DIFFERENTIABILITY OF THE L^q-SPECTRUM AND MULTIFRACTAL DECOMPOSITION BY USING INFINITE GRAPH-DIRECTED IFSs

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ABSTRACT. By constructing an infinite graph-directed iterated iterated function system associated with a finite iterated function system, we develop a new approach for proving the differentiability of the L^q-spectrum and establishing the multifractal formalism of certain self-similar measures with overlaps, especially those defined by similitudes with different contraction ratios. We apply our technique to a well-known class of self-similar measures of generalized finite type.

1. INTRODUCTION

The idea and physical significance of multifractal measures have their origins in the work of Mandelbrot [20] in the 1970's. In the 1980's physicists (see [8–10] and references therein) proposed the so-called multifractal formalism, which allows one to obtain the dimension of the multifractal components of the support of a measure by taking the Legendre transform of its L^q-spectrum.

Let μ be a finite Borel measure on $\mathbb{R}^n$ with compact support. For $q \in \mathbb{R}$, the *L^q-spectrum* of μ is defined as

$$\tau(q) = \tau_\mu(q) := \liminf_{\delta \to 0^+} \frac{\log\big(\sup \sum_i \mu(B_\delta(x_i))^q\big)}{\log \delta}, \tag{1.1}$$

where the supremum is taken over all families of disjoint balls $B_\delta(x_i)$ of radius δ and center $x_i \in \operatorname{supp}(\mu)$.

The L^q-spectrum is one of the basic ingredients in studying multifractal phenomena. Heuristically, if $\tau(q)$ is differentiable, then by varying q, one might be able to extract different

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multifractal components

$$K^{(\alpha)} := \Big\{x \in \text{supp}(\mu) : \lim_{\delta\to 0^+} \frac{\log \mu(B_\delta(x))}{\log \delta} = \alpha\Big\}, \qquad \alpha = \tau'(q),$$

that dominate the sum $\sum_i \mu(B_\delta(x_i))^q$ in (1.1). More precisely, we say that the *multifractal formalism* holds if the Legendre transform of $\tau(q)$, defined as $\tau^*(\alpha) := \inf\{q\alpha - \tau(q) : q \in \mathbb{R}\}$, equals the Hausdorff dimension of $K^{(\alpha)}$, i.e.,

$$\dim_{\rm H} K^{(\alpha)} = \tau^*(\alpha). \tag{1.2}$$

It is well known that if μ is a self-similar measure defined by an *iterated function system (IFS)* of contractive similitudes $\{S_i\}_{i=1}^m$ satisfying the *open set condition (OSC)* [12], $\tau(q)$ can be calculated by an explicit formula and is differentiable on $\mathbb{R}$ [1,3]. We refer the reader to [1,3,7,15,25] for some further properties and results concerning the L^q-spectrum and the multifractal formalism.

We say that an IFS has *overlaps* if it does not satisfy the open set condition. In this case, it is much harder to obtain a formula for $\tau(q)$ and it is not known whether the multifractal formalism holds in general. Nevertheless, Lau and Ngai [15] proved that (1.2) holds if the IFS satisfies the *weak separation condition (WSC)* and $\alpha = \tau'(q)$ where $q \geq 0$. (WSC) is strictly weaker than (OSC) and is satisfied by many interesting IFSs with overlaps. Feng and Lau [7] proved that for IFSs of contractive similitudes satisfying (WSC) there exists an open ball such that the L^q-spectrum obtained by restricting the measure to it behaves more nicely, and using this they proved the multifractal formalism for the original measure in the range $q \geq 0$. It is also worth mentioning that Feng [6] showed, without assuming any separation condition, that if $\alpha = \tau'(q)$ is differentiable at some $q > 1$, then the multifractal formalism holds for the corresponding α.

For iterated function systems with overlaps, the differentiability of $\tau(q)$ remains an interesting and largely unsolved problem. Lau and Ngai showed that for the infinite Bernoulli convolution associated with the golden ratio [14] and a class of convolutions of the Cantor measure [16], $\tau(q)$ is differentiable in the region $q > 0$. Feng [5] showed that for a class of Pisot numbers, $\tau(q)$ is differentiable for all $q \in \mathbb{R}$. The existence of a nondifferentiable point $q_0 < 0$ was proved for the infinite Bernoulli convolution associated by the golden ratio by Feng [5] and for the three-fold convolution of the Cantor measure by Lau and Wang [18]. Feng [4] proved that for equicontractive IFSs on $\mathbb{R}$ satisfying the finite type condition [23], $\tau(q)$ is differentiable on $(0, \infty)$.

In this paper we study the differentiability of $\tau(q)$ for IFSs with overlaps without assuming that the contraction ratios of the IFSs maps are the same. We will assume the generalized finite type condition [13, 17]. Our basic idea is to convert a finite IFS into an infinite *graph-directed iterated function system (GIFS).*

Consider the following multifractal decomposition problem. Let (V, E, M, P) be a GIFS and assume that there exists a unique *family of graph-directed sets* (also called the *attractor*) $\boldsymbol{K} = (K_u)_{u\in V}$ and a unique *family of graph-directed measures* $\boldsymbol{\mu} = (\mu_u)_{u\in V}$ (see Theorem 2.1), where P is a transition probability matrix and M is the associated M-matrix; see Section 2 for more details about GIFSs. Let $B_\delta(x)$ be the ball in $\mathbb{R}^n$ with center x and radius δ, E_u^∞ be the set of all infinite paths associated to the graph (V, E), and $p(\mathbf{e}|k)$ be the weight associated to the prefix $\mathbf{e}|k$ of the infinite path $\mathbf{e}$ (see definitions in Section 2). Define

$$\begin{aligned} K_u^{(\alpha)} &:= \Big\{x \in K_u : \lim_{\delta\to 0^+} \frac{\log \mu_u(B_\delta(x))}{\log \delta} = \alpha\Big\}, \\ \hat{K}_u^{(\alpha)} &:= \Big\{\mathbf{e} \in E_u^\infty : \lim_{k\to\infty} \frac{\log p(\mathbf{e}|k)}{\log r(\mathbf{e}|k)} = \alpha\Big\}. \end{aligned} \tag{1.3}$$

Let $B(q, \hat{\tau}, P) = (b_{uv})$ be the corresponding *weighted incidence matrix* with

$$b_{uv} = b_{uv}(q, \hat{\tau}) := \sum_{e\in E_{uv}} p(e)^q r(e)^{-\hat{\tau}}. \tag{1.4}$$

For each q, let $\hat{\tau}(q)$ be the unique number such that the spectral radius of $B(q, \hat{\tau}(q), P)$ is 1. Clearly, this defines a differentiable function $\hat{\tau}$ when the vertex set V is finite. If V is finite and (OSC) holds, Edgar and Mauldin [3] obtained the multifractal formula

$$\dim_{\mathrm{H}} \pi_u(\hat{K}_u^{(\alpha)}) = \dim_{\mathrm{P}} \pi_u(\hat{K}_u^{(\alpha)}) = \hat{\tau}^*(\alpha),$$

where $\dim_{\mathrm{P}} F$ denotes the packing dimension of a set $F \subseteq \mathbb{R}^n$ and π_u is the natural projection from E_u^∞ onto K_u.

In contrast to [3], we will consider GIFSs (V, E, M, P) with V being countably infinite. The following is one of our main results. We refer the reader to Section 3 for the definitions of the matrix $B(q, \hat{\tau}(q), P)$, the degree $\deg(V)$ of V, and the positive separation condition. Let $\mathbf{v}^t$ denote the transpose of a finite or infinite vector $\mathbf{v}$.

Theorem 1.1. *Let (V, E, M, P) be an infinite GIFS that has a unique attractor $\boldsymbol{K} = (K_u)_{u\in V}^t$ and a unique family of graph-directed measures $\boldsymbol{\mu} = (\mu_u)_{u\in V}^t$. Assume that* $\deg(V) < \infty$*, $B := B(q, \hat{\tau}(q), P)$ is primitive, and (OSC) holds. Let $\mathbf{x}^t = (x_u)_{u\in V}$ and $\mathbf{y} = (y_u)_{u\in V}^t$ be positive 1-invariant measure and vector of B, respectively, with $\mathbf{x}^t\mathbf{y} < \infty$. Assume, in addition, that the following conditions hold:*

(1) *the functions* $\hat{\tau}$, $\mathbf{x}^t$, $\mathbf{y}$, *and* $\mathbf{x}^t\mathbf{y}$ *are differentiable;*
(2) $\frac{\mathrm{d}}{\mathrm{d}q}\sum_u x_u y_u = \sum_u \frac{\mathrm{d}}{\mathrm{d}q}(x_u y_u)$*;*
(3) *for all* q, $0 < \inf_u\{y_u\} \le \sup_u\{y_u\} < \infty$.

Then for all $u \in V$,

$$\dim_{\mathrm{H}} \pi_u(\hat{K}_u^{(\alpha)}) = \dim_{\mathrm{P}} \pi_u(\hat{K}_u^{(\alpha)}) = \hat{\tau}^*(\alpha).$$

Moreover, if (OSC) is replaced by the positive separation condition, then the above dimension formula also holds when $\pi_u(\hat{K}_u^{(\alpha)})$ *is replaced by* $K_u^{(\alpha)}$*, i.e.,*

$$f(\alpha) = \hat{\tau}^*(\alpha).$$

We will apply this theorem to the following class of IFSs with overlaps:

$$S_1(x) = r_1 x, \qquad S_2(x) = r_2 x + r_1(1 - r_2), \qquad S_3(x) = r_2 x + 1 - r_2, \tag{1.5}$$

where the contraction ratios $r_1, r_2 \in (0,1)$ satisfy

$$r_1 + 2r_2 - r_1 r_2 < 1, \tag{1.6}$$

i.e., $S_2(1) < S_3(0)$. Let K be the corresponding self-similar set (see Figure 1).

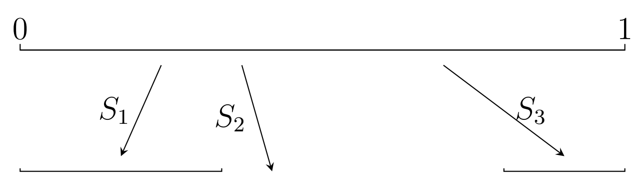


FIGURE 1. The first iteration of $\{S_i\}_{i=1}^3$. The figure is drawn with $r_1 = 1/3$ and $r_2 = 1/5$.

This well-known class of IFSs appeared in the work of [13, 17, 19]. The IFSs satisfy the generalized finite type condition but is not of finite type in the sense of [23], because the contraction ratios are not necessarily exponentially commensurable. The dimension of the self-similar set has been investigated in [13, 17, 19] using various methods; it is the unique number d satisfying

$$r_1^d + 2r_2^d - (r_1 r_2)^d = 1.$$

One of our main objectives is to prove the differentiability of $\tau(q)$. As the contraction ratios of the IFS maps are not equal, previous techniques such as those in [4, 5, 16] cannot be applied. This is in fact a main motivation of this paper.

Let μ be the self-similar measure defined by an IFS in (1.5) and a probability vector (p_1, p_2, p_3). Define

$$K^{(\alpha)} = \left\{ x \in K : \lim_{\delta \to 0^+} \frac{\log \mu(B_\delta(x))}{\log \delta} = \alpha \right\}, \quad \alpha \in [0, \infty). \tag{1.7}$$

From the structure of the self-similar set K, we will construct an infinite GIFS (V, E, M, P) such that K, μ are the first components of the attractor $\boldsymbol{K}$ and the family of graph-directed measures $\boldsymbol{\mu}$ respectively. This infinite GIFS defines a unique $\hat{\tau}(q)$. Concerning this $\hat{\tau}(q)$ we have the following result.

Theorem 1.2. *Let μ be the self-similar measure defined by the IFS* (1.5) *with probability weights p_1, p_2, p_3. Assume that* (1.6) *holds. If $p_2 > p_3$ or $q \geq 0$, then $\hat{\tau}(q)$ is differentiable and*

$$\dim_{\mathrm{H}} K^{(\alpha)} = \dim_{\mathrm{P}} K^{(\alpha)} = q\alpha - \hat{\tau}(q).$$

In the case $p_2 \leq p_3$, it is likely that $\tau(q)$ may have a nondifferentiable point in the region $q < 0$. See Section 9 for additional discussions.

As a consequence of Theorem 1.2 and [11, 22], we have

Corollary 1.3. *Assume the same hypotheses of Theorem 1.2. Then $\tau(q) = \hat{\tau}(q)$ if $q \geq 0$. Moreover,*

$$\dim_{\mathrm{H}}(\mu) = \hat{\tau}'(1).$$

As the proof of Theorem 1.2 is long and complicated, we outline the main steps here. A key in the proof is to show the differentiability of $\hat{\tau}(q)$. First, we use quasi-extension and quasi-truncation techniques to define a sequence of infinite GIFSs (V, E, M, P_n) which converges to (V, E, M, P) in some sense. Each of these infinite GIFSs defines a differentiable function $\hat{\tau}_n(q)$. Second, for a fixed q, we use the boundedness of $\{\hat{\tau}_n(q)\}$ to choose a subsequence $\hat{\tau}_{n_k}(q)$ that converges to $\hat{\tau}(q)$. Let $\mathbf{x}_{n_k}^t, \mathbf{y}_{n_k}$ be the 1-invariant measure and 1-invariant vector of the incidence matrix $B_{n_k}(q, \hat{\tau}) = B(q, \hat{\tau}_{n_k}(q), P_{n_k})$ of (V, E, M, P_{n_k}). By showing that $\mathbf{x}_{n_k}^t, \mathbf{y}_{n_k}$ converge to $\mathbf{x}^t, \mathbf{y}$, the 1-invariant measure and 1-invariant vector of $B(q, \hat{\tau}, P)$ respectively, we conclude that the spectral radius of $B(q, \hat{\tau}(q))$ is 1. This implies, by the monotonicity of $\rho(B(q, \hat{\tau}, P))$ as a function of $\hat{\tau}$, that $\hat{\tau}_n(q)$ converges to $\hat{\tau}(q)$. Third, we use the implicit function theorem to obtain the differentiability of $\hat{\tau}(q)$. Finally, we show that $\mathbf{x}^t\mathbf{y} < \infty$ and

$$\frac{\mathrm{d}}{\mathrm{d}q}(\mathbf{x}^t\mathbf{y}) = \sum_{k \geq 1} \frac{\mathrm{d}}{\mathrm{d}q}(x_k y_k).$$

Now all the conditions in Theorem 1.1 are satisfied, and the conclusion follows.

This paper is organized as follows. In Section 2, we recall the definition of an M-matrix and introduce infinite GIFSs. In Section 3, we present the proof of Theorem 1.1. The definitions of quasi-extension and quasi-truncation are stated in Section 4. We also define a sequence of infinite GIFSs induced by any IFS of generalized finite type in the family (1.5). We prove the convergence of $\hat{\tau}_n(q)$ in Section 5 and the differentiability of the limit function $\hat{\tau}(q) = \lim_{n\to\infty} \hat{\tau}_n(q)$ in Section 6. The proof of Theorem 1.2 is given in Section 7. Finally, we state some comments in Section 9.

2. M-matrices and GIFSs

Let us first recall the definition of an M-matrix and refer the reader to [13] for more details. An *M-matrix* is a matrix each of its entries is a collection of mappings on $\mathbb{R}^n$. We use $\mathfrak{M} = \mathfrak{M}(m)$ (m is finite or ∞) to denote the collection of all $m \times m$ M-matrices. For $M \in \mathfrak{M}$, let M^2 be the M-matrix whose (i, j) entry is $\bigcup_{k\geq 1} M_{ik}M_{kj}$, where the product of two sets of mappings Φ and Ψ is defined by

$$\Phi\Psi := \{\varphi \circ \psi : \varphi \in \Phi, \psi \in \Psi\}.$$

If $\varphi = \emptyset$, we define $\varphi \circ \psi = \psi \circ \varphi = \emptyset$. Inductively, we can define M^k for all $k \geq 1$.

For a set of mappings Φ and a subset $F \subset \mathbb{R}^n$, define $\Phi(F) := \bigcup_{\varphi\in\Phi} \varphi(F)$, where if $\varphi = \emptyset$, we define $\varphi(F) := \emptyset$. Let $M \in \mathfrak{M}(m)$ be an M-matrix, where m is finite or ∞, and let $\mathbf{F} = (F_1, F_2, \dots)^t$ be a vector of sets with m components, where $F_i \subset \mathbb{R}^n$. Denote

$$M\mathbf{F} := \Big(\bigcup_{j=1}^{m} M_{1j}(F_j),\ \bigcup_{j=1}^{m} M_{2j}(F_j), \dots\Big)^t.$$

Let $G = (V, E)$ be a directed graph with a finite or countably infinite vertex set V and edge set E. To each vertex v, we associate a metric space X_v. Throughout this paper, we assume $X_v = \mathbb{R}^n$. Let E_{uv} be the set of all edges from u to v. For $e = (u, v) \in E_{uv}$, we let $\iota(e) := u$ and $\kappa(e) := v$ denote the *initial* and *terminal vertices* of e, respectively. Let $E_u := \bigcup_{v\in V} E_{uv}$ be the set of all edges with initial vertex u. Then $E = \bigcup_u E_u$. To avoid redundancy, we assume $E_u \neq \emptyset$ for any $u \in V$. We call $\mathbf{e} = e_1 \cdots e_k$ a *path* with length k, and denote its length by $|\mathbf{e}|$, if the terminal vertex of each edge e_i $(1 \leq i \leq k-1)$ equals the initial vertex of the edge e_{i+1}. Infinite paths are defined similarly. We denote by E^k_{uv} the set of all paths of length k from vertex u to vertex v. Let $E^*_{uv} := \bigcup_{k\geq 0} E^k_{uv}$, where $E^0_{uv} := \{\emptyset\}$

and let

$$E_u^k := \bigcup_{v\in V} E_{uv}^k, \qquad E^k := \bigcup_{u\in V} E_u^k, \qquad E_u^* := \bigcup_{k\geq 0} E_u^k, \qquad E^* := \bigcup_{k\geq 0} E^k.$$

Let E_u^∞ be the set of all infinite paths which begin at u and $E^\infty := \bigcup_{u\in V} E_u^\infty$ be the set of all infinite paths. We say that a graph $G = (V, E)$ is *strongly connected* if for any $u, v \in V$, there is a path from u to v. For $\mathbf{e} \in E^n \bigcup E^\infty$ and $k \leq n$, let $\mathbf{e}|k := e_1 \cdots e_k$. For $\mathbf{e} \in E^k$, define the *cylinder* with prefix $\mathbf{e}$ as $[\mathbf{e}] := \{\mathbf{e}' \in E^\infty : \mathbf{e}'|k = \mathbf{e}\}$.

To each edge $e \in E_{uv}$, we associate a weight $p(e)$ and a contractive similitude $S_e : X_v \to X_u$ with contraction ratio $r(e) \in (0, 1)$. This defines an M-matrix $M := (M_{uv})$ with $M_{uv} := \{S_e : e \in E_{uv}\}$ and a matrix $P = (p(e))$. For each $\mathbf{e} = e_1 \cdots e_k \in E^k$, use the notation

$$p(\mathbf{e}) := p(e_1)\cdots p(e_k), \qquad S_{\mathbf{e}} := S_{e_1} \circ \cdots \circ S_{e_k}, \qquad \iota(\mathbf{e}) := \iota(e_1), \qquad \kappa(\mathbf{e}) := \kappa(e_k).$$

If $p(e) \in (0, 1)$ for all $e \in E$, and the weights of all edges leaving a given vertex u sum to 1, namely,

$$\sum_{v\in V} \sum_{e\in E_{uv}} p(e) = 1, \tag{2.1}$$

we call $p(e)$ is a *transition probability* and P a *probability matrix.* Unless stated otherwise, we always assume that P is a probability matrix. We remark that M is indexed by the vertices $u \in V$, while P is indexed by the edges $e \in E$. If the cardinality of the E_{uv} does not exceed 1, then P is also indexed by $u \in V$; this is indeed the case when we study the family of IFSs in (1.5). We call (V, E, M), or more broadly, (V, E, M, P) a *graph-directed iterated function system (GIFS).* If V is countably infinite, we also call (V, E, M) an *infinite GIFS.*

If there exists a finite or countably infinite sequence of nonempty compact sets $\boldsymbol{K} = (K_u)_{u\in V}^t$ such that $M\boldsymbol{K} = \boldsymbol{K}$, namely

$$K_u = \bigcup_{v\in V} \bigcup_{e\in E_{uv}} S_e(K_v), \quad u \in V, \tag{2.2}$$

we call $\boldsymbol{K}$ a *family of graph-directed sets* (or an *attractor*). We remark that an attractor need not exist, and even if one exists, it may not be unique. In fact, if we let $V = \{1, 2, \ldots\}$, assume that for each $n \in V$, there is exactly one edge from n to $n+1$, and define $f_e(x) := x/2$ for all $e \in E$, then $\boldsymbol{K}$ is not unique since K_1 can be any nonempty compact set.

Definition 2.1. *We say that a finite or infinite GIFS* (V, E, M) *that has a unique attractor* $\boldsymbol{K} = (K_u)_{u\in V}^t$ *satisfies*

(a) *the* open set condition (OSC) *if there exists a sequence of nonempty bounded open sets* $\{U_u : u \in V\}$ *such that*
  (1) *for each* $u \in V$, $\bigcup_{v\in V} \bigcup_{e\in E_{uv}} S_e(U_v) \subset U_u$, *and*
  (2) $S_e(U_v) \cap S_{e'}(U_{v'}) = \emptyset$ *for all* $e \in E_{uv}$ *and* $e' \in E_{uv'}$ *with* $e \neq e'$;

(b) *the* strong separation condition (SSC) *if, for any* $u \in V$, *the union on the right side of* (2.2) *is disjoint;*

(c) *the* positive separation condition (PSC) *if*

$$d(V) := \inf_u \inf \left\{ d(S_e(K_v), S_{e'}(K_{v'})) : e \neq e', e \in E_{uv}, e' \in E_{uv'}, v, v' \in V \right\} > 0,$$

where $d(X,Y) := \inf\{|x-y| : x \in X, y \in Y\}$ *denotes the distance between two sets* $X, Y \subseteq \mathbb{R}^n$.

We remark that (PSC) is stronger than (SSC) in general. However, they are equivalent if V is finite.

For an attractor $\boldsymbol{K}$ and a probability matrix $P = (p(e))$, we define a vector probability measure $\boldsymbol{\mu}$ supported on $\boldsymbol{K}$ as follows. For each $u \in V$, (2.1) implies that there exists a unique Borel probability measure $\hat{\mu}_u$ supported on E_u^∞ such that

$$\hat{\mu}_u([\mathbf{e}]) = p(\mathbf{e}) = p(e_1)\cdots p(e_k), \quad \mathbf{e} = e_1 \cdots e_k \in E_u^*, \quad k \geq 0.$$

Now we let $\mu_u = \hat{\mu}_u \circ \pi_u^{-1}$, where $\pi_u : E_u^\infty \to K_u$ is the natural surjection, i.e., for each infinite path $\mathbf{e} \in E_u^\infty$, $\pi_u(\mathbf{e})$ is the unique element in the intersection

$$\bigcap_k S_{\mathbf{e}|k}(K_{\kappa(\mathbf{e}|k)}).$$

Finally, let $\boldsymbol{\mu} := (\mu_u)_{u\in V}^t$ and call it the *family of graph-directed measures.*

Let $\deg(u) := \# E_u$ be the number of edges starting from the vertex u, and define the *degree* of V as

$$\deg(V) := \sup_u \deg(u).$$

In order to prove the existence and uniqueness of the attractor $\boldsymbol{K}$ of (V, E, M), we assume that

$$\deg(V) < \infty. \tag{2.3}$$

Similar to the proof of the existence and uniqueness of self-similar sets and measures, we have

Theorem 2.1. *Let (V, E, M, P) be a GIFS with $G = (V, E)$ being strongly connected. Assume*

(1) $\deg(V) < \infty$*;*
(2) $\{S(0) : S \in M_{uv}, u, v \in E\}$ *is a bounded set;*
(3) *there exists a positive constant $C < 1$ such that* $\sup\{r(e) : e \in E\} \leq C$.

Then there eixst a unique attractor $\boldsymbol{K}$ and a unique family of graph-directed measures $\boldsymbol{\mu}$.

The theorem follows from Banach's fixed point theorem. Condition (1) allows one to use the fact that a finite union of compact sets is compact. We omit the details.

3. Proof of Theorem 1.1

Let (V, E, M, P) be a GIFS with a unique attractor $\boldsymbol{K} = (K_u)_{u\in V}^t$, a unique family of graph-directed measures $\boldsymbol{\mu} = (\mu_u)_{u\in V}^t$, and a primitive probability matrix P. Throughout this section, we assume that condition (2.3) holds.

For any $\alpha \in \mathbb{R}$ and $u \in V$, let $K_u^{(\alpha)}$ and $\hat{K}_u^{(\alpha)}$ be defined as in (1.3). We point out that $K_u^{(\alpha)} = \pi_u(\hat{K}_u^{(\alpha)})$ if (PSC) holds. Let $B(q, \hat{\tau}) = B(q, \hat{\tau}, P)$ denote the matrix whose (u, v) entry is given by (1.4), and let $\rho(B(q, \hat{\tau}))$ denote the spectral radius of $B(q, \hat{\tau})$.

Define the *norm* of a matrix $A = (a_{uv})$ as

$$\|A\| := \sup_u \sum_{v\in E_{uv}} |a_{uv}|. \tag{3.1}$$

Let $\mathcal{B}(\ell^\infty, \ell^\infty)$ be the space of all bounded linear operators on ℓ^∞. For any $A \in \mathcal{B}(\ell^\infty, \ell^\infty)$, A^k is well-defined for any $k \geq 1$. Note that the operator norm of $A \in \mathcal{B}(\ell^\infty, \ell^\infty)$ coincides with the norm defined in (3.1). We also recall that the spectral radius of a matrix $A \in \mathcal{B}(\ell^\infty, \ell^\infty)$ can be computed by

$$\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k}. \tag{3.2}$$

We need two mild conditions.

(i) The contraction ratios have a common positive lower bound and a common upper bound less than 1, i.e.,

$$r_{\inf} := \inf\big\{r(e) : e \in E\big\} > 0, \qquad r_{\sup} := \sup\big\{r(e) : e \in E\big\} < 1. \tag{3.3}$$

(ii) $B(q, \hat{\tau})$ is an operator from ℓ^∞ to ℓ^∞. If $r_{\inf} > 0$, this is equivalent to $B(q, 0)$ having a finite norm which is defined by (3.1).

By the assumptions $B\mathbf{y} = \mathbf{y}$ and $\inf_u y_u > 0$ of Theorem 1.1, condition (ii) holds when $\hat{\tau} = \hat{\tau}(q)$. Condition (i) is used to ensure the differentiability of $\mathbf{x}^t, \mathbf{y}, \mathbf{x}^t\mathbf{y}, \hat{\tau}$ and the existence of the attractor $\boldsymbol{K}$.

The following proposition shows that these conditions imply that the spectral radius $\rho(B(q,\hat{\tau}))$ is also a continuous function of $\hat{\tau}$, as when $B(q,\hat{\tau})$ is a finite matrix.

Proposition 3.1. *Assume that conditions (i) and (ii) above hold. For a given q, if $B(q,0)$ has a finite norm, then $\rho_q(\hat{\tau}) = \rho(B(q,\hat{\tau}))$ is a continuous function of $\hat{\tau}$, which is strictly increasing; moreover,*

$$\rho_q(-\infty) := \lim_{\hat{\tau}\to-\infty} \rho_q(\hat{\tau}) = 0, \qquad \rho_q(\infty) := \lim_{\hat{\tau}\to\infty} \rho_q(\hat{\tau}) = \infty.$$

Proof. As $B(q,0)$ has a finite norm and $r_{\inf} > 0$, for any $\hat{\tau}$, $B(q,\hat{\tau})$ also has a finite norm. Hence

$$\rho_q(\hat{\tau}) = \lim_{k\to\infty} \|B(q,\hat{\tau})^k\|^{1/k}$$

is well-defined. For any $h > 0$,

$$\begin{aligned}
\rho_q(\hat{\tau}+h) &= \lim_{k\to\infty} \|B(q,\hat{\tau}+h)^k\|^{1/k} \\
&= \lim_{k\to\infty} \Big(\sup_u \sum_v \sum_{\mathbf{e}\in E_{uv}^k} p(\mathbf{e})^q r(\mathbf{e})^{-\hat{\tau}-h} \Big)^{1/k} \\
&\le \lim_{k\to\infty} \Big(\sup_u \sum_v \sum_{\mathbf{e}\in E_{uv}^k} r_{\inf}^{-kh} p(\mathbf{e})^q r(\mathbf{e})^{-\hat{\tau}} \Big)^{1/k} \\
&= r_{\inf}^{-h} \rho_q(\hat{\tau}).
\end{aligned}$$

Similarly, $\rho_q(\hat{\tau}+h) \ge r_{\sup}^{-h}\rho_q(\hat{\tau}) \ge \rho_q(\hat{\tau})$ for any $h < 0$. This proves that ρ_q is strict increasing and continuous.

Owing again to the fact that $B(q,0)$ has a finite norm, $p(e)^q r(e)^{-\hat{\tau}}$ converges uniformly to zero as $\hat{\tau} \to -\infty$. So $B(q,-\infty) := \lim_{\hat{\tau}\to-\infty} B(q,\hat{\tau})$ is the zero matrix and hence $\rho_q(-\infty) = 0$. Since P is primitive by our assumption, we assume, without loss of generality, that E_{uu} is not empty. Choose $e \in E_{uu}$. Then

$$\rho_q(\infty) = \lim_{\hat{\tau}\to\infty} \lim_{k\to\infty} \|B(q,\hat{\tau})^k\|^{1/k} \ge \lim_{\hat{\tau}\to\infty} p(e)^q r(e)^{-\hat{\tau}} = \infty. \qquad \square$$

Assume the hypotheses of Proposition 3.1, then for all $q \in \mathbb{R}$, there is a unique number $\hat{\tau}$ such that $B(q,\hat{\tau})$ has spectral radius 1. This defines a function $\hat{\tau}_P(q)$, also denoted by $\hat{\tau}(q)$

or $\hat{\tau}$. Notice that $\rho_{q_1}(\hat{\tau}) > \rho_{q_2}(\hat{\tau})$ if $q_1 < q_2$. This implies that $\hat{\tau}(q)$ is increasing with respect to q. We now give the proof of Theorem 1.1 under the condition $\hat{\tau}(q)$ is differentiable.

For any vector $\mathbf{y} = (y_1, y_2, \dots)$, we let

$$\mathbf{y}_{\inf} := \inf_i y_i \qquad \text{and} \qquad \mathbf{y}_{\sup} := \sup_i y_i.$$

Also, for any $F \subseteq \mathbb{R}^n$, $s \geq 0$, and $\varepsilon > 0$, let

$$\mathcal{P}^s_\varepsilon(F) : = \sup \Big\{ \sum_i (2r_i)^s : \{B_{r_i}(x_i)\} \text{ is a countable collection of disjoint balls with centers } x_i \in F \text{ and radius } r_i \leq \varepsilon \Big\}.$$

Proof of Theorem 1.1. The proof is similar to that of [3, Theorem 1.6]; we only give a sketch. Let $\alpha = \alpha(q) := \hat{\tau}'(q)$. We normalize $\mathbf{x}^t$ and $\mathbf{y}$ so that $\sum_u x_u = 1$ and $\sum_u x_u y_u = 1$ for all q. Conditions (1) and (2) imply that the series $\mathbf{x}^t\mathbf{y} = \sum_u x_u y_u$ is differentiable and differentiable termwise. Denote $S := \mathbf{x}^t\mathbf{y} = \mathbf{x}^t B\mathbf{y} = \sum_u \sum_v \sum_{e\in E_{uv}} x_u p(e)^q r(e)^{-\hat{\tau}(q)} y_v$, which is also differentiable termwise since the cardinality of $\{e \in E_{uv} : v \in V\}$ has a common upper bound, a consequence of the assumption $\deg(V) < \infty$. Differentiating $S = 1$ and solving for $\hat{\tau}'(q)$ yields

$$\alpha = \frac{\sum_u \sum_v \sum_{e\in E_{uv}} (x_u p(e)^q r(e)^{-\hat{\tau}(q)} y_v) \log p(e)}{\sum_u \sum_v \sum_{e\in E_{uv}} (x_u p(e)^q r(e)^{-\hat{\tau}(q)} y_v) \log r(e)}.$$

For $e \in E_{uv}$, define $\hat{p}(e) := y_u^{-1} p(e)^q r(e)^{-\hat{\tau}(q)} y_v$. For $\mathbf{e} = e_1 \cdots e_k \in E^k_{uv}$ with $\iota(e_i) = v_i$ and $\kappa(e_i) = v_{i+1}$, define the measure of the cylinder $[\mathbf{e}]$ by

$$\hat{\mu}^{(q)}_u([\mathbf{e}]) := \prod_{i=1}^k \hat{p}(e_i) = \prod_{i=1}^k y_{v_i}^{-1} \hat{p}(e_i)^q r(e_i)^{-\hat{\tau}(q)} y_{v_{i+1}} = y_u^{-1} p(\mathbf{e})^q r(\mathbf{e})^{-\hat{\tau}(q)} y_v.$$

Using a similar argument as [3, Lemma 4.1], and using Lemma 4.1 instead of [26, Theorems 4.1, 4.2] for a countable Markov chain, it can be shown that $\hat{\mu}^{(q)}_u(\hat{K}^{(\alpha)}_u) = 1$; we omit the details.

First, we compute the upper packing dimension of $\pi_u(\hat{K}^{(\alpha)}_u)$. By virtue of condition (3.3), packing dimension can be computed as in the case of a finite GIFS. For each integer $k \geq 1$ and each $\delta > 0$, define

$$\hat{S}^{(k)}_u = \hat{S}^{(k)}_u(\alpha, \delta) := \begin{cases} \Big\{ \mathbf{e} \in E^*_u : \dfrac{\log p(\mathbf{e}|k)}{\log r(\mathbf{e}|k)} \leq \alpha + \dfrac{\delta}{q} \Big\}, & q > 0, \\ \Big\{ \mathbf{e} \in E^*_u : \dfrac{\log p(\mathbf{e}|k)}{\log r(\mathbf{e}|k)} \geq \alpha + \dfrac{\delta}{q} \Big\}, & q < 0. \end{cases}$$

Denote

$$\hat{K}_u^{(N)} = \hat{K}_u^{(N)}(\alpha,\delta) := \bigcap_{k\ge N} \hat{S}_u^{(k)}, \qquad K_u^{(N)} = K_u^{(N)}(\alpha,\delta) := \pi_u \hat{K}_u^{(N)}.$$

Then $\hat{K}_u^{(\alpha)} \subset \bigcup_{N\ge1} \hat{K}_u^{(N)}$ and $K_u^{(\alpha)} \subset \bigcup_{N\ge1} K_u^{(N)}$. Fix $N\in\mathbb{N}$ and fix $\varepsilon>0$ small enough so that $r(\mathbf{e})>\varepsilon$ for all $\mathbf{e}\in E_u^N$. Let $B_{\varepsilon_i}(x_i)$ be a countable disjoint collection of balls with centers $x_i \in K_u^{(N)}$ and radii $\varepsilon_i<\varepsilon$. Let $\mathbf{e}_i$ satisfy $\pi_u(\mathbf{e}_i)=x_i$ and choose $k_i\in\mathbb{N}$ such that $r(\mathbf{e}_i|k_i)<\varepsilon_i\le r(\mathbf{e}_i|(k_i-1))$. The choice of ε implies $\mathbf{e}_i\in \hat{S}_u^{(N)}$ and hence

$$p(\mathbf{e}_i|k_i)^q r(\mathbf{e}_i|k_i)^{-\hat{\tau}(q)} \ge r(\mathbf{e}_i|k_i)^{q\alpha-\hat{\tau}(q)+\delta},$$

which holds for all $q\in\mathbb{R}$. Since $\varepsilon_i \le r(\mathbf{e}_i|k_i)/r_{\inf}$,

$$\begin{aligned}\sum_i (r_{\inf}\varepsilon_i)^{\hat{\tau}^*(\alpha)+\delta} &\le \sum_i r(\mathbf{e}_i|k_i)^{\hat{\tau}^*(\alpha)+\delta} \le \sum_i p(\mathbf{e}_i|k_i)^q r(\mathbf{e}_i|k_i)^{-\hat{\tau}(q)}\\ &\le \frac{\mathbf{y}_{\sup}}{\mathbf{y}_{\inf}} \sum_i \hat{\mu}_u^{(q)}([\mathbf{e}_i|k_i]) \le \frac{\mathbf{y}_{\sup}}{\mathbf{y}_{\inf}},\end{aligned}$$

where the last inequality holds since the cylinders $[\mathbf{e}_i|k_i]$ are disjoint and $\hat{\mu}_u^{(q)}(\hat{K}_u^{(\alpha)})=1$. So $\mathcal{P}_\varepsilon^{\hat{\tau}^*(\alpha)+\delta}(K_u^{(N)}) \le C(2/r_{\inf})^{\hat{\tau}^*(\alpha)+\delta}$, where $C:=\mathbf{y}_{\sup}/\mathbf{y}_{\inf}$. This, together with the arbitrariness of δ, implies that the packing dimension of $\pi_u(\hat{K}_u^{(\alpha)})$ is bounded above by $\hat{\tau}^*(\alpha)$.

Next, we prove the desired lower bound for the Hausdorff dimension of $\pi_u(\hat{K}_u^{(\alpha)})$. We take a Borel set $F\subset\mathbb{R}^n$ such that $2a := \hat{\mu}_u^{(q)}(\pi_u^{-1}F)>0$. Define

$$\hat{S}_u^{(k)} = \hat{S}_u^{(k)}(\alpha,\delta) := \begin{cases} \Big\{\mathbf{e}\in\pi_u^{-1}(F):\ \dfrac{\log p(\mathbf{e}|k)}{\log r(\mathbf{e}|k)} \ge \alpha - \dfrac{\delta}{q}\Big\}, & q>0,\\[2ex] \Big\{\mathbf{e}\in\pi_u^{-1}(F):\ \dfrac{\log p(\mathbf{e}|k)}{\log r(\mathbf{e}|k)} \le \alpha - \dfrac{\delta}{q}\Big\}, & q<0.\end{cases}$$

Denote $\hat{K}_u^{(N)} := \cap_{k\ge N}\hat{S}_u^{(k)}$ and $K_u^{(N)} := \pi_u(\hat{K}_u^{(N)})$. Choose N large enough so that $\hat{\mu}_u^{(q)}(\hat{K}_u^{(N)}) > a$, and take $\varepsilon>0$ small enough such that $\varepsilon<r(\mathbf{e})$ for all $\mathbf{e}\in E_u^N$. Let $\{A_i\}$ be any countable cover of F with $\mathrm{diam}A_i<\varepsilon$. For each A_i, let

$$H_i = \big\{\mathbf{e}\in E_u^*:\ r(\mathbf{e}) < \mathrm{diam}A_i \le r(\mathbf{e}\big||\mathbf{e}|-1), \pi_u([\mathbf{e}])\cap A_i\cap F\cap \pi_u(\hat{K}_u^{(N)}) \ne \emptyset\big\}.$$

Then $[\mathbf{e}_1]\cap[\mathbf{e}_2]=\emptyset$ for any distinct $\mathbf{e}_1,\mathbf{e}_2\in H_i$. Denote $H:=\bigcup_i H_i$. Then $\{[\mathbf{e}]:\ \mathbf{e}\in H\}$ forms a cover of $\hat{K}_u^{(N)}$. Now for each $\mathbf{e}\in H$, we see that

$$p(\mathbf{e})^q r(\mathbf{e})^{-\hat{\tau}(q)} \le r(\mathbf{e})^{\hat{\tau}^*(\alpha)-\delta},$$

which holds for all $q\in\mathbb{R}$. This implies

$$a \le \hat{\mu}_u(\hat{K}_u^{(N)}) \le \sum_{\mathbf{e}\in H} \hat{\mu}_u^{(q)}([\mathbf{e}]) \le \frac{\mathbf{y}_{\sup}}{\mathbf{y}_{\inf}} \sum_{\mathbf{e}\in H} r(\mathbf{e})^{\hat{\tau}^*(\alpha)-\delta} \le C\sum_i (\mathrm{diam}A_i)^{\hat{\tau}^*(\alpha)-\delta}.$$

The last inequality can be obtained from [21, Lemma V]. It follows that $\dim_{\mathrm{H}} \pi_u^{-1}(F) \geq \hat{\tau}^*(\alpha)$. Hence $\dim_{\mathrm{H}} \pi_u(\hat{K}_u^{(\alpha)}) \geq \hat{\tau}^*(\alpha)$.

When (PSC) holds, $K_u^{(\alpha)} = \pi_u(\hat{K}_u^{(\alpha)})$, and this completes the proof of the theorem. □

4. INFINITE MATRICES AND SELF-SIMILAR SETS

4.1. **Infinite matrices.** Let $A = (a_{ij})$ be a non-negative infinite matrix. We assume that $A \in \mathcal{B}(\ell^\infty, \ell^\infty)$, the space of all bounded linear operators on ℓ^∞, and consequently A^k is well-defined for any $k \geq 1$. Note that the operator norm of A coincides with the one defined in (3.1). Denote the maximal real eigenvalue of A, if it exists, by $\lambda(A)$. It is well known that $\lambda(A) \leq \rho(A)$.

We denote the (i,j) entry of the infinite matrix A^k by $a_{ij}^{(k)}$. If for each pair (i,j), there exists k such that $a_{ij}^{(k)} > 0$, we call A *irreducible*. Let d_i be the greatest common divisor of those k for which $a_{ii}^{(k)} > 0$. An irreducible matrix A is said to be *primitive* if $d_i = 1$ for some (and hence all) i.

Let $\beta > 0$. A non-negative row vector $\mathbf{x}^t \neq \mathbf{0}$ is called a *β-invariant measure* if $\beta \mathbf{x}^t A = \mathbf{x}^t$. A non-negative column vector $\mathbf{y} \neq \mathbf{0}$ is called a *β-invariant vector* if $\beta A \mathbf{y} = \mathbf{y}$.

The following lemma is a combination of Theorems 5.5 and 6.4 in [26].

Lemma 4.1 (General Ergodic Theorem)**.** *Let $A = (a_{ij})$ be a primitive (row) stochastic matrix, i.e., $\sum_{j=1}^{\infty} a_{ij} = 1$ for each i. Suppose $\mathbf{x}^t$ is a 1-invariant measure and $\mathbf{y}$ a 1-invariant vector of A. If*

$$\mathbf{x}^t \mathbf{y} = \sum_{i=1}^{\infty} x_i y_i < \infty,$$

then for each j,

$$\lim_{k \to \infty} a_{ij}^{(k)} = c^{-1} x_j,$$

where $c = \sum_{n \geq 1} x_n$. Thus $c^{-1} \mathbf{x}^t$ is the unique stationary distribution of A.

Let us introduce some "extensions" of vectors and matrices; they will play a key role in our theory. Let $\bar{B}_n$ be an $n \times n$ matrix and B be an infinite matrix defined as

$$\bar{B}_n = \begin{pmatrix} B_{11} & \mathbf{b}_1 \\ \mathbf{b}_2 & b_{nn;n} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} B_{11} & \mathbf{b}_1 & & & \\ \mathbf{b}_2 & & b_{n,n+1} & & \\ \mathbf{b}_3 & & & b_{n+1,n+2} & \\ \vdots & & & & \ddots \end{pmatrix}, \tag{4.1}$$

where $b_{nn;n} \in \mathbb{R}$ and the undisplayed entries are zero. We denote the (i, j) entry of $\bar{B}_n$ by $b_{ij;n}$ and that of B by b_{ij}. We call $\bar{B}_n$ the n-th *quasi-truncation* of B if $b_{nn;n} = b_{n,n+1}$ and B the *quasi-extension* of $\bar{B}_n$ if $\mathbf{b}_{2+j} = \mathbf{b}_2$, $b_{n+j,n+j+1} = b_{nn;n}$ for any $j \geq 0$. If $\bar{B}_n$ is the n-th quasi-truncation of B and B_n is the quasi-extension of $\bar{B}_n$, we call B_n the *extended n-th quasi-truncation* of B.

For a column vector $\bar{\mathbf{y}}_n = (y_1, \ldots, y_n)^t$, we call $\mathbf{y} = (y_1, \ldots, y_n, y_n, y_n, \ldots)^t$ the *column extension* of $\bar{\mathbf{y}}_n$. For a row vector $\bar{\mathbf{x}}_n^t = (x_1, \ldots, x_n)^t$ and a number $\zeta \in [0, 1)$, we call

$$\mathbf{x}^t(\zeta) = \big(\zeta_1, \ldots, \zeta_{n-1}, (1-\zeta)x_n, (1-\zeta)\zeta x_n, (1-\zeta)\zeta^2 x_n, \ldots\big)$$

the *row extension* of $\bar{\mathbf{x}}_n^t$ with respect to ζ. Note that

$$\sum_{k=0}^{\infty} (1-\zeta)\zeta^k x_n = x_n \qquad \text{and} \qquad \mathbf{x}^t \mathbf{y} < \infty.$$

It is clear that for any $\bar{\mathbf{x}}_n^t$ and $\bar{\mathbf{y}}_n$, the extensions $\mathbf{x}^t$ and $\mathbf{y}$ belong to ℓ^∞.

The following three lemmas provide some information on the spectral radius of an infinite matrix. Lemma 4.4 will play a key role in defining $\hat{\tau}(q)$.

Lemma 4.2. *Let $A \in \mathcal{B}(\ell^\infty, \ell^\infty)$ be a nonnegative infinite matrix and assume that there exists a 1-invariant vector $\mathbf{y} = (y_1, y_2, \ldots)^t$ of A satisfying $0 < \mathbf{y}_{\inf} \leq \mathbf{y}_{\sup} < \infty$. Then $\rho(A) = 1$.*

Proof. Let $c := \mathbf{y}_{\sup}/\mathbf{y}_{\inf} < \infty$. Since $A^k \mathbf{y} = \mathbf{y}$ for any k, we have

$$\|A^k\| = \sup_i \sum_{j \geq 1} a_{ij}^{(k)} \leq \sup_i \frac{\sum_{j\geq 1} a_{ij}^{(k)} y_j}{\mathbf{y}_{\inf}} = \sup_i \frac{y_i}{\mathbf{y}_{\inf}} = c < \infty.$$

It follows that $\|A^k\|^{1/k} \leq c^{1/k} \to 1$ as $k \to \infty$. An similar argument yields $\|A^k\|^{1/k} \geq c^{-1/k} \to 1$ as $k \to \infty$. Hence $\rho(A) = 1$ by (3.2). $\square$

Lemma 4.3. *Let $B = (b_{ij})$ be a non-negative infinite matrix as in* (4.1) *with $\{b_{n,2}\}$, $\{b_{n,n+1}\}$ being positive for any $n \geq 2$ and $b_{ij} = 0$ for other i, j. Denote*

$$s := \inf\{b_{n,2} : n \geq 2\} \qquad \textit{and} \qquad t := \inf\{b_{n,n+1} : n \geq 2\}.$$

Then $s+t \le \rho(B)$.

Proof. By using induction, we see that the sum of the second row of B^k is no less than $(s+t)^k$, which implies $\rho(B) \ge s+t$. □

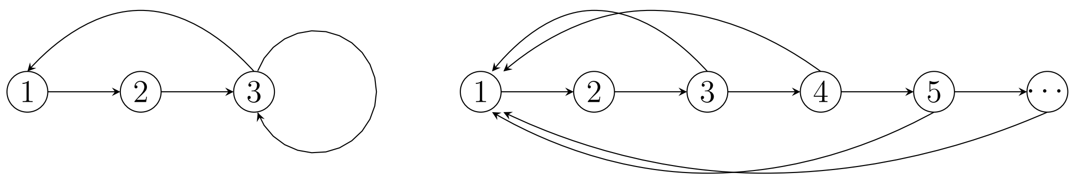


FIGURE 2. Figure for Lemma 4.4 for the case $n=3$. The left one is for $(\bar{V}, \bar{E})$ and the right one is for (V, E).

Lemma 4.4. *Let $\bar{B}_n$ be a finite irreducible non-negative matrix with a positive invariant measure and B_n be its quasi-extension. Then*

$$\rho(B_n) = \lambda(B_n) = \lambda(\bar{B}_n). \tag{4.2}$$

Furthermore, if $\lambda(\bar{B}_n) = 1$, $\bar{\mathbf{x}}_n^t > 0$ is a corresponding positive 1-invariant measure, $b_{nn;n} > 0$, and $\mathbf{x}^t$ is the row extension of $\bar{\mathbf{x}}_n^t$ with respect to $\zeta = b_{nn;n}$, then

$$\mathbf{x}^t B_n = \mathbf{x}^t.$$

Proof. Since $\bar{B}_n$ is a finite irreducible matrix, $\lambda = \lambda(\bar{B}_n)$ is the unique maximal eigenvalue, which is positive by the Perron-Frobenius theorem. Let $\bar{\mathbf{y}}_n = (1, y_2, \dots, y_n)^t$ be a positive λ-eigenvector and $\mathbf{y}$ be its column extension. Then by definition, $B_n\mathbf{y} = \lambda\mathbf{y}$, i.e., λ is a positive eigenvalue of B_n.

Next, we show $\rho(B_n) = \rho(\bar{B}_n)$. We regard $\bar{B}_n$ and B_n as two weighted incidence matrices (not necessarily stochastic) for two directed graphs $\bar{G} = (\bar{V}, \bar{E})$ and $G = (V, E)$ respectively, with $V, \bar{V} \subseteq \mathbb{N}$ (see Figure 2). For $\bar{\mathbf{e}} := \bar{e}_1 \cdots \bar{e}_k \in \bar{E}^*$, denote $b_{\bar{\mathbf{e}};n} := \prod_{j=1}^k b_{\bar{e}_j;n}$. Similarly, we define $b_{\mathbf{e}}$ for $\mathbf{e} \in E^*$. For $u \in V$ and $v \in \bar{V}$, let

$$A_u(k) := \sum_{\mathbf{e} \in E_u^k} b_{\mathbf{e}} \qquad \text{and} \qquad \bar{A}_v(k) := \sum_{\bar{\mathbf{e}} \in \bar{E}_v^k} b_{\bar{\mathbf{e}};n}, \tag{4.3}$$

where we define $A_u(0) := \bar{A}_v(0) := 1$. Define a map φ from V onto $\bar{V}$ as:

$$\varphi(v) := \begin{cases} v, & \text{if } v < n, \\ n, & \text{if } v \ge n. \end{cases}$$

This map induces a surjection, also denoted by φ, from E^* onto $\bar{E}^*$. The definition of B_n implies that $b_{\mathbf{e}} = b_{\varphi(\mathbf{e});n}$.

Claim: For $u < n$, the map φ is a bijection from E_u^* onto $\bar{E}_u^*$.

To prove the claim, note that for $u, v \geq n$, $b_{uv} > 0$ if and only if $(u, v) = (p, p+1)$ for some $p \geq n$. This implies the following:

Observation 1: *For each $k \geq 1$, there exists a unique path in E_n^k that does not pass through any vertex v if $v < n$, but passes through the vertices $n, n+1, \ldots, n+k-1$ in the given order. (See Figure 2.)*

Let $\mathbf{e} = e_1 \cdots e_k$ and $\mathbf{f} = f_1 \cdots f_k$ be two paths in E_u^* satisfying $\varphi(\mathbf{e}) = \varphi(\mathbf{f})$. Let v_i, u_i be the i-th vertex in the paths $\mathbf{e}, \mathbf{f}$, respectively. We will show $u_i = v_i$ for each i, which implies φ is bijective. Fix $i \in \{1, \ldots, k\}$. By the definition of φ, we see $v_i = u_i$ if $v_i < n$ or $u_i < n$. When $v_i \geq n$ (and hence $u_i \geq n$), denote

$$\begin{aligned} c_0 &:= \min\{j : v_p \geq n, j \leq p \leq i\}, \qquad c_1 := \max\{j : v_p \geq n, i \leq p \leq j\}, \\ d_0 &:= \min\{j : u_p \geq n, j \leq p \leq i\}, \qquad d_1 := \max\{j : u_p \geq n, i \leq p \leq j\}. \end{aligned}$$

It follows from $u < n$ that $c_0, d_0 \geq 2$. The definition of φ again yields $c_i = d_i$, $i = 0, 1$. Then from Observation 1, we know $v_j = u_j = n + c_1 - c_0$ for $c_0 \leq j \leq c_1$. Hence, $v_i = u_i$, which completes the proof of the claim.

From the claim above and (4.3), we know for each $u < n$,

$$A_u(k) = \bar{A}_u(k) \quad \text{for all } k \geq 1. \tag{4.4}$$

On the other hand, by the definition of $A_u(k)$ and Observation 1,

$$\begin{aligned} A_{u-1}(k+1) &= \sum_{v \in V} \sum_{e \in E_{u-1,v}} b_e A_v(k) = \sum_{1 \leq v < n} \sum_{e \in E_{u-1,v}} b_e A_v(k) + b_{u-1,u} A_u(k) \\ &= \sum_{1 \leq v < n} b_{u-1,v} A_v(k) + b_{u-1,u} A_u(k), \qquad u \geq n. \end{aligned} \tag{4.5}$$

The first equality holds by the definition of $A_v(k)$. The second one holds since the vertex set V is divided into two parts, one satisfying $v < n$ and the other satisfying $v \geq n$; however, the second part contains a unique vertex u by Observation 1 since $u \geq n$. Rewriting (4.5) gives

$$A_u(k) = b_{u-1,u}^{-1} \Big(A_{u-1}(k+1) - \sum_{1 \leq v < n} b_{u-1,v} A_v(k) \Big), \quad u \geq n. \tag{4.6}$$

Applying the above argument to $(\bar{V}, \bar{E})$ and $\bar{B}_n$ yields

$$\bar{A}_n(k) = b_{n-1,n;n}^{-1} \Big(\bar{A}_{n-1}(k+1) - \sum_{1 \leq v < n} b_{n-1,v;n} \bar{A}_v(k) \Big). \tag{4.7}$$

Substituting $u = n$ into (4.6) and comparing it with (4.7), we conclude from (4.4) and the equality $b_{n-1,n;n} = b_{n-1,n}$ that

$$A_n(k) = \bar{A}_n(k). \tag{4.8}$$

The definition of $\bar{A}_n(k+1)$ yields

$$\bar{A}_n(k+1) = \sum_{1 \le v < n} b_{n,v;n} \bar{A}_v(k) + \bar{b}_{n,n;n} A_n(k). \tag{4.9}$$

Now we compare (4.9) with (4.5) with u substituted by $n+1$, i.e.,

$$A_n(k+1) = \sum_{1 \le v < n} b_{n,v} A_v(k) + b_{n,n+1} A_{n+1}(k).$$

Note that we have $b_{n,v;n} = b_{n,v}$ for $v < n$, $b_{n,n;n} = b_{n,n+1}$, and $A_v(k) = \bar{A}_v(k)$ for $v \le n$. It follows that $A_{n+1}(k) = \bar{A}_n(k)$. Using (4.6) and induction, we get

$$A_u(k) = \bar{A}_n(k), \quad u > n. \tag{4.10}$$

Notice that $\|B_n^k\| = \sup_{u \in V} \sum_{v \ge 1} b_{uv}^{(k)} = \sup_{u \in V} A_u(k)$ and $\|\bar{B}_n^k\| = \sup_{u \in \bar{V}} \bar{A}_u(k)$. By combining (4.4), (4.8), and (4.10), we get $\|B_n^k\| = \|\bar{B}_n^k\|$ for all $k \ge 1$. Thus by (3.2), $\rho(B_n) = \rho(\bar{B}_n)$. Finally, (4.2) follows from the fact that the spectral radius of a finite non-negative matrix is equal to its maximal real eigenvalue.

Now suppose $\bar{\mathbf{x}}_n^t = [x_1, \ldots, x_n]^t$ is a positive 1-invariant measure of $\bar{B}_n$. Then $\zeta := b_{nn;n} \in [0,1)$, as $\lambda(\bar{B}_n) = 1$. Denote the j-th component of $\mathbf{x}^t$ by z_j. By the definition of $\mathbf{x}^t$, $z_j = x_j$ for $j < n$ and $z_j = (1-\zeta)\zeta^{j-n} x_n$ for $j \ge n$. Notice that $\sum_{j \ge 0} z_{n+j} = x_n$. Hence by using the definitions of $\bar{B}_n$ and B_n, we get

$$\sum_{i=1}^{\infty} z_i b_{ij} = \begin{cases} \sum_{i=1}^{n-1} x_i b_{ij;n} + \sum_{i \ge 0} z_{n+i} b_{nj;n} = \sum_{i=1}^{n} x_i b_{ij;n} = x_j = z_j, & \text{if } 1 \le j < n, \\ z_{j-1} b_{j-1,j} = (1-\zeta)\zeta^{j-n-1} x_n b_{nn;n} = (1-\zeta)\zeta^{j-n} x_n = z_j, & \text{if } j > n, \\ \sum_{i=1}^{n} x_i b_{ij;n} - x_j b_{nn;n} = x_j(1 - b_{jj;n}) = z_j, & \text{if } j = n. \end{cases}$$

These equalities yield $\mathbf{x}^t B_n = \mathbf{x}^t$, which completes the proof. □

Remark 4.5. *Under the same conditions as in Lemma 4.4, the invariant measure $\mathbf{x}^t$ is positive if $b_{n,n;n} > 0$.*

4.2. **An infinite GIFS induced by an IFS with overlaps.** Let $\{S_i(x) = r_iO_ix + d_i\}_{i=1}^N$ be an IFS of contractive similitudes, where $r_i \in (0,1)$, O_i is an orthogonal matrix and $d_i \in \mathbb{R}^n$. Then there exists a unique nonempty compact set K satisfying

$$K = \bigcup_{i=1}^N S_i(K).$$

We call K the *self-similar set* generated by the IFS. For a probability vector $(p_1, \dots, p_N)$ (i.e., $\sum_{i=1}^N p_i = 1$ and $p_i > 0$ for all i), there is a unique probability measure μ, called a *self-similar measure*, such that

$$\mu = \sum_{i=1}^N p_i \mu \circ S_i^{-1}. \tag{4.11}$$

Moreover, the support of μ is K. For $\alpha \in [0,\infty)$, let $K^{(\alpha)}$ be defined as in (1.7).

Now we consider the special IFS $\{S_i\}_{i=1}^3$ given as in (1.5) with contraction ratios satisfying (1.6). Let μ be the self-similar measure with respect to the probability vector (p_1, p_2, p_3). To avoid confusion, we denote, in the rest of this section,

$$S_a := S_1, \quad S_b := S_2, \quad S_c := S_3, \quad p_a := p_1, \quad p_b := p_2, \quad p_c := p_3, \quad \Sigma_3 := \{a, b, c\}.$$

We list some properties of K and μ. They are based on the structure of K and will play a key role in the construction of a desired infinite GIFS. Let $v_1 := 1$ and

$$v_n := 1 + \frac{p_a}{p_b}\left(1 + \frac{p_c}{p_b} + \cdots + \left(\frac{p_c}{p_b}\right)^{n-2}\right), \quad n \ge 2. \tag{4.12}$$

Denote $p_{1,1} := p_c$, $p_{1,2} := 1 - p_c$, and

$$p_{n,1} := \frac{p_c}{v_n}, \qquad p_{n,2} = 1 - \frac{p_c}{v_n} - \frac{p_b v_{n+1}}{v_n}, \qquad p_{n,n+1} := \frac{p_b v_{n+1}}{v_n}, \qquad n \ge 2. \tag{4.13}$$

Then $P = (p_{ij})$ is a primitive infinite stochastic matrix if we set all other $p_{ij} := 0$. Let

$$Q_1 := K \quad \text{and} \quad Q_n := S_{b^{n-2}}\big(S_a(K) \bigcup S_b(K)\big) \quad \text{for } n \ge 2$$

(see Figure 3). Then $Q_1 = Q_2 \bigcup S_c(Q_1)$ and $Q_{n+1} = S_b(Q_n)$ for all $n \ge 2$. Also, by using these relations and the equality $S_{ac} = S_{ba}$, we conclude by induction that

$$Q_n = S_{b^{n-2}a}(Q_2) \bigcup Q_{n+1} \bigcup S_{b^{n-1}c}(Q_1) \quad \text{for } n \ge 2. \tag{4.14}$$

Lemma 4.6. *Using the notation above, we have $\mu(Q_n) = p_b^{n-1} v_n$ for all $n \ge 1$. Consequently,*

$$\frac{\mu(S_{b^{n-1}c}(Q_1))}{\mu(Q_n)} = p_{n,1}, \quad \frac{\mu(S_{b^{n-2}a}(Q_2))}{\mu(Q_n)} = p_{n,2}, \quad \frac{\mu(Q_{n+1})}{\mu(Q_n)} = p_{n,n+1}. \tag{4.15}$$

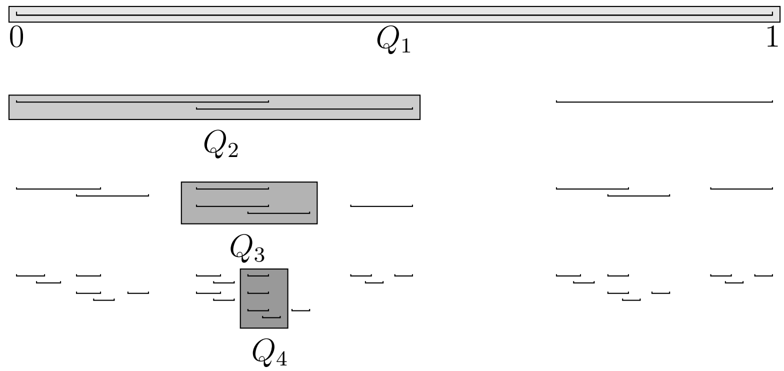


FIGURE 3. The first 4 intervals containing $\{Q_n\}$, drawn with $r_1 = 3^{-1}$ and $r_2 = 2/7$.

Proof. Since $v_1 = 1$ and $p_b > 0$, $\mu(Q_1) = \mu(K) = 1$. For $n \geq 2$, we let $I_{n-1} := \{\gamma \in \Sigma_3^{n-1} : S_\gamma(K) \cap Q_n \neq \emptyset\}$. Then

$$I_{n-1} = \{ac^{n-2},\, bac^{n-3},\, \dots,\, b^{n-3}ac,\, b^{n-2}a,\, b^{n-1}\}.$$

Clearly, $S_\gamma^{-1}(Q_n) \cap K = K$ for each $\gamma \in I_{n-1}$. As $Q_n = S_{b^{n-2}a}(K) \bigcup S_{b^{n-1}}(K)$,

$$S_\gamma^{-1}(Q_n) \supset \begin{cases} S_\gamma^{-1}(S_{b^{n-2}a}(K)) = K, & \text{if } S_\gamma = S_{b^{n-2}a}, \\ S_\gamma^{-1}(S_{b^{n-1}}(K)) = K, & \text{if } S_\gamma = S_{b^{n-1}}. \end{cases}$$

Therefore, by the definition of v_n,

$$\begin{aligned} \mu(Q_n) &= \sum_{\gamma \in I_{n-1}} p_\gamma \mu(S_\gamma^{-1}(Q_n)) \\ &= \big(p_a p_c^{n-2} + p_b p_a p_c^{n-3} + \cdots + p_b^{n-3} p_a p_c + p_b^{n-2} p_a + p_b^{n-1}\big)\mu(K) \\ &= p_b^{n-1} v_n. \end{aligned}$$

Now we show (4.15). The third term follows directly from the equality $\mu(Q_n) = p_b^{n-1} v_n$ and the definition of $p_{n,n+1}$ in (4.13). Notice that there is only one index $\gamma \in \Sigma_3^n$ satisfying $S_\gamma(K) \cap S_{b^{n-1}c}(Q_1) \neq \emptyset$. So, $\mu(S_{b^{n-1}c}(Q_1)) = p_b^{n-1} p_c$. This, together with the definition of $p_{n,1}$, yields the first equality in (4.15). Finally, since the union on the right side of (4.14) is disjoint, and $p_{n,1} + p_{n,2} + p_{n,n+1} = 1$, we get

$$\frac{\mu(S_{b^{n-2}a}(Q_2))}{\mu(Q_n)} = 1 - \frac{\mu(Q_{n+1}) + \mu(S_{b^{n-1}c}(Q_1))}{\mu(Q_n)} = 1 - p_{n,n+1} - p_{n,1} = p_{n,2}.$$

This completes the proof. □

Lemma 4.7. *Let $\theta \in \Sigma_3^*$.*

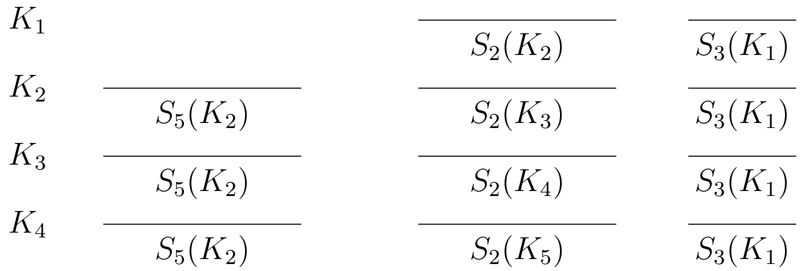


FIGURE 4. The sets $K_i, i = 1, 2, 3, 4$, where $S_5 := S_2^{-1} S_1 S_2$.

(a) *For any μ-measurable set $F \subset Q_2$,*

$$\mu(S_{\theta a}(F)) = \frac{\mu(S_{\theta a}(Q_2))\mu(F)}{\mu(Q_2)} = \frac{\mu(S_{\theta a}(Q_2))\mu(F)}{1 - p_c}. \tag{4.16}$$

(b) *For any μ-measurable set $F \subset Q_1$,*

$$\mu(S_{\theta\theta'}(F)) = \frac{\mu(S_{\theta\theta'}(Q_1))\mu(F)}{\mu(Q_1)} = \mu(S_{\theta\theta'}(Q_1))\mu(F), \tag{4.17}$$

where $\theta' = bc$ or $\theta' = cc$. Furthermore, (4.17) *also holds if $\theta\theta' = c$.*

Proof. (a) Assume $|\theta| = n$ and let $F \subseteq Q_2$. Let $J_{n+1} := \{\gamma \in \Sigma_3^{n+1} : S_\gamma(K) \cap S_{\theta a}(Q_2) \neq \emptyset\}$. Then it follows from the structure of K that $\bigcup_{\gamma \in J_{n+1}} S_\gamma(K) = S_{\theta a}(K)$. Notice that $S_{\theta a}(Q_2) \subset S_\theta(K)$. So

$$\frac{\mu(S_{\theta a}(F))}{\mu(S_{\theta a}(Q_2))} = \frac{\sum_{\gamma \in J_{n+1}} p_\gamma \mu(S_\gamma^{-1} S_{\theta a}(F))}{\sum_{\gamma \in J_{n+1}} p_\gamma \mu(S_\gamma^{-1} S_{\theta a}(Q_2))} = \frac{\mu(F)}{\mu(Q_2)}.$$

We get (4.16).

(b) A similar argument yields (4.17). The last conclusion is clear since $S_c(K)$ does not intersect $S_a(K) \bigcup S_b(K)$. □

Now we construct an infinite GIFS such that K and μ are, respectively, the first components of the attractor and the family of graph-directed measures. First, let

$$K_1 := K \quad \text{and} \quad K_n := S_2^{-n+1}(Q_n) = S_2^{-1} S_1(K) \bigcup K, \quad n \geq 2.$$

Then, by (4.14) (see Figure 4),

$$K_1 = S_3(K_1) \bigcup S_2(K_2), \quad K_n = S_3(K_1) \bigcup S_2^{-1} S_1 S_2(K_2) \bigcup S_2(K_{n+1}), \quad n \geq 2. \tag{4.18}$$

The above recurrent equations, together with $P = (p_{ij})$ defined as in (4.13), form an infinite GIFS (V, E, M, P) with attractor $\boldsymbol{K} = (K_1, K_2, \ldots)^t$, where the (i, j) entry of M is defined as

$$M_{ij} := S_{ij} = \begin{cases} S_c, & j=1, \\ S_b, & j=i+1,\ i\geq 1, \\ S_b^{-1}S_aS_b, & j=2,\ i\geq 2, \\ \emptyset, & \text{otherwise.} \end{cases}$$

That is,

$$M = \begin{pmatrix} S_3 & S_2 & \emptyset & \emptyset & \emptyset & \cdots \\ S_3 & S_2^{-1}S_1S_2 & S_2 & \emptyset & \emptyset & \cdots \\ S_3 & S_2^{-1}S_1S_2 & \emptyset & S_2 & \emptyset & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}.$$

By (1.6), we know (V, E, M) satisfies (PSC). For (V, E, M), (3.3) becomes

$$r_{\inf} = \min\{r_a, r_b\}, \qquad r_{\sup} = \max\{r_a, r_b\}. \tag{4.19}$$

For $\mathbf{e} = e_1 \cdots e_n \in E^n$, we call $e_i \cdots e_j$ a *quasi-circle* of $\mathbf{e}$ if $\iota(e_m)$, $i \leq m \leq j$, are increasing and $\iota(e_{i-1}) \geq \iota(e_i)$ (if e_{i-1} exists), $\kappa(e_j) \leq \iota(e_j)$ (if $j = |\mathbf{e}|$, we allow $\kappa(e_j) \geq \iota(e_j)$). We call $\mathbf{e} \in E^*$ an *n-quasi-circle* if $\mathbf{e}$ has n quasi-circles. For example, $e = (1,1)$ is a quasi-circle and both $(1,1)(1,2)(2,1)$ and $(1,1)(1,2)$ are 2-quasi-circles.

Theorem 4.8. *Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots)^t$ be the family of graph-directed measures of (V, E, M, P). Then the first component of $\boldsymbol{\mu}$ coincides with the self-similar measure μ, namely $\mu_1 = \mu$.*

Proof. The conclusion $K = K_1$ is obvious. To show $\mu_1 = \mu$, we first notice that the set consisting of cylinders $\mathcal{F} = \{S_{\mathbf{e}}(K_i) : \mathbf{e} \in E^*_{1i}, i \geq 1\}$ forms a subbase of K. Hence it suffices to show $\mu_1(S_{\mathbf{e}}(K_{\kappa(\mathbf{e})})) = \mu(S_{\mathbf{e}}(K_{\kappa(\mathbf{e})}))$ for each $\mathbf{e} \in E^*_1$. We use induction. First, we show that the conclusion is true when $\mathbf{e} = e_1 \cdots e_n \in E^*_1$ is a quasi-circle by considering the following three cases.

Case 1. $\mathbf{e} = (1,1)$. In this case $\mu_1(S_{\mathbf{e}}(K_1)) = \hat{\mu}_1([\mathbf{e}]) = p_{1,1} = p_c$. On the other hand, from $S_{\mathbf{e}}(K_1) = S_c(K)$, we see $\mu(S_{\mathbf{e}}(K_1)) = p_c$. So, $\mu_1(S_{\mathbf{e}}(K_1)) = \mu(S_{\mathbf{e}}(K))$.

Case 2. *$\mathbf{e}$ ends with $n+1$.* In this case, from the structure of the infinite directed graph (V, E, M, P) and the definition of quasi-circle, we know the path $\mathbf{e}$ has the form $\mathbf{e} = (1,2)(2,3)\cdots(n, n+1)$. This yields

$$\mu_1(S_{\mathbf{e}}(K_{n+1})) = p_{1,2} \cdots p_{n,n+1} = p_b^n v_{n+1}.$$

The fact $S_{\mathbf{e}}(K_{n+1}) = S_b^n(K_{n+1}) = Q_{n+1}$ implies

$$\mu(S_{\mathbf{e}}(K_{n+1})) = \mu(Q_{n+1}) = p_n^n v_{n+1}. \tag{4.20}$$

So, $\mu_1(S_{\mathbf{e}}(K_{n+1})) = \mu(S_{\mathbf{e}}(K_{n+1}))$.

Case 3. $\mathbf{e} = (1,2)\cdots(n-1,n)(n,i)$ *with* $n \geq 2$*, where* $i = 1$ *or* $i = 2$. In this case $S_{\mathbf{e}} = S_{b^{n-1}} S_\gamma$, where $\gamma = c$ if $i = 1$ and $\gamma = b^{-1}ab$ if $i = 2$. By using $K_2 = S_2^{-1}(Q_2)$, equation (4.20), and the first two equations in (4.15), we get

$$\mu(S_{\mathbf{e}}(K_i)) = \mu(S_{b^{n-1}\gamma}(K_i)) = \mu(Q_n) p_{n,i} = p_b^{n-1} v_n p_{n,i}.$$

The definition of μ_1 implies

$$\mu_1(S_{\mathbf{e}}(K_i)) = p_{1,2}\cdots p_{n-1,n} p_{n,i} = p_b^{n-1} v_n p_{n,i}.$$

Thus we get $\mu_1(S_{\mathbf{e}}(K_i)) = \mu(S_{\mathbf{e}}(K_i))$.

Next, we assume that the conclusion holds for any $(k-1)$-quasi-circle $\mathbf{e} \in E_1^*$ and consider the case $\mathbf{e} = e_1 \cdots e_n \in E_1^*$ being a k-quasi-circle. We divide this into two cases. Let $\mathbf{e}' = e_1 \cdots e_i$ be the first quasi-circle of $\mathbf{e}$ (i could be 1). Denote $j = \kappa(\mathbf{e})$.

Case 1. The terminal vertex of $\mathbf{e}'$ *is 1.* In this case, we choose the unique $\theta \in \Sigma_3^*$ satisfying $S_\theta = S_{e_1\cdots e_i}$. Since S_a is followed by S_b or S_a in $\{S_{\mathbf{e}} : \mathbf{e} \in E^*\}$, θ must end with c and cannot end with ac (since $\kappa(\mathbf{e}') = 1$). We conclude that $S_{e_{i+1}\cdots e_n}(K_j)$ is a subset of $Q_1 = K_1$. Since θ ends with ac or cc, the path $e_{i+1}\cdots e_n$ starts with vertex 1 and $S_{e_{i+1}\cdots e_n}$ maps K_j into K_1. Denote $F = S_{e_{i+1}\cdots e_n}(K_j)$. Then $F \subset K_1$. Now (4.17) implies

$$\mu(S_{\mathbf{e}}(K_j)) = \mu(S_{\theta e_{i+1}\cdots e_n}(K_j)) = \mu(S_\theta(F)) = \mu(S_\theta(K_1))\mu(F) = \mu(S_\theta(K_1))\mu(S_{e_{i+1}\cdots e_n}(K_j)).$$

(where we have used the fact $Q_1 = K_1$). Applying the induction hypothesis and the fact $S_{\mathbf{e}'} = S_\theta$, we get

$$\begin{aligned} \mu(S_{\mathbf{e}}(K_j)) &= \mu(S_{\mathbf{e}'}(K_1))\mu_1(S_{e_{i+1}\cdots e_n}(K_j)) = \mu_1(S_{\mathbf{e}'}(K_1))\mu_1(S_{e_{i+1}\cdots e_n}(K_j)) \\ &= p_{\mathbf{e}'} p_{e_{i+1}\cdots e_n} = \mu_1(S_{\mathbf{e}}(K_j)). \end{aligned}$$

Case 2. The terminal vertex of e_i *is* 2. Note that $e_1 \cdots e_i$ is a quasi-circle in $\mathbf{e}$ and $i < |\mathbf{e}|$, which implies that $2 = \kappa(e_i) \leq \iota(e_i)$, and consequently, $e_i \neq (1,2)$. Hence in this case $S_{e_i} = S_b^{-1} S_a S_b$. So $S_b S_{e_{i+1}\cdots e_n}(K_j) = S_{(1,2)e_{i+1}\cdots e_n}(K_j)$ is a subset of Q_2 and $(1,2)e_{i+1}\cdots e_n$ is a $(k-1)$-quasi-circle. Notice that there exists $\theta \in \Sigma_3^*$ such that $S_\theta = S_{e_1\cdots e_{i-1}} S_b^{-1}$. That

$K_2 = S_b^{-1}(Q_2)$ implies $S_{\theta a}(Q_2) = S_{\theta ab}(K_2) = S_{e_1\cdots e_i}(K_2)$. Hence (4.16) yields

$$\begin{aligned}
\mu(S_{\mathbf{e}}(K_j)) &= \mu(S_\theta S_a S_b S_{e_{i+1}\cdots e_n}(K_j)) \\
&= \mu(S_{\theta a}(Q_2))\frac{\mu(S_b S_{e_{i+1}\cdots e_n}(K_j))}{\mu(Q_2)} \quad (\text{using } e_i \neq (1,2)) \\
&= \mu_1(S_{e_1\cdots e_i}(K_2))\frac{\mu_1(S_{1,2}S_{e_{i+1}\cdots e_n}(K_j))}{1-p_c} \\
&= p_{e_1\cdots e_i}\frac{p_{1,2}p_{e_{i+1}\cdots e_n}}{1-p_c} \\
&= p_{\mathbf{e}} = \mu_1(S_{\mathbf{e}}(K_j)).
\end{aligned}$$

By induction, we see that $\mu_1(S_{\mathbf{e}}(K_{\kappa(\mathbf{e})})) = \mu(S_{\mathbf{e}}(K_{\kappa(\mathbf{e})}))$ for each $\mathbf{e} \in E_1^*$. The proof is complete. □

Let $P_n = (p_{ij;n})$ be the extended n-th quasi-truncation of P. Then P_n converges to P in the ℓ^∞ norm and P_n is another transition probability matrix of the GIFS (V, E, M).

Next, we establish some properties of the matrices P_n and P. Denote

$$\sigma_1 := \frac{p_c}{p_b}, \qquad \sigma_2 := \frac{p_a + p_b - p_c}{p_a}.$$

Lemma 4.9. *Let P and P_n be defined as above. Then the following hold.*

(a) *The sequence $\{p_{n,1} = p_c/v_n\}$ is decreasing for each $p_c > 0$, and*

$$\lim_{n\to\infty} p_{n1} = \begin{cases} 0, & \sigma_1 \geq 1, \\ \dfrac{p_c(p_b - p_c)}{p_a + p_b - p_c}, & \sigma_1 < 1. \end{cases}$$

(b) *For each $n \geq 1$,*

$$p_{n,n+1} = \begin{cases} \dfrac{p_b(\sigma_2 - \sigma_1^n)}{\sigma_2 - \sigma_1^{n-1}}, & \sigma_1 \neq 1, \\ \dfrac{p_b(p_b + np_1)}{p_b + (n-1)p_a}, & \sigma_1 = 1. \end{cases}$$

The sequence $\{p_{n,n+1}\}_{n\geq 1}$ is increasing when $p_c > 1/2$, decreasing when $p_c < 1/2$, and constant when $p_c = 1/2$. Moreover,

$$\lim_{n\to\infty} p_{n,n+1} = \max\{p_b, p_c\}.$$

(c) *The sequence $\{p_{n,2}\}_{n\geq 2}$ is increasing and*

$$\lim_{n\to\infty} p_{n,2} = \begin{cases} 1 - p_c, & \sigma_1 \geq 1, \\ \dfrac{p_a(p_a + p_b)}{2(p_a + p_b) - 1}, & \sigma_1 < 1. \end{cases}$$

(d) $\lim_{n\to\infty} P_n = P$ *in ℓ^∞ norm.*

Proof. Before giving the proof, we observe that v_n can be written as

$$v_n = \begin{cases} 1 + \dfrac{p_a(1-\sigma_1^{n-1})}{p_b - p_c}, & \sigma_1 \neq 1, \\ 1 + (n-1)\dfrac{p_a}{p_b}, & \sigma_1 = 1. \end{cases} \tag{4.21}$$

(a) Since the sequence $\{v_n\}$ is increasing as n increases, $\{p_{n,1} = p_c/v_n\}$ is decreasing. The limit of $p_{n,1}$ follows directly from (4.21).

(b) First, suppose $n \geq 2$. When $\sigma_1 \neq 1$, the definitions of σ_1 and σ_2 imply that

$$p_{n,n+1} = \frac{p_b v_{n+1}}{v_n} = p_b \cdot \frac{\sigma_2 - \sigma_1^n}{\sigma_2 - \sigma_1^{n-1}}.$$

When $\sigma_1 = 1$,

$$p_{n,n+1} = p_b \cdot \frac{p_b + np_1}{p_b + (n-1)p_a}.$$

Notice that when $n = 1$, $p_{1,2} = p_a + p_b$, i.e., the expression for $p_{n,n+1}$ also holds for $n = 1$. So the formula of $p_{n,n+1}$ holds for each $n \geq 1$.

It is clear that $\{p_{n,n+1}\}$ is increasing to p_b when $\sigma_1 = 1$. Now we show the other cases. Notice that if $\sigma_1 \neq 1$ then

$$\frac{p_{n+1,n+2}}{p_{n,n+1}} = \frac{(\sigma_2 - \sigma_1^{n-1})(\sigma_2 - \sigma_1^{n+1})}{(\sigma_2 - \sigma_1^n)^2} = \frac{\sigma_2^2 + \sigma_1^{2n} - \sigma_2\sigma_1^{n-1} - \sigma_2 a^{n+1}}{\sigma_2^2 + \sigma_1^{2n} - 2\sigma_2\sigma_1^n}.$$

The difference of the numerator and the denominator is

$$\sigma_2\sigma_1^{n-1}(2\sigma_1 - \sigma_1^2 - 1) = -\sigma_2\sigma_1^{n-1}(\sigma_1 - 1)^2,$$

which is positive if $\sigma_2 < 0$ and negative if $\sigma_2 > 0$. Since $p_c > 1/2$, $p_c = 1/2$ and $p_c < 1/2$ are equivalent to $\sigma_2 < 0$, $\sigma_2 = 0$ and $\sigma_2 > 0$ respectively, $p_{n,n+1}$ is decreasing if $p_c < 1/2$, a constant if $p_c = 1/2$, and increasing if $p_c > 1/2$.

The conclusion of the limit of $\{p_{n,n+1}\}$ is obvious in view of the formula for $p_{n,1}$.

(c) Notice that $p_{n,2} = 1 - p_{n,1} - p_{n,n+1}$ and is thus increasing when $p_c \leq 1/2$. To show the monotonicity of $\{p_{n,2}\}$ for $p_c > 1/2$, we consider $t_n = p_{n,1} + p_{n,n+1}$. Now $\sigma_1 = p_c/p_b \geq p_c/(1-p_c) > 1$. Thus,

$$\frac{t_n}{t_{n+1}} = \frac{\dfrac{p_c}{1 + \frac{p_a(1-\sigma_1^{n-1})}{p_b - p_c}} + \dfrac{p_b(\sigma_2 - \sigma_1^n)}{\sigma_2 - \sigma_1^{n-1}}}{\dfrac{p_c}{1 + \frac{p_a(1-\sigma_1^n)}{p_b - p_c}} + \dfrac{p_b(\sigma_2 - \sigma_1^{n+1})}{\sigma_2 - \sigma_1^n}} = \frac{p_c(\sigma_1^n - \sigma_2)(p_c^2 - p_b(p_a + p_b - p_a\sigma_1^n))}{(p_b\sigma_1^n - p_c\sigma_2)(p_c^2 - p_b(p_a + p_b) + p_a p_c \sigma_1^n)}.$$

Since $p_b \neq p_c$,

$$t_n - t_{n+1} = \sigma_1^n p_a (p_a + p_b)(p_b - p_c)^2 > 0$$

and hence $p_{n,2}$ is increasing.

Finally, the formula for the limit of $\{p_{n,2}\}$ follows from those for $\{p_{n,1}\}$ and $\{p_{n,n+1}\}$.

(d) Notice that the first n rows of P_n and P are the same. Hence the structure of E implies

$$\|P_n - P\| = \sup_{i>n} \big(|p_{i,1} - p_{n,1}| + |p_{i,2} - p_{n,2}| + |p_{i,i+1} - p_{n,n+1}| \big). \tag{4.22}$$

Parts (a)–(c) imply that the sequences $\{p_{n,1}\}$, $\{p_{n,2}\}$, and $\{p_{n,n+1}\}$ converge. Thus the right side of (4.22) converges to zero as $n \to \infty$, completing the proof. □

5. Convergence

Suppose (V, E, M, P) and (V, E, M, P_n) are given as in Subsection 4.2, and $B(q, \hat{\tau}, P)$, $B_n(q, \hat{\tau}) := B(q, \hat{\tau}, P_n)$ are defined by (1.4). $B_n(q, \hat{\tau})$ is the extended n-th quasi-truncation of $B(q, \hat{\tau}, P)$. The matrices P and $B(q, \hat{\tau}, P)$ are, respectively,

$$\begin{pmatrix} p_{11} & p_{12} & 0 & \cdots & & \\ p_{21} & p_{22} & p_{23} & 0 & \cdots & \\ p_{31} & p_{32} & 0 & p_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad \begin{pmatrix} p_{11}^q r_2^{-\hat{\tau}} & p_{12}^q r_2^{-\hat{\tau}} & 0 & \cdots & & \\ p_{21}^q r_2^{-\hat{\tau}} & p_{22}^q r_1^{-\hat{\tau}} & p_{23}^q r_2^{-\hat{\tau}} & 0 & \cdots & \\ p_{31}^q r_2^{-\hat{\tau}} & p_{32}^q r_1^{-\hat{\tau}} & 0 & p_{34}^q r_2^{-\hat{\tau}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \tag{5.1}$$

Note that $\bar{B}_n(q, \hat{\tau}, P) = \bar{B}_n(q, \hat{\tau}, P_n)$ and hence we can denote the common value unambiguously by $\bar{B}_n(q, \hat{\tau})$. For each q, we denote by $\hat{\tau}_n(q)$ the unique number $\hat{\tau}$ such that $\rho(B_n(q, \hat{\tau})) = 1$. Since $\bar{B}_n(q, \hat{\tau})$ is an $n \times n$ primitive matrix, $\hat{\tau}_n(q)$ is a well-defined differentiable function of the variable q. Moreover, $\rho(B_n(q, \hat{\tau}_n(q))) = 1$ by [3, Proposition 3.2] and Lemma 4.4. Denote

$$p := \max\{p_b, p_c\}. \tag{5.2}$$

Lemma 5.1. *For any fixed $q \geq 0$, the sequence $\{\hat{\tau}_n(q)\}_n$ is bounded. If $p_2 > p_3$, the same result holds for any fixed $q \in \mathbb{R}$.*

Proof. If $q = 0$, all $B_n(0, \hat{\tau})$'s ($n \geq 3$) are the same and $\rho(B_3(0, \hat{\tau})) = \rho(\bar{B}_3(0, \hat{\tau}))$ by (4.2) in Lemma 4.4. If $q = 1$, the fact $\rho(P_n) = 1 = \rho(B_n(1, \hat{\tau}_n(1)))$ implies that $B_n(1, \hat{\tau}_n(1)) = P_n$ and $\hat{\tau}_n(1) = 0$. Hence $\{\hat{\tau}_n(0)\}$ and $\{\hat{\tau}_n(1)\}$ are bounded. Next, we divide the proof into several cases. Fix $q \in \mathbb{R} \setminus \{0, 1\}$.

Case 1. $q > 1$. In this case we know $B_n(q,0) \le P_n$ and hence $\hat\tau_n(q) > 0$. Denote

$$\widetilde\tau := \frac{q \ln p_{22} - \ln 2}{\ln r_2}.$$

The $(2,2)$ entry of $B_n(q,\widetilde\tau)$ is 2, which implies $\rho(B_n(q,\widetilde\tau)) > 1$. Thus the conclusion $\hat\tau_n(q) \le \widetilde\tau$ follows from Proposition 3.1.

Case 2. $0 < q < 1$. By Proposition 3.1 again, the inequality $\rho(B_n(q,0)) \ge \rho(B_n(1,0)) = 1$ implies $\hat\tau_n(q) \le 0$. Denote

$$\widetilde\tau := \frac{\ln 4}{\ln r_{\inf}}.$$

So for each $e \in E$, $p_n(e)^q r(e)^{-\widetilde\tau} < r_{\inf}^{-\widetilde\tau} = 1/4$. Therefore, as $\deg(V) = 3$ we have

$$\sup_u \sum_{e\in E_u} p_n(e)^q r(e)^{-\widetilde\tau} < 1,$$

which implies that $\rho(B_n(q,\widetilde\tau)) \le 1$. Hence $\widetilde\tau \le \hat\tau_n(q) \le 0$.

Case 3. $q < 0$. As in Case 2, we get $\hat\tau_n(q) \le 0$. By Lemma 4.9, the assumption $p_2 > p_3$ implies

$$p_{\inf} := \inf\big\{p(e) : e \in E\big\} > 0.$$

Since $P_n \to P$ in $\mathcal{B}(\ell^\infty,\ell^\infty)$ as $n \to \infty$, for large n, we get $p_n(e) > p_{\inf}/2$ for any edge $e \in E$. Let

$$\widetilde\tau := \frac{(q-1)\log(p_{\inf}/2)}{\log r_{\sup}}.$$

Then for all $e \in E$,

$$\begin{aligned} p_n(e)^q r(e)^{-\widetilde\tau} &= p_n(e)^q \exp\bigg(-\frac{(q-1)\log(p_{\inf}/2)}{\log r_{\sup}} \log r(e)\bigg) \\ &< p_n(e)^q \exp\big((1-q)\log p_n(e)\big) \\ &= p_n(e)^q p_n(e)^{1-q} = p_n(e). \end{aligned}$$

So $\rho(B_n(q,\widetilde\tau)) \le \rho(P_n) = 1$. Proposition 3.1 implies $\hat\tau_n(q) \ge \widetilde\tau$, completing the proof. □

In the following lemmas, we let $\hat\tau_\infty(q)$ be a limit point of $\{\hat\tau_n(q)\}$ at q, where $q \ne 0,1$. Let $\{\hat\tau_{n_k}(q)\}$ be a subsequence converging to $\hat\tau_\infty(q)$. Denote $B_{n_k} := B_{n_k}(q,\hat\tau_{n_k}(q),P)$ and $B_\infty := B(q,\hat\tau_\infty(q),P) =: (b_{ij})$. Clearly, B_∞ belongs to $\mathcal{B}(\ell^\infty,\ell^\infty)$ if and only if $q \ge 0$ or $p_2 > p_3$.

We will study the following limits:

$$a_1 := \lim_{i\to\infty} b_{i1}, \qquad a_2 := \lim_{i\to\infty} b_{i2}, \qquad a_3 := \lim_{i\to\infty} b_{i,i+1}. \tag{5.3}$$

Lemma 5.2. *Assume $p_2 > p_3$ or $q \ge 0$. Let p be defined as in* (5.2) *and a_1, a_2, a_3 be defined as in* (5.3)*. The following hold.*

(a) *$\{B_{n_k}\}$ converges to B_∞ in $\mathcal{B}(\ell^\infty, \ell^\infty)$;*
(b) *the limits defining a_1, a_2, a_3 exist;*
(c) *$b_{11} < 1$ and $b_{22} < 1$;*
(d) *$a_3 = p^q r_2^{-\hat{\tau}_\infty(q)} < 1$.*

Proof. (a) Recall that $\deg(V) = 3$ and $p_{ij;n} = p_{nj;n}$ if $i \ge n$. Note that

$$\|B_{n_k} - B_\infty\| \le \|B_{n_k} - B(q, \hat{\tau}_{n_k}(q), P)\| + \|B(q, \hat{\tau}_{n_k}(q), P) - B_\infty\|. \tag{5.4}$$

The second term on the right side is equal to

$$\sup_u \sum_v \sum_{e \in E_{uv}} p(e)^q \big| r(e)^{-\hat{\tau}_{n_k}(q)} - r(e)^{-\hat{\tau}_\infty(q)} \big|$$

By the mean-value theorem,

$$\sup_{e \in E} p(e)^q \big| r(e)^{-\hat{\tau}_{n_k}(q)} - r(e)^{-\hat{\tau}_\infty(q)} \big| \le C_q \big| \hat{\tau}_{n_k}(q) - \hat{\tau}_\infty(q) \big|, \tag{5.5}$$

where

$$C_q := \begin{cases} \sup_{e \in E} \{ p(e)^q |\ln r_{\inf}| \}, & \text{if } q < 1, \\ r_{\inf}^{-2\hat{\tau}_\infty(q)} |\ln r_{\inf}|, & \text{if } q > 1, \end{cases}$$

which is finite for each q. It follows that

$$\|B(q, \hat{\tau}_{n_k}(q), P) - B_\infty\| \le 3C_q |\hat{\tau}_{n_k}(q) - \hat{\tau}_\infty(q)|. \tag{5.6}$$

Now we estimate the first term on the right side of (5.4), which is equal to

$$\sup_u \sum_v \sum_{e \in E_{uv}} r(e)^{-\hat{\tau}_{n_k}(q)} \big| p_{n_k}(e)^q - p(e)^q \big| < M_q \sup_u \sum_v \sum_{e \in E_{uv}} \big| p_{n_k}(e)^q - p(e)^q \big|,$$

where, by the convergence of $\{\hat{\tau}_{n_k}(q)\}$,

$$M_q := \sup \big\{ r(e)^{-\hat{\tau}_{n_k}(q)} :\ e \in E, n_k \ge 1 \big\} < \infty.$$

We first consider the case $q \le 0$ or $q > 1$. By letting

$$c_q := \begin{cases} |q| (p_{\inf}/2)^{q-1} & \text{if } q \le 0, \\ q & \text{if } q > 1 \end{cases}$$

and applying the mean-value theorem to $p_{n_k}(e)^q - p(e)^q$, we see that

$$\big| p_{n_k}(e)^q - p(e)^q \big| \le c_q \big| p_{n_k}(e) - p(e) \big| \quad \text{for all } e \in E.$$

It follows that

$$\|B_{n_k} - B(q, \hat{\tau}_{n_k}(q), P)\| \le M_q c_q \|P_{n_k} - P\|. \tag{5.7}$$

Next, we consider the remaining case $0 < q < 1$. Fix $\varepsilon > 0$ and choose k_0 large enough such that $\|P_{n_k} - P\| < \varepsilon$ for all $k > k_0$. Then for such k,

$$\left|p_{n_k}(e)^q - p(e)^q\right| \le \begin{cases} (3\varepsilon)^q, & \text{if } p(e) < 2\varepsilon, \\ q\varepsilon^{q-1}|p_{n_k}(e) - p(e)| < \varepsilon^q, & \text{if } p(e) \ge 2\varepsilon. \end{cases}$$

(Note that $q - 1 < 0$, and hence $\max\{p(e)^{q-1}, p_{n_k}(e)^{q-1}\} < \varepsilon^{q-1}$ for $p(e) \ge 2\varepsilon$.) Thus $|p_{n_k}(e)^q - p(e)^q| < 3^q$. The fact $\deg(V) = 3$ implies that

$$\|B_{n_k} - B(q, \hat{\tau}_{n_k}(q), P)\| \le 9M_q\varepsilon^q. \tag{5.8}$$

Then, combining (5.6), (5.7) and (5.8), we obtain

$$\|B_{n_k} - B_\infty\| \to 0 \quad \text{as } k \to \infty.$$

(b) The existence of the limits defining a_1, a_2, a_3 is guaranteed by the monotonicity and boundedness of $\{p_{i1}\}_{i\ge1}$, $\{p_{i2}\}_{i\ge2}$ and $\{p_{i,i+1}\}$ (Lemma 4.9).

(c) First, we see that $c_0 := b_{12}b_{21} = (p_{12}p_{21})^q r_2^{-2\hat{\tau}_\infty(q)} > 0$. For $0 < \varepsilon < c_0/4$, by the convergence of B_{n_k}, we choose k_0 sufficiently large so that $k > k_0$ implies $b_{ij;n_k} > b_{ij} - \varepsilon$ and $b_{12;n_k}b_{21;n_k} > c_0^2/4$. Let $\bar{B}_{n_k,2}$ be the positive matrix $(b_{ij;n_k})_{1\le i,j\le 2}$. The spectral radius of $\bar{B}_{n_k,2}$ is

$$\rho(\bar{B}_{n_k,2}) = \frac{b_{11;n_k} + b_{22;n_k} + \sqrt{(b_{11;n_k} - b_{22;n_k})^2 + 4b_{12;n_k}b_{21;n_k}}}{2}, \tag{5.9}$$

which is no more than $\rho(B_{n_k}) = 1$. This implies $b_{11;n_k} < 1$ or $b_{22;n_k} < 1$, which in turn implies $b_{11} \le 1$ or $b_{22} \le 1$. If $b_{11} = 1$ and $b_{22} = 1$, then for ε and k_0 given as above,

$$\rho(B_{n_k,2}) > \frac{1 - \varepsilon + 1 - \varepsilon + c_0}{2} > 1,$$

a contradiction. Thus $b_{11} < 1$ or $b_{22} < 1$. Without loss of generality, we assume $b_{22} < 1$. Then (5.9) implies

$$b_{11;n_k} \le 1 - (1 - b_{22;n_k})^{-1} b_{12;n_k} b_{21;n_k}.$$

By letting $k \to \infty$, we see that $b_{11} \le 1 - (1 - b_{22})^{-1} b_{12} b_{21} < 1$.

(d) Since $b_{i,i+1} = p_{i,i+1}^q r_2^{-\hat{\tau}_\infty(q)}$ and the limit of $p_{i,i+1}$ is p (Lemma 4.9), we obtain $a_3 = p^q r_2^{-\hat{\tau}_\infty(q)}$, where p is defined as in (5.2). If $\{p_{n,n+1}\}_{n\ge2}$ is increasing, then by Lemma 4.9, $p_3 > p_2$. So $a_3 = p_3^q r_2^{-\hat{\tau}_\infty(q)} = b_{11} < 1$. In the remaining case when $\{p_{n,n+1}\}_{n\ge2}$ is not increasing, we replace each entry lying in the first row and the first column of B_{n_k} by zero and denote the resulting matrix by $B_{n_k}^{0,0}$. Clearly $B_{n_k} \ge B_{n_k}^{0,0}$, and thus $\rho(B_{n_k}) \ge \rho(B_{n_k}^{0,0})$. Assume ε and k_0 are given as in the proof of (c). Since for $k > k_0$, $B_{n_k}^{0,0}$ satisfies the condition of Lemma 4.3 with respect to

$$s := \inf_{k\geq 2}\{b_{n_k,2;n_k}\} = b_{22;n_k} > b_{22} - \varepsilon \quad \text{and} \quad t := \inf_{k\geq 2}\{b_{n_k,n_k+1;n_k}\} > a_3 - \varepsilon,$$

we get $a_3 + b_{22} \leq 1 + 2\varepsilon$. As $b_{22} > 0$, it follows that $a_3 < 1$. □

Lemma 5.3. *Suppose $q > 0$ or $p_2 > p_3$. Let $\mathbf{x}^t_{n_k}$ and $\mathbf{y}_{n_k}$ be 1-invariant measure and 1-invariant vector of B_{n_k} respectively, with the first component of $\mathbf{y}_{n_k}$ being 1. Then $\{\mathbf{x}^t_{n_k}\}$ and $\{\mathbf{y}_{n_k}\}$ converge in ℓ^∞.*

Proof. Recall that $\bar{B}_{n_k}$ is the n_k-th quasi-truncation of B_{n_k} and B_{n_k} is the quasi-extension of $\bar{B}_{n_k}$. We divide the proof into five steps.

Step 1. Existence of $\mathbf{x}_{n_k}$ and $\mathbf{y}_{n_k}$. By the structure of B_{n_k}, we know that $\bar{B}_{n_k}$ is primitive and $b_{n_k n_k;n_k} \in (0,1)$. Theorem 4.4 says that $\rho(B_{n_k}) = \lambda(B_{n_k}) = \lambda(\bar{B}_{n_k}) = 1$. Let $\bar{\mathbf{x}}^t_{n_k}$ and $\bar{\mathbf{y}}_{n_k}$ be 1-invariant measure and 1-invariant vector of $\bar{B}_{n_k}$, respectively. Let $\mathbf{x}^t_{n_k}$ be the row extension of $\bar{\mathbf{x}}^t_{n_k}$ with respect to $b_{n_k n_k;n_k}$ and $\mathbf{y}_{n_k}$ be the column extension of $\bar{\mathbf{y}}_{n_k}$. By Theorem 4.4 and its proof, $\mathbf{x}^t_{n_k}$ and $\mathbf{y}_{n_k}$ are 1-invariant measure and 1-invariant vector of B_{n_k}, respectively. This establishes the existence.

Step 2. Convergence of $\{x_{i;n_k}\}$, the i-th component of $\mathbf{x}_{n_k}$. Before proving this, we see that for any $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that for each i and $k > N$ with $i \geq n_k$, we have

$$\begin{cases} (1-\varepsilon)a_1 \leq b_{i1;n_k} \leq \max\{\varepsilon, (1+\varepsilon)a_1\}, \\ (1-\varepsilon)a_2 \leq b_{i2;n_k} \leq (1+\varepsilon)a_2, \\ (1-\varepsilon)a_3 \leq b_{i,i+1;n_k} \leq (1+\varepsilon)a_3, \end{cases} \tag{5.10}$$

where a_1, a_2, a_3 are defined in (5.3). Equation (5.10) holds because $\{B_{n_k}\}$ and the three sequences $\{b_{n_k,1}\}_{k\geq 1}$, $\{b_{n_k,2}\}_{k\geq 2}$, $\{b_{n_k,n_k+1}\}_{k\geq 1}$ converge.

For convenience, we consider a multiple of the 1-invariant measure

$$\mathbf{z}^t_{n_k} = (z_{1;n_k}, z_{2;n_k}, \dots) = \frac{1}{x_{2;n_k}}\mathbf{x}^t_{n_k}$$

instead of the invariant measure itself, where we have used the fact that $\bar{B}_{n_k}$ is irreducible and thus $\mathbf{x}^t_{n_k}$ is strictly positive (Remark 4.5). Then, by the convergence of B_{n_k}, for $i \geq 3$,

$$z_{i;n_k} = z_{i-1;n_k} b_{i-1,i;n_k} = \prod_{j=2}^{i-1} b_{j,j+1;n_k} \quad \to \quad \prod_{j=2}^{i-1} b_{j,j+1} =: z_i \quad (k \to \infty). \tag{5.11}$$

Since $a_3 < 1$, the series $\sum_{i\geq 2} z_i$ converges and we let c_1 denote the limit. Let

$$0 < \varepsilon < \min\left\{(1-b_{11})/4, (1-a_3)/2\right\}. \tag{5.12}$$

Then there exists $N_1(> N)$ sufficiently large so that

$$\prod_{j=2}^{N_1} b_{j,j+1} < (1+\varepsilon)^{-1}\varepsilon \tag{5.13}$$

and (5.10) holds. We also take N_2 sufficiently large so that $n_k > N_2$ implies

$$(1-\varepsilon)^{1/N_1} b_{j,j+1} \le b_{j,j+1;n_k} \le (1+\varepsilon)^{1/N_1} b_{j,j+1}, \quad j \ge 1, \tag{5.14}$$

and thus by (5.10), (5.12), and (5.14)

$$\begin{aligned} z_{m+1;n_k} &= \prod_{j=2}^{m} b_{j,j+1;n_k} < \prod_{j=2}^{N_1} b_{j,j+1} \prod_{j=N_1+1}^{m} ((1+\varepsilon)a_3), \\ &\le \varepsilon\big((1+\varepsilon)a_3\big)^{m-N_1} \le c\big((1+\varepsilon)a_3\big)^{m}, \qquad m \ge N_1, \end{aligned} \tag{5.15}$$

where c is some positive constant.

Denote

$$\varepsilon' := \frac{2\varepsilon}{1-b_{11}}, \qquad a(\varepsilon) := \frac{1+\varepsilon}{1-\varepsilon'}, \qquad b(\varepsilon) := \frac{1-\varepsilon}{1+\varepsilon'}.$$

Then $a(\varepsilon) < 4$ by the choice of ε. Since $\mathbf{z}_{n_k}^t B_{n_k} = \mathbf{z}_{n_k}^t$,

$$\frac{\sum_{j\ge 2} b_{j1} z_{j;n_k}}{1-b_{11}} b(\varepsilon) \le z_{1;n_k} = \frac{\sum_{j\ge 2} b_{j1;n_k} z_{j;n_k}}{1-b_{11;n_k}} \le \frac{\sum_{j\ge 2} b_{j1} z_{j;n_k}}{1-b_{11}} a(\varepsilon).$$

Let $M := \sup_{j\ge 2} b_{j1} < \infty$. The for all $n_k, n_l > N_2$,

$$\begin{aligned} \frac{(1-b_{11})|z_{1;n_k} - z_{1;n_l}|}{M} &\le \frac{1}{M}\Big|\sum_{j\ge 2} b_{j1} z_{j;n_k} a(\varepsilon) - \sum_{j\ge 2} b_{j1} z_{j;n_l} b(\varepsilon)\Big| \\ &\le \Big(\sum_{2\le j\le N_1} + \sum_{j>N_1}\Big)\Big|z_{j;n_k} a(\varepsilon) - z_{j;n_l} b(\varepsilon)\Big| \\ &\le \big(a(\varepsilon)(1+\varepsilon) - b(\varepsilon)(1-\varepsilon)\big) \sum_{2\le j\le N_1} z_j + a(\varepsilon) \sum_{j>N_1} (z_{j;n_k} + z_{j;n_l}) \\ &\le \frac{16 c_1 \varepsilon}{1-b_{11}} + 8\varepsilon \sum_{j>N_1} \prod_{k=N_1+1}^{j-1} (1+\varepsilon)a_3 \\ &\le \frac{16 c_1 \varepsilon}{1-b_{11}} + 8\varepsilon \cdot \frac{1}{1-(1+\varepsilon)a_3} \\ &\le \Big(\frac{16 c_1}{1-b_{11}} + \frac{16}{1-a_3}\Big)\varepsilon. \end{aligned}$$

The fourth inequality holds by using the inequality $a(\varepsilon) < 4$ and equation (5.15) for $m = N_1$. This implies the convergence of $\{z_{1;n_k}\}_{k\ge 1}$. In view of (5.11) and the fact that $z_{2;n_k} = 1$, we obtain

$$\lim_{k\to\infty} z_{i;n_k} = z_i \quad \text{for all } i. \tag{5.16}$$

Let ε and N_1 be given as above. Choose $N_3 > N_1$ sufficiently large so that $\sum_{i>N_3} z_i < \varepsilon$. Take $N_4 > N_3$ such that $|z_{i;n_k} - z_i| < N_3^{-1}\varepsilon$ for $n_k > N_4$. Now (5.15) yields

$$\begin{aligned}\Big|\sum_{i\geq 1} z_{i;n_k} - \sum_{i\geq 1} z_i\Big| &\leq \sum_{1\leq i\leq N_3} |z_{i;n_k} - z_i| + \sum_{i>N_3} (z_{i;n_k} + z_i) \\ &\leq \varepsilon + \varepsilon + \frac{\varepsilon}{1-(1+\varepsilon)a_3}.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty} \sum_{i\geq 1} z_{i;n_k} = \sum_{i\geq 1} z_i. \tag{5.17}$$

So the sequence $\{\sum_{i\geq 1} z_{i;n_k}\}_k$ converges to $z_1 + c_1 \in (0,\infty)$, and consequently, $x_{2;n_k} = (\sum_{i\geq 1} z_{i;n_k})^{-1}$ converges to $(z_1 + c_1)^{-1} > 0$. It follows that $\{x_{i;n_k} = z_{i;n_k} x_{2;n_k}\}_{k\geq 1}$ converges for each i.

Step 3. Convergence of $\{\mathbf{x}^t_{n_k}\}$. Denote

$$\mathbf{x}^t = (\textstyle\sum_{j\geq 1} z_j)^{-1}(z_1, z_2, \ldots).$$

Then

$$\begin{aligned}\|\mathbf{x}^t_{n_k} - \mathbf{x}^t\| &= \sup_{i\geq 1}\Big|\frac{z_{i;n_k}}{\sum_j z_{j;n_k}} - \frac{z_i}{\sum_j z_j}\Big| \\ &\leq \sup_{i\geq 1}\Big(\Big|\frac{z_{i;n_k}}{\sum_j z_{j;n_k}} - \frac{z_{i;n_k}}{\sum_j z_j}\Big| + \Big|\frac{z_{i;n_k}}{\sum_j z_j} - \frac{z_i}{\sum_j z_j}\Big|\Big),\end{aligned}$$

which, in view of (5.15), (5.16), and (5.17), tends to zero as $k \to \infty$. This implies the convergence of $\{\mathbf{x}_{n_k}\}_{k\geq 1}$.

Step 4. Convergence of each sequence $\{y_{i;n_k}\}_{k\geq 1}$*, the* i*-th component of* $\mathbf{y}_{n_k}$. From the construction of B_{n_k}, we know $y_{i;n_k} = y_{n_k;n_k}$ for any $i > n_k$. For $i = 2$,

$$y_{2;n_k} = \frac{1 - b_{11;n_k}}{b_{12;n_k}} = \frac{1 - p_3^q r_2^{-\hat{\tau}_{n_k}(q)}}{(1-p_3)^q r_2^{-\hat{\tau}_{n_k}(q)}} \quad \to \quad \frac{1 - p_3^q r_2^{-\hat{\tau}_\infty(q)}}{(1-p_3)^q r_2^{-\hat{\tau}_\infty(q)}} \quad \text{as } k \to \infty.$$

Assuming $\{y_{i;n_k}\}_{k\geq 1}$ converges for some $i \geq 2$, we now consider the case $i+1$. Equating the i-th components on both sides of the equation $B_{n_k}\mathbf{y}_{n_k} = \mathbf{y}_{n_k}$, we get

$$y_{i+1;n_k} = \frac{y_{i;n_k} - b_{i+1,1;n_k} - b_{i+1,2;n_k} y_{2;n_k}}{b_{i,i+1;n_k}}.$$

Each term on the right side of the equation is convergent, and therefore the left side converges. By induction, each sequence $\{y_{i;n_k}\}_{k\geq 1}$ converges, and we denoted the limit by y_i.

Step 5. Convergence of $\{\mathbf{y}_{n_k}\}$. Choose $\varepsilon > 0$ small enough such that

$$\varepsilon < \min\big\{a_3^{-1/2} - 1, (2a_3)^{-1} - 1\big\}$$

and thus $(1+\varepsilon)^2 a_3 < 1$. Let $N = N(\varepsilon)$ be sufficiently large so that (5.10) holds and $(1-\varepsilon)y_2 < y_{2,n_k} < (1+\varepsilon)y_2$ for any $k > N$. Denote

$$\xi := \max\{\varepsilon, (1+\varepsilon)a_1\} + (1+\varepsilon)^2 a_2 y_2 \quad \text{and} \quad \eta := (1+\varepsilon)a_3.$$

Then $\eta < (1+\varepsilon)^2 a_3 < 1$. Fix any pair (i, n_k) with $i, k > N$. Since $\mathbf{y}_{n_k}$ is a 1-invariant vector of B_{n_k},

$$\begin{aligned} y_{i;n_k} &= b_{i1;n_k} + b_{i2;n_k} y_{2;n_k} + b_{i,i+1;n_k} y_{i+1;n_k} \\ &\leq \max\{\varepsilon, (1+\varepsilon)a_1\} + (1+\varepsilon)^2 a_2 y_2 + (1+\varepsilon) a_3 y_{n_k+1;n_k} \quad \text{(by (5.10))} \\ &< \xi + \eta y_{i+1;n_k} < \xi + \xi\eta + \eta^2 y_{i+2;n_k} < \cdots \\ &< \frac{\xi}{1-\eta} + \eta^m y_{i+m;n_k}. \end{aligned}$$

This implies that $y_{i;n_k} < \xi/(1-\eta)$ since $y_{i+m;n_k}$ is constant for large m and $\eta^m \to 0$ as $m \to \infty$. Similarly, we get $y_{i;n_k} > \xi'/(1-\eta')$, where

$$\xi' = (1-\varepsilon)a_1 + (1-\varepsilon)^2 a_2 y_2 \qquad \text{and} \qquad \eta' = (1-\varepsilon)a_3.$$

Now in view of Step 4, let $N_5 > N$ be sufficiently large so that $|y_{i;n_k} - y_{i;n_l}| < \varepsilon$ for any $1 \leq i \leq N$ and $k, l > N_5$. It follows that if $k, l > N_5$, then

$$\begin{aligned} \|\mathbf{y}_{n_k} - \mathbf{y}_{n_l}\| &= \sup_{i \geq 1} |y_{i;n_k} - y_{i;n_l}| \\ &= \max\Big\{ \max_{1 \leq i \leq N} \{|y_{i;n_k} - y_{i;n_l}|\}, \sup_{i > N} \{|y_{i;n_k} - y_{i;n_l}|\} \Big\} \\ &\leq \max\Big\{ \varepsilon, \frac{c}{1-d} - \frac{c'}{1-d'} \Big\}. \end{aligned}$$

Let $c_0 := \max\{1, a_1, a_2, y_2\}$. Then the choice of ε yields

$$\frac{c}{1-d} - \frac{c'}{1-d'} \leq \frac{2c_0^2(4 - a_3 + a_3\varepsilon^2)\varepsilon}{(1-a_3)^2 - (a_3\varepsilon)^2} \leq \frac{16 c_0^2 \varepsilon}{(1-a_3)^2}.$$

Hence $\{\mathbf{y}_{n_k}\}_{n \geq 1}$ converges by the completeness of ℓ^∞. □

Lemma 5.4. *Suppose $p_2 > p_3$ or $q \geq 0$. Let $\mathbf{x}^t$, $\mathbf{y}$ be the limits of $\{\mathbf{x}^t_{n_k}\}$, $\{\mathbf{y}_{n_k}\}$ respectively. Then $\mathbf{x}^t$ and $\mathbf{y}$ are respectively the 1-invariant measure and 1-invariant vector of B_∞; moreover, $\mathbf{x}^t\mathbf{y} < \infty$ and $0 < \inf_i y_i \leq \sup_i y_i < \infty$, where y_i is the i-th component of $\mathbf{y}$. In particular, $\rho(B_\infty) = 1$.*

Proof. Note that $B_{n_k}\mathbf{y}_{n_k} = \mathbf{y}_{n_k}$, i.e.,

$$\sum_{j=1}^{\infty} b_{ij;n_k} y_{j;n_k} = y_{i;n_k}, \quad i \geq 1.$$

Since $\deg(V) = 3$, the left sum consists of three terms. By letting $k \to \infty$, we see that

$$\sum_{j=1}^{\infty} b_{ij} y_i = y_j, \quad i \geq 1,$$

i.e., $B_\infty \mathbf{y} = \mathbf{y}$. Clearly, all $y_i \geq 0$, $y_1 = 1 > 0$ and

$$0 < \frac{c'}{1-d'} \leq y_i \leq \frac{c}{1-d} < \infty$$

for i large enough, where c, d, c' and d' are given as in the proof of Lemma 5.3. This and the convergence of each sequence $\{y_{i;n_k}\}_{k\geq 1}$ show that $\sup_i y_i < \infty$. If $\inf_i y_i = 0$, then there would exist some i such that $y_i = 0$. From $B_\infty \mathbf{y} = \mathbf{y}$, we see that

$$y_i = b_{i1} + b_{i2} y_2 + y_{i+1} = 0.$$

This is impossible since, by the definition of B_∞, $b_{i1} = p_{i1}^q r_2^{-\hat{\tau}_\infty(q)} > 0$. So $\inf_i y_i > 0$.

Denote $\mathbf{x}^t = (x_1, x_2, \ldots)$. We have the following estimate

$$\begin{aligned}\Big|\sum_{i\geq 1} b_{ij} x_i - x_j\Big| \quad &\leq \quad \sum_{i\geq 1} x_i |b_{ij} - b_{ij;n_k}| + \sum_{i\geq 1} b_{ij;n_k} |x_i - x_{i;n_k}| \\ &\qquad + \Big|\sum_{i\geq 1} b_{ij;n_k} x_{i;n_k} - x_{j;n_k}\Big| + |x_{j;n_k} - x_j|.\end{aligned}$$

The third term on the right side of the above inequality is zero since $\mathbf{x}_{n_k}^t$ is a 1-invariant measure of B_{n_k}. That B_{n_k} converges to B_∞ and $\sum_{i\geq 1} x_i < \infty$ imply that the first term tends to zero as $k \to \infty$. The convergence of $\{x_{j;n_k}\}$ implies that the fourth term tends to zero as $k \to \infty$. Now we consider the second term. The assumption $p_2 > p_3$ or $q \geq 0$ implies B_∞ has a finite norm. Since B_{n_k} converges to B_∞, $b_{ij;n_k} < 2\|B_\infty\|$ for large k and all i, j. So

$$\sum_{i\geq 1} b_{ij;n_k} |x_i - x_{i;n_k}| \leq 2\|B_\infty\| \sum_{i\geq 1} |x_i - x_{i;n_k}|.$$

An similar argument as in Step 2 of the proof of Lemma 5.3 yields $\sum_{i\geq 1} |x_i - x_{i;n_k}| \to 0$ as $k \to \infty$. This implies $\sum_{i\geq 1} b_{ij} x_i = x_j$ for all j, i.e., $\mathbf{x}^t B_\infty = \mathbf{x}^t$. The conclusion $\sum_i x_i = 1$ follows from the convergence of $\{\sum_{i\geq 1} x_{i;n_k}\}_{k\geq 1}$. If $x_i = 0$ for some $i \geq 2$, then the equality $\mathbf{x}^t B_\infty = \mathbf{x}$ implies that $x_i = 0$ for all $i \geq 2$, and hence $x_1 b_{11} = x_1$. However, by Lemma 5.2(c), the fact $b_{11} < 1$ implies that $x_1 = 0$ as well, which is impossible. Thus all x_i are positive. Therefore, $\mathbf{x}^t$ is a 1-invariant measure of B_∞.

Now,

$$\mathbf{x}^t \mathbf{y} \leq \|\mathbf{y}\| \sum_i x_i = \|\mathbf{y}\| < \infty.$$

So, the last conclusion $\rho(B_\infty) = 1$ follows from Lemma 4.2. □

Theorem 5.5. *Suppose $p_2 > p_3$ or $q \geq 0$. Then $\hat{\tau}_n(q)$ converges.*

Proof. Fix $q \in \mathbb{R}$. From Lemma 5.1, $\hat{\tau}_n(q)$ is bounded. Suppose $\hat{\tau}^{(0)}(q)$ and $\hat{\tau}^{(1)}(q)$ are two distinct limits of $\{\hat{\tau}_n(q)\}$. Then Lemma 5.4 yields $\rho(B_\infty(q, \hat{\tau}^{(0)}(q))) = \rho(B_\infty(q, \hat{\tau}^{(1)}(q))) = 1$. This contradicts the strict monotonicity of $\rho_q(\hat{\tau})$ (Proposition 3.1). So $\hat{\tau}_n(q)$ converges. □

Remark 5.6. *Since $\hat{\tau}_n$ converges, all results in Lemmas 5.2,* (5.3) *and* (5.4) *still hold if we replace n_k by n.*

6. Differentiability of $\hat{\tau}(q)$

Recall that $\hat{\tau}_\infty(q) := \lim_{q\to\infty} \hat{\tau}_n(q)$. For convenience, we denote $\hat{\tau}_\infty(q)$ by $\hat{\tau}(q)$ and B_∞ by B. That is, for each q, $\hat{\tau}(q)$ is the unique number such that $\rho(B) = \rho(B(q, \hat{\tau}(q), P)) = 1$. Our goal in this section is to show the differentiability of $\hat{\tau}(q)$. We do this by showing $\hat{\tau}(q)$ is differentiable on any symmetric interval $[-L, L]$, where L is a fixed positive number. We remark that if q is negative, the condition $p_2 > p_3$ is necessary, as before.

Recall that $B_n(q, \hat{\tau}) = B(q, \hat{\tau}, P_n)$ is the extended n-th quasi-truncation of $B(q, \hat{\tau}) = B(q, \hat{\tau}, P)$ and $\bar{B}_n(q, \hat{\tau}) = (\bar{b}_{ij;n})$ is the n-th quasi-truncation of $B(q, \hat{\tau})$. To avoid confusion, we denote the (i, j) entry of $B(q, \hat{\tau})$, where q and $\hat{\tau}$ are independent, by b_{ij} and that of B by b^0_{ij}. We emphasize that b_{ij} is a function of the variables q and $\hat{\tau}$, while b^0_{ij} is a function of the single variable q.

For any $e = (i, j) \in E$, $b_{ij} = p^q_{ij} r^{-\hat{\tau}}_{ij}$, and r_{ij}, p_{ij} being positive constants. These yield the following facts.

Fact 1. *b_{ij} is continuous on the $(q, \hat{\tau})$ plane, and hence attains its maximum and minimum on any compact set.*

Fact 2. *$\partial b_{ij}/\partial q < 0$, $\partial b_{ij}/\partial \hat{\tau} > 0$ for all q and $\hat{\tau}$, and hence, b_{ij} is monotonically decreasing with respect to q and monotonically increasing with respect to $\hat{\tau}$.*

In order to prove the differentiability of $\hat{\tau}(q)$, we will first establish several lemmas. For any fixed positive number L, let

$$k_{\hat{\tau}} = k_{\hat{\tau}}(L) := \max\left\{\left|\inf_{|q|\leq L} \hat{\tau}(q)\right|, \left|\sup_{|q|\leq L} \hat{\tau}(q)\right|\right\} + 1. \tag{6.1}$$

By the monotonicity of $\hat{\tau}(q)$ and a similar proof as that of Lemma 5.1, we see $k_{\hat{\tau}} < \infty$.

Lemma 6.1. *Assume* $p_2 > p_3$ *or* $q \geq 0$. *Then* $\sup_{|q|\leq L} b_{ii}^0 < 1$ *for* $i = 1, 2$.

Proof. The definition of $k_{\hat{\tau}}$ yields $\inf_{|q|\leq L} b_{ij}^0 \geq p_{ij}^L r_{ij}^{k_{\hat{\tau}}} > 0$ for any $(i,j) \in E$. So $c_1 := \inf_{|q|\leq L} b_{12}^0 b_{21}^0 > 0$. The spectral radius of the positive matrix $\bar{B}_2^0 := (b_{ij}^0)_{1\leq i,j\leq 2}$ satisfies

$$1 \geq \rho(\bar{B}_2^0) \geq \frac{b_{11}^0 + b_{22}^0 + \sqrt{(b_{11}^0 - b_{22}^0)^2 + 4c_1}}{2}, \tag{6.2}$$

because $\rho(\bar{B}_2^0) \leq \rho(B) = 1$. This implies that $b_{11}^0, b_{22}^0 < 1$ and that

$$b_{11}^0 \leq 1 - (1 - b_{22}^0)^{-1} c_1 < 1. \tag{6.3}$$

If $\sup_{|q|\leq L} b_{11}^0 = 1$, there exists a convergent sequence $\{q_i\}$ such that $\lim_{i\to\infty} b_{11}^0(q_i) = 1$. So $\lim_{i\to\infty} b_{22}^0(q_i) = 1$ by (6.3). Now by (6.2), we obtain

$$\varlimsup_{i\to\infty} \rho(B_2^0) \geq \lim_{i\to\infty} \frac{b_{11}^0(q_i) + b_{22}^0(q_i) + \sqrt{c_1}}{2} > 1,$$

which contradicts $\rho(B_2^0) \leq 1$. Thus $\sup_{|q|\leq L} b_{11}^0 < 1$. The conclusion $\sup_{|q|\leq L} b_{22}^0 < 1$ follows from $b_{22}^0 \leq 1 - (1 - b_{11}^0)^{-1} c_1$. □

Recall that $p = \max\{p_b, p_c\}$. Define

$$\underline{a}_3 := \inf_{|q|\leq L} a_3 \qquad \text{and} \qquad \overline{a}_3 := \sup_{|q|\leq L} a_3.$$

Corollary 6.2. *Assume the hypotheses of Lemma 6.1 and let* $a_3 := \lim_{i\to\infty} b_{i,i+1}^0$ *be defined as in* (5.3). *Then* $0 < \underline{a}_3 \leq \overline{a}_3 < 1$.

Proof. That $\underline{a}_3 > 0$ follows from the inequality $a_3 \geq p^L r_2^{k_{\hat{\tau}}}$, where $k_{\hat{\tau}}$ is defined in (6.1). If $\{p_{n,n+1}\}_{n\geq 2}$ is increasing, then $p_3 \geq p_2$ (Lemma 4.9(b)) and hence $a_3 = b_{11}^0$ (Lemma 5.3). The supremum of a_3 is less than 1 by Lemma 6.1. If $\{p_{n,n+1}\}_{n\geq 2}$ is not increasing, then from the proof of Lemma 5.2(d), we get $a_3 + b_{22}^0 \leq 1$. The assertion follows since $\inf_{|q|\leq L} b_{22}^0 > 0$. □

From $\bar{B}_n(q, \hat{\tau})$, we replace $\bar{b}_{nn;n}$ by zero and denote the resulting matrix by $\bar{B}_n^0(q, \hat{\tau})$. Define

$$D_n := \det\left(I_n - \bar{B}_n^0(q, \hat{\tau})\right) \qquad \text{and} \qquad F_n := D_n - b_{n,n+1} D_{n-1},$$

where I_n is the $n \times n$ identity matrix, and let

$$U := \left\{(q, \hat{\tau}) : \frac{q \ln p - \ln(\overline{a}_3/2)}{\ln r_2} \leq \hat{\tau} \leq \frac{q \ln p - \ln((1 + \overline{a}_3)/2)}{\ln r_2},\ |q| \leq L\right\},$$

where $\underline{a}_3, \overline{a}_3$ are given as in Corollary 6.2. Clearly, for each pair $(q, \hat{\tau}) \in U$, we have

$$\frac{\overline{a}_3}{2} < p^q r_2^{-\hat{\tau}} < \frac{1 + \overline{a}_3}{2}. \tag{6.4}$$

It follows from this that the compact region U contains the curve $\{(q, \hat{\tau}(q)) : |q| \leq L\}$.

Lemma 6.3. *Assume the hypotheses of Lemma 6.1. Then there exists a constant γ (independent of n) such that $|\partial D_n/\partial\hat{\tau}| < \gamma$ and $|D_n| < \gamma$ for all $(q, \hat{\tau}) \in U$.*

Proof. By using definition and induction, we have

$$D_n = (b_{11} - 1)\sum_{i=1}^{n} s_{i2} - b_{12}\sum_{i=1}^{n} s_{i1},$$

where $s_{11} = 0$, $s_{12} = -1$, and

$$s_{ij} := b_{i,j}\prod_{k=2}^{i-1} b_{k,k+1} = \Big(p_{i,j}\prod_{k=2}^{i-1} p_{k,k+1}\Big)^{q} \cdot r_2^{-(i-1)\hat{\tau}}\Big(\frac{r_j}{r_2}\Big)^{-\hat{\tau}}, \quad i \geq 2,\ j = 1, 2. \tag{6.5}$$

Notice that b_{11}, b_{21} are continuous on the compact region U. We claim that the series with non-negative terms $\sum_{i\geq 2} s_{ij}$ converges uniformly on U for $j = 1, 2$. If so, we can find a constant γ_1 such that $|D_n| < \gamma_1$ for all $(q, \hat{\tau}) \in U$ and all $n \geq 3$.

To show the claim, we first fix $j \in \{1, 2\}$. From Lemma 4.9, the three sequences $\{p_{n,n+1}\}$, $\{p_{n,1}\}$ and $\{p_{n,2}\}$ are convergent. For any $\varepsilon > 0$ satisfying $(1+\varepsilon)^{2L}(1+\overline{a}_3)/2 < 1$, there exists $N > 0$ such that $n > N$ implies $p_{n+1,j}/p_{n,j} < 1+\varepsilon$ and $p_{n,n+1}/p \leq 1+\varepsilon$. Therefore, if $i > N$, then for each pair $(q, \hat{\tau}) \in U$, by (6.4) and (6.5),

$$\frac{s_{i+1,j}}{s_{ij}} = \frac{p_{i+1,j}^{q}}{p_{i,j}^{q}}\frac{p_{i,i+1}^{q}}{p^{q}} p^{q} r_2^{-\hat{\tau}} \leq (1+\varepsilon)^{2L}\frac{1+\overline{a}_3}{2} < 1. \tag{6.6}$$

Thus the series $\sum_{i=2}^{\infty} s_{ij}$ converges uniformly on the compact region U, as claimed.

Differentiating s_{ij} with respect to $\hat{\tau}$ for $i > 2$,

$$\frac{\partial s_{ij}}{\partial\hat{\tau}} = \begin{cases} s_{ij}\cdot(i-1)|\ln r_2|, & j = 2,\\ s_{ij}\cdot((i-2)|\ln r_2| + |\ln r_1|), & j = 1.\end{cases} \tag{6.7}$$

Then

$$\overline{\lim_{i\to\infty}}\frac{\partial s_{i+1,j}/\partial\hat{\tau}}{\partial s_{ij}/\partial\hat{\tau}} = \overline{\lim_{i\to\infty}}\frac{s_{i+1,j}}{s_{ij}} < 1.$$

This also proves that the series with non-negative terms $\sum_{i=2}^{\infty}\partial s_{ij}/\partial\hat{\tau}$ converges uniformly on the region U. So $\frac{\partial}{\partial\hat{\tau}}\sum_{i\geq 1} s_{ij} = \sum_{i\geq 1}\frac{\partial s_{ij}}{\partial\hat{\tau}}$. Let c satisfy

$$\max_{(q,\hat{\tau})\in U}\Big\{\sum_{j=1}^{2}\Big(1 + \sum_{i\geq 2} s_{ij} + \sum_{i\geq 2}\frac{\partial s_{ij}}{\partial\hat{\tau}}\Big)\Big\} < c. \tag{6.8}$$

From the expression $b_{ij} = p_{ij}^q r_{ij}^{-\hat{\tau}}$ for $e = (i,j) \in E$, we know $\partial b_{ij}/\partial\hat{\tau}$ is continuous on the compact set U, $i, j \in \{1, 2\}$. Differentiating D_n with respect to $\hat{\tau}$ yields

$$\begin{aligned}\Big|\frac{\partial D_n}{\partial\hat{\tau}}\Big| &= \Big|\frac{\partial b_{11}}{\partial\hat{\tau}}\sum_{i=1}^n s_{i2} + (b_{11}-1)\sum_{i=1}^n \frac{\partial s_{i2}}{\partial\hat{\tau}} - \frac{\partial b_{12}}{\partial\hat{\tau}}\sum_{i=1}^n s_{i1} - b_{12}\sum_{i=1}^n \frac{\partial s_{i1}}{\partial\hat{\tau}}\Big| \\ &\le c_1\sum_{j=1}^2\Big(1 + \sum_{i\ge 2} s_{ij} + \sum_{i\ge 2}\frac{\partial s_{ij}}{\partial\hat{\tau}}\Big) < c_1 c =: \gamma_2,\end{aligned}$$

where $c_1 := \max_{(q,\hat{\tau})\in U}\{\max\{|b_{11}-1|, |\partial b_{11}/\partial\hat{\tau}|, |b_{12}|, |\partial b_{12}/\partial\hat{\tau}|\}\}$. The conclusion now follows by letting $\gamma := \max\{\gamma_1, \gamma_2\}$. □

Corollary 6.4. *There exists a constant γ (independent of n) such that both $|F_n|$ and $|\partial F_n/\partial\hat{\tau}|$ are less than γ for any pair $(q,\hat{\tau}) \in U$.*

Proof. The conclusion follows from Lemma 6.3 and the fact that $b_{n,n+1}$ and $\partial b_{n,n+1}/\partial\hat{\tau}$ are continuous on U. □

Now we consider the differentiability of $\hat{\tau}(q)$.

Theorem 6.5. *Assume $p_2 > p_3$ or $q \ge 0$. Then the function $\hat{\tau}(q)$ is differentiable on $(-L, L)$.*

Proof. We use the implicit function theorem. Let

$$D := (b_{11}-1)\sum_{i\ge 1} s_{i2} - b_{12}\sum_{i\ge 1} s_{i1}, \tag{6.9}$$

where s_{ij} is defined as in the proof of Lemma 6.3. Next, we verify the conditions in the implicit function theorem in several steps.

Step 1. $\hat{\tau}(q)$ is a solution of the functional equation $D(q,\hat{\tau}(q)) = 0$. From the proof of Lemma 6.3, we see that D_n converges uniformly to D on U, and thus F_n converges uniformly to $F = D(1 - p^q r_2^{-\hat{\tau}})$ on U. For any $\varepsilon > 0$ and $q \in [-L, L]$, choose N sufficiently large so that $n > N$ implies $|F - F_n| < \varepsilon$ and $|\hat{\tau}_n(q) - \hat{\tau}(q)| < \varepsilon$. Note that $\lambda(\bar{B}_n(q,\hat{\tau}_n(q))) = 1$. So $F_n(q,\hat{\tau}_n(q)) = 0$. Now the mean-value theorem yields

$$\begin{aligned}\big|F(q,\hat{\tau}(q))\big| &\le \big|F(q,\hat{\tau}(q)) - F_n(q,\hat{\tau}(q))\big| + \big|F_n(q,\hat{\tau}(q)) - F_n(q,\hat{\tau}_n(q))\big| \\ &\le \varepsilon + \Big|\frac{\partial F_n}{\partial\hat{\tau}}\Big|\big|\hat{\tau}_n(q) - \hat{\tau}(q)\big| \\ &\le (1+\gamma)\varepsilon.\end{aligned}$$

The last inequality holds by Corollary 6.4. Therefore $F(q,\hat\tau(q))=0$ for all $q\in[-L,L]$. It follows from $1-p^q r_2^{-\hat\tau}\neq 0$ that $D(q,\hat\tau(q))=0$.

Step 2. $\partial D/\partial\hat\tau$ is continuous in the interior of U, denoted by U^o, and $\partial D/\partial\hat\tau\neq 0$ at $(q,\hat\tau(q))$ for $-L<q<L$. Using (6.9), together with the uniform convergence of $\sum_{i\geq 1} s_{ij}$ and $\sum_{i\geq 1}\partial s_{ij}/\partial\hat\tau$, $j=1,2$, we obtain the continuity of $\partial D/\partial\hat\tau$ on U.

Let $q\in(-L,L)$. We state three obvious observations:

- (O1) It follows from Lemma 6.1 that $b_{11}^0(q)=b_{11}(q,\hat\tau(q))<1$.
- (O2) The proof of Lemma 6.3 implies that the series $\sum_{i\geq 1} s_{i1}$ converges to a positive number. (Note that $s_{11}=0$, $s_{i1}>0$ for $i\geq 2$, by the definition of s_{i1} at the point $(q,\hat\tau(q))$);
- (O3) The equality $D(q,\hat\tau(q))=0$ and (6.9) imply that the series $\sum_{i\geq 1} s_{i2}$ is negative at $(q,\hat\tau(q))$.

The partial derivative of D with respect to $\hat\tau$ can be written as

$$\frac{\partial D}{\partial\hat\tau}=\frac{\partial b_{11}}{\partial\hat\tau}\sum_{i\geq 1} s_{i2}+(b_{11}-1)\sum_{i\geq 1}\frac{\partial s_{i2}}{\partial\hat\tau}+\frac{\partial}{\partial\hat\tau}\Big(-b_{12}\sum_{i\geq 1} s_{i1}\Big). \tag{6.10}$$

The first term on the right side of (6.10) is negative at $(q,\hat\tau(q))$ from Fact 2 and observation (O3). The second term is non-positive from the monotonicity of the series $\sum_{i\geq 1} s_{i2}$ with respect to $\hat\tau$ and observation (O1). The monotonicity of the series $b_{12}\sum_{i\geq 1} s_{i1}$ with respect to $\hat\tau$ shows that the last term is non-positive. Hence $\partial D/\partial\hat\tau|_{(q,\hat\tau(q))}\neq 0$.

Step 3. $\partial D/\partial q$ is continuous on U°. First, by (4.13), $p_{i,1}=p_c/v_i$ for $i\geq 2$, where v_i is defined as in (4.12). By Lemma 4.9(a,b), we know

$$p_{i,1}\prod_{k=2}^{i-1} p_{k,k+1}=\frac{p_b^{i-1}p_c}{p_a+p_b}. \tag{6.11}$$

Then

$$\frac{\partial s_{i1}}{\partial q}=s_{i1}\cdot\big((i-1)\ln p_b+\ln p_c-\ln(p_a+p_b)\big).$$

As for s_{i2}, we have

$$\frac{\partial s_{i2}}{\partial q}=s_{i2}\cdot\Big(\ln p_{i,2}+\sum_{k=2}^{i-1}\ln p_{k,k+1}\Big).$$

As both $\{p_{i,2}\}$ and $\{p_{i,i+1}\}$ converge to positive numbers, we have

$$\varlimsup_{i\to\infty}\frac{\partial s_{i+1,j}/\partial q}{\partial s_{ij}/\partial q}=\varlimsup_{i\to\infty}\frac{s_{i+1,j}}{s_{ij}}<1. \tag{6.12}$$

Thus the series $\sum_{i=1}^{\infty} \partial s_{ij}/\partial q$ converges uniformly on the region U, and hence $\frac{\partial}{\partial q}\sum_{i\ge 1} s_{ij} = \sum_{i\ge 1} \frac{\partial s_{ij}}{\partial q}$ is continuous on U^o. Now the continuity of $\partial D/\partial q$ follows from the equation

$$\frac{\partial D}{\partial q} = \frac{\partial b_{11}}{\partial q}\sum_{i\ge 1} s_{i2} + (b_{11}-1)\sum_{i\ge 1}\frac{\partial s_{i2}}{\partial q} - \frac{\partial b_{12}}{\partial q}\sum_{i\ge 1} s_{i1} - b_{12}\sum_{i\ge 1}\frac{\partial s_{i1}}{\partial q}.$$

By the three steps above, the three conditions in the implicit function theorem for $D(q,\hat\tau)$ hold. Hence $\hat\tau(q)$ is differentiable on $(-L,L)$, and

$$\hat\tau'(q) = -\frac{\partial D/\partial q}{\partial D/\partial \hat\tau}.$$

This completes the proof. □

7. Proof of Theorem 1.2

In this section, we denote the (i,j) entry of $B = B(q,\hat\tau(q),P)$ again by b^0_{ij}, which is a function of q. Clearly, for each $(i,j)\in E$, b^0_{ij} is differentiable. To prove Theorem 1.2, we need some notation and lemmas. For $j=1,2$ and $i\ge 2$, denote

$$s^0_{ij} := b^0_{ij}\prod_{k=2}^{i-1} b^0_{k,k+1}. \tag{7.1}$$

We remark that $s^0_{ij}(q) = s_{ij}(q,\hat\tau(q))$, where s_{ij} is given in the proof of Lemma 6.3.

Lemma 7.1. *If $p_2 > p_3$ or $q \ge 0$, then the following series*

$$\sum_{i\ge 2} i\Big(s^0_{i1} + s^0_{i2} + \Big|\frac{\mathrm{d}s^0_{i1}}{\mathrm{d}q}\Big| + \Big|\frac{\mathrm{d}s^0_{i2}}{\mathrm{d}q}\Big|\Big), \qquad \sum_{i\ge 2}\frac{s^0_{i2}}{b^0_{i,2}}, \qquad \sum_{i\ge 2}\frac{\mathrm{d}}{\mathrm{d}q}\Big(\frac{s^0_{i2}}{b^0_{i,2}}\Big) \tag{7.2}$$

converge uniformly on $[-L,L]$. Consequently, the functions

$$\sum_{i\ge 2} s^0_{i1}, \qquad \sum_{i\ge 2} s^0_{i2}, \qquad \sum_{i\ge 2}\frac{s^0_{i2}}{b^0_{i,2}}$$

are differentiable.

Proof. We first observe that for each $q\in[-L,L]$, the limit of $b^0_{i,2}$ is positive. From the proof of Lemma 6.3, there exists a constant $0<c<1$ such that for all $q\in[-L,L]$,

$$\overline{\lim_{i\to\infty}}\,\frac{(i+1)s^0_{i+1,2}/b^0_{i+1,2}}{i\cdot s^0_{i2}/b^0_{i,2}} = \overline{\lim_{i\to\infty}}\,\frac{s^0_{i+1,2}}{s^0_{i2}} < c, \qquad \overline{\lim_{i\to\infty}}\,\frac{(i+1)s^0_{i+1,1}}{i\cdot s^0_{i1}} = \overline{\lim_{i\to\infty}}\,\frac{s^0_{i+1,1}}{s^0_{i1}} < c.$$

The ratio test implies the uniform convergence of $\sum_{i\geq 2} is^0_{i1}$, $\sum_{i\geq 2} is^0_{i2}$ and $\sum_{i\geq 2} s^0_{i2}/b^0_{i,2}$. Differentiating s^0_{ij}, $j=1,2$, with respect to q and using (6.7), we get

$$\frac{\mathrm{d}s^0_{i1}}{\mathrm{d}q}=\frac{\partial s_{i1}}{\partial\hat\tau}\frac{\mathrm{d}\tau}{\mathrm{d}q}+\frac{\partial s_{i1}}{\partial q}=s^0_{i1}t_{i1},\qquad \frac{\mathrm{d}s^0_{i2}}{\mathrm{d}q}=\frac{\partial s_{i2}}{\partial\hat\tau}\frac{\mathrm{d}\tau}{\mathrm{d}q}+\frac{\partial s_{i2}}{\partial q}=s^0_{i2}t_{i2},$$

where

$$t_{i1}:=\big((2-i)\ln r_2-\ln r_1\big)\frac{\mathrm{d}\tau}{\mathrm{d}q}+(i-1)\ln p_2+\ln\frac{p_c}{p_a+p_b},\quad \text{and}$$

$$t_{i2}:=\big((1-i)\ln r_2\big)\frac{\mathrm{d}\tau}{\mathrm{d}q}+\ln p_{i,2}+\sum_{k=2}^{i-1}\ln p_{k,k+1}.$$

Note that $p_{k,k+1}\to p\in(0,1)$ as $k\to\infty$. A simple computation yields

$$\lim_{i\to\infty}\frac{t_{i+1,j}}{t_{ij}}=1,\quad j=1,2.$$

It follows that

$$\overline{\lim_{i\to\infty}}\frac{(i+1)|\,\mathrm{d}s^0_{i+1,j}/\,\mathrm{d}q|}{i\cdot|\,\mathrm{d}s^0_{ij}/\,\mathrm{d}q|}=\overline{\lim_{i\to\infty}}\frac{s^0_{i+1,j}|t_{i+1,j}|}{s^0_{i,j}|t_{ij}|}=\overline{\lim_{i\to\infty}}\frac{s^0_{i+1,j}}{s^0_{i,j}}<c<1.$$

This, together with the uniform convergence of $\sum_{i\geq 2} i\cdot s^0_{i1}$ and $\sum_{i\geq 2} i\cdot s^0_{i2}$, implies the uniform convergence of the first series in (7.2) on the interval $[-L,L]$.

Now we show the convergence of the second series in (7.2). Since $s^0_{ij}/b^0_{i,2}=\prod_{k=2}^{i-1}p_{k,k+1}$, it follows that the derivative of $s^0_{ij}/b^0_{i,2}$ at q is $(s^0_{ij}/b^0_{i,2})t_i$, where

$$t_i:=\big((2-i)\ln r_2\big)\frac{\mathrm{d}\tau}{\mathrm{d}q}+\sum_{k=2}^{i-1}\ln p_{k,k+1}.$$

Also, a simple computation shows that $t_{i+1}/t_i\to 1$ as $i\to\infty$. So

$$\overline{\lim_{i\to\infty}}\frac{\mathrm{d}(s^0_{i+1,2}/b^0_{i+1,2})/\,\mathrm{d}q}{\mathrm{d}(s^0_{i2}/b^0_{i,2})/\,\mathrm{d}q}=\overline{\lim_{i\to\infty}}\frac{s^0_{i+1,2}/b^0_{i+1,2}}{s^0_{i2}/b^0_{i,2}}\cdot\frac{t_{i+1}}{t_i}<c.$$

The ratio test implies the uniform convergence of the second series in (7.2). The last part of the conclusion follows from the uniform convergence of the two corresponding series. □

Lemma 7.2. *If $p_2>p_3$ or $q\geq 0$, then $\mathbf{x}^t$ and $\mathbf{y}$ are differentiable. Moreover,*

$$\frac{\mathrm{d}}{\mathrm{d}q}\sum_{i\geq 1}x_iy_i=\sum_{i\geq 1}\frac{\mathrm{d}}{\mathrm{d}q}(x_iy_i).\tag{7.3}$$

Proof. Since $\mathbf{x}^t$ is a 1-invariant measure of B, $\mathbf{x}^tB=\mathbf{x}^t$. Notice that $x_2\neq 0$ for any q. By letting $\mathbf{z}^t=x_2^{-1}\mathbf{x}^t$, we get $\mathbf{z}^tB=\mathbf{z}^t$. This implies that for $i\geq 2$,

$$z_{i+1}=b^0_{i,i+1}z_i=\prod_{2\leq k\leq i}b^0_{k,k+1}.$$

It follows that z_i is differentiable for all $i \geq 2$. Equating the second components of $\mathbf{z}^t B$ and $\mathbf{z}^t$ yields

$$1 - b_{12}^0 z_1 = \sum_{i\geq 2} z_i b_{i2}^0 = \sum_{i\geq 2} s_{i2}^0,$$

which is differentiable by Lemma 7.1. As $b_{12}^0 > 0$ is differentiable, so is z_1. An analogous argument shows the differentiability of the sum

$$\sum_{i\geq 1} z_i = z_1 + \sum_{i\geq 2} \frac{s_{i2}^0}{b_{i,2}^0}.$$

The formulas $x_2 = (\sum_{i\geq 1} z_i)^{-1}$ and $x_i = x_2 z_i$ now yield the differentiability of $\mathbf{x}^t$.

From $B\mathbf{y} = \mathbf{y}$, we get $y_2 = (b_{12}^0)^{-1}(1 - b_{11}^0)$ and

$$y_{i+1} = \frac{y_i - b_{i1}^0 - b_{i2}^0 y_2}{b_{i,i+1}^0}, \qquad i \geq 2. \tag{7.4}$$

So each y_i, $i \geq 1$, is a differentiable function of q.

Notice that $z_2 = 1$ and $y_1 = 1$. The expressions for y_{i+1} and z_{i+1} imply, by iteration,

$$z_{i+1} y_{i+1} = z_i y_i - (b_{i1}^0 y_1 - b_{i2}^0 y_2) z_i = y_2 - y_2 \sum_{k=2}^{i} s_{k2}^0 - \sum_{k=2}^{i} s_{k1}^0.$$

The above equation can be rewritten as

$$b_{12}^0 z_{i+1} y_{i+1} = (b_{11}^0 - 1)\Big(\sum_{k=2}^{i} s_{k2}^0 - 1\Big) - b_{12}^0 \sum_{k=2}^{i} s_{k1}^0.$$

Since $\hat{\tau}(q)$ is the solution of $D(q, \hat{\tau}(q)) = 0$, we see that

$$-b_{12}^0 z_i y_i = (b_{11}^0 - 1) \sum_{k\geq i} s_{k2}^0 - b_{12}^0 \sum_{k\geq i} s_{k1}^0.$$

By the uniform convergence of the first series in (7.2), we see that $\sum_{i\geq 2} b_{12}^0 z_i y_i$ converges (absolutely) on $[-L, L]$. Notice that

$$\begin{aligned}\Big|\frac{\mathrm{d}}{\mathrm{d}q}(b_{12}^0 z_i y_i)\Big| &= \Big|\frac{\mathrm{d}b_{11}}{\mathrm{d}q}\sum_{k\geq i} s_{k,2}^0 + (b_{11}^0 - 1)\sum_{k\geq i}\frac{\mathrm{d}s_{k,2}^0}{\mathrm{d}q} - \frac{\mathrm{d}b_{12}^0}{\mathrm{d}q}\sum_{k\geq i} s_{k,1} - b_{12}^0 \sum_{k\geq i}\frac{\mathrm{d}s_{i1}}{\mathrm{d}q}\Big| \\ &\leq c_0 \sum_{k\geq i}\Big(s_{k,1}^0 + s_{k,2}^0 + \Big|\frac{\mathrm{d}s_{k,1}^0}{\mathrm{d}q}\Big| + \Big|\frac{\mathrm{d}s_{k,2}^0}{dq}\Big|\Big),\end{aligned}$$

where $c_0 := \max_{|q|\leq L} \max\{|\,\mathrm{d}b_{11}^0/\,\mathrm{d}q|, b_{11}^0 + 1, |\,\mathrm{d}b_{12}^0/\,\mathrm{d}q|, b_{i2}^0\}$. Since

$$\sum_{i\geq 2}\Big|\frac{\mathrm{d}}{\mathrm{d}q}(b_{12}^0 z_i y_i)\Big| \leq c_0 \sum_{i\geq 2} i\Big(s_{i1}^0 + s_{i2}^0 + \Big|\frac{\mathrm{d}s_{i1}^0}{\mathrm{d}q}\Big| + \Big|\frac{\mathrm{d}s_{i2}^0}{\mathrm{d}q}\Big|\Big),$$

$\sum_{i\geq 2}\frac{\mathrm{d}}{\mathrm{d}q}(b_{12}^0 z_i y_i)$ converges uniformly by Lemma 7.1, and so does $\sum_{i\geq 2}\frac{\mathrm{d}}{\mathrm{d}q}(x_2^{-1}z_i y_i)=\sum_{i\geq 1}\frac{\mathrm{d}}{\mathrm{d}q}(x_i y_i)$, because $x_2=(\sum_{i\geq 1}z_i)^{-1}$ and b_{12}^0 are positive. Hence $\sum_{i\geq 1}x_i y_i$ is differentiable and (7.3) holds. □

Theorem 7.3. *Let $K^{(\alpha)}$ be given as above. If $p_2 > p_3$ or $q \geq 0$, then*

$$\dim_{\mathrm{H}} K^{(\alpha)} = \dim_{\mathrm{P}} K^{(\alpha)} = \alpha - \hat{\tau}(q).$$

Proof. We apply Theorem 1.1. Theorem 6.5 guarantees the differentiability of $\hat{\tau}$. Lemma 5.4 and Remark 5.6 show that $\mathbf{x}^t$ and $\mathbf{y}$ are, respectively, the 1-invariant measure and 1-invariant vector of B with $0 < \inf_i y_i \leq \sup_i y_i < \infty$ and $\mathbf{x}^t\mathbf{y} < \infty$. Lemma 7.2 implies the differentiability of $\mathbf{x}^t$ and $\mathbf{y}$, as well as the termwise differentiability of $\mathbf{x}^t\mathbf{y}$. Finally, by equation (1.6), $d(V) = 1 - r_1 - 2r_2 + r_1 r_2 > 0$, and thus (PSC) holds. The conclusion now follows from Theorem 1.1. □

Proof of Corollary 1.3. We first notice that the concavity of $\hat{\tau}_n(q)$ implies that of $\hat{\tau}(q)$. It follows from Theorem 1.2 and [7, Theorem 1.2] that $\hat{\tau}^*(\alpha(q)) = \tau^*(\alpha(q))$. Now, by concavity, $\hat{\tau}(q) = \hat{\tau}^{**}(q) = \tau^{**}(q) = \tau(q)$.

Finally, the equality $\dim_{\mathrm{H}}(\mu) = \tau'(1)$ follows from the differentiability of τ and results in [11, 22] concerning this equality. □

8. Extensions to other IFSs

The technique developed in this paper can be applied to other classes of IFS of generalized finite type. We briefly discuss several families of examples without going into details. In each case, the probability weights defining the self-similar measure must be chosen appropriately as in Theorem 1.2. All of these IFSs can be represented by an infinite graph-directed IFS satisfying (PSC). It is of interest to investigate the extend to which the method in this paper can be generalized.

We first note that the self-similar set K generated by the IFS in (1.5) is of generalized finite type. A graph-directed system satisfying (OSC) is given below (see Fig 5):

$$\begin{cases} K := T_1 = S_3(T_1) \cup S_2(T_2); \\ T_2 = S_5(T_2) \cup S_2(T_2) \cup S_3(T_1). \end{cases} \tag{8.1}$$

We point out that, although T_1 and K have the same Hausdorff dimension and the same Hausdorff measure, the first component of the invariant vector measure $\boldsymbol{\mu} = (\mu_1, \mu_2)$ (for

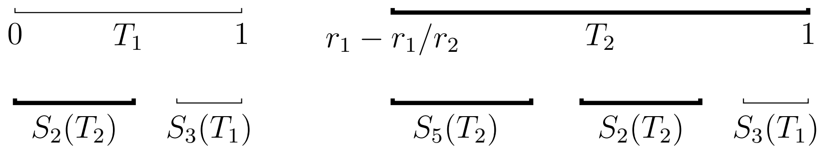


FIGURE 5. A graph-directed system satisfying (OSC) that generates K, where $T_1 = K$ and $S_5 = S_2^{-1}S_1S_2$. The mappings are represented by thin or thick lines depending on whether they originate from T_1 or T_2. The figures are drawn with $r_1 = 1/3$ and $r_2 = 2/7$. .

T_1, T_2) of such graph-directed system does not coincide with the self-similar measure μ (for K). In fact, they are singular to each other.

The graph in Fig 6(a) is the basic directed graph for the above graph-directed system. Our method can be applied to the case when certain additional edges are added to such directed graph (see Fig. 6(b)) so that we can consider a class of IFSs obtained by adding more maps to the IFS in (1.5). The difference between the matrices P and $B = B(q, \hat{\tau}, P)$ for the new graph-directed system and the ones in (5.1) is in the first columns. The m-th components of the first columns of new matrices P and $B(q, \hat{\tau}, P)$ are

$$\begin{cases} \displaystyle\sum_{i=0}^{n} p_{m1,i}, \text{ and} \sum_{i=0}^{n} p_{m1,i}^q r_{uu,i}^{-\hat{\tau}}, & m = 1, \\ \displaystyle\sum_{j=0}^{2n} p_{m1,j}, \text{ and} \sum_{j=0}^{2n} p_{m1,j}^q r_{uv,j}^{-\hat{\tau}}, & m \geq 2, \end{cases}$$

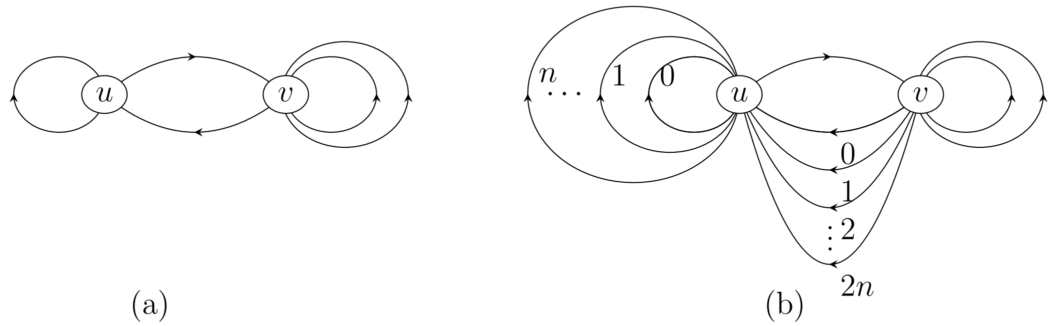
respectively, where $r_{uu,i}$ and $r_{uv,j}$ stand for the ratios of the mappings on the edges $e_{uu,i}$ and $e_{uv,j}$ respectively. Moreover, $p_{11,i}$ and $p_{m1,j}$ $(m \geq 2)$ can be obtained using the same method for p_{11} and p_{m1} $(m \geq 2)$.

The following example is for $n = 2$, obtained by adding two mappings from T_1 to T_1 and four mappings from T_1 to T_2.

Example 8.1. *Add* $S_4(x) = r_1^2 x + r_2 + 0.9r_1$ *and* $S_5(x) = r_1 r_2 x + 1 - 1.5r_2$ *to the IFS in* (1.5) *to obtain a new IFS (see Fig. 6). To guarantee that the new IFS is of general finite type, the conditions* $S_2(1) < S_4(0), S_4(1) < S_5(0), S_5(1) < S_3(0)$ *are required.*

From Figure 6, we see that the new IFS is of generalized finite type. The attractor has positive finite α-Hausdorff measure where α is the unique number satisfying $r_1^\alpha + 2r_2^\alpha + r_2^{2\alpha} = 1$.

We can also handle IFSs in higher dimensions. The following family of IFSs from [17] is a two-dimensional extension of the family in (1.5).



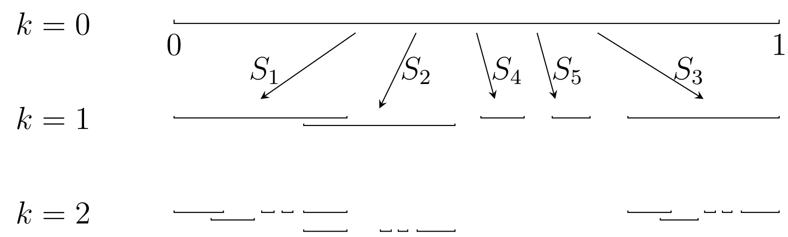


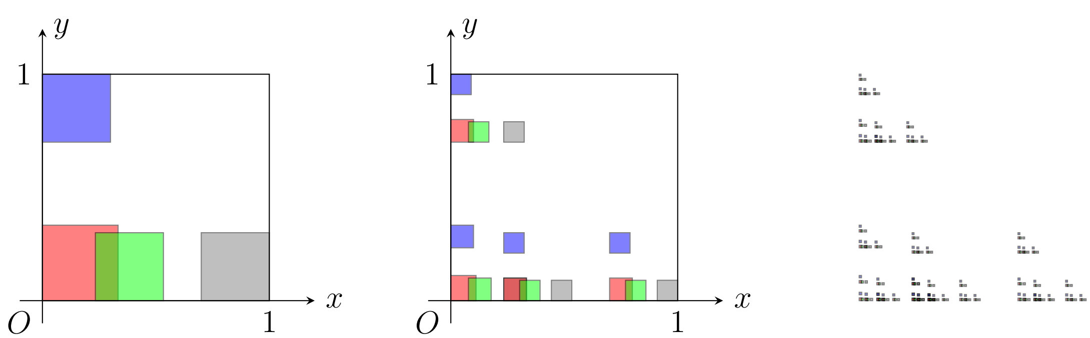
FIGURE 6. An IFS with more added mappings. The figure is drawn with $r_1 = 1/4, r_2 = 2/7$.

Example 8.2. *Let* $\{S_i\}_{i=1}^4$ *be an IFS on* $\mathbb{R}^2$ *defined as*

$$S_1(\boldsymbol{x}) = r_1\boldsymbol{x}, \qquad S_2(\boldsymbol{x}) = r_2\boldsymbol{x} + (r_1 - r_1r_2, 0),$$

$$S_3(\boldsymbol{x}) = r_2\boldsymbol{x} + (1 - r_2, 0), \qquad S_4(\boldsymbol{x}) = r_2\boldsymbol{x} + (0, 1 - r_2),$$

where $0 < r_1 < 1$, $0 < r_2 < 1$, *and* $r_1 + 2r_2 - r_1r_2 \le 1$. *See Figure 7.*



(a) $k = 1$ (b) $k = 2$ (c) The attractor

FIGURE 7. The first two iterations of the IFS in Example 8.2. The attractor is shown in (c). The figures are drawn with $r_1 = 1/3$ and $r_2 = 3/10$.

It is shown in [17] that $\{S_i\}_{i=1}^4$ is of generalized finite type and it does not satisfy the open set condition. Let F be the attractor of the IFS. Then $\dim_{\mathrm{B}}(F) = \dim_{\mathrm{H}}(F) = \alpha$, where α is the unique solution of the equation $r_1^\alpha + 3r_2^\alpha - (r_1r_2)^\alpha = 1$. Moreover, $0 < \mathcal{H}^\alpha(F) < \infty$.

Example 8.2 has a three-dimensional extension as shown in the following example.

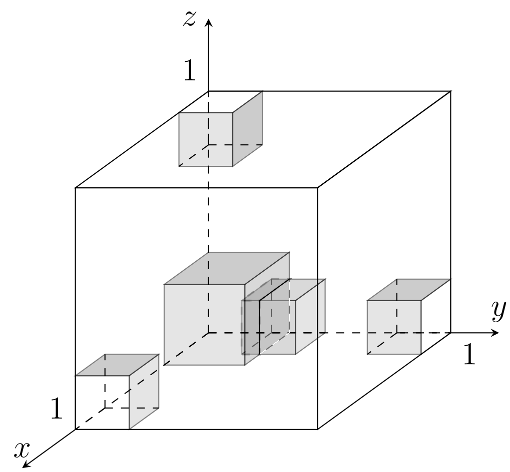


FIGURE 8. The first iteration of the example in $\mathbb{R}^3$. The figure is drawn with $r_1 = 1/3$ and $r_2 = 2/9$.

Example 8.3. *Let* $\{S_i\}_{i=1}^5$ *be an IFS on* $\mathbb{R}^3$ *defined as*

$$S_1(\boldsymbol{x}) = r_1\boldsymbol{x}, \qquad S_2(\boldsymbol{x}) = r_2\boldsymbol{x} + (r_1 - r_1r_2, 0, 0),$$

$$S_3(\boldsymbol{x}) = r_2\boldsymbol{x} + (1 - r_2, 0, 0), \quad S_4(\boldsymbol{x}) = r_2\boldsymbol{x} + (0, 1 - r_2, 0), \quad S_5(\boldsymbol{x}) = r_2\boldsymbol{x} + (0, 0, 1 - r_2),$$

where $0 < r_1 < 1$, $0 < r_2 < 1$, *and* $r_1 + 2r_2 - r_1r_2 \leq 1$. *See Figure 8 for the first iteration.*

It is easy to show that the attractor, denoted by G, given as in Example 8.3 is of generalized finite type, although it does not satisfy (OSC). The attractor G shares some similar properties with the attractor F; for example, $\mathcal{H}^\alpha(G)$ is positive finite, where α the unique number satisfying the equation

$$r_1^\alpha + 4r_2^\alpha - (r_1r_2)^\alpha = 0.$$

Our results can also be extended to certain graph IFSs of generalized finite type. Such IFSs are studied in [2, 24].

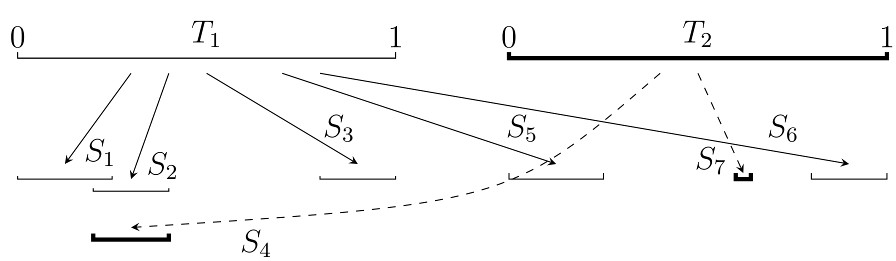


FIGURE 9. The mappings are represented by thick or thin lines depending on whether they originate from T_1 or T_2. The figure is drawn with $r_1 = 1/4$ and $r_2 = 1/5$.

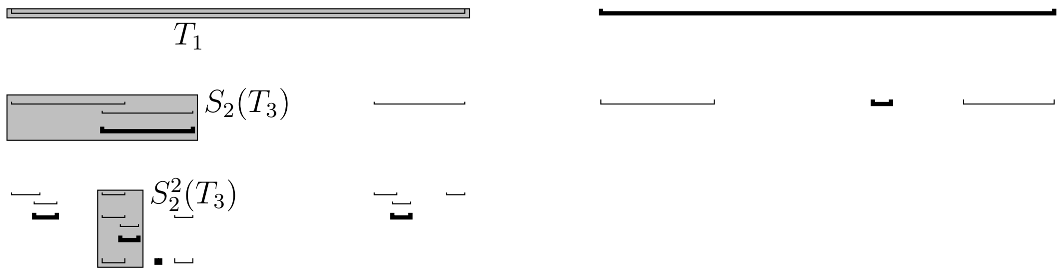


FIGURE 10. The graph-directed system satisfying (SSC).

Example 8.4. *Let* $V=\{1,2\}$, $E_{11}=\{e_1,e_2,e_3\}$, $E_{12}=\{e_4\}$, $E_{21}=\{e_5,e_6\}$, *and* $E_{22}=\{e_7\}$. *The corresponding graph-directed IFS is (after abbreviating* S_{e_i} *as* S_i*):*

$$S_1(x)=S_5(x)=r_1x, \qquad S_2(x)=S_4(x)=r_2x+r_1(1-r_2),$$
$$S_3(x)=S_6(x)=r_2x+(1-r_2), \qquad S_7(x)=r_2^2x+3r_2,$$

where $r_1,r_2\in(0,1)$, $r_1+2r_2-r_1r_2$ *and* $r_1(1-r_2)<3r_2<1-r_2$. *The reader is referred to Fig. 9 for the first iteration.*

The graph-directed system $(V,E,\{S_i\})$ for the vector attractor $\{T_1,T_2\}$ stated above does not satisfy (OSC). However, we can construct another graph-directed system satisfying (OSC) as follows. Let $T_3=S_2^{-1}S_1(T_1)\cup T_1\cup T_2$. Then $\{T_1,T_2,T_3\}$ is the vector attractor of a graph-directed system satisfying (OSC) (see Fig. 10). More precisely, denoting $S_0=S_2^{-1}S_1S_2$, we have

$$\begin{cases} T_1=S_2(T_3)\cup S_3(T_1), \\ T_2=S_5(T_1)\cup S_6(T_1)\cup S_7(T_2), \\ T_3=S_0(T_3)\cup S_2(T_3)\cup S_3(T_1)\cup S_4(T_2). \end{cases}$$

By extending such graph-directed system to an infinite one, we can consider the measure for T_1.

With suitable modification, our method seems applicable to some other IFSs; however, additional work may be needed. We illustrate this by the following family of IFSs.

Example 8.5. *Let an IFS be defined as below*

$$\begin{cases} S_1(x)=r_1x, \quad S_2(x)=r_1^{-1}r_2^2x+(1-r_2)(r_1+r_2), \\ S_3(x)=r_2x+r_1(1-r_2), \quad S_4(x)=r_2x+1-r_2. \end{cases} \tag{8.2}$$

where r_1,r_2 *satisfy* $r_1,r_2\in(0,1)$, *and* $r_1+r_2-r_1r_2<1$ *(see Figure 11 for the first two steps of the iteration).*

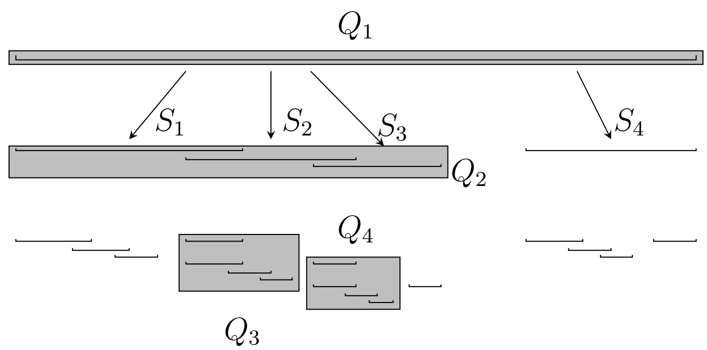


FIGURE 11. Figure for Example 8.5. The figure is drawn with $r_1 = 1/3$ and $r_2 = 1/4$.

It can be checked directly that the IFS above is of generalized finite type. The matrix $B(q, \hat{\tau}, P)$ for this IFS, shown below, can be obtained by using the method for the IFS in (1.5).

$$B = B(q, \hat{\tau}, P) = \begin{pmatrix} p_{11}^q r_2^{-\hat{\tau}} & p_{12}^q (r_2^2/r_1)^{-\hat{\tau}} & 0 & 0 & 0 & \cdots & \\ p_{21}^q r_2^{-\hat{\tau}} & p_{22}^q r_1^{-\hat{\tau}} & p_{23}^q (r_2^2/r_1)^{-\hat{\tau}} & p_{24}^q r_2^{-\hat{\tau}} & 0 & \cdots & \\ p_{31}^q r_2^{-\hat{\tau}} & p_{32}^q r_1^{-\hat{\tau}} & p_{33}^q (r_2^2/r_1)^{-\hat{\tau}} & 0 & p_{35}^q r_2^{-\hat{\tau}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

We refer the reader to Figure 11 for an illustration of the second row of B.

9. Comments and questions

In the case $p_2 < p_3$ in Theorem 1.2, it is likely that there is phase-transition in the region $q < 0$; that is, $\tau(q)$ has a non-differentiable point. In this case there are points with very small μ measure; in fact, $\lim_{n\to\infty} p_{n,1} = 0$.

It is of interest to study, under the assumption of Theorem 1.1, the equality $\hat{\tau}(q) = \tau(q)$.

We do not know whether the method developed in this paper can be applied to IFSs that do not satisfy the generalized finite type condition.

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