EIGENVALUE ASYMPTOTICS AND BOHR'S FORMULA FOR FRACTAL SCHRÖDINGER OPERATORS

SZE-MAN NGAI AND WEI TANG

Abstract. For a fractal Schrödinger operator with a continuous potential and a coupling parameter, we obtain an analog of a semi-classical asymptotic formula for the number of bound states as the parameter tends to infinity. We also study Bohr's formula for Schrödinger operators on blowups of self-similar sets. For a Schrödinger operator defined by a fractal measure and a locally bounded potential that tends to infinity, we derive an analog of Bohr's formula under various assumptions. We demonstrate how these results can be applied to self-similar measures with overlaps, including the infinite Bernoulli convolution associated with the golden ratio, a family of convolutions of Cantor-type measures, and a family of measures that we call essentially of finite type.

Contents

1. Introduction

The eigenvalues of a Schrödinger operator are referred to by physicists as the bound state energies. Let Δ be the Dirichlet Laplacian on $\mathbb{R}^n$ and let $N^-(V)$ be the number of the

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negative eigenvalues, also called *bound states*, of the Schrödinger operator $-\Delta+V$, where V is a potential. In the early 1970's, Birman and Borzov [1], Martin [24], and Tamura [36] proved the following semi-classical asymptotic formula for $N^-(\beta V)$:

$$N^-(\beta V)\sim\frac{\omega_n}{(4\pi)^{n/2}}\beta^{n/2}\int_{D_n^-(V)}\big(-V(x)\big)^{n/2}\,dx,\quad\text{as }\beta\to+\infty,\tag{1.1}$$

where V is a continuous and compactly supported potential, β is called a *coupling parameter*, $D_n^-(V):=\{x\in\mathbb{R}^n: V(x)\le 0\}$, ω_n is the volume of the unit ball in $\mathbb{R}^n$, and throughout this paper, $f\sim g$ means $\lim_{x\to+\infty}f(x)/g(x)=1$. The main ingredients are the *Dirichlet-Neumann bracketing technique* [30] and the *Weyl law* [37]. For fractal sets, Strichartz [34] studied the counting function for the negative eigenvalues of the Schrödinger operator $\Delta+V$ on the product of two copies of an infinite blowup of the Sierpiński gasket, where V is a Coulomb potential. He showed that the number of eigenvalues that are less than $-\epsilon$ is of the order $\epsilon^{-\delta}$ as $\epsilon\to 0^+$, where $\delta=\big(\ln(25/9)\ln 9\big)/\big(\ln(9/5)\ln 5\big)$. A main goal of this paper is to obtain a crude analogue of (1.1) for Schrödinger operators $-\Delta_\mu+\beta V$ defined on domains by a measure μ (see Theorem 1.1).

The classical one-dimensional Bohr's formula states that, under suitable conditions,

$$N(\lambda,-\Delta+V)\sim\frac{1}{\pi}\int_0^\infty(\lambda-V(x))_+^{1/2}\,dx,\qquad\text{as }\lambda\to+\infty,$$

where Δ is the Laplacian in $L^2([0,+\infty),dx)$, and $V(x)\to+\infty$ as $x\to\infty$(see [12]). In the classical setting, various forms of Bohr's formula have been studied extensively (see, e.g., [30]). In the fractal setting, Bohr's formula has been obtained by Chen *et al.* [2] for some unbounded potentials V on several types of unbounded fractal spaces K_∞ supporting a measure μ_∞ and having a well-defined Laplacian Δ_{μ_∞}. K_∞ is obtained by blowing up some fractal K. In [2], sufficient conditions for the following Bohr's formula to hold are obtained:

$$N(\lambda,-\Delta_{\mu_\infty}+V)\sim g(V,\lambda),\qquad\text{as }\lambda\to+\infty,$$

where $N(\lambda,-\Delta_{\mu_\infty}+V):=\#\big\{n:\lambda_n(-\Delta_{\mu_\infty}+V)\le\lambda\big\}$, and

$$g(V,\lambda):=\int_{K_\infty}\Big(\big(\lambda-V(x)\big)_+\Big)^{d_s/2}G\left(\frac{1}{2}\ln\big(\lambda-V(x)\big)_+\right)d\mu_\infty(x),\tag{1.2}$$

$d_s=d_s(-\Delta_{\mu_\infty})$, and $G(\cdot)$ is a periodic function. Moreover, these conditions are verified for fractalfolds and fractal fields based on nested fractals. A key condition they assume is

$$N(\lambda,\Delta^b_{\mu|_K})=\lambda^{d_s/2}\Big(G\big(\frac{1}{2}\ln\lambda\big)+R(\lambda)\Big)\qquad\text{for }b\in\{D,N\},\tag{1.3}$$

where $N(\lambda, \Delta^b_{\mu|_K}) := \#\{n : \lambda_n(-\Delta^b_{\mu|_K}) \le \lambda\}$, $R^b(\lambda)$ denotes the remainder term of order $o(1)$, and $-\Delta^D_{\mu|_K}$ and $-\Delta^N_{\mu|_K}$ are Dirichlet and Neumann Laplacians in $L^2(K, \mu)$, respectively. Unfortunately, fractals with overlaps usually do not, or not known to, satisfy this condition. Thus it is another main goal of this paper to derive an analog of Bohr's formula for such fractals by modifying (1.3).

Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ that is semi-bounded below. If A has pure discrete spectrum, then we define the *eigenvalue counting function* as

$$N(\lambda, A) := \#\{n : \lambda_n(A) \le \lambda\}, \tag{1.4}$$

where $\lambda_n(A)$ is the n-th eigenvalue of A counted according to their multiplicities, and $\#E$ denotes the cardinality of a finite set E. Furthermore, we define the *lower and upper spectral dimensions* of A, respectively, as

$$\underline{d}_s(A) := \varliminf_{\lambda\to\infty} \frac{2\ln N(\lambda, A)}{\ln \lambda} \qquad \text{and} \qquad \overline{d}_s(A) := \varlimsup_{\lambda\to\infty} \frac{2\ln N(\lambda, A)}{\ln \lambda}.$$

If $\underline{d}_s(A) = \overline{d}_s(A)$, the common value, denoted by $d_s(A)$, is called the *spectral dimension* of A; it measures the asymptotic growth rate of the eigenvalue counting function.

Let $X \subset \mathbb{R}^n$ be a compact subset and μ be a positive finite Borel measure on X with $\mu(X^\circ) > 0$ and $\operatorname{supp}(\mu) \subseteq X$. It is known that μ defines a Dirichlet Laplace operator Δ_μ, if the following *Poincaré inequality for a measure (MPI)* holds: There exists a constant $C > 0$ such that

$$\int_{X^\circ} |u|^2\,d\mu \le C \int_{X^\circ} |\nabla u|^2\,dx \qquad \text{for all } u \in C_c^\infty(X^\circ) \tag{1.5}$$

(see, e.g., [13, 26, 27]). In the rest of this section, we assume that μ satisfies (MPI).

The first part of this paper studies the Schrödinger operators $-\Delta_\mu + \beta V$ in $L^2(X, \mu)$ with a continuous potential V and a coupling parameter β, focusing on self-similar measures. Throughout this paper, we let $D^-(V) := \{x \in X : V(x) \le 0\}$ and $N^-_\mu(V)$ be the number of negative eigenvalues of $-\Delta_\mu + V$ for a real-valued continuous function V on X.

Before stating the main results, we introduce some definitions that will be used. A cell is a closed subset of X whose interior has positive μ measure. A *μ-partition* $\mathcal{P}$ of X is a finite family of interior disjoint cells such that $\mu(X) = \sum_{B\in\mathcal{P}} \mu(B)$. Under suitable assumptions, μ also defines a Neumann Laplacian $-\Delta^N_\mu$ (see Subsection 2.3). Roughly speaking, we say a sequence of μ-partitions $(\mathcal{P}_k)_{k\ge1}$ satisfies condition (N), if for each cell $B \in \bigcup_{k=1}^\infty \mathcal{P}_k$, the Neumann Laplacian $-\Delta^N_{\mu|_B}$ is well-defined and has compact resolvent. The precise statements are given in Definition 3.1. Let ν be a positive finite Borel measure on X. A sequence of μ-partitions $(\mathcal{P}_k)_{k\geq 1}$ is said to be *refining* with respect to ν if each member of $\mathcal{P}_{k+1}$ is a subset of some member of $\mathcal{P}_k$, and $\max\{\nu(B) : B \in \mathcal{P}_k\} \to 0$ as $k \to \infty$.

Theorem 1.1. *Let $X \subseteq \mathbb{R}^n$ be a compact subset and μ be a positive finite Borel measure on $\mathbb{R}^n$ with $\operatorname{supp}(\mu) \subseteq X$ and $\mu(X^\circ) > 0$. Assume that μ satisfies (MPI) and V is a real-valued continuous function on X. Let ν be a positive Borel measure on X.*

(a) *If there exist positive constants C and α, and a refining μ-partition $(\mathcal{P}_k)_{k\geq 1}$ of X with respect to ν such that for all $B \in \bigcup_{k=1}^{\infty} \mathcal{P}_k$,*

$$N(\lambda, -\Delta_{\mu|_B}) \geq \lambda^{\alpha/2}\big(C\nu(A) + o(1)\big), \qquad \text{as } \lambda \to +\infty, \tag{1.6}$$

then

$$N_\mu^-(\beta V) \geq \beta^{\alpha/2}\Big(C \int_{D^-(V)} (-V)^{\alpha/2}\, d\nu + o(1)\Big), \quad \text{as } \beta \to +\infty. \tag{1.7}$$

(b) *If there exist positive constants C and α, and a refining μ-partition $(\mathcal{P}_k)_{k\geq 1}$ of X with respect to ν satisfying condition (N), and for all $B \in \bigcup_{k=1}^{\infty} \mathcal{P}_k$,*

$$N(\lambda, -\Delta^N_{\mu|_B}) \leq \lambda^{\alpha/2}\big(C\nu(A) + o(1)\big), \qquad \text{as } \lambda \to +\infty, \tag{1.8}$$

then the reverse inequality in (1.7) *holds.*

We remark that (1.6) and (1.8) are more general than the Weyl law. In the proof of Theorem 1.1, we use a similar method in [30, Theorem XIII.79] with (1.6) and (1.8) replacing the Weyl law. We illustrate Theorem 1.1 by a family of self-similar measures that are so-called *essentially of finite type (EFT)* (see Section 3). The spectral dimension of all these measures are known [29].

The second part of this paper studies the Schrödinger operator for some non-negative, locally bounded, piecewise continuous potentials that tend to infinity, focusing on self-similar measures with overlaps. Let $X \subseteq \mathbb{R}^n$ be a compact subset and μ be a positive finite Borel measure on $\mathbb{R}^n$ with $\operatorname{supp}(\mu) \subseteq X$ and $\mu(X^\circ) > 0$. Assume that μ satisfies (MPI). We first state some Weyl asymptotic properties for Δ_μ, which will be used in Section 4.

Definition 1.2. *Let μ, X and Δ_μ be defined as above. Define the following two* Weyl asymptotic properties.

(a) *We say that* (W1) *holds if there exist positive constants C_1, C_2, d_s such that*

$$C_1\lambda^{d_s/2} \leq N(\lambda, -\Delta_\mu) \leq C_2\lambda^{d_s/2}, \qquad \text{as } \lambda \to +\infty.$$

(b) *We say that* (W2) *holds if there exists a finite collection of closed subsets* $\{Y_j\}_{j\in J}$ *of* X *with nonempty interiors satisfying the following conditions:*

(1) *there exist positive constants* C_0 *and* $(\xi_{j,k})_{j\in J}$, $k=1,2$, *such that for all* $\lambda>0$,

$$\sum_{j\in J} N(\xi_{j,1}\lambda, -\Delta_{\mu|_{Y_j}}) - C_0 \le N(\lambda,-\Delta_\mu) \le \sum_{j\in J} N(\xi_{j,2}\lambda, -\Delta_{\mu|_{Y_j}}) + C_0; \tag{1.9}$$

(2) *for each* $j\in J$, *there exists a periodic or constant function* $G_j:\mathbb{R}\to\mathbb{R}^+$ *such that* $0<\inf G_j\le \sup G_j<\infty$, *and as* $\lambda\to+\infty$,

$$N(\lambda,-\Delta_{\mu|_{Y_j}}) = \lambda^{d_s/2}\Big(G_j\big(\ln\lambda\big)+R_j(\lambda)\Big),$$

where d_s *is a positive constant independent of* j, *and* $R_j(\lambda)$ *denotes the remainder term of order* $o(1)$.

We remark that (W1) implies that $d_s(-\Delta_\mu)=d_s$, and (W2) is stronger than (W1). Condition (2) of (W2) means that $-\Delta_{\mu|_{Y_j}}$ satisfies (1.3) for all $j\in J$. Consequently, (W2) is more general than (1.3), which corresponds to (W2) with $J=\{1\}$, $Y_1=X$, and $G_1(\cdot)$ being a periodic function.

We extend X to an unbounded space X_∞ as follows. Let $X_\infty:=\bigcup_{i\in I}X_i$, where

(C1) I is a countably infinite index set containing 0;
(C2) each $i\in I$ corresponds to a similitude $\tau_i:X\to X_i$ of the form $\tau_i(x)=x+b_i$, with $b_i\in\mathbb{R}^n$ such that τ_0 is the identity map on $\mathbb{R}^n$ and $\tau_i(X)=X_i$;
(C3) for any distinct $i,j\in I$, $X_i\cap X_j=\partial X_i\cap\partial X_j$.

Note each X_i is isometric to X; in particular, $|X_i|=|X|$ for all $i\in I$. Condition (C3) implies that the interiors of any two distinct X_i are disjoint. For each $i\in I$, $\mu_i:=\mu\circ\tau_i^{-1}$ defines a positive finite Borel measure on X_i. Intuitively, μ_i and μ have same measure structure. Also, $\mu_0=\mu$. In a natural way, we can define a glued measure μ_∞ on X_∞ by

$$\mu_\infty(E):=\sum_{i\in I}\mu_i(E\cap X_i) \qquad \text{for all Borel subsets } E\subseteq X_\infty. \tag{1.10}$$

Throughout this paper, we assume that $\mu_\infty(X_i\cap X_j)=0$ for any distinct $i,j\in I$. For a real-valued function f on X_∞ and $\lambda>0$, we define the *distribution function of* f with respect to μ_∞ as:

$$F(\lambda,f):=\mu_\infty\big(\{x\in X_\infty: f(x)\le\lambda\}\big). \tag{1.11}$$

Assume (W2) holds. For any $j \in J$, define

$$X_{\infty,j} := \bigcup_{i \in I} \tau_i(Y_j) \qquad \text{and} \qquad \mu_{\infty,j} := \mu_\infty|_{X_{\infty,j}}.$$

In order to state the precise results, we introduce the following associated *Bohr's asymptotic function*: for any $j \in J$, $\lambda > 0$, and $f \in L^1_{\mathrm{loc}}(X_{\infty,j}, \mu_{\infty,j})$, define

$$g_j(\lambda, f) := \frac{1}{\mu(Y_j)} \int_{X_{\infty,j}} \Big(\big(\lambda - f(x)\big)_+ \Big)^{d_s/2} G_j\Big(\ln \big(\lambda - f(x)\big)_+ \Big)\, d\mu_{\infty,j}(x), \tag{1.12}$$

where $G_j(\cdot)$ is given in (W2). We remark that $g_j(\cdot,\cdot)$ is an analog of the $g(\cdot,\cdot)$ in (1.2), which appears in [2], but slightly different because it is assumed in [2] that $\mu(K) = 1$. Let V be a non-negative, locally bounded, piecewise continuous function on X_∞ so that $V(x) \to +\infty$ as $|x| \to \infty$. Also, let $V^\wedge$ (resp. $V^\vee$) be the piecewise constant function which takes the value $\sup_{x \in X_i} V(x)$ (resp. $\inf_{x \in X_i} V(x)$) on X_i. The theorem below gives the eigenvalue asymptotics of $N(\lambda, -\Delta_{\mu_\infty} + V)$.

Theorem 1.3. *Use the notation above. Let V be a non-negative, locally bounded, piecewise continuous function on X_∞ so that $V(x) \to +\infty$ as $|x| \to \infty$. Assume (MPI) and (W2) hold. Let $F(\cdot,\cdot)$ and $g_j(\cdot,\cdot)$ be defined as in* (1.11) *and* (1.12) *for $j \in J$, respectively. Assume that*

$$F(\lambda, V^\vee)/F(\lambda, V^\wedge) = 1 + o(1), \qquad \text{as } \lambda \to +\infty, \tag{1.13}$$

and that there exists some $C > 0$ such that $F(2\lambda, V^\vee) \le CF(\lambda, V^\wedge)$ for all sufficiently large $\lambda > 0$. Then as $\lambda \to +\infty$,

$$\big(1 + o(1)\big) \sum_{j \in J} g_j(\xi_{j,1}\lambda, \xi_{j,1}V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le \big(1 + o(1)\big) \sum_{j \in J} g_j(\xi_{j,2}\lambda, \xi_{j,2}V),$$

where $(\xi_{j,k})_{j \in J}$, $k = 1, 2$, are the constants in (1.9).

We remark that Theorem 1.3 cannot be deduced from [2, Theorem 2.11], since (W2) is more general than (1.3), which is a key assumptionin Theorem [2, Theorem 2.11]. Theorem 1.3 allows us to obtain eigenvalue asymptotics of Schrödinger operators in the absence of condition (1.3), as illustrated in the examples of IFSs with overlaps in Section 5. It also enables us to draw conclusions on $N(\lambda, -\Delta_{\mu_\infty} + V)$ even though we only have information about the Weyl asymptotics of the Laplacian on a proper subset of X.

We apply Theorem 1.3 to three classes of self-similar measures in Section 5. The infinite Bernoulli convolution associated with the golden ratio and a class of convolutions of Cantor-type measures have been studied extensively (see [9, 11, 15, 16, 19, 28]). They define Laplacians that exhibit many behaviors analogous to Laplacians on p.c.f. fractals, such as sub-Gaussian heat kernel estimates [11] and infinite wave propagation speed [29]. The third class is used in [29] to illustrate self-similar measures satisfying (EFT). We show that all these measures satisfy (W2). However, it is not clear whether these three classes of measures satisfy (1.3).

The rest of this paper is organized as follows. Section 2 summarizes some of the definitions and results that will be needed throughout the paper. In Section 3, we prove Theorem 1.1, and apply it to a class of self-similar measures satisfying (EFT). In Section 4, we study Bohr's formula for Schrödinger operators defined by measures and non-negative locally bounded potentials, and prove Theorem 1.3. Finally, in Section 5, we illustrate Bohr's formula by three classes of self-similar measures with overlaps.

2. Preliminaries

Let X be a metric space. For any subset $E \subseteq X$, let $\overline{E}$, ∂E, $|E|$ and E° (or $\operatorname{int}(E)$) denote, respectively, the closure, boundary, diameter and interior of E in X. For a real-valued function f on X, we define $f_+ := \max\{f, 0\}$ and $f_- := -\min\{f, 0\}$, and let $f|_F$ denote the restriction of the function f to $F \subseteq X$.

Let (X, μ) be a measure space with a σ-finite measure μ. We denote by $\mu|_E$ the restriction of μ to $E \subseteq X$. For $1 \le p \le \infty$, let $\|u\|_{p,\mu} = \|u\|_{L^p(X,\mu)}$ denote the norm in $L^p(X, \mu)$. We denote by $\mathcal{L}^n(U)$ the n-dimensional Lebesgue measure of $U \subseteq \mathbb{R}^n$.

Let $(\mathcal{H}_1, \|\cdot\|_1)$ and $(\mathcal{H}_2, \|\cdot\|_2)$ be Hilbert spaces. Let A_1, A_2 be linear operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. A_1 and A_2 are said to be unitarily equivalent, denoted $A_1 \approx A_2$, if there exists a unitary operator $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\varphi(\mathrm{dom} A_1) = \mathrm{dom} A_2 \quad \text{and} \quad \varphi(A_1(u)) = A_2(\varphi(u)) \quad \text{for all } u \in \mathrm{dom} A_1.$$

Note that u is a λ-eigenvector of A_1 if and only if $\varphi(u)$ is a λ-eigenvector of A_2. In particular, unitarily equivalent operators have the same set of eigenvalues.

Let $(\mathcal{H}_i)_{i \in I}$ be a countably infinite or finite family of Hilbert spaces. Define a Hilbert space

$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i := \Big\{ u = (u_i)_{i \in I} : u_i \in \mathcal{H}_i \text{ for all } i \in I \text{ and } \|u\|_{\mathcal{H}}^2 := \sum_{i \in I} \|u_i\|_{\mathcal{H}_i}^2 < \infty \Big\}.$$

Assume that each A_i is a self-adjoint operator in $\mathcal{H}_i$. We write $A := \bigoplus_{i \in I} A_i$, if $Au := (A_i u_i)_{i \in I}$ with domain $\mathrm{dom}\, A := \{u = (u_i)_{i \in I} \in \mathcal{H} : u_i \in \mathrm{dom}\, A_i \text{ for all } i \in I \text{ and } Au \in \mathcal{H}\}$ (see [31]). We remark that $(A, \mathrm{dom}\, A)$ is a self-adjoint operator in $\mathcal{H}$.

2.1. **Quadratic forms and the min-max principle.** Let $\mathcal{H}$ be a (real or complex) Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We call a symmetric densely defined bilinear form $\mathcal{E}$ in $\mathcal{H}$ a *quadratic form* in $\mathcal{H}$. A quadratic form $(\mathcal{E},\operatorname{dom}\mathcal{E})$ is called *semi-bounded below* if there exists some constant $M\geq 0$ such that

$$\mathcal{E}(u,u)\geq -M\|u\|^2 \qquad \text{for all } u\in\operatorname{dom}\mathcal{E}, \tag{2.1}$$

and that $(\mathcal{E},\operatorname{dom}\mathcal{E})$ is *non-negative* if one may take $M=0$ in (2.1) (see [31]). Then a quadratic form $(\mathcal{E},\operatorname{dom}\mathcal{E})$ is said to be *closed* if it is semi-bounded below and $(\mathcal{E}_{M+1},\operatorname{dom}\mathcal{E})$ is a Hilbert space, where $\mathcal{E}_{M+1}(u,v):=\mathcal{E}(u,v)+(M+1)(u,v)$ for all $u,v\in\operatorname{dom}\mathcal{E}$.

A self-adjoint operator A in $\mathcal{H}$ is said to be *semi-bounded below* if there exists some constant $C\geq 0$ such that $(Au,u)\geq -C\|u\|^2$ for all $u\in\operatorname{dom}A$. It is well known that if a quadratic from $(\mathcal{E},\operatorname{dom}\mathcal{E})$ is closed, then there exists a unique self-adjoint operator A, called the *generator* of $(\mathcal{E},\operatorname{dom}\mathcal{E})$, that is semi-bounded below, such that $\operatorname{dom}A\subseteq\operatorname{dom}\mathcal{E}$, and

$$\mathcal{E}(u,v)=(Au,v) \qquad \text{for all } u\in\operatorname{dom}A \text{ and } v\in\operatorname{dom}\mathcal{E}$$

(see [10, Section 1.3]). On the other hand, any self-adjoint operator $(A,\operatorname{dom}A)$ in $\mathcal{H}$ determines a quadratic form $(\mathcal{E},\operatorname{dom}A)$ by $\mathcal{E}(u,v):=(Au,v)$ for all $u,v\in\operatorname{dom}A$. Moreover, if A is semi-bounded below, then $(\mathcal{E},\operatorname{dom}A)$ is closable, and its closure $(\mathcal{E},\operatorname{dom}\mathcal{E})$ is called the *closed quadratic form associated with* A. We let $Q(A):=\operatorname{dom}\mathcal{E}$ and call it the *form domain of* A. Furthermore, if A is non-negative, then $A^{1/2}$ is well-defined; moreover,

$$\mathcal{E}(u,v)=\big(A^{1/2}u,A^{1/2}v\big) \quad\text{ and }\quad \operatorname{dom}\mathcal{E}=\operatorname{dom}(A^{1/2})$$

(see, e.g., [10, Theorem 1.3.1]). Moreover, for $u\in\operatorname{dom}\mathcal{E}$, we have $u\in\operatorname{dom}A$ if and only if there exists a unique $f\in\mathcal{H}$ such that $\mathcal{E}(u,v)=\big(f,v\big)$ for all $v\in\operatorname{dom}\mathcal{E}$. In this case, $Au=f$.

For $i=1,2$, let $(\mathcal{E}_i,\operatorname{dom}\mathcal{E}_i)$ be a closed quadratic form in a Hilbert space $\mathcal{H}$ with generator A_i. If $\operatorname{dom}\mathcal{E}_1\cap\operatorname{dom}\mathcal{E}_2$ is dense in $\mathcal{H}$, then we denote the generator of the closure of $(\mathcal{E}_1+\mathcal{E}_2,\operatorname{dom}\mathcal{E}_1\cap\operatorname{dom}\mathcal{E}_2)$ by A_1+A_2, and say that A_1+A_2 is an operator *defined as a sum of quadratic forms.*

Let (X,μ) be a measure space with a σ-finite measure μ. For any $V\in L^1_{\text{loc}}(X,\mu)$, the quadratic form $\mathcal{E}_V$ given by

$$\mathcal{E}_V(u,v)=\int_X uvV\,d\mu \qquad \text{for all } u,v\in C_c^\infty(X),$$

is closable on $L^2(X,\mu)$. In this case, we denote the closure of $(\mathcal{E}_V, C_c^\infty(X))$ by $(\mathcal{E}_V, \operatorname{dom}\mathcal{E}_V)$ and regard V as the generator (see [5, 31]).

We now state the min-max principle for self-adjoint operators that are semi-bounded below (see, e.g., [30]).

Theorem 2.1. ([30, Theorems XIII. 1 and XIII.2]) *Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Assume that A is semi-bounded below and let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be the closed quadratic form associated with A. Define*

$$\lambda_n(A) := \sup_{u_1,\dots,u_{n-1}} \inf_{\substack{u\in\operatorname{dom}\mathcal{E}, \|u\|=1\\ (u,u_i)=0,\, i=1,\dots,n-1}} \mathcal{E}(u,u) \qquad \text{for } n \geq 1. \tag{2.2}$$

Note that the u_i are not necessarily independent. Then for each fixed n, either

(a) *there are n eigenvalues (counting multiplicity) below the bottom of the essential spectrum, and $\lambda_n(A)$ is the n-th eigenvalue counting multiplicity;*

or

(b) *$\lambda_n(A)$ is the bottom of the essential spectrum, $\lambda_n(A) = \lambda_{n+1}(A) = \lambda_{n+2}(A) = \cdots$, and there are at most $n-1$ eigenvalues (counting multiplicity) below $\lambda_n(A)$.*

Definition 2.2. *For $i = 1, 2$, let A_i be a self-adjoint operator in a Hilbert space $\mathcal{H}_i$ that is semi-bounded below, and $(\mathcal{E}_i, \operatorname{dom}\mathcal{E}_i)$ be the associated closed quadratic form. We say $A_1 \leq A_2$* (in the sense of quadratic forms) *if $\mathcal{H}_2 \subseteq \mathcal{H}_1$, $\operatorname{dom}\mathcal{E}_2 \subseteq \operatorname{dom}\mathcal{E}_1$, and $\mathcal{E}_1(u,u) \leq \mathcal{E}_2(u,u)$ for all $u \in \operatorname{dom}\mathcal{E}_2$.*

We state a simple proposition. A proof can be found in [30, Section XIII].

Proposition 2.3. *For $i = 1, 2$, let A_i be a self-adjoint operator in a Hilbert space $\mathcal{H}_i$ that is semi-bounded below. Assume $A_1 \leq A_2$. If A_1 has compact resolvent, then so does A_2; moreover, $N(\lambda, A_1) \geq N(\lambda, A_2)$ for all $\lambda \in \mathbb{R}$.*

2.2. **Dirichlet Laplacian defined by a measure.** For convenience, we summarize the definition of the Dirichlet Laplacian on a bounded domain defined by a measure; details can be found in [13]. Let $U \subseteq \mathbb{R}^n$ be a bounded open subset and μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. We assume that μ satisfies (MPI) (see (1.5)). (MPI) implies that each equivalence class $u \in H_0^1(U)$ contains a unique (in the $L^2(U,\mu)$ sense) member $\hat{u}$ that belongs to $L^2(U,\mu)$ and satisfies both conditions below:

(1) there exists a sequence $\{u_n\}$ in $C_c^\infty(U)$ such that $u_n \to \hat{u}$ in $H_0^1(U)$ and $u_n \to \hat{u}$ in $L^2(U,\mu)$;
(2) $\hat{u}$ satisfies inequality (1.5).

We call $\hat{u}$ the $L^2(U,\mu)$-*representative* of u. Define a mapping $\iota : H_0^1(U) \to L^2(U,\mu)$ by $\iota(u) = \hat{u}$. ι is a bounded linear operator, but not necessarily injective. Consider the subspace $\mathcal{N}$ of $H_0^1(U)$ defined as $\mathcal{N} := \{u \in H_0^1(U) : \|\iota(u)\|_{2,\mu} = 0\}$. Now let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $H_0^1(U)$. Then $\iota : \mathcal{N}^\perp \to L^2(U,\mu)$ is injective. Unless explicitly stated otherwise, we will denote the $L^2(U,\mu)$-representative $\hat{u}$ simply by u.

Consider the non-negative bilinear form $\mathcal{E}_D(\cdot,\cdot)$ in $L^2(U,\mu)$ given by

$$\mathcal{E}_D(u,v) := \int_U \nabla u \cdot \nabla v \, dx \tag{2.3}$$

with domain $\operatorname{dom}\mathcal{E}_D = \mathcal{N}^\perp$, or more precisely, $\iota(\mathcal{N}^\perp)$. (MPI) implies that $(\mathcal{E}_D, \operatorname{dom}\mathcal{E}_D)$ is a non-negative, closed quadratic form in $L^2(U,\mu)$. We use $-\Delta_\mu^D$ (or simply $-\Delta_\mu$) denote the generator of $(\mathcal{E}_D, \operatorname{dom}\mathcal{E}_D)$, and call it the *(Dirichlet) Laplacian* with respect to μ.

Some sufficient conditions for (MPI) and the existence of an orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2(U,\mu)$ consisting of the eigenfunctions of $-\Delta_\mu$ can be found in [6, 13, 26]. We remark that if $n = 1$, then (MPI) holds for any such μ, and thus Δ_μ is well-defined; moreover, $-\Delta_\mu$ has compact resolvent.

2.3. **Neumann Laplacian defined by a measure.** By assuming some regularity conditions of the boundary of U, one can define a Neumann Laplacian in a similar fashion. We state a result below that is sufficient for the purpose of this paper. Laplacians with more general boundary conditions are studied in a forthcoming paper by Lau and the first author [17].

Let U be a bounded open subset of $\mathbb{R}^n$. We say that $f \in C^\infty(\overline{U})$ if $f \in C^\infty(U)$ and all of whose partial derivatives can be extended continuously to $\overline{U}$. Suppose U is a bounded open subset in $\mathbb{R}^n$ that has the extension property. Then $C^\infty(\overline{U})$ is dense in $H^1(U)$. All bounded regions in $\mathbb{R}^n$ with piecewise smooth or Lipschitz boundaries have the extension property. Let μ be a finite positive Borel measure on U with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. The following analog of (MPI), which we call *Poincaré inequality* for measures (MPI*)* is crucial: There exists a constant $C > 0$ such that

$$\int_U |u|^2 \, d\mu \le C \left(\int_U |\nabla u|^2 \, dx + \int_U |u|^2 \, dx \right) \qquad \text{for all } u \in C^\infty(\overline{U}). \tag{2.4}$$

We remark that (MPI*) is stronger than (MPI). Similarly, (MPI*) implies that each equivalence class $u \in H^1(U)$ contains a unique (in the $L^2(U,\mu)$ sense) member $\hat{u}$ that belongs to $L^2(U,\mu)$ and satisfies both conditions below:

(1) there exists a sequence $\{u_n\}$ in $C^\infty(\overline{U})$ such that $u_n \to \hat{u}$ in $H^1(U)$ and $u_n \to \hat{u}$ in $L^2(U,\mu)$;
(2) $\hat{u}$ satisfies inequality (2.4).

In view of Subsection 2.2, we need one additional inequality, namely, *Poincaré inequality (PI)*, i.e., there exists some constant $C' > 0$ such that

$$\int_U |u - u^*|^2 \, dx \le C' \int_U |\nabla u|^2 \, dx \qquad \text{for all } u \in H^1(U),$$

where $u^* := (1/\mathcal{L}^n(U)) \cdot \int_U u \, dx$ (see, e.g., [23, Theorem 8.11]).

Define a quadratic form $\mathcal{E}_N(\cdot,\cdot)$ in $L^2(U,\mu)$ by

$$\mathcal{E}_N(u,v) := \int_U \nabla u \cdot \nabla v \, dx,$$

with domain $\operatorname{dom}\mathcal{E}_N := \iota(\mathcal{N}^\perp)$, where $\iota : H^1(U) \to L^2(U,\mu)$ and $\mathcal{N}$ are analogues of those in Subsection 2.2. (MPI*) and (PI) imply that $(\mathcal{E}_N, \operatorname{dom}\mathcal{E}_N)$ is a non-negative closed quadratic form in $L^2(U,\mu)$ (see [13,17]). We denote the generator of $(\mathcal{E}_N, \operatorname{dom}\mathcal{E}_N)$ by $-\Delta_\mu^N$, and call it the *Neumann Laplacian* with respect to μ. We remark that $-\Delta_\mu^N \le -\Delta_\mu$.

Some sufficient conditions for the existence of an orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2(U,\mu)$ consisting of the eigenfunctions of $-\Delta_\mu^N$ can be found in [17]. We remark that if $n = 1$ and $U = (a,b)$, then (MPI*) holds for any such μ and (PI) holds, and thus Δ_μ^N is well-defined. Moreover, Δ_μ^N has compact resolvent.

3. Fractal analog of a semi-classical asymptotic formula for the number of bound states

In this section, we prove Theorem 1.1 and illustrate it by a class of self-similar measures with overlaps.

3.1. **Proof of Theorem 1.1.**

Let μ be a positive finite Borel measure on a compact subset $X \subseteq \mathbb{R}^n$ with $\operatorname{supp}(\mu) \subseteq X$ and $\mu(X^\circ) > 0$. We call a μ-measurable closed subset B of X a *cell (in X)* if $\mu(B^\circ) > 0$. Clearly, X itself is a cell. We say that two cells B and B' are *μ-equivalent*, denoted by $B \simeq_{\mu,\tau,w} B'$ (or simply $B \simeq_\mu B'$), if there exist some similitude $\tau : B \to B'$ of the form $\tau(x) = rx + b$, $r > 0, b \in \mathbb{R}^n$, and some constant $w > 0$ such that $\tau(B) = B'$ and

$$\mu|_{B'} = w \cdot \mu|_B \circ \tau^{-1}. \tag{3.5}$$

It is easy to check that $\simeq_\mu$ is an equivalence relation.

We call a finite family $\mathcal{P}$ of cells a *μ-partition* of X if any two distinct cells in $\mathcal{P}$ have disjoint interiors, and $\mu(X) = \sum_{B\in\mathcal{P}} \mu(B)$.

Definition 3.1. *Let $(\mathcal{P}_k)_{k\geq 1}$ be a sequence μ-partitions of X. We say that $(\mathcal{P}_k)_{k\geq 1}$ satisfies* condition (N) *if for each $k \geq 1$, $\mathcal{L}^n(X^\circ \setminus (\bigcup_{B\in\mathcal{P}_k} B^\circ)) = 0$, and the following conditions are satisfied: for each $B \in \bigcup_{k=1}^\infty \mathcal{P}_k$,*

(1) *B° has the extension property and satisfies (PI);*
(2) *$\mu|_B$ satisfies (MPI*);*
(3) *$-\Delta^N_{\mu|_B}$ has compact resolvent.*

Condition (N) ensures that the Dirichlet-Neumann bracketing technique can be used in the proof of Theorem 1.1 The following theorem from [17] gives a sufficient condition for condition (N). For the definition of $\underline{\dim}_\infty(\mu)$ and other unexplained terms, see [13, 38].

Theorem 3.2. ([17]) *Let $U \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded domain that has the cone property. Assume that U has the uniform cone property or ∂U is minimally smooth. Let μ be a finite positive Borel measure on U with* $\operatorname{supp}(\mu) \subseteq \overline{U}$ *and $\mu(U) > 0$. Assume that $\underline{\dim}_\infty(\mu) > n-2$. Then $B := \overline{U}$ satisfies conditions (1)–(3) in Definition 3.1.*

Consequently, $(\mathcal{P}_k)_{k\geq 1}$ satisfies condition (N) if for each $k \geq 1$, $\mathcal{L}^n(X^\circ \setminus (\bigcup_{B\in\mathcal{P}_k} B^\circ)) = 0$, and for each $B \in \bigcup_{k=1}^\infty \mathcal{P}_k$, $U := B^\circ$ satisfies the hypotheses of Theorem 3.2.

Let $(\mathcal{P}_k)_{k\geq 1}$ be a sequence of μ-partitions of X, and let ν be a positive finite Borel measure on $X \subseteq \mathbb{R}^n$. For each $k \geq 1$, let $\overline{m}_k = \overline{m}_k(\mathcal{P}_k) := \max\{\nu(B) : B \in \mathcal{P}_k\}$. We say that $(\mathcal{P}_k)_{k\geq 1}$ is *refining* with respect to ν if it satisfies the following conditions:

(1) $\{\overline{m}_k\}$ is nonincreasing and $\lim_{k\to\infty} \overline{m}_k = 0$;
(2) for any $B \in \mathcal{P}_k$ and any $B' \in \mathcal{P}_{k+1}$, either $B' \subseteq B$ or $(B')^\circ \cap B^\circ = \emptyset$.

Condition (2) means that each member of $\mathcal{P}_{k+1}$ is a subset of some member of $\mathcal{P}_k$.

We now prove Theorem 1.1 by modifying a method in [30, Theorem XIII 79].

Proof of Theorem 1.1. Since X is compact and V is continuous, $-\Delta_\mu + \beta V$ has discrete spectrum on the negative real line for any $\beta > 0$, i.e., $N^-_\mu(\beta V)$ is finite for any $\beta > 0$. $N^-_\mu(\beta V) \leq N^-_\mu(\beta V_{\min}) = N(-\beta V_{\min}, -\Delta_\mu) < \infty$, where $V_{\min} := \min\{V(x) : x \in X\}$. For each $k \geq 1$, let $\mathcal{P}_k := \{B_{k,\ell}\}_{\ell\in\Pi_k}$ and define $V_k^\vee$ (resp., $V_k^\wedge$) to be the piecewise constant function over each $B_{k,\ell}$ with the value $V_{k,\ell}^\vee := \min\{V(x) : x \in B_{k,\ell}\}$ (resp., $V_{k,\ell}^\wedge := \max\{V(x) : x \in B_{k,\ell}\}$).

(a) For $k \geq 1$, let $-\Delta_\mu^k$ be the Dirichlet Laplacian on the union of the interiors of the cells in $\mathcal{P}_k$. Since $C_c^\infty(\bigcup_{B\in\mathcal{P}_k} B^\circ) \subseteq C_c^\infty(X^\circ)$, we have $-\Delta_\mu \leq -\Delta_\mu^k$ for $k \geq 1$. Combining this inequality with $V \leq V_k^\wedge$, we have $-\Delta_\mu + \beta V \leq -\Delta_\mu^k + \beta V_k^\wedge$ for $k \geq 1$. It follows from

Proposition 2.3 that for all $k \geq 1$,

$$\begin{aligned} N_\mu^-(\beta V) &\geq N(0, -\Delta_\mu^k + \beta V_k^\wedge) = \sum_{\ell \in \Pi_k} N(0, -\Delta_{\mu|_{B_{k,\ell}}} + \beta V_{k,\ell}^\wedge) \\ &= \sum_{\ell \in \Pi_k} N(-\beta V_{k,\ell}^\wedge, -\Delta_{\mu|_{B_{k,\ell}}}) = \sum_{\{\ell \in \Pi_k : V_{k,\ell}^\wedge \leq 0\}} N(-\beta V_{k,\ell}^\wedge, -\Delta_{\mu|_{B_{k,\ell}}}). \end{aligned} \tag{3.6}$$

Combining (1.6) and (3.6) yields, for each $k \geq 1$,

$$N_\mu^-(\beta V) \geq \beta^{\alpha/2}\Big(C \sum_{\{\ell \in \Pi_k : V_{k,\ell}^\wedge \leq 0\}} \big(-V_{k,\ell}^\wedge\big)^{\alpha/2} \nu(B_{k,\ell}) + o(1)\Big), \qquad \text{as } \beta \to +\infty. \tag{3.7}$$

The definition of refining implies that $\lim_{k\to\infty} \max\{\nu(B_{k,\ell}) : \ell \in \Pi_k\} = 0$. Moreover, it follows from the continuity of V that

$$\lim_{k\to\infty} \sum_{\{\ell \in \Pi_k : V_{k,\ell}^b \leq 0\}} \big(-V_{k,\ell}^b\big)^{\alpha/2} \nu(B_{k,\ell}) = \int_{D^-(V)} (-V)^{\alpha/2}\, d\nu \qquad \text{for } b \in \{\vee, \wedge\}, \tag{3.8}$$

which, together with (3.7), yields the desired inequality.

(b) The proof is similar to that of part (a). Since $(\mathcal{P}_k)_{k\geq 1}$ satisfies condition (N), the Neumann Laplacian $-\Delta_\mu^{k,N}$ is well defined on the union of the interiors of the cells in $\mathcal{P}_k$ all $k \geq 1$. Let $u \in C_c^\infty(X^\circ)$. Then the restriction of u to $\bigcup_{B\in\mathcal{P}_k} B^\circ$ is in $\bigoplus_{B\in\mathcal{P}_k} C^\infty(B)$. Moreover, since $\mathcal{L}^n(X^\circ \setminus (\bigcup_{B\in\mathcal{P}_k} B^\circ)) = 0$,

$$\int_{X^\circ} |\nabla u|^2\, dx = \sum_{B\in\mathcal{P}_k} \int_{B^\circ} |\nabla u|^2\, dx.$$

It follows that $-\Delta_\mu^{k,N} \leq -\Delta_\mu$ for $k \geq 1$. Hence $-\Delta_\mu^{k,N} + \beta V_k^\vee \leq -\Delta_\mu + \beta V$, which, together with Proposition 2.3, yields

$$\begin{aligned} N_\mu^-(\beta V) &\leq N(0, -\Delta_\mu^{k,N} + \beta V_k^\vee) = \sum_{\ell \in \Pi_k} N(0, -\Delta^N_{\mu|_{B_{k,\ell}}} + \beta V_{k,\ell}^\vee) \\ &= \sum_{\ell \in \Pi_k} N(-\beta V_{k,\ell}^\vee, -\Delta^N_{\mu|_{B_{k,\ell}}}) = \sum_{\{\ell \in \Pi_k : V_{k,\ell}^\vee \leq 0\}} N(-\beta V_{k,\ell}^\vee, -\Delta^N_{\mu|_{B_{k,\ell}}}) \qquad \text{for all } k \geq 1. \end{aligned}$$

Thus (1.8) implies the following analogue of (3.7), which holds for all $k \geq 1$,

$$N_\mu^-(\beta V) \leq \beta^{\alpha/2}\Big(C \sum_{\{\ell \in \Pi_k : V_{k,\ell}^\vee \leq 0\}} \big(-V_{k,\ell}^\vee\big)^{\alpha/2} \nu(B_{k,\ell}) + o(1)\Big), \qquad \text{as } \beta \to +\infty.$$

Hence, the assertion follows from (3.8). □

It is well-known that if $B \subseteq X$ is a closed interval on $\mathbb{R}$, then $N(\lambda, -\Delta_\mu) \le N(\lambda, -\Delta_\mu^N) \le N(\lambda, -\Delta_\mu) + 2$ for all $\lambda \ge 0$ (see, e.g., [28]). Thus $N(\lambda, -\Delta_\mu)$ and $N(\lambda, -\Delta_\mu^N)$ have the same asymptotically behavior as $\lambda \to \infty$. Consequently, the result in the following remark holds.

Remark 3.3. *Let $X = [a, b]$. If there exist positive constants C and α, and a refining μ-partition $(\mathcal{P}_k)_{k\ge1}$ of X such that all sets in $\bigcup_{k=1}^\infty \mathcal{P}_k$ are closed interval, and* (1.6) *holds with the inequality being reversed, then the conclusion of Theorem 1.1(b) holds.*

Let (X, μ) be a measure space with μ being σ-finite, and let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be a non-negative closed quadratic form with generator A. Levin-Solomyak (see [20, Theorem 1.2]) proved that if following Sobolev's inequality holds for some $q > 2$: there exists some constant $C > 0$ such that $\|u\|_{q,\mu}^2 \le C\mathcal{E}(u, u)$ for all $u \in \operatorname{dom}\mathcal{E}$, then the following *general Cwikel-Lieb-Rosenbljum (CLR) inequality* holds:

$$N(0, A - \beta V) \le e^p C^p \beta^p \int_X V^p \, d\mu, \qquad \text{for all } \beta > 0, \tag{3.9}$$

where $0 \le V \in L^p(X, \mu)$ and $p := q/(q-2) > 1$. In the case $X = \mathbb{R}^n$, $n \ge 3$, μ is Lebesgue measure on $\mathbb{R}^n$, and the generator A is the Dirichlet Laplace $-\Delta$ on $\mathbb{R}^n$, then (3.9) holds with $p = n/2$ and $C^{-1} = \big(n(n-2)/4\big)^{n/2}\omega_{n-1}$, where ω_{n-1} is the volume of the unit $(n-1)$-sphere in $\mathbb{R}^n$. In this case, (3.9) is called the *classical (CLR) inequality* (see [3, 21, 22, 30, 32]).

We give a simple corollary for the general (CLR) inequality (3.9).

Corollary 3.4. *Let μ be a continuous Borel probability measure in $\mathbb{R}$ with $\operatorname{supp}(\mu) \subseteq [a, b]$, and $(\mathcal{E}_D, \operatorname{dom}\mathcal{E}_D)$ be defined as in* (2.3). *Assume $0 \le V \in L^p((a, b), \mu)$ for some $p > 1$. Then*

$$N(0, -\Delta_\mu - \beta V) \le e^p |b-a|^p \beta^p \int_a^b V^p \, d\mu, \qquad \textit{for all } \beta > 0.$$

Proof. For all $u \in H_0^1(a, b)$ and $x \in [a, b]$,

$$\big|u(x)\big| = \big|u(x) - u(a)\big| = \Big|\int_a^x u'(t)\, dt\Big| \le |b-a|^{1/2} \mathcal{E}_D(u, u)^{1/2}.$$

It follows that for all $q > 0$,

$$\Big(\int_a^b |u(x)|^q \, d\mu\Big)^{2/q} \le |b-a| \mathcal{E}_D(u, u),$$

and thus Sobolev's inequality holds with $C := |b - a|$. Setting $q := 2p/(p-1)$. Thus using the discussion above or [20, Theorem 1.2], the desired inequality holds.

□

We remark that Theorem 1.1(b) does not follow from [20, Theorem 1.2], which requires the Sobolev's inequality. For $n = 1$, Corollary 3.4 implies that the general (CLR) inequality (3.9) holds for all $p > 1$. However, Theorem 1.1(b) does not follow from [20, Theorem 1.2] in this case either, since the constant α in Theorem 1.1(b), which corresponds to the constant p in the general (CLR) inequality (3.9), could be less than or equals 1. Precisely, we would like to have $\alpha = d_s(-\Delta_\mu) \le 1$, if $d_s(-\Delta_\mu)$ exists and $n = 1$, as in our examples below.

We now apply Theorem 1.1 to a class of self-similar measures on $\mathbb{R}$ satisfying (OSC). For the convenience of the reader, we first state a slightly modified version of [28, Proposition 2.2(b)] below.

Proposition 3.5. ([28, Proposition 2.2]) *Let $S : \mathbb{R} \to \mathbb{R}$ be a similitude, with Lipschitz constant r, such that $S[a, b] = [c, d]$, $S(a) = c$, and $S(b) = d$. Let ν be a continuous positive finite Borel measure on $[a, b]$ with $\operatorname{supp}(\nu) \subseteq [a, b]$. Assume that $[a, b] \simeq_{\nu,w,\tau} [c, d]$. Then $-\Delta_{\nu|_{[c,d]}} \approx (r_\tau w)^{-1} \cdot \big(- \Delta_{\nu|_{[a,b]}}\big)$, where r_τ is the contraction ratio of τ.*

Let $\{S_i\}_{i=1}^m$, $m \ge 2$, be an IFS on $\mathbb{R}$ satisfying (OSC) with respect to an open set (a, b), and let μ be a self-similar measure defined by $\{S_i\}_{i=1}^m$ and a probability vector $(p_i)_{i=1}^m$. Let $X = [a, b]$, and d_s be the unique solution of

$$\sum_{i=1}^{m} (p_i r_i)^{d_s/2} = 1,$$

where r_i is the contraction ratio of S_i. Solomyak and Verbitsky [33] have studied the asymptotic behavior of the eigenvalue counting function $N(\lambda, -\Delta_\mu)$ as $\lambda \to +\infty$. They proved that there exist some positive constants C_1, C_2 such that

$$C_1 \lambda^{d_s/2} \le N(\lambda, -\Delta_\mu) \le C_2 \lambda^{d_s/2} \qquad \text{as } \lambda \to +\infty. \tag{3.10}$$

In particular, for $c_k := \ln(r_k p_k)$, if at least one of the ratios c_k/c_ℓ is irrational, where $k, \ell \in \{1, \dots, m\}$, then there exists some constant $C > 0$ such that $N(\lambda, -\Delta_\mu) \sim C\lambda^{d_s/2}$. The same holds for the Neumann Laplacian for with the same constant C. We note that $d_s = d_s(-\Delta_\mu)$.

Proposition 3.6. *Use the notation above and assume that $\{S_i\}_{i=1}^m$ satisfies (OSC). Let ν be the self-similar measure defined by the probability vector $((p_i r_i)^{d_s/2})_{i=1}^m$. Then for all continuous functions V on X,*

(a) *there exist positive constants C_1, C_2 such that, as $\beta \to +\infty$,*

$$C_1 \int_{D^-(V)} \big(-V\big)^{d_s/2}\, d\nu \le \frac{N_\mu^-(\beta V)}{\beta^{d_s/2}} \le C_2 \int_{D^-(V)} \big(-V\big)^{d_s/2}\, d\nu. \tag{3.11}$$

(b) *if, in addition, at least one of the ratios c_k/c_ℓ is irrational for any distinct $k, \ell \in \{1, \dots, m\}$, then one may take $C_1 = C_2$ in* (3.11).

Proof. Using the discussion above, we see that (b) follows from (a). Thus, we only prove (a). For $k \ge 1$, define $\mathcal{P}_k := \big\{S_{\boldsymbol{i}}([a,b]) : \boldsymbol{i} \in \{1, \dots, m\}^k\big\}$. It is easy to see that $(\mathcal{P}_k)_{k\ge1}$ is a refining μ-partition of $[a,b]$ with respect to ν, and all cells in $\bigcup_{k\ge1} \mathcal{P}_k$ are closed interval. Thus by Theorem 1.1 and Remark 3.3, it suffices to show that for all $k \ge 1$ and $\boldsymbol{i} \in \{1, \dots, m\}^k$,

$$C_1 \nu\big(S_{\boldsymbol{i}}([a,b])\big)\lambda^{d_s/2} \le N\big(\lambda, -\Delta_{\mu|_{S_{\boldsymbol{i}}([a,b])}}\big) \le C_2 \nu\big(S_{\boldsymbol{i}}([a,b])\big)\lambda^{d_s/2}, \qquad \text{as } \lambda \to +\infty. \tag{3.12}$$

Fix any $k \ge 1$ and any $\boldsymbol{i} \in \{1, \dots, m\}^k$. (OSC) implies that $\mu|_{S_{\boldsymbol{i}}([a,b])} = p_{\boldsymbol{i}} \mu|_{[a,b]} \circ S_{\boldsymbol{i}}^{-1}$ on $S_{\boldsymbol{i}}([a,b])$. It follows that $[a,b] \simeq_{\mu, p_{\boldsymbol{i}}, S_{\boldsymbol{i}}} S_{\boldsymbol{i}}([a,b])$ and $\mu(S_{\boldsymbol{i}}([a,b])) = p_{\boldsymbol{i}}$. In the view of Proposition 3.5, we get $N(\lambda, -\Delta_{\mu|_{S_{\boldsymbol{i}}([a,b])}}) = N(r_{\boldsymbol{i}} p_{\boldsymbol{i}} \lambda, -\Delta_\mu)$, where $r_{\boldsymbol{i}}$ is the contraction ratio of $S_{\boldsymbol{i}}$. Combining this with (3.10), there exist some positive constants C_1, C_2 such that

$$C_1 (r_{\boldsymbol{i}} p_{\boldsymbol{i}})^{d_s/2} \lambda^{d_s/2} \le N(\lambda, -\Delta_{\mu|_{S_{\boldsymbol{i}}([a,b])}}) \le C_2 (r_{\boldsymbol{i}} p_{\boldsymbol{i}})^{d_s/2} \lambda^{d_s/2}, \qquad \text{as } \lambda \to +\infty.$$

Since $(r_{\boldsymbol{i}} p_{\boldsymbol{i}})^{d_s/2} = \nu(S_{\boldsymbol{i}}([a,b]))$, (3.12) follows, which completes the proof. $\square$

3.2. **A class of self-similar measures satisfying (EFT).** In the rest of this section, we consider the following family of IFSs:

$$S_1(x) = r_1 x, \quad S_2(x) = r_2 x + r_1(1 - r_2), \quad S_3(x) = r_2 x + 1 - r_2, \tag{3.13}$$

where the contraction ratios $r_1, r_2 \in (0,1)$ satisfy $r_1 + 2r_2 - r_1 r_2 \le 1$, i.e., $S_2(1) \le S_3(0)$. The Hausdorff dimension of the self-similar sets is computed in [18]. The multifractal properties and spectral dimension of the corresponding self-similar measures are recently studied in [7, 29].

Let μ be a self-similar measure defined by an IFS in (3.13) and a probability vector $(p_i)_{i=1}^3$, and $-\Delta_\mu$ be the associated Dirichlet Laplacian with respect to μ. We note that $\operatorname{supp}(\mu) = X := [0,1]$. Let d_s be the unique solution of

$$\big(1 - (p_2 r_2)^{d_s/2}\big)\big(1 - (p_3 r_2)^{d_s/2}\big) \sum_{k=0}^{\infty} \big(w_1(k) r_1 r_2^k\big)^{d_s/2} + \big(p_2^{d_s/2} + p_3^{d_s/2}\big) r_2^{d_s/2} = 1, \tag{3.14}$$

where $w_1(k) := p_1 \sum_{i=0}^{k} p_2^{k-i} p_3^i$. [29, Theorem 1.2] implies that there exist some positive constants C_1, C_2 such that, for $k = 0, 1$,

$$C_1 \lambda^{d_s/2} \leq N(\lambda, -\Delta_{\mu|_{B_{1,k}}}) \leq C_2 \lambda^{d_s/2}, \qquad \text{as } \lambda \to +\infty, \tag{3.15}$$

where $B_{1,1} := S_1(X) \bigcup S_2(X)$ and $B_{1,0} := S_3(X)$. In particular, $d_s = d_s(-\Delta_\mu)$.

In order to define a sequence of refining μ-partitions of $[0,1]$ with respect to μ, we adopt the definition of an island from [29]. Let $\mathcal{M}_k := \{1,2,3\}^k$ for $k \geq 1$ and $\mathcal{M}_0 := \emptyset$. A closed subset $B \subseteq [0,1]$ is called a *level-k island* with respect to $\{\mathcal{M}_k\}$ if the following conditions hold:

(1) there exists a finite sequence of indexes $\boldsymbol{i}_0, \boldsymbol{i}_1, \ldots, \boldsymbol{i}_n$ in $\mathcal{M}_k$ such that $S_{\boldsymbol{i}_k}(0,1) \cap S_{\boldsymbol{i}_{k+1}}(0,1) \neq \emptyset$ for all $k = 0, \ldots, n-1$, and $B = \bigcup_{k=0}^{n} S_{\boldsymbol{i}_k}([0,1])$;

(2) for any $\boldsymbol{j} \in \mathcal{M}_k \setminus \{\boldsymbol{i}_0, \ldots, \boldsymbol{i}_n\}$ and any $k \in \{0, \ldots, n\}$, $S_{\boldsymbol{j}}(0,1) \cap S_{\boldsymbol{i}_k}(0,1) = \emptyset$.

Intuitively, for each level-k island B, B° is a connected component of $S_{\mathcal{M}_k}(0,1) := \bigcup_{\boldsymbol{i} \in \mathcal{M}_k} S_{\boldsymbol{i}}(0,1)$ (see Figure 1). For $k \geq 1$, define

$$\mathcal{P}_k := \big\{ B : B \text{ is a level-}k \text{ island with respect to } \{\mathcal{M}_k\} \ \big\}. \tag{3.16}$$

We note that $\mathcal{P}_1 = \{B_{1,1}, B_{1,0}\}$ (see Figure 1). It is easy to see that $(\mathcal{P}_k)_{k \geq 1}$ is a sequence of μ-partitions of $[0,1]$. By the proof of [29, Example 3.3], $(\mathcal{P}_k)_{k \geq 1}$ is refining with respect to μ; moreover, for any $k \geq 1$ and any $B \in \mathcal{P}_k$, we can find a unique sequence of μ-partitions $(\mathcal{B}_\ell)_{\ell \geq 1}$ of B satisfying the following conditions:

(i) $\bigcup_{\ell \geq 1} \mathcal{B}_\ell \subseteq \bigcup_{k \geq 1} \mathcal{P}_k$;

(ii) for each $\ell \geq 1$, there exists a unique cell $B_\ell^* \in \mathcal{B}_\ell$ such that B_ℓ^* is not μ-equivalent to any $B_{1,i}$, $i = 0, 1$, and $\mu(B_\ell^*) \to 0$ as $\ell \to \infty$;

(iii) $\mathcal{B}_1 \subseteq \mathcal{P}_{k+1}$, $\mathcal{B}_{\ell+1} \setminus \mathcal{B}_\ell \subseteq \mathcal{P}_{k+\ell+1}$ and $\mathcal{B}_\ell \setminus \{B_\ell^*\} \subseteq \mathcal{B}_{\ell+1}$ for all $\ell \geq 1$.

Notice that $(\mathcal{B}_\ell)_{\ell \geq 1}$ is not refining with respect to μ, since condition (1) in the definition of refining fails. For example, define

$$\mathcal{B}_\ell := \big\{ S_{2^{i-1}1}(B_{1,1}) : 1 \leq i \leq \ell \big\} \bigcup \big\{ S_{2^i(B_{1,0})} : 1 \leq i \leq \ell \big\} \bigcup \big\{ S_{2^\ell}(B_{1,1}) \big\} \qquad \text{for all } \ell \geq 1.$$

Then $(\mathcal{B}_\ell)_{\ell \geq 1}$ is a sequence of μ-partitions of $B_{1,1}$ such that all conditions above hold with $B_\ell^* = S_{2^\ell}(B_{1,1})$ (see Figure 1).

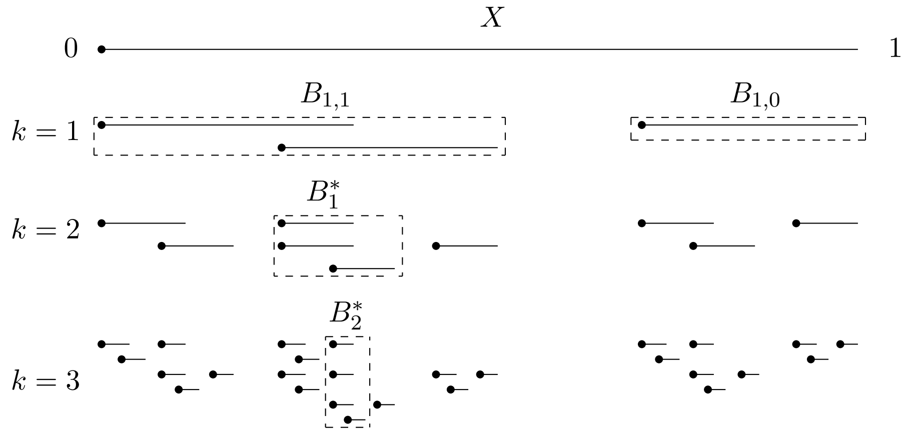

Figure 1. μ-partitions $\mathcal{P}_k$ and $B_\ell^* := S_{2^\ell}(B_{1,1})$ for $k = 1, 2, 3$ and $\ell = 1, 2$, where $\mathcal{P}_k$ is defined as in (3.16). Cells that are labeled consist of line segments enclosed by a box. The figure is drawn with $r_1 = 1/3$ and $r_2 = 2/7$.

Proposition 3.7. *Use the notation above. Let ν be a positive finite Borel measure on $\mathbb{R}$ and assume that $\max\{\nu(B) : B \in \mathcal{P}_k\} \to 0$ as $k \to \infty$. Let d_s be defined as in (3.14) and let $\mathcal{P}_* := \{B \in \mathcal{P}_k : k \geq 1$ and $B \simeq_\mu B_{1,i}$ for some $i \in \{0, 1\}\}$.*

(a) *If there exists some constant $c > 0$ such that*

$$\big(|B|\,\mu(B)\big)^{d_s/2} \geq c\nu(B) \qquad \text{for all } B \in \mathcal{P}_*, \tag{3.17}$$

then there exists some constant $C > 0$ such that

$$N_\mu^-(\beta V) \geq C\beta^{d_s/2}\left(\int_{D^-(V)} (-V)^{d_s/2}\, d\nu + o(1)\right), \qquad \text{as } \beta \to +\infty. \tag{3.18}$$

(b) *The reverse inequality in (3.18) holds if (3.17) holds with the inequality being reversed.*

Proof. Since $(\mathcal{P}_k)_{k\geq 1}$ is refining with respect to μ and $\max\{\nu(B) : B \in \mathcal{P}_k\} \to 0$ as $k \to \infty$, $(\mathcal{P}_k)_{k\geq 1}$ is refining with respect to ν. (a) In view of Theorem 1.1(a), it suffices to show that for all $B \in \bigcup_{k\geq 1} \mathcal{P}_k$,

$$N(\lambda, -\Delta_{\mu|_B}) \geq \lambda^{d_s/2}\big(C\nu(B) + o(1)\big), \qquad \text{as } \lambda \to +\infty. \tag{3.19}$$

Fix any $B \in \bigcup_{k\geq 1} \mathcal{P}_k$. Let $(\mathcal{B}_\ell)_{\ell\geq 1}$ be the unique sequence of μ-partitions of B satisfying conditions (i), (ii), and (iii) in the paragraph preceding this proposition, and B_ℓ^* be the cell satisfying the condition (ii) above. Thus, for each $\ell \geq 1$, we can write

$$\mathcal{B}_\ell := \Big(\bigcup_{i=1}^{\ell} \{B_{i,j} : 0 \leq j \leq m(i)\} \Big) \bigcup \{B_\ell^*\}, \tag{3.20}$$

where $m(i)$ is a positive integer for any $i \geq 1$. Hence, for any $i \geq 1$ and $0 \leq j \leq m(i)$, there exists a unique number $w(i,j) > 0$, a unique $\kappa(i,j) \in \{0,1\}$, and a unique similitude $\tau_{i,j}$ such that $B_{1,\kappa(i,j)} \simeq_{\mu, w(i,j), \tau_{i,j}} B_{i,j}$. It follows from (3.17) that there exists some $c_* > 0$ such that for all $i \geq 1$ and $0 \leq j \leq m(i)$,

$$\big(w(i,j) r(i,j)\big)^{d_s/2} = \left(\frac{\mu(B_{i,j})}{\mu(B_{1,\kappa(i,j)})} \cdot \frac{|B_{i,j}|}{|B_{1,\kappa(i,j)}|} \right)^{d_s/2} \geq c_* \nu(B_{i,j}), \tag{3.21}$$

where $r(i,j)$ is the contraction ratio of $\tau_{i,j}$. By assumption, we have $\nu(B_\ell^*) \to 0$ as $\ell \to \infty$, and thus

$$\nu(B) = \sum_{i=1}^{\infty} \sum_{j=1}^{m(i)} \nu(B_{i,j}). \tag{3.22}$$

Using calculations from [29, Sections 4 and 5], we get, as $\lambda \to +\infty$,

$$\begin{aligned} N(\lambda, -\Delta_{\mu|_B}) &= \sum_{i=1}^{\infty} \sum_{j=1}^{m(i)} N(\lambda, -\Delta_{\mu|_{B_{i,j}}}) + \lambda^{d_s/2} o(1) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{m(i)} N\big(w(i,j) r(i,j) \lambda, -\Delta_{\mu|_{B_{1,\kappa(i,j)}}}\big) + \lambda^{d_s/2} o(1), \end{aligned} \tag{3.23}$$

where the fact $B_{1,\kappa(i,j)} \simeq_{\mu, w(i,j), \tau_{i,j}} B_{i,j}$ and Proposition 3.5 are used in the last equality. Combining (3.23) with (3.15), (3.21), and (3.22), we obtain positive constants C_1, C_2 such that

$$\begin{aligned} N(\lambda, -\Delta_{\mu|_B}) &\geq C_1 \lambda^{d_s/2} \Big(\sum_{i=1}^{\infty} \sum_{j=1}^{m(i)} \big(w(i,j) r(i,j)\big)^{d_s/2} + o(1) \Big) \\ &\geq C_2 \lambda^{d_s/2} \Big(\sum_{i=1}^{\infty} \sum_{j=1}^{m(i)} \nu(B_{i,j}) + o(1) \Big) = C_2 \lambda^{d_s/2} \Big(\nu(B) + o(1) \Big), \qquad \text{as } \lambda \to +\infty, \end{aligned}$$

i.e., (3.19) holds, which completes the proof.

(b) The proof is similar to that of part (b). If (3.17) holds with the inequality being reversed, then the same is true for (3.21). Consequently, the desired inequality holds. □

We now give a sufficient condition for the reverse inequality in (3.17) to hold.

Remark 3.8. *Use the notation in Proposition 3.7. If* $(r_1p_1)^{d_s/2} \le p_1$*,* $p_2 = p_3$ *and* $(r_2p_2)^{d_s/2} \le p_2$*, then there exists some constant* $c > 0$ *such that*

$$\big(|B|\,\mu(B)\big)^{d_s/2} \le c\mu(B) \qquad \text{for all } B \in \mathcal{P}_*.$$

Proof. Let c be a positive constant such that

$$\big(|B|\,\mu(B)\big)^{d_s/2} \le c\mu(B) \qquad \text{for } B \in \{B_{1,0}, B_{1,1}\}. \tag{3.24}$$

By assumption, $w_1(j) = p_1 \sum_{i=0}^{j} p_2^{j-i} p_3^i = p_1(j+1)p_2^j$ for $j \ge 0$. Using the assumptions $(r_1p_1)^{d_s/2} \le p_1$ and $(r_2p_2)^{d_s/2} \le p_2$, we have

$$\begin{aligned}\big(r_1 r_2^j w_1(j)\big)^{d_s/2} &= (r_1p_1)^{d_s/2} \cdot \big((j+1)(p_2r_2)^j\big)^{d_s/2} \le p_1(j+1)^{d_s/2} p_2^j \\ &\le p_1(j+1)p_2^j = w_1(j),\end{aligned} \tag{3.25}$$

where the last inequality uses the fact $d_s/2 < 1$. Fix any $B \in \mathcal{P}_*$. By the definition of $\mathcal{P}_*$, there exist a unique $k_0 \in \{0,1\}$, $w > 0$, and $\boldsymbol{i} \in \bigcup_{k\ge 0}\{1,2,3\}^k$ such that $B_{1,k_0} \simeq_{\mu,w,S_{\boldsymbol{i}}} B$. Let $r_{\boldsymbol{i}}$ be the contraction ratio of $S_{\boldsymbol{i}}$. By the definition of $\simeq_\mu$, $|B| = r_{\boldsymbol{i}}|B_{1,k_0}|$ and $\mu(B) = w\mu(B_{1,k_0})$. From the proofs of [29, Lemma 3.5 and Example 3.3], we see that w can be expressed as $w = w_1(i)p_1^j p_2^\ell p_3^{k-1-i-j-\ell} = w_1(i)p_1^j p_2^{k-1-i-j}$ for some $i, j, \ell \in \{0,1,\dots,k\}$. In this case, $r_{\boldsymbol{i}} = r_1^{j+1} r_2^{k-j-1}$. Hence,

$$\begin{aligned}\big(|B|\,\mu(B)\big)^{d_s/2} &= \big(r_{\boldsymbol{i}}|B_{1,k_0}| \cdot w\mu(B_{1,k_0})\big)^{d_s/2} \\ &= \Big(|B_{1,k_0}|\mu(B_{1,k_0}) \cdot r_1 r_2^i w_1(i) \cdot (p_1r_1)^j \cdot \big((p_2r_2)^{k-1-i-j}\big)\Big)^{d_s/2} \\ &\le c\mu(B_{1,k_0}) w_1(i) p_1^j p_2^{k-1-i-j} = c\mu(B),\end{aligned}$$

where we have use (3.24), (3.25), and the assumptions to get the inequality. This completes the proof. $\square$

4. Bohr's formula for Schrödinger operators with locally bounded potentials

Let $X \subseteq \mathbb{R}^n (n \ge 1)$ be a compact subset, and μ be a continuous positive finite Borel measure on X such that $\mu(X^\circ) > 0$ and $\operatorname{supp}(\mu) \subseteq X$. We extend X to $X_\infty := \bigcup_{i\in I} X_i$ as described in Section 1 so that conditions (C1)–(C3) are satisfied. For each $i \in I$, let τ_i be a similitude of the form $\tau_i(x) = x + b_i$, $b_i \in \mathbb{R}^n$ satisfying condition (C2) in Section 1, and $\mu_i := \mu \circ \tau_i^{-1}$. Also, let μ_∞ be a positive measure on X_∞ defined as in (1.10). Assume that $\mu_\infty(X_i \cap X_j) = 0$ for any distinct $i, j \in I$.

In the rest of section, we assume that μ satisfies (MPI). Let $-\Delta_\mu$ be the Dirichlet Laplacian with respect to μ. We first give a simple proposition.

Proposition 4.1. *Let $(\mu_i)_{i\in I}$, $(X_i)_{i\in I}$, X_∞, and μ_∞ be defined as above. Assume that μ satisfies (MPI). Then*

(a) *for any $i \in I$, the Dirichlet Laplacian $-\Delta_{\mu_i}$ with respect to μ_i is well-defined and $-\Delta_{\mu_i} \approx (-\Delta_\mu)$.*
(b) *$-\Delta_{\mu_\infty} := \bigoplus_{i\in I}(-\Delta_{\mu_i})$ is a non-negative self-adjoint operator in $L^2(X_\infty, \mu_\infty)$.*

Proof. Part (a) can be proved by verifying (MPI) and using a similar argument as that in [28, Lemma 2.1]. Part (b) follows from the facts that $L^2(X_\infty, \mu_\infty)$ is isomorphic to $\bigoplus_{i\in I} L^2(X_i^\circ, \mu_i)$ and that $-\Delta_{\mu_i}$ is a non-negative self-adjoint operator in $L^2(X_i^\circ, \mu_i)$ for all $i \in I$. We omit the details. $\square$

In the rest of this section, we define $-\Delta_{\mu_\infty} := \bigoplus_{i\in I}(-\Delta_{\mu_i})$. Let V be a non-negative, locally bounded, piecewise continuous function on X_∞ such that $V(x) \to \infty$ as $|x| \to \infty$. Then $-\Delta_{\mu_i} + V|_{X_i}$ is a non-negative self-adjoint operator in $L^2(X_i, \mu_i)$ for all $i \in I$.

Theorem 4.2. *Use the notation above and assume that V is a non-negative, locally bounded, piecewise continuous function on X_∞ so that $V(x) \to +\infty$ as $|x| \to \infty$. Then the Schrödinger operator $-\Delta_{\mu_\infty} + V$, defined as a sum of quadratic forms, is a non-negative self-adjoint operator in $L^2(X_\infty, \mu_\infty)$ and has compact resolvent.*

Proof. Let $\mathcal{D} := \{(u_i)_{i\in I} \in L^2(X_\infty, \mu_\infty) : u_i \in C_c^\infty(X_i^\circ) \text{ for all } i \in I\}$. It follows from the fact $\mathcal{D} \subseteq Q(-\Delta_{\mu_\infty}) \cap Q(V)$ is dense in $L^2(X_\infty, \mu_\infty)$ that $-\Delta_{\mu_\infty} + V$, defined as a sum of quadratic forms, is a non-negative self-adjoint operator in $L^2(X_\infty, \mu_\infty)$. The remaining assertion holds by using Theorem 2.1 and the proof of [30, Theorem XIII.16]. $\square$

Let Y be a closed subset of X with $\mu(Y^\circ) > 0$. If μ satisfies (MPI), then so does $\mu|_Y$. Hence, we can obtain analogues of Proposition 4.1 and Theorem 4.2 for $\mu|_Y$, as follows.

Remark 4.3. *Let $(\mu_i)_{i\in I}$, $(X_i)_{i\in I}$, $(\tau_i)_{i\in I}$, X_∞, and μ_∞ be defined as above. Assume that μ satisfies (MPI). Let Y be a closed subsets of X. Define $X_{\infty,Y} := \bigcup_{i\in I} \tau_i(Y)$ and $\mu_{\infty,Y} := \mu_\infty|_{X_{\infty,Y}}$. Then*

(a) *$-\Delta_{\mu_i|_{\tau_i(Y)}} \approx (-\Delta_{\mu|_Y})$ for any $i \in I$;*
(b) *$-\Delta_{\mu_{\infty,Y}} := \bigoplus_{i\in I}(-\Delta_{\mu_i|_{\tau_i(Y)}})$ is a non-negative self-adjoint operator in $L^2(X_{\infty,Y}, \mu_{\infty,Y})$;*
(c) *$-\Delta_{\mu_{\infty,Y}} + V|_{X_{\infty,Y}}$ is a non-negative self-adjoint operator in $L^2(X_{\infty,j}, \mu_{\infty,j})$ with compact resolvent.*

Proposition 4.1(b) implies that $-\Delta_{\mu_\infty} + V = \bigoplus_{i\in I}(-\Delta_{\mu_i} + V|_{X_i})$. It follows that

$$N(\lambda, -\Delta_{\mu_\infty} + V) = \sum_{i\in I} N(\lambda, -\Delta_{\mu_i} + V|_{X_i}) \qquad \text{for all } \lambda > 0. \tag{4.1}$$

Let $V^\wedge$ (resp. $V^\vee$) be the piecewise constant function which takes the value $\sup_{x\in X_i} V(x)$ (resp. $\inf_{x\in X_i} V(x)$) on X_i. Applying Theorem 4.2 to V^b for $b \in \{\vee, \wedge\}$, we see that $-\Delta_{\mu_\infty} + V^b$ is a non-negative self-adjoint operator in $L^2(X_\infty, \mu_\infty)$. Note that σ is an eigenvalue of $-\Delta_\mu|_{X_i} + V^b|_{X_i}$ with eigenvalue φ if and only if $\sigma - V^b|_{X_i}$ is an eigenvalue of $-\Delta_\mu|_{X_i}$ with the same eigenfunction. Hence,

$$N(\lambda, -\Delta_{\mu_i} + V^b|_{X_i}) = N(\lambda - V^b|_{X_i}, -\Delta_{\mu_i}). \tag{4.2}$$

This allows us to relate the eigenvalue counting function of the Schrödinger operator to that of the Laplacian. Since $0 \le V^\vee \le V \le V^\wedge$ in the sense of quadratic forms, we have $-\Delta_{\mu_\infty} + V^\vee \le -\Delta_{\mu_\infty} + V \le -\Delta_{\mu_\infty} + V^\wedge$, and thus, for all $\lambda > 0$,

$$N(\lambda, -\Delta_{\mu_\infty} + V^\wedge) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le N(\lambda, -\Delta_{\mu_\infty} + V^\vee). \tag{4.3}$$

As in (4.1), for $b \in \{\vee, \wedge\}$, by (4.2),

$$\begin{aligned} N(\lambda, -\Delta_{\mu_\infty} + V^b) &= \sum_{i\in I} N(\lambda, -\Delta_{\mu_i} + V^b|_{X_i}) = \sum_{i\in I} N(\lambda - V^b|_{X_i}, -\Delta_{\mu_i}) \\ &= \sum_{\{i\in I : V^b|_{X_i} \le \lambda\}} N(\lambda - V^b|_{X_i}, -\Delta_\mu), \end{aligned} \tag{4.4}$$

where Proposition 4.1(a) is used in the last equality. Replacing Proposition 4.1(b) and Theorem 4.2 by Remark 4.3(b) and (c), respectively, we also can now obtain analogues of (4.1) and (4.4) as follows. For all $j \in J$ and $b \in \{\vee, \wedge\}$, we get

$$N(\lambda, -\Delta_{\mu_{\infty,j}} + V|_{X_{\infty,j}}) = \sum_{i\in I} N(\lambda, -\Delta_{\mu_i|_{\tau_i(Y_j)}} + V|_{\tau_i(Y_j)}), \quad \text{and} \tag{4.5}$$

$$N(\lambda, -\Delta_{\mu_{\infty,j}} + V^b|_{X_{\infty,j}}) = \sum_{\{i\in I : V^b|_{X_i} \le \lambda\}} N(\lambda - V^b|_{X_i}, -\Delta_{\mu|_{Y_j}}). \tag{4.6}$$

Define $B(x, r) := \{y \in X_\infty : |x - y| < r\}$. The following theorem gives the existence of spectral dimension of $-\Delta_{\mu_\infty} + V$. A similar result was obtained by Chen *et al.* [2]. We replace their assumption that μ_∞ is Ahlfors-regular by a more general condition.

Theorem 4.4. *Use the notation above. Assume (W1) holds, and there exist positive constants c_1, c_2, c_3, θ such that*

$$c_1|x|^\theta \le V(x) \le c_2|x|^\theta \qquad \textit{for all } x \in X_\infty, \tag{4.7}$$

and $\mu_\infty(B(0,2r)) \le c_3\mu_\infty(B(0,r))$ *as* $r \to \infty$*. Then there exist positive constants* C, C_1, C_2 *such that, as* $\lambda \to +\infty$*,*

$$F(2\lambda, V^\vee) \le CF(\lambda, V^\wedge) \quad \textit{and} \quad C_1\lambda^{d_s/2}F(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le C_2\lambda^{d_s/2}F(\lambda, V),$$

1 *where* $F(\cdot,\cdot)$ *is defined as in* (1.11) *and* d_s *comes from (W1).*

2 *Proof.* Fix any $b \in \{\vee, \wedge\}$. Since $V^b|_{X_i}$ is a constant for any $i \in I$, we see that

$$F(\lambda, V^b) = \sum_{\{i\in I: V^b|_{X_i}\le\lambda\}} \mu_\infty(X_i) = \mu(X)\cdot\#\{i \in I : V^b|_{X_i} \le \lambda\} \quad \text{for } \lambda > 0. \tag{4.8}$$

By (W1), there exist positive constants c_4, c_5, M_0 such that $c_4\lambda^{d_s/2} \le N(\lambda, -\Delta_\mu) \le c_5\lambda^{d_s/2}$ for all $\lambda > M_0$. Thus

$$N(\lambda - V^b|_{X_i}, -\Delta_\mu) \le N(\lambda, -\Delta_\mu) \le c_5\lambda^{d_s/2} \quad \text{for all } \lambda > M_0 \text{ and any } i \in I,$$

while for all $\lambda > 2M_0$ and $i \in I$ such that $V^b|_{X_i} \le \lambda/2$,

$$N(\lambda - V^b|_{X_i}, -\Delta_\mu) \ge N(\lambda/2, -\Delta_\mu) \ge (c_4 2^{-d_s/2})\cdot\lambda^{d_s/2}.$$

Combining these estimates with (4.4) and (4.8), we get

$$\begin{aligned}
N(\lambda, -\Delta_{\mu_\infty} + V^b) &\le c_5\lambda^{d_s/2}\#\{i \in I : V^b|_{X_i} \le \lambda\} \\
&= (c_5/\mu(X))\cdot\lambda^{d_s/2}F(\lambda, V^b) \qquad \text{for all } \lambda > M_0, \text{ and,} \\
N(\lambda, -\Delta_{\mu_\infty} + V^b) &\ge \sum_{\{i\in I: V^b|_{X_i}\le\lambda/2\}} N(\lambda - V^b|_{X_i}, -\Delta_\mu) \\
&\ge (c_4 2^{-d_s/2})\cdot\lambda^{d_s/2}\#\{i \in I : V^b|_{X_i} \le \lambda/2\} \\
&= \big(c_4 2^{-d_s/2}/\mu(X)\big)\cdot\lambda^{d_s/2}F(\lambda/2, V^b) \qquad \text{for all } \lambda > 2M_0.
\end{aligned}$$

3 It follows that there exist constants $c_6, c_7 > 0$ such that for all $\lambda > 2M_0$,

$$c_6\lambda^{d_s/2}F(\lambda/2, V^b) \le N(\lambda, -\Delta_{\mu_\infty} + V^b) \le c_7\lambda^{d_s/2}F(\lambda, V^b). \tag{4.9}$$

By the definition of $F(\cdot,\cdot)$, $F(\lambda/2, V^\wedge) \le F(\lambda, V) \le F(\lambda, V^\vee)$ for all $\lambda > 0$. Using (4.3), we have

$$\frac{N(\lambda, -\Delta_{\mu_\infty} + V^\wedge)}{\lambda^{d_s/2}F(\lambda, V^\vee)} \le \frac{N(\lambda, -\Delta_{\mu_\infty} + V)}{\lambda^{d_s/2}F(\lambda, V)} \le \frac{N(\lambda, -\Delta_{\mu_\infty} + V^\vee)}{\lambda^{d_s/2}F(\lambda/2, V^\wedge)} \quad \text{for all } \lambda > 0,$$

which, together with (4.9), gives

$$c_6\frac{F(\lambda/2,V^\wedge)}{F(\lambda,V^\vee)}\le\frac{N(\lambda,-\Delta_{\mu_\infty}+V)}{\lambda^{d_s/2}F(\lambda,V)}\le c_7\frac{F(\lambda,V^\vee)}{F(\lambda/2,V^\wedge)}\qquad\text{for all }\lambda>2M_0.\tag{4.10}$$

Using (4.7), we obtain positive constants M_1,c_8,c_9 such that

$$c_8|x|^\theta\le V^\vee(x)\le V^\wedge(x)\le c_9|x|^\theta\qquad\text{for all }x\in X_\infty\text{ with }|x|>M_1.\tag{4.11}$$

Define $M_2:=\sup\{V^\wedge(x):x\in X_\infty\text{ such that }|x|\le M_1\}$. Thus for all $\lambda>2M_2$, we have

$$\begin{aligned}F(\lambda/2,V^\wedge)&\ge\mu_\infty\big(\{x\in X_\infty:c_9|x|^\theta\le\lambda/2\}\big)=\mu_\infty\big(B(0,c_{10}\lambda^{1/\theta})\big),\\ F(\lambda,V^\vee)&\le\mu_\infty(\{x\in X_\infty:c_8|x|^\theta\le\lambda\})=\mu_\infty\big(B(0,c_{11}\lambda^{1/\theta})\big),\end{aligned}$$

where $c_{10}:=(2c_9)^{-1/\theta}$ and $c_{11}:=c_8^{-1/\theta}$. Moreover, since $\mu_\infty(B(0,2r))\le c_3\mu_\infty(B(0,r))$ for all sufficiently large r by assumption, we have

$$\mu_\infty\big(B(0,c_{11}\lambda^{1/\theta})\big)\le c_3^{m_0}\mu_\infty\big(B(0,2^{-m_0}c_{11}\lambda^{1/\theta}\big)\le c_3^{m_0}\mu_\infty\big(B(0,c_{10}\lambda^{1/\theta})\big),\quad\text{as }\lambda\to+\infty,$$

where $m_0:=\min\{i\in\mathbb{Z}:i\ge\ln(c_{11}/c_{10})/\ln 2\}$. Thus $F(\lambda,V^\vee)\le c_3^{m_0}F(\lambda/2,V^\wedge)$ as $\lambda\to+\infty$. It follows from (4.10) that

$$c_3^{-m_0}c_6\lambda^{d_s}F(\lambda,V)\le N(\lambda,-\Delta_{\mu_\infty}+V)\le c_3^{m_0}c_7\lambda^{d_s}F(\lambda,V)\qquad\text{as }\lambda\to+\infty,$$

which completes the proof. □

Assume (W2) holds. Fix $j\in J$ and $b\in\{\vee,\wedge\}$. Define

$$R_j(\lambda,V^b):=\sum_{\{i\in I:V^b|_{X_i}\le\lambda\}}\big(\lambda-V^b|_{X_i}\big)^{d_s/2}R_j\big(\lambda-V^b|_{X_i}\big).\tag{4.12}$$

Let $g_j(\cdot,\cdot)$ be defined as in (1.12) for $j\in J$. We first observed that

$$g_j(\lambda,V^b)=\sum_{\{i\in I:V^b|_{X_i}\le\lambda\}}\big(\lambda-V^b|_{X_i}\big)^{d_s/2}G_j\Big(\ln\big(\lambda-V^b|_{X_i}\big)\Big).\tag{4.13}$$

Thus $\lim_{\lambda\to\infty}R_j(\lambda,V^b)/g_j(\lambda,V^b)=0$, and using (4.6), we have $N(\lambda,-\Delta_{\mu_{\infty,j}}+V^b|_{X_{\infty,j}})=g_j(\lambda,V^b)+R_j(\lambda,V^b)$ as $\lambda\to+\infty$. It follows that

$$\lim_{\lambda\to\infty}\frac{N(\lambda,-\Delta_{\mu_{\infty,j}}+V^b|_{X_{\infty,j}})}{g_j(\lambda,V^b)}=\lim_{\lambda\to\infty}\frac{g_j(\lambda,V^b)+R_j(\lambda,V^b)}{g_j(\lambda,V^b)}=1.\tag{4.14}$$

The following theorem is a slight modification of a similar one in [2], in order to suit our purpose. We include a proof for completeness.

Theorem 4.5. [2, Theorem 2.11] *Let V be a locally bounded non-negative, piecewise continuous function on X_∞ so that $V(x) \to +\infty$ as $|x| \to \infty$. Assume that (W2) and* (1.13) *hold. Let $F(\cdot,\cdot)$ and $g_j(\cdot,\cdot)$ be defined as in* (1.11) *and* (1.12) *for $j \in J$, respectively. Then for each $j \in J$,*

$$N(\lambda, -\Delta_{\mu_{\infty,j}} + V|_{X_{\infty,j}}) \sim g_j(\lambda, V), \qquad \text{as } \lambda \to +\infty. \tag{4.15}$$

Proof. Fix any $j \in J$. We claim that

$$g_j(\lambda, V^\vee)/g_j(\lambda, V^\wedge) = 1 + o(1), \qquad \text{as } \lambda \to +\infty. \tag{4.16}$$

Define $F_j(\lambda, V^b) := \mu_\infty(\{x \in X_{\infty,j} : V^b(x) \le \lambda\})$ for $b \in \{\vee, \wedge\}$. Similar to (4.8), we get $F_j(\lambda, V^b) = \mu(Y_j) \cdot \#\{i \in I : V^b|_{X_i} \le \lambda\}$ for $b \in \{\vee, \wedge\}$ and $\lambda > 0$. This, together with (4.8), yields $F_j(\lambda, V^\vee)/F_j(\lambda, V^\wedge) = F(\lambda, V^\vee)/F(\lambda, V^\wedge)$. By [2, Proposition 4.2], if $F_j(\lambda, V^\vee) \sim F_j(\lambda, V^\wedge)$ as $\lambda \to +\infty$, then (4.16) holds. The claim follows by combining these observations with (1.13). Combining (4.16) and (4.14), we get

$$\lim_{\lambda \to +\infty} \frac{N(\lambda, -\Delta_{\mu_{\infty,j}} + V^\vee|_{X_{\infty,j}})}{g_j(\lambda, V^\wedge)} = \lim_{\lambda \to +\infty} \frac{N(\lambda, -\Delta_{\mu_{\infty,j}} + V^\wedge|_{X_{\infty,j}})}{g_j(\lambda, V^\vee)} = 1. \tag{4.17}$$

We note that $h_j(\lambda) := \lambda^{d_s/2} G_j(\ln \lambda)$ is nondecreasing on $(M, +\infty)$ for some constant $M > 0$. Hence, by the definition of $g_j(\cdot,\cdot)$ in (1.12),

$$g_j(\lambda, V^\wedge) \le g_j(\lambda, V) \le g_j(\lambda, V^\vee) \qquad \text{as } \lambda \to +\infty. \tag{4.18}$$

As in (4.3), we have

$$N(\lambda, -\Delta_{\mu_{\infty,j}} + V^\wedge|_{X_{\infty,j}}) \le N(\lambda, -\Delta_{\mu_{\infty,j}} + V|_{X_{\infty,j}}) \le N(\lambda, -\Delta_{\mu_{\infty,j}} + V^\vee|_{X_{\infty,j}}).$$

It follows that, as $\lambda \to +\infty$,

$$\frac{N(\lambda, -\Delta_{\mu_{\infty,j}} + V^\wedge|_{X_{\infty,j}})}{g_j(\lambda, V^\vee)} \le \frac{N(\lambda, -\Delta_{\mu_{\infty,j}} + V|_{X_{\infty,j}})}{g_j(\lambda, V)} \le \frac{N(\lambda, -\Delta_{\mu_{\infty,j}} + V^\vee|_{X_{\infty,j}})}{g_j(\lambda, V^\wedge)},$$

which, together with (4.17), yields (4.15). □

We now prove Theorem 1.3.

Proof of Theorem 1.3. Proposition 4.1(a) and Remark 4.3(a) imply $N(\lambda, -\Delta_{\mu_i}) = N(\lambda, -\Delta_\mu)$ and $N(\lambda, -\Delta_{\mu_i|_{\tau_i(Y_j)}}) = N(\lambda, -\Delta_{\mu|_{Y_j}})$ for all $i \in I$ and all $j \in J$. Also, (1.9) holds by (W2).

Thus, for all $i \in I$,

$$\sum_{j\in J} N(\xi_{j,1}\lambda, -\Delta_{\mu_i|_{\tau_i(Y_j)}}) - C_0 \le N(\lambda, -\Delta_{\mu_i}) \le \sum_{j\in J} N(\xi_{j,2}\lambda, -\Delta_{\mu_i|_{\tau_i(Y_j)}}) + C_0. \tag{4.19}$$

For all $i \in I$, since $V^\vee|_{X_i} \le V|_{X_i} \le V^\wedge|_{X_i}$, Proposition 2.3 and (4.2) give

$$\begin{aligned} N(\lambda - V^\wedge|_{X_i}, -\Delta_{\mu_i}) &= N(\lambda, -\Delta_{\mu_i} + V^\wedge|_{X_i}) \le N(\lambda, -\Delta_{\mu_i} + V|_{X_i}) \\ &\le N(\lambda, -\Delta_{\mu_i} + V^\vee|_{X_i}) = N(\lambda - V^\vee|_{X_i}, -\Delta_{\mu_i}), \end{aligned}$$

which, together with (4.19), yields

$$\begin{aligned} &\sum_{j\in J} N(\xi_{j,1}(\lambda - V^\wedge|_{X_i}), -\Delta_{\mu_i|_{\tau_i(Y_j)}}) - C_0 \le N(\lambda, -\Delta_{\mu_i} + V|_{X_i}) \\ &\qquad \le \sum_{j\in J} N\big(\xi_{j,2}(\lambda - V^\vee|_{X_i}), -\Delta_{\mu_i|_{\tau_i(Y_j)}}\big) + C_0. \end{aligned}$$

It follows that

$$\begin{aligned} &\sum_{i\in I}\sum_{j\in J} N(\xi_{j,1}(\lambda - V^\wedge|_{X_i}), -\Delta_{\mu_i|_{\tau_i(Y_j)}}) - C_0 \cdot \#\{i \in I : V^\wedge|_{X_i} \le \lambda\} \\ \le& N(\lambda, -\Delta_{\mu_\infty} + V) = \sum_{i\in I} N(\lambda, -\Delta_{\mu_i} + V|_{X_i}) \\ \le& \sum_{i\in I}\sum_{j\in J} N(\xi_{j,2}(\lambda - V^\vee|_{X_i}), -\Delta_{\mu_i|_{\tau_i(Y_j)}}) + C_0 \cdot \#\{i \in I : V^\vee|_{X_i} \le \lambda\}. \end{aligned}$$

Using (4.8) and (4.6), we get

$$\begin{aligned} &\sum_{j\in J} N(\xi_{j,1}\lambda, -\Delta_{\mu_{\infty,j}} + \xi_{j,1}V^\wedge|_{X_{\infty,j}}) - C_1 F(\lambda, V^\wedge) \\ \le& N(\lambda, -\Delta_{\mu_\infty} + V) \\ \le& \sum_{j\in J} N(\xi_{j,2}\lambda, -\Delta_{\mu_{\infty,j}} + \xi_{j,2}V^\vee|_{X_{\infty,j}}) + C_2 F(\lambda, V^\vee), \end{aligned} \tag{4.20}$$

where C_i, $i = 1, 2$ are positive constants. Combining (4.13) and (4.8) with the fact $0 < \inf G_j \le \sup G_j < +\infty$, we have for $b \in \{\vee, \wedge\}$ and all $c > 0$,

$$\begin{aligned} g_j(c\lambda, cV^b) &\ge (c^{d_s/2} \inf G_j) \cdot \sum_{\{i\in I : V^b|_{X_i} \le \lambda\}} \big(\lambda - V^b|_{X_i}\big)^{d_s/2} \\ &\ge (c^{d_s/2} \inf G_j)(\lambda/2)^{d_s/2} \cdot \#\{i \in I : V^b|_{X_i} \le \lambda/2\} \\ &\ge C_3 \lambda^{d_s/2} F(\lambda/2, V^b) \ge C_4 \lambda^{d_s/2} F(\lambda, V^b), \qquad \text{as } \lambda \to +\infty, \end{aligned} \tag{4.21}$$

where $C_i > 0$, $i = 3, 4$, are constants, and the last inequality uses the assumption $F(2\lambda, V^\vee) \leq CF(\lambda, V^\wedge)$ as $\lambda \to +\infty$. Combining (4.21) with (4.18), (4.17), and (4.20), we have

$$\begin{aligned}\lim_{\lambda\to\infty} \frac{N(\lambda, -\Delta_{\mu_\infty} + V)}{\sum_{j\in J} g_j(\xi_{j,1}\lambda, \xi_{j,1}V)} &\geq \lim_{\lambda\to\infty} \frac{\sum_{j\in J} N(\xi_{j,1}\lambda, -\Delta_{\mu_{\infty,j}} + \xi_{j,1}V^\wedge|_{X_{\infty,j}})}{\sum_{j\in J} g_j(\xi_{j,1}\lambda, \xi_{j,1}V^\vee)} \\ &\quad - \lim_{\lambda\to\infty} \frac{C_1 F(\lambda, V^\wedge)}{\sum_{j\in J} g_j(\xi_{j,1}\lambda, \xi_{j,1}V^\vee)} = 1 + 0 = 1.\end{aligned}$$

Similarly, we have

$$\lim_{\lambda\to\infty} \frac{N(\lambda, -\Delta_{\mu_\infty} + V)}{\sum_{j\in J} g_i(\xi_{j,2}\lambda, \xi_{j,2}V)} \leq 1.$$

The proof is complete. □

A sufficient condition for (1.13) is given in [2, Remark 2.9]. We now give a simple sufficient condition for (1.13), which is needed in Section 5.

Proposition 4.6. *Let $X = [0, a] \subseteq \mathbb{R}$ and let X_∞ and μ_∞ be defined as above. Assume that V is a locally bounded non-negative, piecewise continuous function on X_∞ so that $V(x) \sim c|x|^\beta$ for some $\beta > 1$ and $c > 0$. Let $F(\cdot, \cdot)$ be defined as in* (1.11). *Then* (1.13) *holds.*

Proof. By the definition of $F(\cdot, \cdot)$ in (1.11), we have $F(\lambda, V^\vee) \geq F(\lambda, V^\wedge)$. Since $V(x) \sim c|x|^\beta$, for any sufficiently small $\epsilon > 0$, there exists some sufficiently large constant $M > 0$ such that $0 < c(|x| - a)^\beta - \epsilon \leq V^\vee(x) \leq V^\wedge(x) \leq c(|x| + a)^\beta + \epsilon$ for all $x \in X_\infty$ such that $|x| > M$. Define $M_0 := \sup\{V(x) : x \in X_\infty \text{ such that } |x| > M\}$. Thus for all $\lambda > M_0$, we have $\#\{i \in I : V^\vee|_{X_i} \leq \lambda\} \leq \#\{i \in I : V^\wedge|_{X_i} \leq \lambda\} + C(\epsilon, M)$, where $C(\epsilon, M)$ is a positive constant depending only on ϵ and M. It follows that

$$\begin{aligned}1 \leq \lim_{\lambda\to\infty} \frac{F(\lambda, V^\vee)}{F(\lambda, V^\wedge)} &= \lim_{\lambda\to\infty} \frac{\#\{i \in I : V^\vee|_{X_i} \leq \lambda\})}{\#\{i \in I : V^\wedge|_{X_i} \leq \lambda\}} \\ &\leq \lim_{\lambda\to\infty} \frac{\#\{i \in I : V^\wedge|_{X_i} \leq \lambda\} + C(\epsilon, M)}{\#\{i \in I : V^\wedge|_{X_i} \leq \lambda\}} = 1.\end{aligned}$$

Hence, (1.13) holds. □

5. Examples: self-similar measures on $\mathbb{R}$ with overlaps

In this section, we apply Theorem 1.3 to self-similar measures on $\mathbb{R}$ with overlaps. We first prove a simply proposition, which leads to a sufficient condition for Theorem 4.4.

Proposition 5.1. *Let $X := [0, a]$ and μ be a continuous positive finite Borel measure with* $\text{supp}(\mu) \subseteq [0, a]$. *Let $X_\infty := \bigcup_{i\in I} \tau_i(X)$ and μ_∞ be defined as in Section 1 with $I = \mathbb{Z}$ and*

$\tau_i(x) = x + b_i$, where $b_i = a + b_{i-1}$ for all $i \in I$. Then $X_\infty = \mathbb{R}$, and there exist positive constants C_1, C_2 such that $C_1 r \le \mu_\infty(B(x,r)) \le C_2 r$ for all $x \in \mathbb{R}$ and $r \ge 2a$, where $B(x,r) := \{x \in \mathbb{R} : |x| < r\}$. Consequently, under the assumptions of (W1) and (4.7), the conclusions of Theorem 4.4 holds.

Proof. By assumption, $\tau_i(0) = \tau_{i-1}(a)$ and $|\tau_i(X)| = a$ for all $i \in \mathbb{Z}$. Thus $X_\infty = \mathbb{R}$. Let $X_i := \tau_i(X)$. Fix any $x \in \mathbb{R}$ and $r > 2a$. Then there exist positive integers m_0, m_1 such that $m_1 - m_0 \ge 2$ and $\bigcup_{i=m_0}^{m_1} X_i \subseteq B(x,r) \subseteq \bigcup_{i=m_0-1}^{m_1+1} X_i$. Thus

$$a(m_1 - m_0) = \sum_{i=m_0}^{m_1} |X_i| \le 2r \le \sum_{i=m_0-1}^{m_1+1} |X_i| = a(m_1 - m_0 + 2) \le 2a(m_1 - m_0).$$

It follows that

$$\frac{\mu(X)}{a} r \le (m_1 - m_0)\mu(X) \le \mu_\infty(B(x,r)) \le 2(m_1 - m_0)\mu(X) \le \frac{4\mu(X)}{a} r,$$

where the fact $\mu_\infty(X_i) = \mu_i(X_i) = \mu(X)$ for all $i \in I$ is used. Hence the assertion holds. $\square$

The spectral dimension of the examples in Subsections 5.1 and 5.2 are computed in [28]. We will compute the spectral dimension of the example in Subsection 5.3 by using a similar method. The technique is to apply a vector-valued renewal theorem [14, Theorem 4.2] by deriving a system of renewal equations for the eigenvalue counting functions, and express them in vector form as:

$$\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z}, \tag{5.1}$$

where $\alpha \ge 0$, and

$$\begin{aligned}
&\mathbf{f} = \mathbf{f}^\alpha(t) := [f_1^{(\alpha)}(t), \dots, f_n^{(\alpha)}(t)], \quad t \in \mathbb{R};\\
&\mathbf{M}_\alpha := [\mu_{\ell m}^{(\alpha)}] \quad \text{is a } n \times n \text{ matrix of Radon measures on } \mathbb{R};\\
&\mathbf{z} := \mathbf{z}^{(\alpha)}(t) = [z_1^{(\alpha)}(t), \dots, z_n^{(\alpha)}(t)] \quad \text{is some error function.}
\end{aligned}$$

Let

$$\mathbf{M}_\alpha(\infty) := \Big[\mu_{\ell m}^{(\alpha)}(\mathbb{R})\Big]_{\ell,m=1}^n. \tag{5.2}$$

If the error functions decay exponentially to 0 as $t \to \infty$, then $d_s(-\Delta_\mu)$ is given by the unique α such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is equal to 1.

For the examples in this section, the functions G_j in condition (W2) tend to either a constant or a (non-constant) periodic function as $\lambda \to \infty$. This dichotomy is determined by whether a set $\mathbb{R}_M$ in [14] is arithmetic or non-arithmetic, where $M := M_\alpha = [\mu^{(\alpha)}_{\ell m}]^n_{\ell,m=1}$ is an $n \times n$ matrix-valued Radon measure and $\mathbb{R}_M$ is the closed subgroup of $(\mathbb{R},+)$ generated by $\mathcal{G} := \bigcup \big\{\operatorname{supp}(\mu_\gamma) : \ \gamma \text{ is a simple cycle on } \{1,\dots,n\}\big\}$ (see [14]).

5.1. **Infinite Bernoulli convolution associated with the golden ratio.** In this section, we consider the infinite Bernoulli convolution associated with the golden ratio:

$$\mu = \frac{1}{2}\mu \circ S_1^{-1} + \frac{1}{2}\mu \circ S_2^{-1}, \tag{5.3}$$

where

$$S_1(x) = \rho x, \qquad S_2(x) = \rho x + (1-\rho), \qquad \rho = (\sqrt{5}-1)/2.$$

We note that $\operatorname{supp}(\mu) = [0,1]$. Strichartz *et al.* [35] showed that μ satisfies a family of second-order identities with respect to the following auxiliary IFS:

$$T_0(x) := \rho^2 x, \qquad T_1(x) := \rho^3 x + \rho^2, \qquad T_2(x) := \rho^2 x + \rho. \tag{5.4}$$

For any integer $k \ge 0$ and any index $\boldsymbol{j} = (j_1,\dots,j_k) \in \{0,2\}^k$, define

$$c_{\boldsymbol{j}} := \frac{1}{2\cdot 4^{k+1}}\begin{bmatrix}1 & 1\end{bmatrix} P_{\boldsymbol{j}} \begin{bmatrix}1\\1\end{bmatrix}, \quad P_{\boldsymbol{j}} := P_{j_1}\cdots P_{j_k}, \quad P_0 = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \quad \text{and} \quad P_2 = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}.$$

The vector-valued renewal equation (5.1) reduces to the following scalar-valued one:

$$f(t) = \sum_{k=0}^{\infty}\sum_{\boldsymbol{j}\in J_0^k} (\rho^{2k+3}c_{\boldsymbol{j}})^\alpha f\big(t + \ln(\rho^{2k+3}c_{\boldsymbol{j}})\big) + z^\alpha(t),$$

where $f(t) = e^{-\alpha t}N(e^t, -\Delta_{\mu|_{T_1(X)}})$ and $z^\alpha(t) = o(e^{-\sigma t})$, as $t \to \infty$, for some $\sigma > 0$ (see [28, Section 5]). Moreover, $M = [\mu^{(\alpha)}]$ is a 1×1 matrix-valued Radon measure, where $\mu^{(\alpha)}$ is a discrete measure with support $\mathcal{G} := \{-\ln(\rho^{2k+3}c_{\boldsymbol{j}}) : k \ge 0, \boldsymbol{j} \in \{0,2\}^k\}$. Let $\mathbb{R}_M$ be the closed subgroup of $(\mathbb{R},+)$ generated by $\mathcal{G}$.

Let d_s be the unique positive solution of

$$\sum_{k=0}^{\infty}\sum_{\boldsymbol{j}\in J_0^k} (\rho^{2k+3}c_{\boldsymbol{j}})^{d_s/2} = 1. \tag{5.5}$$

[28, Theorem 1.2] shows that $d_s(-\Delta_\mu) = d_s$, and (W1) holds.

Proposition 5.2. *Let μ be the self-similar measure defined as in* (5.3)*, and $-\Delta_\mu$ be the associated Dirichlet Laplacian with respect to μ. Then* (1.9) *holds with $J = \{1\}$ and $Y_1 :=$*

$T_1(X)$, where T_1 is defined as in (5.4). *Moreover, (W2) holds; in particular, the non-arithmetic case holds: there exists a constant $G_1 > 0$ such that*

$$N\big(\lambda, -\Delta_{\mu|_{T_1(X)}}\big) = \lambda^{d_s/2}\big(G_1 + o(1)\big), \qquad \text{as } \lambda \to +\infty,$$

where d_s is defined as in (5.5).

Proof. From [28, paragraph following Proposition 3.2], we see that there exists a constant $\xi > 0$ such that

$$N\big(\lambda, -\Delta_{\mu|_{T_1(X)}}\big) \le N(\lambda, -\Delta_{\mu|_X}) \le N\big(\xi\lambda, -\Delta_{\mu|_{T_1(X)}}\big) + 1 \quad \text{for all } \lambda > 0, \tag{5.6}$$

and hence the first assertion holds. Condition (2) of (W2) holds by combining [28, Theorem 1.2 and Theorem 4.1]. We now use [28, Theorem 4.1] again to show that the non-arithmetic case holds by verifying that $\mathbb{R}_M = \mathbb{R}$. Suppose, on the contrary, that $\mathbb{R}_M \ne \mathbb{R}$. Letting $k = 0$ and 1, we obtain the elements $a := -\ln(\rho^3/4)$ and $b := -\ln(3\rho^5/32)$ in $\mathcal{G}$. Then $b/a = 1 - \ln(3\rho^2/8)/a \in \mathbb{Q}$ and thus there exist $m, n \in \mathbb{Z}$ such that $-\ln(3\rho^2/8)/a = n/m$. Consequently, $3^m = 2^{3m-2n}\beta^{2m-3n}$, where $\beta = 2/(\sqrt{5}-1) = 1/\rho$. Without loss of generality, we assume that $2m - 3n > 0$. Define $h(x) := 2^{3m-2n}x^{2m-3n} - 3^m$. Then $h(\beta) = 0$. Since β is an algebraic integer with $x^2 - x - 1$ being its minimal polynomial, $x^2 - x - 1$ divides $h(x)$, a contradiction. Hence, $\mathbb{R}_M = \mathbb{R}$, which implies the desired result. $\square$

Let $I := \mathbb{Z}$ and define $\tau_i(x) = x + i$ for all $i \in I$. Define $X_\infty := \bigcup_{i\in I} \tau_i(X)$ and let μ_∞ be defined as in (1.10). Thus $X_\infty = \mathbb{R}$. Define $X_{\infty,1} := \bigcup_{i\in I} \tau_i(T_1(X))$ and $\mu_{\infty,1} := \mu_\infty|_{X_{\infty,1}}$.

Corollary 5.3. *Let X_∞, μ_∞, $X_{\infty,1}$ and $\mu_{\infty,1}$ be defined as above. Assume V is a locally bounded non-negative, piecewise continuous function on X_∞ such that $V(x) \sim c|x|^\beta$ for some $\beta > 1$ and $c > 0$. Let d_s be defined as in* (5.5). *Then*

(a) *there exist positive constants C_1, C_2 such that*

$$C_1\lambda^{d_s/2}F(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le C_2\lambda^{d_s/2}F(\lambda, V), \qquad \text{as } \lambda \to +\infty,$$

where $F(\cdot,\cdot)$ is defined as in (1.11)*;*

(b) *as $\lambda \to +\infty$,*

$$\big(1 + o(1)\big)g_1(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le \big(1 + o(1)\big)g_1(\xi\lambda, \xi V),$$

where ξ comes from (5.6)*, and $g_1(\cdot,\cdot)$ is defined as in* (1.12) *with $G_1(\cdot)$ being a constant function in Proposition 5.2.*

Proof. Part (a) follows from Proposition 5.1 and the fact that (W1) holds. Part (b) follows by combining Theorems 1.3 and 4.4 with Propositions 4.6, 5.1 and 5.2. $\square$

5.2. **A class of convolutions of Cantor-type measures.** We study the following family of convolutions of Cantor-type measures studied in [16, 28]. Let

$$S_0(x)=\frac{1}{m}x,\qquad S_1(x)=\frac{1}{m}x+\frac{m-1}{m},\tag{5.7}$$

where $m\ge 3$ is an odd integer. Let ν_m be the self-similar measure defined by the IFS (5.7) with probability weights $p_0=p_1=1/2$. The m-fold convolution μ_m of ν_m^* is the self-similar measure defined by the following IFS with overlaps (see [28]):

$$S_i(x)=\frac{1}{m}x+\frac{m-1}{m}i,\qquad i=0,1,\dots,m,$$

together with probability weights $w_i:=\binom{m}{i}/2^m$, $i=0,1,\dots,m$. That is,

$$\mu_m=\sum_{i=0}^{m} w_i\cdot\mu_m\circ S_i^{-1}.\tag{5.8}$$

Note that $\operatorname{supp}(\mu_m)=[0,m]$. For the rest of this subsection, we fix an odd integer $m\ge 3$, and let $\mu:=\mu_m$ for convenience. It is shown in [16] that μ satisfies a family of second-order identities with respect to the IFS

$$T_j(x)=\frac{1}{m}x+j,\qquad j=0,1,\dots,m-1.\tag{5.9}$$

Similarly, the vector-valued renewal equation (5.1) is given in [28, Section 6] with $f_\ell(t)=e^{-\alpha t}N(e^t,-\Delta_{\mu|_{T_\ell(X)}})$, $\ell=1,\dots,m-2$. Thus $M=[\mu_{k\ell}^{(\alpha)}]_{k,\ell=1}^{m-2}$ is an $(m-2)\times(m-2)$ matrix-valued Radon measure. By the proof of [28, Proposition 6.2], we have

$$\operatorname{supp}\big(\mu_{11}^{(\alpha)}\big)=\big\{\ln(2^m)\big\}\bigcup\big\{-\ln(c_{\boldsymbol{j}}/m^{k+2}):k\ge 0,\boldsymbol{j}\in\{0,2\}^k\big\},\tag{5.10}$$

where for any integer $k\ge 0$ and any index $\boldsymbol{j}=(j_1,\dots,j_k)\in\{0,2\}^k$,

$$P_{\boldsymbol{j}}:=P_{j_1}\cdots P_{j_k},\quad c_{\boldsymbol{j}}:=\frac{1}{2^{2m+mk}}\begin{bmatrix}\binom{m}{2} & \binom{m}{1}\end{bmatrix}P_{\boldsymbol{j}}\begin{bmatrix}1\\1\end{bmatrix},\quad P_0:=\begin{bmatrix}1&0\\1&m\end{bmatrix},\quad P_{m-1}:=\begin{bmatrix}m&1\\0&1\end{bmatrix}.$$

By the definition of $\mathcal{G}$ and [14, Lemma 2.3], we get $\operatorname{supp}\big(\mu_{11}^{(\alpha)}\big)\subseteq\mathcal{G}$. In particular, the equation holds if $m=3$.

An explicit formula for the spectral dimension of $-\Delta_\mu$ is given in [28, Theorem 1.3], which also shows that (W1) holds.

Proposition 5.4. *Let $\mu := \mu_m$ be defined as in* (5.8)*, and $-\Delta_\mu$ be the associated Dirichlet Laplacian. Then* (1.9) *holds with $J = \{j\}$ and $Y_j := T_j(X)$ for any $j = 1, \dots, m-2$, where T_j is defined as in* (5.9)*. Moreover, (W2) holds. In particular, the non-arithmetic case holds: for any $j = 1 \dots, m-2$, there exists a constant $G_j > 0$ such that*

$$N(\lambda, -\Delta_{\mu|_{T_j(X)}}) = \lambda^{d_s/2}\big(G_j + o(1)\big), \qquad \text{as } \lambda \to +\infty,$$

where d_s is the spectral dimension of $-\Delta_\mu$.

Proof. As in the proof of Proposition 5.2, using the discussion in the paragraph following [28, Proposition 3.2], we see that there exist positive constants $(\xi_j)_{j=1}^{m-2}$ such that for each $j = 1, \dots, m-2$,

$$N(\lambda, -\Delta_{\mu|_{T_j(X)}}) \le N(\lambda, -\Delta_{\mu|_X}) \le N(\xi_j\lambda, -\Delta_{\mu|_{T_j(X)}}) + 1. \tag{5.11}$$

Hence, the first result holds. Condition (2) of (W2) follows from [28, Section 6 and Theorem 4.1]. As in Proposition 5.2, we show that $\mathbb{R}_M = \mathbb{R}$. Letting $k = 0$, we get $a := -\ln(m(m+1)/2) + 2\ln(2^m) \in \mathcal{G}^* \subseteq \mathbb{R}_M$. Suppose $\mathbb{R}_M \ne \mathbb{R}$. Since $\ln(2^m) \in \mathcal{G}^*$, we have $-a/\ln(2^m) + 2 = \ln(m(m+1)/2)/\ln(2^m) = s/t$ for some $s, t \in \mathbb{Z}$. Thus $m^s(m+1)^s = 2^{mt+s}$, a contradiction, and the assertion follows. □

Let $Y_j := T_j(X)$ for $j = 1, \dots, m-2$. Let $I := \mathbb{Z}$ and define $\tau_i(x) = x + mi$ for all $i \in I$. Define $X_\infty := \bigcup_{i \in I} \tau_i(X)$ and let μ_∞ be defined as in (1.10). Then $X_\infty = \mathbb{R}$. Define $X_{\infty,j} := \bigcup_{i \in I} \tau_i(Y_j)$ and $\mu_{\infty,j} := \mu_\infty|_{X_{\infty,j}}$ for $j = 1, \dots, m-2$.

Corollary 5.5. *Let X_∞, μ_∞, $X_{\infty,j}$ and $\mu_{\infty,j}$ be defined as above. Assume V is a locally bounded non-negative, piecewise continuous function on X_∞ so that $V(x) \sim c|x|^\beta$ for some $\beta > 1$ and $c > 0$. Let d_s be the spectral dimension of $-\Delta_\mu$. Then*

(a) *there exist positive constants C_1, C_2 such that*

$$C_1\lambda^{d_s/2}F(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le C_2\lambda^{d_s/2}F(\lambda, V), \qquad \text{as } \lambda \to +\infty,$$

where $F(\cdot,\cdot)$ is defined as in (1.11)*;*

(b) *as $\lambda \to +\infty$,*

$$\big(1 + o(1)\big)\sum_{j=1}^{m-2} g_j(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le \big(1 + o(1)\big)\sum_{j=1}^{m-2} g_j(\xi_j\lambda, \xi_j V),$$

where ξ_j comes from (5.11)*, and $g_j(\cdot,\cdot)$ is defined as in* (1.12) *with $G_j(\cdot)$ being the constant function in Proposition 5.4.*

Proof. The proof is similar to that of Corollary 5.3 with Proposition 5.4 replacing Proposition 5.2. □

5.3. **A class of graph-directed self-similar measures satisfying (EFT).** This purpose of this subsection is to illustrates the arithmetic case by constructing a graph-directed self-similar measure.

A *graph-directed iterated function system (GIFS)* of contractive similitudes is an ordered pair $G = (V, E)$ described as follows (see [25]). $V := \{1, \dots, q\}$ is the set of *vertices* and E is the set of *directed edges* with each edge beginning and ending at a vertex. It is possible for an edge to begin and end at the same vertex and we allow more than one edge between two vertices. Let E_{ij} denote the set of all edges that begin at vertex i and end at vertex j. We call $\boldsymbol{e} = e_1 \dots e_k$ a *path* with length k if the terminal vertex of each edge e_i $(1 \le i \le k-1)$ equals the initial vertex of the edge e_{i+1}.

Consider the GIFS $G = (V, E)$ with $V = \{1, 2\}$ and $E = \{e_i : 1 \le i \le 5\}$, where $e_1, e_2 \in E_{11}$, $e_3 \in E_{12}$, $e_4 \in E_{21}$, $e_5 \in E_{22}$. The five similitudes associated with E are defined by

$$S_{e_1}(x) = \frac{1}{4}x, \;\; S_{e_2}(x) = \frac{1}{4}x + \frac{3}{4}, \;\; S_{e_3}(x) = \frac{1}{4}x - \frac{5}{16}, \;\; S_{e_4}(x) = \frac{1}{4}x + 2, \;\; S_{e_5}(x) = \frac{1}{4}x + \frac{9}{4}.$$

The GIFS $G = (V, E)$ is used in [4] as basic example for the graph finite type condition. It is known (see [8, 25]) that if for each edge $e \in E$, there corresponds a *transition probability* p_e, then for each $i \in V$, there exists a unique Borel probability measure μ_i such that

$$\mu_i = \sum_{j=1}^{2} \sum_{e \in E_{ij}} p_e \cdot \mu_j \circ S_e^{-1}.$$

We note that $\operatorname{supp}(\mu_1) = [0, 1]$ and $\operatorname{supp}(\mu_2) = [2, 3]$.

Define $\mu(E) := \mu_1(E \cap [0,1]) + \mu_2(E \cap [2,3])$ for all measurable subset $E \subseteq \mathbb{R}$. We call μ the *graph-directed self-similar measure* defined by $G = (V, E)$ and probability matrix $(p_e)_{e \in E}$. Since μ satisfies (EFT) (see [29, Example 3.6]), we can derive a vector-valued renewal equation by using the same method in [29, Section 4] as follows. Let $Y_1 := S_{e_1}([0,1]) \cup S_{e_3}([2,3])$ and $Y_2 := S_{e_2}([0,1])$. For $\alpha \ge 0$ and $j = 1, 2$, define $f_j(t) = f_j^{(\alpha)}(t) := e^{-\alpha t} N(e^t, -\Delta_{\mu|_{Y_j}})$.

Thus, combining the proof of [29, Example 3.6] and the process of deriving the vector-valued renewal equation in [29, Section 4], we see that (5.1) can be written as

$$
\begin{aligned}
f_1(t) &= \big(\frac{p_{e_1}}{4}\big)^{\alpha} f_1\Big(t+\ln(\frac{p_{e_1}}{4})\Big) + \Big(\frac{p_{e_1}}{4}+\frac{p_{e_3}p_{e_4}}{4p_{e_2}}\Big)^{\alpha} f_2\Big(t+\ln\big(\frac{p_{e_1}}{4}+\frac{p_{e_3}p_{e_4}}{4p_{e_2}}\big)\Big)\\
&\quad + \sum_{k=1}^{\infty}\Big(\frac{p_{e_3}p_{e_5}^k}{4^{k+1}}\cdot\frac{p_{e_4}}{p_{e_2}}\Big)^{\alpha} f_2\Big(t+\ln\big(\frac{p_{e_3}p_{e_5}^k}{4^{k+1}}\cdot\frac{p_{e_4}}{p_{e_2}}\big)\Big) + z_1^{(\alpha)}(t), \qquad (5.12)\\
f_2(t) &= \big(\frac{p_{e_2}}{4}\big)^{\alpha} f_1\Big(t+\ln\big(\frac{p_{e_2}}{4}\big)\Big) + \big(\frac{p_{e_2}}{4}\big)^{\alpha} f_2\Big(t+\ln\big(\frac{p_{e_2}}{4}\big)\Big) + z_2^{(\alpha)}(t),
\end{aligned}
$$

where $z_1^{(\alpha)}(t) := e^{-\alpha t}(N\big(e^t, -\Delta_{\mu|_{B_{n_t}}}\big) + \varepsilon(n_t,1))$, $B_{n_t} := S_{e_3 e_5^{n_t-1}}(B_{1,3})$, and $z_2^{(\alpha)}(t) := e^{-\alpha t}\varepsilon(2,2)$.

For $j,k\in\{1,2\}$, let $\mu_{\ell m}^{(\alpha)}$ be the discrete measure such that

$$
\begin{aligned}
&\mu_{11}^{(\alpha)}\Big(-\ln\big(\frac{p_{e_1}}{4}\big)\Big) := \Big(\frac{p_{e_1}}{4}\Big)^{\alpha}; \qquad \mu_{21}^{(\alpha)}\Big(-\ln\big(\frac{p_{e_2}}{4}\big)\Big) := \Big(\frac{p_{e_2}}{4}\Big)^{\alpha};\\
&\mu_{12}^{(\alpha)}\Big(-\ln\big(\frac{p_{e_1}}{4}+\frac{p_{e_3}p_{e_4}}{4p_{e_2}}\big)\Big) := \Big(\frac{p_{e_1}}{4}+\frac{p_{e_3}p_{e_4}}{4p_{e_2}}\Big)^{\alpha};\\
&\mu_{12}^{(\alpha)}\Big(-\ln\big(\frac{p_{e_3}p_{e_5}^k}{4^{k+1}}\cdot\frac{p_{e_4}}{p_{e_2}}\big)\Big) := \Big(\frac{p_{e_3}p_{e_5}^k}{4^{k+1}}\cdot\frac{p_{e_4}}{p_{e_2}}\Big)^{\alpha} \qquad \text{for } k\geq 1; \qquad (5.13)\\
&\mu_{22}^{(\alpha)}\Big(-\ln\big(\frac{p_{e_2}}{4}\big)\Big) := \Big(\frac{p_{e_2}}{4}\Big)^{\alpha}.
\end{aligned}
$$

Let $\mathbf{M}_\alpha(\infty)$ be defined as in (5.2). Since $\mu_{\ell m}^{(\alpha)}(\mathbb{R})>0$ for all $\ell,m\in\{1,2\}$, $\mathbf{M}_\alpha(\infty)$ is irreducible. The remaining conditions of [29, Theorem 1.1(b)] can be easily checked by using the same method as in [29, Propositions 5.2 and 5.4]. Finally, It follows from [29, Theorem 1.1(b)] that the spectral dimension of $-\Delta_\mu$ exists, and (W1) holds.

Proposition 5.6. *Let μ be the graph-directed self-similar measure defined by the GIFS above and probability vector $(p_e)_{e\in E}$, and $-\Delta_\mu$ be the associated Dirichlet Laplacian. Also, let Y_1 and Y_2 be defined as above. Then* (1.9) *holds with $J=\{j\}$ and $Y_j:=T_j(X)$ for any $j=1,2$. Moreover, (W2) holds. In particular, if $p_{e_1}=p_{e_2}=1/4$ and $p_{e_3}=p_{e_4}=p_{e_5}=1/2$, then the arithmetic case holds: there exist non-constant period functions $G_1(\cdot)$ and $G_2(\cdot)$ such that for $j=1,2$,*

$$
N(\lambda,-\Delta_{\mu|_{Y_j}}) = \lambda^{d_s/2}\big(G_j(\ln\lambda)+o(1)\big), \qquad \textit{as } \lambda\to+\infty, \qquad (5.14)
$$

where d_s is the spectral dimension of $-\Delta_\mu$.

Proof. Combining [29, Example 3.6] and [29, Proposition 4.5], we see that for each $j = 1, 2$, there exists some constant $\xi_j > 0$ such that

$$N(\lambda, -\Delta_{\mu|_{Y_j}}) \le N(\lambda, -\Delta_\mu) \le N\big(\xi_j\lambda, -\Delta_{\mu|_{Y_j}}\big). \tag{5.15}$$

Hence, the first assertion holds. Since all conditions of [29, Theorem 1.1(b)] hold, condition (2) of (W2) follows from [28, Theorem 4.1]. Hence, (W2) holds. Assume that $p_{e_2} = p_{e_4} = 1/4$ and $p_{e_1} = p_{e_3} = p_{e_5} = 1/2$. Using [28, Theorem 4.1] again, we show that the arithmetic case holds by verifying that $\mathbb{R}_M$ can be generated by a real number $a \in \mathbb{R}$. By (5.12), $M = [\mu_{ij}^{(\alpha)}]$ is a 2×2 matrix-valued Radon measure, where $\mu_{ij}^{(\alpha)}$ is defined as in (5.13). It follows from [14, Lemma 2.3] that $\mathbb{R}_M$ is the closed subgroup generated by $\operatorname{supp}(\mu_{11}^{(\alpha)})$, $\operatorname{supp}(\mu_{22}^{(\alpha)})$, and the closure of $\operatorname{supp}(\mu_{12}^{(\alpha)}) + \operatorname{supp}(\mu_{12}^{(\alpha)})$. Combining (5.13) and the assumptions on $(p_e)_{e\in E}$ shows that $\operatorname{supp}(\mu_{11}^{(\alpha)}) = \{\ln(8)\}$, $\operatorname{supp}(\mu_{21}^{(\alpha)}) = \operatorname{supp}(\mu_{22}^{(\alpha)}) = \{\ln(16)\}$, and $\operatorname{supp}(\mu_{12}^{(\alpha)}) = \{\ln(4)\}\bigcup\{\ln(2^{3k+3}) : k \ge 1\}$. Consequently, $\mathbb{R}_M$ can be generated by $\ln(2)$, which completes the proof. $\square$

Let $X = [0, 3]$ and $I := \mathbb{Z}$. Define $\tau_i(x) = x + 3i$ for all $i \in I$. Define $X_\infty := \bigcup_{i\in I} \tau_i(X)$ and let μ_∞ be defined as in (1.10). Then $X_\infty = \mathbb{R}$. Define $X_{\infty,j} := \bigcup_{i\in I} \tau_i(Y_j)$ and $\mu_{\infty,j} := \mu_\infty|_{X_{\infty,j}}$ for $j = 1, 2$.

Corollary 5.7. *Let X_∞, μ_∞, $X_{\infty,j}$ and $\mu_{\infty,j}$ be defined as above. Assume V is a locally bounded non-negative, piecewise continuous function on X_∞ so that $V(x) \sim c|x|^\beta$ for some $\beta > 1$ and $c > 0$. Let d_s be the spectral dimension of $-\Delta_\mu$. Then the following hold.*

(a) *There exist positive constants C_1, C_2 such that*

$$C_1\lambda^{d_s/2}F(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le C_2\lambda^{d_s/2}F(\lambda, V), \qquad \text{as } \lambda \to +\infty,$$

where $F(\cdot,\cdot)$ is defined as in (1.11).

(b) *As $\lambda \to +\infty$,*

$$\big(1 + o(1)\big)\sum_{j=1}^{2} g_j(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le \big(1 + o(1)\big)\sum_{j=1}^{2} g_j(\xi_j\lambda, \xi_j V),$$

where ξ_j comes from (5.15), and $g_j(\cdot,\cdot)$ is defined as in (1.12) with the non-constant period function $G_j(\cdot)$ in (5.14).

Proof. The proof is similar to that of Corollary 5.3 with Proposition 5.6 replacing Proposition 5.2. $\square$

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