu(K_\tau)$ when $\ell = 0$ (for the second case we have $\mu(K_\tau) \le 2\mu(K_\omega)$), and hence (3.11) holds. Meanwhile, using induction, we obtain, for any integer $\ell \ge 1$,

$$M_0^\ell = \frac{1}{8^\ell}\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix}^\ell = \frac{1}{8^\ell}\begin{bmatrix} 2^\ell & 0 & 0 \\ 2^{\ell-1}\cdot\ell & 2^\ell & 0 \\ 2^\ell\cdot(\ell-1) & 2^{\ell+1} & 0 \end{bmatrix}, \tag{3.16}$$

and also

$$M_2^\ell = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\cdot M_0^\ell \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \frac{1}{8^\ell}\begin{bmatrix} 0 & 2^{\ell+1} & 2^\ell\cdot(\ell-1) \\ 0 & 2^\ell & 2^{\ell-1}\cdot\ell \\ 0 & 0 & 2^\ell \end{bmatrix}. \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.14) and (3.15) respectively, we see that

$$\mu(K_\omega) = \frac{1}{8^{\ell+1}}\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} 2^{\ell+1}\cdot(\ell+1) \\ 2^\ell\cdot(2\ell+3) \\ 2^{\ell+1}\cdot(\ell+2) \end{bmatrix},$$

and

$$\mu(K_\tau) = \frac{1}{8^{\ell+1}}\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} 2^\ell\cdot(\ell+2) \\ 2^\ell\cdot(\ell+2) \\ 2^\ell\cdot(\ell+2) \end{bmatrix}.$$

Thus we obtain

$$\frac{\mu(K_\omega)}{\mu(K_\tau)} = \frac{(2\ell+2)a + (2\ell+3)b + (2\ell+4)c}{(\ell+2)(a+b+c)},$$

which implies that

$$1 \le \frac{\mu(K_\omega)}{\mu(K_\tau)} \le 2.$$

Thus (3.11) holds. □

For any $0 \le x < y \le 1$, we define a set $\mathcal{W}(x,y)$ of finite words as follows:

$$\mathcal{W}(x,y) := \Big\{\omega = \omega_1\cdots\omega_n \in \mathcal{J}^* : \quad \omega_n = 1,\ K_\omega \subseteq [x,y], \\ \text{and } \omega \text{ is a father}\Big\}, \tag{3.18}$$

where the notion "$\omega = \omega_1\cdots\omega_n$ is a father" means that no *proper ancestor* $\omega_1\cdots\omega_k$ ($k < |\omega|$) of ω satisfies both of the following conditions:

(1) $\omega_k = 1$;
(2) $K_{\omega_1\cdots\omega_k} \subseteq [x,y]$.

Namely, any word $\omega\xi$ with $\xi \ne \emptyset$ cannot belong to $\mathcal{W}(x,y)$ if $\omega \in \mathcal{W}(x,y)$, and hence ω is of the shortest length among this class, or is a father.

For example, if $[x,y] = [\rho^2,\rho] = K_1$, then $\mathcal{W}(x,y) = \{1\}$, consisting of only one singleton. Note that the word "11" does not belong to $\mathcal{W}(\rho^2,\rho)$ since "1" is its father. Another example is when $[x,y] = [0,\rho^2] = K_0$, then

$$\begin{aligned} \mathcal{W}(x,y) &= \{01, 001, 021, 0001, 0021, 0201, 0221, \dots\} \\ &= \{0J1 : J \in \mathcal{J}_0^*\}, \end{aligned} \tag{3.19}$$

an infinite set of words with finite length (noting that $\mathcal{J}_0^*$ contains the empty word). Note that none of the words in the following set

$$\{011, 0101, 0111, 0121\}$$

is in $\mathcal{W}(0,\rho^2)$, although each word ω in this set ends with the symbol "1", and $K_\omega \subseteq [0,\rho^2]$. The reason is that all of them are offspring of the word "01", which is in $\mathcal{W}(0,\rho^2)$.

Note that for any x, y with $0 \leq x < y \leq 1$, the set $\mathcal{W}(x,y) \neq \emptyset$. This is because $K = [0,1]$ and for each $n \geq 1$, $\bigcup_{\omega\in\mathcal{J}^n} K_\omega = [0,1]$. Let $\omega = \omega_1 \ldots \omega_n$ be the shortest word such that $K_\omega \subseteq [x,y]$ (such a word exists since the midpoint $(x+y)/2$ belongs to at least one nonempty cell $K_\tau \subseteq [x,y]$ and we let ω be the shortest of such τ). If $\omega_n = 1$, then $\omega \in \mathcal{W}(x,y)$; otherwise $\omega 1 \in \mathcal{W}(x,y)$. Therefore $\mathcal{W}(x,y)$ is nonempty.

Define a distance d_* on K as follows: $d_*(x,y) = 0$ if $x = y$, and

$$d_*(x,y) = \sum_{\omega\in\mathcal{W}(x,y)} (r_\omega \mu(K_\omega))^\alpha \tag{3.20}$$

if $0 \leq x < y \leq 1$, where r_0, r_1, r_2 are given by (3.5) and α is the unique solution of the following equation

$$\sum_{k=0}^{\infty} \sum_{J\in\mathcal{J}_0^k} \left(\rho^{2k+3} c_J\right)^\alpha = 1 \tag{3.21}$$

with c_J given by

$$c_J := \frac{1}{4}\begin{bmatrix}0 & 1 & 0\end{bmatrix} M_J \begin{bmatrix}1\\1\\1\end{bmatrix} \text{ where } M_J := M_{j_1}\cdots M_{j_k} \tag{3.22}$$

for any index $J = j_1 \cdots j_k \in \mathcal{J}_0^k := \{0,2\}^k$ and any integer $k \geq 0$. Here we use the convention that $M_\omega := I$, the identity matrix, if ω is the empty word.

Remark 3.3. Let Δ_μ be the Laplacian defined by μ (see [5, 17]). Then

$$\dim_s(\mu) = 2\alpha,$$

where $\dim_s(\mu)$ is the *spectral dimension* of the corresponding Dirichlet and Neumann Laplacians $-\Delta_\mu$ (see [28]). In fact (see [28, p.654]), we have

$$\alpha = \frac{\dim_s(\mu)}{2} \approx \frac{0.998}{2} = 0.499 < 0.5 \tag{3.23}$$

(the value of α is close to but strictly less than 0.5). This sharply contrasts with the classical case where $\alpha = 0.5$ for the Euclidean metric and the Lebesgue measure.

Note that for any word ω, letting $x = T_\omega(0)$, $y = T_\omega(1)$, we have by definition (3.20),

$$d_*(x,y) = (r_\omega \mu(K_\omega))^\alpha \tag{3.24}$$

if ω ends with the symbol "1" since $\mathcal{W}(x,y) = \{\omega\}$, while

$$d_*(x,y) = \sum_{k=0}^{\infty} \sum_{J\in\mathcal{J}_0^k} (r_{\omega J1} \mu(K_{\omega J1}))^\alpha \tag{3.25}$$

if ω ends with the symbols "0" or "2" since $\mathcal{W}(x,y) = \left\{\omega J1 : J \in \mathcal{J}_0^*\right\}$.

Proposition 3.4. *For any $0 \le x < y \le 1$ and for any distinct $\omega, \tau \in \mathcal{W}(x, y)$, we have*

$$K_\omega \cap K_\tau = \emptyset. \tag{3.26}$$

Proof. Assume that there are two distinct words $\omega, \tau \in \mathcal{W}(x, y)$ such that $K_\omega \cap K_\tau \neq \emptyset$. Since both ω and τ end with 1, the only possible case is that one cell is contained in the other (Indeed, if $\omega = \omega_1 \cdots \omega_{n-1}1$ and $\tau = \tau_1 \cdots \tau_{m-1}1$ ($n \le m$), and if $\omega_k \neq \tau_k$ for some $k \le n$, then $K_\omega \cap K_\tau = \emptyset$). Without loss of generality, assume that $K_\omega \subset K_\tau$. Then τ is a father of ω, a contradiction by the definition of $\mathcal{W}(x, y)$. The proposition follows. □

Proposition 3.5. *The quantity d_* is a metric on K, and satisfies*

$$d_*(x, z) = d_*(x, y) + d_*(y, z) \tag{3.27}$$

for any $0 \le x < y < z \le 1$. Consequently, condition (1.12) is satisfied.

Proof. If $d_*(x, y) = 0$ then $x = y$; otherwise there would exist some nonempty word $\omega \in \mathcal{W}(x, y)$ such that $\mu(K_\omega) = 0$, a contradiction by using (1.6).

It suffices to prove (3.27), since this will imply that d_* satisfies the triangle inequality, and thus d_* is a metric on K.

To do this, we first claim that for any $\omega = \omega_1 \cdots \omega_n$ with $\omega_n = 1$, we have

$$(r_\omega \mu(K_\omega))^\alpha = \sum_{J \in \mathcal{J}_0^*} (r_{\omega J1} \mu(K_{\omega J1}))^\alpha = \sum_{k=0}^{\infty} \sum_{J \in \mathcal{J}_0^k} (r_{\omega J1} \mu(K_{\omega J1}))^\alpha . \tag{3.28}$$

The left-hand side of (3.28) is

$$(r_\omega \mu(K_\omega))^\alpha = r_\omega^\alpha \cdot \mu(K_\omega)^\alpha,$$

and in view of (3.5), the right-hand side of (3.28) is:

$$r_\omega^\alpha \cdot \sum_{k=0}^{\infty} \sum_{J \in \mathcal{J}_0^k} \left(\rho^{2k+3}\right)^\alpha \cdot \mu(K_{\omega J1})^\alpha.$$

Thus we only need to show that

$$\mu(K_\omega)^\alpha = \sum_{k=0}^{\infty} \sum_{J \in \mathcal{J}_0^k} \left(\rho^{2k+3}\right)^\alpha \cdot \mu(K_{\omega J1})^\alpha. \tag{3.29}$$

To do this, we use (3.10) to get

$$\mu(K_\omega) = \frac{1}{3} \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} \cdot M_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} a & b & c \end{bmatrix} \cdot M_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{12}(a + b + c), \tag{3.30}$$

where $\begin{bmatrix} a & b & c \end{bmatrix} = \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}}$ is some row vector with nonnegative entries. Similarly, we have

$$\mu(K_{\omega J1}) = \frac{1}{3} \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} \cdot M_1 \cdot M_J \cdot M_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$
\begin{aligned}
&= \frac{1}{12}\begin{bmatrix}0 & a+b+c & 0\end{bmatrix} \cdot M_J \cdot M_1 \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} \\
&= \frac{1}{12}(a+b+c) \cdot \frac{1}{4}\begin{bmatrix}0 & 1 & 0\end{bmatrix} M_J \begin{bmatrix}1\\1\\1\end{bmatrix} \\
&= \frac{1}{12}(a+b+c)c_J. \qquad (3.31)
\end{aligned}
$$

Therefore, by (3.21), (3.30) and (3.31), we obtain (3.29), showing (3.28).

Let $x, y, z \in K$ with $x < y < z$. Observe that $\mathcal{W}(x, y) \cap \mathcal{W}(y, z) = \emptyset$. By the definition of d_* and using (3.28), we see that

$$
d_*(x, y) + d_*(y, z) = \sum_{\omega \in \mathcal{W}(x,y) \cup \mathcal{W}(y,z)} (r_\omega \mu(K_\omega))^\alpha \,, \qquad (3.32)
$$

and that

$$
d_*(x, z) = \sum_{\tau \in \mathcal{W}(x,z)} (r_\tau \mu(K_\tau))^\alpha \,. \qquad (3.33)
$$

Observing that each word $\omega \in \mathcal{W}(x, y) \cup \mathcal{W}(y, z)$ either belongs to $\mathcal{W}(x, z)$ or is an offspring of some word in $\mathcal{W}(x, z)$. For the latter case, write

$$
\omega = \tau J_1 1 J_2 1 \cdots J_k 1
$$

for some $k \geq 1$ and $J_1, J_2, \ldots, J_k \in \mathcal{J}_0^*$, and for some $\tau \in \mathcal{W}(x, z)$. Using (3.28) k times, we have

$$
\begin{aligned}
(r_\tau \mu(K_\tau))^\alpha &= \sum_{J'_1,\ldots,J'_k \in \mathcal{J}_0^*} \left(r_{\tau J'_1 1 J'_2 1 \cdots J'_k 1} \mu(K_{\tau J'_1 1 J'_2 1 \cdots J'_k 1})\right)^\alpha \\
&\geq \sum_{\omega \in \mathcal{W}(x,y) \cup \mathcal{W}(y,z) \text{ and } \omega \text{ has prefix } \tau} (r_\omega \mu(K_\omega))^\alpha \,,
\end{aligned}
$$

and hence by (3.32), (3.33),

$$
d_*(x, y) + d_*(y, z) \leq d_*(x, z). \qquad (3.34)
$$

On the other hand, for any $\varepsilon > 0$, using (3.28) k times again, we have by (3.33),

$$
d_*(x, z) = \sum_{\omega \in \mathcal{W}(x,z)} \sum_{J_1,\ldots,J_k \in \mathcal{J}_0^*} (r_{\omega J_1 1 J_2 1 \cdots J_k 1} \mu(K_{\omega J_1 1 J_2 1 \cdots J_k 1}))^\alpha \,. \qquad (3.35)
$$

We can pick an integer k sufficiently large such that each term

$$
(r_{\omega J_1 1 J_2 1 \cdots J_k 1} \mu(K_{\omega J_1 1 J_2 1 \cdots J_k 1}))^\alpha < \varepsilon.
$$

Observe that there is at most one cell $K_{\omega J_1 1 J_2 1 \cdots J_k 1}$ on the right-hand side of (3.35) containing y since the words in $\mathcal{W}(x, y) \cup \mathcal{W}(y, z)$ represent disjoint cells by Proposition 3.4, whilst each of the remaining words is an offspring of some element in $\mathcal{W}(x, y) \cup \mathcal{W}(y, z)$. Applying these facts to the right-hand side of (3.35), we obtain

$$
d_*(x, z) \leq \sum_{\omega \in \mathcal{W}(x,y)} \sum_{J_1,\ldots,J_k \in \mathcal{J}_0^*} (r_{\omega J_1 1 J_2 1 \cdots J_k 1} \mu(K_{\omega J_1 1 J_2 1 \cdots J_k 1}))^\alpha
$$

$$+\sum_{\omega\in\mathcal{W}(y,z)}\sum_{J_1,\ldots,J_k\in\mathcal{J}_0^*}\left(r_{\omega J_1 1 J_2 1\cdots J_k 1}\mu(K_{\omega J_1 1 J_2 1\cdots J_k 1})\right)^{\alpha}+\varepsilon$$
$$\le d_*(x,y)+d_*(y,z)+\varepsilon. \tag{3.36}$$

Since ε is arbitrary, we have

$$d_*(x,z)\le d_*(x,y)+d_*(y,z). \tag{3.37}$$

Combining (3.34), (3.37), we obtain (3.27). The proof is complete. □

For any word ω, define

$$d_*(K_\omega)=\sup_{x,y\in K_\omega} d_*(x,y)=d_*\left(T_\omega(0),T_\omega(1)\right), \tag{3.38}$$

the *diameter* of the cell K_ω under the metric d_*.

Lemma 3.6. *There exists a constant $C>0$ (depending only on ρ) such that for any two neighbouring words ω and τ,*

$$C^{-1}d_*(K_\tau)\le d_*(K_\omega)\le Cd_*(K_\tau). \tag{3.39}$$

(In fact, one can take $C=\left\{1+2\left(8\rho^{-3}\right)^{\alpha}\right\}\left(10\rho^{-1}\right)^{\alpha}$.)

Proof. We first claim that for any finite word ω, the following holds:

$$\begin{aligned}\left(\frac{\rho^2}{10}\right)^{\alpha} d_*(K_{\omega 1}) &\le d_*(K_{\omega 0})\le\left(\frac{8}{\rho^3}\right)^{\alpha} d_*(K_{\omega 1}),\\ \left(\frac{\rho^2}{10}\right)^{\alpha} d_*(K_{\omega 1}) &\le d_*(K_{\omega 2})\le\left(\frac{8}{\rho^3}\right)^{\alpha} d_*(K_{\omega 1}),\end{aligned} \tag{3.40}$$

that is,

$$d_*(K_{\omega 0})\asymp d_*(K_{\omega 1})\asymp d_*(K_{\omega 2}).$$

We first show the '≤' part of the first line in (3.40). Indeed, by (3.24)

$$d_*(K_{\omega 1})=\mu(K_{\omega 1})^{\alpha} r_{\omega 1}^{\alpha},$$

and using (3.10), we see that

$$\mu(K_{\omega 1})=\frac{1}{3}\mathbf{e}_{\omega_1}M_{\omega_2}\cdots M_{\omega_n}M_1\begin{bmatrix}1\\1\\1\end{bmatrix}=\frac{1}{12}\mathbf{e}_{\omega_1}M_{\omega_2}\cdots M_{\omega_n}\begin{bmatrix}1\\1\\1\end{bmatrix}=\frac{1}{12}(a+b+c),$$

where $\begin{bmatrix}a & b & c\end{bmatrix}:=\mathbf{e}_{\omega_1}M_{\omega_2}\cdots M_{\omega_n}$ and a,b,c are nonnegative numbers. Thus we have

$$d_*(K_{\omega 1})=\left(\frac{a+b+c}{12}\right)^{\alpha}\cdot r_{\omega 1}^{\alpha}=\rho^{3\alpha}\left(\frac{a+b+c}{12}\right)^{\alpha}\cdot r_{\omega}^{\alpha}. \tag{3.41}$$

Again, combining the definition of d_*, (3.10), and (3.25), we get

$$d_*(K_{\omega 0})=\sum_{k=0}^{\infty}\sum_{J\in\mathcal{J}_0^k}\mu(K_{\omega 0J1})^{\alpha} r_{\omega 0J1}^{\alpha}, \tag{3.42}$$

and

$$\begin{aligned}\mu(K_{\omega 0J1}) &= \frac{1}{3}\mathbf{e}_{\omega_1}M_{\omega_2}\cdots M_{\omega_n}M_0M_JM_1\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{12}\begin{bmatrix}a & b & c\end{bmatrix}M_0M_J\begin{bmatrix}1\\1\\1\end{bmatrix}\\ &= \frac{1}{96}\begin{bmatrix}2a+b & 2b+4c & 0\end{bmatrix}M_J\begin{bmatrix}1\\1\\1\end{bmatrix} \le \frac{a+b+c}{48}\begin{bmatrix}1 & 2 & 0\end{bmatrix}M_J\begin{bmatrix}1\\1\\1\end{bmatrix}\\ &= \frac{2(a+b+c)}{3}\cdot\left(\frac{1}{4}\begin{bmatrix}0 & 1 & 0\end{bmatrix}M_0M_J\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \frac{2(a+b+c)}{3}c_{0J}.\end{aligned}$$

Thus, the right-hand side of (3.42) satisfies

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{J\in\mathcal{J}_0^k}\mu(K_{\omega 0J1})^{\alpha}r_{\omega 0J1}^{\alpha} &\le \sum_{k=0}^{\infty}\sum_{J\in\mathcal{J}_0^k}\left(\frac{2(a+b+c)}{3}\right)^{\alpha}c_{0J}^{\alpha}\cdot r_{\omega}^{\alpha}\cdot\rho^{(2k+5)\alpha}\\ &= \left(\frac{2(a+b+c)}{3}\right)^{\alpha}\cdot r_{\omega}^{\alpha}\cdot\sum_{k=0}^{\infty}\sum_{J\in\mathcal{J}_0^k}c_{0J}^{\alpha}\cdot\rho^{(2k+5)\alpha}\\ &\le \left(\frac{2(a+b+c)}{3}\right)^{\alpha}\cdot r_{\omega}^{\alpha}\cdot\sum_{k=0}^{\infty}\sum_{J'\in\mathcal{J}_0^k}c_{J'}^{\alpha}\cdot\rho^{(2(k+1)+3)\alpha} \qquad (\text{let } J'=0J)\\ &\le \left(\frac{2(a+b+c)}{3}\right)^{\alpha}\cdot r_{\omega}^{\alpha}. \qquad (\text{definition of } \alpha)\end{aligned}$$

Using this and comparing with (3.41), we obtain

$$d_*(K_{\omega 0}) \le \left(\frac{8}{\rho^3}\right)^{\alpha}d_*(K_{\omega 1}),$$

thus proving the '$\le$' part of the first line in (3.40).

On the other hand, using Lemma 3.2 and (1.6), for any word θ,

$$\begin{aligned}\mu(K_{\theta}) &= \mu(K_{\theta 0})+\mu(K_{\theta 1})+\mu(K_{\theta 2})\\ &\le 2\mu(K_{\theta 1})+\mu(K_{\theta 1})+2\mu(K_{\theta 1}) = 5\mu(K_{\theta 1}).\end{aligned}\tag{3.43}$$

Applying this inequality with $\theta=\omega 0$ and using Lemma 3.2 again,

$$\mu(K_{\omega 1}) \le 2\mu(K_{\omega 0}) \le 10\mu(K_{\omega 01}),$$

which implies, in view of (3.24) again, that

$$d_*(K_{\omega 01}) = \mu(K_{\omega 01})^{\alpha}r_{\omega 01}^{\alpha} \ge 10^{-\alpha}\mu(K_{\omega 1})^{\alpha}\rho^{2\alpha}r_{\omega 1}^{\alpha} = \left(\frac{\rho^2}{10}\right)^{\alpha}d_*(K_{\omega 1}).$$

Thus, we have

$$d_*(K_{\omega 0}) \ge d_*(K_{\omega 01}) \ge \left(\frac{\rho^2}{10}\right)^{\alpha}d_*(K_{\omega 1}),\tag{3.44}$$

proving the '$\ge$' part of the first line in (3.40). By symmetry, the second line in (3.40) also holds. This proves our claim.

Therefore, for any finite word ω, by (3.27) and (3.40), we have

$$\begin{aligned} d_*(K_{\omega 1}) &\leq d_*(K_\omega) = d_*(K_{\omega 0}) + d_*(K_{\omega 1}) + d_*(K_{\omega 2}) \\ &\leq \left\{1 + 2\left(8\rho^{-3}\right)^{\alpha}\right\} d_*(K_{\omega 1}) := A d_*(K_{\omega 1}), \end{aligned} \tag{3.45}$$

where $A = 1 + 2\left(8\rho^{-3}\right)^{\alpha}$.

Finally, for any neighboring words ω and τ, using (3.5) and (3.12) (or (3.13)), we have

$$\rho r_\tau \leq r_\omega \leq \rho^{-1} r_\tau, \tag{3.46}$$

which implies, by using Lemma 3.2 and (3.43) with $\theta = \tau$, that

$$\begin{aligned} r_{\omega 1}\mu(K_{\omega 1}) &\leq r_{\omega 1}\mu(K_\omega) \leq r_1\left(\rho^{-1} r_\tau\right)(2\mu(K_\tau)) \\ &= 2\rho^{-1} r_{\tau 1}\mu(K_\tau) \leq 10\rho^{-1} r_{\tau 1}\mu(K_{\tau 1}). \end{aligned}$$

Thus, using (3.45), we have

$$\begin{aligned} d_*(K_\omega) &\leq A d_*(K_{\omega 1}) = A\left[r_{\omega 1}\mu(K_{\omega 1})\right]^{\alpha} \\ &\leq A\left(10\rho^{-1}\right)^{\alpha}\left[r_{\tau 1}\mu(K_{\tau 1})\right]^{\alpha} = A\left(10\rho^{-1}\right)^{\alpha} d_*(K_{\tau 1}) \\ &\leq \left\{1 + 2\left(8\rho^{-3}\right)^{\alpha}\right\}\left(10\rho^{-1}\right)^{\alpha} d_*(K_\tau), \end{aligned} \tag{3.47}$$

thus proving the lemma. □

We need the following proposition.

Proposition 3.7. *Let ω be the shortest word such that $K_\omega \subseteq [x, y]$ for $0 \leq x < y \leq 1$. Set $|\omega| = n$. Then there are at most four n-cells between the points x and y.*

Proof. Assume that $\omega \neq \emptyset$; otherwise nothing needs to be proved. Let ω' be the father of ω; that is, ω is one of $\{\omega'0, \omega'1, \omega'2\}$. Without loss of generality assume that $\omega' \neq \emptyset$. Note that x, y cannot be separated apart by any $(n-1)$-cell; otherwise, ω is not the shortest. Namely, both x and y must lie in the union of two neighboring $(n-1)$-cells.

Case (1). $\omega = \omega'0$. The point x must lie to the left of $K_{\omega'}$, since $K_\omega = K_{\omega'0} \subseteq [x, y]$. Let τ' be the left neighboring $(n-1)$-word of ω'. Then

$$x \in K_{\tau'} \ \text{ and } \ y \in K_{\omega'}$$

(see Figure 2.)

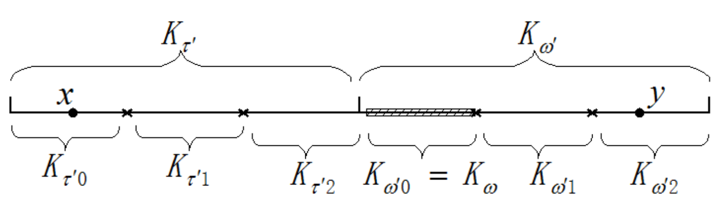

FIGURE 2. Points x, y and cells $K_{\omega'}, K_{\tau'}$, where $\omega = \omega'0$.

Clearly, there are at most four n-cells between the points x and y:

$$x \in K_{\tau'0} \sim K_{\tau'1} \sim K_{\tau'2} \sim K_{\omega'0} = K_\omega \sim K_{\omega'1} \sim K_{\omega'2} \ni y,$$

thus proving our conclusion.

Case (2). $\omega = \omega'2$. Under this assumption, the worst case is the following:

$$x \in K_{\omega'0} \text{ and } y \in K_{\tau''2},$$

where τ'' is the right neighboring $(n-1)$-word of ω' (see Figure 3).

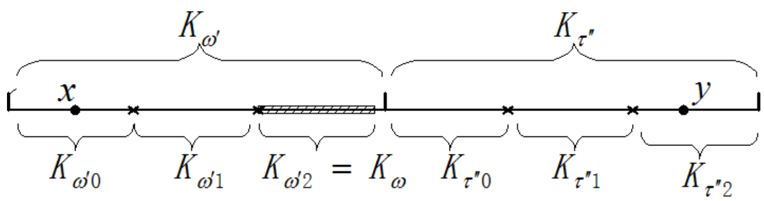

FIGURE 3. Points x, y and cells $K_{\omega'}, K_{\tau''}$, where $\omega = \omega'2$.

Clearly, there are also at most four n-cells between the points x and y:

$$x \in K_{\omega'0} \sim K_{\omega'1} \sim K_{\omega'2} = K_\omega \sim K_{\tau''0} \sim K_{\tau''1} \sim K_{\tau''2} \ni y,$$

and our conclusion is true as well.

Case (3). $\omega = \omega'1$. Since x and y cannot both lie outside of $K_{\omega'}$, we assume, without loss of generality, that $y \in K_{\omega'}$. The other point x lies either in $K_{\omega'}$, or in the left or the right neighboring cell of $K_{\omega'}$, and we assume that $x \in K_{\tau'}$, the left neighboring cell of $K_{\omega'}$ (see Figure 4).

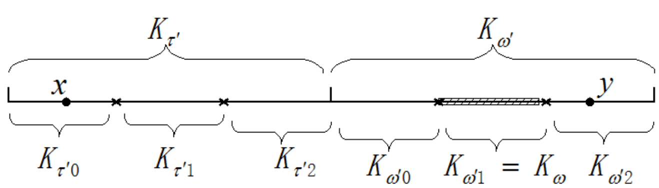

FIGURE 4. Cells $K_{\omega'}, K_{\tau'}$ where $\omega = \omega'1$ and $y \in K_{\omega'}, x \in K_{\tau'}$ with $K_{\tau'}$ being the left neighboring cell.

Clearly our conclusion is also true. The proof is complete. $\square$

Lemma 3.8. *Let ω be the shortest word such that $K_\omega \subseteq [x, y]$ for $0 \le x < y \le 1$. Then*

$$\mu(K_\omega) \le \mu([x, y]) \le 30\mu(K_\omega), \tag{3.48}$$

$$r_\omega \le |x - y| \le \left(1 + \rho^{-1}\right)\rho^{-2} r_\omega, \tag{3.49}$$

$$C^{-1}\left[\mu(K_\omega) r_\omega\right]^\alpha \le d_*(x, y) \le C\left[\mu(K_\omega) r_\omega\right]^\alpha. \tag{3.50}$$

Consequently, condition (1.13) holds with $\beta = 1/\alpha$.

Proof. If $\omega = \emptyset$, nothing needs to be proved. Assume that $\omega \neq \emptyset$.

We first consider the case $\omega = \omega' 0$ for some word ω'. Without loss of generality assume that $\omega' \neq \emptyset$. Then $y \in K_{\omega'}$ and $x \in K_{\tau'}$ for the left neighboring word τ' of ω' (see Figure 2 above), and

$$K_{\omega' 0} = K_\omega \subseteq [x, y] \subseteq K_{\omega'} \cup K_{\tau'}. \tag{3.51}$$

It follows that

$$\begin{aligned} \mu(K_\omega) &\le \mu([x,y]) \le \mu(K_{\omega'}) + \mu(K_{\tau'}) \le 3\mu(K_{\omega'}) \qquad \text{(using (3.11))} \\ &\le 15\mu(K_{\omega' 1}) \qquad \text{(using (3.43))} \\ &\le 30\mu(K_{\omega' 0}) = 30\mu(K_\omega) \qquad \text{(using (3.11)),} \end{aligned}$$

thus proving (3.48).

Observe that by (3.51) and (3.1),

$$r_\omega \le |x - y| \le r_{\omega'} + r_{\tau'}. \tag{3.52}$$

Since $r_\omega = r_{\omega' 0} = \rho^2 r_{\omega'}$, using (3.46) we have

$$r_{\omega'} + r_{\tau'} \le r_{\omega'} + \rho^{-1} r_{\omega'} = \left(1 + \rho^{-1}\right) r_{\omega'} = \left(1 + \rho^{-1}\right)\rho^{-2} r_\omega.$$

Combining this with (3.52) proves (3.49).

Using (3.39) and (3.27), we have from (3.51) that

$$\begin{aligned} C^{-1} d_*(K_{\omega' 1}) &\le d_*(K_{\omega' 0}) = d_*(K_\omega) \le d_*(x, y) \\ &\le d_*(K_{\omega'}) + d_*(K_{\tau'}) \le (1 + C) d_*(K_{\omega'}) \\ &\le (1 + C) A d_*(K_{\omega' 1}) \ \ \text{(using (3.45) with } \omega \text{ replaced by } \omega' \text{).} \end{aligned} \tag{3.53}$$

By (3.11) and (3.5),

$$\begin{aligned} \mu(K_{\omega' 1}) r_{\omega' 1} &\le 2\mu(K_{\omega' 0}) r_{\omega' 1} = 2\rho\mu\left(K_\omega\right) r_\omega, \\ \mu(K_{\omega' 1}) r_{\omega' 1} &\ge 2^{-1}\mu(K_{\omega' 0}) r_{\omega' 1} = 2^{-1}\rho\mu\left(K_\omega\right) r_\omega. \end{aligned}$$

It follows by using (3.24) that

$$\left(2^{-1}\rho\right)^\alpha \left[\mu\left(K_\omega\right) r_\omega\right]^\alpha \le \left[\mu(K_{\omega' 1}) r_{\omega' 1}\right]^\alpha = d_*(K_{\omega' 1}) \le (2\rho)^\alpha \left[\mu\left(K_\omega\right) r_\omega\right]^\alpha.$$

From this and using (3.53), we have

$$C^{-1}\left(2^{-1}\rho\right)^\alpha \left[\mu\left(K_\omega\right) r_\omega\right]^\alpha \le d_*(K_\omega) \le (1 + C) A \left(2\rho\right)^\alpha \left[\mu\left(K_\omega\right) r_\omega\right]^\alpha,$$

thus proving (3.50).

The cases $\omega = \omega' 2$ and $\omega = \omega' 1$ can be treated similarly.

Finally, the formula (1.13) with $\beta = 1/\alpha$ follows directly by combining (3.48), (3.49), and (3.50). The proof is complete. □

Lemma 3.9. *Condition (1.14) is true.*

Proof. It suffices to show that there exists a constant $C > 1$ (depending only on ρ) such that for any $0 \le x < y < z \le 1$ with $d_*(x, y) = d_*(y, z)$, we have

$$C^{-1}\mu\left([y, z]\right) \le \mu\left([x, y]\right) \le C\mu\left([y, z]\right). \tag{3.54}$$

Choose two shortest words ω and τ such that

$$K_\omega \subseteq [x, y] \text{ and } K_\tau \subseteq [y, z].$$

We claim that there exists a universal integer $k \geq 0$ (depending only on ρ) such that

$$\big||\omega| - |\tau|\big| \leq k. \tag{3.55}$$

Indeed, without loss of generality, assume that $|\omega| \geq |\tau| \geq 1$, and let $\omega' \leq \omega$ such that $|\omega'| = |\tau|$, $\omega = \omega'\theta$ for some word θ (possibly $\theta = \emptyset$). Then by applying Proposition 3.7, we see that the number of words with the same length $|\tau|$ and lying between y and z is at most 4. See Figure 5 for the worst case when $\omega' = \omega$. More precisely, the cell $K_{\omega'}$ can be connected to cell K_τ by at most four $|\tau|$-cells.

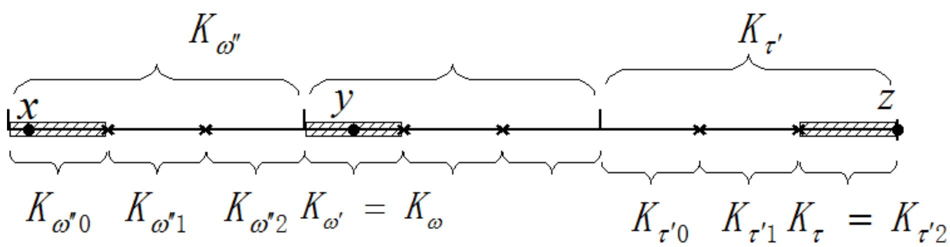

Figure 5. Positions of three points x, y, z when $\omega' = \omega$.

Thus, by repeatedly using Lemma 3.2, we have

$$2^{-5}\mu(K_\tau) \leq \mu(K_{\omega'}) \leq 2^5\mu(K_\tau), \tag{3.56}$$

and repeatedly using (3.46) yields

$$r_{\omega'} \leq \rho^{-5} r_\tau. \tag{3.57}$$

By (3.50), we see that

$$\mu(K_\omega)^\alpha r_\omega^\alpha \asymp d_*(x, y) = d_*(y, z) \asymp \mu(K_\tau)^\alpha r_\tau^\alpha.$$

Combining this with (3.56) and the inclusion $K_\omega \subseteq K_{\omega'}$, we see that there exists some $C_0 > 0$ such that

$$\mu(K_\tau)r_\tau \leq C_0\mu(K_\omega)r_\omega \leq C_0\mu(K_{\omega'})r_\omega \leq C_0 2^5\mu(K_\tau)r_\omega, \tag{3.58}$$

and after dividing by $\mu(K_\tau)$,

$$r_\tau \leq C_0 2^5 r_\omega.$$

Combining this with (3.57), we have

$$r_{\omega'} \leq C_0(2\rho^{-1})^5 r_\omega \leq C_0(2\rho^{-1})^5 r_{\omega'}\rho^{2(|\omega|-|\tau|)},$$

where we have used the following fact from (3.5):

$$r_\omega = r_{\omega'}r_\theta \leq r_{\omega'}\rho^{2|\theta|} = r_{\omega'}\rho^{2(|\omega|-|\omega'|)} = r_{\omega'}\rho^{2(|\omega|-|\tau|)}.$$

This shows that $|\omega| - |\tau|$ is bounded by a universal integer k, proving our claim.

Note that by (3.43) and Lemma 3.2,

$$\mu(K_{\theta i}) \leq \mu(K_\theta) \leq 5\mu(K_{\theta 1}) \leq 10\mu(K_{\theta i}) \tag{3.59}$$

for any word θ and any $i \in \{0, 1, 2\}$. From this and using (3.55), we have

$$\begin{aligned} \mu(K_\omega) &\le \mu(K_{\omega'}) \le 10\mu(K_{\omega'\theta_1}) \le 10^2\mu(K_{\omega'\theta_1\theta_2}) \le \cdots \\ &\le 10^k\mu(K_{\omega'\theta_1\cdots\theta_k}) = 10^k\mu(K_\omega), \end{aligned}$$

which together with (3.56) gives that

$$\mu(K_\omega) \asymp \mu(K_\tau).$$

Finally, from this and using (3.48), we see that

$$\mu([x, y]) \asymp \mu(K_\omega) \asymp \mu(K_\tau) \asymp \mu([y, z]),$$

thus proving (3.54). The lemma follows. □

Lemma 3.10. *Condition (1.15) with $\beta = 1/\alpha$ is satisfied.*

Proof. For any $x \in [0, 1]$, any small $0 < r < d_*(K)/2$ and any integer $\ell \ge 1$, choose z and y_ℓ in $[0, 1]$ such that

$$d_*(x, z) = r = \ell d_*(x, y_\ell). \tag{3.60}$$

Then, letting $\eta = 1/\ell$, we have

$$V(x, r) \asymp \mu([x, z]) \quad \text{and} \quad V(x, \eta r) \asymp \mu([x, y_\ell]).$$

From this and using (1.13), we have

$$\begin{aligned} \frac{V(x, r)}{r^\beta} &\asymp \frac{\mu([x, z])}{r^\beta} \asymp \frac{d_*(x, z)^\beta}{|x - z| r^\beta} = \frac{1}{|x - z|}, \\ \frac{V(x, \eta r)}{(\eta r)^\beta} &\asymp \frac{\mu([x, y_\ell])}{(\eta r)^\beta} \asymp \frac{d_*(x, y_\ell)^\beta}{|x - y_\ell|(\eta r)^\beta} = \frac{1}{|x - y_\ell|}. \end{aligned}$$

Thus, in order to prove (1.15), it suffices to show that

$$\lim_{\ell \to \infty} \frac{|x - y_\ell|}{|x - z|} = 0, \tag{3.61}$$

where the limit is independent of the choice of x, r and y_ℓ, z.

We may assume that $x < y_\ell < z$; the other cases $y_\ell < x < z$, $z < y_\ell < x$ and $z < x < y_\ell$ are similar. Choose a shortest word $\omega := \omega_\ell$ such that

$$K_\omega \subseteq [x, y_\ell].$$

Then by (3.49) we have

$$|x - y_\ell| \asymp r_\omega. \tag{3.62}$$

Consider a chain of $k + 1$ neighbouring words starting from ω and with length $|\omega|$. There exists a constant $C_0 > 1$ such that the total distance of these cells is not more than

$$d_*(K_\omega)(1 + C_0 + \cdots + C_0^k). \tag{3.63}$$

Choose k to be the largest integer such that

$$1 + C_0 + \cdots + C_0^k \le \ell, \tag{3.64}$$

and thus

$$k \asymp \log \ell. \tag{3.65}$$

From (3.60), (3.63) and (3.64), we see that the chain of $k+1$ neighbouring cells are contained in $[x,z]$, see Figure 6. Set $k = 3^{N+1}$ for an integer $N \geq 0$. We can find a word τ

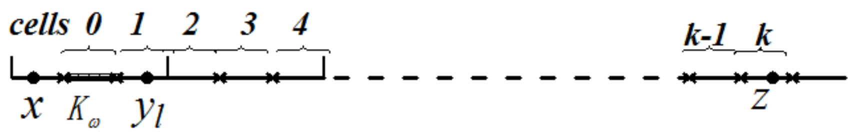

FIGURE 6. A chain of $k+1$ cells with length $|\omega|$ are contained in $[x,z]$.

such that

$$|\omega| - N = |\tau| > 0 \text{ and } K_\tau \subseteq [x,z], \tag{3.66}$$

that is, the points x and z can be separated apart by at least one $(|\omega|-N)$-cell. In fact, if $[x,z]$ does not contain any $(|\omega|-N)$-cell, then the number of $(|\omega|-N)$-cells outside $[x,z]$ is at least $3^{|\omega|-N}-2$; this is because the total number of $(|\omega|-N)$-cells on K is $3^{|\omega|-N}$ whilst the number of $(|\omega|-N)$-cells covering $[x,z]$ is at most 2. As each $(|\omega|-N)$-cell contains 3^N cells of length $|\omega|$, the total number of cells with length $|\omega|$ outside $[x,z]$ is thus at least

$$\left(3^{|\omega|-N}-2\right)\cdot 3^N = 3^{|\omega|} - 2\cdot 3^N.$$

However, inside $[x,z]$ there are $k+1 = 3^{N+1}+1$ cells with length $|\omega|$ from above. Thus, after summing up, the total number of $|\omega|$-cells on K is at least

$$\left(3^{|\omega|} - 2\cdot 3^N\right) + \left(3^{N+1}+1\right) = 3^{|\omega|} + 3^N + 1 > 3^{|\omega|},$$

a contradiction, since the number of $|\omega|$-cells on K is $3^{|\omega|}$.

Let $\omega' < \omega$ such that

$$|\omega'| = |\tau|.$$

Then ω' and τ are neighbouring words or $\omega' = \tau$. We obtain

$$\begin{aligned}
|x-z| &\geq r_\tau \quad (\text{since } K_\tau \subseteq [x,z] \quad \text{by (3.66)}) \\
&\geq \rho r_{\omega'} \qquad (\text{by (3.46)}) \\
&\geq \rho\cdot\rho^{-2N} r_\omega \\
&\geq C^{-1}\rho^{-2N}|x-y_\ell| \qquad (\text{by (3.62)})
\end{aligned} \tag{3.67}$$

for some constant C depending only on ρ, where (3.67) follows by setting $\omega = \omega'\theta$,

$$r_\omega = r_{\omega'}\cdot r_\theta \leq r_{\omega'}\left(\rho^2\right)^{|\theta|} = r_{\omega'}\left(\rho^2\right)^N$$

by virtue of (3.5) and (3.66). Therefore, using (3.65), we have

$$\frac{|x-y_\ell|}{|x-z|} \leq C\rho^{2N} = C\rho^{2\log_3 k} \leq C(\log \ell)^{2\log_3 \rho} \to 0$$

as $\ell \to \infty$, completing the proof. □

Proof of Theorem 1.2. From above, conditions (1.10)–(1.15) are all satisfied with $\beta = 1/\alpha$. By Theorem 1.1 we see that Theorem 1.2 is true. □

4. m-FOLD CONVOLUTION OF CANTOR-TYPE MEASURES

Let $\{S_i\}_{i=0}^{m}$ and μ be defined as in (1.19) and (1.21) respectively, with $m \geq 3$ being an odd integer, and let $K := [0, m]$. In this section we introduce a new metric d_* on K, and again show that conditions (1.10)–(1.15) are all satisfied. Therefore, the conclusion in Theorem 1.3 is true.

Let $\{T_i\}_{i=0}^{m-1}$ be the auxiliary IFS defined by

$$T_i(x) = \frac{1}{m}x + i, \quad i = 0, 1, \ldots, m-1, \tag{4.1}$$

see Figure 7 for $m = 3$.

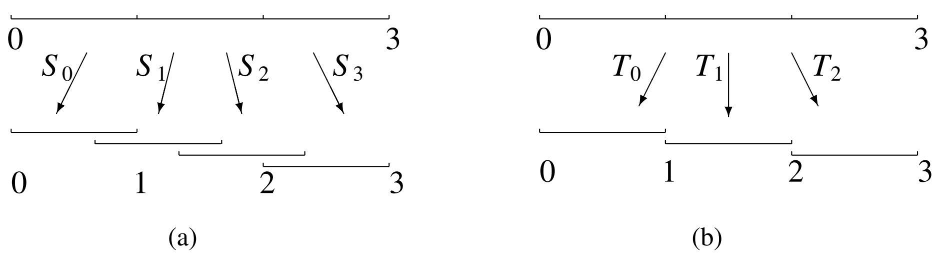

FIGURE 7. (a) The IFS $\{S_i\}_{i=0}^{3}$ has overlaps. (b) The auxiliary IFS $\{T_i\}_{i=1}^{3}$ does not have overlaps.

Proposition 4.1. *Let $\{S_i\}_{i=0}^{m}$, μ, and $\{T_i\}_{i=0}^{m-1}$ be defined as in (1.19), (1.21), and (4.1) respectively. Then for any $u \in \mathcal{F}$,*

$$\mathcal{E}(u) = \sum_{i=0}^{m-1} m\,\mathcal{E}(u \circ T_i). \tag{4.2}$$

Consequently, condition (1.10) is true with each $r_i = \frac{1}{m}$.

Proof. For $0 \leq i \leq m-1$, we have

$$\mathcal{E}(u \circ T_i) = \int_0^m ((u \circ T_i)'(x))^2\,dx = \frac{1}{m^2}\int_0^m (u'(x/m + i))^2\,dx = \frac{1}{m}\int_i^{i+1} (u'(y))^2\,dy.$$

Hence,

$$\sum_{i=0}^{m-1} \mathcal{E}(u \circ T_i) = \frac{1}{m}\sum_{i=0}^{m-1}\int_i^{i+1} (u')^2\,dy = \frac{1}{m}\mathcal{E}(u),$$

thus proving (4.2). $\square$

Let

$$\mathcal{J} = \{0, 1, \ldots, m-1\}, \qquad \mathcal{J}_1 := \{1, \ldots, m-2\}, \qquad \mathcal{J}_0 := \{0, m-1\},$$

and for each $n \in \mathbb{N}$, let

$$\mathcal{J}^n := \{0, 1, \dots, m-1\}^n, \quad \mathcal{J}_0^n := \{0, m-1\}^n, \quad \mathcal{J}^* := \bigcup_{k=0}^{\infty} \mathcal{J}^n, \quad \mathcal{J}_0^* := \bigcup_{k=0}^{\infty} \mathcal{J}_0^n,$$

where $\mathcal{J}^0$ and $\mathcal{J}_0^0$ are defined to be the empty word as before. It is shown in [27] that μ satisfies a family of second-order identities with respect to the IFS $\{T_i\}_{i=0}^{m-1}$. More precisely, for $i, j, k \in \mathcal{J}$, define

$$a_{j,k}^{(i)} := \begin{cases} w_\ell, & \text{if } \exists \ell (0 \le \ell \le m) \text{ such that } i + mj - (m-1)\ell = k \\ 0, & \text{otherwise,} \end{cases}$$

where $\{w_\ell\}_{\ell=0}^m$ is given by (1.20), and let M_i, $0 \le i \le m-1$, be the matrix

$$M_i := \left[a_{p-1,q-1}^{(i)} \right]_{p,q=1}^m . \tag{4.3}$$

For example, for $m = 3$,

$$M_0 = \begin{bmatrix} w_0 & 0 & 0 \\ 0 & w_1 & 0 \\ w_3 & 0 & w_2 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & w_0 & 0 \\ w_2 & 0 & w_1 \\ 0 & w_3 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} w_1 & 0 & w_0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix}.$$

In general,

$$M_0 = \begin{bmatrix} w_0 & 0 & \cdots & 0 & 0 \\ 0 & w_1 & 0 & \ddots & 0 \\ \vdots & 0 & w_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ w_m & \cdots & 0 & 0 & w_{m-1} \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 0 & w_0 & 0 & \cdots & 0 \\ \vdots & 0 & w_1 & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & 0 \\ w_{m-1} & 0 & \cdots & 0 & w_{m-2} \\ 0 & w_m & 0 & \cdots & 0 \end{bmatrix}, \dots$$

$$M_{m-2} = \begin{bmatrix} 0 & \cdots & 0 & w_0 & 0 \\ w_2 & 0 & \cdots & 0 & w_1 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & w_{m-1} & \ddots & 0 \\ 0 & \cdots & 0 & w_m & 0 \end{bmatrix}, \qquad M_{m-1} = \begin{bmatrix} w_1 & 0 & \cdots & 0 & w_0 \\ 0 & w_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & w_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & w_m \end{bmatrix}. \tag{4.4}$$

Then for all $A \subseteq [0, m]$,

$$\begin{bmatrix} \mu(T_0 T_i A) \\ \vdots \\ \mu(T_{m-1} T_i A) \end{bmatrix} = M_i \begin{bmatrix} \mu(T_0 A) \\ \vdots \\ \mu(T_{m-1} A) \end{bmatrix}, \quad i \in \mathcal{J}. \tag{4.5}$$

For $\omega = \omega_1 \cdots \omega_\ell \in \mathcal{J}^\ell$, we use the notation

$$K_\omega := T_{\omega_1} \circ \cdots \circ T_{\omega_\ell}(K)$$

as before. For $i \in \mathcal{J}$, let $\mathbf{e}_i$ denote the unit vector in $\mathbb{R}^m$ whose $(i+1)$-st coordinate is 1. Applying (4.5) repeatedly yields

$$\mu(T_\omega A) = \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_\ell} \begin{bmatrix} \mu(T_0 A) \\ \vdots \\ \mu(T_{m-1} A) \end{bmatrix} \quad \text{for all } A \subseteq [0, m]. \tag{4.6}$$

Proposition 4.2. *Condition (1.11) is true.*

Proof. Assume, without loss of generality, that K_ω is on the left of K_τ. Then exactly one of the following relationships holds for $i = 0, 1, \dots, m-2$: (θ could be empty)

$$\omega = \theta i \underbrace{(m-1)\cdots(m-1)}_{k} \qquad \text{and} \qquad \tau = \theta(i+1)\underbrace{0\cdots 0}_{k}.$$

Assume that $\theta = \theta_1 \cdots \theta_\ell$ for $\ell \geq 1$ and $i = 0$; other cases are similar. Then

$$\mu(K_\omega) = \mathbf{e}_{\theta_1} M_{\theta_2} \cdots M_{\theta_\ell} M_0 M_{m-1}^k \begin{bmatrix} \mu(T_0 K) \\ \vdots \\ \mu(T_{m-1}K) \end{bmatrix} =: \begin{bmatrix} a_0 & \cdots & a_{m-1} \end{bmatrix} M_0 M_{m-1}^k \boldsymbol{\mu}^t, \tag{4.7}$$

where $\begin{bmatrix} a_0 & \cdots & a_{m-1} \end{bmatrix} = \mathbf{e}_{\theta_1} M_{\theta_2} \cdots M_{\theta_\ell}$ and $\boldsymbol{\mu} = \begin{bmatrix} \mu(T_0 K) & \cdots & \mu(T_{m-1}K) \end{bmatrix}$.
Similarly,

$$\mu(K_\tau) =: \begin{bmatrix} a_0 & \cdots & a_{m-1} \end{bmatrix} M_1 M_0^k \boldsymbol{\mu}^t. \tag{4.8}$$

A direct calculation shows

$$M_{m-1}^k = \begin{bmatrix} w_1^k & 0 & \cdots & 0 & w_0 \sum_{i=1}^k w_1^{k-i} w_m^{i-1} \\ 0 & w_2^k & 0 & \ddots & 0 \\ 0 & 0 & w_3^k & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & w_m^k \end{bmatrix} \asymp \begin{bmatrix} w_1^k & 0 & \cdots & 0 & w_1^k \\ 0 & w_2^k & 0 & \ddots & 0 \\ 0 & 0 & w_3^k & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & w_m^k \end{bmatrix},$$

$$M_0^k = \begin{bmatrix} w_0^k & 0 & \dots & 0 & 0 \\ 0 & w_1^k & 0 & \ddots & 0 \\ 0 & 0 & w_2^k & \dots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ w_m \sum_{i=1}^k w_0^{k-i} w_{m-1}^{i-1} & 0 & \dots & 0 & w_{m-1}^k \end{bmatrix} \asymp \begin{bmatrix} w_0^k & 0 & \dots & 0 & 0 \\ 0 & w_1^k & 0 & \ddots & 0 \\ 0 & 0 & w_2^k & \dots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ w_{m-1}^k & 0 & \dots & 0 & w_{m-1}^k \end{bmatrix}.$$

Thus we obtain

$$M_0 M_{m-1}^k \boldsymbol{\mu}^t \asymp \begin{bmatrix} w_1^k & 0 & \cdots & 0 & w_1^k \\ 0 & w_2^k & 0 & \ddots & 0 \\ 0 & 0 & w_3^k & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ w_1^k & \cdots & 0 & 0 & w_1^k \end{bmatrix} \boldsymbol{\mu}^t \asymp \begin{bmatrix} w_1^k \\ w_2^k \\ \vdots \\ w_{m-1}^k \\ w_1^k \end{bmatrix}, \tag{4.9}$$

and similarly, we have

$$M_1 M_0^k \boldsymbol{\mu}^t \asymp \begin{bmatrix} 0 & w_1^k & 0 & \cdots & 0 \\ 0 & 0 & w_2^k & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ w_{m-1}^k & 0 & \ddots & 0 & w_{m-1}^k \\ 0 & w_1^k & \cdots & 0 & 0 \end{bmatrix} \boldsymbol{\mu}^t \asymp \begin{bmatrix} w_1^k \\ w_2^k \\ \vdots \\ w_{m-1}^k \\ w_1^k \end{bmatrix}. \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.7) and (4.8) separately, we obtain

$$\mu(K_\omega) \asymp \begin{bmatrix} a_0 & \cdots & a_{m-1} \end{bmatrix} \cdot \begin{bmatrix} w_1^k \\ w_2^k \\ \vdots \\ w_{m-1}^k \\ w_1^k \end{bmatrix} \asymp \mu(K_\tau).$$

The assertion follows. □

Recall that for $-(m-1) \le k \le m-1$, the *k-diagonal* of an $m \times m$ matrix $A = (a_{\ell j})$ consists of the entries $j = \ell + k$. The main diagonal is the 0-diagonal. We say that A is of *Type* 0 (or *Type* $m-1$) if all its k-diagonals are zero, except possibly for $k = 0$ or $k = \pm(m-1)$, and if is of *Type* i $(1 \le i \le m-2)$ if all its k-diagonals are zero, except possibly for $k = i$ or $k = -i'$, where

$$i' = m - 1 - i. \tag{4.11}$$

An entry $a_{\ell j}$ of A belongs to the k-diagonal $(-(m-1) \le k \le m-1)$ if and only $j - \ell = k$. Note that for each $i \in \{0, 1, \dots, m-1\}$, the matrix M_i defined in (4.3) is of Type i. Denote the transpose of a matrix A by A^t.

Proposition 4.3. *Let A be an $m \times m$ matrix of Type i for $0 \le i \le m-1$. Then*

(a) *A^t, the transpose of A, is of Type i';*
(b) *the $(m-i)$-th row of A is of the form $[*, 0, \dots, 0, *]$;*
(c) *the $(i+1)$-st column of A is of the form $[*, 0, \dots, 0, *]^t$.*

Proof. These are obvious if $i = 0$ or $i = m-1$. So we assume that $1 \le i < m-2$.

(a) The possibly nonzero entries of A are

(i) the i-diagonal: $(1, i+1),\ (2, i+2),\ \dots,\ (m-i, m)$, and
(ii) the $-i'$-diagonal: $(m-i, 1),\ (m-i+1, 2),\ \dots,\ (m, i+1)$.

Hence, the possibly nonzero entries of A^t are:

(i') $(1, m-i),\ (2, m-i+1),\ \dots,\ (i+1, m)$, and
(ii') $(i+1, 1),\ (i+2, 2),\ \dots,\ (m, m-i)$,

which are respectively the i' and $-i = -(i')'$ diagonals of A^t. Hence A^t is of Type i'.

(b) By (i) and (ii), the entries of the two diagonals belong to the same row if and only if the row number is $m-i$ and the entries are $(m-i, m)$ and $(m-i, 1)$, proving (b).

Finally, (c) follows from (b) by taking transpose and using (a). □

We now study products of such matrices.

Proposition 4.4. *Let A, B be two $m \times m$ matrices of Types i and j, respectively. Then AB is of Type $i + j \mod (m-1)$.*

Proof. Case 1. $i = 0$ or $j = 0$. It is obvious that if both A and B are of Type 0, then so is AB. Now assume that $i = 0$ and $1 \le j \le m-2$. Using (i) and (ii) in the proof of Proposition 4.3, we have:

(1) the nonzero entries corresponding to multiplication of A and the j-diagonal of B are $(\ell, j+\ell)$, $1 \le \ell \le m-j$, which belong to the j-diagonal, and $(m, j+1)$, which belongs to the $(-j')$ -diagonal;
(2) the nonzero entries corresponding to multiplying A with the $(-j')$-diagonal of B are $(m-j+\ell, \ell+1)$, $0 \le \ell \le j$, which belong to the $(-j')$-diagonal, and $(1, j+1)$, which belongs to the j-diagonal.

Hence AB is of Type j.

Next, we assume $1 \le i \le m-2$ and $j = 0$. Then $AB = (B^tA^t)^t$. By Proposition 4.3(a), B^t is of Types 0 and A^t is of Type i'. By what we have just proved above, B^tA^t if of Type i'. By Proposition 4.3(a) again, we see that AB is of Type i.

Case 2. $1 \le i, j \le m-2$. The i-diagonal of A is: $(1, i+1), (2, i+2), \ldots, (m-i, m)$; the $(-i')$-diagonal of A is: $(m-i, 1), (m-i+1, 2), \ldots, (m, i+1)$; the j-diagonal of B is: $(1, j+1), (2, j+2), \ldots, (m-j, m)$; the $(-j')$-diagonal of B is: $(m-j, 1), (m-j+1, 2), \ldots, (m, j+1)$. We divide the proof into four subcases.

Subcase I. i-diagonal of A times j-diagonal of B. An entry corresponding to such a product can be nonzero only if it is of the form:

$$(\ell, j+k), \quad \text{where } i+\ell = k, \quad 1 \le \ell \le m-i \quad \text{and} \quad 1 \le k \le m-j.$$

If $i + j \le m-1$, it is of the form $(\ell, i+j+\ell)$, $1 \le \ell \le m-i-j$, which lies in the $(i+j)$-diagonal. If $i+j \le m$, then $i+\ell = k \le m-j$ would imply that $i+j+\ell \le m$, which is impossible.

Subcase II. i-diagonal of A times $(-j')$ -diagonal of B. An entry in the product is nonzero only if it is of the form:

$$(\ell, k+1), \quad \text{where } i+\ell = m-j+k, \quad 1 \le \ell \le m-i, \quad 0 \le k \le j.$$

Consider the case $i+j \le m-1$. Equivalently, $m-\ell+k \le m-1$, or $k-\ell \ge 1$. The entry is of the form $(m-1-(i+j)+k+1, k+1)$, $0 \le k \le j$, which lies on the $-(i+j)'$-diagonal. Next, we consider the case $i+j \ge m$. The entry is of the form $(\ell, i+j-(m-1)+\ell)$, which lies on the $(i+j-(m-1))$-diagonal.

Subcase III. $(-i')$-diagonal of A times j-diagonal of B. An entry is nonzero only if it is of the form:

$$(m-i+\ell-1, j+k), \quad \text{where } \ell = k, \quad 1 \le \ell \le i+1, \quad 1 \le k \le m-j.$$

If $i+j \le m-1$, then the entry is of the form $(m-1-(i+j)+j+k, j+k)$, $1 \le k \le \min\{i+1, m-j\}$, which is on the $-(i+j)'$-diagonal. If $i+j \ge m$, then, as $j+k-(m-i+\ell-1) = i+j-(m-1)$, the entry is on the $(i+j-(m-1))$-diagonal.

Subcase IV. $(-i')$-diagonal of A times $(-j')$-diagonal of B. An entry is nonzero only if it is of the form:

$$(m-i+\ell-1, k+1), \quad \text{where } \ell = m-j+k, \quad 1 \le \ell \le i+1, \quad 0 \le k \le j.$$

First, assume $i+j \le m-1$. Since $(k+1)-(m-i+\ell-1) = i+j-2(m-1) < 0$, the entry $(m-i+\ell, k+1)$ does not belong to the upper triangle of AB. On the other hand,

$$m-i-\ell-1-(k+1) = 2(m-1)-(i+j) \ge m-1.$$

Hence $(m-i+\ell-1, k+1)$ can be a nonzero entry only if $i+j = m-1$ and the nonzero entry is $(m, 0)$, which belongs to the $-(i+j)'$-diagonal.

Next, assume $i+j \ge m$. Then the entry is

$$(m-i+m-j+k-1, k+1) = (m-1-(i+j-m+1)+k+1, k+1),$$

which belongs to the $-(i+j-m+1)'$-diagonal of AB.

To summarize, we see that if $i+j \le m-1$, then the nonzero entries of AB are along the $(i+j)$-diagonal or the $-(i+j)'$-diagonal. If $i+j \ge m$, then the nonzero entries of AB are along the $(i+j-m+1)$ -diagonal or the $-(i+j-m+1)'$-diagonal. In the former case, AB is of Type $i+j$; in the latter case, it is of Type $i+j-m+1$. This completes the proof. □

Proposition 4.5. *Let $i, \ell \in \mathcal{J}$ and let A be an $m \times m$ matrix of Type ℓ.*

(a) *If $i+\ell \equiv 0 \mod (m-1)$, then $\mathbf{e}_i A$ is of the form $[*, 0, \dots, 0, *]$.*
(b) *If $i+\ell \equiv k \mod (m-1)$, where $1 \le k \le m-2$, then $\mathbf{e}_i A$ is of the form $a\mathbf{e}_k$ for some $a \ge 0$.*

Proof. (a) Since A is of Type ℓ, by Proposition 4.3, the $(m-\ell)$-th row of A is of the form $[*, 0, \dots, 0, *]$. It follows from $i+\ell \equiv 0 \mod (m-1)$ that $i = m-1-\ell$ or $i = 2(m-1)-\ell$. In the former case, $\mathbf{e}_i = \mathbf{e}_{m-\ell-1}$ and the assertion follows. In the latter case $i = \ell = m-1$ and thus $\mathbf{e}_i = \mathbf{e}_{m-1}$ and A is of Type 0. Again the assertion follows.

(b) Let $i+\ell \equiv k \mod (m-1)$ and $1 \le k \le m-2$. Since the unique nonzero entry of A in row $(i+1)$ falls in the column $(k+1)$,

$$\mathbf{e}_i A = [0, \dots, 0, *, 0 \dots, 0] = a\mathbf{e}_k$$

for some $a \ge 0$. □

For $i \in \{0, m-1\}$, let $\widetilde{M}_i$ be the matrix formed from M_i by keeping its first and last rows and assigning 0 to all other entries. For $i \in \mathcal{J}_1$, let $\widetilde{M}_i$ denote the matrix formed from M_i by keeping its $(m-i)$−th row and assigning 0 to all other entries. For $J = (j_1, \dots, j_k)$, where $k \ge 0$ and $j_\ell \in \{0, m-1\}$, define (see [27])

$$c_{i,J} := [w_{i+1}, \mathbf{0}, w_i] M_J \begin{bmatrix} w_0 \\ \mathbf{0} \\ w_m \end{bmatrix} = [w_{i+1}, \mathbf{0}, w_i] \widetilde{M}_J \begin{bmatrix} w_0 \\ \mathbf{0} \\ w_m \end{bmatrix} = \mathbf{e}_i M_{i'} \widetilde{M}_J \begin{bmatrix} w_0 \\ \mathbf{0} \\ w_m \end{bmatrix}, \; i \in \mathcal{J}_1, \qquad (4.12)$$

where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^{m-2}$.

Define a distance d_* on K as follows. For any $x, y \in [0, m]$ with $x < y$, let

$$\mathcal{W}(x,y) := \Big\{\omega = \omega_1 \cdots \omega_n \in \mathcal{J}^n : \omega_n \in \mathcal{J}_1, K_\omega \subseteq [x,y], \sum_{i=1}^n \omega_i \equiv 0 \mod (m-1) \text{ and } \omega \text{ is a father}\Big\},$$

where the notion "$\omega = \omega_1 \cdots \omega_n$ is a father" means that none of the proper *ancestors* (or *prefixes*) $\omega_1 \cdots \omega_k$ ($k < n$) satisfies all of the following conditions:

- $\omega_k \in \mathcal{J}_1$;
- $K_{\omega_1 \cdots \omega_k} \subseteq [x,y]$;
- $\sum_{i=1}^k \omega_i \equiv 0 \mod (m-1)$.

For example, let $m = 3$. If $[x,y] = [0,1] = K_0$, then

$$\mathcal{W}(x,y) = \{011, 0011, 0211, 0101, 0121, \dots\}.$$

If $[x,y] = \left[\frac{4}{3}, \frac{5}{3}\right] = K_{11}$, then $\mathcal{W}(x,y) = \{11\}$.

Similar to Proposition 3.4, we have

Proposition 4.6. *For any $0 \le x < y \le m$ and any distinct $\omega, \tau \in \mathcal{W}(x,y)$, we have*

$$K_\omega \cap K_\tau = \emptyset. \tag{4.13}$$

Define a symbol set $\mathcal{S}$ by

$$\mathcal{S} = \Big\{\omega_1 \cdots \omega_n : \omega_1 \in \mathcal{J}_1, \omega_n \in \mathcal{J}_1, \sum_{i=1}^n \omega_i \equiv 0 \mod (m-1), \text{ and } \sum_{i=1}^k \omega_i \not\equiv 0 \mod (m-1) \text{ for each } k = 1, \dots, n-1\Big\}.$$

Now define $d_*(x,y) := 0$ if $x = y$, and if $x < y$, define

$$d_*(x,y) := \sum_{\omega \in \mathcal{W}(x,y)} \sum_{J \in \mathcal{J}_0^*, \sigma \in \mathcal{S}} \left(r_{\omega J \sigma} \mu(K_{\omega J \sigma})\right)^\alpha, \tag{4.14}$$

where α is the unique solution of the equation

$$\frac{1}{m^\alpha} \sum_{i=1}^{m-1} w_i^\alpha + \sum_{J \in \mathcal{J}_0^*} \frac{1}{m^{(|J|+2)\alpha}} \sum_{i=1}^{m-2} c_{i,J}^\alpha = 1, \tag{4.15}$$

where $c_{i,J}$ is given by (4.12). We remark that 2α is the spectral dimension of the Laplacian $-\Delta_\mu$ defined by μ [28]. For example if $m = 3$, then $\alpha \approx$0.4985 < 0.5 (this value is close to but strictly less than 0.5).

Proposition 4.7. *Condition (1.12) is true with d_* defined above.*

Proof. Following the same spirit in the proof of Proposition 3.5, we need only to show that for any $\omega = \omega_1 \cdots \omega_n$ with $\omega_n \in \mathcal{J}_1$, $\sum_{i=1}^n \omega_i \equiv 0 \mod (m-1)$,

$$\sum_{J \in \mathcal{J}_0^*, \sigma \in \mathcal{S}} \left(r_{\omega J \sigma} \mu(K_{\omega J \sigma})\right)^\alpha = \sum_{J \in \mathcal{J}_0^*, \sigma \in \mathcal{S}} \sum_{J' \in \mathcal{J}_0^*, \sigma' \in \mathcal{S}} \left(r_{\omega J \sigma J' \sigma'} \mu(K_{\omega J \sigma J' \sigma'})\right)^\alpha. \tag{4.16}$$

Indeed, assume that (4.16) is true. Let $x, y, z \in K$ with $x < y < z$. Since $\mathcal{W}(x, y) \cap \mathcal{W}(y, z) = \emptyset$, by the definition of d_*, we see that

$$d_*(x,y) + d_*(y,z) = \sum_{\omega \in \mathcal{W}(x,y) \cup \mathcal{W}(y,z)} \sum_{J \in \mathcal{J}_0^*, \sigma \in \mathcal{S}} (r_{\omega J\sigma}\mu(K_{\omega J\sigma}))^\alpha, \tag{4.17}$$

and that

$$d_*(x,z) = \sum_{\omega \in \mathcal{W}(x,z)} \sum_{J \in \mathcal{J}_0^*, \sigma \in \mathcal{S}} (r_{\omega J\sigma}\mu(K_{\omega J\sigma}))^\alpha. \tag{4.18}$$

Observing that each word in $\mathcal{W}(x, y) \cup \mathcal{W}(y, z)$ either belongs to $\mathcal{W}(x, z)$ or is an offspring of some word in $\mathcal{W}(x, z)$, and repeatedly using (4.16) to the words in $\mathcal{W}(x, z)$, we obtain

$$d_*(x,y) + d_*(y,z) \le d_*(x,z). \tag{4.19}$$

For any $\varepsilon > 0$, similar to (3.36), we have

$$d_*(x,z) \le d_*(x,y) + d_*(y,z) + \varepsilon.$$

Since ε is arbitrary, we see

$$d_*(x,z) \le d_*(x,y) + d_*(y,z). \tag{4.20}$$

We conclude from (4.19) and (4.20) that

$$d_*(x,z) = d_*(x,y) + d_*(y,z),$$

thus proving that condition (1.12) is holds with this d_*.

We now turn to show that (4.16) is true. Indeed, by (4.6) with $A = K$,

$$\mu(K_{\omega J\sigma}) = \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} M_{\omega_n} \cdot M_J M_\sigma \cdot \begin{bmatrix} \mu(T_0 K) \\ \vdots \\ \mu(T_{m-1}K) \end{bmatrix}. \tag{4.21}$$

On the other hand, we show that

$$\mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} M_{\omega_n} = \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} \, \mathbf{e}^t_{\omega'_n} \cdot \begin{bmatrix} w_{\omega'_n+1} & \mathbf{0} & w_{\omega'_n} \end{bmatrix}, \tag{4.22}$$

where $\omega'_n = m-1-\omega_n$. In fact, applying Proposition 4.5(*b*) with $i = \omega_1$, $\ell = \omega_2+\cdots+\omega_{n-1}$, $k = i + \ell = \omega_1 + \cdots + \omega_{n-1} \equiv m - 1 - \omega_n = \omega'_n \mod (m-1)$,

$$\mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} = a\mathbf{e}_{\omega'_n} \tag{4.23}$$

for some $a \ge 0$. Recall that

$$M_{\omega_n} = \begin{bmatrix} 0 & \cdots & w_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{\omega'_n+1} & \cdots & 0 & \cdots & w_{\omega'_n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & w_m & \cdots & 0 \end{bmatrix},$$

and hence, using (4.23) twice,

$$\begin{aligned} \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} M_{\omega_n} &= a\mathbf{e}_{\omega'_n} M_{\omega_n} = a\begin{bmatrix} w_{\omega'_n+1} & \mathbf{0} & w_{\omega'_n} \end{bmatrix} \\ &= a\mathbf{e}_{\omega'_n} \cdot \mathbf{e}^t_{\omega'_n} \begin{bmatrix} w_{\omega'_n+1} & \mathbf{0} & w_{\omega'_n} \end{bmatrix} \\ &= \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} \cdot \mathbf{e}^t_{\omega'_n} \begin{bmatrix} w_{\omega'_n+1} & \mathbf{0} & w_{\omega'_n} \end{bmatrix}, \end{aligned}$$

thus showing (4.22).

We show that for any $J \in \mathcal{J}_0^*$ and any $\sigma_1 \in \mathcal{J}_1 = \{1, 2, \ldots, m-2\}$,

$$\begin{bmatrix} w_{\omega_n'+1} & \mathbf{0} & w_{\omega_n'} \end{bmatrix} M_J M_{\sigma_1} = c_{\omega_n',J} \mathbf{e}_{\sigma_1}. \tag{4.24}$$

In fact, the matrix M_J is of Type-0 by Proposition 4.4, since so is each M_{J_k} $(1 \le k \le \ell)$ if $J = J_1 \cdots J_\ell \in \mathcal{J}_0^*$. Hence

$$\begin{bmatrix} w_{\omega_n'+1} & \mathbf{0} & w_{\omega_n'} \end{bmatrix} M_J = \begin{bmatrix} w_{\omega_n'+1} & \mathbf{0} & w_{\omega_n'} \end{bmatrix} \begin{bmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{bmatrix} = \begin{bmatrix} a & \mathbf{0} & b \end{bmatrix} \tag{4.25}$$

for some numbers $a, b \ge 0$. As M_{σ_1} is of Type-σ_1, its (σ_1+1)-th column looks like $\begin{bmatrix} w_0 \\ * \\ w_m \end{bmatrix}$, and thus,

$$\begin{aligned} \begin{bmatrix} a & \mathbf{0} & b \end{bmatrix} \cdot M_{\sigma_1} &= \begin{bmatrix} 0 & \cdots & aw_0 + bw_m & \cdots & 0 \end{bmatrix} = (aw_0 + bw_m)\, \mathbf{e}_{\sigma_1} \\ &= \begin{bmatrix} a & \mathbf{0} & b \end{bmatrix} \begin{bmatrix} w_0 \\ \mathbf{0} \\ w_m \end{bmatrix} \mathbf{e}_{\sigma_1} = \begin{bmatrix} w_{\omega_n'+1} & \mathbf{0} & w_{\omega_n'} \end{bmatrix} M_J \begin{bmatrix} w_0 \\ \mathbf{0} \\ w_m \end{bmatrix} \mathbf{e}_{\sigma_1}. \end{aligned}$$

Combining this with (4.25) and (4.12), we obtain

$$\begin{aligned} \begin{bmatrix} w_{\omega_n'+1} & \mathbf{0} & w_{\omega_n'} \end{bmatrix} M_J M_{\sigma_1} &= \begin{bmatrix} a & \mathbf{0} & b \end{bmatrix} M_{\sigma_1} \\ &= \begin{bmatrix} w_{\omega_n'+1} & \mathbf{0} & w_{\omega_n'} \end{bmatrix} M_J \begin{bmatrix} w_0 \\ \mathbf{0} \\ w_m \end{bmatrix} \mathbf{e}_{\sigma_1} = c_{\omega_n',J} \mathbf{e}_{\sigma_1}, \end{aligned}$$

thus showing (4.24).

For any $\sigma = \sigma_1 \cdots \sigma_\ell$ with $\sigma_1 \in \mathcal{J}_1$, it follows by using (4.22) and (4.24) that

$$\begin{aligned} & \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} M_{\omega_n} \cdot M_J M_\sigma \\ = \;& \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} \mathbf{e}_{\omega_n'}^t \cdot \begin{bmatrix} w_{\omega_n'+1} & \mathbf{0} & w_{\omega_n'} \end{bmatrix} M_J M_{\sigma_1} \cdot M_{\sigma_2} \cdots M_{\sigma_\ell} \\ = \;& \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} \mathbf{e}_{\omega_n'}^t \cdot c_{\omega_n',J} \mathbf{e}_{\sigma_1} \cdot M_{\sigma_2} \cdots M_{\sigma_\ell}. \end{aligned}$$

From this and the fact that

$$\mathbf{e}_{\sigma_1} \cdot M_{\sigma_2} \cdots M_{\sigma_\ell} \begin{bmatrix} \mu(T_0 K) \\ \vdots \\ \mu(T_{m-1} K) \end{bmatrix} = \mu(K_\sigma) \;\; \text{(using (4.6))}$$

we obtain

$$\mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} M_{\omega_n} \cdot M_J M_\sigma \begin{bmatrix} \mu(T_0 K) \\ \vdots \\ \mu(T_{m-1} K) \end{bmatrix} = \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} \mathbf{e}_{\omega_n'}^t \cdot c_{\omega_n',J} \mu(K_\sigma). \tag{4.26}$$

Thus by (4.21), for any $J \in \mathcal{J}_0^*$ and any $\sigma \in \mathcal{J}^*$ with initial letter $\sigma_1 \in \mathcal{J}_1$, and for any $\omega = \omega_1 \cdots \omega_n$ with $\omega_n \in \mathcal{J}_1$, $\sum_{i=1}^n \omega_i \equiv 0 \mod (m-1)$,

$$\mu(K_{\omega J \sigma}) = \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} \mathbf{e}_{\omega_n'}^t \cdot c_{\omega_n',J} \mu(K_\sigma). \tag{4.27}$$

From this, we only need to prove (4.16) without ω and summation of J; that is,

$$\sum_{\sigma\in\mathcal{S}}(r_\sigma\mu(K_\sigma))^\alpha=\sum_{\sigma\in\mathcal{S}}\sum_{J'\in\mathcal{J}_0^*,\sigma'\in\mathcal{S}}(r_{\sigma J'\sigma'}\mu(K_{\sigma J'\sigma'}))^\alpha\,. \tag{4.28}$$

We first claim that for any integer $k\geq 3$ and any $\theta\in\mathcal{J}^*$,

$$\sum_{\sigma\in\mathcal{S},|\sigma|=k}\mu(K_{\sigma\theta})^\alpha=\sum_{j=1}^{m-1}w_j^\alpha\cdot\sum_{\sigma\in\mathcal{S},|\sigma|=k-1}\mu(K_{\sigma\theta})^\alpha, \tag{4.29}$$

Indeed, take any $\sigma=\sigma_1\sigma_2\cdots\sigma_{k-1}\in\mathcal{S}$ with $|\sigma|=k-1$. Let $\mathcal{S}_\sigma$ be a collection of $m-1$ *words* with length k which are formed by replacing σ_1 in σ by one of the elements

$$\begin{aligned}&1(\sigma_1-1),2(\sigma_1-2),\dots,(\sigma_1-1)1,\sigma_10,\\&\sigma_1(m-1),(\sigma_1+1)(m-2),\dots,(m-2)(\sigma_1+1),\end{aligned}$$

whilst keeping the remaining symbols $\sigma_2\cdots\sigma_{k-1}$ unchanged. It is not hard to see that

$$\bigcup_{\sigma\in\mathcal{S},|\sigma|=k-1}\mathcal{S}_\sigma=\{\sigma:\sigma\in\mathcal{S},|\sigma|=k\}. \tag{4.30}$$

We first look at the element $1(\sigma_1-1)\sigma_2\cdots\sigma_{k-1}$ in $\mathcal{S}_\sigma$. By (4.6), we have

$$\begin{aligned}\mu(K_{1(\sigma_1-1)\sigma_2\cdots\sigma_{k-1}\theta})&=\mathbf{e}_1M_{\sigma_1-1}M_{\sigma_2}\cdots M_{\sigma_{k-1}}M_\theta\begin{bmatrix}\mu(T_0K)\\\vdots\\\mu(T_{m-1}K)\end{bmatrix}\\&=w_1\mathbf{e}_{\sigma_1}M_{\sigma_2}\cdots M_{\sigma_{k-1}}M_\theta\begin{bmatrix}\mu(T_0K)\\\vdots\\\mu(T_{m-1}K)\end{bmatrix}=w_1\mu(K_{\sigma\theta}).\end{aligned}$$

We similarly treat the other elements $\tau\in\mathcal{S}_\sigma$. Raising to the power α and then summing up, we obtain that

$$\sum_{\tau\in\mathcal{S}_\sigma}\mu(K_{\tau\theta})^\alpha=\sum_{j=1}^{m-1}w_j^\alpha\cdot\mu(K_{\sigma\theta})^\alpha.$$

After summing up over $\{\sigma:\sigma\in\mathcal{S},|\sigma|=k-1\}$ and using (4.30), we obtain (4.29), thus proving our claim.

From the claim, we have

$$\sum_{\sigma\in\mathcal{S},|\sigma|=k}\frac{1}{m^{|\sigma|\alpha}}\mu(K_{\sigma\theta})^\alpha=\left(\frac{1}{m^\alpha}\sum_{j=1}^{m-1}w_j^\alpha\right)^{k-2}\cdot\sum_{\sigma\in\mathcal{S},|\sigma|=2}\frac{1}{m^{|\sigma|\alpha}}\mu(K_{\sigma\theta})^\alpha. \tag{4.31}$$

From this we have

$$\begin{aligned}\sum_{\sigma\in\mathcal{S}}\frac{1}{m^{|\sigma|\alpha}}\mu(K_{\sigma\theta})^\alpha&=\sum_{k=2}^\infty\sum_{\sigma\in\mathcal{S},|\sigma|=k}\frac{1}{m^{|\sigma|\alpha}}\mu(K_{\sigma\theta})^\alpha\\&=\sum_{\sigma\in\mathcal{S},|\sigma|=2}\frac{1}{m^{|\sigma|\alpha}}\mu(K_{\sigma\theta})^\alpha\cdot\sum_{\ell=0}^\infty\Big(\frac{1}{m^\alpha}\sum_{j=1}^{m-1}w_j^\alpha\Big)^\ell\end{aligned}$$

$$= \sum_{\sigma\in\mathcal{S},|\sigma|=2} \frac{1}{m^{|\sigma|\alpha}}\mu(K_{\sigma\theta})^\alpha \cdot \left(1 - \frac{1}{m^\alpha}\sum_{j=1}^{m-1} w_j^\alpha\right)^{-1},$$

where we have used the fact that $(1/m^\alpha)\sum_{j=1}^{m-1} w_j^\alpha < 1$. By taking $\theta = \emptyset$ and using the fact that $r_\sigma = 1/m^{|\sigma|}$, we see that the left-hand side of (4.28) is

$$\sum_{\sigma\in\mathcal{S}} \frac{1}{m^{|\sigma|\alpha}}\mu(K_\sigma)^\alpha = \frac{\sum_{i=1}^{m-2}\mu(K_{ii'})^\alpha}{m^{2\alpha}} \cdot \left(1 - \frac{1}{m^\alpha}\sum_{j=1}^{m-1} w_j^\alpha\right)^{-1}. \tag{4.32}$$

On the other hand, using (4.31) with $\theta = J'\sigma'$ and summing up, we can write the right-hand side of (4.28) as

$$\begin{aligned}&\sum_{\sigma\in\mathcal{S}}\sum_{J'\in\mathcal{J}_0^*,\sigma'\in\mathcal{S}} (r_{\sigma J'\sigma'}\mu(K_{\sigma J'\sigma'}))^\alpha \\ &= \left(1 - \frac{1}{m^\alpha}\sum_{j=1}^{m-1} w_j^\alpha\right)^{-1} \sum_{\sigma\in\mathcal{S},|\sigma|=2}\sum_{J'\in\mathcal{J}_0^*,\sigma'\in\mathcal{S}} (r_{\sigma J'\sigma'}\mu(K_{\sigma J'\sigma'}))^\alpha, \end{aligned} \tag{4.33}$$

Observing that the set of all the σ with $|\sigma| = 2$ is $\{1(m-2), 2(m-3), \dots, (m-2)1\}$, and for each $\sigma = i(m-1-i) = ii'$ with $i \in \mathcal{J}_1$, we have by (4.27) that

$$\mu(K_{\sigma J'\sigma'}) = \mu(K_{ii'J'\sigma'}) = \mathbf{e}_i\mathbf{e}_{(i')'}^t \cdot c_{(i')',J'}\mu(K_{\sigma'}) = c_{i,J'}\mu(K_{\sigma'}). \tag{4.34}$$

Substituting (4.34) and (4.32) into (4.33), we have

$$\begin{aligned}&\sum_{\sigma\in\mathcal{S}}\sum_{J'\in\mathcal{J}_0^*,\sigma'\in\mathcal{S}} (r_{\sigma J'\sigma'}\mu(K_{\sigma J'\sigma'}))^\alpha \\ &= \left(1 - \frac{1}{m^\alpha}\sum_{j=1}^{m-1} w_j^\alpha\right)^{-2} \cdot \frac{\sum_{i=1}^{m-2}\mu(K_{ii'})^\alpha}{m^{2\alpha}} \cdot \sum_{i=1}^{m-2}\sum_{J'\in\mathcal{J}_0^*} \frac{1}{m^{(|J'|+2)\alpha}} c_{i,J'}^\alpha. \end{aligned} \tag{4.35}$$

Comparing (4.32) and (4.35), we see that (4.28) is equivalent to

$$1 = \left(1 - \frac{1}{m^\alpha}\sum_{j=1}^{m-1} w_j^\alpha\right)^{-1} \cdot \sum_{i=1}^{m-2}\sum_{J'\in\mathcal{J}_0^*} \frac{1}{m^{(|J'|+2)\alpha}} c_{i,J'}^\alpha,$$

which is true by using the definition of α. □

For $j \in \mathcal{J}_1$, let

$$\mathcal{S}^j = \Big\{\omega_1\cdots\omega_n : \omega_1 \in \mathcal{J}_1, \omega_n \in \mathcal{J}_1, \sum_{i=1}^n \omega_i \equiv j \ (\mathrm{mod}\ m-1),$$
$$\text{and} \sum_{i=1}^k \omega_i \not\equiv j \ (\mathrm{mod}\ m-1) \text{ for each } k = 1,\dots,n-1\Big\}.$$

Lemma 4.8. *There exists a constant $C > 0$ such that for any finite word ω,*

$$C^{-1}d_*(K_\omega) \le r_\omega^\alpha\mu(K_\omega)^\alpha \le Cd_*(K_\omega), \tag{4.36}$$

where $d_*(K_\omega)$ is the diameter of K_ω under the metric d_*.

Proof. We first claim that for any finite word ω, and any $b \in \mathcal{J} = \{0, 1, \ldots, m-1\}$,

$$d_*(K_{\omega b}) \asymp r_\omega^\alpha \mu(K_\omega)^\alpha. \tag{4.37}$$

Case 1. $\omega = \omega_1 \cdots \omega_n$ and $\sum_{k=1}^{n} \omega_k \equiv m-1-j \pmod{m-1}$ for some $j \in \mathcal{J}_1$.

If $b \neq j$, then $\mathcal{W}(K_{\omega b})$ is the set of all elements of the form $\omega b J \tau$ with $J \in \mathcal{J}_0^*$ and $\tau \in \mathcal{S}^{j-b}$. Thus by the definition of d_*, we have

$$d_*(K_{\omega b}) = \sum_{J\in\mathcal{J}_0^*, \tau\in\mathcal{S}^{j-b}} \sum_{J'\in\mathcal{J}_0^*, \sigma\in\mathcal{S}} r^\alpha_{\omega bJ\tau J'\sigma}\mu(K_{\omega bJ\tau J'\sigma})^\alpha. \tag{4.38}$$

Since $\omega bJ\tau \equiv 0 \pmod{m-1}$, we obtain that, using (4.27),

$$\mu(K_{\omega aJ\tau J'\sigma}) = \mathbf{e}_{\omega_1} \cdot M_{\omega_2} \cdots M_{\omega_n} M_b \cdot M_J \cdot M_{\tau_1} \cdots M_{\tau_{s-1}} \cdot \mathbf{e}^t_{\tau'_s} \cdot c_{\tau'_s, J'} \mu(K_\sigma). \tag{4.39}$$

On the other hand, by Proposition 4.5(b), we see that $\mathbf{e}_{\omega_1} \cdot M_{\omega_2} \cdots M_{\omega_n} M_b \cdot M_J \cdot M_{\tau_1} \cdots M_{\tau_{s-1}}$ can be written as $a\mathbf{e}_{\tau'_s}$. Then by using (4.6),

$$\begin{aligned}\mu(K_{\omega bJ\tau}) &= \mathbf{e}_{\omega_1} \cdot M_{\omega_2} \cdots M_{\omega_n} M_b \cdot M_J \cdot M_{\tau_1} \cdots M_{\tau_{s-1}} \cdot M_{\tau_s} \begin{bmatrix} \mu(T_0K) \\ \vdots \\ \mu(T_{m-1}K) \end{bmatrix} \\ &= a\mathbf{e}_{\tau'_s} \cdot c\mathbf{e}^t_{\tau'_s} \asymp \mathbf{e}_{\omega_1} \cdot M_{\omega_2} \cdots M_{\omega_n} M_b \cdot M_J \cdot M_{\tau_1} \cdots M_{\tau_{s-1}} \mathbf{e}^t_{\tau'_s},\end{aligned}$$

where c is the $(\tau'_s + 1)$-th entry of $M_{\tau_s} \begin{bmatrix} \mu(T_0K) \\ \vdots \\ \mu(T_{m-1}K) \end{bmatrix}$, thus

$$\mu(K_{\omega bJ\tau J'\sigma}) \asymp \mu(K_{\omega bJ\tau}) \cdot c_{\tau'_s, J'} \cdot \mu(K_\sigma) \tag{4.40}$$

where $\tau = \tau_1 \cdots \tau_s$.

Using (4.32) and (4.40), we have

$$\begin{aligned}\sum_{J'\in\mathcal{J}_0^*, \sigma\in\mathcal{S}} r^\alpha_{\omega bJ\tau J'\sigma}\mu(K_{\omega bJ\tau J'\sigma})^\alpha &\asymp r^\alpha_{\omega bJ\tau}\mu(K_{\omega bJ\tau})^\alpha \sum_{J',\sigma} r^\alpha_{J'\sigma} c^\alpha_{\tau, J'} \mu(K_\sigma)^\alpha \\ &= r^\alpha_{\omega bJ\tau}\mu(K_{\omega bJ\tau})^\alpha \cdot \frac{\sum_{i=1}^{m-2} \mu(K_{ii'})^\alpha}{m^{2\alpha}} \cdot \left(1 - \frac{1}{m^\alpha}\sum_{j=1}^{m-1} w_j^\alpha\right)^{-1} \cdot \sum_{J'\in\mathcal{J}_0^*} \frac{1}{m^{|J'|\alpha}} c^\alpha_{\tau, J'}.\end{aligned}$$

From the identity (4.15) and the fact that $c_{i,J'} \asymp c_{j,J'}$ for any $i, j \in \mathcal{J}_1$, we have

$$\sum_{J'\in\mathcal{J}_0^*, \sigma\in\mathcal{S}} r^\alpha_{\omega bJ\tau J'\sigma}\mu(K_{\omega bJ\tau J'\sigma})^\alpha \asymp r^\alpha_{\omega bJ\tau}\mu(K_{\omega bJ\tau})^\alpha. \tag{4.41}$$

For $J \in \mathcal{J}_0^*$, denote by $|J(0)|$ the number of '0' in J and $|J(m-1)|$ the number of '$m-1$' in J. Using Proposition 4.5(b), we see that $\mathbf{e}_{\omega_1} \cdot M_{\omega_2} \cdots M_{\omega_n} \cdot M_b$ is of the form $a\mathbf{e}_{b-j}$. Thus

$$\begin{aligned}&\mu(K_{\omega aJ\tau}) \\ =&\mathbf{e}_{\omega_1} \cdot M_{\omega_2} \cdots M_{\omega_n} \cdot M_b \cdot M_0(b-j, b-j)^{|J(0)|} \cdot M_{m-1}(b-j, b-j)^{|J(m-1)|} \cdot M_\tau \begin{bmatrix} \mu(T_0K) \\ \vdots \\ \mu(T_{m-1}K) \end{bmatrix} \\ =&\mu(K_{\omega b\tau}) \cdot M_0(b-j, b-j)^{|J(0)|} \cdot M_{m-1}(b-j, b-j)^{|J(m-1)|},\end{aligned}$$

where $M(b-j,b-j)$ denotes the $(b-j+1,b-j+1)$-th entry of a matrix M. Using the fact that

$$w_0^{|J|} \le M_0(b-j,b-j)^{|J(0)|} \cdot M_{m-1}(b-j,b-j)^{|J(m-1)|} \le w_{(m+1)/2}^{|J|},$$

we have

$$r^{\alpha}_{\omega b\tau}\mu(K_{\omega b\tau})^{\alpha} \cdot \sum_{k\ge 0} 2^k \cdot \left(\frac{w_0}{m}\right)^{\alpha k} \le \sum_{J\in\mathcal{J}_0^*} r^{\alpha}_{\omega bJ\tau}\mu(K_{\omega bJ\tau})^{\alpha} \le r^{\alpha}_{\omega b\tau}\mu(K_{\omega b\tau})^{\alpha} \cdot \sum_{k\ge 0} 2^k \cdot \left(\frac{w_{(m+1)/2}}{m}\right)^{\alpha k},$$

which, by observing that $2\left(\frac{w_{(m+1)/2}}{m}\right)^{\alpha} < 1$, implies that

$$\sum_{J\in\mathcal{J}_0^*} r^{\alpha}_{\omega bJ\tau}\mu(K_{\omega bJ\tau})^{\alpha} \asymp r^{\alpha}_{\omega b\tau}\mu(K_{\omega b\tau})^{\alpha}. \tag{4.42}$$

Using Proposition 4.5(b), we see that $\mathbf{e}_{\omega_1} \cdot M_{\omega_2} \cdots M_{\omega_n}$ can be written as $a\mathbf{e}_{j'}$. Therefore, we have by (4.6),

$$\begin{aligned} \mu(K_{\omega b\tau}) &= a\mathbf{e}_{j'} \cdot M_b \cdot M_\tau \begin{bmatrix} \mu(T_0K) \\ \vdots \\ \mu(T_{m-1}K) \end{bmatrix} = a\cdot c\cdot \mathbf{e}_{b-j}\cdot M_\tau \begin{bmatrix} \mu(T_0K) \\ \vdots \\ \mu(T_{m-1}K) \end{bmatrix} \\ &\asymp a\mathbf{e}_{j'} \begin{bmatrix} \mu(T_0K) \\ \vdots \\ \mu(T_{m-1}K) \end{bmatrix} \cdot \mathbf{e}_{b-j}\cdot M_\tau \begin{bmatrix} \mu(T_0K) \\ \vdots \\ \mu(T_{m-1}K) \end{bmatrix} = \mu(K_\omega)\cdot\mu(K_{(b-j)\tau}), \end{aligned}$$

where c is the only nonzero entry in the j'-th row of M_b.

Finally, we consider the summation

$$\sum_{\tau\in\mathcal{S}^{j-b}} r^{\alpha}_{\omega b\tau}\mu(K_{\omega b\tau})^{\alpha} \asymp \mu(K_\omega)^{\alpha}\cdot r^{\alpha}_{\omega}\cdot \sum_{\tau\in\mathcal{S}^{j-b}} r^{\alpha}_{\tau}\mu(K_{(b-j)\tau})^{\alpha}.$$

By using the formula (4.32), we see that

$$\sum_{\tau\in\mathcal{S}^{j-b}} r^{\alpha}_{\tau}\mu(K_{(b-j)\tau})^{\alpha} \le m^{\alpha}\cdot \sum_{|s|\ge 2}\frac{1}{m^{|s|\alpha}}\mu(K_s)^{\alpha} = \frac{\sum_{i=1}^{m-2}\mu(K_{ii'})^{\alpha}}{m^{2\alpha}}\cdot\left(1-\frac{1}{m^{\alpha}}\sum_{j=1}^{m-1} w_j^{\alpha}\right)^{-1}.$$

Thus, $\sum_{\tau\in\mathcal{S}^{j-b}} r^{\alpha}_{\tau}\mu(K_{(b-j)\tau})^{\alpha}$ has an upper bound, and also a lower bound and these bounds are independent of ω or b. We conclude that

$$\sum_{\tau\in\mathcal{S}^{j-a}} r^{\alpha}_{\omega b\tau}\mu(K_{\omega b\tau})^{\alpha} \asymp \mu(K_\omega)^{\alpha}\cdot r^{\alpha}_{\omega}. \tag{4.43}$$

Combining (4.38), (4.41), (4.42), and (4.43), we obtain (4.37) as desired.

If $b=j$, we have

$$d_*(K_{\omega j}) = \sum_{J\in\mathcal{J}_0^*,\sigma\in\mathcal{S}} r^{\alpha}_{\omega jJ\sigma}\mu(K_{\omega jJ\sigma})^{\alpha}.$$

Using (4.41), we obtain

$$d_*(K_{\omega j}) \asymp \mu(K_{\omega j})^{\alpha}\cdot r^{\alpha}_{\omega j} \asymp \mu(K_\omega)^{\alpha}\cdot r^{\alpha}_{\omega},$$

where the second asymptotic relation follows from Proposition 4.2. Hence we have shown that (4.37) is true in *Case* 1.

Case 2. $\omega = \omega_1 \cdots \omega_n$ and $\sum_{k=1}^{n} \omega_k \equiv 0 \pmod{m-1}$.

If $b \in \mathcal{J}_1$, then we can use a similar strategy as in *Case* 1 to show that (4.37) is true, we omit the details. If $b \in \mathcal{J}_0$, we can assume, without loss of generality, that $b = 0$. Then

$$d_*(K_{\omega 0}) = \sum_{J \in \mathcal{J}_0^*, \sigma \in \mathcal{S}} \sum_{J' \in \mathcal{J}_0^*, \sigma' \in \mathcal{S}} r^{\alpha}_{\omega 0 J \sigma J' \sigma'} \mu(K_{\omega 0 J \sigma J' \sigma'})^{\alpha}. \tag{4.44}$$

Similar to (4.41) in *Case* 1, we can drop the summation of J', σ'; that is

$$d_*(K_{\omega 0}) \asymp \sum_{J \in \mathcal{J}_0^*} \sum_{\sigma \in \mathcal{S}} r^{\alpha}_{\omega 0 J \sigma} \mu(K_{\omega 0 J \sigma})^{\alpha}.$$

Using (4.6), we have

$$\begin{aligned}
\mu(K_{\omega 0 J \sigma}) &= \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_n} \cdot M_0 M_J M_{\sigma} \begin{bmatrix} \mu(T_0 K) \\ \vdots \\ \mu(T_{m-1} K) \end{bmatrix} \\
&\leq C \mathbf{e}_{\omega_1} M_{\omega_2} \cdots M_{\omega_n} \cdot \begin{bmatrix} \mu(T_0 K) \\ \vdots \\ \mu(T_{m-1} K) \end{bmatrix} \cdot \begin{bmatrix} w_1 & \mathbf{0} & w_0 \end{bmatrix} M_{0J} M_{\sigma} \cdot \begin{bmatrix} \mu(T_0 K) \\ \vdots \\ \mu(T_{m-1} K) \end{bmatrix} \\
&\leq C \mu(K_{\omega}) \cdot c_{1,0J} \mu(K_{\sigma}).
\end{aligned}$$

Substituting this into (4.44), and using (4.32), we obtain

$$\begin{aligned}
d_*(K_{\omega 0}) &\leq C \sum_{J \in \mathcal{J}_0^*} r^{\alpha}_{\omega 0 J} \mu(K_{\omega})^{\alpha} \cdot c^{\alpha}_{1,0J} \sum_{\sigma \in \mathcal{S}} r^{\alpha}_{\sigma} \mu(K_{\sigma}) \leq C' \sum_{J \in \mathcal{J}_0^*} r^{\alpha}_{\omega 0 J} \mu(K_{\omega})^{\alpha} \cdot c^{\alpha}_{1,0J} \\
&\leq C r^{\alpha}_{\omega} \mu(K_{\omega})^{\alpha} \sum_{|J'| \geq 0} \frac{1}{m^{|J'|\alpha}} c^{\alpha}_{1,J'} \leq C r^{\alpha}_{\omega} \mu(K_{\omega})^{\alpha}.
\end{aligned} \tag{4.45}$$

On the other hand, since $\omega 01$ with $b = 0$ belongs to *Case* 1, we have

$$d_*(K_{\omega 0}) \geq d_*(K_{\omega 010}) \geq C^{-1} r^{\alpha}_{\omega 01} \mu(K_{\omega 01})^{\alpha} \geq C^{-1} r^{\alpha}_{\omega} \mu(K_{\omega})^{\alpha}, \tag{4.46}$$

where we obtain the last inequality by using Proposition 4.2 . We conclude from (4.45) and (4.46) that (4.37) is true in *Case* 2. Therefore our claim holds.

It follows from Proposition 4.7 and (4.37) that

$$d_*(K_{\omega}) = \sum_{j=0}^{m-1} d_*(K_{\omega j}) \asymp r^{\alpha}_{\omega} \mu(K_{\omega})^{\alpha},$$

which completes the proof. □

Lemma 4.9. *Condition (1.13) holds with $\beta = 1/\alpha$, where α is given by (4.15).*

Proof. Choose one of the shortest words, say ω', such that $K_{\omega'} \subseteq [x, y]$. Without loss of generality, we assume that ω' is non-empty. Let $\omega' = \omega j$, where $0 \leq j \leq m-1$ and ω may be empty. Then there exists a neighbor τ of ω such that

$$K_{\omega j} \subseteq [x, y] \subseteq K_{\omega} \cup K_{\tau}. \tag{4.47}$$

Using Proposition 4.2 and (4.47), we have

$$C^{-1}\mu(K_{\omega j}) \le \mu([x,y]) \le \mu(K_\omega) + \mu(K_\tau) \le (C'+1)\mu(K_\omega) \le C\mu(K_{\omega j}). \tag{4.48}$$

Thus,

$$\mu([x,y]) \asymp \mu(K_{\omega j}). \tag{4.49}$$

Also, by (4.47), $m \cdot r_{\omega j} \le |x-y| \le m \cdot (r_\omega + r_\tau)$, which yields

$$|x-y| \asymp r_{\omega j}. \tag{4.50}$$

From (4.47) and Proposition 4.7, we also get

$$d_*(K_{\omega j}) \le d_*(x,y) \le d_*(K_\omega) + d_*(K_\tau), \tag{4.51}$$

Hence, using (4.36) and Proposition 4.2,

$$\begin{aligned} C^{-1}\mu(K_{\omega j})^\alpha r_{\omega j}^\alpha \le d_*(x,y) &\le C\left(\mu(K_\omega)^\alpha r_\omega^\alpha + \mu(K_\tau)^\alpha r_\tau^\alpha\right) \\ &\le C'\mu(K_\omega)^\alpha r_\omega^\alpha \le C\mu(K_{\omega j})^\alpha r_{\omega j}^\alpha. \end{aligned} \tag{4.52}$$

Finally, combining (4.52), (4.50) and (4.49), we have

$$d_*(x,y)^{1/\alpha} \asymp \mu(K_{\omega j}) r_{\omega j} \asymp \mu(K_{\omega j})|x-y| \asymp \mu([x,y])|x-y|,$$

which yields (1.13) with $\beta = 1/\alpha$ and completes the proof. □

Lemma 4.10. *Condition (1.14) is satisfied.*

Proof. Using condition (1.12), it suffices to show that there exists a constant $c > 1$ such that for all x, y, z with $0 \le x < y < z \le m$ and $d_*(x,y) = d_*(y,z)$, we have

$$c^{-1}\mu([y,z]) \le \mu([x,y]) \le c\mu([y,z]). \tag{4.53}$$

Choose two shortest words ω and τ such that

$$K_\omega \subseteq [x,y] \qquad \text{and} \qquad K_\tau \subseteq [y,z],$$

and that the point y is closest to K_ω and K_τ.

Claim. There exists a constant $L \ge 0$ such that

$$\big||\omega| - |\tau|\big| \le L. \tag{4.54}$$

To prove the claim we assume, without loss of generality, that $|\omega| - |\tau| \ge 0$, and let $\omega' \le \omega$ such that $|\omega'| = |\tau|$. Then

$$r_{\omega'} = r_\tau. \tag{4.55}$$

The number of words lying between $K_{\omega'}$ and K_τ with length $|\omega'|$ is less than some constant $c_1 > 0$. Thus by Proposition 4.2,

$$c_2^{-1}\mu(K_\tau) \le \mu(K_{\omega'}) \le c_2\mu(K_\tau). \tag{4.56}$$

From (4.52), we have

$$d_*(x,y) \asymp \mu(K_\omega)^\alpha r_\omega^\alpha$$

and that

$$d_*(y,z) \asymp \mu(K_\tau)^\alpha r_\tau^\alpha.$$

Using this and the equality $d_*(x,y) = d_*(y,z)$, we see that there exists some constant $c_3 > 0$ such that

$$\mu(K_\tau) r_\tau \le c_3\mu(K_\omega) r_\omega \le c_3\mu(K_{\omega'}) r_\omega. \tag{4.57}$$

Combining (4.56), (4.55), and (4.57), we have

$$r_{\omega'} = r_\tau \le \frac{c_3\mu(K_{\omega'})r_\omega}{\mu(K_\tau)} \le c_3c_2r_\omega = c_3c_2r_{\omega'}\left(\frac{1}{m}\right)^{|\omega|-|\omega'|},$$

which implies that $1 \le c_3c_2/m^{(|\omega|-|\omega'|)}$, proving the claim.

Now by (4.54) and (4.56),

$$\mu(K_\omega) \asymp \mu(K_{\omega'}) \asymp \mu(K_\tau).$$

It now follows from (4.49) that $\mu([x,y]) \asymp \mu([y,z])$. □

We say that two words τ and σ with equal length are *consecutive* if $K_\tau \cap K_\omega \neq \emptyset$.

Lemma 4.11. *Condition (1.15) holds with $\beta = 1/\alpha$.*

Proof. For any $x \in (0,m)$, any small $r \in (0,1)$ and any integer $\ell \ge 1$, choose z and y_ℓ in $[0,m]$ such that

$$d_*(x,z) = r = \ell d_*(x,y_\ell) \tag{4.58}$$

We need to show that

$$\lim_{\ell\to\infty} \frac{|x-y_\ell|}{|x-z|} = 0, \tag{4.59}$$

where the limit is independent of x and r. We may assume that $x < y_\ell < z$; the other cases are similar. Choose a shortest word ω such that $K_\omega \subseteq [x,y_\ell]$.

Consider a chain of $k+1$ consecutive words starting from ω and with length $|\omega|$, where k will be determined later. By Lemma 4.8 and Proposition 4.2, $d_*(K_\omega) \asymp d_*(K_\tau)$ if τ is neighbor of ω. Hence there exists a constant $c_0 > 1$ such that the total distance of the corresponding cells is no more than

$$d_*(K_\omega)(1 + c_0 + \cdots + c_0^k). \tag{4.60}$$

Let k be the largest integer such that

$$1 + c_0 + \cdots + c_0^k \le \ell, \tag{4.61}$$

and thus

$$k \asymp \log \ell. \tag{4.62}$$

Using (4.58), (4.60), and the inclusion $K_\omega \subseteq [x,y_\ell]$, we see that there exists a chain of $k+1$ consecutive cells starting from K_ω that are contained in $[x,z]$. Let $N := [\log_m k]-1$, where $[\cdot]$ denotes the greatest integer function. Since the Euclidean length of K_ω is $m\left(\frac{1}{m}\right)^{|\omega|}$ and since $k \ge m^{N+1}$, we see that

$$|x-z| \ge k\cdot m\left(\frac{1}{m}\right)^{|\omega|} \ge m^2\left(\frac{1}{m}\right)^{|\omega|-N} > 2m\left(\frac{1}{m}\right)^{|\omega|-N}.$$

This implies that there exists a word τ such that

$$|\tau| = |\omega| - N \qquad \text{and} \qquad K_\tau \subseteq [x,z].$$

Let $\omega' \le \omega$ such that $|\omega'| = |\tau|$. Hence

$$|x-z| \ge m\cdot r_\tau = m\cdot r_{\omega'} = m^{N+1}r_\omega. \tag{4.63}$$

Note that by (4.50),

$$|x - y_\ell| \asymp r_\omega. \tag{4.64}$$

Combining (4.64), (4.62), and (4.63), we have

$$\frac{|x-y_\ell|}{|x-z|} \le \frac{c' r_\omega}{m^{N+1} r_\omega} = \frac{c'}{m^{N+1}} = \frac{c'}{m^{[\log_m k]}} \le \frac{mc'}{k} \le \frac{c}{\log \ell} \to 0 \quad \text{as } \ell \to \infty.$$

This proves (4.59). Finally, letting $\eta = \frac{1}{\ell}$ and using (2.22),

$$\begin{aligned} \sup_{x\in K, 0<r<1} \frac{\eta^{1/\alpha} V(x,r)}{V(x,\eta r)} &= \sup_{x\in K, 0<r<1} \frac{(\eta r)^\beta / \mu(B_{d_*}(x,\eta r))}{r^\beta / \mu(B_{d_*}(x,r))} \\ &\asymp \sup_{x\in K, 0<r<1} \frac{|x-y_\ell|}{|x-z|}, \end{aligned}$$

which tends to 0 as $\eta \to 0^+$ by (4.59). □

Proof of Theorem 1.3. From above, conditions (1.10)–(1.15) are all satisfied with $\beta = 1/\alpha$. Theorem 1.3 now follows from Theorem 1.1. □

Acknowledgements. QG and SN thank the Department of Mathematical Sciences of Tsinghua University, Hunan Normal University, the Chinese University of Hong Kong, and Harvard University, where part of this research was carried out. SN is grateful to Professor Shing-Tung Yau for the opportunity to visit CMSA and thanks the center for its hospitality and support. The authors are indebted to Ka-Sing Lau for his constant support.

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Department of Mathematics, The Chinese University of Hong Kong, Hong Kong, China

E-mail address: qsgu@math.cuhk.edu.hk

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

E-mail address: hujiaxin@mail.tsinghua.edu.cn

Key Laboratory of High Performance Computing and Stochastic Information Processing (HPCSIP) (Ministry of Education of China), College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, P. R. China, and Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460-8093, USA.

E-mail address: smngai@georgiasouthern.edu