WAVE PROPAGATION SPEED ON FRACTALS

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ABSTRACT. We study the wave propagation speed problem on metric measure spaces, emphasizing on self-similar sets that are not postcritically finite. We prove that a sub-Gaussian lower heat kernel estimate leads to infinite propagation speed, extending a result of Y.-T. Lee [32] to include bounded and unbounded generalized Sierpiński carpets as well as some fractal blowups. We also formulate conditions under which a Gaussian upper heat kernel estimate leads to finite propagation speed, and apply this result to two classes of iterated function systems with overlaps, including those defining the classical infinite Bernoulli convolutions.

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1. Introduction

Strichartz [44] conjectured in 1999 that on certain fractals, such as the Sierpiński gasket, waves may propagate with infinite speed, due to the difference in time and Laplacian scalings. This prediction shows that fractals could exhibit behaviors that differ fundamentally from classical smooth objects. Y.-T. Lee [32] recently proved that on a class of self-similar sets satisfying the post-critically finite (p.c.f.) condition, including the Sierpiński gasket, the conjecture is true. The first objective of this paper is to extend Lee's result to fractals that are non-p.c.f., such as generalized Sierpiński carpets.

Let $\mathcal{H}$ be a Hilbert space and A be a non-negative self-adjoint operator on $\mathcal{H}$. The *wave equation* is defined as

$$\begin{cases} u_{tt}(t) = -Au(t), & t \geq 0, \\ u(0) = f, \\ u_t(0) = g. \end{cases} \tag{1.1}$$

The *heat equation* is defined as

$$\begin{cases} v_t(t) = -Av(t), & t \geq 0, \\ v(0) = f. \end{cases} \tag{1.2}$$

It is well known that each of these equations has a unique solution.

There are two main ingredients in Lee's proof, namely, sub-Gaussian heat kernel estimates and a relation between the wave and heat equations. Using these we obtain the following generalizations of Lee's theorem on bounded or unbounded sets. We refer the reader to Section 2 for the definitions of the bounded, finite, and infinite propagation speed properties, abbreviated (BPS), (FPS), and (IPS), respectively. Let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be a closed quadratic form on $L^2(X,\mu)$, we say that the *Sobolev-type inequality* holds on the metric space (X,d) if there exist some constants $c>0$ and $\alpha \in (0,1]$ such that

$$\big|u(x)-u(y)\big| \leq cd(x,y)^\alpha \sqrt{\mathcal{E}(u,u)} \quad \text{for all } u \in \operatorname{dom}\mathcal{E} \text{ and all } x,y \in X. \tag{1.3}$$

Let $C(X,d)$ denote the space of all real-valued continuous functions on X, $\|u\|_p$ denote the L^p-norm in $L^p(X,\mu)$, $1 \leq p \leq \infty$. Let $\operatorname{supp}(\mu)$ and $\operatorname{supp}(f)$ denote the support of a measure μ and a function f respectively.

Theorem 1.1. ***(General form of Lee's theorem on bounded metric spaces)*** *Let (X,d) be a bounded metric space, μ be a finite positive Borel measure on (X,d), A be a non-negative self-adjoint operator on $L^2(X,\mu)$, and $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be the closed quadratic form on $L^2(X,\mu)$ associated with A. Assume that the Sobolev-type inequality* (1.3) *holds, and that the heat kernel $p(t,x,y)$ of $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ exists and satisfies the following sub-Gaussian lower estimate: there exist $\epsilon>0, c_1>0, c_2 \geq 0$, $\eta \in \mathbb{R}$, and $\gamma<1$ such that*

$$p(t,x,y) \geq c_1 t^\eta \exp\Big(-\frac{c_2}{t^\gamma}\Big) \quad \text{for all } x,y \in X \text{ and all } t \in (0,\epsilon).$$

(a) *If* $f \in \operatorname{dom}\mathcal{E}$, $f \geq 0$, *and* $\|f\|_1 > 0$, *then for any* $x \in X$ *and any* $\delta > 0$, *there exists some* $t < \delta$ *such that* $\cos(t\sqrt{A})f(x) > 0$.

(b) *If, in addition, there exists a non-negative* $f \in \operatorname{dom}\mathcal{E}$ *with* $\operatorname{supp}(\mu) \setminus \operatorname{supp}(f) \neq \emptyset$, *then* A *satisfies (IPS).*

We remark that if $c_2 = 0$, the heat kernel estimate in Theorem 1.1 is not sub-Gaussian. Moreover, if $f \in \operatorname{dom} A$ and $g = 0$, then $\cos(t\sqrt{A})f$ is the solution of the wave equation (1.1). Also, if $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a regular Dirichlet form, then the additional assumption in Theorem 1.1(b) is satisfied.

Lee [32] considered Laplacians A defined on p.c.f. fractals with a regular harmonic structure and thus the associated closed form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a resistance form with resistance metric $R^{1/2}$ (compatible with the original topology), as well as a regular Dirichlet form (see [26, Chapter 3]). Hence the Sobolev-type inequality (1.3) holds on $(X, R^{1/2})$ with $\alpha = 1$ and there exists some $f \in \operatorname{dom}\mathcal{E}$ satisfying the assumption in Theorem 1.1(b). Moreover, the sub-Gaussian heat kernel estimate there corresponds to $\eta = 0$ and $c_2 = 1$ in Theorem 1.1. Thus, Theorem 1.1 generalizes [32, Theorem 8] and we will see that it can be applied to certain non-p.c.f fractals.

A main motivation of this work is to study Strichartz wave propagation speed conjecture on fractals defined by iterated function systems with overlaps together with the associated self-similar measures. Let $K = [a, b]$ and $H^1(a, b)$ be the usual Sobolev space on (a, b). Consider the bilinear form $\mathcal{E}$ defined as

$$\mathcal{E}(u, v) = \int_a^b u'(x)v'(x)\,dx \qquad \text{for all } u, v \in \operatorname{dom}\mathcal{E} := H_0^1(a, b). \tag{1.4}$$

It is well known that $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a Dirichlet form in $L^2((a, b), \mu)$ (see [17]) and the Sobolev-type inequality (1.3) holds on $(K, d_{|\cdot|})$ with $\alpha = 1/2$, where $d_{|\cdot|}$ is the Euclidean metric. The first measure we study is the infinite Bernoulli convolution associated with the golden ratio. Let

$$S_0(x) = \rho x, \qquad S_1(x) = \rho x + (1 - \rho), \qquad \rho = \frac{\sqrt{5} - 1}{2} \tag{1.5}$$

and let μ be the self-similar measure with $\operatorname{supp}(\mu) = [0, 1]$ satisfying:

$$\mu = \frac{1}{2}\mu \circ S_0^{-1} + \frac{1}{2}\mu \circ S_1^{-1}. \tag{1.6}$$

Combining Theorem 1.1 we the sub-Gaussian heat kernel estimate obtained recently by Gu *et al.* [20], we have

Corollary 1.2. *Let* $K = [0, 1]$ *and* μ *be defined as in* (1.5) *and* (1.6). *Let* $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ *be defined by* (1.4) *and* A *be the associated non-negative self-adjoint operator. Then* $(A, K, d_{|\cdot|}, \mu)$ *satisfies (IPS), where* $d_{|\cdot|}$ *is the Euclidean metric.*

We also study a family of convolutions of Cantor-type measures. Let

$$S_0(x) = \frac{1}{m}x, \qquad S_1(x) = \frac{1}{m}x + \frac{m-1}{m}, \tag{1.7}$$

where $m \geq 3$ is an odd integer. Let ν_m be the self-similar measure defined by the IFS (1.7) with probability weights $p_0 = p_1 = 1/2$. The m-fold convolution μ_m of ν_m^{*m} is the self-similar measure defined by the following IFS with overlaps (see[38]):

$$S_i(x) = \frac{1}{m}x + \frac{m-1}{m}i, \qquad i = 0, 1, \ldots, m,$$

together with probability weights

$$w_i := \frac{1}{2^m}\binom{m}{i}, \qquad i = 0, 1, \ldots, m.$$

That is,

$$\mu_m = \sum_{i=0}^{m} \frac{1}{2^m}\binom{m}{i}\mu_m \circ S_i^{-1}. \tag{1.8}$$

Note that $\operatorname{supp}(\mu_m) = [0, m]$.

Corollary 1.3. *For any odd integer $m \geq 3$, let $K = [0, m]$ and μ be the m-fold convolution of the Cantor measure defined as* (1.8)*. Let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be defined by* (1.4)*, A be the associated non-negative self-adjoint operator, and $d_{|\cdot|}$ be the Euclidean metric. Then $(A, K, d_{|\cdot|}, \mu_m)$ satisfies (IPS).*

In order to study unbounded fractals, we also prove a general form of Lee's theorem on locally compact metric spaces.

Theorem 1.4. ***(General form of Lee's theorem on locally compact metric spaces)*** *Let (X, d) be a locally compact metric space, μ be a σ-finite Borel measure on (X, d), and A be a non-negative self-adjoint operator on $L^2(X, \mu)$. Assume the corresponding heat kernel $p(t, x, y)$ exists and there exist an open subset U of X, some constants $\epsilon > 0, c_1 > 0, c_2 \geq 0$, $\eta \in \mathbb{R}$, and $\gamma < 1$ such that the heat kernel $p(t, x, y)$ satisfies*

$$p(t, x, y) \geq c_1 t^{\eta} \exp\Big(-\frac{c_2}{t^{\gamma}}\Big) \qquad \text{for } \mu\text{-a.e. } x \in U \text{ and all } t \in (0, \epsilon). \tag{1.9}$$

Then the following hold:

(a) *Let $f \in L^2(U, \mu)$, $f \geq 0$, and $\|f\|_{L^1(U,\mu)} > 0$. Assume that for any $t > 0$, $\cos(t\sqrt{A})f$ is continuous on X and there exists some constant $C > 0$ such that $\|\cos(t\sqrt{A})f\|_\infty \leq C$. Then for any $x \in U$ and any $\delta > 0$, there exists some $t \in (0, \delta)$ such that $\big(\cos(t\sqrt{A})f\big)(x) > 0$.*

(b) *A satisfies (IPS).*

Theorems 1.1 and 1.4 allow us to prove (IPS) for certain fractal blowups, which are unbounded, as well as certain non-p.c.f. fractals.

We first describe fractal blowups. Let $\Sigma = \{1, \dots, N\}$ for an integer $N \geq 2$ and $\{S_i\}_{i=1}^N$ be an IFS on $\mathbb{R}^n$. For any $m \geq 0$ and any word $I = i_1 \cdots i_m \in \Sigma^m$, we use $|I| = m$ to denote the *length* of I, and I is the empty word if $|I| = 0$. Denote by

$$S_I := S_{i_1} \circ \cdots \circ S_{i_m}.$$

Definition 1.1. *Let $N \geq 2$ and K be the self-similar set associated with an IFS $\{S_i\}_{i=1}^N$. Fix an infinite word $\theta = i_1 i_2 \cdots \in \Sigma^\infty$. For each $m \geq 1$, let*

$$K^m := S^{-1}_{i_1 \cdots i_m}(K) := S^{-1}_{i_1} \circ \cdots \circ S^{-1}_{i_m}(K).$$

Define

$$K_\infty := \bigcup_{m=1}^\infty K^m. \tag{1.10}$$

K_∞ is called a fractal blowup.

We remark that K_∞ is determined by the choice of the infinite word θ. It is easy to check that K_∞ is unbounded.

Example 1.5. *Let $K = [0, 1]$ and $S_i(x) = (x - a_i)/3 + a_i$ for $i = 1, 2, 3$, where $a_1 = 0$, $a_2 = 1/2$, $a_3 = 1$. Let μ be the self-similar measure with corresponding probability weight $\{p_i\}_{i=1}^3$. Let K_∞ be the fractal blowup given by* (1.10) *with $\theta = 1313\cdots$.*

We note that $K_\infty = S^{-1}_{1313\cdots}([0, 1]) = \mathbb{R}$. Let d_s be the unique positive number satisfying

$$\Big(\frac{p_1}{3}\Big)^{d_s/2} + \Big(\frac{p_2}{3}\Big)^{d_s/2} + \Big(\frac{p_3}{3}\Big)^{d_s/2} = 1.$$

Denote by

$$w_i = \Big(\frac{p_i}{3}\Big)^{d_s/2}, \qquad i = 1, 2, 3. \tag{1.11}$$

For any $m \in \mathbb{Z}$ and any word $I = i_1 \cdots i_k j_1 \cdots j_{k+m}$, denote by

$$p_I = p^{-1}_{i_1 \cdots i_k} p_{j_1 \cdots j_{k+m}} \qquad \text{and} \qquad w_I = w^{-1}_{i_1 \cdots i_k} w_{j_1 \cdots j_{k+m}}.$$

We extend μ to K_∞ by

$$\mu(K_I) = p_I \qquad \text{for any } K_I = S^{-1}_{i_1 \cdots i_k} \circ S_{j_1 \cdots j_{k+m}}(K). \tag{1.12}$$

For $x, y \in K_\infty$ with $x \leq y$ and any $m \geq 1$, let

$$W_m(x, y) = \{I : K_I \subseteq [x, y]\}. \tag{1.13}$$

Define

$$d_m(x, y) := \sum_{I \in W_m(x,y)} w_I \qquad \text{and} \qquad d_*(x, y) := \lim_{m \to \infty} d_m(x, y). \tag{1.14}$$

Kigami [27, Section 5] constructed a regular local Dirichlet form $(\mathcal{E}, \operatorname{dom} \mathcal{E})$ on $L^2(K, \mu)$. Beginning with $(\mathcal{E}, \operatorname{dom} \mathcal{E})$, Gu and Hu [19] constructed a regular local conservative Dirichlet form $(\widetilde{\mathcal{E}}, \operatorname{dom} \widetilde{\mathcal{E}})$ on $L^2(K_\infty, \mu)$.

Corollary 1.6. *Let K, K_∞ be defined as in Example 1.5 and μ be the Radon measure defined in* (1.12)*. Let $A, \widetilde{A}$ be the non-negative self-adjoint operators associated with the Dirichlet forms $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ in* [27] *and $(\widetilde{\mathcal{E}}, \operatorname{dom}\widetilde{\mathcal{E}})$ in* [19]*, respectively, and $p_1 = p_3 \neq p_2$. Then $(\widetilde{A}, K_\infty, d_*, \mu)$ and $(\widetilde{A}, K_\infty, d_{|\cdot|}, \mu)$ satisfies (IPS), where $d_{|\cdot|}$ is the Euclidean metric. The same holds with A, K replacing $\widetilde{A}, K_\infty$ respectively.*

We remark that wave equations defined by the operator A in Corollary 1.6 has recently been studied by Andrews *et al.* [1].

Theorems 1.1 and 1.4 also allow us to prove that waves propagate with infinite speed on generalized Sierpiński carpets, which are not p.c.f. self-similar sets. The definitions of these carpets, as well as the corresponding Laplacians, are given in Section 5.

Corollary 1.7. *Let F denote a generalized Sierpiński carpet and $\widetilde{F}$ denote the corresponding unbounded Sierpiński carpet. Let A and $\widetilde{A}$ be the Laplacians on F and $\widetilde{F}$ respectively given in* [5, 6] *or* [30]*. Then*

(a) *A and $\widetilde{A}$ satisfy (IPS).*
(b) *Let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be the regular Dirichlet form on $L^2(F, \mu)$ associated with A. Assume that $d_s < 2$. Then for any non-negative and non-zero function $f \in \operatorname{dom}\mathcal{E}$, any $x \in F$ and any $\delta > 0$, there is $t \in (0, \delta)$ such that $\big(\cos(t\sqrt{A})f\big)(x) > 0$.*

We do not know whether Corollary 1.7(b) holds for $\widetilde{F}$ and $\widetilde{A}$. Generalized Sierpiński carpets in $\mathbb{R}^n$ have been studied in [2, 3, 4, 6, 7]. It is known that $d_s \leq d_f < n$ and thus $d_s < 2$ in $\mathbb{R}^2$. Generalized Sierpiński carpets with $n \geq 3$ and $d_s < 2$ can be found in [6, 30].

In view of Lee's theorem, as well as its more general forms above, it is natural to ask whether a Gaussian upper heat kernel estimate will imply finite propagation speed. The second objective of this paper is to prove that, under suitable conditions, this is true.

The relationship between wave propagation speed and heat kernel estimates is well known. Cheeger, Gromov, and Taylor [11] obtained unit propagation speed (UPS) (see Definition 2.2(a)) for Laplacians defined on complete Riemannian manifolds and used it to study heat kernel estimates. Coulhon and Sikora [12, 41] showed that (UPS) is equivalent to the Davies-Gaffney estimate (see (6.1)), and obtained heat kernel estimates by assuming (UPS). We remark that in the literature (UPS) is called finite propagation speed. Our definition of finite propagation speed (FPS) (see Definition 2.2(b)) is a weaker notion.

Let (X, d, μ) be a metric measure space and A be a nonnegative self-adjoint operator in $L^2(X, \mu)$. It is well known that the following Gaussian upper heat kernel estimate implies the Davies-Gaffney estimate (6.1): there exists $c > 0$ such that

$$p(t,x,y) \le c\exp\Big(-\frac{d(x,y)}{4t}\Big) \qquad \text{for all } x,y \in X \text{ and all } t > 0. \tag{1.15}$$

Thus, (1.15) implies (UPS). There is also an analogue of this for bounded propagation speed (BPS) (see Definition 2.2(a) and Corollary 6.2). In order to prove our main result on finite propagation speed, we will first weaken the assumptions of these results and establish (BPS) (see Theorem 6.6), and then use strong regularity to obtain (FPS) (see Theorem 1.8).

In Section 7 we study Laplacians defined by measures on a bounded subsets of $\mathbb{R}^n$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset of $\mathbb{R}^n$, μ be a positive finite Borel measure on $\mathbb{R}^n$ with $\text{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. It is known that μ defines a Dirichlet Laplace operator Δ_μ, if the following *Poincaré inequality for a measure (MPI)* holds: There exists a constant $C > 0$ such that for all $u \in C_c^\infty(\Omega)$,

$$\int_\Omega |u|^2\,d\mu \le C\int_\Omega |\nabla u|^2\,dx \tag{1.16}$$

(see, e.g., [34, 35, 21]). Here $C_c^\infty(\Omega)$ denotes the space of all C^∞ functions on Ω with compact support. We write $V \subset\subset \Omega$ if V is compactly contained in Ω, i.e., $\overline{V} \subset \Omega$ and $\overline{V}$ is compact. We call an open connected subset of $\mathbb{R}^n$ a *domain*.

In the following theorem, ρ and $d_{|\cdot|}$ stand for the intrinsic metric (see Definition 7.1) and the Euclidean metric respectively.

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that μ is equivalent to the restriction of Lebesgue measure on $\overline{\Omega}$ with density $d\mu/dx = f \in L^\infty(\Omega,\mu)$ and let $-\Delta_\mu$ be the Dirichlet Laplacian with respect to μ. Also, assume that for every $V \subset\subset \Omega$, there exists some constant $\varepsilon(V)$ such that $f \ge \varepsilon(V) > 0$ Lebesgue a.e. on V. Then $(\Omega,\rho,\mu,-\Delta_\mu)$ satisfies (BPS) and $(\Omega,d_{|\cdot|},\mu,-\Delta_\mu)$ satisfies (FPS).*

Here we outline the main ideas of the proof of Theorem 1.8. First, the assumption $f \ge \varepsilon(V) > 0$ allows us to prove that the intrinsic metric ρ is topologically equivalent to the Euclidean metric and hence the Dirichlet form in question is strongly regular. It also leads to the completeness and the volume doubling properties (see Section 7). Second, we use the boundedness of f to prove (MPI) and the strong Poincaré inequality (see (7.10)). Third, we use these properties to invoke a theorem of Sturm [46] and establish a desired upper heat-kernel estimate with respect to the intrinsic metric, which leads to (BPS). Finally, we use strong regularity to obtain (FPS) with respect to the Euclidean metric.

In Section 8, we apply Theorem 1.8 to two classes of self-similar measures on $\mathbb{R}$. Let μ be a self-similar measure defined by an iterated function systems (IFS) $\{S_i\}_{i=0}^N$ on $\mathbb{R}$. It is known that if the attractor K is not a singleton, then μ is upper s-regular for some $s > 0$, and hence μ satisfies (MPI) (see, e.g., [21]).

The first family of IFSs we study is:

$$S_i(x) = \frac{1}{2}x + \frac{i}{2}, \qquad i = 0, 1, \ldots, N, \tag{1.17}$$

where $N \geq 3$. For each $N \geq 3$, let μ be the self-similar measure defined by the IFS together with probability weights $p_i = 1/(N+1)$.

The second family consists of the well-known *infinite Bernoulli convolutions*, which are defined by the following class of IFSs on $\mathbb{R}$:

$$S_0(x) = rx, \qquad S_1(x) = rx + 1 - r, \qquad 0 < r < 1, \tag{1.18}$$

together with probability weights $p_0 = p_1 = 1/2$.

Theorem 1.9. *Let μ be a self-similar measure defined by the IFS $\{S_i\}_{i=0}^N$ on $\mathbb{R}$ associated with probability weights $p_i = 1/(N+1)$ for all $i = 0, 1, \ldots, N$.*

(a) *Let $\{S_i\}_{i=0}^N$ be defined by* (1.17) *with N being odd. Then $-\Delta_\mu$ satisfies (FPS).*
(b) *Let $\{S_0, S_1\}$ be defined by* (1.18)*. Assume μ is absolutely continuous with respect to Lebesgue measure and $r \in (2/3, 1)$. Then $-\Delta_\mu$ satisfies (FPS).*

This rest of this paper is organized as follows. Section 2 summarizes some of the definitions and results that will be needed throughout the paper. Several results concerning the abstract wave and heat equations are proved in Section 3. Section 4 is devoted to the proof of Theorems 1.1 and 1.4. In Section 5, we apply these two theorems to generalized Sierpiński carpets and unbounded Sierpiński carpets and prove Corollary 1.7. Section 6 studies (BPS) and (FPS) in general metric measure spaces. In Section 7 we study (FPS) for Laplacians defined by measures on $\mathbb{R}^n$ and prove Theorem 1.8. In Section 8, we provide examples of finite propagation speed, including infinite Bernoulli convolutions, and prove Theorem 1.9. Finally, we state some open questions and comments in Section 9.

2. Preliminaries

In this section, we summarize some notation, definitions, and preliminary results that will be used throughout this paper.

2.1. **Strong derivative.**

Definition 2.1. *Let $\mathcal{B}$ be a Banach space, $u : (a, b) \subseteq \mathbb{R} \to \mathcal{B}$, and $t_0 \in (a, b)$. Then u is said to be* differentiable *at t_0 if there exists $v_0 \in \mathcal{B}$ such that*

$$\lim_{h\to 0} \left\| \frac{u(t_0 + h) - u(t_0)}{h} - v_0 \right\|_{\mathcal{B}} = 0.$$

v_0 is called the strong derivative *of u at t_0, and and we write*

$$v_0 = u_t(t_0) = \lim_{h\to 0} \frac{u(t_0 + h) - u(t_0)}{h}.$$

u is said to be differentiable *on (a,b) if it is differentiable at every point in (a,b). Higher-order strong derivatives are defined similarly.*

Note that if u is differentiable at t_0, then it is strongly continuous at t_0.

2.2. **Closed quadratic form and semigroup.** Let $\mathcal{H}$ be a (real or complex) Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We call a non-negative definite symmetric densely defined bilinear form on $\mathcal{H}$ a *quadratic form* on $\mathcal{H}$. Given a quadratic form $\mathcal{E}$ on $\mathcal{H}$ with domain $\operatorname{dom}\mathcal{E}$, define $\mathcal{E}_*(u,v) := \mathcal{E}(u,v) + (u,v)$ for any $u,v \in \operatorname{dom}\mathcal{E}$. A quadratic form $(\mathcal{E},\operatorname{dom}\mathcal{E})$ is said to be *closed* if $(\mathcal{E}_*,\operatorname{dom}\mathcal{E})$ is a Hilbert space.

For a closed quadratic form $(\mathcal{E},\operatorname{dom}\mathcal{E})$ on $\mathcal{H}$, the generator of $(\mathcal{E},\operatorname{dom}\mathcal{E})$ will be denoted by $(A,\operatorname{dom}A)$.

A family $\{T_t\}_{t\geq 0}$ of symmetric bounded linear operators $T_t : \mathcal{H} \to \mathcal{H}$ is called a *semigroup* if (1) $T_0 = I$, the identity, and (2) $T_{s+t} = T_s T_t$ for all $s \geq 0$ and $t \geq 0$. A semigroup $\{T_t\}_{t\geq 0}$ is said to be *strongly continuous* if

$$\lim_{t\to 0^+} \|T_t u - u\| = 0 \qquad \text{for all } u \in \mathcal{H},$$

and is said to be *contractive* if

$$\|T_t u\| \leq \|u\| \qquad \text{for all } u \in \mathcal{H} \text{ and all } t \geq 0.$$

The generator A of a closed quadratic form gives rise to a strongly continuous contractive semigroup $\{T_t\}_{t\geq 0}$ defined by

$$T_t = \exp(-tA) \quad \text{ and } \quad \operatorname{dom} T_t = \mathcal{H} \quad \text{for } t \geq 0.$$

2.3. **Dirichlet form.** Let (X,μ) be a measure space with a σ-finite measure μ. Let $L^2(X,\mu) = \{u : X \to \mathbb{R} : \int_X |u|^2\,d\mu < \infty\}$ with inner product $(u,v)_\mu = \int_X uv\,d\mu$. Also, let $\|B\|_{p\to q}$ denote the operator norm of a bounded linear operator $B : L^p(X,\mu) \to L^q(X,\mu)$.

A semigroup $\{T_t\}_{t\geq 0}$ on $L^2(X,\mu)$ is said to be *ultracontractive* if T_t can be extended to a bounded operator from $L^2(X,\mu)$ to $L^\infty(X,\mu)$ for any $t \geq 0$. A sufficient condition for ultracontractivity is the Nash inequality (see, e.g., [26]).

A *Dirichlet form* is a closed quadratic form $\mathcal{E}$ with domain $\operatorname{dom}\mathcal{E} \subseteq L^2(X,\mu)$ which satisfies the *Markov property*, namely, $u \in \operatorname{dom}\mathcal{E}$ implies $\bar{u} = (u \vee 0) \wedge 1 \in \operatorname{dom}\mathcal{E}$ and $\mathcal{E}(\bar{u},\bar{u}) \leq \mathcal{E}(u,u)$. It is called *strongly local* if for any $u,v \in \operatorname{dom}\mathcal{E}$ with compact support, $\mathcal{E}(u,v) = 0$ whenever u is constant on an open neighborhood of the support of v. Let $C_c(X)$ be the collection of all continuous functions on X with compact support. A Dirichlet form $(\mathcal{E},\operatorname{dom}\mathcal{E})$ is said to be *regular* if $C_c(X) \cap \operatorname{dom}\mathcal{E}$ is dense both in $\operatorname{dom}\mathcal{E}$ with respect to the $\mathcal{E}_*$-norm and in $C_c(X)$ with respect to the supremum norm.

2.4. **Heat kernel.** Let X be a Hausdorff topological space. A positive Borel measure μ on X is called a *Radon measure* if it is (1) *inner regular*, i.e., for each measurable set A, $\mu(A) = \sup\{\mu(K) : K \subseteq A,\ K \text{ is compact}\}$ and (2) *locally finite*, i.e., each point in X has a neighborhood U such that $\mu(U) < \infty$. A signed Borel measure μ on X is called a *signed Radon measure* if its positive and negative parts are Radon measures.

Let (X, d, μ) be a metric measure space, A be a non-negative self-adjoint operator on $L^2(X, \mu)$, and $\{T_t\}_{t\geq 0}$ be the associated semigroup on $L^2(X, \mu)$. Also let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be the closed quadratic form on $L^2(X, \mu)$ associated with A. A non-negative measurable function $p(t, x, y)$ on $(0, \infty) \times X \times X$ is called the *heat kernel* of the semigroup $\{T_t\}_{t\geq 0}$ (or of the quadratic form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$) if $p(t, x, y)$ is the integral kernel of the operator T_t, i.e., for any $t \geq 0$ and any $f \in L^2(X, \mu)$,

$$(T_t f)(x) = \int_X p(t, x, y) f(y)\, d\mu(y) \qquad \text{for } \mu\text{-a.e. } x \in X.$$

The heat kernel may not exist in general. However, it is known to exist in many cases such as Brownian motions on Euclidean spaces, Riemannian manifolds, and certain classes of fractals (see [13] for a sufficient condition). If it exists then it is unique (up to a set of measure zero).

2.5. **Wave propagation speed.** Let (X, d, μ) be a metric measure space, i.e., μ is a Borel measure with respect to the topology defined by the metric d and $B_d(x, r) := \{y \in X : d(x, y) < r\}$ denote an open ball with radius r and center x. Let $C(X, d)$ denote the collection of all real-valued continuous functions on X.

For any measurable subset $U \subseteq X$, we denote $L^2(U, \mu|_U)$ simply by $L^2(U, \mu)$. Let $d(U_1, U_2) := \inf\{d(x, y) : x \in U_1, y \in U_2\}$ denote the distance between $U_1, U_2 \subseteq X$, and write $d(x, U) := d(\{x\}, U)$.

Definition 2.2. *Let (X, d, μ) be a metric measure space and A be a non-negative self-adjoint operator on $L^2(X, \mu)$. Regarding solutions of the corresponding wave equation, we say (X, d, μ, A) (or simply A) has the*

(a) bounded propagation speed property (BPS) *(resp.* unit propagation speed property (UPS) *) if there exists some $s > 0$ (resp. $0 < s \leq 1$) such that*

$$\big(\cos(t\sqrt{A}) f_1, f_2\big)_\mu = 0$$

for all $t \in (0, r/s)$, all open subsets $U_i \subseteq X$, and all $f_i \in L^2(U_i, \mu)$, $i = 1, 2$, where $r := d(U_1, U_2) > 0$ (c.f. [41]);

(b) finite propagation speed property (FPS) *if for any open subsets* $U_i \subseteq X$ $(i = 1,2)$ *with* $r := d(U_1, U_2) > 0$*, there exists some* $s > 0$ *(may depend on* U_1, U_2*) such that*

$$\big(\cos(t\sqrt{A})f_1, f_2\big)_\mu = 0$$

for all $0 < t < r/s$ *and all* $f_i \in L^2(U_i, \mu)$*;*

(c) infinite propagation speed property (IPS) *if there exist open subsets* $U_i \subseteq X$ $(i = 1,2)$ *with* $d(U_1, U_2) > 0$ *such that for any* $s > 0$*, there exists some* $t \in (0, s)$ *and* $f_i \in L^2(U_i, \mu)$ *satisfying*

$$\big(\cos(t\sqrt{A})f_1, f_2\big)_\mu \neq 0.$$

From (BPS) one obtains (UPS) by a simple change of the metric d, and vice versa. It follows from (2.2) that (BPS) implies that wave propagation speed is less than s. (FPS) and (IPS) are negations of each other.

The following theorem compares wave propagation speeds in two different metrics; part (b) will be needed in the proof of Theorem 1.8. We say that two metric spaces (X, d_1) and (X, d_2) are *strongly equivalent* if there exists two positive constants c_1 and c_2 such that for all $x, y \in X$, $c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_1(x,y)$.

Proposition 2.1. *Let* (X, d_i, μ)*,* $i = 1,2$*, be two metric measure spaces and let* A *be a non-negative self-adjoint operator on* $L^2(X, \mu)$*.*

(a) *If* d_1 *and* d_2 *are strongly equivalent and* (X, d_1, μ, A) *satisfies (BPS), then so does* (X, d_2, μ, A)*.*

(b) *If* d_1 *is topologically equivalent to* d_2 *and* (X, d_1, μ, A) *satisfies (FPS), then* (X, d_2, μ, A) *satisfies (FPS).*

Proof. The proof of (a) is straightforward; we only prove (b). Let $U_i \subseteq X$ $(i = 1,2)$ be two open subsets on the metric space (X, d_2) with $d_2(U_1, U_2) > 0$. Since d_1 is topologically equivalent to d_2, U_1 and U_2 are two open subsets on the metric space (X, d_1) and there exist some constants $C_1, C_2 > 0$, depending on U_1, U_2, such that $C_1 d_2(U_1, U_2) \leq d_1(U_1, U_2) \leq C_2 d_2(U_1, U_2)$. By assumption, there exists some constant $s_1 > 0$, may depend on U_i, such that $(\cos(t\sqrt{A})f_1, f_2)_\mu = 0$ for all $0 < t < d_1(U_1, U_2)/s_1$ and all $f_i \in L^2(U_i, \mu)$. It follows that $(\cos(t\sqrt{A})f_1, f_2)_\mu = 0$ for all $0 < t < C_1 d_2(U_1, U_2)/s_1$ and all $f_i \in L^2(U_i, \mu)$. Hence (X, d_2, μ, A) has (FPS). □

3. Wave and heat equations on Hilbert spaces

Let $\mathcal{H}$ be (real or complex) Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, A be a non-negative self-adjoint operator on $\mathcal{H}$ with the domain $\operatorname{dom} A$, and $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be closed quadratic form on $\mathcal{H}$ associated with A. Let $A = \int_0^\infty \lambda\, dE_\lambda$ be the unique spectral representation of A. Then

$$\operatorname{dom}\mathcal{E} = \Big\{u \in \mathcal{H} : \int_0^\infty \lambda\, d(E_\lambda u, v) < \infty\Big\} \tag{3.1}$$

and

$$\operatorname{dom} A = \Big\{u \in \mathcal{H} : \int_0^\infty \lambda^2\, d(E_\lambda u, v) < \infty\Big\}. \tag{3.2}$$

The the formula

$$\big(\phi(A)u, \psi(A)v\big) = \int_0^\infty \phi(\lambda)\psi(\lambda)\, d(E_\lambda u, v), \quad u \in \operatorname{dom}\phi(A), v \in \operatorname{dom}\psi(A),$$

holds for any continuous functions ϕ and ψ on $[0, +\infty)$ (see, [17, Section 1.3]). In particular,

$$(Au, v) = \int_0^\infty \lambda\, d(E_\lambda u, v), \quad u \in \operatorname{dom} A, v \in \mathcal{H}$$

and

$$\mathcal{E}(u, v) = \int_0^\infty \lambda\, d(E_\lambda u, v), \quad u, v \in \operatorname{dom}\mathcal{E}.$$

Definition 3.1. *A function $u : \mathbb{R} \to \mathcal{H}$ is called a* solution *of* (1.1) *if it has two strong derivatives with respect to t, $u(t) \in \operatorname{dom} A$ for any $t \in \mathbb{R}$, and equation* (1.1) *is satisfied.*

The existence and uniqueness of solution of the abstract wave equation (1.1) is well known (see, e.g., [40]).

Theorem 3.1. *Let $\mathcal{H}$ be a complex Hilbert space and A be a non-negative self-adjoint operator on $\mathcal{H}$ with domain $\operatorname{dom} A$. Then for any $f \in \operatorname{dom} A$ and $g \in \operatorname{dom}\sqrt{A}$, the initial value problem*

$$u_{tt}(t) = -Au(t), \quad u(0) = f, \quad u_t(0) = g,$$

has a unique solution $u : [0, \infty) \to \mathcal{H}$ given by

$$u(t) = \int_0^\infty \cos(t\sqrt{\lambda})\, dE_\lambda f + \int_0^\infty \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\, dE_\lambda g,$$

where $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family associated with A.

Definition 3.2. *A function $v : [0, \infty) \to \mathcal{H}$ is called a* solution *of* (1.2) *if $v(t)$ is strongly continuous at $t = 0$, $v(t)$ is differentiable on $(0, \infty)$, $v(t) \in \operatorname{dom} A$ for any $t > 0$ and satisfies equation* (1.2).

Let $\{T_t\}_{t\geq 0}$ be the strongly continuous contractive semigroup associated with A. It is well known that for any $f \in \mathcal{H}$, there exists a unique solution $v : [0, \infty) \to \mathcal{H}$ of the heat equation (1.2), given by $v(t) = T_t f$ (see, e.g., [26]).

The following relation between the wave equation (1.1) and the heat equation (1.2) is known. We include a proof for completeness.

Lemma 3.2. *Let $\mathcal{H}$ be a separable Hilbert space and let A be a non-negative self-adjoint operator on $\mathcal{H}$ with domain* $\operatorname{dom} A$*. Then for any $f \in \mathcal{H}$, the function $v(t)$ defined by*

$$v(t) := \begin{cases} f, & t = 0, \\ \dfrac{1}{\sqrt{\pi t}} \displaystyle\int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big)\cos(s\sqrt{A})f\,ds, & t \in (0,\infty), \end{cases} \tag{3.3}$$

is the solution of the heat equation (1.2) *with initial data f.*

Proof. Let

$$A = \int_0^\infty \lambda\, dE_\lambda$$

be the unique spectral representation of A. Then for any $f \in \mathcal{H}$ and $s \geq 0$, $\|\cos(s\sqrt{A})f\| \leq \|f\|$. Let $T_t = \exp(-tA)$, $t \geq 0$, be the semigroup associated with A. Since the heat equation (1.2) has the unique solution given by $T_t f = \exp(-tA)f$ for any $f \in \mathcal{H}$, it suffices to show that $v(t) = \exp(-tA)f$ for any $f \in \mathcal{H}$. For any $t > 0$ and $f \in \mathcal{H}$,

$$\begin{aligned} \frac{1}{\sqrt{\pi t}}\int_0^\infty\int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big)\cos(s\sqrt{\lambda})\,ds\,dE_\lambda f &= \int_0^\infty \exp(-\lambda t)\,dE_\lambda f \\ &= \exp(-tA)f = T_t f. \end{aligned} \tag{3.4}$$

By using Bochner's Theorem (see, e.g., [48, Section V.5]) and Fubini's Theorem, we obtain, for any $t > 0$ and $w \in \mathcal{H}$,

$$\begin{aligned} \big(v(t), w\big) &= \frac{1}{\sqrt{\pi t}}\int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big)\big(\cos(s\sqrt{A})f, w\big)\,ds && \text{(Bochner)} \\ &= \frac{1}{\sqrt{\pi t}}\int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big)\int_0^\infty \cos(s\sqrt{\lambda})\,d(E_\lambda f, w)\,ds \\ &= \frac{1}{\sqrt{\pi t}}\int_0^\infty\int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big)\cos(s\sqrt{\lambda})\,ds\,d(E_\lambda f, w) && \text{(Fubini)} \\ &= \Big(\frac{1}{\sqrt{\pi t}}\int_0^\infty\int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big)\cos(s\sqrt{\lambda})\,ds\,dE_\lambda f, w\Big) \\ &= \big(T_t f, w\big), && \text{(by (3.4))} \end{aligned}$$

which completes the proof. □

In the rest of this section, we include a brief discussion of the case when A has compact resolvent, even though the results are not needed in the paper. Let $(\varphi_n)_{n\geq 1}$ be an orthonormal basis of $\mathcal{H}$ consisting of the eigenfunctions of A such that $A\varphi_n = \lambda_n\varphi_n$ for $n \geq 1$, $0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots$, and $\lim_{n\to\infty}\lambda_n = \infty$. The domains $\operatorname{dom}\mathcal{E}$ and $\operatorname{dom} A$ can be expressed by using eigenfunctions and eigenvalues as

$$\operatorname{dom}\mathcal{E} = \Big\{\sum_{n=1}^\infty \alpha_n\varphi_n : \sum_{n=1}^\infty \alpha_n^2\lambda_n < \infty\Big\}$$

and

$$\operatorname{dom} A=\Big\{\sum_{n=1}^{\infty}\alpha_n\varphi_n:\sum_{n=1}^{\infty}\alpha_n^2\lambda_n^2<\infty\Big\}$$

(c.f. (3.1) and (3.2)). Moreover, for $u=\sum_{n=1}^{\infty}\alpha_n\varphi_n\in\operatorname{dom} A$,

$$Au=\sum_{n=1}^{\infty}\alpha_n\lambda_n\varphi_n.$$

In the wave equation (1.1), let

$$f=\sum_{n=1}^{\infty}\alpha_n\varphi_n \quad\text{and}\quad g=\sum_{n=1}^{\infty}\beta_n\varphi_n. \tag{3.5}$$

Let

$$\begin{aligned} u(t) &:= \sum_{n=1}^{\infty}\alpha_n\cos(t\sqrt{\lambda_n})\varphi_n+\sum_{n=1}^{\infty}\beta_n\frac{\sin(t\sqrt{\lambda_n})}{\sqrt{\lambda_n}}\varphi_n, \qquad (3.6)\\ G(t) &:= -\sum_{n=1}^{\infty}\alpha_n\sqrt{\lambda_n}\sin(t\sqrt{\lambda_n})\varphi_n+\sum_{n=1}^{\infty}\beta_n\cos(t\sqrt{\lambda_n})\varphi_n,\\ K(t) &:= -\sum_{n=1}^{\infty}\alpha_n\lambda_n\cos(t\sqrt{\lambda_n})\varphi_n-\sum_{n=1}^{\infty}\beta_n\sqrt{\lambda_n}\sin(t\sqrt{\lambda_n})\varphi_n. \end{aligned}$$

Theorem 3.1 leads to the following result for the wave equation (1.1). The proof is standard; we omit the details.

Theorem 3.3. *Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space with norm $\|\cdot\|$, A be a non-negative self-adjoint operator on $\mathcal{H}$ with the domain* $\operatorname{dom} A$, *and $(\mathcal{E},\operatorname{dom}\mathcal{E})$ be the associated closed quadratic form. Assume that A has compact resolvent. Let $(\varphi_n)_{n\geq 1}$ be an orthonormal basis of $\mathcal{H}$ consisting of the eigenfunctions of A such that $A\varphi_n=\lambda_n\varphi_n$ for $n\geq 1$, $0\leq\lambda_1\leq\cdots\leq\lambda_n\leq\lambda_{n+1}\leq\cdots$, and $\lim_{n\to\infty}\lambda_n=\infty$. Then for any $f\in\operatorname{dom} A$ and $g\in\operatorname{dom}\sqrt{A}$, where f and g are given by* (3.5), *$u(t)$ defined in* (3.6) *is the unique solution of the wave equation* (1.1). *Moreover, $u_t=G(t)\in\operatorname{dom}\sqrt{A}$ and $u_{tt}=K(t)\in\mathcal{H}$ for any $t\in\mathbb{R}$.*

4. Infinite propagation speed

Let (X,d,μ) be a metric measure space, A be a non-negative self-adjoint operator on $L^2(X,\mu)$, and $(\mathcal{E},\operatorname{dom}\mathcal{E})$ be a closed quadratic form on $L^2(X,\mu)$ associated with A. We prove Theorems 1.1 and 1.4 in this section. Some key ideas of Lee [32] are used.

Proof of Theorem 1.1. (a) Since the Sobolev-type inequality (1.3) holds, $\operatorname{dom}\mathcal{E}\subseteq C(X,d)$. Let $v(x,t)$ be the solution of the heat equation (1.2) with initial data f. Then for all $x\in X$

and $t \in (0, \epsilon)$,

$$\begin{aligned} v(x,t) &= \int_X p(t,x,y) f(y)\, d\mu(y) \geq c_1 \int_X t^\eta \exp\Big(-\frac{c_2}{t^\gamma}\Big) f(y)\, d\mu(y) \\ &\geq c_1 \|f\|_1 t^\eta \exp\Big(-\frac{c_2}{t^\gamma}\Big). \end{aligned} \tag{4.1}$$

Define $u(x,t) := \cos(t\sqrt{A})f(x)$. Fix any $x \in X$. Suppose, on the contrary, that there exists $\delta > 0$ such that

$$u(x,t) \leq 0 \quad \text{for all} \quad t < \delta. \tag{4.2}$$

Using the self-adjointness of $\cos(t\sqrt{A})$, we have, for $t \geq 0$,

$$\begin{aligned} \mathcal{E}(u,u) &= \mathcal{E}\big(\cos(t\sqrt{A})f, \cos(t\sqrt{A})f\big) = \big(A\big(\cos^2(t\sqrt{A})f\big), f\big) = \int_0^\infty \lambda \cos^2(t\sqrt{\lambda})\, d(E_\lambda f, f) \\ &\leq \int_0^\infty \lambda\, d(E_\lambda f, f) = (Af, f) = \mathcal{E}(f,f) =: c_3 < \infty. \end{aligned} \tag{4.3}$$

Combining this with the Sobolev-type inequality (1.3), we see that there exist constants $c_4 > 0$ and $\alpha \in (0, 1]$ such that for any $t \geq 0$ and any $x, y \in X$,

$$|u(x,t) - u(y,t)| \leq c_4 d(x,y)^\alpha \sqrt{\mathcal{E}(u,u)} \leq c_4 \sqrt{c_3} d(x,y)^\alpha \leq c_5,$$

where $c_5 > 0$ is a constant. The inequality

$$\sup_{x\in X} |u(x,t)| \leq \sup_{x \in X} |u(x,t) - u(y,t)| + |u(y,t)| \qquad \text{for any } t \geq 0 \text{ and any } y \in X,$$

implies that for any $t > 0$,

$$\begin{aligned} \sup_{x\in X} |u(x,t)| &= \frac{1}{\mu(X)} \int_X \sup_{x\in X} |u(x,t)|\, d\mu(y) \\ &\leq \frac{1}{\mu(X)} \int_X \sup_{x\in X} |u(x,t) - u(y,t)|\, d\mu(y) + \frac{1}{\mu(X)} \int_X |u(y,t)|\, d\mu(y) \\ &\leq c_5 + \frac{\|u(x,t)\|_2}{\mu(X)^{1/2}} \leq c_5 + \frac{\|f\|_2}{\mu(X)^{1/2}} \leq c_6. \end{aligned} \tag{4.4}$$

Lemma 3.2 and (4.2) imply that

$$\begin{aligned} v(x,t) &= \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big) u(x,s)\, ds \\ &\leq \frac{1}{\sqrt{\pi t}} \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big) |u(x,s)|\, ds \\ &\leq \frac{c_6}{\sqrt{\pi t}} \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big)\, ds \qquad \text{for all } t \geq 0. \end{aligned} \tag{4.5}$$

Letting $\omega = (s - \delta)/\sqrt{4t}$ in (4.5), we see that

$$\begin{aligned} v(x,t) &\leq \frac{2c_6}{\sqrt{\pi}} \int_0^\infty \exp\Big(-\big(\frac{\delta}{\sqrt{4t}} + \omega\big)^2\Big)\, d\omega = \frac{2c_6}{\sqrt{\pi}} \exp\Big(-\frac{\delta^2}{4t}\Big) \int_0^\infty \exp\Big(-\omega^2 - \frac{\delta}{\sqrt{t}}\omega\Big)\, d\omega \\ &= \frac{2c_6}{\sqrt{\pi}} \exp\Big(-\frac{\delta^2}{4t}\Big)\Big(\frac{\sqrt{t}}{c_7} + O\big(t^{3/2}\big)\Big) \qquad \text{as } t \to 0^+. \end{aligned} \tag{4.6}$$

where c_7 is a positive constant. Combining (4.1) and (4.6) yields

$$c_1\|f\|_1 t^\eta \exp\Big(-\frac{c_2}{t^\gamma}\Big) \le \frac{2c_6}{\sqrt{\pi}} \exp\Big(-\frac{\delta^2}{4t}\Big)\Big(\frac{\sqrt{t}}{c_7} + O\big(t^{3/2}\big)\Big),$$

As $\|f\|_1 > 0$, we get

$$t^\eta \exp\Big(t^{-\gamma}\big(-c_2 + \frac{\delta^2}{4t^{1-\gamma}}\big)\Big) \le c_8\Big(\frac{\sqrt{t}}{c_7} + O\big(t^{3/2}\big)\Big),$$

where c_8 is a positive constant. Letting $t \to 0^+$, the left side tends to ∞, while the right side tends to 0, a contradiction.

(b) Let $y \in \operatorname{supp}(\mu) \setminus \operatorname{supp}(f)$ and let $U \subseteq X$ be an open subset such that $\operatorname{supp}(f) \subseteq U$ and $y \notin \overline{U}$. By (a), for any $\delta > 0$, there exists $t_0 < \delta$ such that $u(y, t_0) > 0$. By the continuity of $u(x, t_0)$, there exists a neighborhood V_y of y such that $d(U, V_y) > 0$ and $u(x, t_0) > 0$ for all $x \in V_y$. Define $h(x) := u(x, t_0)$ on V_y and $h(x) := 0$ otherwise. Thus,

$$\big(\cos(t_0\sqrt{A})f, h\big)_\mu = \big(u(x, t_0), h\big)_\mu = \int_{V_y} h^2(y)\, d\mu(y) > 0.$$

Therefore, A satisfies (IPS). □

We now consider the case X is a locally compact, but not necessarily bounded.

Proof of Theorem 1.4. The proof of part (a) is similar to that of Theorem 1.1(a); the main difference is that inequality (4.4) is replaced by the assumption $\|\cos(t\sqrt{A})f\|_\infty \le C$.

(b) Suppose, on the contrary, that A satisfies (FPS), i.e., for any open subsets $U_i \subseteq X$ $(i = 1, 2)$ with $d(U_1, U_2) > 0$, there exists $\delta > 0$ such that

$$\big(\cos(t\sqrt{A})f_1, f_2\big)_\mu = 0 \qquad \text{for all } 0 < t < \delta \text{ and all } f_i \in L^2(U_i, \mu).$$

Let V, V_* be two open subsets of U such that $d(V, V_*) > 0$, $\mu(V) > 0$ and $0 < \mu(V_*) < +\infty$. Fix a non-negative $f \in L^2(V, \mu)$ and $\|f\|_1 > 0$. Then there exists $\delta > 0$ such that

$$\big(\cos(t\sqrt{A})f, f_*\big)_\mu = 0 \qquad \text{for all } 0 < t < \delta \text{ and all } f_* \in L^2(V_*, \mu).$$

It follows that $\cos(t\sqrt{A})f(x) = 0$ for μ-a.e. $x \in V_*$ and all $t \in (0, \delta)$.

Let $v(t)$ be the solution of the heat equation (1.2) with initial data f. Since $f \geq 0$, $v(t) = T_t f \geq 0$ for all $t \geq 0$. By Lemma 3.2, for all $t \geq 0$,

$$\begin{aligned}
\int_{V_*} |v(t)|\,d\mu = \int_{V_*} v(t)\,d\mu &= \frac{1}{\sqrt{\pi t}} \int_{V_*} \int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big) \cos(s\sqrt{A}) f\,ds\,d\mu \\
&= \frac{1}{\sqrt{\pi t}} \int_{V_*} \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big) \cos(s\sqrt{A}) f\,ds\,d\mu \\
&= \frac{1}{\sqrt{\pi t}} \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big) \int_{V_*} \cos(s\sqrt{A}) f\,d\mu\,ds \\
&\leq \frac{1}{\sqrt{\pi t}} \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big) \big\| \cos(s\sqrt{A}) f \big\|_2 \sqrt{\mu(V_*)}\,ds \\
&\leq \frac{\sqrt{\mu(V_*)}}{\sqrt{\pi t}} \|f\|_2 \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big)\,ds \\
&\leq \frac{C_1}{\sqrt{\pi t}} \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big)\,ds. \qquad (4.7)
\end{aligned}$$

The rest of the proof is similar to that of Theorem 1.1. First, letting $\omega = (s-\delta)/\sqrt{4t}$ in the left of (4.7) yields, for all $t > 0$,

$$\int_{V_*} |v(t)|\,d\mu \leq \frac{2C_3}{\sqrt{\pi}} \exp\Big(-\frac{\delta^2}{4t}\Big)\Big(\frac{\sqrt{t}}{C_4} + O\big(t^{3/2}\big)\Big). \qquad (4.8)$$

where C_3, C_4 are positive constants.

On the other hand, the lower heat kernel estimate gives, for all $t \in (0, \epsilon)$,

$$v(t) \geq c_1 \|f\|_1 t^\eta \exp\big(-\frac{c_2}{t^\gamma}\big) \qquad \text{on } U.$$

It follows that

$$\int_{V_*} |v(t)|\,d\mu \geq c_1 \mu(V_*) \|f\|_1 t^\eta \exp\big(-\frac{c_2}{t^\gamma}\big),$$

for all $t \in (0, \epsilon)$. Together with (4.8) this implies an analogue of (4) and a contradiction. Hence, A satisfies (IPS). □

5. Fractals with infinite propagation speed

5.1. **Iterated function systems with overlaps.**

Proof of Corollary 1.2. Gu *et al.* [20, Theorem 1.2] obtained the following lower heat kernel estimate:

$$p(t,x,y) \geq \frac{c_1}{V(x,t^{1/\beta})} \exp\left(-c_2 \left(\frac{d_*(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right), \qquad (5.1)$$

for all $t \in (0,1)$ and $x, y \in K$, where $\beta > 2$, d_* is a metric on K (see [20, Section 3]), and $V(x,t^{1/\beta}) := \mu(B_{d_*}(x,t^{1/\beta}))$. Notice that $\gamma := 1/(\beta-1) < 1$. It is also easy to see that for any $t \in (0,1)$, $V(x,t^{1/\beta}) \leq \mu(K) = 1$. Let $c_3 := \sup_{x,y\in K} d_*(x,y)^{\beta/(\beta-1)}$. Using the result $|x-y|\mu([x,y]) \asymp d_*(x,y)^\beta$ in [20, Lemma 3.8] (where $\asymp$ means that the two quantities dominate each other by two constants), we have $0 < c_3 < \infty$. Combining these with (5.1) we get

$$p(t,x,y) \geq c_1 \exp\left(-\frac{c_3}{t^\gamma}\right),$$

for all $t \in (0,1)$ and $x, y \in K$. Thus the corollary follows from Theorem 1.1. □

Proof of Corollary 1.3. The following the lower heat kernel estimate is obtained in [20, Theorem 1.3]:

$$p(t,x,y) \geq \frac{c_1}{V(x,t^{1/\beta})} \exp\left(-c_2\left(\frac{d_*(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right),$$

for all $t \in (0,1)$ and $x, y \in K$, where $\beta > 2$, d_* is a metric on K defined in [20, Section 4], and $V(x,t^{1/\beta}) := \mu_m(B_{d_*}(x,t^{1/\beta}))$. The rest of proof is similar to that of Corollary 1.2. □

5.2. **Fractal blowups.** In this subsection, we will apply our results on infinite propagation speed to a time change Brownian motion on $\mathbb{R}$, which is constructed by blowing up a given self-similar set (see [19]).

Proof of Corollary 1.6. Gu and Hu [19] obtained the lower estimates of the heat kernel:

$$p(t,x,y) \geq \frac{c_1}{V(x,t^{1/\beta})} \exp\left(-c_2\left(\frac{d_*(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right), \tag{5.2}$$

for all $t > 0$ and $x, y \in K_\infty$, where $\beta = 2/d_s > 2$. Thus $\gamma := 1/(\beta-1) < 1$. Let U be a bounded subset on (K_∞, d_*). Then for any $x \in U$, there exists some open subset $V \subseteq K_\infty$ such that $B_{d_*}(x,1) \subseteq V$. Thus for any $t \in (0,1)$, $V(x,t^{1/\beta}) = \mu(B_{d_*}(x,t^{1/\beta})) \leq \mu(B_{d_*}(x,1)) \leq \mu(V) := c_3$. Together with (5.2) this implies that

$$p(t,x,y) \geq c_4 \exp\left(-\frac{c_5}{t^\gamma}\right), \tag{5.3}$$

for all $t \in (0,1)$ and $x, y \in U$, where $c_5 := \sup_{x,y\in U} d_*(x,y)^{\beta/(\beta-1)}$ and $\gamma < 1$. Thus (1.9) holds. Using Theorem 1.4, $(\widetilde{A}, K_\infty, d_*, \mu)$ has (IPS). Since there exist constants $c_6, c_7 > 0$ such that for any $x, y \in K_\infty$,

$$c_6 \frac{d_*(x,y)^{2/d_s}}{V(x,d_*(x,y))} \leq |x-y| \leq c_7 \frac{d_*(x,y)^{2/d_s}}{V(x,d_*(x,y))}$$

(see [19]), d_* is topologically equivalent to the Euclidean metric $d_{|\cdot|}$. Proposition 2.1 implies that $(\widetilde{A}, K_\infty, d_{|\cdot|}, \mu)$ has (IPS). In particular, we can choose $U = K$. Since $\sup_{x,y\in K} d_*(x,y) = 1$, the proof for A and K is similar. □

5.3. **Generalized Sierpiński carpets.** In this section, we illustrate Theorems 1.1 and 1.4 by using both the classes of bounded and unbounded generalized Sierpiński carpets. The following definition is given in [6] and [8].

Let $n \geq 2$, $F_0 = [0,1]^n$, and let $l_F \in \mathbb{N}$ with $l_F \geq 3$ be fixed. For $k \in \mathbb{Z}$, let $\mathcal{Q}_k$ be the collection of closed cubes with side length l_F^{-k} and with vertices at $l_F^{-k}\mathbb{Z}^n$. For $E \subseteq \mathbb{R}^n$, let ∂E and E° be the boundary and interior of E respectively, and let

$$\mathcal{Q}_k(E) := \{Q \in \mathcal{Q}_k : Q^\circ \cap E \neq \emptyset\}. \tag{5.4}$$

For $Q \in \mathcal{Q}_k$, let Ψ_Q be the orientation preserving affine map (i.e., similitude with no rotation part) which maps F_0 onto Q. Define a decreasing sequence $\{F_k\}$ of closed subsets of F_0. Let $1 \leq m_F \leq l_F^n$ be an integer, and F_1 be the union of m_F distinct elements of $\mathcal{Q}_1(F_0)$. We impose the following conditions on F_1.

- (H1) (*Symmetry*) F_1 is preserved by all the isometries of the unit cube F_0.
- (H2) (*Connectedness*) F_1° is connected.
- (H3) (*Non-diagonality*) Let $m \geq 1$ and $B \subseteq F_0$ be a cube of side length $2l_F^{-m}$, which is the union of 2^n distinct elements of $\mathcal{Q}_m$. Then if $\mathrm{int}(F_1 \cap B)$ is non-empty, it is connected.
- (H4) (*Border included*) F_1 contains the line segment $\{x : 0 \leq x_1 \leq 1, x_2 = \cdots = x_n = 0\}$.

One may think of F_1 as being derived from F_0 by removing the interiors of $l_F^n - m_F$ cubes in $\mathcal{Q}_1(F_0)$. Iterating, we obtain a sequence $\{F_k\}$, where F_k is the union of m_F^k cubes in $\mathcal{Q}_k(F_0)$. Formally, we define

$$F_{k+1} = \bigcup_{Q \in \mathcal{Q}_k(F_k)} \Psi_Q(F_1) = \bigcup_{Q \in \mathcal{Q}_1(F_1)} \Psi_Q(F_k), \quad k \geq 1. \tag{5.5}$$

We call the set $F := \bigcap_{k=0}^\infty F_k$ a *generalized Sierpiński carpet (GSC).*

Example 5.1. (Sierpiński carpet, [28]) *Let* $p_1 = 0$, $p_2 = 1/2$, $p_3 = 1$, $p_4 = 1 + \sqrt{-1}/2$, $p_5 = 1 + \sqrt{-1}$, $p_6 = 1/2 + \sqrt{-1}$, $p_7 = \sqrt{-1}$ *and* $p_8 = \sqrt{-1}/2$. *Define* $S_i : \mathbb{C} \to \mathbb{C}$ *as* $S_i(z) = (z - p_i)/3 + p_i$ *for* $i \in \{1, \ldots, 8\}$. *Then there exists a unique nonempty compact subset* F, *which satisfies* $F = \bigcup_{i=1}^8 S_i(F)$. F *is called the standard Sierpiński carpet.*

The standard Sierpiński carpet in the above Example is a GSC with $n = 2$, $l_F = 3$, $m_F = 8$ and with F_1 being obtained from F_0 by removing the middle cube.

We also consider a related set, which has a large-scale structure similar to the small-scale structure of F. Set $F_k := F_0$ for $k < 0$ and for $i \in \mathbb{Z}$, let

$$\widetilde{F}_i = \bigcup_{r=0}^\infty l^r F_{i+r},$$

and $\widetilde{F} = \bigcap_{i=0}^\infty \widetilde{F}_i$, is called an *unbounded generalized Sierpiński carpets.* Let $\mu_k(dx) = m_F^k 1_{F_k} dx$ and μ be the weak limit of the μ_k. Then μ is a constant multiple of the Hausdorff $x^{\log m_F / \log l_F}$-measure on $\widetilde{F}$.

For $n = 2$, Barlow and Bass [2] constructed a Brownian motion on the standard Sierpiński carpet, i.e., a strong Markov process with state space F that has continuous paths and is invariant under an appropriate class of transformation. They later extended the strong Markov process to unbounded Sierpiński carpets $\widetilde{F}$ and obtained upper and lower bounds for the transition densities $p(t, x, y)$ on $\widetilde{F}$ (see [5]). Subsequently, Kusuoka and Zhou [30] gave a different construction of a continuous strong Markov process on the standard Sierpiński carpet, which also has the invariance properties of the Brownian motion constructed in [2]. In [6] the results of [2, 5] are extended to GSC and unbounded Sierpiński carpets embedded in $\mathbb{R}^n$ for $n \geq 3$. Furthermore, the following sub-Gaussian heat kernel estimates are obtained:

$$\begin{aligned}\widetilde{c}_1 \cdot t^{-d_s/2} \exp\left(-\widetilde{c}_2 \left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right) &\leq p(t,x,y) \\ &\leq \widetilde{c}_3 \cdot t^{-d_s/2} \exp\left(-\widetilde{c}_4 \left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right)\end{aligned} \tag{5.6}$$

for any $t > 0$ and any $x, y \in \widetilde{F}$, where d_s is the spectral dimension of F, d_f is the Hausdorff dimension of F, and $d_w := 2d_f/d_s > 2$ (see [6]). Barlow *et al.* [8] showed that, up to scalar multiples of the time parameter, there exists only one such Brownian motion on a generalized Sierpiński carpet. Hence, the Laplacian on GSC is uniquely defined.

Let A be the Laplacian in $L^2(F, \mu)$ associated with the process X_t constructed in [5, 6] or [30] with domain $\operatorname{dom}\mathcal{E}$. Write $\widetilde{X}_t$ for the extension of X_t to the corresponding unbounded Sierpiński carpet $\widetilde{F}$, and let $(\widetilde{A}, \operatorname{dom}\widetilde{A})$ be the associated Laplacian of $\widetilde{X}_t$ in $L^2(\widetilde{F}, \mu)$.

Proof of Corollary 1.7. (a) Let $U \subseteq \widetilde{F}$ be a bounded subset. By virtue of the sub-Gaussian heat kernel estimates (5.6), inequality (1.9) holds with $\epsilon = 1$, $\eta = -d_s/2$, $c_2 = \widetilde{c}_2 \sup_{x,y\in F} |x-y|^{d_w/(d_w-1)} < \infty$, and $\gamma := 1/(d_w - 1) < 1$. It follows from Theorem 1.4 that $\widetilde{F}$ satisfy (IPS). In particular, by choosing $U \subseteq A$, we see that the proof for A and F is similar.

(b) Let d be the Euclidean metric on $\mathbb{R}^n$. Since the Nash inequality holds, i.e., there exists some constant $c_1 > 0$ such that

$$\|u\|_2^{2+4/d_s} \leq c_1 \mathcal{E}(u,u) \|u\|_1^{4/d_s}, \qquad u \in \operatorname{dom}\mathcal{E},$$

(see [6, Theorem 7.1]), we conclude that $\{T_t\}_{t>0}$ is ultracontractive and there exists $c > 0$ such that $\|T_t\|_{1\to\infty} \leq ct^{-d_s/2}$ for any $t \in (0, 1]$. Moreover, it follows from $d_s < 2$ that there exists $M > 0$ such that

$$\|u\|_\infty^2 \leq M\mathcal{E}_*(u,u)$$

for any $u \in \operatorname{dom}\mathcal{E}$ and that $\operatorname{dom}\mathcal{E} \subseteq C(F,d)$ (see [28, Theorem A.6]). Let $f \in \operatorname{dom}\mathcal{E}$ be a non-negative and non-zero function. As in (4.3), for any $t \geq 0$,

$$\mathcal{E}\big(\cos(t\sqrt{A})f, \cos(t\sqrt{A})f\big) \leq \mathcal{E}(f,f) =: C_1 < \infty.$$

Hence $\cos(t\sqrt{A})f \in \operatorname{dom}\mathcal{E} \subseteq C(F,d)$ and

$$\begin{aligned}\mathcal{E}_*\big(\cos(t\sqrt{A})f, \cos(t\sqrt{A})f\big) &= \mathcal{E}\big(\cos(t\sqrt{A})f, \cos(t\sqrt{A})f\big) + \big(\cos(t\sqrt{A})f, \cos(t\sqrt{A})f\big)_\mu \\ &\leq \mathcal{E}(f,f) + \|f\|_2 = \mathcal{E}_*(f,f) < \infty.\end{aligned}$$

Thus $\|\cos(t\sqrt{A})f\|_\infty \leq M\mathcal{E}_*(\cos(t\sqrt{A})f, \cos(t\sqrt{A})f) \leq M\mathcal{E}_*(f,f) < \infty$. Theorem 1.4(a) now implies that for any non-negative and non-zero function $f \in \operatorname{dom}\mathcal{E}$, any $x \in F$, and any $\delta > 0$, there is $t \in (0,\delta)$ such that $\big(\cos(t\sqrt{A})f\big)(x) > 0$. □

6. Finite propagation speed

In this section, we let (X,d,μ) be a metric measure space, A be a non-negative self-adjoint operator on $L^2(X,\mu)$, and $T_t = \exp(-tA)$, $t \geq 0$, be the associated semigroup on $L^2(X,\mu)$.

Definition 6.1. *Let (X,d,μ) be a metric measure space and A be a non-negative self-adjoint operator on $L^2(X,\mu)$. We say that (X,d,μ,A) (or simply A) satisfies the*

(a) Davies-Gaffney estimate *if*

$$\big|(T_t f_1, f_2)_\mu\big| \leq \|f_1\|_2\|f_2\|_2 \exp\big(-r^2/(4t)\big) \tag{6.1}$$

for $t > 0$, open subsets $U_i \subseteq X$ and $f_i \in L^2(U_i,\mu)$, $i = 1,2$, where $r := d(U_1,U_2) > 0$;

(b) generalized Davies-Gaffney estimate *if there exists a constant $c > 0$ such that*

$$\big|(T_t f_1, f_2)_\mu\big| \leq \|f_1\|_2\|f_2\|_2 \exp\big(-r^2/(ct)\big) \tag{6.2}$$

for $t > 0$, open subsets $U_i \subseteq X$ and $f_i \in L^2(U_i,\mu)$, $i = 1,2$, where $r := d(U_1,U_2) > 0$.

It is well known that the Davies-Gaffney estimate holds for essentially all self-adjoint, elliptic, or sub-elliptic second-order differential operators (see [12, 14, 41]). The following theorem shows that the Davies-Gaffney estimate is equivalent to (UPS).

Theorem 6.1. ([12, Theorem 3.4]) *Assume that (X,d,μ) is a metric measure space and A is a non-negative self-adjoint operator on $L^2(X,\mu)$. Then (UPS) and the Davies-Gaffney estimate* (6.1) *are equivalent.*

As a corollary, we show that (BPS) is equivalent to the generalized Davies-Gaffney estimate.

Corollary 6.2. *Assume the hypotheses of Theorem 6.1. Then (X, d, μ, A) satisfies (BPS) if and only if the generalized Davies-Gaffney estimate* (6.2) *holds.*

Proof. Assume that (X, d, μ, A) satisfies (BPS) and let $s > 0$ be as in the definition of (BPS). Define a metric d_* on X as $d_*(x, y) := (1/s)d(x, y)$ for any $x, y \in X$. As $d_*(U_1, U_2) = d(U_1, U_2)/s$, we see that (X, d_*, μ, A) satisfies (UPS). By Theorem 6.1, (X, d_*, μ, A) satisfies the Davies-Gaffney estimate. Using the definition of d_*, we have, for $r := d(U_1, U_2) > 0$,

$$\big|(T_t f_1, f_2)_\mu\big| \le \|f_1\|_2 \|f_2\|_2 \exp\big(-r^2/(4s^2 t)\big) \qquad \text{for all } t > 0.$$

Thus (6.2) holds with $C = 4s^2$.

Conversely, assume that (6.2) holds. Define a metric $\widetilde{d}_*$ on X as $\widetilde{d}_*(x, y) := 2/\sqrt{C} \cdot d(x, y)$ for any $x, y \in X$. Then for open subsets $U_i \subseteq X$, $(i = 1, 2)$, we have $r_* := \widetilde{d}_*(U_1, U_2) = 2r/\sqrt{C}$ if $r = d(U_1, U_2)$. By definition, $(X, \widetilde{d}_*, \mu, A)$ satisfies the Davies-Gaffney estimate. By Theorem 6.1, $(X, \widetilde{d}_*, \mu, A)$ satisfies (UPS). Thus there exists $s := \sqrt{C}/2$ such that for $i = 1, 2$, $U_i \subseteq X$ open, $f_i \in L^2(U_i, \mu)$, and $d(U_1, U_2) > 0$,

$$\big(\cos(t\sqrt{A}) f_1, f_2\big)_\mu = 0 \quad \text{for all } 0 < t < d(U_1, U_2)/s.$$

Hence (X, d, μ, A) satisfies (BPS). □

An analogous relation holds for (FPS), as shown in the following theorem. It can be proved by modifying that of [12, Theorem 3.4]; we omit the details.

Theorem 6.3. *Assume the hypotheses of Theorem 6.1. Then (X, d, μ, A) satisfies (FPS) if and only if for any open subsets $U_i \subseteq X$, $i = 1, 2$, there exists a constant $c > 0$, which may depend on U_1, U_2, such that*

$$\big|(T_t f_1, f_2)_\mu\big| \le \|f_1\|_2 \|f_2\|_2 \exp\big(-r^2/(ct)\big) \tag{6.3}$$

for $t > 0$ and $f_i \in L^2(U_i, \mu)$, where $r := d(U_1, U_2) > 0$.

We now study the relationship between upper heat kernel estimate and wave propagation speed. For any complex number $z \in \mathbb{C}$, we denote the real and imaginary parts of z by $\mathscr{R}(z)$ and $\mathscr{I}(z)$, respectively. Let $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathscr{R}(z) > 0\}$ denote the right half of the complex plane. Define $T_z f := \exp(-zA)f$ for any $f \in L^2(X, \mu)$ and $z \in \mathbb{C}_+$. Since A is a non-negative self-adjoint operator, we have

$$\|T_z f\|_2 \le \|f\|_2 \quad \text{for all } f \in L^2(X, \mu) \text{ and all } z \in \mathbb{C}_+. \tag{6.4}$$

In fact, if $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family associated with A, then for any $z \in \mathbb{C}_+$ and any $f \in L^2(X,\mu)$,

$$\begin{aligned}\|T_z f\|_2^2 &= \Big\|\int_0^\infty \exp(-z\lambda)\, dE_\lambda f\Big\|_2^2 = \int_0^\infty \big|\exp(-z\lambda)\big|^2\, d\|E_\lambda f\|_2^2 \\ &= \int_0^\infty \exp\big(-\Re(z)\lambda\big)^2\, d\|E_\lambda f\|_2^2 = \big\|T_{\Re(z)} f\big\|_2^2 \le \|f\|_2^2,\end{aligned}$$

where the last inequality is because the semigroup $\{T_t\}_{t\ge 0}$ is contractive.

The following lemma is a slight modification of a similar one in [12]; the proof is the same.

Lemma 6.4. ([12, Proposition 2.2]) *Let u be an analytic function on $\mathbb{C}_+$. Assume that there exist positive numbers c_1, γ and a positive number $c_2 = c_2(\gamma)$ (which may depend on γ) such that*

$$|u(z)| \le c_1 \qquad \text{for all } z \in \mathbb{C}_+$$

and

$$|u(t)| \le c_2 \exp(-\gamma/t) \qquad \text{for all } t > 0.$$

Then

$$|u(z)| \le c_1 \exp\big(-\mathscr{R}(\gamma/z)\big) \qquad \text{for all } z \in \mathbb{C}_+.$$

In the following lemma we modify a result in [12] by allowing the constant C to depend on f_i, in order to suit our purposes. We include a proof for completeness.

Lemma 6.5. ([12, Section 3]) *Let (X, d, μ) be a separable metric space. Assume that for $f_i \in L^2(X,\mu)$, $\operatorname{supp}(f_i) \subseteq B_d(x_i, r_i)$, $i = 1, 2$, and $r := d(B_d(x_1, r_1), B_d(x_2, r_2)) > 0$, there exists a constant $c := c(f_1, f_2, B_d(x_1, r_1), B_d(x_2, r_2)) > 0$ such that*

$$\big|(T_t f_1, f_2)_\mu\big| \le c\|f_1\|_2\|f_2\|_2 \exp\big(-r^2/(4t)\big) \qquad \text{for } t > 0. \tag{6.5}$$

Then the Davies-Gaffney estimate (6.1) *holds.*

Proof. Define $u(z) := (T_z f_1, f_2)_\mu$ for $z \in \mathbb{C}_+$. (6.4) implies that

$$\big|u(z)\big| = \big|(T_z f_1, f_2)_\mu\big| \le \|T_z f_1\|_2\|f_2\|_2 \le \|f_1\|_2\|f_2\|_2. \tag{6.6}$$

Combining this with (6.5) and Lemma 6.4 yields

$$\big|u(z)\big| = \big|(T_z f_1, f_2)_\mu\big| \le \|f_1\|_2\|f_2\|_2 \exp\big(-r^2\mathscr{R}(1/(4z)\big) \quad \text{for } z \in \mathbb{C}_+.$$

In particular,

$$\big|(T_t f_1, f_2)_\mu\big| \le \|f_1\|_2\|f_2\|_2 \exp\big(-r^2/(4t)\big) \qquad \text{for } t > 0. \tag{6.7}$$

Now let U_1, U_2 be arbitrary open subsets of X such that $r := d(U_1, U_2) > 0$. Let $f := \sum_{i=1}^k f_i$, where for all $1 \le i \le k$, $f_i \in L^2(B_d(x_i, r_i), \mu)$, $B_d(x_i, r_i) \subset U_1$, and $f_{i_1} f_{i_2} = 0$

in $L^2(X,\mu)$ for all $1\le i_1<i_2\le k$. Similarly, let $g:=\sum_{j=1}^{\ell} g_j$, where $g_j\in L^2(B_d(y_j,s_j),\mu)$, $B_d(y_j,s_j)\subset U_2$ for all $1\le j\le \ell$, and $g_{j_1}g_{j_2}=0$ in $L^2(X,\mu)$ for all $1\le j_1<j_2\le \ell$. It is obvious that $d(B_d(x_i,r_i),B_d(y_j,s_j))\ge d(U_1,U_2)=r$. Thus (6.7) implies that for $t>0$,

$$\begin{aligned}\big|(T_tf,g)_\mu\big| &\le \sum_{i=1}^{k}\sum_{j=1}^{\ell}\big|(T_tf_i,g_j)_\mu\big| \le \sum_{i=1}^{k}\sum_{j=1}^{\ell}\|f_i\|_2\|g_j\|_2\exp\Big(-\frac{r^2}{4t}\Big)\\ &= \Big(\sum_{i=1}^{k}\|f_i\|_2\Big)\Big(\sum_{j=1}^{\ell}\|g_j\|_2\Big)\exp\Big(-\frac{r^2}{4t}\Big)\\ &\le \sqrt{k\ell}\|f\|_2\|g\|_2\exp\Big(-\frac{r^2}{4t}\Big). \qquad (6.8)\end{aligned}$$

Combining this with (6.6) and Lemma 6.4 gives

$$\big|(T_tf,g)_\mu\big|\le \|f\|_2\|g\|_2\exp\big(-r^2/(4t)\big) \qquad \text{for } t>0.$$

To finish the proof of the lemma, it suffices to note that, since (X,d,μ) is a separable metric space, the space of all possible finite linear combinations of functions h of the form $\operatorname{supp}(h)\subset B(x,r)\subset U$ is dense in $L^2(U,\mu)$. Moreover, if $h:=\sum_{i=1}^m h_i$ and $h_i\in L^2(B(x_i,r_i),\mu)$ for all $1\le i\le m$, then there exist functions $\widetilde h_i\in L^2(B(x_i,r_i),\mu)$ such that $h:=\sum_{i=1}^m\widetilde h_i$ and $\widetilde h_{i_1}\widetilde h_{i_2}=0$ in $L^2(X,\mu)$ for all $1\le i_1<i_2\le k$. □

Theorem 6.6. *Let (X,d) be a locally compact separable metric space, μ be a finite Radon measure on (X,d), A be a non-negative self-adjoint operator in $L^2(X,\mu)$, and $(\mathcal{E},\operatorname{dom}\mathcal{E})$ be the closed quadratic form on $L^2(X,\mu)$ associated with A. Assume that the heat kernel $p(t,x,y)$ of $(\mathcal{E},\operatorname{dom}\mathcal{E})$ exists and there exist positive constants c_1,c_2,γ such that for any $Y\subset\subset X$, there exist positive constants $\widetilde c_1:=\widetilde c_1(c_1,Y),\widetilde c_2:=\widetilde c_2(c_2,Y)$ and $\delta:=\delta(Y)>0$ satisfying*

$$p(t,x,y)\le \widetilde c_1\exp\Big(-\frac{d(x,y)^2}{c_1t}\Big) \qquad \textit{for all } t>\delta \textit{ and } x,y\in Y, \qquad (6.9)$$

and

$$p(t,x,y)\le \widetilde c_2t^{-\gamma}\exp\Big(-\frac{d(x,y)^2}{c_2t}\Big) \qquad \textit{for } 0<t\le\delta \textit{ and } x,y\in Y. \qquad (6.10)$$

Then (X,d,μ,A) satisfies (BPS).

Proof. Let $\{T_t\}_{t\ge0}$ be the semigroup on $L^2(X,\mu)$ associated with A. Let $f_i\in L^2(X,\mu)$ with $\operatorname{supp}(f_i)\subseteq B_d(x_i,r_i)$, $i=1,2$, and with $r:=d(B_d(x_1,r_1),B_d(x_2,r_2))>0$. For any $x\in\operatorname{supp}(f_1)$ and $y\in\operatorname{supp}(f_2)$, $d(x,y)^2\ge r^2$ and thus for $t>0$ and $i=1,2$,

$$\exp\Big(-\frac{d(x,y)^2}{c_it}\Big)\le\exp\Big(-\frac{r^2}{c_it}\Big). \qquad (6.11)$$

Choose a subset $Y \subset\subset X$ such that $\mathrm{supp}(f_1) \cup \mathrm{supp}(f_2) \subset Y$. Then

$$\begin{aligned}\big(T_t f_1, f_2\big)_\mu &= \int_X (T_t f_1)(y) f_2(y)\, d\mu(y) = \int_{X\times X} p(t,x,y) f_1(x) f_2(y)\, d\mu(x) d\mu(y) \\ &= \int_{\mathrm{supp}(f_1)\times \mathrm{supp}(f_2)} p(t,x,y) f_1(x) f_2(y)\, d\mu(x) d\mu(y). \qquad (6.12)\end{aligned}$$

Together with (6.9) and (6.12), this implies, for $t > \delta$,

$$\begin{aligned}|(T_t f_1, f_2)_\mu| &\le \widetilde{c}_1 \int_{\mathrm{supp} f_1 \times \mathrm{supp} f_2} \exp\Big(-\frac{r^2}{c_1 t}\Big) \big|f_1(x) f_2(y)\big|\, d\mu(x) d\mu(y) \\ &\le \widetilde{c}_1 \mu(X) \|f_1\|_2 \|f_2\|_2 \exp\Big(-\frac{r^2}{c_1 t}\Big). \qquad (6.13)\end{aligned}$$

Similarly, for $0 < t \le \delta$, combining (6.10), (6.12) and (6.11) yields

$$\begin{aligned}|(T_t f_1, f_2)_\mu| &\le \widetilde{c}_2 t^{-\gamma} \int_{\mathrm{supp} f_1 \times \mathrm{supp} f_2} \exp\Big(-\frac{r^2}{c_2 t}\Big) \big|f_1(x) f_2(y)\big|\, d\mu(y) d\mu(x) \\ &\le \widetilde{c}_2 \mu(X) \|f_1\|_2 \|f_2\|_2 t^{-\gamma} \exp\Big(-\frac{r^2}{c_2 t}\Big). \qquad (6.14)\end{aligned}$$

Since $\lim_{t\to 0^+} t^{-\gamma} \exp(-c/t) = 0$ for any constant $c > 0$, there exists some constant $\widetilde{c}_3 := \widetilde{c}_3(b_2, r, \gamma, \delta) > 0$ such that for $0 < t \le \delta$,

$$t^{-\gamma} \exp\Big(-\frac{r^2}{c_2 t}\Big) = t^{-\gamma} \exp\Big(-\frac{r^2}{c_2(c_2+1)t}\Big) \exp\Big(-\frac{r^2}{(c_2+1)t}\Big) \le \widetilde{c}_3 \exp\Big(-\frac{r^2}{(c_2+1)t}\Big).$$

Together with (6.14), this implies

$$|(T_t f_1, f_2)_\mu| \le \widetilde{c}_2 \widetilde{c}_3 \mu(X) \|f_1\|_2 \|f_2\|_2 \exp\Big(-\frac{r^2}{(c_2+1)t}\Big) \quad \text{for } 0 < t \le \delta. \qquad (6.15)$$

Combining (6.13) and (6.15), we see that there exist constants $c_3 := c_3(c_1, c_2) > 0$ (independent on Y) and $\widetilde{c}_4 := \widetilde{c}_4(\widetilde{c}_1, \widetilde{c}_2, \widetilde{c}_3) > 0$ such that

$$\big|(T_t f_1, f_2)_\mu\big| \le \widetilde{c}_4 \|f_1\|_2 \|f_2\|_2 \exp\Big(-\frac{r^2}{c_3 t}\Big) \quad \text{for } t > 0. \qquad (6.16)$$

Define a metric d_* on X as $d_*(x,y) := (2/\sqrt{b_3}) d(x,y)$ for $x, y \in X$. Then

$$r_* := d_*\big(B(x_1, r_1), B(x_2, r_2)\big) = 2r/\sqrt{c_3}$$

and thus it follows from (6.16) that

$$|(T_t f_1, f_2)_\mu| \le \widetilde{c}_4 \|f_1\|_2 \|f_2\|_2 \exp\Big(-\frac{r_*^2}{4t}\Big) \quad \text{for } t > 0. \qquad (6.17)$$

Lemma 6.5 implies that (X, d_*, μ, A) satisfies the Davies-Gaffney estimate and thus (X, d, μ, A) satisfies the generalized Davies-Gaffney estimate. The assertion now follows by using Corollary 6.2. □

7. Laplacians defined by measures

In this section, let $\mathcal{L}^n$ (and dx) be the Lebesgue measure on $\mathbb{R}^n$, $d_{|\cdot|}$ denote the Euclidean metric, $B(x,r)=\{y:|x-y|<r\}$ denote the Euclidean open ball with radius r and center x, and $\operatorname{diam}(U):=\sup\{|x-y|:x,y\in U\}$ denote the diameter of $U\subseteq\mathbb{R}^n$. Let $\Omega\subseteq\mathbb{R}^n$, $n\geq 1$, be a bounded open subset, $H^1(\Omega)$ be the Sobolev space with inner product

$$\langle u,v\rangle_{H^1(\Omega)}:=\int_\Omega uv\,dx+\int_\Omega \nabla u\nabla v\,dx,$$

and let $H_0^1(\Omega)$ denote the completion of $C_c^\infty(\Omega)$ in the $H^1(\Omega)$ norm.

Let μ be a positive finite Borel measure with $\operatorname{supp}(\mu)\subseteq\overline{\Omega}$ and $\mu(\Omega)>0$. For convenience, we summarize the definition of the Dirichlet Laplacian with respect to a measure μ; details can be found in [21]. We further suppose μ satisfies (MPI) (see (1.16)), which implies that each equivalence class $u\in H_0^1(\Omega)$ contains a unique (in $L^2(\Omega,\mu)$ sense) member $\hat{u}$ that belongs to $L^2(\Omega,\mu)$ and satisfies both conditions below:

(1) there exists a sequence $\{u_n\}$ in $C_c^\infty(\Omega)$ such that $u_n\to\hat{u}$ in $H_0^1(\Omega)$ and $u_n\to\hat{u}$ in $L^2(\Omega,\mu)$;
(2) $\hat{u}$ satisfies inequality (1.16).

We call $\hat{u}$ the $L^2(\Omega,\mu)$-representative of u. Define a mapping $\iota:H_0^1(\Omega)\to L^2(\Omega,\mu)$ by

$$\iota(u)=\hat{u}.$$

ι is a bounded linear operator, but not necessarily injective. Consider the subspace $\mathcal{N}$ of $H_0^1(\Omega)$ defined as

$$\mathcal{N}:=\big\{u\in H_0^1(\Omega):\|\iota(u)\|_2=0\big\}.$$

Now let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $H_0^1(\Omega)$. Then $\iota:\mathcal{N}^\perp\to L^2(\Omega,\mu)$ is injective. Unless explicitly stated otherwise, we will denote the $L^2(\Omega,\mu)$-representative $\hat{u}$ simply by u and identify $\iota(\mathcal{N}^\perp)$ with $\mathcal{N}^\perp$.

Consider a non-negative bilinear form $\mathcal{E}(\cdot,\cdot)$ on $L^2(\Omega,\mu)$ given by

$$\mathcal{E}(u,v):=\int_\Omega \nabla u\cdot\nabla v\,dx \tag{7.1}$$

with domain $\operatorname{dom}\mathcal{E}=\mathcal{N}^\perp$. (MPI) implies that $(\mathcal{E},\operatorname{dom}\mathcal{E})$ is a closed quadratic form on $L^2(\Omega,\mu)$. Hence there exists a non-negative self-adjoint operator A on $L^2(\Omega,\mu)$. We write $\Delta_\mu=-A$, and call it the *(Dirichlet) Laplacian* with respect to μ.

For $n=1$ and $\Omega=(a,b)$. If $\operatorname{supp}(\mu)=[a,b]$, then $\operatorname{dom}\mathcal{E}=H_0^1(a,b)$ (see [10]). For $n\geq 2$, we have the following result.

Proposition 7.1. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, be a bounded open subset and μ be a positive finite Borel measure. Assume that μ satisfies* (MPI) *and let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be the closed quadratic form defined in* (7.1).

(a) *If μ is equivalent to the restriction of Lebesgue measure on $\overline{\Omega}$, then* $\operatorname{dom}\mathcal{E} = H_0^1(\Omega)$.
(b) *If* $\operatorname{dom}\mathcal{E} = H_0^1(\Omega)$, *then $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a regular and strongly local Dirichlet form on $L^2(\Omega, \mu)$ with $C_c^\infty(\Omega)$ being a core.*

Proof. (a) Since $\operatorname{dom}\mathcal{E} = \mathcal{N}^\perp$, it suffices to show that $\mathcal{N} = \{0\}$. Assume that $u \in \mathcal{N}$. Then $\|\iota(u)\|_2 = 0$. Thus $\iota(u) = 0$ μ-a.e., and since μ is equivalent to Lebesgue measure, we have $u = 0 \in H_0^1(\Omega)$. Hence $\mathcal{N} = \{0\}$ and thus $\operatorname{dom}\mathcal{E} = H_0^1(\Omega)$.

(b) Since $(\mathcal{E}, H_0^1(\Omega))$ is a Dirichlet form on $L^2(\Omega, dx)$, $(\mathcal{E}, H_0^1(\Omega))$ has the Markov property. Thus $(\mathcal{E}, H_0^1(\Omega))$ is a Dirichlet form on $L^2(\Omega, \mu)$.

Next, we show that $C_c^\infty(\Omega)$ is a core of $(\mathcal{E}, \operatorname{dom}\mathcal{E})$. Since $C_c^\infty(\Omega)$ is dense in $C_c(\Omega)$ with respect to the supremum norm, it suffices to show that $C_c^\infty(\Omega)$ is dense in $\operatorname{dom}\mathcal{E}$ with respect to the $\mathcal{E}_*$-norm, where $\mathcal{E}_*(u,v) := \mathcal{E}(u,v) + (u,v)_\mu$. Let $u \in \operatorname{dom}\mathcal{E}$. Then there exists a sequence $u_m \in C_c^\infty(\Omega)$ such that $u_m \to u$ in $H_0^1(\Omega)$, i.e.,

$$\mathcal{E}(u_m - u, u_m - u) = \|u_m - u\|_{H_0^1(\Omega)} \to 0 \quad \text{as } m \to +\infty.$$

By (1.16), we have $\|u_m - u\|_2 \leq C\|u_m - u\|_{H_0^1(\Omega)} \to 0$, as $m \to +\infty$. Hence, $u_m \to u$ with respect to the $\mathcal{E}_*$-norm. Therefore, $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is regular.

Finally, let $u \in \operatorname{dom}\mathcal{E}$ be constant on a neighborhood of the support of some $v \in \operatorname{dom}\mathcal{E}$. Then $\nabla u \equiv 0$ on a neighborhood of $\operatorname{supp}(v)$. Thus $\operatorname{supp}(\nabla u) \cap \operatorname{supp}(\nabla v) = \emptyset$ and hence $\mathcal{E}(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx = 0$. It follows that the Dirichlet form is strongly local. □

Hereafter, assume that μ satisfies (MPI) and is equivalent to the restriction of Lebesgue measure on $\overline{\Omega}$, and let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be defined as in (7.1). Then $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a regular and strongly local Dirichlet form on $L^2(\Omega, \mu)$ with domain $\operatorname{dom}\mathcal{E} = H_0^1(\Omega)$. Thus for every $u \in \operatorname{dom}\mathcal{E} \cap L^\infty(\Omega, \mu)$, there exists a uniquely positive finite Radon measure $\mu_{\langle u \rangle}$ on Ω uniquely defined by the formula

$$\int_\Omega v \, d\mu_{\langle u \rangle} \quad = \quad \mathcal{E}(u, vu) - \frac{1}{2}\mathcal{E}(u^2, v) \quad \text{for all } v \in \operatorname{dom}\mathcal{E} \cap C_c(\Omega)$$

(see, e.g., [17]). $\mu_{\langle u \rangle}$ is called the *energy measure* of u. Define a bounded signed measure $\mu_{\langle u,v \rangle}$, $u, v \in \mu_{\langle u,v \rangle}$, by

$$\mu_{\langle u,v \rangle} := \frac{1}{2}(\mu_{\langle u+v \rangle} - \mu_{\langle u \rangle} - \mu_{\langle v \rangle}).$$

Hence, we can write

$$\mathcal{E}(u,v) = \int_\Omega d\mu_{\langle u,v \rangle}.$$

For any open set $U \subset \Omega$, let $\mathcal{F}_{\rm loc}(U)$ denote the set of all μ-measurable functions u on U such that for every $V \subset\subset U$, there exists a function $v \in \operatorname{dom}\mathcal{E}$ with $v = u$ μ-a.e. on V. As $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is local, one can extend the definition of the energy measure to $\mathcal{F}_{\rm loc}(U)$. For $u, v \in \mathcal{F}_{\rm loc}(U)$ the signed Radon measure $\mu_{\langle u,v\rangle}$ on U will be defined via its restriction to $V \subset\subset U$ by $1_V\, d\mu_{\langle u,v\rangle} := 1_V\, d\mu_{\langle u',v'\rangle}$, where u' and v' are suitably chosen functions in $\operatorname{dom}\mathcal{E}$ which coincide on V with u and v, respectively.

Definition 7.1. *The energy measure $\mu_{\langle\cdot,\cdot\rangle}$ defines a pseudo metric ρ on Ω by*

$$\rho(x,y) := \sup\big\{u(x) - u(y) : u \in \mathcal{F}_{\rm loc}(\Omega) \cap C(\Omega), \mu_{\langle u\rangle} \leq \mu \text{ on } \Omega\big\},$$

called the intrinsic *(or* Carathéodory*)* metric*.*

The condition $\mu_{\langle u\rangle} \leq \mu$ in (7.1) means that the energy measure $\mu_{\langle u\rangle}$ is absolutely continuous with respect to μ with Radon-Nikodym derivative $d\mu_{\langle u\rangle}/d\mu \leq 1$ μ-a.e. on Ω. The intrinsic metric is a generalization of the classical notion

$$\rho(x,y) = \sup\big\{u(x) - u(y) : f \in C^1,\ |\nabla f| \leq 1\big\};$$

those induced by strongly local regular Dirichlet forms were first studied by Biroli, Mosco and Sturm (see [16, 36, 9, 22, 24] and the references therein).

Definition 7.2. *A strongly local Dirichlet form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ on $L^2(\Omega,\mu)$ is said to be* strongly regular *if it is regular and if ρ (defined by* (7.1)*) is a metric on Ω whose topology coincides with the original one.*

We say that a measure μ is *upper s-regular* for $s > 0$, if there exists some $c > 0$ such that for all $x \in \operatorname{supp}(\mu)$ and all $0 \leq r \leq \operatorname{diam}(\operatorname{supp}(\mu))$,

$$\mu(B(x,r)) \geq cr^s.$$

Proposition 7.2. *Let $\Omega = (a,b) \subset \mathbb{R}$ and μ be a positive finite Borel measure. Assume that μ is upper s-regular for some $s > 0$ and is equivalent to the restriction of Lebesgue measure on $\overline{\Omega}$ with $d\mu/dx = f \in L^\infty(\Omega,\mu)$. Then ρ is a metric on Ω and is topologically equivalent to the Euclidean metric.*

Proof. It is well known that (MPI) holds on $\mathbb{R}$ (see, e.g., [34, 35, 21] for this and related results). Propositions 7.1 implies that $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a regular, strongly local Dirichlet form on $L^2(\Omega,\mu)$ with domain $\operatorname{dom}\mathcal{E} = H_0^1(\Omega)$. Define a map $u : \Omega \to \mathbb{R}$ by

$$u(x) := \|f\|_\infty^{-1/2} \int_0^x f(t)\, dt.$$

Then u is absolutely continuous and $u'(x) = \|f\|_\infty^{-1/2} f(x) \in L^\infty(\Omega, dx)$. Thus $u \in H^1(\Omega) \subset \mathcal{F}_{\rm loc}(\Omega)$. Since μ is upper s-regular for some $s > 0$, there exists a constant $c_1 > 0$ such that, for any $[x,y]\subseteq\overline{\Omega}$

$$\big|u(x)-u(y)\big| = \|f\|_\infty^{-1/2}\mu[x,y] \geq c_1|x-y|^s. \tag{7.2}$$

Moreover, since $[x,y]\subseteq\overline{\Omega}$,

$$\int_x^y d\mu_{\langle u\rangle} = \int_x^y (u'(t))^2\,dt = \int_x^y \frac{f(t)}{\|f\|_\infty}\,d\mu,$$

we have $d\mu_{\langle u\rangle}/d\mu = f(t)/\|f\|_\infty \leq 1$ μ-a.e. on $\overline{\Omega}$. Combining this with the definition of ρ and (7.2), we obtain

$$c_1|x-y|^s \leq \rho(x,y) \quad \text{for all} \quad x,y\in\overline{\Omega}. \tag{7.3}$$

On the other hand, assume $u\in\mathcal{F}_{\rm loc}(\Omega)\cap C(\Omega)$ with $\mu_{\langle u\rangle}\leq\mu$ on Ω. By the definition of $\mathcal{F}_{\rm loc}(\Omega)$, for any $x<y\in\Omega$, there exists $v\in H_0^1(\Omega)=\operatorname{dom}\mathcal{E}$ such that $v=u$ on $[x,y]$. Thus

$$\begin{aligned}\big|u(x)-u(y)\big| &= \int_x^y u'(t)\,dt \leq |x-y|^{1/2}\Big(\int_x^y \big(u'(t)\big)^2\,dt\Big)^{1/2} = |x-y|^{1/2}\Big(\int_x^y d\mu_{\langle u\rangle}\Big)^{1/2}\\ &\leq |x-y|^{1/2}\Big(\int_x^y f(t)\,dt\Big)^{1/2} \leq \|f\|_\infty^{1/2}|x-y|.\end{aligned}$$

Hence, for any $x,y\in\Omega$,

$$\rho(x,y)\leq\|f\|_\infty^{1/2}|x-y|. \tag{7.4}$$

Combining (7.3) and (7.4) proves that ρ is a metric and is topologically equivalent to the Euclidean metric. □

Proposition 7.3. *Let $\Omega\subset\mathbb{R}^n$, $n\geq 1$, be a bounded domain. Assume μ is equivalent to the restriction of Lebesgue measure on $\overline{\Omega}$ with $d\mu/dx=f\in L^\infty(\Omega,\mu)$ and for every $V\subset\subset\Omega$, there exists some constant $\varepsilon(V)$ such that $f\geq\varepsilon(V)>0$ μ-a.e. on V. Then ρ is a metric on Ω and the topology induced by ρ coincides with the original one.*

Proof. We use the method in [47, Theorem 4.1]. Fix any $w=(w_1,\dots,w_n)\in\mathbb{R}^n$ and define a map $g_w:\Omega\to\mathbb{R}$ by

$$g_w(x) := w\cdot x = \sum_{i=1}^n w_i x_i.$$

Then $g_w\in C^\infty(\Omega)$ with $\nabla g_w = w$. Fix any open subset $V\subset\subset\Omega$. Choose an open subset U such that $V\subset\subset U\subset\subset\Omega$. Let $\xi=\xi(V,U)\in C_c^\infty(\Omega)$ be a cut-off function such that

$$\begin{cases}\xi\equiv 1 & \text{on } V,\\ \xi\equiv 0 & \text{on } \Omega\setminus U,\\ 0\leq\xi\leq 1 & \text{on } U\setminus V.\end{cases}$$

Then $\xi g_w\in C_c^\infty(\Omega)\subset H_0^1(\Omega)=\operatorname{dom}\mathcal{E}$, $\xi g_w = g_w$ on V, and $\xi g_w = 0$ on $\Omega\setminus U$. For all $x\in\Omega$, using Cauchy-Schwarz inequality and the boundedness of Ω, we have

$$\big|\nabla(\xi g_w)(x)\big| \;\leq\; c_1(V)|g_w(x)|+|w| \leq c_1(V)|x|\cdot|w|+|w| \leq c_2(V)|w| := c_3(V,w).$$

Let $v := (\sqrt{\varepsilon(U)}/c_3(V,w))\xi g_w$. Then

$$|\nabla v|^2 = \frac{\varepsilon(U)}{(c_3(V,w))^2}\big|\nabla(\xi g_w)\big|^2 \le \varepsilon(U).$$

Since $f(x) \ge \varepsilon(U)$ for any $x \in U$, for any $F \subset\subset \Omega$,

$$\int_F d\mu_{\langle v\rangle} = \int_F |\nabla v|^2\,dx = \int_{F\cap U} |\nabla v|^2\,dx \le \int_{F\cap U} \varepsilon(U)\,dx \le \int_{F\cap U} f\,dx \le \int_F d\mu.$$

Thus $d\mu_{\langle v\rangle}/d\mu \le 1$ μ-a.e. on Ω and hence $\mu_{\langle v\rangle} \le \mu$. For any distinct $y, y' \in V$, choose $w = (y-y')/(c_2(V)|y-y'|)$. Then $c_3(V,w) = c_2(V)|w| = 1$. As $\xi \equiv 1$ on $V \subseteq U$, we have

$$v(y) - v(y') = \frac{\sqrt{\varepsilon(U)}}{c_3(V,w)}\big(g_w(y) - g_w(y')\big) = \sqrt{\varepsilon(U)}w\cdot(y-y') = \frac{\sqrt{\varepsilon(U)}}{c_2(V)}|y-y'|.$$

By the definition of ρ, for any distinct $y, y' \in V$,

$$\frac{\sqrt{\varepsilon(U)}}{c_2(V)}|y-y'| \le \rho(y,y').$$

Thus for any $V \subset\subset \Omega$, there exists $c_4(V) > 0$ such that for any $x, y \in V$

$$c_4(V)|x-y| \le \rho(x,y). \tag{7.5}$$

On the other hand, assume $u \in \mathcal{F}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ with $\mu_{\langle u\rangle} \le \mu$. It follows that for any $F \subset\subset \Omega$,

$$\int_F f\,dx = \int_F d\mu \ge \int_F d\mu_{\langle u\rangle} = \int_F |\nabla u|^2\,dx.$$

Thus for Lebesgue a.e. $x \in \Omega$,

$$\big|\nabla u(x)\big|^2 \le \|f\|_\infty. \tag{7.6}$$

If $n = 1$, then by (7.4),

$$\rho(x,y) \le \|f\|_\infty^{1/2}|x-y| \quad \text{for all} \quad x, y \in \Omega.$$

For $n \ge 2$, we use an argument in [47]. We fix arbitrary $x, y \in \Omega$. Without loss of generality, let $x = (0,\dots,0)$ and $y = (R,0,\dots,0)$. Let $B'(0,\epsilon)$ be the Euclidean ball in $\mathbb{R}^{n-1}$ with radius ϵ and centered at the origin. Let $C_\epsilon = [0,R] \times B'(0,\epsilon) = \{w = (r,w') \in \mathbb{R}^n : 0 \le r \le R, |w'| < \epsilon\}$. Moreover, let $\mathcal{L}^{n-1}$ (and $d'w$) be the $(n-1)$-dimensional Lebesgue measure. Since u is continuous, by Lebesgue's density theorem,

$$u(x) - u(y) = \lim_{\epsilon\to 0^+} \frac{1}{\mathcal{L}^{n-1}(B'(0,\epsilon))}\int_{B'_d(0,\epsilon)} \big(u(0,w') - u(R,w')\big)\,d'w'. \tag{7.7}$$

By using a similar argument as that in [47, Theorem 4.1], we get

$$u(x) - u(y) = \lim_{\epsilon\to 0^+} \frac{1}{\mathcal{L}^{n-1}(B'(0,\epsilon))}\int_{C_\epsilon} \frac{\partial}{\partial x_1}u(w)\,dw. \tag{7.8}$$

Combining (7.8) and (7.6) implies

$$\begin{aligned}\big|u(x)-u(y)\big| &\leq \lim_{\epsilon\to 0}\frac{1}{\mathcal{L}^{n-1}(B'(0,\epsilon))}\int_{C_\epsilon}\Big|\frac{\partial}{\partial x_1}u(w)\Big|\,dw\\ &\leq \lim_{\epsilon\to 0}\frac{1}{\mathcal{L}^{n-1}(B'(0,\epsilon))}\int_{C_\epsilon}\|f\|_\infty^{1/2}\,dw=\|f\|_\infty^{1/2}R=\|f\|_\infty^{1/2}|x-y|.\end{aligned}$$

In other words, for all those $x,y\in\Omega$ that can be connected by a straight line in Ω, we have $|u(x)-u(y)|\leq\|f\|_\infty^{1/2}|x-y|$ and hence $\rho(x,y)\leq\|f\|_\infty^{1/2}|x-y|$. Combining this with (7.5) proves that ρ is a metric and is topologically equivalent to the Euclidean metric. □

Following [46], we state and discuss several properties of the associated Dirichlet form $(\mathcal{E},\operatorname{dom}\mathcal{E})$ on U.

Definition 7.3. *Assume $(\mathcal{E},\operatorname{dom}\mathcal{E})$ is a strongly regular Dirichlet form on $L^2(\Omega,\mu)$. Fix an arbitrary subset $U\subset\Omega$.*

1. *Completeness property (C): For any ball $B_\rho(x,2r)\subset U$, the closed ball $\overline{B}_\rho(x,r)$ is complete (or, equivalently, compact) on the metric space (Ω,ρ), where $\overline{B}_\rho(x,r):=\{y\in\Omega:\rho(x,y)\leq r\}$.*
2. *Doubling property (VD): There exists a constant $N:=N(U)$ such that for all balls $B_\rho(x,2r)\subset U$,*
$$\mu\big(B_\rho(x,2r)\big)\leq N\mu\big(B_\rho(x,r)\big). \tag{7.9}$$
3. *Strong Poincaré inequality (SPI): There exists a constant $C_P:=C_P(U)$ such that for all balls $B_\rho(x,r)\subseteq U$ and all $u\in\operatorname{dom}\mathcal{E}$,*
$$\int_{B_\rho(x,r)}|u-u_{B_\rho(x,r),\mu}|^2\,d\mu\leq C_P\cdot r^2\int_{B_\rho(x,r)}d\mu_{\langle u\rangle}, \tag{7.10}$$
where $u_{B_\rho(x,r),\mu}:=\int_{B_\rho(x,r)}u\,d\mu/\mu(B_\rho(x,r))$.

Theorem 7.4. ([46, Theorem 4.1]) *Let X be a locally compact separable Hausdorff space and μ be a Radon measure with* $\operatorname{supp}(\mu)=X$. *Assume $(\mathcal{E},\operatorname{dom}\mathcal{E})$ is a strongly regular Dirichlet form on $L^2(X,\mu)$. Assume (C), (VD), and (SPI) are simultaneously satisfied on the open set $Y\subset X$. Then for every $\epsilon>0$, there exists a constant $C>0$, depending only on ϵ, $N=N(Y)$ and $c_P=c_P(Y)$ (in* (7.9) *and* (7.10) *respectively), such that the following estimate holds for all $x,y\in Y$ and $t>0$:*

$$p(t,x,y)\leq C\mu\big(B_\rho(x,\sqrt{\tau})\big)^{-1/2}\mu\big(B_\rho(y,\sqrt{\tau})\big)^{-1/2}\exp\Big(-\frac{\rho(x,y)^2}{(4+\epsilon)t}\Big),$$

where $\tau=\inf\{t,R^2\}$ with $R:=\inf\{\rho(x,X\setminus Y),\rho(y,X\setminus Y)\}$ ($R:=+\infty$ if $X=Y$).

In fact, Theorem 7.4 is a special case of Theorem 4.1 in [46] with $\mathcal{E}_t\equiv\mathcal{E}$, $\kappa=1$ and $p(t,y,s,x)=p(t-s,y,x)$.

Proposition 7.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that μ is equivalent to the restriction of Lebesgue measure on $\overline{\Omega}$ with $d\mu/dx = f \in L^\infty(\Omega,\mu)$ and $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ defined in* (7.1) *is strongly regular Dirichlet form with domain* $\operatorname{dom}\mathcal{E} = H_0^1(\Omega)$. *Then (SPI) holds.*

Proof. By a result of Jerison [22], there exists some constant $c_1 > 0$ such that

$$\int_{B_\rho(x,r)} (u - u_{B_\rho(x,r)})^2\, dy \le c_1 r^2 \int_{B_\rho(x,r)} |\nabla u|^2\, dy, \tag{7.11}$$

for all $u \in C^\infty(\overline{B}_\rho(x,r))$, where $u_{B_\rho(x,r)} := \int_{B_\rho(x,r)} u\, dy/\mathcal{L}^n(B_\rho(x,r))$. Since $C^\infty(B_\rho(x,r))$ is dense in $H^1(B_\rho(x,r))$, (7.11) holds for $u \in H^1(B_\rho(x,r))$ or $u \in H_0^1(\Omega)$. Thus, for $u \in H_0^1(\Omega)$,

$$\begin{aligned}
\int_{B_\rho(x,r)} (u - u_{B_\rho(x,r),\mu})^2\, d\mu(y) &= \min_{a\in\mathbb{R}} \int_{B_\rho(x,r)} (u-a)^2\, d\mu(y) \\
&\le \int_{B_\rho(x,r)} (u - u_{B_\rho(x,r)})^2\, d\mu(y) \\
&= \int_{B_\rho(x,r)} (u - u_{B_\rho(x,r)})^2 f\, dy \\
&\le \|f\|_\infty \int_{B_\rho(x,r)} (u - u_{B_\rho(x,r)})^2\, dy \\
&\le c_1 \|f\|_\infty r^2 \int_{B_\rho(x,r)} |\nabla u|^2\, dy.
\end{aligned}$$

Therefore, (SPI) holds. □

Proof of Theorem 1.8. Since $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and μ is equivalent to the restriction of Lebesgue measure on $\overline{\Omega}$ with $d\mu/dx = f \in L^\infty(\Omega,\mu)$, μ is a positive finite Radon measure with $\operatorname{supp}(\mu) = \overline{\Omega}$ and satisfies (MPI). Let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be the closed quadratic form defined in (7.1) and Δ_μ be the associated Dirichlet Laplacian with respect to μ. Proposition 7.1 implies that $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a regular, strongly local Dirichlet form on $L^2(\Omega,\mu)$ with domain $\operatorname{dom}\mathcal{E} = H_0^1(\Omega)$; moreover, Proposition 7.3 implies that ρ is a metric and is topologically equivalent to the Euclidean metric. Hence, $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a strongly regular Dirichlet form on $L^2(\Omega,\mu)$ with domain $\operatorname{dom}\mathcal{E} = H_0^1(\Omega)$.

Fix any $V \subset\subset \Omega$. Since ρ is topologically equivalent to the Euclidean metric $d_{|\cdot|}$, $\overline{B}_\rho(x,r)$ is closed in $(\Omega, d_{|\cdot|})$ for any ball $B_\rho(x,2r) \subset V$. Thus $\overline{B}_\rho(x,r)$ is compact in $(\Omega, d_{|\cdot|})$ and hence $\overline{B}_\rho(x,r)$ is compact in (Ω,ρ). Therefore, Property (C) holds on V. Proposition 7.3 implies that there exists $c_1 := c_1(V) > 0$ such that for any $x, y \in V$

$$c_1|x-y| \le \rho(x,y) \le \|f\|_\infty^{1/2}|x-y|. \tag{7.12}$$

It follows that for any ball $B_\rho(x,r) \subseteq V$,

$$B(x, r/\|f\|_\infty^{1/2}) \subseteq B_\rho(x,r) \subseteq B(x, r/c_1), \tag{7.13}$$

and thus there exist constants $c_3 := c_3(V) > 0$ such that

$$w_n r^n = \mathcal{L}^n(B(x, r/\|f\|_\infty^{1/2})) \le \mathcal{L}^n(B_\rho(x,r)) \le \mathcal{L}^n(B(x, r/c_1)) = c_3 r^n,$$

where $w_n := \mathcal{L}^n(B(x,1))$. The inequality

$$\mu(B_\rho(x,r)) \le \|f\|_\infty \mathcal{L}^n(B_\rho(x,r)) \le c_3 \|f\|_\infty r^n$$

and

$$\mu(B_\rho(x,r)) \ge \varepsilon(V)\mathcal{L}^n(B_\rho(x,r)) \ge \varepsilon(V) w_n r^n \tag{7.14}$$

imply that there exists $N := N(V) > 0$ such that for all balls $B_\rho(x, 2r) \subset V$

$$\mu(B_\rho(x,2r)) \le N\mu(B_\rho(x,r)).$$

Thus (VD) holds on V. Proposition 7.5 shows that (SPI) holds on Ω. Hence, Theorem 7.4 implies that the heat kernel $p(t,x,y)$ of $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ exists; moreover, for every $\epsilon > 0$, there exists a constant $c_4 := c_4(\epsilon, V)$ such that the following estimate holds for all $x, y \in V$ and $t > 0$:

$$p(t,x,y) \le c_4 \mu\big(B_\rho(x,\sqrt{\tau_V})\big)^{-1/2} \cdot \mu\big(B_\rho(y,\sqrt{\tau_V})\big)^{-1/2} \cdot \exp\Big(-\frac{\rho(x,y)^2}{(4+\epsilon)t}\Big), \tag{7.15}$$

where $\tau_V := \inf\{t, R^2\}$ with $R := \inf\{\rho(x,\Omega\backslash V), \rho(y,\Omega\backslash V)\}$ ($R := +\infty$ if $\Omega = V$). It is easy to see that $B_\rho(x,\sqrt{\tau_V}) \subseteq V$ and $B_\rho(y,\sqrt{\tau_V}) \subseteq V$. Using (7.14), there exists $c_5 := c_5(V)$ such that

$$\mu\big(B_\rho(x,\sqrt{\tau_V})\big)^{-1/2} \cdot \mu\big(B_\rho(y,\sqrt{\tau_V})\big)^{-1/2} \le c_5 \tau_V^{-n/2}.$$

Together with (7.15), this implies there exists $c_6 := c_6(\epsilon, V) > 0$ such that

$$p(t,x,y) \le c_6 \tau_V^{-n/2} \exp\Big(-\frac{\rho(x,y)^2}{(4+\epsilon)t}\Big) \qquad \text{for all } x,y \in V \text{ and all } t > 0.$$

Choose a set U such that $V \subset\subset U \subset\subset \Omega$. We observe that there exists $c_7 := c_7(\epsilon, V) > 0$

$$p(t,x,y) \le c_7 \tau_U^{-n/2} \exp\Big(-\frac{\rho(x,y)^2}{(4+\epsilon)t}\Big) \qquad \text{for all } x,y \in U \text{ and all } t > 0.$$

Let $\delta := \rho^2(V, \Omega\setminus U) > 0$. Then $\tau_U = t$ for $0 < t \le \delta$ and $x, y \in V$; $\tau_U \ge \delta$ for $t > \delta$ and $x, y \in V$. Thus for $x, y \in V$ and $0 < t \le \delta$,

$$p(t,x,y) \le c_7 t^{-n/2} \exp\Big(-\frac{\rho(x,y)^2}{(4+\epsilon)t}\Big),$$

while for $x, y \in V$ and $t > \delta$,

$$p(t,x,y) \le c_7 \delta^{-n/2} \exp\Big(-\frac{\rho(x,y)^2}{(4+\epsilon)t}\Big).$$

By Theorem 6.6 and Proposition 2.1, $(\Omega, \rho, \mu, -\Delta_\mu)$ has (BPS) and thus $(\Omega, d_{|\cdot|}, \mu, -\Delta_\mu)$ has (FPS). $\square$

8. Self-similar measures with overlaps

In this section, let $B(x,r)=\{y:|x-y|<r\}$ denote an open ball with radius r and center x. We consider self-similar measures on $\mathbb{R}$ defined by IFSs of contractive similitudes of the form

$$S_i(x)=\rho_i x+b_i, \qquad i=0,1,\dots,N,$$

where for each i, $0<\rho_i<1$ and $b_i\in\mathbb{R}$. Let K be the unique attractor (or self-similar set) and let μ be the self-similar measure associated with probability weights $\{p_i\}_{i=0}^N$.

Our purpose in this section is to study self-similar measures μ for which the Dirichlet Laplacian $-\Delta_\mu$ has finite propagation speed. Assume μ is equivalent to the restriction of Lebesgue measure to K with density $d\mu/dx =: f(x)$. Then $f(x)=\lim_{r\to 0}\mu(B(x,r))/(2r)$ for Lebesgue a.e. $x\in\mathbb{R}$ and thus

$$f(x)=\sum_{i=0}^N \frac{p_i}{\rho_i} f\circ S_i^{-1}(x) \qquad \text{for Lebesgue a.e. } x\in\mathbb{R}. \tag{8.1}$$

It follows that $f=d\mu/dx$ if and only if f satisfies (8.1) and $\int_K f\,dx=1$.

8.1. **A family of scaling functions.** In this subsection we consider the family of IFSs defined in (1.17). It is known (see, e.g., [31, 37]) that if N is odd, then μ is absolutely continuous with respect to Lebesgue measure with density $f\in L^2(\mathbb{R})$.

Define $\widetilde{f}$ by

$$\widetilde{f}(x):=\begin{cases} x^{\log_2(N+1)-1}, & x\in[0,1],\\ 0, & x\in(-\infty,0),\end{cases} \tag{8.2}$$

and

$$\widetilde{f}(x)=\frac{N+1}{2}\widetilde{f}\Big(\frac{x}{2}\Big)-\sum_{i=1}^N \widetilde{f}(x-i) \quad \text{for all } x\in\mathbb{R}. \tag{8.3}$$

The following proposition shows that the definitions in (8.2) and (8.3) are compatible and $\widetilde{f}$ is well defined.

Proposition 8.1. *Let $N\ge 3$ be an odd integer, μ be the self-similar measure defined by the IFS (1.17) together with probability weights $p_i=1/(N+1)$, and f be the density of μ.*

(a) *$\widetilde{f}$ is well defined and $f=c\widetilde{f}(x)$, where $c^{-1}:=\int_0^N \widetilde{f}\,dx$.*
(b) *f is continuous, bounded on $\mathbb{R}$, and positive on $(0,N)$.*

Proof. (a) We first notice that for $x\in(-\infty,1]$, $\widetilde{f}(x)$ defined by (8.2) satisfies (8.3). In fact,

$$\widetilde{f}(x)=0=\frac{N+1}{2}\widetilde{f}\Big(\frac{x}{2}\Big)-\sum_{i=0}^N \widetilde{f}(x-i) \quad \text{for any } x\in(-\infty,0)$$

and for any $x \in [0,1]$,

$$\widetilde{f}(x) = x^{\log_2(N+1)-1} = \frac{N+1}{2}\Big(\frac{x}{2}\Big)^{\log_2(N+1)-1} = \frac{N+1}{2}\widetilde{f}\Big(\frac{x}{2}\Big) - \sum_{i=0}^{N}\widetilde{f}(x-i).$$

Next, we show that for $x \in (1,+\infty)$, the value $\widetilde{f}(x)$ is uniquely defined. For $x \in (1,2]$, we have $x/2 \in (0,1]$ and $x - i \in (-\infty,1]$ for any $i = 1,\dots,N$. Combining (8.2) and (8.3), we see for $x \in (1,2]$, $\widetilde{f}(x)$ is uniquely defined as

$$\widetilde{f}(x) = \frac{N+1}{2}\widetilde{f}\Big(\frac{x}{2}\Big) - \sum_{i=1}^{N}\widetilde{f}(x-i) = x^{\log_2(N+1)-1}.$$

By induction, for any $x \in (1,+\infty)$, the value $\widetilde{f}(x)$ is uniquely defined, proving that $\widetilde{f}$ is well defined.

To complete the proof of (a), notice that by (8.3),

$$\widetilde{f}(x) = \frac{2}{N+1}\sum_{i=0}^{N}\widetilde{f}(2x-i) \tag{8.4}$$

for any $x \in \mathbb{R}$. Thus $f = c\widetilde{f}(x)$, where $c^{-1} := \int_0^N \widetilde{f}\,dx$.

(b) By (a), it suffices to show that $\widetilde{f}$ is continuous, bounded, and positive on $(0,N)$. First, it follows from definition that $\widetilde{f}$ is continuous on $(-\infty,1)$. Next, assume that $\widetilde{f}$ is continuous on some interval of the form $[0,q]$ with $q \geq 1/2$. Since $x/2, x-i \in (-\infty,q]$ for all $x \in [q,q+1/2]$ and $i = 1,\dots,N$, (8.3) implies the continuity of $\widetilde{f}$ on $[q,q+1/2]$. Thus $\widetilde{f}$ is continuous on $\mathbb{R}$. As $\widetilde{f} = (1/c)f$, we also conclude that $\widetilde{f}(x) \geq 0$ for all $x \in \mathbb{R}$.

To show that $\widetilde{f}$ is bounded, take any $x \in (0,N)$ and let $\ell > 0$ be a positive integer such that $x/2^\ell \in (0,1)$. Using (8.3), we get

$$\begin{aligned}\widetilde{f}(x) &= \frac{N+1}{2}\widetilde{f}\Big(\frac{x}{2}\Big) - \sum_{i=1}^{N}\widetilde{f}(x-i) \leq \frac{N+1}{2}\widetilde{f}\Big(\frac{x}{2}\Big) \leq \Big(\frac{N+1}{2}\Big)^{\ell}\widetilde{f}\Big(\frac{x}{2^\ell}\Big) \\ &= \Big(\frac{N+1}{2}\Big)^{\ell}\Big(\frac{x}{2^\ell}\Big)^{\log_2(N+1)-1} = x^{\log_2(N+1)-1} \leq C,\end{aligned}$$

where $C > 0$ is a constant and the second inequality follows by iterating the first one.

Finally, to show that $\widetilde{f}$ is positive on $(0,N)$, we first observe from definition that $\widetilde{f}(x) > 0$ for all $x \in (0,1]$. Since $\widetilde{f}$ is symmetric about $x = N/2$, $\widetilde{f}(x) > 0$ for all $x \in [N-1,N)$. For $x \in (1,N-1)$, let $j \in \{0,1,\dots,N\}$ such that $2x - j \in (0,1] \cup [N-1,N)$. Since $\widetilde{f}(x) \geq 0$ for all $x \in \mathbb{R}$, (8.4) implies that $\widetilde{f}(x) \geq 2/(N+1)\ \widetilde{f}(2x-j) > 0$, which completes the proof. □

When $N = 3$, we can derive an explicit formula for the density f.

Corollary 8.2. *For the case $N = 3$ in Proposition 8.1, the density of μ is*

$$f(x) = \begin{cases} x/2, & 0 \le x < 1, \\ 1/2, & 1 \le x \le 2, \\ (3-x)/2, & 2 < x \le 3, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. By Proposition 8.1 and symmetric, $\widetilde{f}$ is defined by $\widetilde{f}(x) = 2\widetilde{f}(x/2) - \sum_{i=1}^{3} \widetilde{f}(x-i)$ for any $x \in \mathbb{R}$ and

$$\widetilde{f}(x) = \begin{cases} x, & x \in [0,1], \\ 3-x, & x \in [2,3], \\ 0, & x \in (-\infty, 0) \cup (3, \infty). \end{cases}$$

For any $x \in (1,2)$, $\widetilde{f}(x) = 2\widetilde{f}(x/2) - \widetilde{f}(x-1) - \widetilde{f}(x-2) - \widetilde{f}(x-3) = 1$. Thus $c = 2$, which completes the proof. $\square$

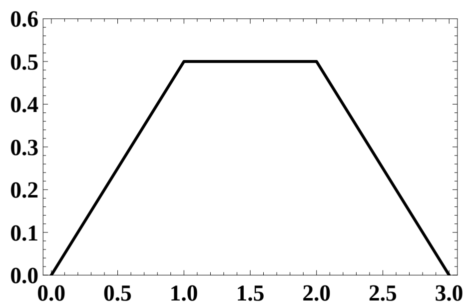


FIGURE 1. Density of the measure μ in Corollary 8.2.

8.2. **Infinite Bernoulli convolutions.** In this subsection we study the infinite Bernoulli convolutions μ defined in (1.18). It is known that for $0 < r < 1/2$, μ is a Cantor-type measure with Hausdorff dimension $\ln 2/\ln r$. If $r = 1/2$, μ is the restriction of Lebesgue measure on $[0,1]$. We are mainly interested in the case $1/2 < r < 1$. Erdős [15] proved that if r^{-1} is a Pisot number, then μ is singular. On the other hand, Wintner [49] proved that μ is absolutely continuous for $r = 2^{-1/k}$, for $k \ge 1$, and Grasia [18] found a family of algebraic integers with the corresponding μ being absolutely continuous. Solomyak [42] proved that for Lebesgue a.e. $r \in (1/2,1)$, μ is absolutely continuous; in particular, for Lebesgue a.e. $r \in (1/\sqrt{2},1)$, μ has bounded density. Mauldin and Simon [33] proved that Bernoulli convolutions are either singular or equivalent to Lebesgue measure. It follows that absolutely continuous Bernoulli convolutions are equivalent to Lebesgue measure. This result was extended by Peres, Schlag, and Solomyak [39] to any self-similar measure. The reader is referred to the survey by Peres, Schlag and Solomyak [39] for a more complete account of these interesting measures.

For $2/3 \le r < 1$, define $\widetilde{f}$ by the following dilation equation:

$$\widetilde{f}(x) = \begin{cases} x^{-\log_r 2 - 1}, & x \in [0, r^{-1} - 1], \\ 0, & x \in (-\infty, 0), \end{cases} \tag{8.5}$$

and

$$\widetilde{f}(x) = 2r\widetilde{f}(rx) - \widetilde{f}(x + 1 - r^{-1}), \qquad x \in \mathbb{R}. \tag{8.6}$$

Note that the condition $2/3 \le r < 1$ implies that

$$1 - r < r^{-1} - 1 \le 1/2 \le 2 - r^{-1} < r \qquad \text{and} \qquad 1 - r \le r^2 < 2 - r^{-1}. \tag{8.7}$$

The following proposition shows that the definitions in (8.5) and(8.6) are compatible and $\widetilde{f}$ is well defined. We remark that Jordan, Shmerkin and Solomyak [23] showed that for Lebesgue a.e. $r \in (1/\sqrt{2}, 1)$, the density is continuous on $\mathbb{R}$ and positive on $(0, 1)$. Proposition 8.3 below enlarges the interval on which absolutely continuous measures are known to have positive density on (0,1); moreover, it gives an explicit expression for the density on part of the domain.

Proposition 8.3. *Let μ be a self-similar measure defined by an IFS in the family* (1.18)*, together with probability weights $p_0 = p_1 = 1/2$. Assume $r \in [2/3, 1)$ and μ is absolutely continuous with respect to Lebesgue measure with density f. Let $\widetilde{f}$ be defined as in* (8.5) *and* (8.6)*. Then*

(a) *$\widetilde{f}$ is well defined, and $f(x) = c\widetilde{f}(x)$, where $c^{-1} := \int_0^N \widetilde{f}\, dx$;*
(b) *f is continuous and bounded on $\mathbb{R}$ and is positive on* $(0, 1)$*.*

Proof. (a) The proof of part (a) is similar to that of Proposition 8.1(a); we omit the details.

(b) By definition, $\widetilde{f}$ is continuous on $(-\infty, r^{-1} - 1)$. Assume that $\widetilde{f}$ is continuous on $(-\infty, r_0]$, where $0 < r_0 < 1$. For any $x \in [r_0, r_0 r^{-1}]$,

$$r_0 r^{-1} + 1 - r^{-1} - r_0 = r_0(r^{-1} - 1) + (1 - r^{-1}) = (r_0 - 1)(r^{-1} - 1) < 0,$$

and thus

$$rx \in [r_0 r, r_0] \subset (0, r_0] \quad \text{and} \quad x + 1 - r^{-1} \in [r_0 + 1 - r^{-1}, r_0 r^{-1} + 1 - r^{-1}] \subseteq (-\infty, r_0].$$

(8.6) implies that $\widetilde{f}$ is continuous on $[r_0, r_0 r^{-1}]$. By induction, $\widetilde{f}$ is continuous on $\mathbb{R}$ and hence $\widetilde{f}(x) \ge 0$ for all $x \in \mathbb{R}$.

For any $x \in (0, 1)$, there exists a constant $\ell > 0$ such that $r^\ell x \in (0, r^{-1} - 1)$. Thus for all $x \in (0, 1)$, we get

$$\begin{aligned} \widetilde{f}(x) &= 2r\widetilde{f}(rx) - \widetilde{f}(x + 1 - r^{-1}) \le 2r\widetilde{f}(rx) \le (2r)^\ell \widetilde{f}(r^\ell x) \\ &= (2r)^\ell (r^\ell x)^{-\log_r 2 - 1} = x^{-\log_r 2 - 1} \le C, \end{aligned}$$

where $C > 0$ is a constant and, again, the second inequality follows by iterating the first one. Hence $\widetilde{f}$ is bounded on $\mathbb{R}$.

To show the positivity of $\widetilde{f}$ on $(0,1)$, we first notice that by definition, $\widetilde{f}(x) > 0$ for any $x \in (0, r^{-1} - 1]$. Since $\widetilde{f}$ is symmetric about $x = 1/2$, $\widetilde{f}(x) > 0$ for all $x \in [2 - r^{-1}, 1)$. Fix $x \in (r^{-1} - 1, 2 - r^{-1})$. (8.7) implies that there exists some $m \in \mathbb{N}$ such that $r^{-m}x \in [2 - r^{-1}, 1)$. Rewrite (8.6) as

$$\widetilde{f}(x) = (2r)^{-1}\widetilde{f}(r^{-1}x) + (2r)^{-1}\widetilde{f}(r^{-1}x + 1 - r^{-1}),$$

and using the fact that $\widetilde{f}$ is non-negative on $\mathbb{R}$, we have $\widetilde{f}(x) \geq (2r)^{-m}\widetilde{f}(r^{-m}x) > 0$, which completes the proof. □

For $r = 2^{-1/k} \in [2/3, 1)$, $k = 2, 3, \ldots$, μ is absolutely continuous with respect to Lebesgue measure [49]. In [43], a numerical method is described to compute the density of μ with $r = 1/\sqrt{2}$. Here we give explicit formulas for f when $r = 1/\sqrt{2}$ and $r = 1/\sqrt[3]{2}$ (see Figure 2).

Example 8.4. *For the case $r = 1/\sqrt{2}$ in Proposition 8.3,*

$$f(x) = \begin{cases} (3/\sqrt{2} + 2)x, & 0 \leq x < \sqrt{2} - 1, \\ 1 + 1/\sqrt{2}, & \sqrt{2} - 1 \leq x \leq 2 - \sqrt{2}, \\ -(3/\sqrt{2} + 2)(x - 1), & 2 - \sqrt{2} < x \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. By Proposition 8.3 and symmetry, $\widetilde{f}$ is defined by

$$\widetilde{f}(x) = \sqrt{2}\widetilde{f}(x/\sqrt{2}) - \widetilde{f}(x + 1 - \sqrt{2});$$

moreover,

$$\widetilde{f}(x) = \begin{cases} x, & x \in [0, \sqrt{2} - 1], \\ 1 - x, & x \in [2 - \sqrt{2}, 1], \\ 0, & x \in (-\infty, 0) \cup (1, +\infty). \end{cases}$$

Hence for any $x \in (\sqrt{2} - 1, 2 - \sqrt{2})$,

$$\widetilde{f}(x) = \sqrt{2}\widetilde{f}(x/\sqrt{2}) - \widetilde{f}(x + 1 - \sqrt{2}) = \sqrt{2}(x/\sqrt{2}) - (x + 1 - \sqrt{2}) = \sqrt{2} - 1.$$

Thus $c = 3\sqrt{2} - 4$, which gives the formula for f. □

To state the next example, we introduce the following abbreviations: Let

$$a := r^{-1}, \qquad \alpha_i := a^{i-1}(a - 1), \quad i = 1, 2, 3, 4.$$

Example 8.5. *For the case $r = 1/\sqrt[3]{2}$ in Proposition 8.3, $f = c\widetilde{f}$, where $\widetilde{f}$ is given by*

$$\widetilde{f}(x) = \begin{cases} x^2, & 0 \le x < \alpha_1, \\ 2(a-1)x - (a-1)^2, & \alpha_1 \le x < \alpha_2, \\ -x^2 + 2(a^2-1)x + 3 - 2a^2, & \alpha_2 \le x < \alpha_3, \\ -2x^2 + 2x - (a^2-1)^2, & \alpha_3 \le x < 1 - \alpha_3, \\ -x^2 + 2(3-a^2)x + 1, & 1 - \alpha_3 \le x < 1 - \alpha_2, \\ -2(a-1)x - (a-1)(3-a), & 1 - \alpha_2 \le x < 1 - \alpha_1, \\ (1-x)^2, & 1 - \alpha_1 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases} \tag{8.8}$$

and $c = \int_0^1 \widetilde{f}\, dx$.

Proof. By Proposition 8.3 and symmetric, $\widetilde{f}(x)$ is defined by

$$\widetilde{f}(x) = a^2 \widetilde{f}(x/a) - \widetilde{f}(x + 1 - a); \tag{8.9}$$

moreover,

$$\widetilde{f}(x) = \begin{cases} x^2, & x \in [0, \alpha_1], \\ (1-x)^2, & x \in [1 - \alpha_1, 1], \\ 0, & x \in (-\infty, 0) \cup (1, +\infty). \end{cases}$$

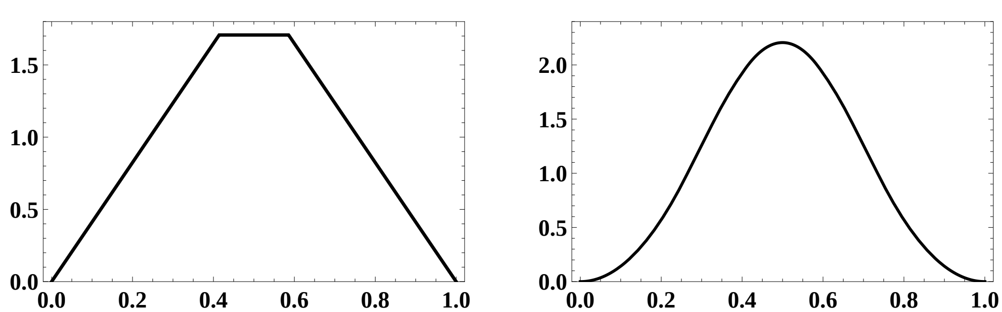
For $x \in [\alpha_1, \alpha_2)$, $x/a, x + 1 - a \in [0, \alpha_1]$ and thus by (8.9),

$$\widetilde{f}(x) = 2(a-1)x - (a-1)^2.$$

Similarly, for $2 \le i \le 3$ and $x \in [\alpha_i, \alpha_{i+1})$, we noticed that $x/a \in [\alpha_{i-1}, \alpha_i)$ and $x + 1 - a \in [0, \alpha_1]$ and thus (8.9) implies

$$\widetilde{f}(x) = \begin{cases} -x^2 + 2(a^2-1)x + 3 - 2a^2, & x \in [\alpha_2, \alpha_3), \\ -2x^2 + 2x - (a^2-1)^2, & x \in [\alpha_3, \alpha_4). \end{cases}$$

Since $1/2 < \alpha_4 < 1 - \alpha_3$, for any $x \in [\alpha_4, 1 - \alpha_3)$, $1 - x \in (\alpha_3, 1 - \alpha_4] \subseteq [\alpha_3, \alpha_4)$ and thus $\widetilde{f}(x) = \widetilde{f}(1-x) = -2x^2 + 2x - (a^2-1)^2$. Hence, (8.8) holds, by using the symmetry of $\widetilde{f}$. □



(a) $n = 2$ (b) $n = 3$

FIGURE 2. Density of the Bernoulli convolution with $r = 1/\sqrt[n]{2}$, $n = 2, 3$.

It is known (see [23]) that for $r \in (1/2, (\sqrt{5}-1)/2)$, there exist infinitely many $x \in (0,1)$ such that $\lim_{r\to 0^+} \mu(B(x,r))/(2r) = 0$. If the corresponding measure is absolutely continuous with a continuous density, the following remark finds an explicit family of zeros of the density.

Remark 8.6. *Let μ be a self-similar measure defined by an IFS in* (1.18) *together with probability weights $p_0 = p_1 = 1/2$. Assume that μ is absolutely continuous with respect to Lebesgue measure with continuous density f and $r \in (1/2, (\sqrt{5}-1)/2)$. Then $f(r^m/(r+1)) = 0$ for $m \geq 0$.*

Proof. For all $x \in [0,1]$, $f(x) = (2r)^{-1} f(r^{-1}x) + (2r)^{-1} f(r^{-1}x + 1 - r^{-1})$. Since $S_2^{-1}(x) = r^{-1}x + 1 - r^{-1} \leq 0$ for $x \in [0, 1-r]$, $f(x) = (2r)^{-1} f(r^{-1}x)$ and thus

$$f(x) = 2rf(rx) \qquad \text{for all } x \in [0, r^{-1} - 1]. \tag{8.10}$$

The inequality

$$1/(r+1) - (r^{-1} - 1) = (r^2 + r - 1)/(r(r+1)) < 0 \qquad \text{for } r \in (1/2, (\sqrt{5}-1)/2),$$

implies $1/(1+r) \in [0, r^{-1} - 1]$. Since f is symmetric about $x = 1/2$, (8.10) implies that

$$f\big(1/(1+r)\big) = 2rf\big(r/(1+r)\big) = 2r\widetilde{f}\big(1 - r/(1+r)\big) = 2rf\big(1/(1+r)\big).$$

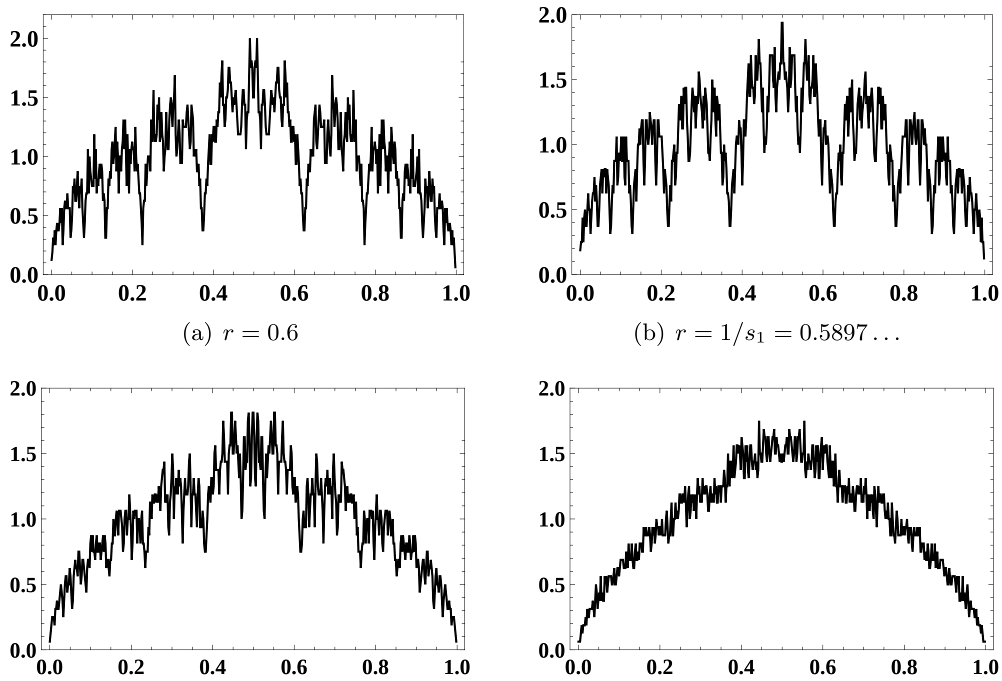
It follows that $f(1/(1+r)) = 0$ and hence $f(r^m/(1+r)) = 0$ for $m \geq 0$. ☐

Figure 3 shows numerical approximations to the densities f for two numbers r in the interval $(1/2, (\sqrt{5}-1)/2)$ and two in the interval $((\sqrt{5}-1)/2, 2/3)$. The numbers s_1 and s_2 are the solutions in $(1,2)$ of the equations $x^3 - x^2 - 2 = 0$ and $x^3 - 2x^2 + 2x - 2 = 0$ respectively; the corresponding measures are shown by Garsia [18] to be absolutely continuous. It is unknown whether the measures in (a) and (c) are absolutely continuous or singular. According to Proposition 8.3, the density function in (d) is positive on $(0,1)$, while according to [23] (see also Remark 8.6), the one in (b) has countably infinitely many zeros in $(0,1)$.

Proof of Theorem 1.9. Again, (MPI) holds since μ is supported on $\mathbb{R}$.

(a) Since N is odd, μ is equivalent to the restriction of Lebesgue measure to $[0, N]$. By Proposition 8.1, f is continuous and bounded on $[0, N]$ and $f(x) > 0$ on $(0, N)$. Theorem 1.8 now implies (FPS).

(b) Similar to that of (a). Use Proposition 8.3 instead. ☐



(a) $r = 0.6$

(b) $r = 1/s_1 = 0.5897\ldots$

(c) $r = 0.62$

(d) $r = 1/s_2 = 0.6477\ldots$

Figure 3. Numerical approximations to the densities of some infinite Bernoulli convolutions with contraction ratio r.

9. Comments and open questions

For a class of p.c.f. fractals, including the Sierpiński gasket, Strichartz [45, Theorem 6.1] obtained two-sided sub-Gaussian heat kernel estimates for the product of these p.c.f. fractals, which are not p.c.f.. Theorem 1.1 can be applied to these products. It is of interest to determine wave propagation speed for other rationally ramified fractals (see, [28]) and other non-p.c.f. fractals such as the diamond fractal (see, [29]).

We do not know whether the condition $f \geq \varepsilon(V) > 0$ in Theorem 1.8 can be removed. In view of Propositions 8.3 and 8.6, it is of interest to know whether the density function of those absolutely continuous infinite Bernoulli convolutions with $r \in ((\sqrt{5}-1)/2, 2/3)$ has a zero in $(0, 1)$, and whether (FPS) or (IPS) holds.

Our result on finite propagation speed can also be applied to the Sierpinski gasket equipped with the Kusuoka measure and the so-called harmonic geodesic metric, as two-sided Gaussian heat kernel estimates have been obtained by Kajino [25].

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