DIMENSIONS IN INFINITE ITERATED FUNCTION SYSTEMS CONSISTING OF BI-LIPSCHITZ MAPPING

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ABSTRACT. We study infinite iterated functions systems (IIFSs) consisting of bi-Lipschitz mappings instead of conformal contractions, focusing on IFSs that do not satisfy the open set condition. By assuming the logarithmic distortion property and some cardinality growth condition, we obtain a formula for the Hausdorff, box, and packing dimensions of the limit set in terms of certain topological pressure. By assuming, in addition, the weak separation condition, we show that these dimensions are equal to the growth dimension of the limit set.

1. Introduction

The study of infinite iterated function systems form an important branch of the theory of IFSs. Mauldin and Urbanski [10] showed that fractal phenomena not exhibited by finite IFSs can appear in IIFSs, such as the existence of dimensionless limit sets, and that the box and packing dimensions are strictly greater than the Hausdorff dimension. They also showed that IIFSs can be used to study complex continued fractions. Fernau [7] showed that IIFSs have stronger descriptive power than finite ones; in fact, any closed set in a separable metric space is the attractor of some IIFS, while there exists a closed and bounded subset of a complete metric space that is the attractor of an IIFS but not of a finite IFS.

The open set condition (OSC) is assumed in [10]. Dimensions of IIFSs consisting of conformal contractions, but not satisfying (OSC), are studied [11] by Tong and the second author. The conformality condition is relaxed in [3], but the IFSs studied are finite.

The main purpose of this paper is to weaken all three assumptions, namely, (OSC), finiteness of the IFS, and conformality. We assume, as in [3], that the IFS maps are bi-Lipschitz essential contractions. By assuming an IIFS satisfies the logarithmic distortion property and a cardinality growth condition (CGC) (see definitions in Section 2), we obtain a formula for the Hausdorff, box, and packing dimensions of the limit set, extending a theorem in [3].

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For FIFSs of contractive similitudes satisfying (WSC), Zerner proved in [12] that if the attractor K is not contained in a hyperplane, then the Hausdorff dimension of K is equal to the growth dimension of the IFS. This result is extended by Deng and Ngai [2] to conformal FIFSs and by Ngai and Tong [11] to IIFSs of contractive similitudes. In this paper, we generalize this result to IFSs of essential contractions (see Theorem 2.18).

This paper is organized as follows. In Section 2, we first introduce some basic definitions, preliminary results, and assumptions, and then state the main results of this paper. In Section 3, we study the logarithmic distortion property (LDP) and a variant called the *logarithmic* distortion property* (L*DP). We also study two topological pressure functions. Sections 4 is devoted to the proof of the main theorems on dimension formula. In Section 5, we study the growth dimension and prove that, under suitable assumptions, it equals the other dimensions. Finally in Section 6, we illustrate our main theorems by a number examples.

2. Preliminaries and Statements of Main Results

In this section we first introduce some basic definitions and notation. Then we state the main theorems in the paper.

Definition 2.1. Let X be a non-empty compact subset of $\mathbb{R}^d$, equipped with the Euclidean metric, and let I be a countable set with at least two elements, $S_i : X \to X$, $i \in I$, be a countable family of mappings. If there exists a metric ρ on X that is topologically equivalent to the Euclidean metric and a constant $\eta \in (0,1)$ such that

$$\rho(S_i(x), S_i(y)) \le \eta\rho(x,y) \quad \text{for all } i \in I \text{ and } x,y \in X, \tag{2.1}$$

then we call $\{S_i\}_{i\in I}$ an *iterated function system (IFS) of essential contractions* with respect to ρ. We say that $\{S_i\}_{i\in I}$ is an *infinite iterated function system (IIFS) of essential contractions* if I is countably infinite, and a *finite iterated function system (FIFS) of essential contractions* if I is finite.

Let

$$I^0 := \{\emptyset\}, \qquad I^k := \{(i_1,\dots,i_k) : i_1,\dots,i_k \in I\} \text{ for } k \ge 1,$$

$$I^* := \bigcup_{k=0}^{\infty} I^k, \qquad I^\infty := \{(i_1, i_2, \dots) : i_j \in I \text{ for all } j \ge 1\}.$$

We call I^* the *space of finite words* with alphabets $i \in I$, and I^∞ the *coding space*. For $\mathbf{i} = (i_1,\dots,i_k) \in I^k$, we denote by $|\mathbf{i}| = k$ the *length* of $\mathbf{i}$ and write $S_{\mathbf{i}} := S_{i_1} \circ \cdots \circ S_{i_k}$ ($S_\emptyset$ is defined to be the identity). We often denote $\mathbf{i} = (i_1,\dots,i_k)$ simply by $\mathbf{i} = i_1 \dots i_k$ and let $\mathbf{i}^{-m} := i_1 \dots i_{k-m}$ be the word obtained from $\mathbf{i}$ by deleting its last $m (1 \le m \le k)$ alphabets. If $\mathbf{i} \in I^* \bigcup I^\infty$ and $n \ge 1$ does not exceed the length of $\mathbf{i}$, we denote by $\mathbf{i}|_n$ the word $i_1 \cdots i_n$ consisting of the first n alphabets of $\mathbf{i}$.

Since ρ is topologically equivalent to the Euclidean metric, (X, ρ) is a compact metric space in $\mathbb{R}^d$ and $\{S_i\}_{i\in I}$ is a family of contraction mapping on (X, ρ). For $E \subset \mathbb{R}^d$, we denote the diameter of E by $\operatorname{diam}(E) = \sup_{x,y\in E} \rho(x, y)$. For each $\mathbf{i} = i_1 i_2 \cdots \in I^\infty$, the compact sets $S_{\mathbf{i}|_n}(X)$, $n \geq 1$, are decreasing and their diameters converge to zero. In fact, by (2.1),

$$\operatorname{diam}(S_{\mathbf{i}|_n}(X)) \leq \eta^n \operatorname{diam}(X). \tag{2.2}$$

This implies that the set

$$\pi(\mathbf{i}) = \bigcap_{n=1}^{\infty} S_{\mathbf{i}|_n}(X)$$

is a singleton and therefore this formula defines a map $\pi : I^\infty \to X$, called the *coding map*, which, in view of (2.2), is continuous. Following [10] and [7], we define the limit set and attractor of an IFS as follows.

Definition 2.2. Let $\{S_i\}_{i\in I}$ be an IFS described as above. Define the *limit set* as

$$K := \bigcup_{\mathbf{i}\in I^\infty} \bigcap_{n=1}^{\infty} S_{\mathbf{i}|_n}(X).$$

If K is closed, it is called the *attractor* (or *fixed point*) of the IFS.

The limit set K is a Souslin set and is hence measurable with respect to Hausdorff measure (see, [4]). K is independent of the metric ρ. Moreover, K is the attractor if and only if it satisfies $K = \overline{\bigcup_{i\in I} S_i(K)}$. Since $S_j(\pi(\mathbf{i})) = \pi(j\mathbf{i})$ for any $j \in I$ and $\mathbf{i} \in I^\infty$, we have

$$\begin{aligned} K = \pi(I^\infty) = \pi\Big(\bigcup_{j\in I}\Big(\bigcup_{\mathbf{i}\in I^\infty} j\mathbf{i}\Big)\Big) &= \bigcup_{j\in I}\bigcup_{\mathbf{i}\in I^\infty} \pi(j\mathbf{i}) = \bigcup_{j\in I}\bigcup_{\mathbf{i}\in I^\infty} S_j(\pi(\mathbf{i})) \\ = \bigcup_{j\in I} S_j\Big(\bigcup_{\mathbf{i}\in I^\infty} \pi(\mathbf{i})\Big) = \bigcup_{j\in I} S_j(\pi(I^\infty)) &= \bigcup_{j\in I} S_j(K). \end{aligned}$$

Thus the limit set K satisfies

$$K = \bigcup_{i\in I} S_i(K).$$

Notice that if I is finite, then K is compact. Unlike FIFSs, K is not necessarily compact if I is countably infinite. Also, if K is not compact, it is not necessarily the unique non-empty bounded subset of $\mathbb{R}^d$ satisfying $K = \bigcup_{i\in I} S_i(K)$ (see, e.g., [11, Example 2.2]). It follows from Banach's fixed point theorem and uniform contractivity that $\{S_i\}_{i\in I}$ has a unique fixed point $\overline{K}$, and if $U \subset X$ is a nonempty *invariant set* (i.e., $\bigcup_{i\in I} S_i(U) \subset U$), then $K \subset \overline{K} \subset \overline{U}$. It is well known that if I is finite, then

$$K = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i}\in I^n} S_{\mathbf{i}}(X), \tag{2.3}$$

and thus K is a Borel set. When I is countably infinite, Mauldin and Urbański in [10] showed that (2.3) also holds under the assumption that the system $\{S_i : i \in I\}$ is *pointwise finite* (meaning that each element of X belongs to at most finitely many elements of $S_i(X)$). In this case, K is an $F_{\sigma\delta}$ and is thus a Borel set. As pointed out in [10], K may even have a much more complicated descriptive set-theoretic structure when the system is not assumed to be pointwise finite. In this paper, we will consider such IFSs of essential contractions.

Let $|\cdot|$ denote the Euclidean metric. We write

$$r_{\mathbf{i}} := \inf_{x\neq y\in X} \frac{|S_{\mathbf{i}}(x)-S_{\mathbf{i}}(y)|}{|x-y|}, \qquad R_{\mathbf{i}} := \sup_{x\neq y\in X} \frac{|S_{\mathbf{i}}(x)-S_{\mathbf{i}}(y)|}{|x-y|}, \qquad \mathbf{i}\in I^*, \tag{2.4}$$

$$r := \inf_{i\in I} r_i, \qquad R := \sup_{i\in I} R_i, \tag{2.5}$$

We also write $r_\varphi := r_{\mathbf{i}}$ and $R_\varphi := R_{\mathbf{i}}$ if $\varphi = S_{\mathbf{i}}$ for some $\mathbf{i}\in I^*$. Since for any $\mathbf{i},\mathbf{j}\in I^*$,

$$\frac{|S_{\mathbf{ij}}(x)-S_{\mathbf{ij}}(y)|}{|x-y|} = \frac{|S_{\mathbf{ij}}(x)-S_{\mathbf{ij}}(y)|}{|S_{\mathbf{j}}(x)-S_{\mathbf{j}}(y)|}\cdot\frac{|S_{\mathbf{j}}(x)-S_{\mathbf{j}}(y)|}{|x-y|},$$

the following inequalities hold (see [2, 3, 9]):

$$\begin{aligned} &R_{\mathbf{ij}} \le R_{\mathbf{i}}R_{\mathbf{j}}, \qquad && r_{\mathbf{ij}} \le R_{\mathbf{i}}r_{\mathbf{j}}, \qquad && r_{\mathbf{ij}} \le r_{\mathbf{i}}R_{\mathbf{j}},\\ &r_{\mathbf{ij}} \ge r_{\mathbf{i}}r_{\mathbf{j}}, && R_{\mathbf{ij}} \ge r_{\mathbf{i}}R_{\mathbf{j}}, && R_{\mathbf{ij}} \ge R_{\mathbf{i}}r_{\mathbf{j}}, \end{aligned} \tag{2.6}$$

which will be used repeatedly in the rest of this paper.

Assumption A. *Throughout this paper we assume that $r_i > 0$ for each i, equivalently, S_i, $i\in I$, are bi-Lipschitz mappings with respect to Euclidean metric. In particular, $r > 0$ when I is finite.*

Remark 2.3. It is possible that $R \ge 1$. Since S_i, $i\in I$, are essential contractions, $R_{\mathbf{i}}$ converges uniformly to 0 as $|\mathbf{i}|$ tends to infinity. As a consequence, we also have $0 \le r < 1$ and $R < +\infty$.

For any $E\subset\mathbb{R}^d$, we use $\dim_{\mathrm{H}}(E), \dim_{\mathrm{P}}(E), \dim_{\mathrm{B}}(E), \mathscr{H}^s(E), \mathcal{L}^d, |E|$, and E° to denote, respectively, the Hausdorff dimension, packing dimension, box dimension, s-dimensional Hausdorff measure, d-dimensional Lebesgue measure, Euclidean diameter, and interior of E. For any set A, we let $\#A$ denote its cardinality. A set $U\subset X$ is said to be open if it is open in the relative Euclidean topology of X.

Fix an invariant set $U\subset X$ and let $0<b<1$. Define

$$\begin{aligned} &\mathscr{S}_{I,b} := \{\mathbf{i}=(i_1,\dots,i_n)\in I^* : R_{\mathbf{i}} \le b < R_{\mathbf{i}^-}\},\\ &\mathscr{S}^*_{I,b}(U) := \{\mathbf{i}=(i_1,\dots,i_n)\in I^* : \mathcal{L}^d(S_{\mathbf{i}}(U)) \le b^d\mathcal{L}^d(U) < \mathcal{L}^d(S_{\mathbf{i}^-}(U))\},\\ &\mathscr{A}_{I,b} := \{S_{\mathbf{i}} : \mathbf{i}\in\mathscr{S}_{I,b}\},\\ &\mathscr{A}^*_{I,b}(U) := \{S_{\mathbf{i}} : \mathbf{i}\in\mathscr{S}^*_{I,b}(U)\}. \end{aligned} \tag{2.7}$$

Remark 2.4. Since R can be greater than 1, for $(i_1, i_2, \dots)\in I^\infty$, there could be more than one prefix $\mathbf{i}=(i_1,\dots,i_n)\in I^*$ such that $\mathbf{i}\in\mathscr{S}_{I,b}$ (resp. $\mathscr{S}^*_{I,b}(U)$). However, by Remark 2.3, the number of such prefixes must be finite.

Remark 2.5. It is possible that $S_{\mathbf{i}} = S_{\mathbf{i}'}$ for distinct $\mathbf{i},\mathbf{i}'\in I^*$; we identify such $S_{\mathbf{i}}$ and $S_{\mathbf{i}'}$. For IFSs of contractive similitudes, $\mathscr{S}_{I,b} = \mathscr{S}^*_{I,b}(U)$ and so $\mathscr{A}_{I,b} = \mathscr{A}^*_{I,b}(U)$. In general, however, they need not be the same. For an IIFS, both $\mathscr{A}_{I,b}$ and $\mathscr{A}^*_{I,b}(U)$ are countably infinite sets.

Definition 2.6. Let $X \subset \mathbb{R}^d$ be nonempty compact subset with $X^\circ \neq \emptyset$ and let $S_i : X \to X, i \in I$, be bi-Lipschitz essential contractions.

(a) We say that $\{S_i\}_{i\in I}$ has the *logarithmic distortion property (LDP)* if there is a constant $\xi > 0$ such that

$$\lim_{b\to 0^+} \sup_{\mathbf{i}\in \mathscr{S}_{I,b}} \frac{b}{r_{\mathbf{i}} |\ln b|^{\xi}} = 0.$$

(b) Let $U \subset X$ be an invariant set. We say that $\{S_i\}_{i\in I}$ has the *logarithmic* distortion property (L*DP)* if there is a constant $\zeta > 0$ such that

$$\lim_{b\to 0^+} \sup_{\mathbf{i}\in \mathscr{S}^*_{I,b}(U)} \frac{b}{r_{\mathbf{i}} |\ln b|^{\zeta}} = 0 \qquad \text{and} \qquad \lim_{b\to 0^+} \sup_{\mathbf{i}\in \mathscr{S}^*_{I,b}(U)} \frac{R_{\mathbf{i}}}{r_{\mathbf{i}} |\ln b|^{\zeta}} = 0. \tag{2.8}$$

Mauldin and Urbański [10] defined the *bounded distortion property (BDP)* of an IFS of injective C^1 conformal contractions. We extend this definition directly to IFSs of essential contractions as follows.

Definition 2.7. An IFS $\{S_i\}_{i\in I}$ of essential contractions as in Definition 2.1 is said to have the *bounded distortion property (BDP)* if there exists a constant $c > 0$ such that

$$\frac{R_{\mathbf{i}}}{r_{\mathbf{i}}} \le c \quad \text{for all } \mathbf{i} \in I^*.$$

Remark 2.8. In the above definitions, we do not assume that the IFS maps are differentiable.

Definition 2.9. Let X and $\{S_i\}_{i\in I}$ satisfy the hypotheses of Definition 2.6, $U \subset X$ be a bounded invariant set that is open in the relative topology of X with $\mathcal{L}^d(U) > 0$, and Φ be a subset of $\{S_{\mathbf{i}} : \mathbf{i} \in I^*\}$. We call a subcollection $\{\varphi_j\}_{j\in J} \subset \Phi$ a *packing family* of Φ with respect to U if it satisfies the following two conditions:

(i) $\varphi_j(U), j \in J$, are pairwise disjoint;

(ii) for any $\varphi \in \Phi$, $\varphi(U)$ intersects at least one $\varphi_j(U)$.

We denote by $\mathscr{P}_{I,U}(b)$ the class of all packing families of $\mathscr{A}_{I,b}$ with respect to U, and denote by $\mathscr{P}^*_{I,U}(b)$ the class of all packing families of $\mathscr{A}^*_{I,b}(U)$.

Remark 2.10. The definition of packing familiy here is a generalization of that in [3] to the case I is countably infinite.

For an index set I, we define, throughout the rest of this paper,

$$\mathcal{F} = \mathcal{F}(I) := \{J \subset I : J \text{ is finite}\}. \tag{2.9}$$

Let $J \in \mathcal{F}$ and $\{S_i\}_{i\in J}$ be the associated FIFS. For any $0 < b < 1$, let $\mathscr{P}_{J,U}(b)$ and $\mathscr{P}^*_{J,U}(b)$ be the corresponding packing families as in Definition 2.9.

Definition 2.11. Let $X, \{S_i\}_{i\in I}$ and U satisfy the hypotheses of Definition 2.9. We say that $\{S_i\}_{i\in I}$ satisfies *cardinality growth condition (CGC)* with respect to U if

$$\gamma := \sup_{J\in\mathcal{F}} \overline{\lim_{b\to 0+}} \frac{\ln(\sup_{\Phi\in\mathscr{P}_{J,U}(b)} \#\Phi)}{-\ln b} < \infty. \tag{2.10}$$

We generalize the definition of topological pressure functions in [3] as follows.

Definition 2.12. Let $X, \{S_i\}_{i\in I}$ and U satisfy the hypotheses of Definition 2.9 and fix $\lambda \in (0,1)$. Define

$$\underline{\mathcal{Q}}_\lambda(s) := \sup_{J\in\mathcal{F}} \Big\{ \underline{\lim_{n\to\infty}} \frac{1}{n} \ln\Big(\inf_{\Phi\in\mathscr{P}_{J,U}(\lambda^n)} \sum_{\varphi\in\Phi} R_\varphi^s \Big) \Big\}, \quad s\in\mathbb{R},$$

$$\overline{\mathcal{Q}}_\lambda(s) := \sup_{J\in\mathcal{F}} \Big\{ \overline{\lim_{n\to\infty}} \frac{1}{n} \ln\Big(\sup_{\Phi\in\mathscr{P}_{J,U}(\lambda^n)} \sum_{\varphi\in\Phi} R_\varphi^s \Big) \Big\}, \quad s\in\mathbb{R}.$$

We call $\underline{\mathcal{Q}}_\lambda$ (resp. $\overline{\mathcal{Q}}_\lambda$) the *lower* (resp. *upper*) *topological pressure function with scale* λ. If $\underline{\mathcal{Q}}_\lambda = \overline{\mathcal{Q}}_\lambda$, we denote by $\mathcal{Q}_\lambda$ the common function and call it a *topological pressure function with scale* λ. Note that λ is fixed and s is the variable of the functions $\underline{\mathcal{Q}}_\lambda(s)$ and $\overline{\mathcal{Q}}_\lambda(s)$.

Similar to Definition 2.12, we define

$$\underline{\mathcal{Q}}^*_\lambda(s) := \sup_{J\in\mathcal{F}} \Big\{ \underline{\lim_{n\to\infty}} \frac{1}{n} \ln\Big(\inf_{\Phi\in\mathscr{P}^*_{J,U}(\lambda^n)} \sum_{\varphi\in\Phi} [\mathcal{L}^d(\varphi(U))]^{s/d} \Big) \Big\}, \quad s\in\mathbb{R},$$

$$\overline{\mathcal{Q}}^*_\lambda(s) := \sup_{J\in\mathcal{F}} \Big\{ \overline{\lim_{n\to\infty}} \frac{1}{n} \ln\Big(\sup_{\Phi\in\mathscr{P}^*_{J,U}(\lambda^n)} \sum_{\varphi\in\Phi} [\mathcal{L}^d(\varphi(U))]^{s/d} \Big) \Big\}, \quad s\in\mathbb{R}.$$

Definition 2.13. Let X and $\{S_i\}_{i\in I}$ satisfy the hypotheses of Definition 2.6. We say that $\{S_i\}_{i\in I}$ satisfies the *weak separation condition (WSC)* if there exist an invariant subset $D\subset X$ with $D^\circ \neq \emptyset$, called a *WSC-region*, and a constant $\kappa\in\mathbb{N}$ such that

$$\sup_{x\in X} \#\big\{\varphi\in\mathscr{A}_{I,b} : x\in\varphi(D)\big\} \le \kappa \quad \text{for all}\, b\in(0,1). \tag{2.11}$$

If $E\subset X$ is an invariant subset and (2.11) holds, we call E a *WSC-set*.

For any $J\in\mathcal{F}$, we denote by K^J the attractor of the FIFS $\{S_i\}_{i\in J}$ and write $\alpha_J := \dim_{\mathrm{H}} K^J$.

Definition 2.14. Let X and $\{S_i\}_{i\in I}$ satisfy the hypotheses of Definition 2.6. We call K a *quasi s-set* if $\mathcal{H}^{\alpha_J}(K^J) < \infty$ for any $J\in\mathcal{F}$.

Remark 2.15. If $\{S_i\}_{i\in I}$ satisfies (BDP) and (WSC), then it follows from [9, Theorem 3.2] that K is a quasi s-set.

We now state the first two main results of this paper.

Theorem 2.16. *Let $X,\{S_i\}_{i\in I},U$ satisfy the hypotheses of Definition 2.9. Assume that (LDP) and (CGC) hold. Then for any $\lambda\in(0,1)$ and any sequence of packing families $\{S_{\mathbf{i}_{n,j}}\}_{j=1}^{k_n^J}\in\mathscr{P}_{J,U}(\lambda^n)$, $n\in\mathbb{N}$, the following hold:*

(a) *for all $s\in\mathbb{R}$,*

$$\underline{\mathcal{Q}}_\lambda(s)=\overline{\mathcal{Q}}_\lambda(s)=\sup_{J\in\mathcal{F}}\bigg\{\lim_{n\to\infty}\frac{1}{n}\ln\Big(\sum_{j=1}^{k_n^J}R^s_{\mathbf{i}_{n,j}}\Big)\bigg\}=(s-\dim_{\mathrm{H}}K)\ln\lambda;$$

(b) *let γ be defined in Definition 2.11, then*

$$\dim_{\mathrm{H}}K=\dim_{\mathrm{P}}K=\dim_{\mathrm{B}}K=\gamma=\sup_{J\in\mathcal{F}}\bigg\{\lim_{n\to\infty}\frac{\ln k_n^J}{-n\ln\lambda}\bigg\}.$$

Theorem 2.17. *Let $X,\{S_i\}_{i\in I},U$ satisfy the hypotheses of Definition 2.9. Assume that (LDP), (L*DP) and (CGC) hold. Then for any $\lambda\in(0,1)$ and any sequence of packing families $\{S_{\mathbf{i}_{n,j}}\}_{j=1}^{k_n^J}\in\mathscr{P}^*_{J,U}(\lambda^n)$, $n\in\mathbb{N}$, we have*

$$\begin{aligned}\underline{\mathcal{Q}}^*_\lambda(s)&=\overline{\mathcal{Q}}^*_\lambda(s)=\sup_{J\in\mathcal{F}}\bigg\{\lim_{n\to\infty}\frac{1}{n}\ln\Big(\sum_{j=1}^{k_n^J}[\mathcal{L}^d(S_{\mathbf{i}_{n,j}}(U)]^{s/d}\Big)\bigg\}=\mathcal{Q}_\lambda(s)\\&=\big(s-\dim_{\mathrm{H}}K\big)\ln\lambda\qquad\text{for all } s\in\mathbb{R}.\end{aligned}$$

Let $X\subset\mathbb{R}^d$ be nonempty compact subset with $X^\circ\neq\emptyset$, and $\{S_i\}_{i\in I}$ be a IFS of essential contractions on X (I is finite or countably infinite). Following Zerner [12], we define the *growth dimension* of an FIFS $\{S_i\}_{i\in I}$ as

$$d_G=\lim_{b\to0+}\frac{\ln\#\mathscr{A}_{I,b}}{-\ln b}.$$

For each finite set $J\subset I$, we denote by d_G^J the growth dimension of the associated FIFS $\{S_i\}_{i\in J}$. Following [11], we define the growth dimension of the IIFS $\{S_i(x)\}_{i\in I}$ (In this case, $\#(\mathscr{A}_{I,b})=\infty$ for any $b\in(0,1)$) as

$$d_G=\sup\{d_G^J: J\subset I\text{ is finite}\}.\tag{2.12}$$

Theorem 2.18. *Let $X,\{S_i\}_{i\in I},U$ satisfy the hypotheses of Definition 2.9 and let K be the associated limit set. Assume that (LDP), (WSC) and (CGC) hold, and K is both a WSC-set and a quasi s-set. Then*

$$\dim_{\mathrm{H}}K=\dim_{\mathrm{P}}K=\dim_{\mathrm{B}}K=\gamma=d_G,$$

where d_G is the growth dimension of K.

3. (LDP), (L*DP), and Topological Pressure Functions

3.1. **Properties of (LDP) and (L*DP).** For any $E\subset X$ and any $\mathbf{i}\in I^*$, the following inequalities will be used repeatedly,

$$\begin{gathered}r_{\mathbf{i}}\,|E|\le|S_{\mathbf{i}}(E)|\le R_{\mathbf{i}}\,|E|\,,\\ r_{\mathbf{i}}(\mathcal{L}^d(E))^{1/d}\le(\mathcal{L}^d(S_{\mathbf{i}}(E)))^{1/d}\le R_{\mathbf{i}}(\mathcal{L}^d(E))^{1/d}.\end{gathered}\tag{3.1}$$

Lemma 3.1. *Assume the same hypotheses on X and $\{S_i\}_{i\in I}$ as in Definition 2.9, and assume that (LDP) holds. Let $R_{\mathbf{i}}$ and $r_{\mathbf{i}}$ be defined as in* (2.4)*, and $\xi>0$ be defined as in Definition 2.6(a). Then for all $b_0>0$ sufficiently small, there exists a constant $c_1=c_1(b_0)>0$ such that*

$$\frac{b}{c_1\,|\ln b|^{\xi}}\le r_{\mathbf{i}}\le R_{\mathbf{i}}\le b \text{ for all } \mathbf{i}\in\mathscr{S}_{I,b} \text{ and } b\in(0,b_0). \tag{3.2}$$

Moreover, $c_1(b_0)\to 0$ as $b_0\to 0$.

Proof. By Definition 2.6, for sufficiently small $b_0>0$,

$$0<c_1:=\sup_{b\in(0,b_0)}\sup_{\mathbf{i}\in\mathscr{S}_{I,b}}\frac{b}{r_{\mathbf{i}}\,|\ln b|^{\xi}}<\infty.$$

This implies that $\frac{b}{r_{\mathbf{i}}|\ln b|^{\xi}}\le c_1$ for any $b\in(0,b_0)$. Thus the conclusion follows from the definition of $\mathscr{S}_{I,b}$ and the fact $r_{\mathbf{i}}\le R_{\mathbf{i}}$. □

Lemma 3.2. *Assume the same hypotheses on X, $\{S_i\}_{i\in I}$ and U as in Definition 2.9, and assume that (L*DP) holds. Let $R_{\mathbf{i}}$ and $r_{\mathbf{i}}$ be as in* (2.4)*, and $\zeta>0$ be defined as in Definition 2.6(b). Then for all $b_0>0$ sufficiently small, there is a constant $c_2=c_2(b_0)>0$ such that*

$$\frac{b}{c_2\,|\ln b|^{\zeta}}\le r_{\mathbf{i}}\le R_{\mathbf{i}}\le c_2\,|\ln b|^{\zeta}\,b \quad \text{for all } \mathbf{i}\in\mathscr{S}^*_{I,b}(U) \text{ and } b\in(0,b_0). \tag{3.3}$$

Moreover, $c_2(b_0)\to 0$ as $b_0\to 0$.

Proof. By Definition 2.6(b), there exists a constant $c_3>0$ such that for sufficiently small $b_0>0$,

$$0<c_3=\sup_{b\in(0,b_0)}\sup_{\mathbf{i}\in\mathscr{S}^*_{I,b}(U)}\frac{b}{r_{\mathbf{i}}\,|\ln b|^{\zeta}}<\infty$$

and a constant $c_4>0$ such that

$$0<c_4=\sup_{b\in(0,b_0)}\sup_{\mathbf{i}\in\mathscr{S}^*_{I,b}(U)}\frac{R_{\mathbf{i}}}{r_{\mathbf{i}}\,|\ln b|^{\zeta}}<\infty.$$

This implies that for any $b\in(0,b_0)$,

$$\frac{b}{r_{\mathbf{i}}\,|\ln b|^{\zeta}}\le c_3 \qquad\text{and}\qquad \frac{R_{\mathbf{i}}}{r_{\mathbf{i}}\,|\ln b|^{\zeta}}\le c_4. \tag{3.4}$$

By the definition of $\mathscr{S}^*_{I,b}(U)$ and (3.1), $\mathbf{i}\in\mathscr{S}^*_{I,b}(U)$ implies that

$$r_{\mathbf{i}}\le\left(\frac{\mathcal{L}^d(S_{\mathbf{i}}(U))}{\mathcal{L}^d(U)}\right)^{1/d}\le b. \tag{3.5}$$

By letting $c_2:=\max\{c_3,c_4\}$ and combining (3.4), (3.5), and the fact that $r_{\mathbf{i}}\le R_{\mathbf{i}}$, we obtain (3.3). □

Remark 3.3. If $\{S_i\}_{i\in I}$ satisfies (BDP), then there is a constant $c>0$ such that $R_{\mathbf{i}}/r_{\mathbf{i}}\le c$ for all $\mathbf{i}\in\mathscr{S}_{I,b}\cup\mathscr{S}^*_{I,b}(U)$. Thus from (3.2) and (3.3), respectively, (LDP) and (L*DP) hold. Examples of IFSs satisfying (LDP) and (L*DP) but not (BDP) will be given in Section 6.

Let $\sigma : I^\infty \to I^\infty$ denote the left shift map on I^∞, i.e., $\sigma(\mathbf{i}) = i_2 i_3 \cdots$ for $\mathbf{i} = i_1 i_2 \cdots$. Since it is possible that $R_{\mathbf{ij}} \geq R_{\mathbf{i}}$ for some $\mathbf{i}, \mathbf{j} \in I^*$ (see Remark 2.4), we have the following property.

Proposition 3.4. *Let $r < 1$ and $\mathscr{S}_{I,R_{\mathbf{i}}}$ be defined as in* (2.5) *and* (2.7) *respectively. Assume that $r > 0$. Then there exists an integer $k_0 > 0$ such that for any $\mathbf{i} \in I^*$ with $|\mathbf{i}| > k_0$, there is a decomposition $\mathbf{i} = \mathbf{i}_1 \mathbf{i}_2$ with $|\mathbf{i}_2| \leq k_0$ such that $\mathbf{i}_1 \in \mathscr{S}_{I,R_{\mathbf{i}}}$.*

Proof. As mentioned in Remark 2.3, $R_{\mathbf{i}} \to 0$ uniformly as $|\mathbf{i}| \to \infty$. Thus there exists an integer $k_0 > 0$ such that $R_{\mathbf{i}} \leq r$ for all $\mathbf{i} \in I^*$ with $|\mathbf{i}| \geq k_0$. Let $\mathbf{i} \in I^*$ with $|\mathbf{i}| = n > k_0$. First we check $R_{\mathbf{i}^{-1}}$. If $R_{\mathbf{i}} < R_{\mathbf{i}^{-1}}$, then we take $\mathbf{i}_1 = \mathbf{i}$ and $\mathbf{i}_2 = \emptyset$, and thus the conclusion holds. If not, we check $R_{\mathbf{i}^{-2}}$. If $R_{\mathbf{i}} < R_{\mathbf{i}^{-2}}$, then $R_{\mathbf{i}^{-1}} \leq R_{\mathbf{i}} < R_{\mathbf{i}^{-2}}$. Thus the conclusion holds by letting $\mathbf{i}_1 = \mathbf{i}^{-1}$ and $\mathbf{i}_2 = \sigma^{n-1}(\mathbf{i})$. Continue. Note that there exists some $1 \leq k \leq n-1$ such that $R_{2^{-k+1}} \leq R_{\mathbf{i}} < R_{\mathbf{i}}^{-k}$, for otherwise, one would get $r \leq R_{\mathbf{i}^{n-1}} \leq R_{\mathbf{i}} < r$, a contradiction. We further claim that there exists some $k \in \mathbb{N}$ with $1 \leq k \leq k_0$ such $R_{\mathbf{i}^{-k+1}} \leq R_{\mathbf{i}} < R_{\mathbf{i}^{-k}}$. To see this we suppose on the contrary that there exists some $\ell \in \mathbb{N}$ satisfying $k_0+1 \leq \ell \leq n-1$ such that $R_{\mathbf{i}^{-(\ell-1)}} \leq R_{\mathbf{i}} < R_{\mathbf{i}^{-\ell}}$. Then one would get

$$R_{\mathbf{i}^{-(\ell-1)}} \leq R_{\mathbf{i}} \leq R_{\mathbf{i}^{-(\ell-1)}} R_{\sigma^{n-(\ell-1)}(\mathbf{i})} \leq R_{\mathbf{i}^{-(\ell-1)}} r < R_{\mathbf{i}^{-(\ell-1)}},$$

where the third inequality is because $|\sigma^{n-(\ell-1)}(\mathbf{i})| = \ell - 1 \geq k_0$. This contradiction proofs the claim. The asserted result now follows by letting $\mathbf{i}_1 = \mathbf{i}^{-k}$ and $\mathbf{i}_2 = \sigma^{n-k}(\mathbf{i})$. □

Lemma 3.5. *Assume the same hypotheses on X, $\{S_i\}_{i\in I}$ and U as in Definition 2.9 and r be defined as in* (2.5)*. If $r > 0$, then (LDP) implies (L*DP).*

Proof. Let k_0 be as in Proposition 3.4. For $\mathbf{t} \in I^*$ with $|\mathbf{t}| = n \geq k_0$, let $n = lk_0 + t$ with $0 \leq t < k_0$. Then $r^n \leq r_{\mathbf{t}} \leq R_{\mathbf{t}} \leq r^l$. Taking logarithm, we have

$$n \ln r \leq \ln r_{\mathbf{t}} \leq \ln R_{\mathbf{t}} \leq l \ln r \leq \frac{n \ln r}{k_0}, \quad |\mathbf{t}| \geq k_0.$$

This implies that

$$\frac{\ln R_{\mathbf{t}}}{\ln r_{\mathbf{t}}} \geq \frac{1}{k_0} > 0. \tag{3.6}$$

Let $\xi > 0$ be as in Definition 2.6(a) and $b_0 \in (0,1)$ be sufficiently small. Since (LDP) holds, we have from Lemma 3.1 that there is a constant $c_1 > 0$ such that

$$\frac{b}{c_1 |\ln b|^\xi} \leq r_{\mathbf{i}} \leq R_{\mathbf{i}} \leq b \quad \text{for all } \mathbf{i} \in \mathscr{S}_{I,b} \text{ and } b \in (0, b_0). \tag{3.7}$$

To prove (2.8), we assume, without loss of generality, that $|\mathbf{j}| > k_0$ for any $\mathbf{j} \in \mathscr{S}^*_{I,b}(U)$ and $R_{\mathbf{j}} < b_0$, since we need only consider sufficiently small $b > 0$. By the definition of $\mathscr{S}^*_{I,b}(U)$, we have

$$\left(\frac{\mathcal{L}^d(S_{\mathbf{j}}(U))}{\mathcal{L}^d(U)}\right)^{1/d} \leq b < \left(\frac{\mathcal{L}^d(S_{\mathbf{j}^{-1}}(U))}{\mathcal{L}^d(U)}\right)^{1/d},$$

and thus $r_{\mathbf{j}} \leq b < R_{\mathbf{j}^{-1}}$. As $R_{\mathbf{j}} \geq R_{\mathbf{j}^{-1}} r$, we get

$$r_{\mathbf{j}} \leq b < r^{-1} R_{\mathbf{j}}. \tag{3.8}$$

Combining (3.6) and (3.8), we see that exists some constant $\tilde{c} \geq 1$ such that

$$\tilde{c}^{-1} \leq \frac{\ln R_{\mathbf{j}}}{\ln b} \leq \tilde{c}. \tag{3.9}$$

As $|\mathbf{j}| > k_0$, Proposition 3.4 implies that there exists a decomposition $\mathbf{j} = \mathbf{j}_1\mathbf{j}_2$ with $|\mathbf{j}_2| \leq k_0$ such that $\mathbf{j}_1 \in \mathscr{S}_{I,R_{\mathbf{j}}}$. As $R_{\mathbf{j}} < b_0$, substituting b and $\mathbf{i}$ in (3.7) by $R_{\mathbf{j}}$ and $\mathbf{j}_1$ respectively yields

$$\frac{R_{\mathbf{j}}}{r_{\mathbf{j}_1} |\ln R_{\mathbf{j}}|^{\xi}} \leq c_1.$$

Using (2.6) and the fact that $\mathbf{j}_1 \in \mathscr{S}_{R_{\mathbf{j}}}$ and $|\mathbf{j}_2| \leq k_0$, we have

$$r^{k_0} r_{\mathbf{j}_1} \leq r_{\mathbf{j}_1} r_{\mathbf{j}_2} \leq r_{\mathbf{j}} \leq r_{\mathbf{j}_1} R^{|\mathbf{j}_2|}. \tag{3.10}$$

Combining (3.8)–(3.10), we get

$$\frac{b}{r_{\mathbf{j}} |\ln b|^{\xi}} \leq \frac{r^{-1} R_{\mathbf{j}} \tilde{c}^{\xi}}{r^{k_0} r_{\mathbf{j}_1} |\ln R_{\mathbf{j}}|^{\xi}} \leq \frac{c_1 \tilde{c}^{\xi}}{r^{k_0+1}} =: c_3 < \infty \tag{3.11}$$

and

$$\frac{R_{\mathbf{j}}}{b |\ln b|^{\xi}} \leq \frac{R_{\mathbf{j}}}{r_{\mathbf{j}} |\ln b|^{\xi}} \leq \frac{R_{\mathbf{j}} \tilde{c}^{\xi}}{r^{k_0} r_{\mathbf{j}_1} |\ln R_{\mathbf{j}}|^{\xi}} \leq \frac{c_1 \tilde{c}^{\xi}}{r^{k_0}} \leq \frac{c_1 \tilde{c}^{\xi}}{r^{k_0+1}} = c_3 < \infty. \tag{3.12}$$

Furthermore, we can see from the proof of Lemma 3.1 that $c_1 \to 0$ as $b_0 \to 0$. Therefore $c_3 \to 0$ as $b_0 \to 0$, and thus the (L*DP) holds from the (3.11), (3.12) and the fact $b \in (0, b_0)$. □

3.2. **Properties of topological pressures.**

Proposition 3.6. *Let $X, \{S_i\}_{i \in I}$ and U satisfy the hypotheses of Definition 2.9. Let $\lambda \in (0,1)$ and r be defined as in* (2.5). *Assume that $r > 0$ and (CGC) holds. Then both $\underline{\mathcal{Q}}_{\lambda}(s)$ and $\overline{\mathcal{Q}}_{\lambda}(s)$ are real-valued, strictly decreasing, and continuous functions on $\mathbb{R}$ that tend to $-\infty$ and ∞ as s tends to ∞ and $-\infty$, respectively. Moreover, $\overline{\mathcal{Q}}_{\lambda}(0) \geq \underline{\mathcal{Q}}_{\lambda}(0) \geq 0$ and $\overline{\mathcal{Q}}_{\lambda}(s)$ is convex on $\mathbb{R}$.*

Proof. Let k_0 be described as in Proposition 3.4. Write $C := \sup\{R_{\mathbf{i}} : |\mathbf{i}| < k_0\}$. Then $C < \infty$. For any $J \in \mathcal{F}$, let $\varphi = S_{i_1 \cdots i_k} \in \Phi \in \mathscr{P}_{J,U}(\lambda^n)$. Let $k - 1 = lk_0 + m$ with $0 \leq m < k_0$, where $l \in \mathbb{N} \cup \{0\}$. Then we have $R_{i_1 \cdots i_{k-1}} \leq Cr^l \leq C' r^{k/k_0}$ for some constant C'. Hence

$$r^k \leq r_{\varphi} \leq R_{\varphi} \leq \lambda^n < R_{i_1 \cdots i_{k-1}} \leq C' r^{k/k_0}. \tag{3.13}$$

It follows that

$$k < nk_0 \log_r \lambda - k_0 \log_r C', \tag{3.14}$$

and thus

$$R_{\varphi} \geq r^{nk_0 \log_r \lambda - k_0 \log_r C'}. \tag{3.15}$$

It follows that from (3.13) and (3.15) that when $s \geq 0$,

$$r^{(nk_0 \log_r \lambda - k_0 \log_r C')s} \leq \sum_{\varphi \in \Phi} R_{\varphi}^s \leq \#\Phi \lambda^{ns},$$

and when $s < 0$,

$$\lambda^{ns} \leq \sum_{\varphi \in \Phi} R_\varphi^s \leq \#\Phi r^{(nk_0 \log_r \lambda - k_0 \log_r C')s}.$$

Thus if $s \geq 0$,

$$\begin{aligned} sk_0 \ln \lambda - \frac{sk_0 \ln C'}{n} &\leq \frac{\ln(\inf_{\Phi \in \mathscr{P}_{J,U}(\lambda^n)} \sum_{\varphi \in \Phi} R_\varphi^s)}{n} \\ &\leq \frac{\ln(\sup_{\Phi \in \mathscr{P}_{J,U}(\lambda^n)} \sum_{\varphi \in \Phi} R_\varphi^s)}{n} \\ &\leq \frac{\ln(\sup_{\Phi \in \mathscr{P}_{J,U}(\lambda^n)} \#\Phi)}{n} + s \ln \lambda \\ &\leq \frac{\ln(\sup_{\Phi \in \mathscr{P}_{J,U}(\lambda^n)} \#\Phi)}{-\ln \lambda^n}(-\ln \lambda) + s \ln \lambda; \end{aligned}$$

if $s < 0$,

$$\begin{aligned} s \ln \lambda &\leq \frac{\ln(\inf_{\Phi \in \mathscr{P}_{J,U}(\lambda^n)} \sum_{\varphi \in \Phi} R_\varphi^s)}{n} \\ &\leq \frac{\ln(\sup_{\Phi \in \mathscr{P}_{J,U}(\lambda^n)} \sum_{\varphi \in \Phi} R_\varphi^s)}{n} \\ &\leq \frac{\ln(\sup_{\Phi \in \mathscr{P}_{J,U}(\lambda^n)} \#\Phi)}{n} + sk_0 \log_r \lambda \ln r - \frac{sk_0 \log_r C' \ln r}{n} \\ &= \frac{\ln(\sup_{\Phi \in \mathscr{P}_{J,U}(\lambda^n)} \#\Phi)}{-\ln \lambda^n}(-\ln \lambda) + sk_0 \ln \lambda - \frac{sk_0 \ln C'}{n}. \end{aligned}$$

Letting $n \to \infty$ and using (2.10) and the definitions of $\overline{\mathcal{Q}}_\lambda(s)$ and $\underline{\mathcal{Q}}_\lambda(s)$, we get

$$\begin{aligned} sk_0 \ln \lambda \leq \underline{\mathcal{Q}}_\lambda(s) \leq \overline{\mathcal{Q}}_\lambda(s) \leq (s - \gamma) \ln \lambda, &\quad \text{if } s \geq 0; \\ s \ln \lambda \leq \underline{\mathcal{Q}}_\lambda(s) \leq \overline{\mathcal{Q}}_\lambda(s) \leq (sk_0 - \gamma) \ln \lambda, &\quad \text{if } s < 0. \end{aligned}$$

Hence $\overline{\mathcal{Q}}_\lambda(s)$ and $\underline{\mathcal{Q}}_\lambda(s)$ are real-valued, $\overline{\mathcal{Q}}_\lambda(0) \geq \underline{\mathcal{Q}}_\lambda(0) \geq 0$. Moreover, since $0 < \lambda < 1$, we have $\lim_{s\to\infty} \overline{\mathcal{Q}}_\lambda(s) = \lim_{s\to\infty} \underline{\mathcal{Q}}_\lambda(s) = -\infty$ and $\lim_{s\to-\infty} \overline{\mathcal{Q}}_\lambda(s) = \lim_{s\to-\infty} \underline{\mathcal{Q}}_\lambda(s) = \infty$.

Next, since $\overline{\mathcal{Q}}_\lambda(s)$ and $\underline{\mathcal{Q}}_\lambda(s)$ are real-valued, for any $\delta > 0$, we have

$$\begin{aligned} \overline{\mathcal{Q}}_\lambda(s + \delta) &\leq \sup_{J \in \mathcal{F}} \left\{ \overline{\lim_{n\to\infty}} \frac{1}{n} \ln \left(\sup_{\Phi \in \mathscr{P}_{J,U} \lambda^n} \sum_{\varphi \in \Phi} R_\varphi^s \lambda^{n\delta} \right) \right\} \\ &= \overline{\mathcal{Q}}_\lambda(s) + \delta \ln \lambda < \overline{\mathcal{Q}}_\lambda(s), \end{aligned} \tag{3.16}$$

and by (3.15),

$$\begin{aligned} \overline{\mathcal{Q}}_\lambda(s + \delta) &\geq \sup_{J \in \mathcal{F}} \left\{ \overline{\lim_{n\to\infty}} \frac{1}{n} \ln \left(\sup_{\Phi \in \mathscr{P}_{J,U} \lambda^n} \sum_{\varphi \in \Phi} R_\varphi^s r^{\delta(nk_0 \log_r \lambda - k_0 \log_r C')} \right) \right\} \\ &= \overline{\mathcal{Q}}_\lambda(s) + \delta k_0 \ln \lambda, \end{aligned}$$

that is,

$$\overline{\mathcal{Q}}_\lambda(s) + \delta k_0 \ln \lambda \leq \overline{\mathcal{Q}}_\lambda(s + \delta) \leq \overline{\mathcal{Q}}_\lambda(s) + \delta \ln \lambda < \overline{\mathcal{Q}}_\lambda(s). \tag{3.17}$$

Moreover, for any $\delta > 0$, we also have

$$\begin{aligned}\overline{\mathcal{Q}}_\lambda(s-\delta) &\le \sup_{J\in\mathcal{F}}\left\{\overline{\lim_{n\to\infty}}\frac{1}{n}\ln\left(\sup_{\Phi\in\mathscr{P}_{J,U}\lambda^n}\sum_{\varphi\in\Phi}R_\varphi^s r^{-\delta(nk_0\log_r\lambda - k_0\log_r C')}\right)\right\}\\ &\le \overline{\mathcal{Q}}_\lambda(s)) - \delta k_0\ln\lambda,\end{aligned}$$

$$\begin{aligned}\overline{\mathcal{Q}}_\lambda(s-\delta) &\ge \sup_{J\in\mathcal{F}}\left\{\overline{\lim_{n\to\infty}}\frac{1}{n}\ln\left(\sup_{\Phi\in\mathscr{P}_{J,U}\lambda^n}\sum_{\varphi\in\Phi}R_\varphi^s\lambda^{-n\delta}\right)\right\}\\ &\ge \overline{\mathcal{Q}}_\lambda(s) - \delta\ln\lambda > \overline{\mathcal{Q}}_\lambda(s),\end{aligned}$$

that is,

$$\overline{\mathcal{Q}}_\lambda(s) < \overline{\mathcal{Q}}_\lambda(s) - \delta\ln\lambda \le \overline{\mathcal{Q}}_\lambda(s-\delta) \le \overline{\mathcal{Q}}_\lambda(s)) - \delta k_0\ln\lambda. \tag{3.18}$$

From (3.17) and (3.18), we see that $\overline{\mathcal{Q}}_\lambda(s)$ is strictly decreasing and continuous on $\mathbb{R}$. We can get analogous results for $\underline{\mathcal{Q}}_\lambda(s)$ in the same way.

Finally, for any $0 < t < 1, s_1, s_2 \in \mathbb{R}$, we have by using Young's inequality and the fact that $\ln x$ is convex,

$$\begin{aligned}\overline{\mathcal{Q}}_\lambda(ts_1+(1-t)s_2) &= \sup_{J\in\mathcal{F}}\left\{\overline{\lim_{n\to\infty}}\frac{1}{n}\ln\left(\sup_{\Phi\in\mathscr{P}_{J,U}\lambda^n}\sum_{\varphi\in\Phi}R_\varphi^{ts_1}R_\varphi^{(1-t)s_2}\right)\right\}\\ &= \sup_{J\in\mathcal{F}}\left\{\overline{\lim_{n\to\infty}}\frac{1}{n}\ln\left(\sup_{\Phi\in\mathscr{P}_{J,U}\lambda^n}\sum_{\varphi\in\Phi}(R_\varphi^{s_1})^t(R_\varphi^{s_2})^{1-t}\right)\right\}\\ &\le t\sup_{J\in\mathcal{F}}\left\{\overline{\lim_{n\to\infty}}\frac{1}{n}\ln\left(\sup_{\Phi\in\mathscr{P}_{J,U}\lambda^n}\sum_{\varphi\in\Phi}R^{s_1}\right)\right\}\\ &+ (1-t)\sup_{J\in\mathcal{F}}\left\{\overline{\lim_{n\to\infty}}\frac{1}{n}\ln\left(\sup_{\Phi\in\mathscr{P}_{J,U}\lambda^n}\sum_{\varphi\in\Phi}R^{s_2}\right)\right\}\\ &= t\overline{\mathcal{Q}}_\lambda(s_1)+(1-t)\overline{\mathcal{Q}}_\lambda(s_2).\end{aligned} \tag{3.19}$$

Thus $\overline{\mathcal{Q}}_\lambda(s)$ is convexity on $\mathbb{R}$. This completes the proof. □

We define the effective domains of $\underline{\mathcal{Q}}_\lambda$ as

$$\mathrm{dom}\underline{\mathcal{Q}}_\lambda := \{s\in\mathbb{R} : -\infty < \underline{\mathcal{Q}}_\lambda(s) < \infty\}.$$

The effective domain of $\overline{\mathcal{Q}}_\lambda$ is defined analogously, denoted by $\mathrm{dom}(\overline{\mathcal{Q}}_\lambda)$. We have the following property without assuming $r > 0$.

Proposition 3.7. *Let $X, \{S_i\}_{i\in I}$ and U satisfy the hypotheses of Definition 2.9. Assume that $\lambda\in(0,1)$ and (CGC) holds. Then $\overline{\mathcal{Q}}_\lambda(s)$ (resp. $\underline{\mathcal{Q}}_\lambda(s)$) is strictly decreasing on $\mathrm{dom}\,\overline{\mathcal{Q}}_\lambda$ (resp. $\mathrm{dom}\,\underline{\mathcal{Q}}_\lambda$) that tends to $-\infty$ as s tends to ∞. Moreover, $\overline{\mathcal{Q}}_\lambda(s)$ is convex on $\mathrm{dom}\,\overline{\mathcal{Q}}_\lambda$, and hence continuous on $(\mathrm{dom}\,\overline{\mathcal{Q}}_\lambda)^\circ$, $0 \le \underline{\mathcal{Q}}_\lambda(0) \le \overline{\mathcal{Q}}_\lambda(0) = -\gamma\ln\lambda < \infty$, where γ is defined as in (2.10).*

Proof. It follows from (CGC) holds and the definitions of $\overline{\mathcal{Q}}_\lambda(s)$ and $\underline{\mathcal{Q}}_\lambda(s)$ that

$$\begin{aligned} 0 &\le \underline{\mathcal{Q}}_\lambda(0) \le \overline{\mathcal{Q}}_\lambda(0) = \sup_{J\in\mathcal{F}}\left\{ \varlimsup_{n\to\infty} \frac{\ln(\sup_{\Phi\in\mathscr{P}_{J,U}(\lambda^n)} \#\Phi)}{n}\right\} \\ &= \sup_{J\in\mathcal{F}}\left\{ \varlimsup_{n\to\infty} \frac{\ln(\sup_{\Phi\in\mathscr{P}_{J,U}(\lambda^n)} \#\Phi)}{-\ln\lambda^n}(-\ln\lambda)\right\} = -\gamma\ln\lambda < \infty. \end{aligned}$$

Next, for any $s \in \operatorname{dom}\overline{\mathcal{Q}}_\lambda$ and $\delta > 0$ with $s+\delta \in \operatorname{dom}\overline{\mathcal{Q}}_\lambda$, we can see from the proof of Proposition 3.6 that (3.16) holds for $\overline{\mathcal{Q}}_\lambda$, and exactly the same inequality as (3.16) holds for $\underline{\mathcal{Q}}_\lambda$. Moreover, we have from the proof of Proposition 3.6 that (3.19) holds for $\overline{\mathcal{Q}}_\lambda$. Hence the rest of conclusions hold. □

The next Lemma follows from [3, Proposition 18] and Assumption A.

Lemma 3.8. *Let X and $\{S_i\}_{i\in I}$ satisfy the hypotheses of Theorem 2.16. Then for any nonempty invariant open set $U \subset X$ with $\mathcal{L}^d(U) > 0$ and sequence of packing families $\{S_{\mathbf{i}_{n,j}}\}_j^{k_n^J}$ of $\mathscr{A}_{J,\lambda_n}$, we have*

$$\underline{\mathcal{Q}}_\lambda(s) = \sup_{J\in\mathcal{F}}\left\{ \varliminf_{n\to\infty} \frac{1}{n}\ln\sum_{j=1}^{k_n^J} R^s_{\mathbf{i}_{n,j}}\right\} = \sup_{J\in\mathcal{F}}\left\{ \varliminf_{n\to\infty} \frac{1}{n}\ln\sum_{j=1}^{k_n^J} r^s_{\mathbf{i}_{n,j}}\right\}, \quad s\in\mathbb{R}$$

$$\overline{\mathcal{Q}}_\lambda(s) = \sup_{J\in\mathcal{F}}\left\{ \varlimsup_{n\to\infty} \frac{1}{n}\ln\sum_{j=1}^{k_n^J} R^s_{\mathbf{i}_{n,j}}\right\} = \sup_{J\in\mathcal{F}}\left\{ \varlimsup_{n\to\infty} \frac{1}{n}\ln\sum_{j=1}^{k_n^J} r^s_{\mathbf{i}_{n,j}}\right\}, \quad s\in\mathbb{R}.$$

The above Lemma implies that for IFSs satisfying (LDP) and (CGC), the definitions of the topological pressures are independent of the choice of the invariant open set U and the packing families of $\mathscr{A}_{J,\lambda_n}$.

4. Proof of Theorem 2.16 and Theorem 2.17

Let X, $\{S_i\}_{i\in I}$ and ρ be described as in Definition 2.1. We assume that there exists a set $\mathbf{p} = \{p_i : i \in I\}$ with $0 < p_i < 1$ and $\sum_{i\in I} p_i = 1$. Let $\mathcal{S} = \{S_i\}_{i\in I}$, $\mathcal{M}$ be the set of Borel regular probability measures having bounded support on X. Let

$$\mathcal{BC}(X) = \{f : X \to \mathbb{R} : f \text{ is continuous and bounded on bounded subsets}\}.$$

For $\mu \in \mathcal{M}, \phi \in \mathcal{BC}(X)$, define $\mu(\phi) = \int \phi d\mu$. For $\nu \in \mathcal{M}$, we define $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ by

$$S(\nu)(A) = \sum_{i\in I} p_i \nu(S_i^{-1}(A)), \quad \text{for any Borel set } A \subset X.$$

We say that ν is variant with respect to $(\mathcal{S}, \mathbf{p})$ if $\mathcal{S}(\nu) = \nu$. Using a proof similar to that of [8, Theorem 4.4(1)], we have

Lemma 4.1. *Let X, $\{S_i\}_{i\in I}$ and ρ be described as in Definition 2.1. Then there exists a unique Borel regular probability measure μ such that $\mu = \sum_{i\in I} p_i\mu \circ S_i^{-1}$.*

Let X and $\{S_i\}_{i\in I}$ be described as in Definition 2.1. We say that $\{S_i\}_{i\in I}$ satisfies the *open set condition (OSC)* if there exists a nonempty bounded invariant open set $O \subset X$ (in the relative Euclidean topology of X), called an *OSC-set*, such that $S_i(O) \cap S_j(O) = \emptyset$ for all $i \neq j$.

Using Lemma 4.1 and a similar proof as that of [3, Proposition 20], we have the following result.

Proposition 4.2. *Let K be the attractor of an FIFS $\{S_i\}_{i\in I}$ satisfying the hypotheses of Definition 2.6, and assume that (LDP) holds. If (OSC) holds with an OSC-set $U \supseteq K$ and $\sum_{j\in I} r_j^s > 1$ with $r_j < 1$, then $\dim_{\mathrm{H}} K \geq s$.*

Proposition 4.3. *Let K be the limit set of the IFS $\{S_i\}_{i\in I}$, and $\mathscr{A}_{I,b}$ be defined as in* (2.7)*. Then $K = \bigcup_{\varphi\in\mathscr{A}_{I,b}} \varphi(K)$.*

Proof. For any $i \in I$, we have $S_i(K) \subset K$ since $K = \bigcup_{i\in I} S_i(K)$. This implies that for any $\varphi = S_{i_1\cdots i_n} \in \mathscr{A}_{I,b}$, $\varphi(K) \subset K$. Thus $\bigcup_{\varphi\in\mathscr{A}_{I,b}} \varphi(K) \subset K$. Notice that the following consequences:

$$\begin{aligned} K &= \bigcup_{i\in I} S_i(K) = \bigcup_{i\in I} S_i(\bigcup_{j\in I} S_j(K)) = \bigcup_{i,j\in I} S_{ij}(K) = \cdots \\ &= \bigcup_{i_1,\dots,i_p\in I} S_{i_1\cdots i_p}(K). \end{aligned}$$

Similarly,

$$S_{i_1\cdots i_p}(K) \;=\; S_{i_1\cdots i_p}\Bigg(\bigcup_{i_{p+1}\in I} S_{i_{p+1}}(K)\Bigg) = \bigcup_{i_{p+1}\in I} S_{i_1\cdots i_p i_{p+1}}(K).$$

Thus if $x \in K$, then there exists $\mathbf{i} = i_1 i_2 \cdots \in I^\infty$ such that $K \supset S_{i_1}(K) \supset S_{i_1 i_2}(K) \supset \cdots \supset \{x\}$. Since there exists n such that $\mathbf{i}|_n \in \mathscr{S}_{I,b}$, we have $x \in S_{\mathbf{i}|_n}(K)$, and thus $K \subset \bigcup_{\varphi\in\mathscr{A}_{I,b}} \varphi(K)$. This completes this proof. □

Proposition 4.4. *Let K be the limit set of the IFS $\{S_i\}_{i\in I}$ and $\mathcal{F}$ be defined as in* (2.9)*. Then $\overline{K} = \overline{\bigcup_{J\in\mathcal{F}} K^J}$.*

Proof. Since $K^J \subset K$, we have $\overline{\bigcup_{J\in\mathcal{F}} K^J} \subset \overline{K}$. Next we prove reverse inclusion. For any $x \in K$, it follows from the definition of π and Definition 2.2 that there exists $i_1 i_2 \cdots \in I^\infty$ such that

$$x = \bigcap_{p=1}^{\infty} S_{i_1\cdots i_p}(X).$$

We write $\{x\} = x_{i_1 i_2 \dots}$. Let $\widehat{i_1\cdots i_p}$ denote the infinite sequence $i_1\cdots i_p i_1 \cdots i_p \cdots \in I^\infty$ with recurrent block $i_1 \cdots i_p$. Since

$$S_{j_1\cdots j_q}(x_{i_1 i_2\dots}) \in S_{j_1\cdots j_q}(\bigcap_{p=1}^{\infty} S_{i_1\cdots i_p}(X)) = \bigcap_{p=1}^{\infty} S_{j_1\cdots j_q i_1\cdots i_p}(X) = x_{j_1\cdots j_q i_1\cdots i_p\cdots},$$

we have $S_{i_1\cdots i_p}(x_{\widehat{i_1\cdots i_p}}) = x_{\widehat{i_1\cdots i_p}}$. It follows that $x_{\widehat{i_1\cdots i_p}}$ is the unique fixed point $s_{i_1\cdots i_p}$ of $S_{i_1\cdots i_p}$. Thus both $s_{i_1\cdots i_p}$ and $x_{i_1\cdots i_p\cdots}$ belong to $S_{i_1\cdots i_p}(X)$. Since $\lim_{p\to\infty} \operatorname{diam}(S_{i_1\cdots i_p}(X)) = 0$, we have $\lim_{p\to\infty} s_{i_1\cdots i_p} = x_{i_1\cdots i_p\cdots}$. For any $p \geq 1$, we choose $J \in \mathcal{F}$ such that $(i_1, \dots, i_p) \in J^p = \{(i_1, \dots, i_p) : i_j \in J,\, 1 \leq j \leq p\}$. Then $s_{i_1\cdots i_p} \in K^J$, and thus $x \in \overline{\bigcup_{J\in\mathcal{F}} K^J}$. Notice that $\overline{\bigcup_{J\in\mathcal{F}} K^J}$ is a close set, and for any $y \in \overline{K}$, there exists a sequence $\{x^n\}$ of points in K such that $\lim_{n\to\infty} x^n = y$. We see that $\overline{K} \subset \overline{\bigcup_{J\in\mathcal{F}} K^J}$. This completes the proof. □

Proof of Theorem 2.16. By Lemma 3.8, we first require $K \subset U$. By Proposition 3.7, $\underline{\mathcal{Q}}_\lambda(0)$ and $\overline{\mathcal{Q}}_\lambda(0)$ are real numbers. We first prove that $\operatorname{dom}\underline{\mathcal{Q}} = \operatorname{dom}\overline{\mathcal{Q}} = \mathbb{R}$, and for any $s \in \mathbb{R}$,

$$\begin{aligned} \underline{\mathcal{Q}}_\lambda(s) &= \underline{\mathcal{Q}}_\lambda(0) + s\ln\lambda = (s-\alpha)\ln\lambda, \\ \overline{\mathcal{Q}}_\lambda(s) &= \overline{\mathcal{Q}}_\lambda(0) + s\ln\lambda = (s-\beta)\ln\lambda, \end{aligned} \tag{4.1}$$

where α, β are the unique zeroes of $\underline{\mathcal{Q}}_\lambda(s)$ and $\overline{\mathcal{Q}}_\lambda(s)$, respectively. Substituting $b = \lambda^n$ and $\mathbf{i} = \mathbf{i}_{n,j}$ into (3.2) yields

$$\frac{\lambda^n}{c_1(n\,|\ln\lambda|)^\xi} \leq r_{\mathbf{i}_{n,j}} \leq R_{\mathbf{i}_{n,j}} \leq \lambda^n,\ n = 1, 2, \dots,\ j = 1, \dots, k_n^J.$$

Hence

$$\begin{aligned} k_n^J\lambda^{ns} &\geq \textstyle\sum_{j=1}^{k_n^J} R_{\mathbf{i}_{n,j}}^s \geq k_n^J\big(\frac{1}{c_1(n|\ln\lambda|)^\xi}\big)^s\lambda^{ns}, \text{ if } s \geq 0; \\ k_n^J\lambda^{ns} &\leq \textstyle\sum_{j=1}^{k_n^J} R_{\mathbf{i}_{n,j}}^s \leq k_n^J\big(\frac{1}{c_1(n|\ln\lambda|)^\xi}\big)^s\lambda^{ns}, \text{ if } s < 0. \end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{1}{c_1(n\,|\ln\lambda|)}\right)^s = 0,$$

by using Proposition 3.8 and the fact that $\underline{\mathcal{Q}}_\lambda(0)$ and $\overline{\mathcal{Q}}_\lambda(0)$ are real numbers, we have

$$\begin{aligned} \underline{\mathcal{Q}}_\lambda(s) &= \sup_{J\in\mathcal{F}}\left\{\varliminf_{n\to\infty}\frac{1}{n}\ln\Big(\sum_{j=1}^{k_n^J} R_{\mathbf{i}_{n,j}}^s\Big)\right\} = \sup_{J\in\mathcal{F}}\left\{\varliminf_{n\to\infty}\frac{1}{n}\ln(k_n^J\lambda^{ns})\right\} \\ &= \underline{\mathcal{Q}}_\lambda(0) + s\ln\lambda \\ \overline{\mathcal{Q}}_\lambda(s) &= \sup_{J\in\mathcal{F}}\left\{\varlimsup_{n\to\infty}\frac{1}{n}\ln\Big(\sum_{j=1}^{k_n^J} R_{\mathbf{i}_{n,j}}^s\Big)\right\} = \sup_{J\in\mathcal{F}}\left\{\varlimsup_{n\to\infty}\frac{1}{n}\ln(k_n^J\lambda^{ns})\right\} \\ &= \overline{\mathcal{Q}}_\lambda(0) + s\ln\lambda. \end{aligned} \tag{4.2}$$

This implies that $\operatorname{dom}\underline{\mathcal{Q}} = \operatorname{dom}\overline{\mathcal{Q}} = \mathbb{R}$. Moreover, by the properties of $s\ln\lambda$, $\underline{\mathcal{Q}}_\lambda(s)$ and $\overline{\mathcal{Q}}_\lambda(s)$ are strictly decreasing and continuous on $\mathbb{R}$ that tend to $-\infty$ and ∞ as s tends to ∞ and $-\infty$, respectively. So, $\underline{\mathcal{Q}}_\lambda(s)$ and $\overline{\mathcal{Q}}_\lambda(s)$ have unique zero, denote by α and β, respectively. Thus we have $\underline{\mathcal{Q}}_\lambda(0) = -\alpha\ln\lambda$ and $\overline{\mathcal{Q}}_\lambda(0) = -\beta\ln\lambda$, and thus (4.1) holds.

Next, we prove

$$\beta \leq \dim_{\mathrm{H}} K.$$

Suppose, on the contrary, $\beta > \dim_{\mathrm{H}} K$. Assume $\dim_{\mathrm{H}} K < s < \beta$. By (4.1), $\overline{\mathcal{Q}}_\lambda(s) = (s-\beta)\ln\lambda > 0$. This implies, by Proposition 3.8, that there exists $J \in \mathcal{F}$ such that

$$\varlimsup_{n\to\infty}\frac{1}{n}\ln\sum_{j=1}^{k_n^J} r_{\mathbf{i}_{n,j}}^s \geq \frac{\overline{\mathcal{Q}}_\lambda(s)}{2} > 0,$$

Thus there exists an integer $n>0$ depending on J, such that

$$\frac{1}{n}\ln\sum_{j=1}^{k_n^J} r_{\mathbf{i}_{n,j}}^s \ge \frac{\overline{\mathcal{Q}}_\lambda(s)}{2} > 0.$$

Let K_n^J denote the attractor of the new FIFS $\{S_{\mathbf{i}_{n,j}}\}_{j=1}^{k_n^J}$. Then this FIFS satisfies (OSC) with U being an OSC-set. Since $U \supset K_n^J$, by applying Proposition 4.2 to the new FIFS $\{S_{\mathbf{i}_{n,j}}\}_{j=1}^{k_n^J}$, we get $\dim_{\mathrm{H}} K_n^J \ge s$. Notice that $K_n^J \subset K^J$ and the fact

$$K^J = \bigcup_{\mathbf{i}\in J^\infty}\bigcap_{m=1}^\infty S_{\mathbf{i}|_m}(X) \subset \bigcup_{\mathbf{i}\in I^\infty}\bigcap_{n=1}^\infty S_{\mathbf{i}|_n}(X) = K,$$

we get $\dim_{\mathrm{H}} K \ge \dim_{\mathrm{H}} K^J \ge \dim_{\mathrm{H}} K_n^J \ge s$, a contradiction. Thus $\dim_{\mathrm{H}} K \ge \beta$.

Now, we prove

$$\underline{\mathcal{Q}}_\lambda(s) = \overline{\mathcal{Q}}_\lambda(s) = (s - \dim_{\mathrm{H}} K)\ln\lambda. \tag{4.3}$$

To this end we first prove $\alpha \ge \dim_{\mathrm{H}} K$. Let $s > \alpha$. Then by (4.1), $\underline{\mathcal{Q}}_\lambda(s) < 0$, and thus $s>0$ by Proposition 3.7. By assumption, for each integer $n>0$, $\{S_{\mathbf{i}_{n,j}}\}_{j=1}^{k_n^J}$ is a packing family of $\mathscr{A}_{J,\lambda^n}$ with respect to U, where $J \in \mathcal{F}$. This implies from Definition 2.9 that for each $\varphi \in \mathscr{A}_{J,\lambda^n}$, there is at least one j such that $\varphi(U)\cap S_{\mathbf{i}_{n,j}}(U) \ne \emptyset$. Choose $x_{n,j} \in S_{\mathbf{i}_{n,j}}(U)$. By (3.1), both $|\varphi(U)|$ and $\left|S_{\mathbf{i}_{n,j}}(U)\right|$ are less than $\lambda^n\,|U|$. Thus $\varphi(U)$ is contained in the ball $B_{2\lambda^n|U|}(x_{n,j})$ with radius $2\lambda^n\,|U|$. It follows from Proposition 4.3 and the fact $K^J \subset U$ that

$$K^J = \bigcup_{\varphi\in\mathscr{A}_{J,\lambda^n}}\varphi(K^J) \subset \bigcup_{\varphi\in\mathscr{A}_{J,\lambda^n}}\varphi(U) \subset \bigcup_{j=1}^{k_n^J} B_{2\lambda^n|U|}(x_{n,j}).$$

This implies that $\{B_{2\lambda^n|U|}(x_{n,j})\}_{j=1}^{k_n^J}$ is a $4\lambda^n\,|U|$ cover of K^J. Thus

$$\{B_{2\lambda^n|U|}(x_{n,j}) : j = 1,2,\dots,k_n^J, J\in\mathcal{F}\}$$

is a $4\lambda^n\,|U|$ cover of $\bigcup_{J\in\mathcal{F}} K^J$. Notice that $\overline{K} = \overline{\bigcup_{J\in\mathcal{F}} K^J}$ (Proposition 4.4). This implies that

$$\{B_{3\lambda^n|U|}(x_{n,j}) : j = 1,2,\dots,k_n^J, J\in\mathcal{F}\}$$

is a $6\lambda^n\,|U|$ cover of $\overline{K}$. By the compactness of $\overline{K}$, there exist $J_1,\dots,J_p$ such that

$$\overline{K} \subset \bigcup_{i=1}^{p}\bigcup_{j=1}^{k_n^{J_i}} B_{3\lambda^n|U|}(x_{n,j}). \tag{4.4}$$

It follows from Proposition 3.8 and (4.2) that

$$\sup_{J\in\mathcal{F}}\left\{\varliminf_{n\to\infty}\frac{1}{n}\ln[k_n^J(6\lambda^n\,|U|)^s]\right\} = \underline{\mathcal{Q}}_\lambda(0) + s\ln\lambda = \underline{\mathcal{Q}}_\lambda(s) < 0.$$

There exists infinitely many integers n such that $k_n^J(6\lambda^n\,|U|)^s < 1$. Notice that $K \subset \overline{K}$ and there exists $J^* \in \mathcal{F}$ such that

$$k_n^{J_1} + \cdots + k_n^{J_p} \le k_n^{J^*}. \tag{4.5}$$

So we get

$$\mathscr{H}^s(K)=\lim_{n\to\infty}\mathscr{H}^s_{\delta_n}(K)\leq \varliminf_{n\to\infty}(k_n^{J^*})(6\lambda^n\,|U|)^s<1<+\infty,$$

and thus $s\geq \dim_{\mathrm{H}} K$. Since $s>\alpha$ is arbitrary, we conclude that $\alpha\geq\dim_{\mathrm{H}} K$.

Since $\underline{\mathcal{Q}}_\lambda(s)\leq\overline{\mathcal{Q}}_\lambda(s)$ implies that $\alpha\leq\beta$, the above discussion implies that $\alpha=\beta=\dim_{\mathrm{H}} K$. Now equation (4.3) follows by substituting this into (4.1). Combining (4.3) and the fact that $\overline{\mathcal{Q}}_\lambda(0)=-\gamma\ln\lambda$ (Proposition 3.7), we get $\dim_{\mathrm{H}} K=\gamma$.

Finally, we prove $\dim_{\mathrm{H}} K=\dim_{\mathrm{B}} K=\dim_{\mathrm{P}} K$. For any integer $n>0$, let

$$\mathscr{B}_n:=\left\{\prod_{i=1}^d[(m_i-1)\lambda^n,m_i\lambda^n):m_i\in\mathbb{Z},K\bigcap\prod_{i=1}^d[(m_i-1)\lambda^n,m_i\lambda^n)\neq\emptyset\right\}$$

and let $N_n:=\#\mathscr{B}_n$, the cardinality of $\mathscr{B}_n$. According to (4.4), we define

$$\begin{aligned}\mathscr{B}_{n,j}: \ &= \ \Bigg\{\prod_{i=1}^d[(m_i-1)\lambda^n,m_i\lambda^n):m_i\in\mathbb{Z},\\ &\qquad B_{3\lambda^n|U|}(x_{n,j})\bigcap\prod_{i=1}^d[(m_i-1)\lambda^n,m_i\lambda^n)\neq\emptyset\Bigg\}.\end{aligned}$$

Then (4.4) implies that $\mathscr{B}_n\subset\bigcup_{i=1}^p\bigcup_{j=1}^{k_n^{J_i}}\mathscr{B}_{n,j}$. Since $B_{3\lambda^n|U|}(x_{n,j})\bigcap\prod_{i=1}^d[(m_i-1)\lambda^n,m_i\lambda^n)\neq\emptyset$, we have $\prod_{i=1}^d[(m_i-1)\lambda^n,m_i\lambda^n)$ is contained in the ball $B_{3\lambda^n|U|+\sqrt{d}\lambda^n}(x_{n,j})$. Note also that $\prod_{i=1}^d[(m_i-1)\lambda^n,m_i\lambda^n)$ are disjoint. Hence

$$\#\mathscr{B}_{n,j}\leq\frac{\mathcal{L}^d(B_{3\lambda^n|U|+\sqrt{d}\lambda^n}(x_{n,j}))}{\mathcal{L}^d(\prod_{i=1}^d[(m_i-1)\lambda^n,m_i\lambda^n))}=\mathcal{L}^d(B_{3|U|+\sqrt{d}}(0)).$$

Therefore,

$$N_n\leq\sum_{i=1}^p\sum_{j=1}^{k_n^{J_i}}\#\mathscr{B}_{n,j}\leq(k_n^{J_1}+\cdots+k_n^{J_p})\mathcal{L}^d(B_{3|U|+\sqrt{d}}(0))\leq(k_n^{J^*})\mathcal{L}^d(B_{3|U|+\sqrt{d}}(0)),$$

where $k_n^{J^*}$ is defined as in (4.5). Hence

$$\begin{aligned}\overline{\dim}_{\mathrm{B}}K \ &= \ \varlimsup_{n\to\infty}\frac{\ln N_n}{-\ln\lambda^n}\leq\varlimsup_{n\to\infty}\frac{\ln(k_n^{J^*})}{-n\ln\lambda}\\ &\leq \ \frac{\overline{Q}_\lambda(0)}{-\ln\lambda}=\gamma=\dim_{\mathrm{H}} K.\end{aligned}$$

Since $\dim_{\mathrm{H}} K\leq\underline{\dim}_{\mathrm{B}}K\leq\overline{\dim}_{\mathrm{B}}K$ and $\dim_{\mathrm{H}} K\leq\dim_{\mathrm{P}} K\leq\overline{\dim}_{\mathrm{B}}K$, the assertion follows immediately. This completes this proof.

Proof of Theorem 2.17. For any $J\in\mathcal{F}$, in view of (3.1) , we have

$$\begin{aligned}\sum_{j=1}^{k_n^J}r_{\mathbf{i}_{n,j}}^s \ &\leq \ \sum_{j=1}^{k_n^J}\frac{[\mathcal{L}^d(S_{\mathbf{i}_{n,j}}(U))]^{s/d}}{[\mathcal{L}^d(U)]^{s/d}}\\ &\leq \ \sum_{j=1}^{k_n^J}R_{\mathbf{i}_{n,j}}^s\leq\sup_{\Psi\in\mathscr{P}^*_{J,U}(\lambda^n)}\sum_{\psi\in\Psi}R_\psi^s,\ \ s\geq0,\end{aligned}\tag{4.6}$$

$$\begin{aligned}\sum_{j=1}^{k_n^J} r_{\mathbf{i}_{n,j}}^s &\geq \sum_{j=1}^{k_n^J} \frac{[\mathcal{L}^d(S_{\mathbf{i}_{n,j}}(U))]^{s/d}}{[\mathcal{L}^d(U)]^{s/d}} \\ &\geq \sum_{j=1}^{k_n^J} R_{\mathbf{i}_{n,j}}^s \geq \inf_{\Psi\in\mathscr{P}^*_{J,U}(\lambda^n)} \sum_{\psi\in\Psi} R_\psi^s, \quad s<0. \end{aligned}\tag{4.7}$$

For any $b\in(0,\lambda)$ and any two packing families $\{S_{\mathbf{i}_i}\}_{i=1}^k \in \mathscr{P}_{J,U}(b)$ and $\{S_{\mathbf{j}_j}\}_{j=1}^m \in \mathscr{P}^*_{J,U}(b)$. Let $\Psi^i=\{S_{\mathbf{j}_j} : S_{\mathbf{j}_j}(U)\cap S_{\mathbf{i}_i}(U)\neq\emptyset\}$ and $\Phi^j=\{S_{\mathbf{i}_i} : S_{\mathbf{i}_i}(U)\cap S_{\mathbf{j}_j}(U)\neq\emptyset\}$. Using (LDP) and (L*DP) (Lemmas 3.1 and 3.2), we have

$$\begin{aligned}\left(\frac{R_{\mathbf{j}_j}}{c_1c_2^2\,|\ln b|^{\xi+2\zeta}}\right)^d\mathcal{L}^d(U) &\leq \left(\frac{r_{\mathbf{j}_j}}{c_1\,|\ln b|^{\xi}}\right)^d\mathcal{L}^d(U)\\ &\leq \left(\frac{1}{c_1\,|\ln b|^{\xi}}\right)^d\mathcal{L}^d(S_{\mathbf{j}_j}U)\\ &\leq \left(\frac{b}{c_1\,|\ln b|^{\xi}}\right)^d\mathcal{L}^d(U)\\ &\leq r_{\mathbf{i}_i}^d\mathcal{L}^d(U)\leq\mathcal{L}^d(S_{\mathbf{i}_i}(U))\\ &\leq R_{\mathbf{i}_i}^d\mathcal{L}^d(U).\end{aligned}$$

Let $c_3:=c_1c_2^2$. It follows that

$$\frac{R_{\mathbf{j}_j}}{c_3\,|\ln b|^{\xi+2\zeta}}\leq R_{\mathbf{i}_i}\quad\text{for all } 1\leq i\leq k,\ 1\leq j\leq m.\tag{4.8}$$

By using (3.2) and (3.3) we see that for $b>0$ with $|\ln b|>1$, $\cup_{S_{\mathbf{j}_j}\in\Psi^i}S_{\mathbf{j}_j}(U)$ is contained in a ball with center in $S_{\mathbf{i}_i}(U)$ and radius $(1+c_2)\,|\ln b|^{\zeta}\,b\,|U|$. Hence it follows from (3.3) again that

$$\begin{aligned}\#\Psi^i\mathcal{L}^d(U)\left(\frac{b}{c_2\,|\ln b|^{\zeta}}\right)^d &\leq \sum_{S_{\mathbf{j}_j}\in\Psi^i}\mathcal{L}^d(S_{\mathbf{j}_j}(U))\\ &\leq ((1+c_2)\,|\ln b|^{\zeta}\,b\,|U|)^d\mathcal{L}^d(B_1(0))\end{aligned}$$

Therefore, there is a constant $c_4>0$ such that

$$m\leq\sum_{i=1}^k\#\Psi^i\leq c_4\,|\ln b|^{2d\zeta}\,k.\tag{4.9}$$

By interchanging the roles of the two packing families, it can be proved in the same way that there exist constants $c_5>0$ and $c_6>0$ such that

$$\frac{R_{\mathbf{i}_i}}{c_5\,|\ln b|^{\zeta+\xi}}\leq R_{\mathbf{j}_j}\ \text{ for all } i,j \text{ satisfying } 1\leq i\leq k \text{ and } 1\leq j\leq m,\tag{4.10}$$

$$k\leq\sum_{j=1}^m\#\Phi^j\leq c_6\,|\ln b|^{d(\xi+\zeta)}\,m.\tag{4.11}$$

Now, let $b=\lambda^n$ with $|\ln b|>1$. By combining the inequalities (4.8)–(4.11), we have

$$\begin{aligned}\sum_{i=1}^{k} R_{\mathbf{i}_i}^s &\le k\max\{R_{\mathbf{i}_i}^s : 1\le i\le k\}\\ &\le (c_6\,|\ln\lambda^n|^{d(\xi+\zeta)}\,m)(c_5\,|\ln\lambda^n|^{\zeta+\xi})^s\min\{R_{\mathbf{j}_j}^s : 1\le j\le m\}\\ &\le (c_6\,|\ln\lambda^n|^{d(\xi+\zeta)})(c_5\,|\ln\lambda^n|^{\zeta+\xi})^s\sum_{j=1}^{m} R_{\mathbf{j}_j}^s\\ &\le (c_6\,|\ln\lambda^n|^{d(\xi+\zeta)})(c_5\,|\ln\lambda^n|^{\zeta+\xi})^s(c_4\,|\ln\lambda^n|^{2d\zeta})(c_3\,|\ln\lambda^n|^{\xi+2\zeta})^s\\ &\quad\times\sum_{i=1}^{k} R_{\mathbf{i}_i}^s,\quad s\ge 0. \end{aligned}\tag{4.12}$$

Similarly, we have

$$\begin{aligned}\sum_{i=1}^{k} R_{\mathbf{i}_i}^s &\le k\max\{R_{\mathbf{i}_i}^s : 1\le i\le k\}\\ &\le (c_6\,|\ln\lambda^n|^{d(\xi+\zeta)}\,m)\left(\frac{1}{c_3\,|\ln\lambda^n|^{\xi+2\zeta}}\right)^s\min\{R_{\mathbf{j}_j}^s : 1\le j\le m\}\\ &\le \frac{c_6\,|\ln\lambda^n|^{d(\xi+\zeta)}}{(c_3\,|\ln\lambda^n|^{\xi+2\zeta})^s}\sum_{j=1}^{m} R_{\mathbf{j}_j}^s\\ &\le \frac{(c_4\,|\ln\lambda^n|^{2d\zeta})(c_6\,|\ln\lambda^n|^{d(\xi+\zeta)})}{(c_3\,|\ln\lambda^n|^{\xi+2\zeta})^s(c_5\,|\ln\lambda^n|^{\zeta+\xi})^s}\sum_{i=1}^{k} R_{\mathbf{i}_i}^s,\quad s<0. \end{aligned}\tag{4.13}$$

From (4.12), (4.13), and Theorem 2.16, we get

$$\begin{aligned}Q_\lambda(s) &= \sup_{J\in\mathcal{F}}\left\{\underline{\lim}_{n\to\infty}\frac{1}{n}\ln\left(\inf_{\Psi\in\mathscr{P}^*_{J,U}(\lambda^n)}\sum_{\psi\in\Psi}R_\Psi^s\right)\right\}\\ &= \sup_{J\in\mathcal{F}}\left\{\overline{\lim}_{n\to\infty}\frac{1}{n}\ln\left(\sup_{\Psi\in\mathscr{P}^*_{J,U}(\lambda^n)}\sum_{\psi\in\Psi}R_\Psi^s\right)\right\}.\end{aligned}\tag{4.14}$$

The conclusion now follows from (4.6), (4.7), (4.12), (4.13), (4.14), Proposition 3.8 and Theorem 2.16. This completes this proof. □

Using the fact $\overline{Q}_\lambda(0)=-\gamma\ln\lambda$ and (4.14), we have

$$\sup_{J\in\mathcal{F}}\left\{\overline{\lim}_{b\to 0+}\frac{\ln\left(\sup_{\Psi\in\mathscr{P}^*_{J,U}(b)}\#\Psi\right)}{-\ln b}\right\}=\gamma.$$

The following proposition follows from (4.12), (4.13) and Definition 2.10.

Proposition 4.5. *Let $X,\{S_i\}_{i\in I}$ and U satisfy the hypotheses of Definition 2.9. If (LDP) and (L*DP) hold, then*

$$\sup_{J\in\mathcal{F}}\left\{\overline{\lim}_{b\to 0+}\frac{\ln(\sup_{\Phi\in\mathscr{P}_{J,U}(b)}\#\Phi)}{-\ln b}\right\}=\gamma$$

if and only if

$$\sup_{J\in\mathcal{F}}\left\{\varlimsup_{b\to 0+}\frac{\ln(\sup_{\Psi\in\mathscr{P}^*_{J,U}(b)}\#\Psi)}{-\ln b}\right\}=\gamma,$$

where γ *is defined as in Definition 2.10.*

5. Growth dimension and proof of Theorem 2.18

We study the growth dimension in this section and prove Theorem 2.18.

Proposition 5.1. *Let* $\{S_i\}_{i\in I}$ *be an FIFS of essential contractions on* X *that has (LDP), and* ξ *be defined as in Definition 2.6(a). Then the following hold.*

(a) $\lim_{b\to 0+}\frac{\ln\#\mathscr{A}_{I,b}}{-\ln b}$ *exists.*
(b) *Let* α_b *be the unique nonnegative number satisfying*

$$\sum_{\varphi\in\mathscr{A}_{I,b}}R_\varphi^{\alpha_b}=1.$$

If $\lim_{b\to 0+}\alpha_b$ *exists, then*

$$(1+\xi)^{-1}\lim_{b\to 0+}\frac{\ln\#\mathscr{A}_{I,b}}{-\ln b}\le\lim_{b\to 0+}\alpha_b\le\lim_{b\to 0+}\frac{\ln\#\mathscr{A}_{I,b}}{-\ln b}. \tag{5.1}$$

Proof. Let $b_1,b_2\in(0,1)$. For any $\mathbf{i}\in\mathscr{S}_{I,b_1b_2}$, the definition of $\mathscr{S}_{I,b_1b_2}$ shows that there is a decomposition $\mathbf{i}=\mathbf{i}_1\mathbf{i}_2$ with $\mathbf{i}_1=i_1\cdots i_k$ satisfying $\mathbf{i}_1\in\mathscr{S}_{I,b_1}$ and $\mathbf{i}_2=i_2\cdots i_n$. Notice that $\mathbf{i}_2$ could be the empty word. Let $S_{\mathbf{i}_2}$ be the identity and thus $R_{\mathbf{i}_2}=1$ if that happens. It follows from (3.2) that

$$R_{\mathbf{i}}\ge\frac{b_1b_2}{c_1\left|\ln b_1b_2\right|^\xi}, \tag{5.2}$$

and we have by using (3.2) and (2.6) that if $n>k$,

$$b_2<R_{i_{k+1}\cdots i_{n-1}}. \tag{5.3}$$

In fact, inequality (5.3) follows from

$$b_1b_2<R_{i_1\cdots i_{n-1}}\le R_{i_1\cdots i_k}R_{i_{k+1}\cdots i_{n-1}}\le b_1R_{i_{k+1}\cdots i_{n-1}}.$$

Now we can proceed as in [2, Proposition 4.1], and conclude that $\lim_{b\to 0+}\frac{\ln\#\mathscr{A}_{I,b}}{-\ln b}$ exists. By (5.2), we have

$$1=\sum_{\varphi\in\mathscr{A}_{I,b}}R_\varphi^{\alpha_b}\ge(\#\mathscr{A}_{I,b})\left(\frac{b}{c_1\left|\ln b\right|^\xi}\right)^{\alpha_b}. \tag{5.4}$$

It follows from (5.4) and the fact $1=\sum_{\varphi\in\mathscr{A}_{I,b}}R_\varphi^{\alpha_b}\le(\#\mathscr{A}_{I,b})b^{\alpha_b}$ that for $0<b<1$,

$$\frac{-\ln b}{\ln\#(\mathscr{A}_{I,b})}\le\frac{1}{\alpha_b}\le\frac{-\ln(\frac{b}{c_1\left|\ln b\right|^\xi})}{\ln\#\mathscr{A}_{I,b}}=\frac{-\ln b+\ln(c_1\left|\ln b\right|^\xi)}{\ln\#\mathscr{A}_{I,b}}\le\frac{-(1+\xi)\ln b+\ln c_1}{\ln\#\mathscr{A}_{I,b}}.$$

Letting $b\to 0$ yields (5.1). □

Proposition 5.2. *Let $\{S_i\}_{i\in I}$ be an FIFS of essential contractions on X that has (LDP) and satisfies (WSC). Assume that K is the associated attractor with $\dim_{\rm H} K=\alpha$. Then $\mathcal{H}^\alpha(K)>0$.*

Proof. Note that for any $E\subset K$, we have

$$E\subseteq\bigcup\{S(K): S\in\mathscr{A}_{I,|E|}, S(K)\cap E\neq\emptyset\}.$$

Since $\{S_i\}_{i\in I}$ satisfies (WSC), we have from [9, Proposition 3.1(c)] that

$$\#\{S(K): S\in\mathscr{A}_{I,|E|}, S(K)\cap E\neq\emptyset\}<\infty.$$

For each $S\in\mathscr{A}_{I,|E|}$ with $S(K)\cap E\neq\emptyset$, consider the map $S^{-1}: S(K)\to K$. For any $x,y\in S(K)$, let $x',y'\in K$ such that $x=S(x')$ and $y=S(y')$. By (2.4) and (2.7), we get

$$|x-y|=\left|S(x')-S(y')\right|\le R_S\left|x'-y'\right|\le|E|\left|S^{-1}(x)-S^{-1}(y)\right|.$$

It follows from [5, Theorem 2] that $\mathcal{H}^\alpha(K)>0$.

□

Proposition 5.3. *Let α, K and $\{S_i\}_{i\in I}$ satisfy the hypotheses of Propositions 5.1 and 5.2. If $\mathcal{H}^\alpha(K)<\infty$, then there exists a constant $c_3>0$ such that for any $0<b<1$,*

$$c_3^{-1}b^{-\alpha}\le\#(\mathscr{A}_{I,b})\le c_3\Big(\frac{b}{|\ln b|^\xi}\Big)^{-\alpha}. \tag{5.5}$$

Consequently,

$$\alpha=\dim_{\rm H}K=\lim_{b\to0+}\frac{\ln\#\mathscr{A}_{I,b}}{-\ln b}. \tag{5.6}$$

Proof. For any $0<b<1$, by Proposition 4.3, we have $K=\bigcup_{\varphi\in\mathscr{A}_{I,b}}\varphi(K)$. It follows from (WSC) and [9, Proposition 3.1] that each $x\in K$ is covered by at most κ elements of $\{\varphi(K):\varphi\in\mathscr{A}_{I,b}\}$. Hence

$$\mathcal{H}^\alpha(K)\le\sum_{\varphi\in\mathscr{A}_{I,b}}\mathcal{H}^\alpha(\varphi(K))\le\kappa\mathcal{H}^\alpha(K). \tag{5.7}$$

For each $\varphi\in\mathscr{A}_{I,b}$, by using (3.2), we get

$$\mathcal{H}^\alpha(\varphi(K))\ge r_\varphi^\alpha\mathcal{H}^\alpha(K)\ge\Big(\frac{b}{c_1\,|\ln b|^\xi}\Big)^\alpha\mathcal{H}^\alpha(K) \tag{5.8}$$

and

$$\mathcal{H}^\alpha(\varphi(k))\le R_\varphi^\alpha\mathcal{H}^\alpha(K)\le b^\alpha\mathcal{H}^\alpha(K).$$

It follows by summing the inequalities over $\varphi\in\mathscr{A}_{I,b}$ and using (5.7) that

$$\Big(\frac{b}{c_1\,|\ln b|^\xi}\Big)^\alpha\mathcal{H}^\alpha(K)\#(\mathscr{A}_{I,b})\le\kappa\mathcal{H}^\alpha(K)$$

and

$$\mathcal{H}^\alpha(K)\le b^\alpha\mathcal{H}^\alpha(K)\#(\mathscr{A}_{I,b}).$$

By the assumption $\mathcal{H}^\alpha(K) < \infty$ and Proposition 5.2, we have $0 < \mathcal{H}^\alpha(K) < \infty$. Thus (5.5) holds. Taking logarithm on both sides of (5.5), we get

$$-\ln c_3 - \alpha \ln b \le \ln \#(\mathscr{A}_{I,b}) \le \ln c_3 - \alpha \ln\Big(\frac{b}{|\ln b|^\xi}\Big) = \ln c_3 - \alpha \ln b + \alpha\xi \ln|\ln b|.$$

The dimension formula (5.6) follows since $\lim_{b\to 0+}(\ln|\ln b|)/\ln b = 0$. □

Let $\{S_i\}_{i\in I}$ be an FIFS of essential contractions on X and let $\lambda \in (0,1)$. Following [2], we define

$$\underline{\mathcal{Q}}^*_\lambda(s) := \varliminf_{n\to\infty} \frac{1}{n} \ln\Big(\sum_{\varphi\in\mathscr{A}_{I,\lambda^n}} R^s_\varphi\Big), \quad s \in \mathbb{R},$$

$$\overline{\mathcal{Q}}^*_\lambda(s) := \varlimsup_{n\to\infty} \frac{1}{n} \ln\Big(\sum_{\varphi\in\mathscr{A}_{I,\lambda^n}} R^s_\varphi\Big), \quad s \in \mathbb{R},$$

and denote by $\mathcal{Q}^*_\lambda(s)$ the common value if $\underline{\mathcal{Q}}^*_\lambda(s) = \overline{\mathcal{Q}}^*_\lambda(s)$.

Proposition 5.4. *Let $\{S_i\}_{i\in I}$ be an FIFS of essential contractions on X, and $\underline{\mathcal{Q}}^*_\lambda$, $\overline{\mathcal{Q}}^*_\lambda$ and $\mathcal{Q}^*_\lambda$ be defined as above. Let K be associate attractor with $\mathcal{H}^s(K) < \infty$, where $s = \dim_{\rm H} K$. Assume that $\{S_i\}_{i\in I}$ has (LDP) and satisfies (WSC). Then for any $\lambda \in (0,1)$, $\dim_{\rm H} K = \alpha$, where α is unique zero of $\underline{\mathcal{Q}}^*_\lambda(s)$ or $\overline{\mathcal{Q}}^*_\lambda(s)$. In particular, $\underline{\mathcal{Q}}^*_\lambda(\alpha) = \overline{\mathcal{Q}}^*_\lambda(\alpha) = \mathcal{Q}^*_\lambda(\alpha) = 0$.*

Proof. Similar to the proof of [3, Proposition 14], we have that [2, Proposition 2.3] holds for FIFS of essential contractions.

For any positive integer n, we can see by combining (3.1), the definition of $\mathscr{A}_{I,\lambda^n}$, and Proposition 4.3 that $\{\varphi(K) : \varphi \in \mathscr{A}_{I,\lambda^n}\}$ is a δ_n-cover of K with $\delta_n = \lambda^n |K| \to 0$ as $n \to \infty$. Let r be defined as in (2.5). Then $r > 0$ here. Combining the above discussion and the fact that $\lambda^n < R_{\mathbf{j}^{-1}} \le r^{-1} R_{\mathbf{j}}$ for $\mathbf{j} \in \mathscr{S}_{I,\lambda^n}$, we have

$$\mathcal{H}^s_{\delta_n}(K) \le \sum_{\varphi\in\mathscr{A}_{I,\lambda^n}} \lambda^{ns} |K|^s \le \sum_{\varphi\in\mathscr{A}_{I,\lambda^n}} r^{-s} R^s_\varphi |K|^s.$$

Furthermore, we have from (5.7) and (5.8) that

$$\mathcal{H}^\alpha(K) \ge \kappa^{-1} \sum_{\varphi\in\mathscr{A}_{I,\lambda^n}} \mathcal{H}^\alpha(\varphi(K)) \ge \kappa^{-1}\Big(\frac{1}{c_1 |\ln \lambda^n|^\xi}\Big)^\alpha \mathcal{H}^\alpha(K) \sum_{\varphi\in\mathscr{A}_{I,\lambda^n}} R^\alpha_\varphi$$

Next, using Proposition 5.2 and the assumption $\mathcal{H}^s(K) < \infty$, and a similar derivation as that in [2, Theorem 3.3], we can see that the assertions hold. □

Lemma 5.5. *Let $\{S_i\}_{i\in I}$ be a IIFS of essential contractions on X and $0 < b_0 < 1$. Assume that (LDP) and (WSC) hold, and K is both a WSC-set and a quasi s-set. Then for all $b \in (0, b_0)$,*

$$\sum_{\varphi\in\mathscr{A}_{J,b}} R^{\alpha_J}_\varphi \le |\ln b|^{\xi\alpha_J} c_1^{\alpha_J} \kappa \ \text{ for all } J \in \mathcal{F} \quad \text{and} \quad \sup_{J\in\mathcal{F}} \sum_{\varphi\in\mathscr{A}_{J,b}} R^*_\varphi \le |\ln b|^{\xi\alpha^*} c_1^{\alpha^*} \kappa, \tag{5.9}$$

where $\alpha_J = \dim_{\rm H} K^J$, $\alpha^* = \sup\{\alpha_J : J \in \mathcal{F}\}$, and $\mathcal{F}$, c_1, and κ are defined as in (2.9), (3.2), and (2.11) respectively.

Proof. Let $J \in \mathcal{F}$. By the assumption, each $x \in K \supseteq K_J$ is covered by no more than κ sets of the form $\varphi(K) \supseteq \varphi(K_J)$, where $\varphi \in \mathscr{A}_{J,b}$. It follows from the assumption that K is a WSC-set and (3.2) that for all $b \in (0, b_0)$,

$$\begin{aligned}\kappa \mathcal{H}^{\alpha_J}(K_J) &\geq \sum_{\varphi \in \mathscr{A}_{J,b}} \mathcal{H}^{\alpha_J}(\varphi(K_J)) \geq \sum_{\varphi \in \mathscr{A}_{J,b}} r_\varphi^{\alpha_J} \mathcal{H}^{\alpha_J}(K_J) \\ &\geq \sum_{\varphi \in \mathscr{A}_{J,b}} \Big(\frac{b}{c_1 |\ln b|^\xi}\Big)^{\alpha_J} \mathcal{H}^{\alpha_J}(K_J) \\ &\geq \sum_{\varphi \in \mathscr{A}_{J,b}} \Big(\frac{1}{c_1 |\ln b|^\xi}\Big)^{\alpha_J} R_\varphi^{\alpha_J} \mathcal{H}^{\alpha_J}(K_J).\end{aligned}$$

Thus (5.9) follows from Proposition 5.2 and the assumption that K is a quasi s-set. □

Note that for each $b \in (0, b_0)$, $\{\varphi(K) : \varphi \in \mathscr{A}_{I,b}\}$ is a $\delta := b\,|K|$-cover for K. Similar to the proof of [2, Theorem 1.2], the following Lemma 5.6 follows from Proposition 5.1 and Proposition 5.3.

Lemma 5.6. *Let $\{S_i\}_{i\in I}$ be an FIFS of essential contractions on X that has (LDP). Then*

(a) $\dim_{\mathrm{H}} K \leq d_G$.

(b) If, in addition, (WSC) is satisfied, then $\dim_{\mathrm{H}} K = d_G$

Let $\{S_i\}_{i\in I}$ be an FIFS of essential contractions on X. Using the notation in [2], we define the *topological pressure function* $P : \mathbb{R} \to \mathbb{R}$ as

$$P(s) = \lim_{n\to\infty} \frac{1}{n} \ln \sum_{\phi \in \{S_J : J \in I^n\}} R_\phi^s. \tag{5.10}$$

The upper and lower topological pressure functions, denoted by $\overline{P}(s)$ and $\underline{P}(s)$, respectively, are defined by using lim inf and lim sup, respectively.

Proposition 5.7. *Let $\{S_i\}_{i\in I}$ be an FIFS of essential contractions on X and $P(s)$ be described as above. Then $P(s)$ exists and is finite. Moreover, $P(s)$ strictly decreasing, convex and continuous on $[0,\infty)$, and $P(0) \geq 0$.*

Proof. Whether (BDP) is satisfied or not, we see from the proof of [2, Proposition 2.2] that $P(s)$ exists and is finite, convex and continuous on $[0,\infty)$, and $P(0) \geq 0$. It follows from (2.6) that for any $\delta > 0$,

$$\frac{1}{n} \ln \sum_{\phi \in \{S_J : J \in I^n\}} R_\phi^{s+\delta} \leq \frac{1}{n} \ln \sum_{\phi \in \{S_J : J \in I^n\}} R_\phi^s + \delta \ln R.$$

Letting $n \to \infty$ and using (5.10), we get

$$P(s+\delta) \leq P(s) + \delta \ln R < P(s), \text{ if } s \geq 0,$$

i.e, $P(s)$ is strictly deceasing on $[0,\infty)$. □

We extend Lemma 5.6 to IIFS.

Lemma 5.8. *Let* $\{S_i\}_{i\in I}$ *be an IIFS of essential contractions on* X *that has (LDP) and (WSC). If* K *is both a WSC-set and a quasi s-set, then* $\dim_{\rm H} K = d_G$.

Proof. Using the notation in [11]. Let $t > \alpha^*$ and $b \in (0, b_0)$, where b_0 is described as in Lemma 5.5. Then by Lemma 5.5,

$$\phi_b(t) = \sup_{J\in\mathcal{F}} \sum_{\tau\in\mathscr{A}_{J,b}} R_\tau^t \le \sup_{J\in\mathcal{F}} \sum_{\tau\in\mathscr{A}_{J,b}} b^{t-\alpha^*} R_\tau^{\alpha^*} \le b^{t-\alpha^*} |\ln b|^{\xi\alpha^*} c_1^{\alpha^*}\kappa. \tag{5.11}$$

Since $\lim_{b\to 0+}(\ln|\ln b|)/\ln b = 0$, (5.11) implies $\underline{Q}(t) \le \overline{Q}(t) \le \alpha^* - t < 0$, where

$$\underline{Q}(t) =: \varliminf_{b\to 0+} \frac{1}{-\ln b} \ln \sum_{\tau\in\mathscr{A}_{I,b}} R_\tau^t \qquad \text{and} \qquad \overline{Q}(t) =: \varlimsup_{b\to 0+} \frac{1}{-\ln b} \ln \sum_{\tau\in\mathscr{A}_{I,b}} R_\tau^t.$$

Note that according to [2, Lemma 3.4], for fixed real numbers p and q, there is an integer $l > 0$ (independent of n) such that

$$\{S_{\mathbf{j}} : \mathbf{j} \in I^*, r^{n+q} \le R_{\mathbf{j}} \le r^{n+p}\} \subseteq \left\{S_{\mathbf{i}} \circ S_{\mathbf{k}} : S_{\mathbf{i}} \in \mathscr{A}_{I,r^n}, \mathbf{k} \in \bigcup_{k=0}^{l} I^k\right\}.$$

This result also holds for FIFS of essential contractions. Using Proposition 5.4, Proposition 5.7, and a similar derivation as that in [2, Theorem 1.1], we conclude that α_J is the unique zero of the associated topological pressure function P. Similar to the proof of [11, Theorem 1.2(a)], we get $\dim_{\rm H} K = \alpha^*$. The assertions follow by using a proof similar to that of [11, Corollary 5.2(b)].

□

Proof of Theorem 2.18. Theorem 2.18 follows from Theorem 2.16(b) and Lemma 5.8.

6. Example

In this section we illustrate the applications of our results by some examples.

Example 6.1. Let $X = [0, 1]$, $0 < r \le 1/2$, and K be the limit set of the following IIFS of similitudes:

$$S_i(x) = \begin{cases} r^{k+1}x, & i = 2k+1; \\ r^{k+1}x + (1 - r^{k+1}), & i = 2k+2, \end{cases}$$

where $k = 0, 1, 2, \ldots$. Then (LDP), (L*DP), (BDP) and (CGC) hold. Moreover, $\dim_{\rm H} K = \frac{\ln 2}{-\ln r}$.

Proof. (LDP), (L*DP) and (BDP) hold since all S_i are contractive similitudes. Let $J^* = \{1, 2\}$. Take $U = (0, 1)$ and $b = r^n$ with $n \ge 1$. It is easy to see that $\sup_{\Phi\in\mathscr{P}_{J^*,U}(r^n)} \#\Phi = 2^n$. For any $J \in \mathcal{F}$ with $J \ne J^*$, since $S_1[(0, 1)]$ and $S_2[(0, 1)]$ are disjoint, $S_{2k+1}(x) =$

$S_{\underbrace{11\dots1}_{k+1}}(x)$, and $S_{2k+2}(x) = S_{\underbrace{22\dots2}_{k+1}}(x)$ for $k \geq 0$, we have $\#\Phi \leq 2^n$ for any $\Phi \in \mathscr{P}_{J,U}(r^n)$. Thus

$$\sup_J \left\{ \overline{\lim_{n\to\infty}} \frac{\ln(\sup_{\Phi\in\mathscr{P}_{J,U}(r^n)} \#\Phi)}{-\ln r^n} \right\} = \overline{\lim_{n\to\infty}} \frac{n\ln 2}{-n\ln r} = \frac{\ln 2}{-\ln r},$$

and thus (CGC) holds. Moreover, it follows from Theorem 2.16 that $\dim_{\mathrm{H}} K = \ln 2/(-\ln r)$. □

The following example shows that (LDP) is strictly weaker than (BDP).

Example 6.2. Let A be a $d\times d$ real matrix, and let $S_j(x) = \rho_j A^{k_j}(x + d_j)$, $j = 1, 2, \dots$, be an IIFS with $k_j \in \{0\} \cup \mathbb{N}$, and $0 < |\rho_j| < 1$. Assume that all eigenvalues of A have moduli 1 and $\sup_{j\geq 1}\{k_j\} < \infty$. Then

(a) (LDP) and (L*DP) are satisfied;

(b) (BDP) holds if and only if there is a real invertible matrix B and a real orthogonal matrix Q such that $A = BQB^{-1}$. In this case, the attractor is similar to a self-similar set generated by the IFS with A replaced by Q.

Proof. The proof is similar to that of [3, Example 23]; we omit the details. □

Example 6.3. Any FIFS of essential contractions satisfies (CGC).

Proof. Assume that $\{S_i\}_{i\in I}$ is an FIFS of essential contractions on X. Then $r > 0$. For any $J \in \mathcal{F}$, it follows from (3.14) that

$$\#\Phi \leq (\#I)^{nk_0 \log_r \lambda - k_0 \log_r C'},$$

where $\Phi \in \mathscr{P}_{J,U}(\lambda^n)$ and C' is described as in the proof of Proposition 3.6. Since

$$\overline{\lim_{n\to\infty}} \frac{(nk_0 \log_r \lambda - k_0 \log_r C') \ln(\#I)}{-n\ln\lambda} = -k_0 \log_r(\#I),$$

we get

$$\overline{\lim_{n\to\infty}} \frac{\ln(\sup_{\Phi\in\mathscr{P}_{J,U}(\lambda^n)} \#\Phi)}{-\ln\lambda^n} \leq -k_0 \log_r(\#I) < \infty.$$

Thus (CGC) holds. □

Example 6.4. Let $S_k : [0,1] \to [0,1]$, $k = 1, 2, \dots$, be an IIFS of contractions as follows:

$$S_k(x) = \begin{cases} \rho_{2k-1}(x + a_k), & x \leq 1/2; \\ \rho_{2k}(x - 1/2) + \rho_{2k-1}(1/2 + a_k), & x > 1/2. \end{cases}$$

where $0 < \rho_k \leq 1/2$. Assume the following conditions hold:

(a) $a_2 = 0$, $(\rho_1 + \rho_2)/2 + \rho_1 a_1 = 1$, $\rho_1 + \rho_2 = \rho_3 + \rho_4$, and

$$c := \sup\left\{ \frac{\max\{\rho_{2k-1}, \rho_{2k}\}}{\min\{\rho_{2k-1}, \rho_{2k}\}} : k = 1, 2, \dots \right\} < \infty;$$

(b) for each k, either $S_k(0,1) \subset (0,1/2)$ or $S_k(0,1) \subset (1/2,1)$;
(c) for each $k \geq 2$, $S_k(1) < S_1(0)$, $S_k(0) \leq S_{k+1}(0)$;
(d) for each $k \geq 1$, $S_{2k}([0,1]) \cap S_{2k+1}([0,1]) \neq \emptyset$ with $S_{2k}S_1 = S_{2k+1}S_2$, and no other $S_j([0,1])$ intersects the union $S_{2k}([0,1]) \cup S_{2k+1}([0,1])$.

Then (LDP), (L*DP), (BDP) and (CGC) hold, and K is both a WSC-set and a quasi s-set.

Proof. By Remark 3.3, we need only prove that $R_{\mathbf{i}}/r_{\mathbf{i}} \leq c$ for all $\mathbf{i} \in I^*$, i.e., (BDP) holds, where c is described as in condition (a). We use induction on the length of $\mathbf{i}$. Since $\rho_k > 0$, it is easy to see that $R_k = \max\{\rho_{2k-1}, \rho_{2k}\}$ and $r_k = \min\{\rho_{2k-1}, \rho_{2k}\}$ for $k = 1, 2, \ldots$, and thus the assertion is true when $|\mathbf{i}| = 1$. Assume it is true for the case $|\mathbf{i}| \leq n$. To check the case $|\mathbf{i}| = n+1$, let $\mathbf{i} = i_1 \cdots i_{n+1}$. If $S_{i_2}[0,1] \subseteq [0,1/2]$, then $S_{\mathbf{i}}(x) = \rho_{2i_1-1}(S_{i_2 \ldots i_{n+1}}(x) + a_{i_1})$ for all $x \in [0,1]$. Hence $R_{\mathbf{i}} = \rho_{2i_1-1}R_{i_2\ldots i_{n+1}}$ and $r_{\mathbf{i}} = \rho_{2i_1-1}r_{i_2\ldots i_{n+1}}$. If $S_{i_2}[0,1] \subseteq [1/2,1]$, using the same argument, we have $R_{\mathbf{i}} = \rho_{2i_1}R_{i_2\ldots i_{n+1}}$ and $r_{\mathbf{i}} = \rho_{2i_1}r_{i_2\ldots i_{n+1}}$. In both cases, it follows by induction hypothesis that $R_{\mathbf{i}}/r_{\mathbf{i}} \leq c$. Thus (LDP) and (L*DP) hold.

From conditions (c) and (d), for $\mathbf{i} \in I^*$, it is easy to see by induction on the length of the composition that all overlapping iterates can be expressed in the form $S_{\mathbf{i}}S_{2k}[0,1] \cap S_{\mathbf{i}}S_{2k+1}[0,1]$, with no other iterates $S_{\mathbf{j}}[0,1]$ with $|\mathbf{j}| = |\mathbf{i}|+1$ intersecting the union $S_{\mathbf{i}}S_{2k}[0,1] \cup S_{\mathbf{i}}S_{2k+1}[0,1]$. Thus it follows by induction on the length of $\mathbf{i}$ with $R_{\mathbf{i}} \leq b < R_{\mathbf{i}^{-1}}$ that $\#\{\tau \in \mathscr{S}_{I,b} : x \in \tau(X)\} \leq 2$ for any $b \in (0,1)$, that is, (WSC) holds and X is a WSC-region.

Taking $U = (0,1)$. For any $b \in (0,1)$ and $J \in \mathcal{F}$, we have

$$\frac{\ln(\sup_{\Phi \in \mathscr{P}_{J,U}(b)} \#\Phi)}{-\ln b} \leq \frac{\ln(\#\mathscr{A}_{J,b})}{-\ln b} \tag{6.1}$$

Let K be the limit set of $\{S_i(x)\}_{i \in \mathbb{N}}$. Then $K \subset X$, and is a WSC-set since X is a WSC-region. Moreover, by Remark 2.15, K is a quasi s-set. It follows from (2.12), (6.1) and Lemma 5.8 that

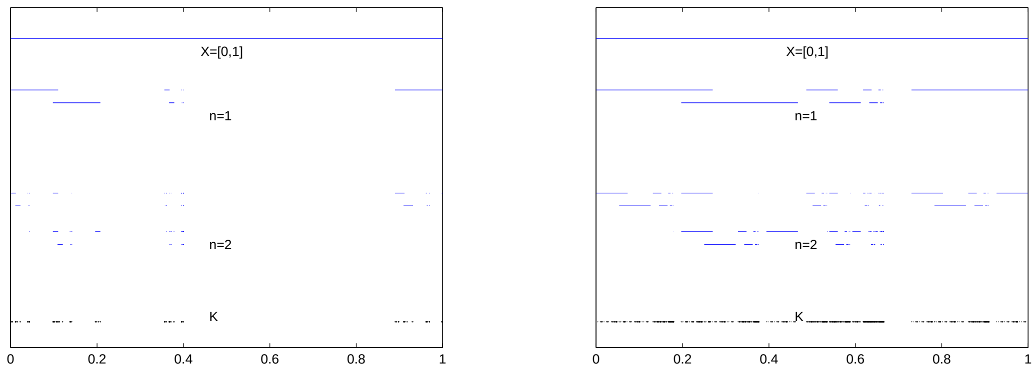
$$\sup_{J \in \mathcal{F}} \left\{ \varlimsup_{b \to 0+} \frac{\ln(\sup_{\Phi \in \mathscr{P}_{J,U}(b)} \#\Phi)}{-\ln b} \right\} \leq d_G = \dim_{\mathrm{H}} K,$$

where d_G is the growth dimension of $\{S_i\}_{i \in \mathbb{N}}$. Thus (CGC) holds. This completes the proof. □

The following is a class of special IIFSs from Example 6.4; see Figure 1(a).

Example 6.5. Let $X = [0,1]$, $0 < r + r^2 < (5 - \sqrt{17})/4 \approx 0.21922...$, and $(r + r^2)(2 - r - r^2)/(1 - r - r^2) < t < 1/2$. Let

$$S_1(x) = \begin{cases} 2rx + (1 - r - r^2), & x \leq 1/2; \\ 2r^2(x - 1/2) + 1 - r^2, & x > 1/2, \end{cases}$$



(a) (b)

FIGURE 1. (a) First two iterations of $[0,1]$ of the IIFS in Example 6.5, with $r = 1/10$ and $t = 2/5$. The limit set K is also shown. (b) Example 6.6, drawn with $r = 0.27$ and $t = 2/3$.

and

$$S_{2k}(x) = (r + r^2)^k x + t(1 - (r + r^2)^{k-1}),$$

$$S_{2k+1}(x) = (r + r^2)^k x + t(1 - (r + r^2)^{k-1}) + (r + r^2)^k (1 - r - r^2)$$

for any $k \geq 1$. Then (LDP), (L*DP), (BDP), and (CGC) hold, and K is both a WSC-set and a quasi s-set.

The following example is different from Example 6.4 in that condition (b) need not be satisfied; see Figure 1(b).

Example 6.6. Let $X = [0,1]$, $0 < r < (2-\sqrt{2})/2 \approx 0.292893...$, $r(2-r)/(1-r) < t < 1-r$. Let $S_1(x) = rx + (1-r)$, and let

$$S_i(x) = \begin{cases} r^k x + t(1 - r^{k-1}), & i = 2k; \\ r^k x + t(1 - r^{k-1}) + r^k(1-r), & i = 2k+1, \end{cases}$$

for any $k \geq 1$. Then (LDP), (L*DP), (BDP), and (CGC) hold. Moreover, K is both a WSC-set and is a quasi s-set. Finally, $\dim_{\mathrm{H}} K = \ln((2-\sqrt{2})/2)/\ln r$.

Proof. Using a proof similar to that of Example 6.4, we conclude that (CGC) holds. The other assertions follow from [11, Example 6.1]. □

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