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HEAT EQUATIONS DEFINED BY A CLASS OF FRACTAL MEASURES

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ABSTRACT. We set up a framework to study one-dimensional heat equations defined by fractal Laplacians associated with self-similar measures with overlaps. We show that for a class of such self-similar measures, a heat equation can be discretized and the finite element method can be applied to yield a system of linear differential equations. We show that the numerical solutions converge to the actual solution and obtain the rate of convergence. We also study some properties of the solutions of the heat equation.

CONTENTS

1. INTRODUCTION

Let $U \subseteq \mathbb{R}^d$, $d \geq 1$, be a bounded open subset, and let μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. It is known (see, e.g., [11]) that μ defines a Dirichlet Laplace operator Δ_μ, if the following *Poincaré inequality for a measure*

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(PI) holds: There exists some constant $C > 0$ such that

$$\int_U |u|^2 \, d\mu \leq C \int_U |\nabla u|^2 \, dx \qquad \text{for all } u \in C_c^\infty(U) \tag{1.1}$$

(see, e.g., [11, 15, 18]). The main purpose of this paper is to study heat equations defined by Dirichlet Laplacians Δ_μ. More precisely, we study the following non-homogeneous *parabolic initial/boundary value problem (IBVP)*:

$$\begin{cases} u_t - \Delta_\mu u = f & \text{on } U \times [0, T], \\ u = 0 & \text{on } \partial U \times [0, T], \\ u = g & \text{on } U \times \{t = 0\}. \end{cases} \tag{1.2}$$

The existence and uniqueness results (see Definition 2.6 for definition of a weak solution) follows easily from the general theory for heat equations in Hilbert spaces (see Section 2.3 and Theorem 2.4 for details). In this paper, we study the solution of equation (1.2) both theoretically and numerically.

We call a μ-measurable subset I of $\overline{U}$ a *cell (in $\overline{U}$)* if $\mu(I) > 0$. Clearly, $\overline{U}$ itself is a cell. Two cells I, J in U are *measure disjoint* with respect to μ if $\mu(I \cap J) = 0$. Let $I \subseteq \overline{U}$ be a cell. We call a finite family $\mathbf{P}$ of measure disjoint cells a *μ-partition* of I if $J \subseteq I$ for all $J \in \mathbf{P}$, and $\mu(I) = \sum_{J \in \mathbf{P}} \mu(J)$. A sequence of μ-partitions $(\mathbf{P}_k)_{k \geq 1}$ is *refining* if for any $J_1 \in \mathbf{P}_k$ and any $J_2 \in \mathbf{P}_{k+1}$, either $J_2 \subseteq J_1$ or they are measure disjoint, i.e., each member of $\mathbf{P}_{k+1}$ is a subset of some member of $\mathbf{P}_k$. Throughout this paper, $|E|$ denotes the diameter of a subset $E \subseteq \mathbb{R}^n$.

In order to discretize (1.2) and obtain numerical approximations of the weak solution, we will often impose the following additional conditions on μ: there exists a sequence of refining μ-partitions $(\mathbf{P}_k)_{k \geq 1} = (\{I_{k,\ell}\}_{\ell=0}^{N(k)})_{k \geq 1}$ of $\overline{U}$ such that

- (P1) there exist some constant $\rho \in (0, 1)$ and some integer m_0 satisfying $\max\{|J| : J \in \mathbf{P}_k\} \leq \rho^{k - m_0}$ for all $k \geq 1$;
- (P2) for any $k \geq 1$, each cell $I \in \mathbf{P}_k$ is closed and connected;
- (P3) for any $k \geq 2$ and any $0 \leq \ell \leq N(k)$, there exist similitudes $(\tau_{k,\ell,i})_{i=0}^{N(1)}$ of the form $\tau_{k,\ell,i}(x) = r_{k,\ell,i} x + b_{k,\ell,i}$ and constants $(c_{k,\ell,i})_{i=0}^{N(1)}$ such that $\tau_{k,\ell,i}(I_{1,i}) \subseteq I_{k,\ell}$, and
$$\mu|_{I_{k,\ell}} = \sum_{i=0}^{N(1)} c_{k,\ell,i} \cdot \mu|_{I_{1,i}} \circ \tau_{k,\ell,i}^{-1}. \tag{1.3}$$

Under assumption (P3), the μ measure of each subinterval in the partition can be computed by using (1.3), making it possible to discretizing the heat equation (1.2). Finally, we assume that μ is a measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a, b]$.

Let $f(x, t) \equiv 0$. Multiplying the first equation in (1.2) by $v \in \operatorname{dom} \mathcal{E}_D$ (see Section 2.1 for definition), integrating both sides with respect to $d\mu$, and then integrating by parts, we obtain

$$-\int_a^b u_x(x,t)v'(x)\,dx = \int_a^b u_t(x,t)v(x)\,d\mu, \tag{1.4}$$

where $u_x(x,t)$ is the weak partial derivative of u with respect to x and $u_t(x,t)$ is the weak partial derivative with respect to t.

Theorem 1.1. *Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a,b]$. Assume that there exists a sequence of refining partitions $(\mathbf{P}_k)_{k\geq 1} = (\{I_{k,\ell}\}_{\ell=0}^{N(k)})_{k\geq 1}$ satisfying conditions (P1)–(P3), and $\int_{I_{1,\ell}} x^i\,d\mu$, $0\leq \ell\leq N(1)$, $i=0,1,2$, can be evaluated explicitly. Then the finite element method for equation (1.4) can be discretized into a system of first-order ordinary differential equations, which has a unique solution that can be solved numerically.*

We are mainly interested in fractal measures μ. Let X be a non-empty compact subset of $\mathbb{R}^d$. Throughout this paper, an *iterated function system (IFS)* refers to a finite family of contractive similitudes $\{S_i\}_{i=1}^q$ defined on X, i.e.,

$$S_i(x) = \rho_i x + b_i, \qquad i = 1,\ldots,q, \tag{1.5}$$

where $0<\rho_i<1$, and $b_i\in\mathbb{R}^d$. It is well-known that for each IFS $\{S_i\}_{i=1}^q$, there exists a unique non-empty compact subset $F\subseteq X$, called the *self-similar set*, such that

$$F = \bigcup_{i=1}^q S_i(F);$$

moreover, associated to each set of probability weights $\{w_i\}_{i=1}^q$ (i.e., $w_i>0$ and $\sum_{i=1}^q w_i = 1$), there is a unique probability measure, called the *self-similar measure*, satisfying the following identity

$$\mu = \sum_{i=1}^q w_i \mu\circ S_i^{-1} \tag{1.6}$$

(see [8, 12]). An IFS $\{S_i\}_{i=1}^q$ is said to satisfy the *open set condition (OSC)* if there exists a non-empty bounded open set O such that $\bigcup_i S_i(O)\subseteq O$ and $S_i(O)\cap S_j(O)=\emptyset$ for all $i\neq j$. IFSs that do not satisfy (OSC), as well as all associated self-similar measures, are said to have overlaps.

It is worth pointing out that for general self-similar measures with overlaps, it does not seem possible to discretize the heat equations (1.2) in the way described in the paper, and thus it is not clear how numerical approximations of the weak solution can be obtain. Theorem 1.1 provides a framework under which discretization can be performed.

Based on Theorem 1.1, we solve the homogeneous IBVP (1.2) numerically for three different one-dimensional self-similar measures with overlaps, namely, the inifinite

Bernoulli convolution associated with the golden ratio, the three-fold convolution of the Cantor measure, and a class of self-similar measures that we call *essentially of finite type (EFT) (see [17]).* These measures share the common property that the support can be partitioned into a sequence of arbitrarily small intervals whose measures can be computed explicitly.

In Section 5, we show that Theorem 1.1 also holds for heat equations defined by Neumann Laplacians (see (5.1)), and study some properties of the solutions.

The following theorem shows that the approximate solutions converge to the actual weak solution and we also obtain a rate of convergence. See Subsection 2.1 and Definition 2.1 for the definitions of $\|\cdot\|_{\operatorname{dom}\mathcal{E}_D}$ and $\|\cdot\|_{2,\operatorname{dom}\mathcal{E}_D}$, respectively.

Theorem 1.2. *Assume the hypotheses of Theorem 1.1, let $f=0$ in equation (1.2), and fix $t\in[0,T]$. Then the approximate solutions u^m obtained by the finite element method converge in $L^2((a,b),\mu)$ to the actual weak solution u. Moreover,*

$$\|u^m-u\|_\mu \le 2\big(\sqrt{T}\|u_t\|_{2,\operatorname{dom}\mathcal{E}_D}+\|u\|_{\operatorname{dom}\mathcal{E}_D}\big)\rho^{m/2}, \tag{1.7}$$

where ρ is any constant in condition (P1).

We remark that Theorem 1.2 also holds for heat equations defined by Neumann Laplacians. Details are given in Section 6.

The rest of this paper is organized as follows. Section 2 summarizes some notation, definitions and results that will be needed throughout the paper, and gives the existence and unique results of the heat equation (1.2). Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we apply Theorem 1.1 to three different self-similar measures with overlaps. We study some properties of the solutions of heat equations defined by Neumann Laplacians in Section 5. Finally, the proof of Theorem 1.2 is given in Section 6.

2. Preliminaries

In this section, we summarize some notation, definitions and facts that will be used throughout the rest of the paper. For a Banach space X, we denote its topological dual by X'. For $v\in X'$ and $u\in X$, we let $\langle v,u\rangle=\langle v,u\rangle_{X',X}:=v(u)$ denote the *dual pairing* of X' and X.

A function $s:[0,T]\to X$ is called *simple* if it has the form

$$s(t)=\sum_{m=1}^{N}\chi_{E_m}(t)u_m \quad \text{for} \in[0,T], \tag{2.1}$$

where for each $m = 1, \ldots, N$, E_m is a Lebesgue measurable subset of $[0, T]$, $u_m \in X$ and χ_{E_m} is the characteristic function on E_m. A function $u : [0, T] \to X$ is called *strongly measurable* if there exist simple functions $s_n : [0, T] \to X$ such that

$$s_n(t) \to u(t) \quad \text{for Lebesgue a.e. } t \in [0, T] \text{ as } n \to \infty.$$

A function $u : [0, T] \to X$ is *weakly measurable* if for each $v \in X'$, the mapping $t \mapsto \langle v, u(t) \rangle$ is Lebesgue measurable.

A function $u : [0, T] \to X$ is *almost separably valued* if there exists a subset $E \subseteq [0, T]$ with zero Lebesgue measure such that the set $\{u(t) : t \in [0, T] \setminus E\}$ is separable. By a theorem of Petti [20], a function $u : [0, T] \to X$ is strongly measurable if and only if it is weakly measurable and almost separably valued. Since any subset of a separable Banach space X is separable, the two concepts of measurability coincide and we can use the term *measurable* without ambiguity.

Definition 2.1. *Let X be a separable Banach space with norm $\|\cdot\|_X$. Denote by $L^p(0, T; X)$ the space of all measurable functions $u : [0, T] \to X$ satisfying*

(1) $\|u\|_{L^p(0,T;X)} := \Big(\int_0^T \|u(t)\|_X^p \, dt \Big)^{1/p} < \infty$, *if* $1 \le p < \infty$, *and*
(2) $\|u\|_{L^\infty(0,T;X)} := \operatorname{esssup}_{0 \le t \le T} \|u(t)\|_X < \infty$, *if* $p = \infty$.

If the interval $[0, T]$ is understood, we will abbreviate these norms as $\|u\|_{p,X}$ and $\|u\|_{\infty,X}$, respectively.

Let $U \subseteq \mathbb{R}^d$, $d \ge 1$, be a bounded open subset. For a function $\varphi : U \to \mathbb{R}$, we let φ' denote both its classical and weak derivatives. If $u \in L^2(0, T; X)$, where X is $H_0^1(U)$ or $L^2(U, \mu)$ etc., then for each fixed t, we denote by $u_x(x, t)$ (or ∇u) the classical or weak derivatives of u with respect to x.

Remark 2.1. *For each $1 \le p \le \infty$, $L^p(0, T; X)$ is a Banach space; moreover, $L^{p_2}(0, T; X) \subseteq L^{p_1}(0, T; X)$ if $1 \le p_1 \le p_2 \le \infty$. Let X be a separable Banach space with inner product $(\cdot, \cdot)_X$. If $(X, (\cdot, \cdot)_X)$ is a separable Hilbert space, then $L^2(0, T; X)$ is a Hilbert space with the inner product*

$$(u, v)_{L^2(0,T;X)} := \int_0^T (u(t), v(t))_X \, dt.$$

Definition 2.2. *Let X be a Banach space and $u \in L^1(0, T; X)$. We say that $v \in L^1(0, T; X)$ is the* weak derivative *of u with respect to t, written $u_t = v$, if*

$$\int_0^T \phi_t(t) u(t) \, dt = - \int_0^T \phi(t) v(t) \, dt$$

for all scalar test functions $\phi \in C_c^\infty(0, T)$.

Definition 2.3. *Let X be a Banach space and X' its dual. We say that a sequence* $\{u_m\}_{m=1}^{\infty} \subseteq X$ converges weakly *to $u \in X$, written $u_m \rightharpoonup u$, if $\langle v, u_m\rangle \to \langle v, u\rangle$ for each bounded linear functional $v \in X'$.*

For the more general definition of devatives of distributions with values in a Hilbert space, we refer the reader to [23, Section 25].

2.1. **Dirichlet Laplacian defined by a measure.** For convenience, we summarize the definition of the Dirichlet Laplacian on a bounded domain defined by a measure; details can be found in [11]. Let $U \subseteq \mathbb{R}^d, d \geq 1$, be a bounded open subset and μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. Let

$$L^2(U,\mu) := \Big\{ f : U \to \mathbb{R} : \|f\|_\mu := \Big(\int_U |f|^2 \, d\mu \Big)^{1/2} < \infty \Big\}.$$

In particular, if μ is Lebesgue measure, we denote $L^2(U,\mu)$ simply by $L^2(U)$. Let $H^1(U)$ be the Sobolev space with inner product

$$\langle u, v\rangle_{H^1(U)} := \int_U uv \, dx + \int_U \nabla u \cdot \nabla v \, dx,$$

and let $H_0^1(U)$ denote the completion of $C_c^\infty(U)$ in the $H^1(U)$ norm, which admits the equivalent inner product defined by

$$\langle u, v\rangle_{H_0^1(U)} := \int_U \nabla u \cdot \nabla v \, dx.$$

Note that both $H^1(U)$ and $H_0^1(U)$ are Hilbert spaces.

We assume that μ satisfies (PI) (see (1.1)). Then each equivalence class $u \in H_0^1(U)$ contains a unique (in the $L^2(U,\mu)$ sense) member $\hat{u}$ that belongs to $L^2(U,\mu)$ and satisfies both conditions below:

(1) there exists a sequence $\{u_n\}$ in $C_c^\infty(U)$ such that $u_n \to \hat{u}$ in $H_0^1(U)$ and $u_n \to \hat{u}$ in $L^2(U,\mu)$;
(2) $\hat{u}$ satisfies inequality (1.1).

We call $\hat{u}$ the $L^2(U,\mu)$-*representative* of u. Define a mapping $\iota : H_0^1(U) \to L^2(U,\mu)$ by $\iota(u) = \hat{u}$. ι is a bounded linear operator, but not necessarily injective. Consider the subspace $\mathcal{N}$ of $H_0^1(U)$ defined as $\mathcal{N} := \{u \in H_0^1(U) : \|\iota(u)\|_\mu = 0\}$. Now let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $H_0^1(U)$. Then $\iota : \mathcal{N}^\perp \to L^2(U,\mu)$ is injective. Unless explicitly stated otherwise, we will denote the $L^2(U,\mu)$-representative $\hat{u}$ simply by u.

Consider the non-negative bilinear form $\mathcal{E}(\cdot,\cdot)$ in $L^2(U,\mu)$ defined by

$$\mathcal{E}(u, v) := \int_U \nabla u \cdot \nabla v \, dx \tag{2.2}$$

with domain $\operatorname{dom}\mathcal{E}_D = \mathcal{N}^\perp$, or more precisely, $\iota(\mathcal{N}^\perp)$. (PI) implies that $(\mathcal{E}, \operatorname{dom}\mathcal{E}_D)$ is a closed quadratic form in $L^2(U,\mu)$. Hence there exists a non-negative self-adjoint operator A in $L^2(U,\mu)$ such that

$$\mathcal{E}(u,v) = \big(A^{1/2}u, A^{1/2}v\big) \quad \text{and} \quad \operatorname{dom}\mathcal{E}_D = \operatorname{dom}(A^{1/2})$$

(see, e.g., [9, Theorem 1.3.1]). We write $\Delta_\mu = -A$, and call it the *(Dirichlet) Laplacian* with respect to μ. Let $u \in \operatorname{dom}\mathcal{E}_D$. Then $u \in \operatorname{dom}\Delta_\mu$ if and only if there exists a unique $f \in L^2(U,\mu)$ such that $\mathcal{E}(u,v) = (f,v)_\mu$ for all $v \in \operatorname{dom}\mathcal{E}_D$. In this case, $-\Delta_\mu u = f$. Throughout this paper, we let

$$\operatorname{dom}\mathcal{E}_D := \mathcal{N}^\perp \quad \text{and} \quad \|\cdot\|_{\operatorname{dom}\mathcal{E}_D} := \|\cdot\|_{H_0^1(U)}.$$

Some sufficient conditions for (PI) and the existence of an orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2(U,\mu)$ consisting of the eigenfunctions of $-\Delta_\mu$ can be found in [4,11,15]. We remark that if $n=1$, then (PI) holds for any such μ, and thus Δ_μ is well-defined; moreover, $-\Delta_\mu$ has compact resolvent.

2.2. **Gelfand triple.**

The notion of Gelfand triple, defined below, plays an important role in our investigation of the heat equation.

Definition 2.4. *Let V, H be separable Hilbert spaces with a continuous dense embedding $\iota : V \hookrightarrow H$. By identifying H with its dual H', we obtain the following continuous and dense embedding $V \hookrightarrow H \cong H' \hookrightarrow V'$. Assume, in addition, that the dual pairing between V and V' is compatible with the inner product on H, in the sense that*

$$\langle v, u\rangle_{V',V} = (v,u)_H \qquad \text{for all } u \in V \subset H \text{ and } v \in H \cong H' \subset V'.$$

The triple (V, H, V') is called a Gelfand triple.

We remark that since V is itself a Hilbert space, it is isomorphic with its dual V'. However, this isomorphism is in general not the same as the composition $\iota^*\iota : V \subset H = H' \hookrightarrow V'$, where ι^* is the adjoint of ι.

Let U, μ and $\operatorname{dom}\mathcal{E}_D$ be given as in Section 2.1. Then the spaces $\operatorname{dom}\mathcal{E}_D$, $L^2(U,\mu)$, $(\operatorname{dom}\mathcal{E}_D)'$ form a *Gelfand triple*:

$$\operatorname{dom}\mathcal{E}_D \hookrightarrow L^2(U,\mu) \cong (L^2(U,\mu))' \hookrightarrow (\operatorname{dom}\mathcal{E}_D),$$

where we identify $L^2(U,\mu)$ with $(L^2(U,\mu))'$. The embedding $L^2(U,\mu) \hookrightarrow (\operatorname{dom}\mathcal{E}_D)'$ is given by

$$w \in L^2(U,\mu) \mapsto (w,\cdot)_\mu \in (L^2(U,\mu))' \subset (\operatorname{dom}\mathcal{E}_D)'.$$

2.3. **Existence and uniqueness of weak solution.** In this subsection, we consider the existence and uniqueness of weak solution of equation (1.2).

Definition 2.5. *Let V be a Hilbert space, and $0 < T < \infty$. For each integer $k \geq 0$, define the Sobolev space*

$$W_2^k(0,T;V) := \Big\{ u : (0,T) \to V \text{ measurable} : \frac{d^n u}{dt^n} \in L^2(0,T;V) \text{ for } 0 \leq n \leq k \Big\},$$

where the differentiation is in the distributional sense. Equip $W_2^k(0,T;V)$ with the norm

$$\|u\|_{k,V}^2 := \sum_{n=0}^{k} \int_0^T \Big\| \frac{d^n u}{dt^n} \Big\|_V^2 \, dt.$$

Let V, H be separable Hilbert spaces. Assume that the embedding $V \hookrightarrow H$ is continuous, injective, and dense, such that $V \hookrightarrow H \hookrightarrow V'$ is a Gelfand triple. Let $0 < T < \infty$. For $t \in [0,T]$ and $\varphi, \psi \in V$, assume that $a(t;\varphi,\psi)$ is a sesquilinear form in φ, ψ. Let $\Re(z)$ denote the real part of a complext number z. We also need the following conditions:

(C1) For fixed $\varphi, \psi \in V$, $a(t;\varphi,\psi)$ is measurable on $[0,T]$.

(C2) There exists some constant $c > 0$, independent of t, such that

$$|a(t;\varphi,\psi)| \leq c\|\varphi\|_V \cdot \|\psi\|_V \qquad \text{for all } t \in [0,T] \text{ and } \varphi, \psi \in V.$$

(C3) There exist real $k_0, \alpha \geq 0$, independent of t and φ, such that

$$\Re(a(t;\varphi,\varphi)) + k_0\|\varphi\|_H^2 \geq \alpha\|\varphi\|_V \qquad \text{for all } t \in [0,T] \text{ and } \varphi \in V.$$

It follows from condition (C2) that there exists a representation operator $L(t) : V \to V'$, such that for each t, $L(t)$ is linear and continuous, with $a(t;\varphi,\psi) = (L(t)\varphi,\psi)_H$.

The proofs of Theorems 2.2 and 2.3 below can be found in [23, Sections 26–27].

Theorem 2.2. *Let V and H be separable Hilbert spaces. Assume that the embedding $V \hookrightarrow H$ is continuous, injective, and dense, such that $V \hookrightarrow H \hookrightarrow V'$ is a Gelfand triple. Assume that the sesquilinear form $a(t;\varphi,\psi)$ satisfies conditions (C1)–(C3) above. Then for any $T \in (0,\infty)$, $f \in L^2(0,T;V')$, and $u_0 \in H$, there exists a unique function $u(t) \in L^2(0,T;V)$ with $du/dt \in L^2(0,T;V')$ such that*

$$\frac{du}{dt} + L(t)u = f \quad \text{for } t \in [0,T], \quad \text{and } u(0) = u_0, \tag{2.3}$$

in the sense that

$$\Big(\frac{du}{dt}, \varphi\Big)_H + (L(t)u,\varphi)_H = (f,\varphi)_H \qquad \text{for all } \varphi \in V.$$

We will also need the following additional assumption on $a(t;\varphi,\psi)$:

(C4) for fixed $\varphi, \psi \in V$, let $a(t; \varphi, \psi)$ be k-fold differentiable with respect to t in $[0, T]$. We assume that $(d^j/dt^j)a(t; \varphi, \psi)$ is continuous in $[0, T]$ for $j = 0, \ldots, k-1$, $(d^k/dt^k)a(t; \varphi, \psi)$ exists and is measurable in $[0, T]$, and there exists a constant c_0, independent of t, such that

$$\left| \frac{d^j}{dt^j} a(t; \varphi, \psi) \right| \le c_0 \|\varphi\|_V \cdot \|\psi\|_V \quad \text{for all } j = 0, \ldots, k.$$

The following theorem shows that the smoothness of the solution of equation (2.3) increases with that of f.

Theorem 2.3. *Assume the hypotheses of Theorem 2.2, and that condition* (C4) *above holds. Consider the parabolic equation*

$$\frac{du}{dt} + L(t)u = f \quad \text{for } t \in [0, T], \quad \text{and} \quad u(0) = u_0. \tag{2.4}$$

Assume, in addition, that $f \in W_2^k(0, T; V')$, $u_0 \in H$ *for* $k = 0$; $u_0 \in V$, $f(0) - L(0)u_0 \in H$ *for* $k = 1$; $u_0 \in V$, $f(0) - L(0)u_0 \in V$, $f^{(1)}(0) - L(0)f(0) \in H$ *for* $k = 2$; *whereas for* $k \ge 3$, *we assume* $u_0 \in V$, $f(0) - L(0)u_0 \in V$,

$$f^{(i)}(0) - L(0)f^{(i-1)}(0) \in V, \quad i = 1, \cdots, k-2, \ \text{and} \quad f^{(k-1)}(0) - L(0)f^{(k-2)}(0) \in H.$$

Then the solution u *of* (2.4) *satisfies*

$$u \in W_2^k(0, T; V) \quad \text{and} \quad \frac{d^{k+1}u(t)}{dt^{k+1}} \in L^2(0, T; V').$$

We will apply Theorems 2.2 and 2.3 to the heat equation (1.2). Let $U \subseteq \mathbb{R}^d$, $d \ge 1$, be a bounded open subset, and let μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. Assume μ satisfies (PI). Let $(\mathcal{E}, \operatorname{dom} \mathcal{E}_D)$ be defined as in (2.2), and $-\Delta_\mu$ be the Dirichlet Laplacian with respect to μ.

Definition 2.6. *Use the notation above. Let* $0 < T < \infty$. *Assume* $f \in L^2(0, T; (\operatorname{dom} \mathcal{E}_D)')$ *and* $g \in L^2(U, \mu)$. *A function* $u \in L^2(0, T; \operatorname{dom} \mathcal{E}_D)$ *with* $u_t \in L^2(0, T; (\operatorname{dom} \mathcal{E}_D)')$ *is a* weak solution *of IBVP* (1.2) *if the following conditions are satisfied:*

(1) $\langle u_t, v \rangle + \mathcal{E}(u, v) = \langle f, v \rangle$ *for each* $v \in \operatorname{dom} \mathcal{E}_D$ *and Lebesgue a.e.* $t \in [0, T]$;
(2) $u(x, 0) = g(x)$ *for all* $x \in U$.

Here $\langle \cdot, \cdot \rangle$ *denotes the pairing between* $(\operatorname{dom} \mathcal{E}_D)'$ *and* $\operatorname{dom} \mathcal{E}_D$.

Theorem 2.4. *Use the notation in Definition 2.6. Assume* $f \in L^2(0, T; (\operatorname{dom} \mathcal{E}_D)')$ *and* $g \in L^2(U, \mu)$. *Then equation* (1.2) *has a unique weak solution.*

Proof. In order to apply Theorem 2.2, we let $V = \operatorname{dom} \mathcal{E}_D$, $H = L^2(U, \mu)$, and set $a(t; u, v) := \mathcal{E}(u, v)$, which is independent of t. Conditions (C1) and (C4) clearly hold, because the map $t \mapsto a(t;u,v) = \mathcal{E}(u,v)$ is constant in time and real-valued. Since for all $u, v \in \operatorname{dom}\mathcal{E}_D$,

$$\begin{aligned}\big|a(t;u,v)\big| &= \left|\int_U \nabla u \cdot \nabla v\,dx\right| \leq \left(\int_U |\nabla u|^2\,dx\right)^{1/2} \cdot \left(\int_U |\nabla v|^2\,dx\right)^{1/2} \\ &= \|u\|_{\operatorname{dom}\mathcal{E}_D} \cdot \|v\|_{\operatorname{dom}\mathcal{E}_D},\end{aligned}$$

condition (C2) holds. Thus there exists a representation operator $L : \operatorname{dom}\mathcal{E}_D \to (\operatorname{dom}\mathcal{E}_D)'$ such that $\mathcal{E}(u,v) = (Lu, v)_\mu$, which satisfies $L = -\Delta_\mu$ on $\operatorname{dom}\Delta_\mu$.

Finally, for all $t \in [0,T]$ and $u \in \operatorname{dom}\mathcal{E}_D$,

$$a(t;u,u) + \|u\|_\mu^2 = \mathcal{E}(u,u) + \|u\|_\mu^2 \geq \mathcal{E}(u,u) = \|u\|^2_{\operatorname{dom}\mathcal{E}_D},$$

and thus condition (C3) holds with $k_0 = \alpha = 1$. Consequently, the assertion follows from Theorem 2.2. □

As a consequence of Theorem 2.3, we have the following regularity result for solutions of homogeneous heat equations in our setting.

Theorem 2.5. *Use the notation in Theorem 2.4 and assume the hypotheses of Theorem 2.4. Assume in addition that $f = 0$. Then for all $k \geq 1$, the solution of the homogeneous equation* (1.2) *satisfies*

$$u \in W_2^k(0,T;\operatorname{dom}\mathcal{E}_D) \quad \textit{and} \quad \frac{d^{k+1}u(t)}{dt^{k+1}} \in L^2\big(0,T;(\operatorname{dom}\mathcal{E}_D)'\big).$$

3. The finite element method

In this section, we let $f = 0$ in equation (1.2), and use the finite element method to solve the homogeneous IBVP. Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a,b]$. Assume that there exists a sequence of refining partitions $(\mathbf{P}_k)_{k\geq 1} = (\{I_{k,\ell}\}_{\ell=0}^{N(k)})_{k\geq 1}$ satisfying conditions (P1)–(P3) in Section 1. Without loss of generality, we can write $I_{m,\ell} = [x_{m,\ell}, x_{m,\ell+1}]$ for $m \geq 1$ and $0 \leq \ell \leq N(m) - 1$. It is easy to see that $x_{m,0} = a$ and $x_{m,N(m)} = b$ for all $m \geq 1$.

We apply the finite element method to approximate the weak solution $u(x,t)$ satisfying (1.4) by

$$u^m(x,t) = \sum_{\ell=0}^{N(m)} \beta_{m,\ell}(t)\phi_{m,\ell}(x), \tag{3.1}$$

where, for $\ell = 0, 1, \ldots, N(m)$, $\beta_{m,\ell}(t)$ are functions to be determined and $\phi_{m,\ell}(x)$ are the standard piecewise linear *finite element basis functions* (also called *tent functions*) defined by

$$\phi_{m,\ell}(x) = \begin{cases} \dfrac{x - x_{m,\ell-1}}{x_{m,\ell} - x_{m,\ell-1}} & \text{if } x \in I_{m,\ell-1},\ \ell = 1, 2, \dots, N(m), \\ \dfrac{x - x_{m,\ell+1}}{x_{m,\ell} - x_{m,\ell+1}} & \text{if } x \in I_{m,\ell},\ \ell = 0, 1, \dots, N(m) - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

We require $u^m(x,t)$ to satisfy the integral form of the homogeneous heat equation

$$\int_a^b u_t^m(x,t)\phi_{m,i}(x)\,d\mu = -\int_a^b u_x^m(x,t)\cdot\phi'_{m,i}(x)\,dx \qquad \text{for } i = 1, \dots, N(m)-1, \tag{3.3}$$

and the Dirichlet boundary condition $u^m(a,t) = u^m(b,t) = 0$, where $u_t^m := (u^m)_t$ and $u_x^m := (u^m)_x$. We note that $\phi_{m,i}(a) = \phi_{m,i}(x_{m,0}) = 0$ and $\phi_{m,j}(b) = \phi_{m,j}(x_{m,N(m)}) = 0$ for all $i = 1, \dots, N(m)$, $j = 0, 1, \dots, N(m) - 1$, and $m \geq 1$. Thus $\beta_{m,0}(t) = \beta_{m,N(m)}(t) = 0$ for all $m \geq 1$. Using this and substituting (3.1) into (3.3) gives

$$\sum_{\ell=1}^{N(m)-1} \beta'_{m,\ell} \int_a^b \phi_{m,\ell}(x)\phi_{m,i}(x)\,d\mu = -\sum_{\ell=1}^{N(m)-1} \beta_{m,\ell} \int_a^b \phi'_{m,\ell}(x)\phi'_{m,i}(x)\,dx \tag{3.4}$$

for $1 \leq i \leq N(m) - 1$. We define the mass matrix $\mathbf{M} = \mathbf{M}^{(m)} = (M_{ij}^{(m)})$ and stiffness matrix $\mathbf{K} = \mathbf{K}^{(m)} = (K_{ij}^{(m)})$, respectively, by

$$M_{ij}^{(m)} = \int_a^b \phi_{m,i}(x)\phi_{m,j}(x)\,d\mu \quad \text{and} \quad K_{ij}^{(m)} = \int_a^b \phi'_{m,i}(x)\phi'_{m,j}(x)\,dx,$$

where $1 \leq i, j \leq N(m) - 1$. It follows from the definition of $\phi_{m,j}(x)$ that both $\mathbf{M}$ and $\mathbf{K}$ are tridiagonal. Let

$$\mathbf{w}(t) = \mathbf{w}_m(t) := \begin{bmatrix} w_{m,1}(t) \\ \vdots \\ w_{m,N(m)-1}(t) \end{bmatrix} = \begin{bmatrix} \beta_{m,1}(t) \\ \vdots \\ \beta_{m,N(m)-1}(t) \end{bmatrix}. \tag{3.5}$$

Then (3.4) can be put into matrix form as

$$\mathbf{M}\mathbf{w}' = -\mathbf{K}\mathbf{w}. \tag{3.6}$$

This gives us a system of first-order linear ODEs with constant coefficients. To solve it, we need to impose initial conditions. The initial condition $u(x,0) = g(x)$ for $a \leq x \leq b$ can be approximated by its linear interpolant $\widetilde{g}(x) = \sum_{i=1}^{N(m)-1} g(x_{m,i})\phi_{m,i}(x)$. Therefore, we set $w_{m,i}(0) = g(x_{m,i})$. This leads to the initial condition

$$\mathbf{w}(0) = \mathbf{w}_m(0) := \mathbf{w}_{m,0} = \begin{bmatrix} g(x_{m,1}) \\ \vdots \\ g(x_{m,N(m)-1}) \end{bmatrix}. \tag{3.7}$$

Consequently, we get the linear system

$$\begin{cases} \mathbf{M}\dfrac{d\mathbf{w}}{dt} = -\mathbf{K}\mathbf{w}, & t > 0, \\ \mathbf{w}(0) = \mathbf{w}_{m,0}. \end{cases} \tag{3.8}$$

Since $\operatorname{supp}(\mu) = [a,b]$, [2, Proposition 3.1] implies that $\mathbf{M}$ is invertible. Thus the system in (3.8) has a unique solution. Hence, the following proposition holds.

Proposition 3.1. *Assume that* $\operatorname{supp}(\mu) = [a,b]$. *Then* (3.8) *has a unique solution* $\mathbf{w}(t)$. *Moreover,* $\beta_{m,j}(t) \in C(0,T)$ *for* $m \geq 1$ *and* $j = 1, \ldots, N(m) - 1$.

Proof of Theorem 1.1. The assertions hold by combining the derivations above and Proposition 3.1. □

We note that the matrix $\mathbf{K}$ can be computed directly; we only describe how to compute $\mathbf{M}$. In the following, the constants $c_{m,i,j}$ and similitudes $\tau_{m,i,j}$ come from condition (P3) in Section 1. From the definition of the $\phi_{m,i}$'s and (1.3), we have, for $1 \leq i \leq N(m) - 1$,

$$\begin{aligned} M_{i,i}^{(m)} &= \frac{1}{(x_{m,i} - x_{m,i-1})^2} \sum_{j=0}^{N(1)} c_{m,i-1,j} \int_{I_{m,i-1}} (x - x_{m,i-1})^2 \, d\mu|_{I_{1,j}} \circ \tau_{m,i-1,j}^{-1} \\ &\quad + \frac{1}{(x_{m,i} - x_{m,i+1})^2} \sum_{j=0}^{N(1)} c_{m,i,j} \int_{I_{m,i}} (x - x_{m,i+1})^2 \, d\mu|_{I_{1,j}} \circ \tau_{m,i,j}^{-1} \\ &= (x_{m,i} - x_{m,i-1})^{-2} \cdot \sum_{j=0}^{N(1)} c_{m,i-1,j} \int_{I_{1,j}} \big(\tau_{m,i-1,j}(x) - x_{m,i-1}\big)^2 \, d\mu \\ &\quad + (x_{m,i} - x_{m,i+1})^{-2} \cdot \sum_{j=0}^{N(1)} c_{m,i,j} \int_{I_{1,j}} \big(\tau_{m,i,j}(x) - x_{m,i+1}\big)^2 \, d\mu. \end{aligned} \tag{3.9}$$

For $2 \leq i \leq N(m) - 1$, we obtain

$$\begin{aligned} M_{i,i-1}^{(m)} &= \frac{-1}{(x_{m,i} - x_{m,i-1})^2} \sum_{j=0}^{N(1)} c_{m,i-1,j} \int_{I_{m,i-1}} (x - x_{m,i-1})(x - x_{m,i}) \, d\mu|_{I_{1,j}} \circ \tau_{m,i-1,j}^{-1} \\ &= -(x_{m,i} - x_{m,i-1})^{-2} \cdot \sum_{j=0}^{N(1)} c_{m,i-1,j} \int_{I_{1,j}} \big(\tau_{m,i-1,j}(x) - x_{m,i-1}\big)\big(\tau_{m,i-1,j}(x) - x_{m,i}\big) \, d\mu, \end{aligned} \tag{3.10}$$

and $M_{i-1,i}^{(m)} = M_{i,i-1}^{(m)}$.

Define

$$\mathcal{J}_{k,j} := \int_{I_{1,j}} x^k \, d\mu, \quad k = 0, 1, 2, \text{ and } j = 0, \ldots, N(1). \tag{3.11}$$

Since each $\tau_{k,\ell,i}$ is of the form $\tau_{k,\ell,i}(x) = r_{k,\ell,i}x + b_{k,\ell,i}$, we can see that the matrix $\mathbf{M}$ is completely determined by the integrals $\mathcal{J}_{k,j}$, where $k = 0, 1, 2$ and $j = 0, \ldots, N(1)$. Hereafter, we assume that the constant $\mathcal{J}_{k,j}$ can be evaluated explicitly for $k = 0, 1, 2$ and $j = 0, \ldots, N(1)$.

Next, we discuss the solution of the linear system (3.6). Let $\mathbf{w}_n := \mathbf{w}(t_n)$, $n \geq 0$, and use the central difference method to solve the equation (3.8). We approximate the derivative as

$$\mathbf{w}'(t_n) \approx \frac{\mathbf{w}_{n+1} - \mathbf{w}_n}{\delta} \quad \text{and} \quad \mathbf{w}(t_n) \approx \frac{\mathbf{w}_{n+1} + \mathbf{w}_n}{2}. \tag{3.12}$$

Substituting (3.12) into (3.6) yields

$$\mathbf{M}\frac{\mathbf{w}_{n+1} - \mathbf{w}_n}{\delta} = -\mathbf{K}\frac{\mathbf{w}_{n+1} + \mathbf{w}_n}{2}.$$

That is, $\mathbf{w}_{n+1} = (2\mathbf{M} + \delta\mathbf{K})^{-1}(2\mathbf{M} - \delta\mathbf{K})\mathbf{w}_n$. Therefore, equation (3.6) becomes

$$\begin{cases} \mathbf{w}_{n+1} = (2\mathbf{M} + \delta\mathbf{K})^{-1}(2\mathbf{M} - \delta\mathbf{K})\mathbf{w}_n, & n = 0, 1, 2, \ldots, \\ \mathbf{w}_0 = \mathbf{w}(t_0) = \mathbf{w}(0), \\ t_n = n\delta. & . \end{cases} \tag{3.13}$$

To solve this system, fix δ and substitute the initial condition $\mathbf{w}_0$ from (3.7) into the first equation in (3.13) to get $\mathbf{w}_1$. Then $\mathbf{w}_{n+1}$ can be computed recursively.

4. Fractal measures defined by iterated function systems

In this section, we solve the homogeneous IBVP (1.2) numerically for three different measures, namely, the infinite Bernoulli convolution associated with the golden ratio, the three-fold convolution of the Cantor measure, and a class of self-similar measures satisfying (EFT). These measures are defined by IFSs with overlaps. In the first and second cases, the measures satisfy a family of second-order self-similar identities (see definition below). These identities were first introduced by Strichartz *et al.* [22] to approximate the density of the infinite Bernoulli convolution associated with the golden ratio.

Let $\{S_i\}_{i=1}^q$ be an IFS of contractive similitudes on $\mathbb{R}$, and let μ be the associated self-similar measure. Assume that $\mathrm{supp}(\mu) = [a, b]$. Define an auxiliary IFS

$$T_j(x) = r_j x + d_j, \qquad j = 1, 2, \ldots, N,$$

where $n_j \in \mathbb{N}$, $d_j \in \mathbb{R}$. We say that μ satisfies a family of *second-order (self-similar) identities* with respect to $\{T_j\}_{j=1}^N$ (see [13]) if

(1) $\mathrm{supp}(\mu) \subseteq \bigcup_{j=1}^N T_j(\mathrm{supp}(\mu))$, and

(2) for each Borel subset $A \subseteq \operatorname{supp}(\mu)$ and $0 \le i, j \le N$, $\mu(T_iT_jA)$ can be expressed as a linear combination of $\{\mu(T_kA) : k = 1, \dots, N\}$. In matrix form,

$$\begin{bmatrix} \mu(T_1T_jA) \\ \vdots \\ \mu(T_NT_jA) \end{bmatrix} = M_j \begin{bmatrix} \mu(T_1A) \\ \vdots \\ \mu(T_NA) \end{bmatrix}, \quad j = 1, \dots, N;$$

or, equivalently,

$$\mu(T_iT_jA) = \mathbf{e}_i M_j \begin{bmatrix} \mu(T_1A) \\ \vdots \\ \mu(T_NA) \end{bmatrix}, \quad i, j = 1, \dots, N, \tag{4.14}$$

where $\mathbf{e}_i$ is the ith row of the $N \times N$ identity matrix and M_j is some $N \times N$ matrix independent of A.

Proposition 4.1. *Let μ be a self-similar measure defined by an IFS of contractive similitudes on $\mathbb{R}$. Assume that $\operatorname{supp}(\mu) = [a, b]$, μ satisfies a family of second-order (self-similar) identities with respect to $\{T_j\}_{j=1}^N$, and $\{T_j\}_{j=1}^N$ satisfies (OSC). Define*

$$\mathbf{P}_k := \left\{ T_{\boldsymbol{j}}([a, b]) : \boldsymbol{j} \in \{1, \dots, N\}^k \right\} \qquad \text{for } k \ge 1.$$

Then $(\mathbf{P}_k)_{k \ge 1}$ is a sequence of refining μ-partitions of $[a, b]$ satisfying conditions (P1)–(P3) in Section 1. Moreover, the matrix $\mathbf{M}$ is completely determined by the integrals $\int_a^b x^k \, d\mu \circ T_j$, $k = 0, 1, 2$, $j = 1, \dots, N$.

Proof. It is easy to see that $(\mathbf{P}_k)_{k \ge 1}$ is a sequence of refining μ-partitions of $[a, b]$ satisfying conditions (P1) and (P2). For $\boldsymbol{j} = (j_1, \dots, j_m) \in \{1, \dots, N\}^m$, iterating (4.14) shows that for any Borel subset $A \subseteq \operatorname{supp}(\mu)$,

$$\mu(T_{\boldsymbol{j}}A) = c_{\boldsymbol{j}} \begin{bmatrix} \mu(T_1A) \\ \vdots \\ \mu(T_NA) \end{bmatrix}, \tag{4.15}$$

where $c_{\boldsymbol{j}} := [c_{\boldsymbol{j}}^1, \dots, c_{\boldsymbol{j}}^N] := \mathbf{e}_{j_1} M_{j_2} \cdots M_{j_m}$, i.e.,

$$\mu(T_{\boldsymbol{j}}A) = \sum_{i=1}^N c_{\boldsymbol{j}}^i \cdot \mu(T_iA).$$

It follows that

$$\mu|_{T_{\boldsymbol{j}}([a,b])} = \sum_{i=1}^N c_{\boldsymbol{j}}^i \cdot \mu|_{T_i([a,b])} \circ \tau_{\boldsymbol{j},i}^{-1},$$

where $\tau_{\boldsymbol{j},i}^{-1} = T_i \circ T_{\boldsymbol{j}}^{-1}$. Hence, condition (P3) holds. Since

$$\int_a^b x^k \, d\mu \circ T_j = \int_{T_j[a,.b]} (T_j^{-1}x)^k \, d\mu \quad \text{and} \quad \int_{T_j[a,b]} x^k \, d\mu = \int_a^b (T_jx)^k \, d\mu \circ T_j,$$

$\mathbf{M}$ is also determined by the intervals $\int_a^b x^k \, d\mu \circ T_j$, $k = 0, 1, 2$, $j = 1, \dots, N$. $\square$

Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\mathrm{supp}(\mu) = [a,b]$. Assume that there exists a sequence of refining partitions $(\mathbf{P}_k)_{k\geq 1} = (\{I_{k,\ell}\}_{\ell=0}^{N(k)})_{k\geq 1}$ satisfying conditions (P1)–(P3) in Section 1. In order to solve (3.6) or (3.13), we need to compute the integrals $\mathcal{J}_{k,j}$, $k = 0,1,2$, $j = 0,\dots,N(1)$, as defined in (3.11). The exact values of these integrals for the first and second measures in this section have been obtained in [2,3]. We will find the exact values of these integrals for the third measure. The following integration formula is used repeatedly. For every continuous function f on $[a,b]$,

$$\int_a^b f\,d\mu = \sum_{i=1}^q w_i \int_a^b f\circ S_i\,d\mu. \tag{4.16}$$

Substituting the values of $\mathcal{J}_{k,j}$ into (3.9), we obtain the matrix $\mathbf{M}$. This allows us to solve equation (3.13).

4.1. **Infinite Bernoulli convolution associated with the golden ratio.** We consider the infinite Bernoulli convolution associated with the golden ratio:

$$\mu = \frac{1}{2}\mu\circ S_1^{-1} + \frac{1}{2}\mu\circ S_2^{-1}, \tag{4.17}$$

where

$$S_1(x) = \rho x, \qquad S_2(x) = \rho x + (1-\rho), \qquad \rho = (\sqrt{5}-1)/2.$$

We note that $\mathrm{supp}(\mu) = [0,1]$. Strichartz *et al.* [22] showed that μ satisfies a family of second-order identities with respect to the following auxiliary IFS:

$$T_1(x) := \rho^2 x, \qquad T_2(x) := \rho^3 x + \rho^2, \qquad T_3(x) := \rho^2 x + \rho. \tag{4.18}$$

Moreover, μ satisfies the following second-order identities [13]: for each Borel $A \subseteq [0,1]$,

$$\begin{bmatrix}\mu(T_1T_jA)\\ \mu(T_2T_jA)\\ \mu(T_3T_jA)\end{bmatrix} = M_j \begin{bmatrix}\mu(T_1A)\\ \mu(T_2A)\\ \mu(T_3A)\end{bmatrix}, \quad j = 1,2,3, \tag{4.19}$$

where

$$M_1 = \frac{1}{8}\begin{bmatrix}2&0&0\\1&2&0\\0&4&0\end{bmatrix}, \quad M_2 = \frac{1}{4}\begin{bmatrix}0&1&0\\0&1&0\\0&1&0\end{bmatrix}, \quad M_3 = \frac{1}{8}\begin{bmatrix}0&4&0\\0&2&1\\0&0&2\end{bmatrix}.$$

This can be used to compute the measure of suitable subintervals of $[0,1]$. In fact, if we let $J = (j_1,\dots,j_m)$, $j_i \in \{1,2,3\}$, then for each Borel $A \subseteq [0,1]$,

$$\mu(T_J(A)) = c_J \begin{bmatrix}\mu(T_1A)\\ \mu(T_2A)\\ \mu(T_3A)\end{bmatrix}, \quad \text{where } c_J = \mathbf{e}_{j_1}M_{j_2}\cdots M_{j_m} = (c_J^1, c_J^2, c_J^3). \tag{4.20}$$

The integrals $\int_0^1 x^k\,d\mu\circ T_j$, $k = 0,1,2$, $j = 1,2,3$, have been calculated in [3, Section 5]. We can thus calculate the entries of the mass matrix $\mathbf{M}$ and solve the linear system (3.6). The result is shown in Figure 1.

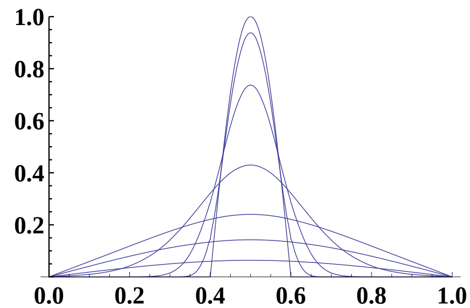


FIGURE 1. Figure for numerical solutions of the heat equation defined by the infinite Bernoulli convolution associated with the golden ratio. The initial condition is given by the function $g(x) := \sin((5\pi(x-0.4))$ for $x \in (0.4, 0.6)$, and $g(x) := 0$ otherwise. Here $\delta = 0.0001$. From top to bottom, the values of t are 0.0, 0.0004, 0.002, 0.01, 0.04, 0.1, 0.2.

4.2. **Three-fold convolution of the Cantor measure.** We consider the following three-fold convolution of the Cantor measure studied in [13, 16, 17]. The three-fold convolution of the Cantor measure μ is the self-similar measure defined by the following IFS with overlaps (see [16]):

$$S_i(x) = \frac{1}{3}x + \frac{2}{3}(i-1), \qquad i = 1, 2, 3, 4,$$

together with probability weights $\{1/8, 3/8, 3/8, 1/8\}$. That is,

$$\mu = \frac{1}{8}\mu \circ S_1^{-1} + \frac{3}{8}\mu \circ S_2^{-1} + \frac{3}{8}\mu \circ S_3^{-1} + \frac{1}{8}\mu \circ S_4^{-1}. \tag{4.21}$$

Note that $\operatorname{supp}(\mu) = [0, 3]$. It is shown in [13] that μ satisfies a family of second-order identities with respect to the following auxiliary IFS

$$T_1(x) := \frac{1}{3}x, \qquad T_2(x) = \frac{1}{3}x + 1, \quad T_3(x) = \frac{1}{3}x + 2. \tag{4.22}$$

In fact, for each Borel $A \subseteq [0, 3]$,

$$\begin{bmatrix} \mu(T_1T_jA) \\ \mu(T_2T_jA) \\ \mu(T_3T_jA) \end{bmatrix} = M_j \begin{bmatrix} \mu(T_1A) \\ \mu(T_2A) \\ \mu(T_3A) \end{bmatrix}, \quad j = 1, 2, 3,$$

where M_1, M_2, M_3 are given by

$$M_1 = \frac{1}{8}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad M_2 = \frac{1}{8}\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_3 = \frac{1}{8}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly, if $J = (j_1, \dots, j_m)$, $j_i \in \{1, 2, 3\}$, then for each Borel $A \subseteq [0, 3]$,

$$\mu(T_J(A)) = c_J \begin{bmatrix} \mu(T_1A) \\ \mu(T_2A) \\ \mu(T_3A) \end{bmatrix}, \quad \text{where } c_J = \mathbf{e}_{j_1} M_{j_2} \cdots M_{j_m} = (c_J^1, c_J^2, c_J^3). \tag{4.23}$$

The integrals $\int_0^1 x^k\, d\mu \circ T_j$, $k = 0, 1, 2$, $j = 1, 2, 3$, have been calculated in [2, Section 4.3]. We can thus calculate the entries of the mass matrix $\mathbf{M}$ and solve the linear system (3.6). The result is shown in Figure 2.

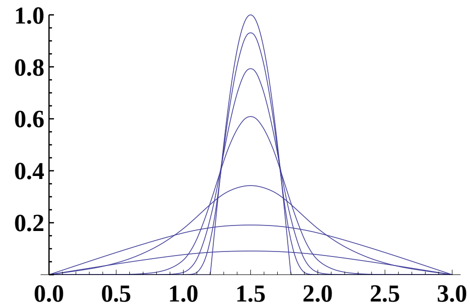


FIGURE 2. Figure for numerical solutions of the heat equation defined by the three-fold convolution of the Cantor measure. The initial condition is given by the function $g(x) = \sin((5\pi(x/3 - 0.4)), x \in (1.2, 1.8)$, and $g(x) = 0$ otherwise. Here $\delta = 0.0001$. From top to bottom, the values of t are 0.0, 0.0012, 0.004, 0.01, 0.04, 0.18, 0.5.

4.3. **A class of self-similar measures satisfying (EFT).** In this subsection, we consider the following family of IFSs:

$$S_1(x) = r_1 x, \quad S_2(x) = r_2 x + r_1(1 - r_2), \quad S_3(x) = r_2 x + 1 - r_2, \tag{4.24}$$

where the contraction ratios $r_1, r_2 \in (0, 1)$ satisfy $r_1 + 2r_2 - r_1 r_2 \le 1$, i.e., $S_2(1) \le S_3(0)$. The Hausdorff dimension of the self-similar sets is computed in [14]. The multifractal properties and spectral dimension of the corresponding self-similar measures are recently studied in [6, 17, 19].

Let μ be a self-similar measure defined by an IFS in (4.24) and a probability vector $(p_i)_{i=1}^3$. Let $I_{1,1} := S_1(X) \bigcup S_2(X)$ and $I_{1,0} := S_3(X)$, where $X = [0, 1]$. In order to define a sequence of refining μ-partitions of $[0, 1]$, we adopt the definition of an island from [17]. Let $\mathcal{M}_k := \{1, 2, 3\}^k$ for $k \ge 1$ and $\mathcal{M}_0 := \emptyset$. A closed subset $I \subseteq [0, 1]$ is called a *level-k island* with respect to $\{\mathcal{M}_k\}$ if the following conditions hold:

(1) there exists a finite sequence of indexes $\boldsymbol{i}_0, \boldsymbol{i}_1, \dots, \boldsymbol{i}_n$ in $\mathcal{M}_k$ such that $S_{\boldsymbol{i}_k}(0, 1) \cap S_{\boldsymbol{i}_{k+1}}(0, 1) \ne \emptyset$ for all $k = 0, \dots, n-1$, and $I = \bigcup_{k=0}^n S_{\boldsymbol{i}_k}([0, 1])$;
(2) for any $\boldsymbol{j} \in \mathcal{M}_k \setminus \{\boldsymbol{i}_0, \dots, \boldsymbol{i}_n\}$ and any $k \in \{0, \dots, n\}$, $S_{\boldsymbol{j}}(0, 1) \cap S_{\boldsymbol{i}_k}(0, 1) = \emptyset$.

Intuitively, for each level-k island I, I° is a connected component of $S_{\mathcal{M}_k}(0, 1) := \bigcup_{\boldsymbol{i} \in \mathcal{M}_k} S_{\boldsymbol{i}}(0, 1)$ (see Figure 3). For $k \ge 1$, define

$$\mathbf{P}_k := \big\{ I : I \text{ is a level-}k \text{ island with respect to } \{\mathcal{M}_k\} \big\}. \tag{4.25}$$

We note that $\mathbf{P}_1 = \{I_{1,1}, I_{1,0}\}$ (see Figure 3). It is easy to see that $(\mathbf{P}_k)_{k \ge 1}$ is a sequence of refining μ-partitions of $[0, 1]$ satisfying condition (P2). By [17, Lemma

3.5] and the proof of [17, Example 3.3], $(\mathbf{P}_k)_{k\geq 1}$ satisfies conditions (P1) and (P3) with $\rho = |I_{1,1}|$ and $m_0 = 0$.

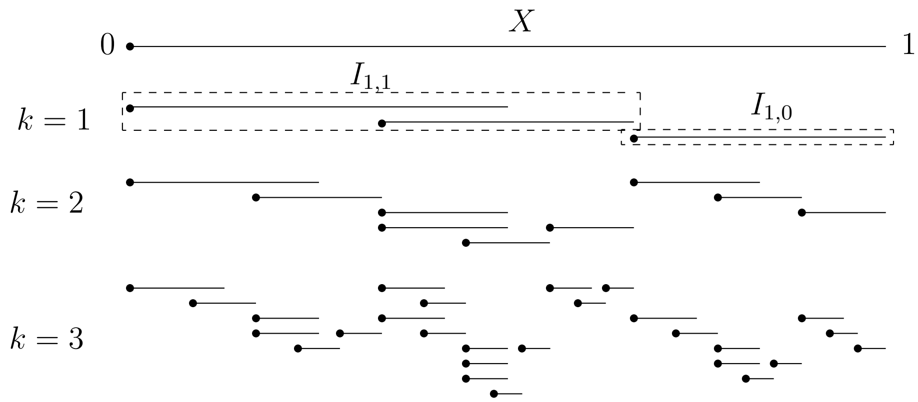


FIGURE 3. μ-partitions $\mathbf{P}_k$ for $k = 1, 2, 3$, where $\mathbf{P}_k$ is defined as in (4.25). Cells that are labeled consist of line segments enclosed by a box. The figure is drawn with $r_1 = 1/2$ and $r_2 = 1/3$.

If $S_2(1) = S_3(0)$, then $\operatorname{supp}(\mu) = [0, 1]$. In particular, we let $r_1 = 1/2$, $r_2 = 1/3$. We can calculate the integrals $\int_{I_{1,i}} x^k\, d\mu$, where $i = 0, 1$ and $k = 0, 1, 2$. If $p_1 = p_2 = p_3 = 1/3$, then

$$\int_{I_{1,0}} d\mu = 1/3, \qquad \int_{I_{1,0}} x\, d\mu = \frac{28}{99}, \qquad \int_{I_{1,0}} x^2\, d\mu = \frac{8}{33},$$
$$\int_{I_{1,1}} d\mu = 2/3, \qquad \int_{I_{1,1}} x\, d\mu = \frac{26}{99}, \qquad \int_{I_{1,1}} x^2\, d\mu = \frac{4}{33}.$$

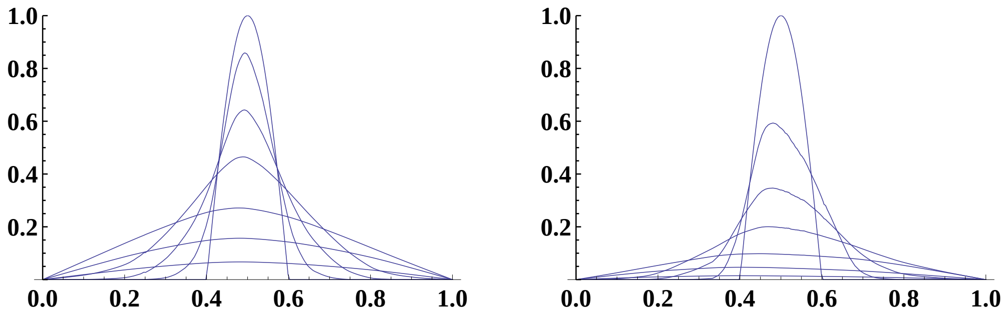
Similarly, if $p_1 = 2/3, p_2 = p_3 = 1/6$, then

$$\int_{I_{1,0}} d\mu = 1/6, \qquad \int_{I_{1,0}} x\, d\mu = \frac{23}{180}, \qquad \int_{I_{1,0}} x^2\, d\mu = \frac{64}{645},$$
$$\int_{I_{1,1}} d\mu = 5/6, \qquad \int_{I_{1,1}} x\, d\mu = \frac{31}{180}, \qquad \int_{I_{1,1}} x^2\, d\mu = \frac{38}{645}.$$

We can thus calculate the entries of the mass matrix $\mathbf{M}$ and solve the linear system (3.6). The result is shown in Figure 4.

5. Neumann boundary condition and additional properties of solutions of heat equations

Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a, b]$. It is well known (see, e.g., [1]) that μ defines a Neumann Laplacian Δ_μ^N such that $\Delta_\mu^N u = f$ if and



(a) $p_1 = p_2 = p_3 = 1/3$ (b) $p_1 = 2/3, p_2 = p_3 = 1/6$

FIGURE 4. Numerical solutions of the heat equation corresponding to the self-similar measure defined by the IFS in (4.24) with probability vector $(p_i)_{i=1}^3$. The initial condition is given by $g(x) := \sin((5\pi(x-0.4))$ for $x \in (0.4, 0.6)$, and $g(x) := 0$ otherwise. Again $\delta = 0.0001$. From top to bottom, the values of t are 0.0, 0.0012, 0.004, 0.01, 0.04, 0.1, 0.2.

only if

$$\mathcal{E}(u,v) = -\int_a^b fv\,d\mu \qquad \text{for all } v \in H^1(a,b),$$

where $\mathcal{E}(\cdot,\cdot)$ is defined as in (2.2). In this section, we consider the following heat equation defined by Neumann Laplacians Δ_μ^N:

$$\begin{cases} u_t - \Delta_\mu^N u = f & \text{on } (a,b) \times [0,T], \\ u_x' = 0 & \text{on } \{a,b\} \times [0,T], \\ u = g & \text{on } (a,b) \times \{t=0\}. \end{cases} \tag{5.1}$$

We first note that the existence and uniqueness results of the heat equation 5.1 can be proved by using a method similar to that for equation (1.2).

Let $f(x,t) \equiv 0$ in (5.1). Similar to (1.4), we obtain

$$-\int_a^b u_x(x,t)v'(x)\,dx = \int_a^b u_t(x,t)v(x)\,d\mu \qquad \text{for all } v \in \operatorname{dom}\mathcal{E}_N. \tag{5.2}$$

In the rest of this section, we assume the hypotheses of Section 3 and use the notation in Section 3. As in Section 3, we use the finite element method to approximate the weak solution $u(x,t)$ satisfying (5.2) by

$$u^m(x,t) = \sum_{\ell=0}^{N(m)} \beta_{m,\ell}(t)\phi_{m,\ell}(x). \tag{5.3}$$

Thus we require $u^m(x,t)$ to satisfy equation (3.3) and the Neumann boundary condition $u_x^m(a,t) = u_x^m(b,t) = 0$. The Neumann boundary condition implies that $\beta_{m,0} = \beta_{m,1}$ and $\beta_{m,N(m)} = \beta_{m,N(m)-1}$, which is different from the Dirichlet case.

Consequently, we can obtain an analogue of (3.4):

$$
\begin{aligned}
&\beta'_{m,1}\Big(\int_a^b \phi_{m,0}(x)\phi_{m,i}(x)\,d\mu + \int_a^b \phi_{m,1}(x)\phi_{m,i}(x)\Big) + \sum_{\ell=2}^{N(m)-2} \beta'_{m,\ell}\int_a^b \phi_{m,\ell}(x)\phi_{m,i}(x)\,d\mu\\
&\quad + \beta'_{m,N(m)-1}\Big(\int_a^b \phi_{m,N(m)-1}(x)\phi_{m,i}(x)\,d\mu + \int_a^b \phi_{m,N(m)}(x)\phi_{m,i}(x)\Big)\\
&= -\beta_{m,1}\Big(\int_a^b \phi'_{m,0}(x)\phi'_{m,i}(x)\,d\mu + \int_a^b \phi'_{m,1}(x)\phi'_{m,i}(x)\Big)\\
&\quad - \sum_{\ell=2}^{N(m)-2} \beta_{m,\ell}\int_a^b \phi'_{m,\ell}(x)\phi'_{m,i}(x)\,dx\\
&\quad - \beta_{m,N(m)-1}\Big(\int_a^b \phi'_{m,N(m)-1}(x)\phi'_{m,i}(x)\,d\mu + \int_a^b \phi'_{m,N(m)}(x)\phi'_{m,i}(x)\Big)
\end{aligned}
$$

for $1 \le i \le N(m)-1$. This system can also be put into a matrix form as (3.6). Using the same derivations as in Scetion 3, we can conclude that (5.2) can be discretized into a system of first-order ordinary differential equations, which has a unique solution that can be solved numerically. That is, Theorem 1.1 holds for the heat equation in (5.1). The numerical solutions of the heat equation (5.1) corresponding to the three different self-similar measures in Section 4 are shown in Figures 5–7, respectively.

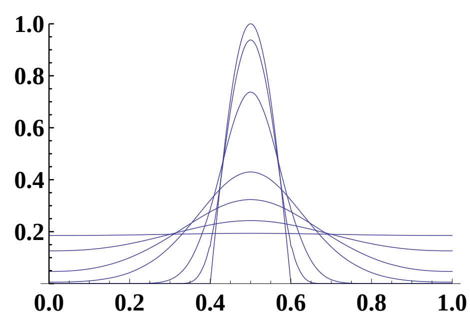


FIGURE 5. Numerical solutions of the heat equation (5.1) defined by the infinite Bernoulli convolution associated with the golden ratio. The initial condition is given by the function $g(x) := \sin((5\pi(x-0.4))$ for $x \in (0.4, 0.6)$, and $g(x) := 0$ otherwise. Here $\delta = 0.0001$. From top to bottom, the values of t are 0.0, 0.0004, 0.002, 0.01, 0.02 0.04, 0.1.

The following proposition shows that for the system (5.1), energy is conserved.

Proposition 5.1. *Let μ be a positive finite Borel measure on $\mathbb{R}$ with* $\operatorname{supp}(\mu) = [a, b]$*, and let u be the weak solution of the heat equation* (5.1)*. Then*

$$h(t) := \int_a^b u(x,t)\,d\mu(x)$$

is a constant function of t.

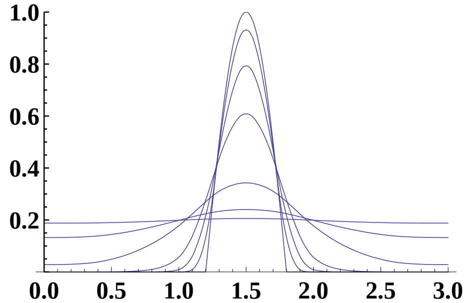
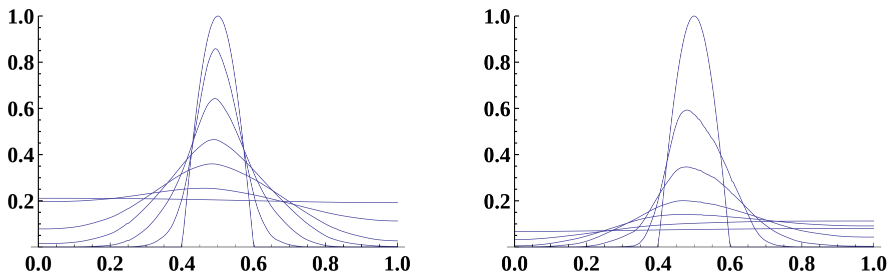


FIGURE 6. Numerical solutions of the heat equation defined by the three-fold convolution of the Cantor measure in (4.21). The initial condition is given by the function $g(x) = \sin((5\pi(x/3-0.4)), x \in (1.2, 1.8)$, and $g(x) = 0$ otherwise. Here $\delta = 0.0001$. From top to bottom, the values of t are 0.0, 0.0012, 0.004, 0.01, 0.04, 0.1, 0.2.



(a) $p_1 = p_2 = p_3 = 1/3$ (b) $p_1 = 2/3, p_2 = p_3 = 1/6$

FIGURE 7. Numerical solutions of the heat equation corresponding to the self-similar measure defined by the IFS in (4.24) with probability vector $(p_i)_{i=1}^3$. The initial condition is given by $g(x) := \sin((5\pi(x-0.4))$ for $x \in (0.4, 0.6)$, and $g(x) := 0$ otherwise. Again $\delta = 0.0001$. From top to bottom, the values of t are 0.0, 0.0012, 0.004, 0.01, 0.02, 0.05, 0.2.

Proof. Fix any $t \in [0, T]$. Then for any $\delta > 0$, there exists $\xi \in (0, \delta)$ such that

$$\begin{aligned}\frac{h(t+\delta)-h(t)}{\delta} &= \int_a^b \frac{u(x,t+\delta)-u(x,t)}{\delta}\, d\mu(x) \\ &= \int_a^b u_t(x, t+\xi)\, d\mu(x).\end{aligned} \tag{5.4}$$

Since u is the weak solution of the heat equation (5.1), (5.4) implies that

$$\begin{aligned}\frac{h(t+\delta)-h(t)}{\delta} &= \int_a^b \Delta_\mu^N u(x, t+\xi)\, d\mu(x) \\ &= -\int_a^b \frac{\partial u}{\partial x}(x, t+\xi) \cdot \frac{\partial}{\partial x}(1)\, dx = 0,\end{aligned}$$

where in the second equality, we have used the fact that 1 belongs to the domain of the form corresponding to $-\Delta_\mu^N$. Hence, the assertion follows. $\square$

The following result is well-known in the classical setting. It holds for both the Dirichlet and Neumann cases.

Proposition 5.2. *Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a,b]$, and let u be the weak solution of the heat equation (1.2) or (5.1). Assume that the heat kernel $p(t,x,y)$ exists and satisfies the following lower estimate for some $\epsilon > 0$:*

$$p(t,x,y) > 0 \qquad \text{for all } x \neq y \in (a,b), \quad \text{and } t \in (0,\epsilon).$$

If $f = 0$, the initial function $g(x)$ is non-negative, continuous, and nonzero on $[a,b]$, then for any $t \in (0,\epsilon)$ and all $x \in (a,b)$, $u(x,t) > 0$.

Proof. We first note that

$$u(x,t) = \int_a^b p(t,x,y)g(y)\,d\mu(y)$$

(see, e.g., [5]). Thus the result follows from the assumptions on $p(t,x,y)$ and $g(x)$. □

It follows from Proposition 5.2 that lower heat kernel implies infinite heat propagation speed. In [10], the following two-sided heat kernel estimates have been obtained for the Neumann Laplacian defined by a class of self-similar measures with full support $[a,b]$:

$$\frac{C_1}{V(x,t^{1/\beta})} \exp\left(-c_1\left(\frac{d_*(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right) \leq p(t,x,y)$$
$$\leq \frac{C_2}{V(x,t^{1/\beta})} \exp\left(-c_2\left(\frac{d_*(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right)$$

for all $t \in (0,1)$ and all $x,y \in K$. Here $\beta > 1$, d_* is some metric, $V(x,r) := \mu(B_{d_*}(x,r))$, and c_i, C_i $(i = 1,2)$ are positive constants. This class includes the infinite Bernoulli convolution associated with the golden ratio and the three-fold convolution of the Cantor measure, but does not include the class defined by (4.24). Hence we have the following corollary of Proposition 5.2.

Corollary 5.3. *Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a,b]$, and let u be the weak solution of the heat equation (5.1). Assume that $f = 0$, and that the initial function $g(x)$ is non-negative, continuous, and nonzero on $[a,b]$.*

(a) *If μ is the infinite Bernoulli convolution associated with the golden ratio in (4.17). Then for any $t \in (0,1)$ and all $x \in (0,1)$, $u(x,t) > 0$.*
(b) *If μ is the three-fold convolution of the Cantor measure in (4.21). Then for any $t \in (0,1)$ and all $x \in (0,3)$, $u(x,t) > 0$.*

Consequenctly, in both cases (a) and (b), the heat propagation speed is infinite.

For the class of examples in (4.24), an analogous lower bound for the heat kernel has not been obtained and so we are not able to conclude that heat propagation speed is infinite.

6. Convergence of numerical approximations

In this section, we prove the convergence of numerical approximations of the homogeneous IBVP (1.2). Some of our results are obtained by modifying similar ones in [21].

Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a, b]$. Assume that there exists a sequence of refining partitions $(\mathbf{P}_k)_{k\geq 1}$ satisfying conditions (P1)–(P3) in Section 1. Let V_m be the set of end-points of all level-m subintervals, and arrange its element so that $V_m = \{x_{m,i} : i = 0, \dots, N(m)\}$ with $x_{m,i} < x_{m,i+1}$ for $i = 0, 1, \dots, N(m) - 1$, $x_{m,0} = a$ and $x_{m,N(m)} = b$. Let S^m be the space of continuous piecewise linear functions on $[a, b]$ with nodes V_m, and let

$$S_D^m = \{u \in S^m : u(a) = u(b) = 0\} \quad \text{and} \quad S_N^m = \{u \in S^m : u'(a) = u'(b) = 0\}$$

be the subspaces of S^m consisting of functions satisfying the Dirichlet boundary condition and Neumann boundary condition, respectively.

We choose the basis of S^m and S_N^m consisting of the tent functions $\{\phi_{m,i}\}_{i=0}^{N(m)}$ defined in (3.2) and choose the basis $\{\phi_{m,i}\}_{i=1}^{N(m)-1}$ for S_D^m. The linear map $\mathcal{P}_m$: $\operatorname{dom}\mathcal{E}_D \to S_D^m$ defined by

$$\mathcal{P}_m v = \sum_{i=1}^{N^m-1} v(x_i)\phi_{m,i}(x), \quad v \in \operatorname{dom}\mathcal{E}_D,$$

is called the *Rayleigh-Ritz projection* with respect to V_m. Let

$$\|V_m\| := \max\{|x_{m,i} - x_{m,i-1}| : 1 \leq i \leq N(m)\}$$

be the *norm* of V_m for $m \geq 1$.

Lemma 6.1. *For $m \geq 1$, let V_m and $\mathcal{P}_m$ be defined as above. Then for any $u \in \operatorname{dom}\mathcal{E}_D$, $\mathcal{P}_m u$ is the component of u in the subspace S_D^m, $u - \mathcal{P}_m u$ vanishes on the boundary $\{a, b\}$, and*

$$\mathcal{E}(u - \mathcal{P}_m u, v) = 0 \qquad \text{for all } v \in S_D^m. \tag{6.1}$$

Proof. The proof can be found in [21]. □

Throughout the rest of this section, let $g \in \operatorname{dom}\mathcal{E}_D$, $f = 0$, and u be the solution of the corresponding homogeneous IBVP (1.2). By Theorem 2.3, $u \in W_2^k(0, T; \operatorname{dom}\mathcal{E}_D)$

for all $k \geq 0$. In particular, $u_t \in \operatorname{dom} \mathcal{E}_D$ and

$$(u_t, v)_\mu + \mathcal{E}(u, v) = 0 \qquad \text{for all } v \in \operatorname{dom} \mathcal{E}_D. \tag{6.2}$$

As in Section 3,

$$u^m(x,t) = \sum_{i=1}^{N(m)-1} \beta_{m,i}(t)\phi_{m,i}(x).$$

Thus u^m satisfies

$$(u_t^m, v^m)_\mu + \mathcal{E}(u^m, v^m) = 0 \qquad \text{for all } v^m \in S_D^m, \tag{6.3}$$

and $u^m(x,0) = \sum_{i=1}^{N(m)-1} g(x_{m,i})\phi_{m,i}(x)$. Finally, define

$$e(x,t) := e^m(x,t) = \mathcal{P}_m u(x,t) - u^m(x,t).$$

Lemma 6.2. *Let u, u^m, e be as above. Then*

$$(e_t, e)_\mu + \mathcal{E}(e, e) = (\mathcal{P}_m u_t - u_t, e)_\mu. \tag{6.4}$$

Proof. By definition and the fact that $u \in W_2^k(0, T; \operatorname{dom} \mathcal{E}_D)$ for $k \geq 0$, the functions e, e_t and $(\mathcal{P}_m u)_t = \mathcal{P}_m u_t$ all belong to S_D^m. Thus substituting e for v in (6.2) and for v^m in (6.3), we get

$$(u_t, e)_\mu + \mathcal{E}(u, e) = 0 \quad \text{and} \quad (u_t^m, e)_\mu + \mathcal{E}(u^m, e) = 0.$$

Subtracting these equations gives $(u_t - u_t^m, e)_\mu + \mathcal{E}(u - u^m, e) = 0$. Equivalently,

$$(u_t - \mathcal{P}_m u_t + \mathcal{P}_m u_t - u_t^m, e)_\mu + \mathcal{E}(u - \mathcal{P}_m u + \mathcal{P}_m u - u^m, e) = 0,$$

which, together with the fact $\mathcal{E}(u - \mathcal{P}_m u, e) = 0$ (Lemma 6.1), yields

$$(\mathcal{P}_m u_t - u_t^m, e)_\mu + \mathcal{E}(\mathcal{P}_m u - u^m, e) = (\mathcal{P}_m u_t - u_t, e)_\mu.$$

□

Lemma 6.3. ([2, Lemma 5.3]) *Assume the hypotheses of Lemma 6.1, and let $v \in \operatorname{dom} \mathcal{E}_D$. Then*

$$\|\mathcal{P}_m v - v\|_\mu \leq 2\|V_m\|^{1/2}\|v\|_{\operatorname{dom} \mathcal{E}_D} \qquad \textit{for all } m \geq 1. \tag{6.5}$$

Theorem 6.4. *Assume the hypotheses of Lemma 6.2, and let ρ be the constant in condition (P1). Then*

$$\|\mathcal{P}_m u - u^m\|_\mu \leq 2\sqrt{T}\rho^{m/2}\|u_t\|_{2,\operatorname{dom} \mathcal{E}_D}.$$

Proof. The first term in (6.4) can be rewritten as

$$(e_t, e)_\mu = \frac{1}{2}\big(\|e\|_\mu^2\big)_t = \|e\|_\mu \cdot \big(\|e\|_\mu\big)_t,$$

and the term $\mathcal{E}(e, e) \geq 0$. Hence (6.4) leads to

$$\|e\|_\mu \cdot \big(\|e\|_\mu\big)_t \leq (\mathcal{P}_m u_t - u_t, e)_\mu \leq \|\mathcal{P}_m u_t - u_t\|_\mu \cdot \|e\|_\mu.$$

Cancelling the common factor $\|e\|_\mu$, we have

$$\big(\|e\|_\mu\big)_t \le \|\mathcal{P}_m u_t - u_t\|_\mu. \tag{6.6}$$

Integrating the left-side of (6.6) with respect to τ from 0 to t, we get

$$\int_0^t \big(\|e(\tau)\|_\mu\big)_\tau \, d\tau = \|e(t)\|_\mu - \big(\|e(0)\|_\mu\big) = \|e(t)\|_\mu, \tag{6.7}$$

where the fact $e(0) = \mathcal{P}_m u(x,0) - u^m(x,0) = \mathcal{P}_m g(x) - \sum_{i=1}^{N(m)-1} g(x_{m,i})\phi_{m,i}(x) = 0$ is used in the last equality. Combining (6.6), (6.7), Lemma 6.3 and Hölder's inequality, we have

$$\begin{aligned}
\|e(t)\|_\mu &\le \int_0^t \|\mathcal{P}_m u_t(\tau) - u_t(\tau)\|_\mu \, d\tau \qquad \text{(by (6.6) and (6.7))}\\
&\le \int_0^T 2\|V_m\|^{1/2}\|u_t\|_{\operatorname{dom}\mathcal{E}_D}\, d\tau \qquad \text{(by Lemma 6.3)}\\
&\le 2\sqrt{T}\|V_m\|^{1/2}\|u_t\|_{2,\operatorname{dom}\mathcal{E}_D} \qquad \text{(by Hölder's inequality)}\\
&\le 2\sqrt{T}\rho^{m/2}\|u_t\|_{2,\operatorname{dom}\mathcal{E}_D}.
\end{aligned}$$

□

Proof of Theorem 1.2. Combining Theorem 6.4 and Lemma 6.3, we have, for each fixed $t \in [0,T]$,

$$\begin{aligned}
\|u^m - u\|_\mu &\le \|u^m - \mathcal{P}_m u\|_\mu + \|\mathcal{P}_m u - u\|_\mu\\
&\le 2\sqrt{T}\rho^{m/2}\|u_t\|_{2,\operatorname{dom}\mathcal{E}_D} + 2\rho^{m/2}\|u\|_{\operatorname{dom}\mathcal{E}_D}\\
&\le 2\big(\sqrt{T}\|u_t\|_{2,\operatorname{dom}\mathcal{E}_D} + \|u\|_{\operatorname{dom}\mathcal{E}_D}\big)\rho^{m/2},
\end{aligned}$$

which completes the proof. □

Using the same method as above, we can prove that Theorem 1.2 also holds for the Neumann heat equation (5.1) as follows. First, define a linear map $\widetilde{\mathcal{P}}_m : \operatorname{dom}\mathcal{E}_N \to S_N^m$ by

$$\widetilde{\mathcal{P}}_m v = v(x_1)\phi_{m,0}(x) + \sum_{i=1}^{N(m)-1} v(x_i)\phi_{m,i}(x) + v(x_{N(m)-1})\phi_{m,N(m)}(x), \quad v \in \operatorname{dom}\mathcal{E}_N.$$

Thus,

$$\mathcal{E}(u - \widetilde{\mathcal{P}}_m u, v) = 0 \qquad \text{for all } v \in S_N^m$$

(see [21]), which is an analog of (6.1). Second, as in Section 5, we observe that, in the Neunmann case,

$$u^m(x,t) = \beta_{m,1}(t)\phi_{m,0}(x) \sum_{i=1}^{N(m)-1} \beta_{m,i}(t)\phi_{m,i}(x) + \beta_{m,N(m)-1}(t)\phi_{m,N(m)}(x).$$

Moreover, u^m satisfies

$$(u_t^m, v^m)_\mu + \mathcal{E}(u^m, v^m) = 0 \qquad \text{for all } v^m \in S_N^m, \tag{6.8}$$

and

$$u^m(x,0) = g(x_{m,1})\phi_{m,0}(x) + \sum_{i=1}^{N(m)-1} g(x_{m,i})\phi_{m,i}(x) + g(x_{m,N(m)-1})\phi_{m,N(m)}(x).$$

Define

$$\widetilde{e}(x,t) := \widetilde{e}^m(x,t) = \widetilde{\mathcal{P}}_m u(x,t) - u^m(x,t).$$

Third, we can verify that Lemmas 6.2 and 6.3, and Theorem 6.4 hold with $\widetilde{\mathcal{P}}_m$, $\widetilde{e}(x,t)$, S_N^m, $\operatorname{dom}\mathcal{E}_N$ replacing $\mathcal{P}_m$, $e(x,t)$, S_D^m, $\operatorname{dom}\mathcal{E}_D$, repectively. We note that $\|\cdot\|_{\operatorname{dom}\mathcal{E}_N} = \|\cdot\|_{\operatorname{dom}\mathcal{E}_D}$. Finally, we can prove that Theorem 1.2 also holds for the heat equation in (5.1).

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