SPECTRAL ASYMPTOTICS OF ONE-DIMENSIONAL GRAPH-DIRECTED SELF-SIMILAR MEASURES WITH OVERLAPS

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Abstract. For the class of graph-directed self-similar measures on $\mathbb{R}$, which could have overlaps but are essentially of finite type, we set up a framework for deriving a closed formula for the spectral dimension of these measures. For the class of finitely ramified graph-directed self-similar sets, the spectral dimension of the associated Laplace operators has been obtained by Hambly and Nyberg [7]. The main novelty of our result is that the graph-directed self-similar measures we consider do not need to satisfy the graph open set condition.

Contents

1. Introduction

Let $U \subseteq \mathbb{R}^d$ be a bounded domain with smooth boundary, Δ be the Dirichlet Laplacian on U, $\{\lambda_n\}$ be the eigenvalues of $-\Delta$, and $N(\lambda, -\Delta)$ be the number of eigenvalues that do not

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exceed λ. Weyl [25] proved the following asymptotic formula for the Dirichlet Laplacian:

$$N(\lambda,-\Delta)=\frac{\mathcal{B}_d}{(2\pi)^d}|U|\lambda^{d/2}+o(\lambda^{d/2})=\frac{1}{(4\pi)^{d/2}\Gamma(d/2+1)}|U|\lambda^{d/2}+o(\lambda^{d/2}), \tag{1.1}$$

where $|U|$ denotes the d-dimensional volume of U and $\mathcal{B}_d$ is the volume of the unit ball in $\mathbb{R}^d$. We point out that in [20, Equation (1.1)], the factor $(2\pi)^d$ is incorrectly typed as $(4\pi)^{d/2}$.

There has been considerable interest in studying spectral dimension on various domains. McKean and Ray [16] computed the spectral dimension of the Cantor measure. Fujita [5], Naimark and Solomyak [17] studied the spectral dimension of the self-similar measures satisfying the open set condition (OSC) (see [9]). Kigami and Lapidus [12] obtained the spectral dimension of Laplacians on post-critically finite (p.c.f.) self-similar sets with a harmonic structure. Croydon and Hambly [2,6] studied the spectral dimension on the continuum random tree and random recursive affine nested fractals. For finitely ramified graph-directed self-similar sets, Hambly and Nyberg [7] studied the spectral dimension of the associated Laplace operators. Freiberg [4] investigated spectral asymptotics of generalized measure geometric Laplacians on Cantor like sets. Kajino [10,11] studied asymptotics of the partition functions associated with self-similar sets. Alonso-Ruiz and Freiberg [1] obtained the spectral dimension of Laplacians on Hanoi attractors.

We say that an iterated function system (IFS) or a graph-directed iterated function system (GIFS), as well as any associated self-similar measure or graph-directed self-similar measure, have *overlaps*, if (OSC) or the graph open set condition (GOSC) (see Section 2.1) fails. In this case, it is much harder to compute the spectral dimension. For a class of IFSs on $\mathbb{R}$ with overlaps and satisfying second-order identities (see [23]), the first author [18] computed the spectral dimension of the corresponding measures. Tang and authors [20] defined measures that are essentially of finite type (EFT), a property describing the finiteness of basic measure types, and computed the spectral dimension of the Laplacian defined by a self-similar measure satisfying (EFT). The first author and Tang [19] computed the spectral dimension of a special class of graph-directed self-similar measures with overlaps. This paper studies the eigenvalue asymptotics of the Dirichlet Laplacians defined by graph-directed self-similar measure with overlaps in much greater generality.

Let $\Omega\subseteq\mathbb{R}^d$ be a bounded open set, and μ be a positive finite Borel measure on $\mathbb{R}^d$ with $\operatorname{supp}(\mu)\subseteq\overline{\Omega}$ and $\mu(\Omega)>0$. We assume that the *Poincaré inequality* (PI) for μ holds: There exists a constant $C>0$ such that

$$\int_\Omega |u|^2\,d\mu\le C\int_\Omega|\nabla u|^2\,dx, \qquad \text{for all } u\in C_c^\infty(\Omega). \tag{1.2}$$

(see, e.g., [8,15,17]) (PI) implies that each equivalence class $u\in H_0^1(\Omega)$ contains a unique (in the $L^2(\Omega,\mu)$ sense) member $\overline{u}$ that belongs to $L^2(\Omega,\mu)$ and satisfies both conditions below:

(1) there exists a sequence $\{u_n\}$ in $C_c^\infty(\Omega)$ such that $u_n\to\overline{u}$ in $H_0^1(\Omega)$ and $u_n\to\overline{u}$ in $L^2(\Omega,\mu)$;
(2) $\overline{u}$ satisfies (1.2).

We call $\overline{u}$ the $L^2(\Omega,\mu)$-*representative* of u. Define a mapping $I : H_0^1(\Omega) \to L^2(\Omega,\mu)$ by $I(u) = \overline{u}$. It is easy to see that I is a bounded linear operator, but not necessarily injective. Consider the subspace $\mathcal{N}$ of $H_0^1(\Omega)$ defined as

$$\mathcal{N} := \Big\{u \in H_0^1(\Omega) : \|I(u)\|_{L^2(\Omega,\mu)} = 0\Big\}.$$

It follows from the continuity of I that $\mathcal{N}$ is a closed subspace of $H_0^1(\Omega)$. Let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $H_0^1(\Omega)$. Then $I : \mathcal{N}^\perp \to L^2(\Omega,\mu)$ is injective. With a slight abuse of notation, we will denote $\overline{u}$ by u.

Consider a nonnegative bilinear form $\mathcal{E}(\cdot,\cdot)$ in $L^2(\Omega,\mu)$ given by

$$\mathcal{E}(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx \tag{1.3}$$

with domain $\operatorname{dom}\mathcal{E} = \mathcal{N}^\perp$. (PI) implies that $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a closed quadratic form on $L^2(\Omega,\mu)$. Hence these exists a nonnegative self-adjoint operator on $L^2(\Omega,\mu)$, which we denote by $-\Delta_\mu$ and call the *(Dirichlet) Laplacian* with respect to μ, such that $\operatorname{dom}\mathcal{E} = \operatorname{dom}(-\Delta_\mu)^{1/2}$ and $\mathcal{E}(u,v) = \langle(-\Delta_\mu)^{1/2}u, (-\Delta_\mu)^{1/2}v\rangle_{L^2(\Omega,\mu)}$ for all $u,v \in \operatorname{dom}\mathcal{E}$. Let $u \in \operatorname{dom}\mathcal{E}$. Then $u \in \operatorname{dom}\Delta_\mu$ holds if and only if there exists $f \in L^2(\Omega,\mu)$ such that $\mathcal{E}(u,v) = \langle f, v\rangle_{L^2(\Omega,\mu)}$ for all $v \in \operatorname{dom}\mathcal{E}$, where $-\Delta_\mu u = f$. We remark that if $d = 1$, then (PI) holds for any such μ, and thus $-\Delta_\mu$ is well defined.

We assume that $L^2(\Omega,\mu)$ is infinite dimensional. It is known (see, e.g., [8]) that there exists an orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2(\Omega,\mu)$ consisting of the eigenfunctions of $-\Delta_\mu$. The eigenvalues $\lambda_n = \lambda_n(-\Delta_\mu)$ satisfy $0 < \lambda_1 \le \lambda_2 \le \cdots$ and $\lim_{n\to\infty} \lambda_n = \infty$.

Let $N(\lambda, -\Delta_\mu)$ be the number of eigenvalues of $-\Delta_\mu$ (counting multiplicity) which do not exceed λ, i.e.,

$$N(\lambda, -\Delta_\mu) := \#\{n : \lambda_n \le \lambda\}, \tag{1.4}$$

where $\#A$ denotes the cardinality of a set A. Define the *lower and upper spectral dimensions* of $-\Delta_\mu$ (or μ), respectively, as

$$\underline{d}_s(-\Delta_\mu) := \varliminf_{\lambda\to\infty} \frac{2\ln N(\lambda,-\Delta_\mu)}{\ln\lambda} \quad \text{and} \quad \overline{d}_s(-\Delta_\mu) := \varlimsup_{\lambda\to\infty} \frac{2\ln N(\lambda,-\Delta_\mu)}{\ln\lambda}.$$

If $\underline{d}_s(-\Delta_\mu) = \overline{d}_s(-\Delta_\mu)$, the common value, denoted $d_s(-\Delta_\mu)$ (or $d_s(\mu)$), is called the *spectral dimension* of $-\Delta_\mu$ (or μ); it measures the asymptotic growth rate of the eigenvalue counting function as well as the magnitude of the n-th eigenvalue.

(EFT) is introduced in [20]. Let $\mu = \sum_{i=1}^q \mu_i$ be the graph-directed self-similar measure defined by the GIFS $G = (V,E)$ on $\mathbb{R}^d$, where $V = \{1,\dots,q\}$ and $E = \{e_i : i \in V\}$. We say that μ satisfies (EFT) (see Definition 2.1) if there exist a family of bounded open subsets $\{\Omega_i\}_{i=1}^q$ with $\Omega_i \subseteq \mathbb{R}^d$, $\operatorname{supp}(\mu_i) \subseteq \overline{\Omega}_i$, and $\mu(\Omega_i) > 0$, and a finite family $\mathbf{B} := \{B_{1,\ell} : \ell \in \Gamma\}$ of measure disjoint cells, $B_{1,\ell} \subseteq \Omega_{i_\ell}$ for some $i_\ell \in V$, such that for any $\ell \in \Gamma$, there is a family of μ-partitions $\{\mathbf{P}_{k,\ell}\}_{k\ge1}$ of $B_{1,\ell}$ satisfying the following conditions: (1) $\mathbf{P}_{1,\ell} = \{B_{1,\ell}\}$, and there exists some $B \in \mathbf{P}_{2,\ell}^1$ such that $B \neq B_{1,\ell}$; (2) for any $k \ge 2$, $\mathbf{P}_{k+1,\ell}^1$ contains all cells in $\mathbf{P}_{k,\ell}^1$ that are μ-equivalent to some cell in $\mathbf{B}$; (3) the sum of the μ-measures of those cells $B \in \mathbf{P}_{k,\ell}$

that are not μ-equivalent to any cell in $\mathbf{B}$ tends to 0 as $k\to\infty$. In this case, we call $\{\Omega_i\}_{i=1}^q$ an *EFT-family*, $\mathbf{B}$ a *basic family of cells*, and $(\mathbf{B},\mathbf{P}) := (\{B_{1,\ell}\},\{\mathbf{P}_{k,\ell}\}_{k\geq 1})_{\ell\in\Gamma}$ a *basic pair*. We say that $(\mathbf{B},\mathbf{P})$ is *regular* if each cell $B\in\bigcup_{k\geq 1,\ell\in\Gamma}\mathbf{P}_{k,\ell}$ is connected, and for any $\ell\in\Gamma$, there exist some similitude τ_ℓ, some Ω_{j_ℓ}, and some constant $w(\ell)>0$ such that $\tau_\ell(\Omega_{j_\ell})\subseteq B_{1,\ell}$ and $\mu\geq w(\ell)\mu\circ\tau_\ell^{-1}$ on $\tau_\ell(\Omega_{j_\ell})$.

Let $\mu=\sum_{i=1}^q\mu_i$ be the graph-directed self-similar measure defined by the GIFS $G=(V,E)$ on $\mathbb{R}$. Assume that μ satisfies (EFT) with $\{\Omega_i\}_{i=1}^q$ being an EFT-family and assume that there exists a regular basic pair $(\mathbf{B},\mathbf{P}) := (\{B_{1,\ell}\},\{\mathbf{P}_{k,\ell}\}_{k\geq 1})_{\ell\in\Gamma}$. Then we can derive renewal equations for the eigenvalue counting functions, and express them in vector form as:

$$\mathbf{f}=\mathbf{f}*\mathbf{M}_\alpha+\mathbf{z},$$

where $\alpha\in\mathbb{R}$, and

$$\begin{aligned}&\mathbf{f}=\mathbf{f}^{(\alpha)}(t)=[f_\ell^{(\alpha)}(t)]_{\ell\in\Gamma}, \quad t\in\mathbb{R};\\ &\mathbf{M}_\alpha=[\mu_{\ell'\ell}^{(\alpha)}]_{\ell,\ell'\in\Gamma} \quad \text{is a finite matrix of Borel measures on } \mathbb{R};\\ &\mathbf{z}=\mathbf{z}^{(\alpha)}(x)=[z_\ell^{(\alpha)}(x)]_{\ell\in\Gamma} \quad \text{is a vector of error functions.}\end{aligned} \tag{1.5}$$

Let

$$\mathbf{M}_\alpha(\infty) := \Big[\mu_{\ell'\ell}^{(\alpha)}(\mathbb{R})\Big]_{\ell,\ell'\in\Gamma}. \tag{1.6}$$

For each $\ell\in\Gamma$ and $\alpha\geq 0$, define

$$F_\ell(\alpha) := \sum_{\ell'\in\Gamma}\mu_{\ell'\ell}^{(\alpha)}(\mathbb{R}), \qquad D_\ell := \{\alpha\geq 0: F_\ell(\alpha)<\infty\}, \qquad \widetilde{\alpha}_\ell := \inf D_\ell. \tag{1.7}$$

If the error functions decay exponentially to 0 as $t\to\infty$, then $d_s(\mu)$ is given by the unique α such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is equal to 1.

Theorem 1.1. *Let $\mu=\sum_{i=1}^q\mu_i$ be a graph-directed self-similar measure defined by a strongly connected GIFS $G=(V,E)$ on $\mathbb{R}$. Assume that μ satisfies (EFT) with $\{\Omega_i\}_{i=1}^q$ being an EFT-family and there exists a regular basic pair. Let $\Omega=\bigcup_{i=1}^q\Omega_i$, Δ_μ be the Dirichlet Laplacian defined by μ, and let $\mathbf{M}_\alpha(\infty), F_\ell(\alpha)$ and $\tilde{\alpha}_\ell$ be defined as in* (1.6) *and* (1.7)*. Assume that for each $\ell\in\Gamma$, $\lim_{\alpha\to\tilde{\alpha}_\ell^+}F_\ell(\alpha)>1$.*

(a) *There exists a unique $\alpha>0$ such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ equals 1.*
(b) *If we assume, in addition, that for the unique α in (a), there exists $\sigma>0$ such that for all $\ell\in\Gamma$, $z_\ell^{(\alpha)}(t)=o(e^{-\sigma t})$ as $t\to\infty$, then $d_s(\mu)=2\alpha$; moreover, if $\mathbf{M}_\alpha(\infty)$ is irreducible, then there exist positive constants C_1 and C_2 such that $C_1\lambda^\alpha\leq N(\lambda,-\Delta_\mu)\leq C_2\lambda^\alpha$ for all sufficiently large λ.*

In section 5, we illustrate Theorem 1.1 by the strongly connected GIFS $G=(V,E)$ with $V=\{1,2\}$ and $E=\{e_i: 1\leq i\leq 5\}$, where $e_1,e_3\in E^{1,1}, e_2\in E^{1,2}, e_4\in E^{2,2}, e_5\in E^{2,1}$. The six similitudes are defined by

$$\begin{aligned}&S_{e_1}(x)=\rho x, && S_{e_2}(x)=rx+\rho(1-r), && S_{e_3}(x)=rx+(1-r),\\ &S_{e_4}(x)=rx+(1-r), && S_{e_5}(x)=\rho x,\end{aligned} \tag{1.8}$$

where

$$\rho + 2r - \rho r \leq 1, \tag{1.9}$$

i.e., $S_{e_2}(1) \leq S_{e_3}(0)$ (see Figure 1).

Corollary 1.2. *Let $\mu = \mu_1 + \mu_2$ be a graph-directed self-similar measure defined by a GIFS $G = (V, E)$ in (1.8) together with a probability matrix $(p_e)_{e\in E}$. Then there exists a unique positive real number α satisfying*

$$\big(1 - (p_{e_4} r)^\alpha\big)\Big((1 - (p_{e_1}\rho)^\alpha)(1 - (p_{e_3} r)^\alpha) - \big((p_{e_1e_3} + p_{e_2e_5})\rho r\big)^\alpha\Big) - \big(p_{e_2e_4e_5}\rho r^2\big)^\alpha = 0. \tag{1.10}$$

Moreover, $d_s(\mu) = 2\alpha$.

Theorem 1.3. *Let $\mu = \sum_{i=1}^q \mu_i$ be a graph-directed self-similar measure on $\mathbb{R}$ defined by a GIFS $G = (V, E)$ that is not strongly connected. Assume that G has γ strongly connected components. For $m = 1, \ldots, \gamma$, let $2\alpha_m$ be the spectral dimension of the graph-directed self-similar measure corresponding to the m^{th} strongly connected component, and let*

$$SC_m := \{i \in V : i \text{ is contained in the } m^{th} \text{ strongly connected component}\}. \tag{1.11}$$

Assume that μ satisfies (EFT) with $\{\Omega_i\}_{i=1}^q$ being an EFT-family and there exists a regular basic pair. Let $\Omega = \bigcup_{i=1}^q \Omega_i$ and Δ_μ be the Dirichlet Laplacian defined by μ. Let $\mathbf{M}_\alpha(\infty), F_\ell(\alpha)$ and $\tilde{\alpha}_\ell$ be defined as in (1.6) and (1.7). Assume that for $m = 1, \ldots, \gamma$, each $i \in SC_m$, and each $\ell \in \Gamma_i$, $\lim_{\alpha_m \to \tilde{\alpha}_\ell^+} F_\ell(\alpha_m) > 1$.

(a) *There exists a unique $\alpha = \max\{\alpha_m : m = 1, \ldots, \gamma\} > 0$ such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ equals 1.*
(b) *If we assume, in addition, that for the unique α in (a), there exists $\sigma > 0$ such that for all $\ell \in \Gamma, z_\ell^{(\alpha)}(t) = o(e^{-\sigma t})$ as $t \to \infty$, then we have $d_s(\mu) = 2\alpha$; moreover, if $\mathbf{M}_\alpha(\infty)$ is irreducible, then there exist positive constants C_1 and C_2 such that $C_1\lambda^\alpha \leq N(\lambda, -\Delta_\mu) \leq C_2\lambda^\alpha$ for all sufficiently large λ.*

In section 6, we illustrate Theorem 1.3 by a GIFS $G = (V, E)$ that is not strongly connected, with $V = \{1, 2\}$ and $E = \{e_i : 1 \leq i \leq 5\}$, where $e_1, e_2, e_3 \in E^{1,1}, e_4 \in E^{2,2}, e_5 \in E^{2,1}$. The five similitudes are defined by

$$\begin{aligned} &S_{e_1}(x) = \rho x, && S_{e_2}(x) = rx + \rho(1 - r), && S_{e_3}(x) = rx + (1 - r), \\ &S_{e_4}(x) = rx + (1 - r), && S_{e_5}(x) = \rho x, && \end{aligned} \tag{1.12}$$

where $\rho + 2r - \rho r \leq 1$, i.e., $S_{e_2}(1) \leq S_{e_3}(0)$ (see Figure 5). For a probability matrix $(p_e)_{e\in E}$, we define

$$w(k) := p_{e_1} \sum_{j=0}^k p_{e_2}^j p_{e_3}^{k-j}, \qquad k \geq 0, \tag{1.13}$$

Corollary 1.4. *Let $\mu = \mu_1 + \mu_2$ be a graph-directed self-similar measure defined by the GIFS $G = (V, E)$ in (1.12) together with a probability matrix $(p_e)_{e\in E}$, and let $w(k)$ be defined as in*

(1.13). *Then there exists a unique positive real number* α *satisfying*

$$(p_{e_2}r)^\alpha + (p_{e_3}r)^\alpha + \big(1-(p_{e_2}r)^\alpha\big)\big(1-(p_{e_3}r)^\alpha\big)\sum_{k=0}^{\infty}(w(k)\rho r^k)^\alpha = 1. \tag{1.14}$$

Moreover, $d_s(\mu) = 2\alpha$.

This paper is organized as follows. In Section 2, we give a modified version of the definition of (EFT). In Section 3, we introduce some properties of the eigenvalue counting function. In Section 4, we derive renewal equations and prove Theorem 1.1 and Theorem 1.3. Section 5 illustrates Theorem 1.1 by a class of one-dimensional strongly connected GIFSs defined as in (1.8); we also prove Corollary 1.2. In Section 6, we study the one-dimensional GIFSs in (1.12), which are not strongly connected, and prove Corollary 1.4.

2. Graph-directed iterated function systems and measures essentially of finite type

2.1. **Graph-directed iterated function systems.** A *graph-directed iterated function system* (GIFS) of contractive similitudes is an ordered pair $G = (V, E)$ described as follows (see [14]). V is a set of *vertices* labeled by $\{1, \dots, q\}$ and E is a set of *directed edges* with each begining and ending at a vertex. It is possible for an edge to begin and end at the same vertex and we allow more than one edge between two vertices. To each edge $e \in E$, there corresponds a contractive similitude $S_e(x) : \mathbb{R}^d \to \mathbb{R}^d$ defined as

$$S_e(x) = \rho_e R_e x + b_e,$$

where $\rho_e \in (0,1)$ is the contraction ratio, R_e is an orthogonal transformation, and $b_e \in \mathbb{R}^d$. Let $E^{i,j}$ denote the set of all edges that begin at vertex i and end at vertex j. We call $\boldsymbol{e} = e_1 \dots e_k$ a *path* with length k, if the terminal vertex of each edge e_i $(1 \le i \le k-1)$ equals the initial vertex of the edge e_{i+1}. It is well known that there exists a unique family of nonempty compact sets $K_1, \dots, K_q$ satisfying

$$K_i = \bigcup_{j=1}^{q} \bigcup_{e \in E^{i,j}} S_e(K_j), \qquad i = 1, \dots, q. \tag{2.15}$$

Define

$$K := \bigcup_{i=1}^{q} K_i. \tag{2.16}$$

We call K the *graph self-similar set* associated with $G = (V, E)$. Assume that for each edge $e \in E$, there corresponds a transition probability $p_e > 0$, and the weights of all edges leaving a given vertex i sum to 1, namely,

$$\sum_{j \in V} \sum_{e \in E^{i,j}} p_e = 1. \tag{2.17}$$

Then for each $i \in V$, there exists a unique Borel probability measures μ_i such that

$$\mu_i = \sum_{j=1}^{q} \sum_{e \in E^{i,j}} p_e \mu_j \circ S_e^{-1}. \tag{2.18}$$

We note that $\text{supp}(\mu_i) = K_i$ for all $i \in V$. Finally, let $\mu := \sum_{i=1}^q \mu_i$ and call it a *graph-directed self-similar measure*. We say that $G = (V, E)$ satisfies *the graph open set condition* (GOSC) (see [24]) if there exists a family $\{O_i\}_{i=1}^q \subseteq \mathbb{R}^d$ of nonempty bounded open sets such that for all $i = 1, \ldots, q$,

$$\bigcup_{e \in E^{i,j}} S_e(O_j) \subseteq O_i \quad \text{and} \quad S_e(O_j) \cap S_{e'}(O_j) = \emptyset \quad \text{for all distinct } e, e' \in E^{i,j}.$$

It is obvious that $K_i \subseteq \overline{O}_i$, i.e., $\text{supp}(\mu_i) \subseteq \overline{O}_i$. A GIFS, as well as any associated graph-directed self-similar measure, are said to be have *overlaps* if (GOSC) fails. Let $\{\Omega_i\}_{i=1}^q$ be a family of nonempty bounded open subsets of $\mathbb{R}^d$. We say that $\{\Omega_i\}_{i=1}^q$ is *invariant* under the GIFS $G = (V, E)$ if $\bigcup_{e \in E^{i,j}} S_e(\Omega_j) \subseteq \Omega_i$ for $i = 1, \ldots, q$. We say G is *connected* if for each pair of vertices $i, j \in V$, there is a (non-directed) path between them. G is said to be *strongly connected* if for each pair of vertices $i, j \in V$, there is a directed path from i to j. A strongly connected component of G is a maximal subgraph H of G such that H is strongly connected. Strongly connected components are pairwise disjoint and do not necessarily cover G. A single vertex may be a strongly connected component if it loops to itself. In this paper, we assume that each graph has at least one strongly connected component.

2.2. **The essentially finite type condition for the graph-directed self-similar measure.** Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open subset and μ be a positive finite Borel measure with $\text{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. We call a μ-measurable subset U of Ω is a *cell (in Ω)* if $\mu(U) > 0$. Clearly, Ω itself is a cell.

We say that two cells U and V are *μ-equivalent*, denoted by $U \simeq_{\mu,\tau,w} V$ (or simply $U \simeq_\mu V$), if there exist some similitude $\tau : U \to V$ and some constant $w > 0$ such that $\tau(U) = V$ and

$$\mu|_V = w\mu|_U \circ \tau^{-1}. \tag{2.19}$$

It is easy to check that $\simeq_\mu$ is an equivalence relation.

Let $U \subseteq \Omega$ be a cell. Two cells V, W in U are *measure disjoint* with respect to μ if $\mu(V \cap W) = 0$. We call a finite family $\mathbf{P}$ of measure disjoint cells a *μ-partition* of U if $V \subseteq U$ for all $V \in \mathbf{P}$, and $\mu(U) = \sum_{V \in \mathbf{P}} \mu(V)$. A sequence of μ-partitions $\{\mathbf{P}_k\}_{k\geq 1}$ is *refining* if for any $V \in \mathbf{P}_k$ and any $W \in \mathbf{P}_{k+1}$, either $W \subseteq V$ or they are measure disjoint, i.e., each member of $\mathbf{P}_{k+1}$ is a subset of some member of $\mathbf{P}_k$.

Let $\mathbf{B} := \{B_{1,\ell}\}_{\ell\in\Gamma}$ be a finite family of measure disjoint cells in Ω, and for each $\ell \in \Gamma$, let $\{\mathbf{P}_{k,\ell}\}_{k\geq 1}$ be a family of refining μ-partitions of $B_{1,\ell}$ with $\mathbf{P}_{1,\ell} := \{B_{1,\ell}\}$. We divide each $\mathbf{P}_{k,\ell}$, $k \geq 2$, into two (possibly empty) subcollections, $\mathbf{P}^1_{k,\ell}$ and $\mathbf{P}^2_{k,\ell}$, with respect to $\mathbf{B}$, defined as follows:

$$\begin{aligned}\mathbf{P}^1_{k,\ell} &:= \big\{B \in \mathbf{P}_{k,\ell} : B \simeq_\mu B_{1,i} \text{ for some } i \in \Gamma\big\},\\ \mathbf{P}^2_{k,\ell} &:= \mathbf{P}_{k,\ell} \setminus \mathbf{P}^1_{k,\ell} = \big\{B \in \mathbf{P}_{k,\ell} : B \notin \mathbf{P}^1_{k,\ell}\big\}.\end{aligned} \tag{2.20}$$

Definition 2.1. *We say that a graph-directed self-similar measure $\mu = \sum_{i=1}^q \mu_i$ on $\mathbb{R}^d$ is* essentially of finite type (EFT) *if there exist a family of bounded open subsets $\{\Omega_i\}_{i=1}^q$ with $\Omega_i \subseteq \mathbb{R}^d$, $\text{supp}(\mu_i) \subseteq \overline{\Omega}_i$ and $\mu(\Omega_i) > 0$, and a finite family $\mathbf{B} := \{B_{1,\ell}\}_{\ell\in\Gamma}$ of measure disjoint* *cells, $B_{1,\ell}\subseteq\Omega_{i_\ell}$ for some $i_\ell=1,\dots,q$, such that for any $\ell\in\Gamma$, there is a family of μ-partitions $\{\mathbf{P}_{k,\ell}\}_{k\ge1}$ of $B_{1,\ell}$ satisfying the following conditions:*

(1) $\mathbf{P}_{1,\ell}=\{B_{1,\ell}\}$, *and there exists some* $B\in\mathbf{P}^1_{2,\ell}$ *such that* $B\neq B_{1,\ell}$;
(2) *if for some* $k\ge2$, *there exists some* $B\in\mathbf{P}^1_{k,\ell}$, *then* $B\in\mathbf{P}^1_{k+1,\ell}$ *and hence* $B\in\mathbf{P}^1_{m,\ell}$ *for all* $m\ge k$;
(3) $\lim_{k\to\infty}\sum_{B\in\mathbf{P}^2_{k,\ell}}\mu(B)=0.$

Here $\mathbf{P}^1_{k,\ell}$ *and* $\mathbf{P}^2_{k,\ell}$ $(k\ge2)$ *are defined as in* (2.20). *In this case, we call* $\{\Omega_i\}_{i=1}^q$ *an* EFT-family, $\mathbf{B}$ *a* basic family of cells, *and* $(\mathbf{B},\mathbf{P}):=(\{B_{1,\ell}\},\{\mathbf{P}_{k,\ell}\}_{k\ge1})_{\ell\in\Gamma}$ *a* basic pair.

For $k\ge2$ and $\ell\in\Gamma$, let $\mathbf{P}_{k,\ell}=\{B_{k,\ell,i},i=1,2,\dots\}$.

Definition 2.2. *Assume that a graph-directed self-similar measure $\mu=\sum_{i=1}^q\mu_i$ satisfies (EFT) with $\{\Omega_i\}_{i=1}^q$ being an EFT-family and $(\mathbf{B},\mathbf{P}):=(\{B_{1,\ell}\},\{\mathbf{P}_{k,\ell}\}_{k\ge1})_{\ell\in\Gamma}$ being a basic pair. We say that $(\mathbf{B},\mathbf{P})$ is* regular *if each cell $B\in\bigcup_{k\ge1,\ell\in\Gamma}\mathbf{P}_{k,\ell}$ is connected, and for any $\ell\in\Gamma$, there exist some similitude τ_ℓ, some Ω_{j_ℓ} and some constant $w(\ell)>0$ such that $\tau_\ell(\Omega_{j_\ell})\subseteq B_{1,\ell}$ and $\mu\ge w(\ell)\mu\circ\tau_\ell^{-1}$ on $\tau_\ell(\Omega_{j_\ell})$. In this case, we call $\mathbf{B}$ a* regular basic family of cells.

3. Eigenvalue counting function

3.1. **Eigenvalue counting function on $\mathbb{R}$.** ([20, Section 4.1]) In this subsection, we only consider the one-dimensional eigenvalue counting function. Let $(\mathscr{E},\operatorname{dom}\mathscr{E})$ be defined as in (1.3) with $\Omega=(a,b)$ and let $-\Delta_\mu$ be the associated Dirichlet Laplacian on $L^2((a,b),\mu)$. Let $\mathscr{P}=\{a_i\}_{i=0}^{n+1}$ be a partition of $[a,b]$ satisfying

$$a_0:=a<a_1<\cdots<a_{n+1}=:b \qquad \text{for } i\in\{0,\dots,n+1\}.$$

Define $\mathcal{F}:=\mathcal{F}(\mathscr{P})=\{u\in\operatorname{dom}\mathscr{E}:u(a_i)=0\text{ for all }i=0\dots,n+1\}$. Then $\mathcal{F}$ is a closed subspace of $\operatorname{dom}\mathscr{E}$. Define a relation $\sim_{\mathscr{E}}$ on $\operatorname{dom}\mathscr{E}$, induced by $\mathcal{F}$, by $u\sim_{\mathscr{E}}v$ if and only if $u-v\in\mathcal{F}$. Then $\sim_{\mathscr{E}}$ is an equivalence relation on $\operatorname{dom}\mathscr{E}$. Define the quotient space

$$\operatorname{dom}\mathscr{E}/\mathcal{F}:=\{[u]_{\mathscr{E}}:u\in\operatorname{dom}\mathscr{E}\},$$

where $[u]_{\mathscr{E}}$ is the equivalence class of u. Define addition and scalar multiplication on $\operatorname{dom}\mathscr{E}/\mathcal{F}$ as usual. For each $i=1,\dots,n$, let f_i be a function in $\operatorname{dom}\mathscr{E}$ that satisfies

$$f_i(a_j)=\delta_{ij},\qquad i,j=1,\dots,2n,$$

where δ_{ij} is the Kronecker delta. Such an f_i clearly exists. It is easy to prove that

$$\operatorname{dom}\mathscr{E}/\mathcal{F}=\text{ span }\{[f_i]_{\mathscr{E}}:i=1,\dots,n\}\qquad\text{and}\qquad\dim(\operatorname{dom}\mathscr{E}/\mathcal{F})=n.$$

Let $-\Delta^{\mathcal{F}}_{\mu|_{(a,b)}}$ be the Laplacian defined by the Dirichlet form (1.3) with $\operatorname{dom}\mathcal{E}=\mathcal{F}$, and let $N(\lambda,-\Delta^{\mathcal{F}}_{\mu|_{(a,b)}}):=\#\{n:\lambda_n(-\Delta^{\mathcal{F}}_{\mu|_{(a,b)}})\le\lambda\}$ be the associated *eigenvalue counting function.* If

$\mathcal{F} = \mathcal{N}^{\perp}$, where $\mathcal{N}$ is defined as in Section 1, then $N(\lambda, -\Delta^{\mathcal{F}}_{\mu|_{(a,b)}})$ reduces to $N(\lambda, -\Delta_{\mu|_{(a,b)}})$. It follows from the variational formula that

$$N(\lambda, -\Delta^{\mathcal{F}}_{\mu|_{(a,b)}}) \le N(\lambda, -\Delta_{\mu|_{(a,b)}}) \le N(\lambda, -\Delta^{\mathcal{F}}_{\mu|_{(a,b)}}) + \#\mathscr{P} - 2. \tag{3.21}$$

If $\mathrm{supp}(\mu) = [a,b]$, then $N(\lambda, -\Delta^{\mathcal{F}}_{\mu|_{(a,b)}}) = \sum_{i=0}^{n} N(\lambda, -\Delta_{\mu|_{(a_i,a_{i+1})}})$. Next, we state a similar formula. A proof can be found in [20, Proposition 4.1].

Proposition 3.1. *Let μ be a continuous positive finite Borel measure on $[a,b]$ with* $\mathrm{supp}(\mu) \subseteq [a,b]$*. Suppose there exists a nonempty subset $\Lambda \subseteq \{0,1,\dots,n\}$ such that $\mu(a_i,a_{i+1}) > 0$ for any $i \in \Lambda$ and $\mu(a_j,a_{j+1}) = 0$ for any $j \notin \Lambda$. Then*

$$N(\lambda, -\Delta^{\mathcal{F}}_{\mu|_{(a,b)}}) = \sum_{i\in\Lambda} N(\lambda, -\Delta_{\mu|_{(a_i,a_{i+1})}}).$$

3.2. **Unitarily equivalent operators.** In this subsection, we state a slightly modified version of [18, Propositions 2.2 and 2.3] below.

Proposition 3.2. ([18, Proposition 2.2]) *Let $S : \mathbb{R} \to \mathbb{R}$ be a similitude, with Lipschitz constant r, such that $S(a,b) = (c,d)$. Let μ be a continuous positive finite Borel measure on $[a,b]$ with* $\mathrm{supp}(\mu) \subseteq [a,b]$*. Then*

(a) $-\Delta_{\mu\circ S^{-1}|_{(c,d)}}$ *and* $r^{-1} \cdot \big(-\Delta_{\mu|_{(a,b)}}\big)$ *are unitarily equivalent.*
(b) *If, in addition, $\mu|_{(c,d)} = w\mu \circ S^{-1}$ on (c,d) for some constant $w > 0$, then $-\Delta_{\nu|_{(c,d)}}$ and $(rw)^{-1} \cdot \big(-\Delta_{\mu|_{(a,b)}}\big)$ are unitarily equivalent.*

Note that unitarily equivalent operators have the same set of eigenvalues.

Proposition 3.3. ([18, Proposition 2.3]) *Let μ, ν be continuous positive finite Borel measures on $[a,b]$ and assume that there exists some constant $w > 0$ such that $\mu \le w\nu$ on $[a,b]$. Then for any $n \ge 1$, $\lambda_n(-\Delta_\mu) \ge w^{-1} \cdot \lambda_n(-\Delta_\nu)$.*

The following result follows by combining Propositions 3.2 and 3.3. The proof can be found in [20].

Proposition 3.4. *Let μ be a continuous positive finite Borel measure on $\mathbb{R}$ and assume that there exist a similitude S with Lipschitz constant r, and a constant $w > 0$ such that $S([a,b]) = [c,d]$ and $\mu \ge w\mu \circ S^{-1}$ on $[c,d]$. Then $N(wr\lambda, -\Delta_{\mu|_{(a,b)}}) \le N(\lambda, -\Delta_{\mu|_{(c,d)}})$.*

4. Renewal equation and Proof of Theorem 1.1 and Theorem 1.3

4.1. **Renewal equation.** Let $\mu = \sum_{i=1}^{q} \mu_i$ be a graph-directed self-similar measure defined by $G = (V,E)$ on $\mathbb{R}$. In the rest of this section, assume that μ satisfies (EFT) with $\{\Omega_i\}_{i=1}^{q}$ being an EFT-family, with $(\mathbf{B},\mathbf{P}) := (\{B_{1,\ell}\}, \{\mathbf{P}_{k,\ell}\}_{k\ge1})_{\ell\in\Gamma}$ being a regular basic pair. The regularity of $(\mathbf{B},\mathbf{P})$ implies that each cell $B \in \bigcup_{k\ge1,\ell\in\Gamma} \mathbf{P}_{k,\ell}$ is an interval. This allows us to apply Propositions 3.1–3.4. Let

$$\Gamma_i := \{\ell \in \Gamma : B_{1,\ell} \subseteq \Omega_i\} \quad \text{and} \quad \mathbf{B}_i := \{B_{1,\ell} : \ell \in \Gamma_i\} \qquad \text{for } i \in V. \tag{4.22}$$

Then $\Gamma = \bigcup_{i=1}^{q}\Gamma_i$ and $\mathbf{B} = \bigcup_{i=1}^{q}\mathbf{B}_i$. Note that Γ_i and $\mathbf{B}_i$ maybe empty. The following Proposition has been modified from [20, Proposition 4.5] to suit our purpose. The proof is similar.

Proposition 4.1. *Let $\mu = \sum_{i=1}^{q}\mu_i$ be a graph-directed self-similar measure defined by a GIFS $G = (V, E)$ on $\mathbb{R}$. Assume that μ satisfies (EFT) with $\{\Omega_i\}_{i=1}^{q}$ being an EFT-family and with $\mathbf{B} := \{B_{1,\ell} : \ell \in \Gamma\}$ being a regular basic family of cells. Let $\Omega = \bigcup_{i=1}^{q}\Omega_i$, and Γ_i, $\mathbf{B}_i$ defined as in* (4.22)*. Then for $i \in V$ and any $\ell \in \Gamma_i$, there exists some constant $c_\ell > 0$ such that*

$$N(\lambda, -\Delta_{\mu_i|_{B_{1,\ell}}}) \le N(\lambda, -\Delta_{\mu|_\Omega}) \le N\big(c_\ell\lambda, -\Delta_{\mu_i|_{B_{1,\ell}}}\big). \tag{4.23}$$

Proposition 4.1 implies that the asymptotic behavior of $N(\lambda, -\Delta_\mu)$ is controlled by that of $N(\lambda, -\Delta_{\mu_i|_{B_{1,\ell}}})$ for $i \in V$ and $\ell \in \Gamma_i$.

Step 1. Derivation of functional equations. For $\ell \in \Gamma_i$ and $k \ge 2$, let $\mathbf{P}^1_{k,\ell}$ and $\mathbf{P}^2_{k,\ell}$ be defined as in (2.20) with respect to $\mathbf{B}$, where $i \in V$. Without loss of generality, we may assume that Γ_i can be partitioned into two (possibly empty) sub-collections, Γ'_i and Γ^*_i, defined as follows. An index $\ell \in \Gamma_i$ belongs to Γ'_i if there exists some integer k satisfying $\mathbf{P}^2_{k,\ell} = \emptyset$. Let $\kappa_\ell \ge 2$ (depending on ℓ) denote the smallest of such k. Define $\Gamma^*_i := \Gamma_i\backslash\Gamma'_i$ and let $\kappa_\ell := \infty$ for $\ell \in \Gamma^*_i$. Let $\Gamma' = \bigcup_{i=1}^{q}\Gamma'_i$ and $\Gamma^* = \bigcup_{i=1}^{q}\Gamma^*_i$. Then $\Gamma = \Gamma' \cup \Gamma^*$.

For $i \in V$, fix any $\ell \in \Gamma_i$. The definition of (EFT) implies that for any $2 \le k \le \kappa_\ell$, there exist two finite disjoint $G^1_{k,\ell}, G^2_{k,\ell} \subseteq \mathbb{N}$ such that

$$\mathbf{P}^1_{k,\ell} = \bigcup_{m=2}^{k}\big\{B_{m,\ell,p} : p \in G^1_{m,\ell}\big\} \quad \text{and} \quad \mathbf{P}^2_{k,\ell} = \big\{B_{k,\ell,p} : p \in G^2_{k,\ell}\big\}.$$

Condition (1) of (EFT) implies that $G^1_{2,\ell} \ne \emptyset$. If $\ell \in \Gamma^*$, condition (3) of (EFT) implies that $\lim_{k\to\infty}\sum_{p\in G^2_{k,\ell}}\mu(B_{k,\ell,p}) = 0$.

Proposition 4.2. *Assume that μ satisfies (EFT). Let $\ell \in \Gamma_i$, and*

$$J_\ell := \big\{j \in V : S_e(\Omega_j) \subseteq B_{1,\ell} \textit{ for } e \in E^{i,j}\big\}, \tag{4.24}$$

where $i \in V$. Let $2 \le k \le \kappa_\ell$. If $G^1_{k,\ell} \ne \emptyset$, then for each $p \in G^1_{k,\ell}$, there exist some $\xi(k,\ell,p) > 0$ and $c(k,\ell,p) \in \Gamma_j$, $j \in J_\ell$, such that

$$N(\lambda, -\Delta_{\mu_i|_{B_{k,\ell,p}}}) = N(\xi(k,\ell,p)\lambda, -\Delta_{\mu_j|_{B_{1,c(k,\ell,p)}}}). \tag{4.25}$$

Proof. For any $p \in G^1_{k,\ell}$, by the definition of $\mathbf{P}^1_{k,\ell}$, there exist some similitude $S_{e(k,\ell,p)}$ with Lipschitz constant $r_{e(k,\ell,p)}$, as well as constants $w(k,\ell,p) > 0$ and $c(k,\ell,p) \in \Gamma_j$ such that $\mu_i|_{B_{k,\ell,p}} = w(k,\ell,p)\mu_j|_{B_{1,c(k,\ell,p)}} \circ S^{-1}_{e(k,\ell,p)}$, where $j \in J_\ell$. Combining this with Proposition 3.2(b), we obtain (4.25) with $\xi(k,\ell,p) := w(k,\ell,p)r_{e(k,\ell,p)}$. □

For all $i \in V$, each $\ell \in \Gamma_i$, and $1 \le n \le \kappa_\ell$, we define a partition $\mathcal{P}_{n,\ell}$ of $B_{1,\ell}$ as follows:

$$\mathcal{P}_{n,\ell} := \big\{x : x \text{ is an end-point of some interval in } \mathbf{P}_{n,\ell}\big\},$$

and let $\mathcal{F}_{n,\ell} := \mathcal{F}(\mathcal{P}_{n,\ell})$. Note that for all $i \in V$, any $\ell \in \Gamma_i$ and $2 \leq n \leq \kappa_\ell$, we have $\#\mathcal{P}_{n,\ell} \leq 2\#\mathbf{P}_{n,\ell}$. It follows from Proposition 3.1 that for $i \in V$, $\ell \in \Gamma_i$, and $2 \leq n \leq \kappa_\ell$,

$$N(\lambda, -\Delta^{\mathcal{F}_{n,\ell}}_{\mu_i|_{B_{1,\ell}}}) = \sum_{k=2}^{n} \sum_{p \in G^1_{k,\ell}} N(\lambda, -\Delta_{\mu_i|_{B_{k,\ell,p}}}) + \sum_{p \in G^2_{n,\ell}} N(\lambda, -\Delta_{\mu_i|_{B_{n,\ell,p}}}).$$

Combining this with (3.21) and Proposition 4.2, for any $\ell \in \Gamma'_i$,

$$N(\lambda, -\Delta_{\mu_i|_{B_{1,\ell}}}) = \sum_{k=2}^{\kappa_\ell} \sum_{p \in G^1_{k,\ell}} N(\xi(k,\ell,p)\lambda, -\Delta_{\mu_j|_{B_{1,c(k,\ell,p)}}}) + \epsilon(\kappa_\ell, \ell), \tag{4.26}$$

where $0 \leq \epsilon(\kappa_\ell, \ell) \leq 2\#\mathbf{P}_{\kappa_\ell,\ell} - 2$. Similarly, for $\ell \in \Gamma^*_i$ and $n \geq 2$, we have

$$\begin{aligned} N(\lambda, -\Delta_{\mu_i|_{B_{1,\ell}}}) = &\sum_{k=2}^{n} \sum_{p \in G^1_{k,\ell}} N(\xi(k,\ell,p)\lambda, -\Delta_{\mu_j|_{B_{1,c(k,\ell,p)}}}) \\ &+ \sum_{p \in G^2_{n,\ell}} N(\lambda, -\Delta_{\mu_i|_{B_{n,\ell,p}}}) + \epsilon(n, \ell), \end{aligned} \tag{4.27}$$

where $0 \leq \epsilon(n, \ell) \leq 2\#\mathbf{P}_{n,\ell} - 2$.

Step 2. Derivation of the vector-valued equation.

Case (1). G is strongly connected. For all $i \in V$, each $\ell \in \Gamma_i$, and $\alpha > 0$, define

$$f_\ell(t) = f^{(\alpha)}_\ell(t) := e^{-\alpha t} N(e^t, -\Delta_{\mu_i|_{B_{1,\ell}}}), \qquad t \in \mathbb{R}. \tag{4.28}$$

Let $\lambda = e^t$. Then $e^{-\alpha t} N(\beta\lambda, -\Delta_{\mu_i|_{B_{1,\ell}}}) = \beta^\alpha f_\ell(t + \ln \beta)$ for any $\beta > 0$. Now, multiply both sides of (4.26) and (4.27) by $e^{-\alpha t}$. Then for $\ell \in \Gamma'_i$, we have

$$f_\ell(t) = \sum_{k=2}^{\kappa_\ell} \sum_{p \in G^1_{k,\ell}} \xi(k,\ell,p)^\alpha f_{c(k,\ell,p)}\big(t + \ln \xi(k,\ell,p)\big) + z^{(\alpha)}_\ell(t), \tag{4.29}$$

where $z^{(\alpha)}_\ell(t) := e^{-\alpha t}\epsilon(\kappa_\ell, \ell)$. For $\ell \in \Gamma^*_i$ and $n \geq 2$, we obtain

$$\begin{aligned} f_\ell(t) = &\sum_{k=2}^{\infty} \sum_{p \in G^1_{k,\ell}} \xi(k,\ell,p)^\alpha f_{c(k,\ell,p)}\big(t + \ln \xi(k,\ell,p)\big) + z^{(\alpha)}_\ell(t) \\ &- \sum_{k=n+1}^{\infty} \sum_{p \in G^1_{k,\ell}} \xi(k,\ell,p)^\alpha f_{c(k,\ell,p)}\big(t + \ln \xi(k,\ell,p)\big), \end{aligned} \tag{4.30}$$

where

$$z^{(\alpha)}_\ell(t) := e^{-\alpha t}\bigg(\sum_{p \in G^2_{n,\ell}} N(\lambda, -\Delta_{\mu_i|_{B_{n,\ell,p}}}) + \epsilon(n, \ell) \bigg). \tag{4.31}$$

Since $\lambda_1(-\Delta_{\mu_i|_{B_{1,\ell}}}) > 0$ for any $i \in V$ and any $\ell \in \Gamma_i$, there exists $t_0 \in \mathbb{R}$ such that $f_\ell(t) = 0$ for any $t < t_0$ and any $\ell \in \Gamma_i$. For each $t \in \mathbb{R}$, all $i \in V$ and all $\ell \in \Gamma^*_i$, let $n_t := n_t(\ell)$ be the positive integer such that

$$t + \max\big\{\ln \xi(k,\ell,p) : p \in G^1_{k,\ell}\big\} < t_0 \quad \text{ for all } k > n_t. \tag{4.32}$$

Let $n=n_t$ in (4.30). Then

$$f_\ell(t)=\sum_{k=2}^{\infty}\sum_{p\in G^1_{k,\ell}}\xi(k,\ell,p)^\alpha f_{c(k,\ell,p)}\big(t+\ln\xi(k,\ell,p)\big)+z^{(\alpha)}_\ell(t)\quad\text{for } \ell\in\Gamma_i^*,\tag{4.33}$$

where $z^{(\alpha)}_\ell(t)$ is obtained from that in (4.31) by replacing n with n_t. For $i\in V$, $\ell\in\Gamma_i$, let $\mu^{(\alpha)}_{\ell'\ell}$ be the discrete measure such that

$$\mu^{(\alpha)}_{\ell'\ell}\big(-\ln\xi(k,\ell,p)\big):=\xi(k,\ell,p)^\alpha \text{ for } 2\le k\le\kappa_\ell,\ p\in G^1_{k,\ell},\ \ell'=c(k,\ell,p)\in\Gamma_j \text{ and } j\in J_\ell.\tag{4.34}$$

Then (see (1.7))

$$\mu^{(\alpha)}_{\ell'\ell}(\mathbb{R})=\sum_{k=2}^{\kappa_\ell}\sum_{p\in G^1_{k,\ell}}\xi(k,\ell,p)^\alpha$$

and

$$F_\ell(\alpha)=\sum_{\ell'\in\Gamma}\sum_{k=2}^{\kappa_\ell}\sum_{p\in G^1_{k,\ell}}\xi(k,\ell,p)^\alpha.$$

Case (2). G is not strongly connected. If $G=(V,E)$ is not strongly connected, then there exists some $i,j\in V$ satisfying $E^{i,j}=\emptyset$. That is, $\bigcup_{\ell\in\Gamma_i}J_\ell=\emptyset$. Assume that G has γ strongly connected components. For $m=1,\dots,\gamma$, let SC_m be defined as in (1.11).

For $m=1,\dots,\gamma$, each $i\in SC_m$ and each $\ell\in\Gamma_i$, define

$$f_\ell(t)=f^{(\alpha_m)}_\ell(t):=e^{-\alpha_m t}N(e^t,-\Delta_{\mu_i|_{B_{1,\ell}}}),\qquad \alpha_m>0, t\in\mathbb{R}.\tag{4.35}$$

Let $\lambda=e^t$. Then $e^{-\alpha_m t}N(\beta\lambda,-\Delta_{\mu_i|_{B_{1,\ell}}})=\beta^{\alpha_m}f_\ell(t+\ln\beta)$ for any $\beta>0$. Now, multiply both sides of (4.26) by $e^{-\alpha_m t}$. Then for $\ell\in\Gamma_i'$, we have

$$f_\ell(t)=\sum_{k=2}^{\kappa_\ell}\sum_{p\in G^1_{k,\ell}}\xi(k,\ell,p)^{\alpha_m}f_{c(k,\ell,p)}\big(t+\ln\xi(k,\ell,p)\big)+z^{(\alpha_m)}_\ell(t),\tag{4.36}$$

where $z^{(\alpha_m)}_\ell(t):=e^{-\alpha_m t}\epsilon(\kappa_\ell,\ell)$. Similarly, for $\ell\in\Gamma_i^*$ and $n\ge 2$, we obtain

$$\begin{aligned}f_\ell(t)=&\sum_{k=2}^{\infty}\sum_{p\in G^1_{k,\ell}}\xi(k,\ell,p)^{\alpha_m}f_{c(k,\ell,p)}\big(t+\ln\xi(k,\ell,p)\big)+z^{(\alpha_m)}_\ell(t)\\&-\sum_{k=n+1}^{\infty}\sum_{p\in G^1_{k,\ell}}\xi(k,\ell,p)^{\alpha_m}f_{c(k,\ell,p)}\big(t+\ln\xi(k,\ell,p)\big),\end{aligned}\tag{4.37}$$

where

$$z^{(\alpha_m)}_\ell(t):=e^{-\alpha_m t}\Big(\sum_{p\in G^2_{n,\ell}}N(\lambda,-\Delta_{\mu_i|_{B_{n,\ell,p}}})+\epsilon(n,\ell)\Big).\tag{4.38}$$

Since $\lambda_1(-\Delta_{\mu_i|_{B_{1,\ell}}})>0$ for all $i\in V$ and all $\ell\in\Gamma_i$, there exists $t_0\in\mathbb{R}$ such that $f_\ell(t)=0$ for any $t<t_0$ and any $\ell\in\Gamma_i$. For each $t\in\mathbb{R}$, all $i\in V$ and all $\ell\in\Gamma_i^*$, let $n_t:=n_t(\ell)$ be the positive integer such that

$$t+\max\big\{\ln\xi(k,\ell,p):p\in G^1_{k,\ell}\big\}<t_0\qquad\text{for all } k>n_t.\tag{4.39}$$

Let $n = n_t$ in (4.37). Then for $m = 1, \dots, \gamma$, each $i \in SC_m$ and $\ell \in \Gamma_i^*$, we have

$$f_\ell(t) = \sum_{k=2}^{\infty} \sum_{p \in G^1_{k,\ell}} \xi(k,\ell,p)^{\alpha_m} f_{c(k,\ell,p)}\big(t + \ln \xi(k,\ell,p)\big) + z_\ell^{(\alpha_m)}(t), \tag{4.40}$$

where $z_\ell^{(\alpha_m)}(t)$ is obtained from that in (4.38) by replacing n with n_t. For $m = 1, \dots, \gamma$, $i \in SC_m$, $\ell \in \Gamma_i$, let $\mu_{\ell'\ell}^{(\alpha_m)}$ be the discrete measure such that

$$\mu_{\ell'\ell}^{(\alpha_m)}\big(-\ln \xi(k,\ell,p)\big) := \xi(k,\ell,p)^{\alpha_m} \text{ for } 2 \le k \le \kappa_\ell,\ p \in G^1_{k,\ell},\ \ell' = c(k,\ell,p) \in \Gamma_j \text{ and } j \in J_\ell. \tag{4.41}$$

Then (see (1.7))

$$\mu_{\ell'\ell}^{(\alpha_m)}(\mathbb{R}) = \sum_{k=2}^{\kappa_\ell} \sum_{p \in G^1_{k,\ell}} \xi(k,\ell,p)^{\alpha_m}$$

and

$$F_\ell(\alpha_m) = \sum_{\ell' \in \Gamma} \sum_{k=2}^{\kappa_\ell} \sum_{p \in G^1_{k,\ell}} \xi(k,\ell,p)^{\alpha_m}.$$

We summarize the above derivations in the following theorem.

Theorem 4.3. *Let $\mu = \sum_{i=1}^q \mu_i$ be a graph-directed self-similar measure on $\mathbb{R}$. Assume that μ satisfies (EFT) with $\{\Omega_i\}_{i=1}^q$ being an EFT-family and there exists a regular basic pair. Let $\Omega = \bigcup_{i=1}^q \Omega_i$ and Δ_μ is defined on Ω. Let $\mathbf{f}, \mathbf{M}_\alpha$ and $\mathbf{z}$ be defined as in* (1.5)*. Then $\mathbf{f}$ satisfies the vector-valued renewal equation $\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z}$.*

4.2. **Proof of Theorem 1.1 and Theorem 1.3.**

Proof of Theorem 1.1. (a) Since each $F_\ell(\alpha)$ is a strictly decreasing positive continuous function of α, $\lim_{\alpha \to \infty} F_\ell(\alpha) = 0$, and $\lim_{\alpha \to \widetilde{\alpha}_\ell^+} F_\ell(\alpha) > 1$, there exists a unique α such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is 1.

(b) Let α be the unique number in part (a). Let $\mathbf{m} := [m_{\ell'\ell}^{(\alpha)}] = [\int_0^\infty x\, d\mu_{\ell'\ell}^{(\alpha)}]$ be the moment matrix. Following the proof of [18, Theorem 1.1(b)], we need to show that some moment condition holds, and it suffices to show that $0 < \sum_{\ell' \in \Gamma} m_{\ell'\ell}^{(\alpha)} < \infty$. It is easy to check that for $\ell \in \Gamma$, $\sum_{\ell' \in \Gamma} m_{\ell'\ell}^{(\alpha)}$ takes the following values:

$$\sum_{\ell' \in \Gamma} \sum_{k=2}^{\kappa_\ell} \sum_{p \in G^1_{k,\ell}} \xi(k,\ell,p)^{\alpha} \big| \ln(\xi(k,\ell,p))\big|.$$

It follows from $\lim_{\alpha\to\widetilde{\alpha}_\ell^+} F_\ell(\alpha) > 1$ that there exists $\epsilon > 0$ such that $0 < F_\ell(\alpha-\epsilon) < \infty$. Thus

$$\begin{aligned} 0 &< \sum_{\ell'\in\Gamma}\sum_{k=2}^{\kappa_\ell}\sum_{p\in G^1_{k,\ell}} \xi(k,\ell,p)^\alpha \big|\ln(\xi(k,\ell,p))\big| \\ &= \sum_{\ell'\in\Gamma}\sum_{k=2}^{\kappa_\ell}\sum_{p\in G^1_{k,\ell}} \xi(k,\ell,p)^{\alpha-\epsilon}\xi(k,\ell,p)^{\epsilon}\big|\ln(\xi(k,\ell,p))\big| \\ &< \infty. \end{aligned}$$

The last inequality follows from the fact $\lim_{t\to 0^+} t^\epsilon \ln t = 0$. By (4.34), we have $\sum_{\ell'\in\Gamma}\mu^{(\alpha)}_{\ell'\ell}(0) = 0 < \sum_{\ell'\in\Gamma}\mu^{(\alpha)}_{\ell'\ell}(\infty)$, i.e., each column of $\mathbf{M}_\alpha$ is nondegenerate at 0. From Theorem 4.3, $\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z}$, where, by assumption, $\mathbf{z}$ is directly Riemann integrable on $\mathbb{R}$.

We first consider the case $\mathbf{M}_\alpha(\infty)$ is irreducible. It follows from the above observations and [18, Theorem 4.1] that there exist positive constants C_1 and C_2 such that $0 < C_1 \le \underline{\lim}_{t\to\infty} f_\ell(t) \le \overline{\lim}_{t\to\infty} f_\ell(t) \le C_2 < \infty$ for all $\ell \in \Gamma$. The definition in (4.28) implies that $C_1 \le \lambda^{-\alpha} N(\lambda, -\Delta_{\mu_i|_{B_{1,\ell}}}) \le C_2$, which, together with (4.23), yields $C_1\lambda^\alpha \le N(\lambda, -\Delta_{\mu|_\Omega}) \le C_2\lambda^\alpha$. Combining this, part (a), and the definition of $d_s(\mu)$, we get $d_s(\mu) = 2\alpha$.

It remains to consider the case $\mathbf{M}_\alpha(\infty)$ is reducible. As in the proof of [18, Theorem 1.1(b), Case 2], we have

$$\lim_{t\to\infty} f_\ell^{(\beta)}(t) = 0 \qquad \text{for all } \ell\in\Gamma_i,\ i\in\{1,\dots,q\} \text{ and all } \beta<\alpha.$$

Moreover, there exists some $\ell_0 \in \Gamma_i$ such that $\underline{\lim}_{t\to\infty} f_{\ell_0}^{(\alpha)}(t) > 0$. The definition of $f_\ell(t)$ implies that $N(\lambda, -\Delta_{\mu_i|_{B_{1,\ell}}}) = o(\lambda^\beta)$ for $\ell \in \Gamma_i$ and $\beta < \alpha$ and $\underline{\lim}_{\lambda\to\infty} \lambda^{-\alpha} N(\lambda, -\Delta_{\mu_i|_{B_{1,\ell_0}}}) > 0$. Thus $N(\lambda, -\Delta_\mu) = o(\lambda^\beta)$ for any $\beta < \alpha$ and $\underline{\lim}_{\lambda\to\infty} \lambda^{-\alpha} N(\lambda, -\Delta_\mu) > 0$. Hence $\overline{d}_s(-\Delta_\mu) \le 2\alpha$ and $\underline{d}_s(-\Delta_\mu) \ge 2\alpha$, which completes the proof.

□

Proof of Theorem 1.3. (a) We observe that each $F_\ell(\alpha_m)$ is a strictly decreasing positive continuous function of α_m, $\lim_{\alpha_m\to\infty} F_\ell(\alpha_m) = 0$, and $\lim_{\alpha_m\to\widetilde{\alpha}_\ell^+} F_\ell(\alpha_m) > 1$. Thus there exists a unique $\alpha = \max\{\alpha_m : m = 1,\dots,\gamma\}$ such that the spectral radius of $\mathbf{M}_{\alpha_m}(\infty)$ is 1.

(b) The proof is similar to that of Theorem 1.1(b). □

5. Strongly connected GIFSs on $\mathbb{R}$

In this section, we compute the spectral dimension of some graph-directed self-similar measures defined by strongly connected GIFSs $G = (V, E)$, which have overlaps.

5.1. **An example of strongly connected GIFS.** We first give an example of a class of graph-directed self-similar measures, defined by strongly connected GIFSs with overlaps, which satisfy (EFT).

Example 5.1. *Let* $\mu = \mu_1 + \mu_2$ *be a graph-directed self-similar measure defined by a GIFS* $G = (V, E)$ *in* (1.8) *together with a probability matrix* $(p_e)_{e\in E}$. *Then* μ *satisfies (EFT) with* $\{\Omega_1, \Omega_2\} = \{(0,1),(0,1)\}$ *being an EFT-family and there exists a regular basic pair.*

FIGURE 1. The first iteration of the GIFS defined in (5.1), where $\Omega_1 = \Omega_2 = (0,1)$. The figure is drawn with $\rho = 1/3$ and $r = 2/7$.

To prove Example 5.1, we first summarize some elementary properties. Throughout this subsection, we let $\Omega_1 = \Omega_2 := (0,1)$. Define

$$\begin{aligned} B_{1,1} &:= S_{e_1}(\Omega_1) \cup S_{e_2}(\Omega_2) = (0, r + \rho\gamma_1), \qquad & B_{1,2} &:= S_{e_3}(\Omega_1) = (\gamma_1, 1), \\ B_{1,3} &:= S_{e_5}(\Omega_1) = (0,\rho), & B_{1,4} &:= S_{e_4}(\Omega_2) = (\gamma_1, 1). \end{aligned} \tag{5.42}$$

Using (2.17) and (2.18), we see that

$$p_{e_1} + p_{e_2} + p_{e_3} = 1, \qquad p_{e_4} + p_{e_5} = 1. \tag{5.43}$$

and

$$\begin{aligned} \mu_1 &= p_{e_1}\mu_1 \circ S_{e_1}^{-1} + p_{e_2}\mu_2 \circ S_{e_2}^{-1} + p_{e_3}\mu_1 \circ S_{e_3}^{-1}, \\ \mu_2 &= p_{e_4}\mu_2 \circ S_{e_4}^{-1} + p_{e_5}\mu_1 \circ S_{e_5}^{-1}. \end{aligned} \tag{5.44}$$

Moreover $\mu = \mu_1 + \mu_2$.

Proposition 5.2. *Let* $\{S_{e_i}\}_{i=1}^5$ *be as in* (1.8). *Then* $S_{e_1e_3} = S_{e_2e_5}$.

Lemma 5.3. *Assume the hypotheses of Example 5.1. Let* $B_{1,\ell}$, $\ell = 1,2,3,4$, *defined as in* (5.42). *Then*

(a) $\mu_1|_{S_{e_1}(B_{1,1})} = p_{e_1}\mu_1|_{B_{1,1}} \circ S_{e_1}^{-1}$;
(b) $\mu_1|_{S_{e_1}(B_{1,2})} = (p_{e_1e_3} + p_{e_2e_5})p_{e_5}^{-1} \cdot \mu_2|_{B_{1,3}} \circ (S_{e_1e_3}S_{e_5}^{-1})^{-1}$;
(c) $\mu_1|_{S_{e_2}(B_{1,4})} = p_{e_2}\mu_2|_{B_{1,4}} \circ S_{e_2}^{-1}$;
(d) *for* $\ell = 1,2$, $\mu_1|_{S_{e_3}(B_{1,\ell})} = p_{e_3}\mu_1|_{B_{1,\ell}} \circ S_{e_3}^{-1}$;
(e) *for* $\ell = 3,4$, $\mu_2|_{S_{e_4}(B_{1,\ell})} = p_{e_4}\mu_2|_{B_{1,\ell}} \circ S_{e_4}^{-1}$;
(f) *for* $\ell = 1,2$ *and* $k \geq 0$, $\mu_2|_{S_{e_5e_3^k}(B_{1,\ell})} = p_{e_5e_3^k}\mu_1|_{B_{1,\ell}} \circ S_{e_5e_3^k}^{-1}$.

Proof. (a) It follows from (1.8) and (5.42) that $S_{e_1}(\Omega_1) = (0,\rho), S_{e_2}(\Omega_2) = (\rho\gamma_1, r + \rho\gamma_1)$ and $S_{e_1}(B_{1,1}) = (0, \rho r + \rho^2\gamma_1)$. Since $\rho + 2r - \rho r \leq 1$, we have $\rho r + \rho^2\gamma_1 \leq \rho\gamma_1$, and hence $S_{e_1}(B_{1,1}) \subseteq S_{e_1}(\Omega_1) \backslash S_{e_2}(\Omega_2)$. So for any $A \subseteq S_{e_1}(B_{1,1})$, i.e., $S_{e_1}^{-1}(A) \subseteq B_{1,1}$, we get $\mu_1(A) = p_{e_1}\mu_1|_{B_{1,1}} \circ S_{e_1}^{-1}(A)$.

(b) By Proposition 5.2 we have $S_{e_1e_3}(\Omega_1) = S_{e_2e_5}(\Omega_1)$. Then for any $A \subseteq S_{e_1}(B_{1,2})$, $S^{-1}_{e_1e_3}(A) = S^{-1}_{e_2e_5}(A) \subseteq \Omega_1$, and hence

$$\mu_1(A) = (p_{e_1e_3} + p_{e_2e_5})\mu_1|_{\Omega_1} \circ S^{-1}_{e_1e_3}(A). \tag{5.45}$$

For any $B \subseteq B_{1,3}$, $S^{-1}_{e_5}(B) \subseteq \Omega_1$, so $\mu_2(B) = p_{e_5}\mu_1|_{\Omega_1} \circ S^{-1}_{e_5}(B)$, and hence

$$\mu_1|_{\Omega_1} = p^{-1}_{e_5}\mu_2|_{B_{1,3}} \circ S_{e_5}. \tag{5.46}$$

Combining this with (5.45), we have $\mu_1|_{S_{e_1}(B_{1,2})} = (p_{e_1e_3} + p_{e_2e_5})p^{-1}_{e_5}\mu_2|_{B_{1,3}} \circ S_{e_5}S^{-1}_{e_1e_3}$.

(c) Using (1.8) and (5.42), we get $S_{e_2}(B_{1,4}) = ((\rho + r)\gamma_1, r + \rho\gamma_1)$. Since $\rho < (\rho + r)\gamma_1$, $S_{e_2}(B_{1,4}) \subseteq S_{e_2}(\Omega_2)\backslash S_{e_1}(\Omega_1)$, and hence $\mu_1|(A) = p_{e_2}\mu_2|_{B_{1,4}} \circ S^{-1}_{e_2}(A)$ for any $A \subseteq S_{e_2}(B_{1,4})$.

(d) and (e) follow from (5.44) and the facts $S_{e_3}(B_{1,\ell}) \subseteq S_{e_3}(\Omega_1)$, $S_{e_4}(B_{1,\ell'}) \subseteq S_{e_4}(\Omega_2)$ for $\ell = 1, 2$ and $\ell' = 3, 4$.

(f) First, we show that for $k \geq 1$ and $\ell = 1, 2$,

$$\mu_1|_{S_{e_3^k}(B_{1,\ell})} = p_{e_3^k}\mu_1|_{B_{1,\ell}} \circ S^{-1}_{e_3^k}. \tag{5.47}$$

It follows from (d) that (5.47) holds for $k = 1$. Assume that (5.47) holds for $k = m$, i.e., $\mu_1|_{S_{e_3^m}(B_{1,\ell})} = p_{e_3^m}\mu_1|_{B_{1,\ell}} \circ S^{-1}_{e_3^m}$. For $k = m + 1$, note that $S_{e_3^m}(B_{1,\ell}) \subseteq B_{1,2}$ for $m \geq 1$. Combining this with (d), we have

$$\begin{aligned}
\mu_1|_{S_{e_3^{m+1}}(B_{1,\ell})} &= \mu_1|_{S_{e_3}(S_{e_3^m}(B_{1,\ell}))} \\
&= p_{e_3}\mu_1|_{S_{e_3^m}(B_{1,\ell}))} \circ S^{-1}_{e_3} \\
&= p_{e_3^{m+1}}\mu_1|_{B_{1,\ell}} \circ S^{-1}_{e_3^{m+1}}.
\end{aligned}$$

It is obvious that $S_{e_5}(B_{1,\ell}) \subseteq S_{e_5}(\Omega_1)$ for $\ell = 1, 2$. Hence

$$\mu_2|_{S_{e_5}(B_{1,\ell})} = p_{e_5}\mu_1|_{B_{1,\ell}} \circ S^{-1}_{e_5} \quad \text{for } \ell = 1, 2. \tag{5.48}$$

Thus (f) holds for $k = 0$. Note that $S_{e_3^m}(B_{1,\ell}) \subseteq B_{1,2}$ for $m \geq 1$. Combining this with (5.48) and (5.47), we have for $k \geq 1$,

$$\begin{aligned}
\mu_2|_{S_{e_5e_3^k}(B_{1,\ell})} &= \mu_2|_{S_{e_5}(S^k_{e_3}(B_{1,\ell}))} \\
&= p_{e_5}\mu_1|_{S_{e_3^k}(B_{1,\ell})} \circ S^{-1}_{e_5} \\
&= p_{e_5e_3^k}\mu_1|_{B_{1,\ell}} \circ S^{-1}_{e_5e_3^k}.
\end{aligned}$$

This completes the proof. □

Proof of Example 5.1. Note that $\{\Omega_1, \Omega_2\}$ is invariant under $G = (V, E)$. It is obvious that any two elements of $\{B_{1,1}, B_{1,3}, B_{1,4}\}$ are measure disjoint. Let $\Gamma := \{1, 3, 4\}$ and $\mathbf{B} := \{B_{1,\ell} : \ell \in \Gamma\}$. Define $\mathbf{P}_{1,\ell} := \{B_{1,\ell}\}$ for $\ell \in \Gamma$. If for some $k \geq 2$ and $\ell \in \Gamma$, $\mathbf{P}_{k,\ell}$ (see Figure 2) is a well-defined μ-partition of $B_{1,\ell}$, then we let $\mathbf{P}^1_{k,\ell}$ and $\mathbf{P}^2_{k,\ell}$ be defined as in (2.20) with respect to $\mathbf{B}$.

For $\ell = 1$, define $\mathbf{P}_{2,1} := \{S_{e_1}(B_{1,1}), S_{e_1}(B_{1,2}), S_{e_2}(B_{1,4})\}$ (see Figure 3). Note that any two elements of $\mathbf{P}_{2,1}$ are measure disjoint. Then $\mathbf{P}_{2,1}$ is a refining μ-partition of $B_{1,1}$. It follows from Proposition 5.3(a,b,c) that $\mathbf{P}^1_{2,1} = \{S_{e_1}(B_{1,1}), S_{e_1}(B_{1,2}), S_{e_2}(B_{1,4})\}$ and $\mathbf{P}^2_{2,1} = \emptyset$. Hence condition (1) of Definition 2.1 holds for $\ell = 1$. For $k \geq 3$, define $\mathbf{P}_{k,1} = \mathbf{P}_{2,1}$. Then conditions (2) and (3) of Definition 2.1 hold for $\ell = 1$.

For $\ell = 3$, define $\mathbf{P}_{2,3} = \{S_{e_5}(B_{1,1}), S_{e_5}(B_{1,2})\}$ (see Figure 3). It is easy to see that $S_{e_5}(B_{1,1}), S_{e_5}(B_{1,2})$ are measure disjoint, and $S_{e_5}(B_{1,i}) \subseteq B_{1,3}$ for $i = 1, 2$. Then $\mathbf{P}_{2,3}$ is a refining μ-partition of $B_{1,3}$. By Proposition 5.3(f), we have $\mathbf{P}^1_{2,3} = \{S_{e_5}(B_{1,1})\}$ and $\mathbf{P}^2_{2,3} = \{S_{e_5}(B_{1,2})\}$. Hence condition (1) of Definition 2.1 holds for $\ell = 3$. For $k \geq 3$, define $\mathbf{P}_{k,3} = \mathbf{P}^1_{k-1,3} \cup \{S_{e_5 e_3^{k-2}}(B_{1,1}), S_{e_5 e_3^{k-2}}(B_{1,2})\}$, Proposition 5.3(f) implies that $\mathbf{P}^1_{k,3} = \mathbf{P}^1_{k-1,3} \cup \{S_{e_5 e_3^{k-2}}(B_{1,1})\}$ and $\mathbf{P}^2_{k,3} = \{S_{e_5 e_3^{k-2}}(B_{1,2})\}$. Then condition (2) of Definition 2.1 holds for $\ell = 3$. Since the closure of $S_{e_5 e_3^{k-2}}(B_{1,2})$ converges to a point as $k \to \infty$, we get $\lim_{k\to\infty} \mu(S_{e_5 e_3^{k-2}}(B_{1,2})) = 0$. Thus condition (3) of Definition 2.1 holds.

For $\ell = 4$, define $\mathbf{P}_{2,4} := \big\{S_{e_4}(B_{1,3}), S_{e_4}(B_{1,4})\big\}$ (see Figure 3). Note that $S_{e_4}(B_{1,3}), S_{e_4}(B_{1,4})$ are measure disjoint, and $S_{e_4}(B_{1,i}) \subseteq B_{1,4}$ for $i = 3, 4$. Then $\mathbf{P}_{2,4}$ is a refining μ-partition of $B_{1,4}$. Proposition 5.3(e) implies that $\mathbf{P}^1_{2,4} = \mathbf{P}_{2,4}$ and $\mathbf{P}^2_{2,4} = \emptyset$. Hence condition (1) of Definition 2.1 holds for $\ell = 4$. For $k > 2$, define $\mathbf{P}_{k,4} := \mathbf{P}_{2,4}$. It follows that conditions (2) and (3) of Definition 2.1 hold for $\ell = 4$. Hence the first assertion follows. Finally, the regularity of $(\{B_{1,\ell}\}, \{\mathbf{P}_{k,\ell}\}_{k\geq 1})_{\ell \in \Gamma}$ is obvious. □

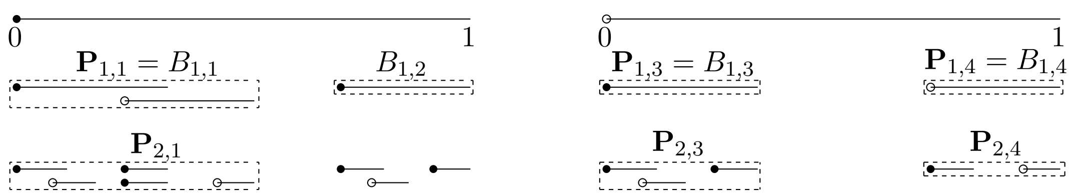

FIGURE 2. μ-partitions $\mathbf{P}_{k,\ell}$ of $B_{1,\ell}$ for the GIFS defined in (5.1). The figure is drawn with $\rho = 1/3$ and $r = 2/7$.

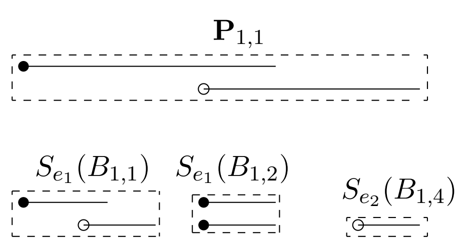

FIGURE 3. μ-partitions $\mathbf{P}_{2,1}$.

FIGURE 4. μ-partitions $\mathbf{P}_{2,3}$ and $\mathbf{P}_{2,4}$.

5.2. **Spectral dimension of μ in Example 5.1.** In this subsection, we derive the vector-valued renewal equations and compute the spectral dimension of μ defined by the strongly connected GIFS in Example 5.1.

Let $\{S_{e_i}\}_{i=1}^5$ be a GIFS in (1.8), $(p_e)_{e\in E}$ be a probability matrix, and μ be the associated graph-directed self-similar measure. For $\ell \in \Gamma := \{1,3,4\}$ and $k \geq 1$, let $B_{1,\ell}$ be defined as in (5.42), and $\mathbf{P}_{k,\ell}$ be in the proof of Example 5.1. Then μ satisfies (EFT) with $\{\Omega_1,\Omega_2\} = \{(0,1),(0,1)\}$ being an EFT-family, and $(\mathbf{B},\mathbf{P}) := (\{B_{1,\ell}\},\{\mathbf{P}_{k,\ell}\}_{k\geq 1})_{\ell\in\Gamma}$ being a regular basic pair.

In the rest of this subsection, we use the notation defined in Section 4.1. For $\ell\in\Gamma$, $i=1,2$ and $k\geq 2$, let $\mathbf{P}_{k,\ell}^i$ be defined as in (2.20). For $i=1,2$, let Γ_i and $\mathbf{B}_i$ be defined as in (4.22). Since $B_{1,1}\subseteq\Omega_1$ and $B_{1,\ell}\subseteq\Omega_2$ for $\ell=3,4$, we have $\Gamma_1=\{1\}$, $\Gamma_2=\{3,4\}$, $\mathbf{B}_1=\{B_{1,1}\}$ and $\mathbf{B}_2=\{B_{1,3},B_{1,4}\}$. Let

$$\begin{aligned} &B_{2,1,1}=S_{e_1}(B_{1,1}), && B_{2,1,2}=S_{e_1}(B_{1,2}), && B_{2,1,3}=S_{e_2}(B_{1,4}),\\ &B_{2,3,1}=S_{e_5}(B_{1,1}), && B_{2,3,2}=S_{e_5}(B_{1,2}),\\ &B_{2,4,1}=S_{e_4}(B_{1,3}), && B_{2,4,2}=S_{e_4}(B_{1,4}). \end{aligned}$$

Then

$$\begin{aligned} &\mathbf{P}_{2,1}^1=\{B_{2,1,1},B_{2,1,2},B_{2,1,3}\}, && \mathbf{P}_{2,1}^2=\emptyset,\\ &\mathbf{P}_{2,3}^1=\{B_{2,3,1}\}, && \mathbf{P}_{2,3}^2=\{B_{2,3,2}\},\\ &\mathbf{P}_{2,4}^1=\{B_{2,4,1},B_{2,4,2}\}, && \mathbf{P}_{2,4}^2=\emptyset. \end{aligned}$$

Define

$$B_{k,3,1}=\{S_{e_5e_3^{k-2}}(B_{1,1})\},\quad B_{k,3,2}=\{S_{e_5e_3^{k-2}}(B_{1,2})\}.$$

Then

$$\mathbf{P}_{k,3}^1=\{B_{k,3,1}\},\quad \mathbf{P}_{k,3}^2=\{B_{k,3,2}\}.$$

Hence $\Gamma_1'=\{1\}$, $\Gamma_1^*=\emptyset$, $\Gamma_2'=\{4\}$, $\Gamma_2^*=\{3\}$, $\kappa_1=\kappa_4=2$, $\kappa_3=\infty$, and

$$\begin{aligned} &G_{2,1}^1=\{1,2,3\}, && G_{2,1}^2=\emptyset,\\ &G_{k,3}^1=\{1\}, && G_{k,3}^2=\{2\},\ \text{for } k\geq 2,\\ &G_{2,4}^1=\{1,2\}, && G_{2,4}^2=\emptyset. \end{aligned}$$

For $\ell\in\Gamma$, let J_ℓ defined as in (4.24). Then $J_1=\{1,2\}$, $J_3=\{1\}$, $J_4=\{2\}$.

Proposition 5.4. *Let $\xi(\cdot,\cdot,\cdot)$ and $c(\cdot,\cdot,\cdot)$ defined as in Proposition 4.2. Then*

(a) $\xi(2,1,1)=p_{e_1}\rho,\ c(2,1,1)=1$;
(b) $\xi(2,1,2)=(p_{e_1e_3}+p_{e_2e_5})p_{e_5}^{-1}\cdot r,\ c(2,1,2)=3$;
(c) $\xi(2,1,3)=p_{e_4}r,\ c(2,1,3)=4$;
(d) *for* $k\geq 2$, $\xi(k,3,1)=p_{e_5e_3^{k-2}}\rho r^{k-2},\ c(k,3,1)=1$;
(e) $\xi(2,4,1)=p_{e_4}r,\ c(2,1,1)=3$;
(f) $\xi(2,4,2)=p_{e_4}r,\ c(2,1,1)=4$.

Proof. (a)–(f) Lemma 5.3 implies that

$$\begin{aligned}
&B_{1,1}\simeq_{\mu,Se_1,p_{e_1}} B_{2,1,1}, &&B_{1,3}\simeq_{\mu,S_{e_1e_3}\circ S_{e_5}^{-1},(p_{e_1e_3}+p_{e_2e_5})/p_{e_5}} B_{2,1,2},\\
&B_{1,4}\simeq_{\mu,S_{e_2},p_{e_2}} B_{2,1,3}, &&B_{1,1}\simeq_{\mu,S_{e_5e_3^{k-2}},p_{e_5e_3^{k-2}}} B_{k,3,1},\ \text{for } k\geq 2,\\
&B_{1,3}\simeq_{\mu,S_{e_4},p_{e_4}} B_{2,4,1}, &&B_{1,4}\simeq_{\mu,S_{e_4},p_{e_4}} B_{2,4,2}.
\end{aligned}$$

The results follows. □

Using Proposition 5.4 and the discussions preceding it, we can express the vector-valued renewal equations (4.29) and (4.33) precisely as

$$\begin{aligned}
f_1(t)&=(p_{e_1}\rho)^\alpha f_1(t+\ln(p_{e_1}\rho))+\big((p_{e_1e_3}+p_{e_2e_5})p_{e_5}^{-1}r\big)^\alpha f_3\big(t+\ln\big((p_{e_1e_3}+p_{e_2e_5})p_{e_5}^{-1}r\big)\big)\\
&\quad+(p_{e_2}r)^\alpha f_4(t+\ln(p_{e_2}r))+z_1^{(\alpha)}(t),\\
f_3(t)&=\sum_{k=0}^{\infty}(p_{e_5e_3^k}\rho r^k)^\alpha f_1(t+\ln(p_{e_5e_3^k}\rho r^k))+z_3^{(\alpha)}(t),\\
f_4(t)&=\sum_{\ell=3}^{4}(p_{e_4}r)^\alpha f_\ell(t+\ln(p_{e_4}r))+z_4^{(\alpha)}(t),
\end{aligned}$$

where

$$\begin{aligned}
&z_1^{(\alpha)}(t):=e^{-\alpha t}\epsilon(2,1), \qquad z_4^{(\alpha)}(t):=e^{-\alpha t}\epsilon(2,4)\\
&z_3^{(\alpha)}(t):=e^{-\alpha t}\big(N(\lambda,-\Delta_{\mu 2|B_{n_t,3,2}})+\epsilon(n_t,3)\big).
\end{aligned}$$

For $\ell,\ell'\in\Gamma$, let $\mu_{\ell'\ell}^{(\alpha)}$ be the discrete measure defined as in (4.34). Then

$$\mathbf{M}_\alpha(\infty)=\begin{pmatrix} a & b & 0\\ c & 0 & e\\ d & 0 & e\end{pmatrix},$$

where

$$\begin{aligned}
&a:=(p_{e_1}\rho)^\alpha, && b:=\frac{(p_{e_5}\rho)^\alpha}{1-(p_{e_3}r)^\alpha},\\
&c:=((p_{e_1e_3}+p_{e_2e_5})p_{e_5}^{-1}r)^\alpha, && d:=(p_{e_2}r)^\alpha, \qquad e:=(p_{e_4}r)^\alpha.
\end{aligned}$$

Proposition 5.5. *For* $\ell=1,3,4$, *let* F_ℓ *and* $\widetilde{\alpha}_\ell$ *be defined as in* (1.7). *Then* $\widetilde{\alpha}_\ell=0$ *and* $F_\ell(0)>1$ *for* $\ell=1,3,4$.

Proof. By the definition of F_ℓ, we see that

$$\begin{aligned}
F_1(\alpha)&=(p_{e_1}\rho)^\alpha+((p_{e_1e_3}+p_{e_2e_5})p_{e_5}^{-1}r)^\alpha+(p_{e_2}r)^\alpha,\\
F_3(\alpha)&=\frac{(p_{e_5}\rho)^\alpha}{1-(p_{e_3}r)^\alpha} \qquad F_4(\alpha)=2(p_{e_4}r)^\alpha.
\end{aligned}$$

It is obvious that $\widetilde{\alpha}_\ell = 0$ for $\ell = 1, 2, 3$, and $F_1(0) = 3$, $\lim_{\alpha\to 0} F_3(\alpha) = \infty$, $F_4(0) = 2$. □

It follows from the Proposition 5.5 that there exists a unique $\alpha > 0$ such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is 1. That is, α is the unique number satisfying (1.10).

Finally, we show that there exists some $\sigma > 0$ such that $z_\ell^{(\alpha)}(t) = o(e^{-\sigma t})$ as $t \to \infty$ for $\ell = 1, 3, 4$. We will first show that $N(\lambda, -\Delta_{\mu_2|_{B_{n_t,3,2}}})$ is bounded.

Proposition 5.6. *There exists $C > 0$ such that*

$$N(\lambda, -\Delta_{\mu_2|_{B_{n_t,3,2}}}) \le C.$$

Proof. Let $A \subseteq B_{n_t,3,2} = S_{e_5 e_3^{n_t-2}}(B_{1,2}) = S_{e_5 e_3^{n_t-1}}(\Omega_1)$. Then $S^{-1}_{e_5 e_3^{n_t-1}}(A) \subseteq \Omega_1$. Hence

$$\mu_1(A) = p_{e_5 e_3^{n_t-1}} \mu_1|_{\Omega_1} \circ S^{-1}_{e_5 e_3^{n_t-1}}. \tag{5.49}$$

Note that $S_{e_5} S^{-1}_{e_5 e_3^{n_t-1}}(A) \subseteq B_{1,3}$ and for any $B \subseteq B_{1,3}$, we have $S_{e_5}^{-1}(B) \subseteq \Omega_1$, and $\mu_2(B) = p_{e_5}\mu_1|_{\Omega_1} \circ S_{e_5}^{-1}$, and hence

$$\mu_1|_{\Omega_1} = p_{e_5}^{-1}\mu_2|_{B_{1,3}} \circ S_{e_5}.$$

Combining this with (5.49), we get

$$\mu_1(A) = p_{e_5 e_3^{n_t-1}} p_{e_5}^{-1} \mu_2|_{B_{1,3}} \circ S_{e_5} \circ S^{-1}_{e_5 e_3^{n_t-1}} = p_{e_3^{n_t-1}}\mu_2|_{B_{1,3}} \circ (S_{e_5 e_3^{n_t-1}} S_{e_5}^{-1})^{-1}.$$

So

$$\mu_1|_{S_{e_5 e_3^{n_t-2}}(B_{1,2})} = p_{e_3^{n_t-1}}\mu_2|_{B_{1,3}} \circ (S_{e_5 e_3^{n_t-1}} S_{e_5}^{-1})^{-1} \quad \text{on } S_{e_5 e_3^{n_t-2}}(B_{1,2}).$$

It follows that

$$\begin{aligned} N(e^t, -\Delta_{\mu_1|_{S_{e_5 e_3^{n_t-2}}(B_{1,2})}}) &= N\Big(e^t, -\Delta_{p_{e_3^{n_t-1}}\mu_2|_{B_{1,3}} \circ (S_{e_5 e_3^{n_t-1}} S_{e_5}^{-1})^{-1}}\Big) \\ &= N((p_{e_3} r)^{n_t-1} e^t, -\Delta_{\mu_2|_{B_{1,3}}}). \end{aligned}$$

(4.32) implies that $(p_{e_3} r)^{n_t-1} e^t < (p_{e_5}\rho)^{-1} e^{t_0}$. Hence

$$N(e^t, -\Delta_{\mu_1|_{S_{e_5 e_3^{n_t-2}}(B_{1,2})}}) \le N((p_{e_5}\rho)^{-1} e^{t_0}, -\Delta_{\mu_2|_{B_{1,3}}}) := C.$$

□

Proposition 5.7. *Let α be defined as in* (1.10). *Then there exists some $\sigma > 0$ such that $z_\ell^{(\alpha)}(t) = o(e^{-\sigma t})$ as $t \to \infty$ for $\ell = 1, 3, 4$.*

Proof. It follows from Proposition 5.6 that there exists some constant $C > 0$ such that $z_3^{(\alpha)}(t) \le (C + 4n_t - 4)e^{-\alpha t}$. Moreover, since $z_\ell^{(\alpha)}(t) \le 2e^{-\alpha t}$ for $\ell = 1, 4$, it suffices to show that for any $0 < \sigma < \alpha$, $n_t e^{-\alpha t} = o(e^{-\sigma t})$ as $t \to \infty$. It follows from (4.32) that $t + \ln\big(p_{e_5 e_3^{n_t-1}} \rho r^{n_t-1}\big) < t_0$, and hence $n_t \le 1 + (\ln(p_{e_3} r))^{-1}(t_0 - t - \ln(p_{e_5}\rho))$. Thus for any $0 < \sigma < \alpha$,

$$\frac{n_t e^{-\alpha t}}{e^{-\sigma t}} \le \big(1 + (\ln(p_{e_3} r))^{-1}(t_0 - \ln(p_{e_5}\rho))\big) e^{(\sigma-\alpha)t} - (\ln(p_{e_3} r))^{-1} \frac{t}{e^{(\alpha-\sigma)t}} \to 0 \qquad \text{as } t \to \infty.$$

This complete the proof. □

Proof of Corollary 1.2. We can apply Propositions 5.5 and 5.7, and Theorem1.1 to get the result. □

6. GIFSs that are not strongly connected

In this section, we compute the spectral dimension of some graph-directed self-similar measures defined by the GIFSs $G=(V,E)$ which have overlaps and are not strongly connected.

6.1. **An example of a GIFS that is not strongly connected.** We now give an example of a class of graph-directed self-similar measures that satisfy (EFT) and are defined by GIFSs with overlaps, which are not strongly connected.

Example 6.1. *Let $\mu=\mu_1+\mu_2$ be a graph-directed self-similar measure defined by a GIFS $G=(V,E)$ in (1.12) together with a probability matrix $(p_e)_{e\in E}$. Then μ satisfies (EFT) with $\Omega=\{\Omega_1,\Omega_2\}=\{(0,1),(0,1)\}$ being an EFT-family and there exists a regular basic pair.*

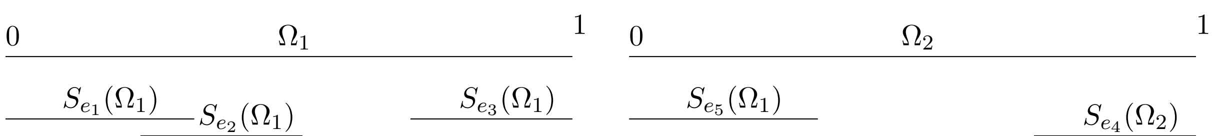

Figure 5. The first iteration of the GIFS defined in (6.1), where $\Omega_1=\Omega_2=(0,1)$. The figure is drawn with $\rho=1/3$ and $r=2/7$.

To prove Example 6.1, we first summarize some elementary properties.

Throughout this subsection, we let $\Omega_1=\Omega_2:=(0,1)$. To simplify notation we let

$$\gamma_k:=1-r^k,\qquad k\ge 0. \tag{6.50}$$

Define

$$\begin{aligned} &B_{1,1}:=S_{e_1}(\Omega_1)\cup S_{e_2}(\Omega_1)=(0,r+\rho\gamma_1), && B_{1,2}:=S_{e_3}(\Omega_1)=(\gamma_1,1),\\ &B_{1,3}:=S_{e_5}(\Omega_1)=(0,\rho), && B_{1,4}:=S_{e_4}(\Omega_2)=(\gamma_1,1). \end{aligned} \tag{6.51}$$

We denote

$$W(k)=\{e_2^j e_1 e_3^{k-j}:j=0,1,\dots,k\},\quad k\ge 0. \tag{6.52}$$

We remark that for $k\ge 0$,

$$p_{e_1}p_{e_3}^{k+1}+p_{e_2}w(k)=w(k+1). \tag{6.53}$$

Using (2.17) and (2.18), we see that

$$p_{e_1}+p_{e_2}+p_{e_3}=1,\qquad p_{e_4}+p_{e_5}=1, \tag{6.54}$$

and

$$\begin{aligned} \mu_1&=p_{e_1}\mu_1\circ S_{e_1}^{-1}+p_{e_2}\mu_1\circ S_{e_2}^{-1}+p_{e_3}\mu_1\circ S_{e_3}^{-1},\\ \mu_2&=p_{e_4}\mu_2\circ S_{e_4}^{-1}+p_{e_5}\mu_1\circ S_{e_5}^{-1}. \end{aligned} \tag{6.55}$$

Moreover, $\mu = \mu_1 + \mu_2$.

Proposition 6.2(a) below implies that all multi-indices in $W(k)$ correspond to the same vertex.

Proposition 6.2. *Let* $\{S_{e_i}\}_{i=1}^5$ *be as in* (1.12). *Then*

(a) $S_{e_1e_3}(\Omega_1) = S_{e_2e_1}(\Omega_1)$. *Moreover, for any* $\boldsymbol{e}, \boldsymbol{e}' \in W(k), S_{\boldsymbol{e}} = S_{\boldsymbol{e}'}$;
(b) *for* $k \ge 1$, $S_{e_2^ke_1}(\Omega_1) = (\rho\gamma_k, \rho)$;
(c) *for* $k \ge 1$, $S_{e_2^k}(\Omega_1) = (\rho\gamma_k, r^k + \rho\gamma_k)$;
(d) *for* $k \ge 1$, $S_{e_2^ke_3}(\Omega_1) = (r^k\gamma_1 + \rho\gamma_k, r^k + \rho\gamma_k)$;
(e) *for* $k \ge 1$, $S_{e_1}(\Omega_1) \cap S_{e_2^k}(\Omega_1) = S_{e_2^ke_1}(\Omega_1)$.

Proof. (a) can be proved directly by a calculation and (b)–(e) can be proved by induction; we omit the details. □

For any $k \ge 0$, $w(k)$ denotes the sum of probability weights of all multi-indices in $W(k)$. Part (d) of the following lemma explains the meaning of the factor $w(k)$.

Lemma 6.3. *Assume the hypotheses of Example 6.1. Then*

(a) *for* $\ell = 1, 2$, $\mu_1|_{S_{e_3}(B_{1,\ell})} = p_{e_3}\mu_1|_{B_{1,\ell}} \circ S_{e_3}^{-1}$;
(b) *for* $\ell = 3, 4$, $\mu_2|_{S_{e_4}(B_{1,\ell})} = p_{e_4}\mu_2|_{B_{1,\ell}} \circ S_{e_4}^{-1}$;
(c) *for* $\ell = 1, 2$, $\mu_2|_{S_{e_5}(B_{1,\ell})} = p_{e_5}\mu_1|_{B_{1,\ell}} \circ S_{e_5}^{-1}$;
(d) *for* $k \ge 0$, $\mu_1|_{S_{e_2^ke_1}(B_{1,1})} = w(k)\mu_1|_{B_{1,1}} \circ S_{e_2^ke_1}^{-1}$;
(e) *for* $k \ge 1$, $\mu_1|_{S_{e_2^k}(B_{1,1})} = w(k-1)\mu_1|_{B_{1,2}} \circ S_{e_2^{k-1}e_1}^{-1} + p_{e_2}^k\mu_1|_{B_{1,1}} \circ S_{e_2^k}^{-1}$;
(f) *for* $k \ge 1$, $\mu_1|_{S_{e_2^k}(B_{1,2})} = p_{e_2}^k\mu_1|_{B_{1,2}} \circ S_{e_2^k}^{-1}$.

Proof. (a,b,c) follows from (6.55) and $S_{e_i}(B_{1,\ell}) \subseteq S_{e_i}(\Omega_1)$, $S_{e_4}(B_{1,\ell'}) \subseteq S_{e_4}(\Omega_2)$, for $i = 3, 5$, $\ell = 1, 2$ and $\ell' = 3, 4$.

(d)–(f) can be proved directly by induction, we only prove (d) as an example. It follows from (1.12) and (6.51) that $S_{e_1}(\Omega_1) = (0, \rho)$, $S_{e_2}(\Omega_2) = (\rho\gamma_1, r + \rho\gamma_1)$, and $S_{e_1}(B_{1,1}) = (0, \rho r + \rho^2\gamma_1)$. Since $\rho + 2r - \rho r \le 1$, we have $\rho r + \rho^2\gamma_1 \le \rho\gamma_1$, and hence $S_{e_1}(B_{1,1}) \subseteq S_{e_1}(\Omega_1)\backslash S_{e_2}(\Omega_2)$. So for any $A \subseteq S_{e_1}(B_{1,1})$, i.e., $S_{e_1}^{-1}(A) \subseteq B_{1,1}$, we get $\mu_1(A) = p_{e_1}\mu_1|_{B_{1,1}} \circ S_{e_1}^{-1}(A)$. Assume that the stated equality holds for $k = m$, i.e., $\mu_1|_{S_{e_2^me_1}(B_{1,1})} = w(m)\mu_1|_{B_{1,1}} \circ S_{e_2^me_1}^{-1}$. For $k = m+1$, by Proposition 6.2, we have $S_{e_2^{m+1}e_1}(B_{1,1}) = S_{e_1e_3^{m+1}}(B_{1,1})$. Then $S_{e_1}^{-1}(A) \subseteq S_{e_3^{m+1}}(B_{1,1})$ and $S_{e_2}^{-1}(A) \subseteq S_{e_2^me_1}(B_{1,1})$ for any $A \subseteq S_{e_2^{m+1}e_1}(B_{1,1})$. It follows that $\mu_1(S_{e_1}^{-1}(A)) = p_{e_3}^{m+1}\mu_1|_{B_{1,1}} \circ S_{e_3^{m+1}}^{-1}(S_{e_1}^{-1}(A))$ and $\mu_1(S_{e_2}^{-1}(A)) = w(m)\mu_1|_{B_{1,1}} \circ S_{e_2^me_1}^{-1}(S_{e_2}^{-1}(A))$. Thus,

$$\begin{aligned}
\mu_1(A) &= p_{e_1}\mu_1 \circ S_{e_1}^{-1}(A) + p_{e_2}\mu_1 \circ S_{e_2}^{-1}(A) \\
&= p_{e_1}p_{e_3}^{m+1}\mu_1|_{B_{1,1}} \circ S_{e_3^{m+1}}^{-1}(S_{e_1}^{-1}(A)) + p_{e_2}w(m)\mu_1|_{B_{1,1}} \circ S_{e_2^me_1}^{-1}(S_{e_2}^{-1}(A)) \\
&= (p_{e_1}p_{e_3}^{m+1} + p_{e_2}w(m))\mu_1|_{B_{1,1}} \circ S_{e_2^{m+1}e_1}^{-1}(A) \\
&= w(m+1)\mu_1|_{B_{1,1}} \circ S_{e_2^{m+1}e_1}^{-1}(A).
\end{aligned}$$

The last equality follows from (6.53). This proves part (d). □

Proof of Example 6.1. Note that $\{\Omega_1, \Omega_2\}$ is invariant under $G = (V, E)$. It is obvious that any elements of $\{B_{1,i}\}_{i=1}^4$ are measure disjoint. Let $\Gamma := \{1, 2, 3, 4\}$ and $\mathbf{B} := \{B_{1,\ell} : \ell \in \Gamma\}$. Define $\mathbf{P}_{1,\ell} := \{B_{1,\ell}\}$ for $\ell \in \Gamma$. If for some $k \geq 2$ and $\ell \in \Gamma$, $\mathbf{P}_{k,\ell}$(see Figure 6) is a well-defined μ-partition of $B_{1,\ell}$, then we let $\mathbf{P}^1_{k,\ell}$ and $\mathbf{P}^2_{k,\ell}$ be defined as in (2.20) with respect to $\mathbf{B}$.

For $\ell = 1$, define $\mathbf{P}_{2,1} := \{S_{e_1}(B_{1,1}), S_{e_2}(B_{1,1}), S_{e_2}(B_{1,2})\}$ (see Figure 7). Lemma 6.3(d,e,f) implies that $\mathbf{P}^1_{2,1} = \{S_{e_1}(B_{1,1}), S_{e_2}(B_{1,2})\}$ and $\mathbf{P}^2_{2,1} = \{S_{e_2}(B_{1,1})\}$. Hence condition (1) of (EFT) holds for $\ell = 1$. For $k \geq 3$, define $\mathbf{P}_{k,1} := \mathbf{P}_{k-1,1} \cup \{S_{e_2^{k-2}e_1}(B_{1,1}), S_{e_2^{k-1}}(B_{1,1}), S_{e_2^{k-1}}(B_{1,2})\}$. Lemma 6.3(d,e,f) again implies that $\mathbf{P}^1_{k,1} = \mathbf{P}^1_{k-1,1} \cup \{S_{e_2^{k-2}e_1}(B_{1,1}), S_{e_2^{k-1}}(B_{1,2})\}$ and $\mathbf{P}^2_{k,1} = \{S_{e_2^{k-1}}(B_{1,1})\}$. Then condition (2) of (EFT) holds for $\ell = 1$. Since the closure of $S_{e_2^k}(B_{1,1})$ converges to a point as $k \to \infty$, we have $\lim_{k\to\infty} \mu(S_{e_2^k}(B_{1,1})) = 0$. Thus condition (3) of (EFT) holds.

For $\ell = 2$, define $\mathbf{P}_{2,2} := \big\{S_{e_3}(B_{1,1}), S_{e_3}(B_{1,2})\big\}$ (see Figure 7). Note that $S_{e_3}(B_{1,1}), S_{e_3}(B_{1,2})$ are measure disjoint, and $S_{e_3}(B_{1,i}) \subseteq B_{1,2}$ for $i = 1, 2$. Hence $\mathbf{P}_{2,2}$ is a refining μ-partition of $B_{1,2}$. It follows from Lemma 6.3(a) that $\mathbf{P}^1_{2,2} = \mathbf{P}_{2,2}$ and $\mathbf{P}^2_{2,2} = \emptyset$. Thus condition (1) of (EFT) holds for $\ell = 2$. For $k > 2$, define $\mathbf{P}_{k,2} := \mathbf{P}_{2,2}$. It follows that conditions (2) and (3) of (EFT) hold for $\ell = 2$.

For $\ell = 3$, define $\mathbf{P}_{2,3} := \big\{S_{e_5}(B_{1,1}), S_{e_5}(B_{1,2})\big\}$ (see Figure 8). Note that $S_{e_5}(B_{1,1}), S_{e_5}(B_{1,2})$ are measure disjoint, and $S_{e_5}(B_{1,i}) \subseteq B_{1,3}$ for $i = 1, 2$. Hence $\mathbf{P}_{2,3}$ is a refining μ-partition of $B_{1,3}$. It follows from Lemma 6.3(c) that $\mathbf{P}^1_{2,3} = \mathbf{P}_{2,3}$ and $\mathbf{P}^2_{2,3} = \emptyset$. Thus condition (1) of (EFT) holds for $\ell = 3$. For $k > 2$, define $\mathbf{P}_{k,3} := \mathbf{P}_{2,3}$. It follows that conditions (2) and (3) of (EFT) hold for $\ell = 3$.

For $\ell = 4$, define $\mathbf{P}_{2,4} := \big\{S_{e_4}(B_{1,3}), S_{e_4}(B_{1,4})\big\}$ (see Figure 8). Note that $S_{e_4}(B_{1,3}), S_{e_4}(B_{1,4})$ are measure disjoint, and $S_{e_4}(B_{1,i}) \subseteq B_{1,4}$ for $i = 3, 4$. Then $\mathbf{P}_{2,4}$ is a refining μ-partition of $B_{1,4}$. It follows from Lemma 6.3(b) that $\mathbf{P}^1_{2,4} = \mathbf{P}_{2,4}$ and $\mathbf{P}^2_{2,4} = \emptyset$. Thus condition (1) of (EFT) holds for $\ell = 4$. For $k > 2$, define $\mathbf{P}_{k,4} := \mathbf{P}_{2,4}$. It follows that conditions (2) and (3) of (EFT) hold for $\ell = 4$. □

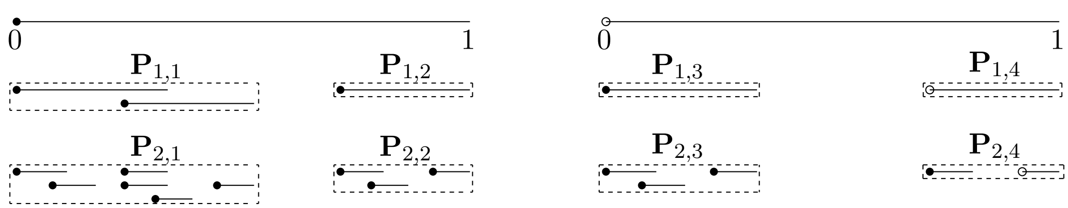

FIGURE 6. μ-partitions $\mathbf{P}_{k,\ell}$ of $B_{1,\ell}$ for the GIFS defined in (5.1). The figure is drawn with $\rho = 1/3$ and $r = 2/7$.

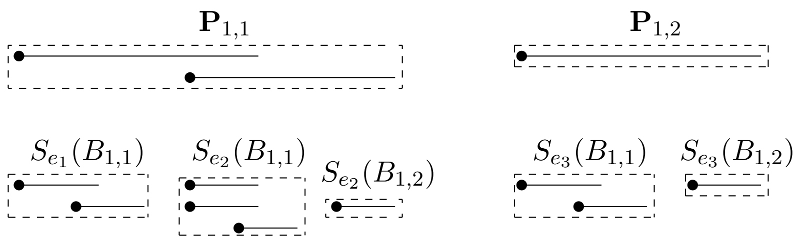

FIGURE 7. μ-partitions $\mathbf{P}_{2,1}$ and $\mathbf{P}_{2,2}$.

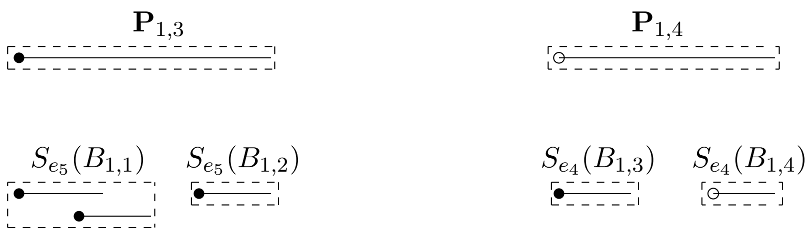

FIGURE 8. μ-partitions $\mathbf{P}_{2,3}$ and $\mathbf{P}_{2,4}$.

6.2. **Spectral dimension of μ in Example 6.1.** In this subsection, we derive the vector-valued renewal equations and compute the spectral dimension of μ defined by the GIFS in Example 6.1.

Let $\{S_{e_i}\}_{i=1}^5$ be a GIFS in (1.12), $(p_e)_{e\in E}$ be a probability matrix, and μ be the associated graph-directed self-similar measure. For $\ell \in \Gamma := \{1,2,3,4\}$ and $k \geq 1$, let $B_{1,\ell}$ be defined as in (6.51), and $\mathbf{P}_{k,\ell}$ be in the proof of Example 6.1. Then μ satisfies (EFT) with $\{\Omega_1, \Omega_2\} = \{(0,1),(0,1)\}$ being an EFT-family, and $(\mathbf{B},\mathbf{P}) := (\{B_{1,\ell}\}, \{\mathbf{P}_{k,\ell}\}_{k\geq 1})_{\ell\in\Gamma}$ being a regular basic pair.

In the rest of this subsection, we use the notation defined in Section 4.1. For $\ell \in \Gamma$, $i = 1,2$ and $k \geq 2$, let $\mathbf{P}^i_{k,\ell}$ be defined as in (2.20). For $i = 1,2$, let Γ_i and $\mathbf{B}_i$ be defined as in (4.22). Since $B_{1,\ell} \subseteq \Omega_1$ and $B_{1,\ell'} \subseteq \Omega_2$ for $\ell = 1,2$, $\ell' = 3,4$, we have $\Gamma_1 = \{1,2\}$, $\Gamma_2 = \{3,4\}$, $\mathbf{B}_1 = \{B_{1,1}, B_{1,2}\}$ and $\mathbf{B}_2 = \{B_{1,3}, B_{1,4}\}$. For $k \geq 2$, let $B_{k,1,1} = S_{e_2^{k-2}e_1}(B_{1,1}), B_{k,1,2} = S_{e_2^{k-1}}(B_{1,1}), B_{k,1,3} = S_{e_2^{k-1}}(B_{1,2})$, and let

$$B_{2,2,p} = S_{e_3}(B_{1,p}), \qquad B_{2,3,p} = S_{e_5}(B_{1,p}), \text{ for } p = 1,2,$$
$$B_{2,4,1} = S_{e_4}(B_{1,3}), \qquad B_{2,4,2} = S_{e_4}(B_{1,4}).$$

Then

$$\mathbf{P}^1_{k,1} = \{B_{k,1,1}, B_{k,1,3}\}, \qquad \mathbf{P}^2_{k,1} = \{B_{k,1,2}\}, \text{ for } k \geq 2$$
$$\mathbf{P}^1_{2,\ell} = \{B_{2,\ell,1}, B_{2,\ell,2}\}, \qquad \mathbf{P}^2_{2,\ell} = \emptyset, \text{ for } \ell = 2,3,4.$$

Hence $\Gamma'_1 = \{2\}$, $\Gamma^*_1 = \{1\}$, $\Gamma'_2 = \{3,4\}$, $\Gamma^*_2 = \emptyset$, $\kappa_2 = \kappa_3 = \kappa_4 = 2$, $\kappa_1 = \infty$, and

$$G^1_{k,1} = \{1,3\}, \qquad G^2_{k,1} = \{2\}, \text{ for } k \geq 2,$$
$$G^1_{2,\ell} = \{1,2\}, \qquad G^2_{2,\ell} = \emptyset \text{ for } \ell = 2,3,4.$$

For $\ell \in \Gamma$, let J_ℓ defined as in (4.24). Then $J_1 = J_2 = J_3 = \{1\}, J_4 = \{2\}$.

Proposition 6.4. *Let $\xi(\cdot,\cdot,\cdot)$ and $c(\cdot,\cdot,\cdot)$ defined as in Proposition 4.2. Then*

(a) *for $k \ge 2$, $\xi(k,1,1) = w(k-2)\rho r^{k-2}$, $c(k,1,1) = 1$, $\xi(k,1,3) = (p_{e_2}r)^{k-1}$, and $c(k,1,3) = 2$;*
(b) *for $p = 1,2$, $\xi(2,2,p) = p_{e_3}r$, $c(2,2,p) = p$;*
(c) *for $p = 1,2$, $\xi(2,3,p) = p_{e_5}\rho, c(2,3,p) = p$;*
(d) *for $p = 1,2$, $\xi(2,4,p) = p_{e_4}r, c(2,4,p) = p+2$.*

Proof. (a)–(f) Lemma 6.3 implies that for $k \ge 2$,

$$B_{1,1} \simeq_{\mu,S_{e_2^{k-2}e_1},w(k-2)} B_{k,1,1}, \quad B_{1,2} \simeq_{\mu,S_{e_2^{k-1}},p_{e_2}^{k-1}} B_{k,1,3},$$

and for $p = 1,2$,

$$B_{1,p} \simeq_{\mu,S_{e_3},p_{e_3}} B_{2,2,p}, \quad B_{1,p} \simeq_{\mu,S_{e_5},p_{e_5}} B_{2,3,p}, \quad B_{1,p+2} \simeq_{\mu,S_{e_4},p_{e_4}} B_{2,4,p}.$$

The results follows. □

It is obvious that G has two strongly connected components, i.e., $\gamma = 2$. For $m = 1,2$, let SC_m be defined as in (1.11). Then $SC_1 = \{1\}$ and $SC_2 = \{2\}$.

Using Proposition 6.4 and the discussions preceding it, we can express the vector-valued renewal equations (4.36) and (4.40) precisely as

$$f_1(t) = \sum_{k=0}^{\infty}(w(k)\rho r^k)^{\alpha_1} f_1(t + \ln(w(k)\rho r^k)) + \sum_{k=1}^{\infty}(p_{e_2}r)^{k\alpha_1} f_2(t + \ln(p_{e_2}r)^k) + z_1^{(\alpha_1)}(t),$$

$$f_2(t) = (p_{e_3}r)^{\alpha_1}\sum_{\ell=1}^{2} f_\ell(t + \ln(p_{e_3}r)) + z_2^{(\alpha_1)}(t),$$

$$f_3(t) = (p_{e_5}\rho)^{\alpha_2}\sum_{\ell=1}^{2} f_3(t + \ln(p_{e_5}\rho)) + z_3^{(\alpha_2)}(t),$$

$$f_4(t) = (p_{e_4}r)^{\alpha_2}\sum_{\ell=3}^{4} f_4(t + \ln(p_{e_4}r)) + z_4^{(\alpha_2)}(t),$$

where

$$z_1^{(\alpha_1)}(t) = e^{-\alpha_1 t}N(\lambda, -\Delta_{\mu_1|_{B_{n_t,1,2}}}) + e^{-\alpha_1 t}\epsilon(n_t, 1),$$
$$z_2^{(\alpha_1)}(t) = e^{-\alpha_1 t}\epsilon(2,2), \qquad z_\ell^{(\alpha_2)}(t) = e^{-\alpha_2 t}\epsilon(2,\ell) \quad \text{for } \ell = 3,4.$$

For $\ell', \ell \in \Gamma$ and $m = 1,2$, let $\mu_{\ell'\ell}^{(\alpha_m)}$ be the discrete measure defined as in (4.41). Then

$$\mathbf{M}_\alpha(\infty) = \begin{pmatrix} \sum_{k=0}^{\infty}(w(k)\rho r^k)^{\alpha_1} & (p_{e_3}r)^{\alpha_1} & (p_{e_5}\rho)^{\alpha_2} & 0 \\ \frac{(p_{e_2}r)^{\alpha_1}}{1-(p_{e_2}r)^{\alpha_1}} & (p_{e_3}r)^{\alpha_1} & (p_{e_5}\rho)^{\alpha_2} & 0 \\ 0 & 0 & 0 & (p_{e_4}r)^{\alpha_2} \\ 0 & 0 & 0 & (p_{e_4}r)^{\alpha_2} \end{pmatrix}$$

Proposition 6.5. *For $\ell = 1,2,3,4$, let F_ℓ and $\tilde{\alpha}_\ell$ be defined as in (1.7). Then $\tilde{\alpha}_\ell = 0$ and $F_\ell > 1$ for $\ell = 1,2,3,4$.*

Proof. By the definition of F_ℓ, we see that

$$F_1(\alpha_1) = \sum_{k=0}^{\infty}(w(k)\rho r^k)^{\alpha_1} + \frac{(p_{e_2}r)^{\alpha_1}}{1-(p_{e_2}r)^{\alpha_1}}, \qquad F_2(\alpha_1) = 2(p_{e_3}r)^{\alpha_1},$$
$$F_3(\alpha_2) = 2(p_{e_5}\rho)^{\alpha_2}, \qquad F_4(\alpha_2) = 2(p_{e_4}r)^{\alpha_2}.$$

For any $\alpha_1 > 0$, since $\sum_{k=0}^{\infty}(w(k)\rho r^k)^{\alpha_1}$ converges, we get $F_1(\alpha_1) < \infty$. It follows from (1.7) that $\tilde{\alpha}_1 = 0$ and $F_1(0) = \infty$. It is obvious that $\tilde{\alpha}_\ell = 0$ and $F_\ell(0) = 2$ for $\ell = 2, 3, 4$. □

Let $|I_4 - \mathbf{M}_\alpha(\infty)| = 0$, where I_4 is the identity $(4,4)$-matrix. Then

$$\big(1-(p_{e_4}r)^{\alpha_2}\big)\Big(1-(p_{e_2}r)^{\alpha_1}-(p_{e_3}r)^{\alpha_1}-\big(1-(p_{e_2}r)^{\alpha_1}\big)\big(1-(p_{e_3}r)^{\alpha_1}\big)\sum_{k=0}^{\infty}(w(k)\rho r^k)^{\alpha_1}\Big) = 0,$$

and hence $\alpha_1 > \alpha_2 = 0$. It follows from the Proposition 6.5 that there exists a unique $\alpha = \max\{\alpha_1, \alpha_2\} = \alpha_1 > 0$ such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is 1. More precisely, α is the unique number satisfying (1.14).

Finally, we show that there exists some $\sigma > 0$ such that $z_\ell^{(\alpha)}(t) = o(e^{-\sigma t})$ as $t \to \infty$ for $\ell = 1, 2, 3, 4$. We will first show that $N(\lambda, -\Delta_{\mu_1|_{B_{n_t,1,2}}})$ is bounded. The proof is the same as that of [20, Proposition 5.3].

Proposition 6.6. *There exists $C > 0$ such that $N(\lambda, -\Delta_{\mu_1|_{B_{n_t,1,2}}}) \le C$.*

Proof. Let $A \subseteq B_{n_t,1,2} = S_{e_2^{n_t-1}}(B_{1,1})$. Then $S^{-1}_{e_2^{n_t-2}}(A) \subseteq S_{e_2}(B_{1,1}) = S_{e_2e_1}(\Omega_1) \cup S_{e_2e_2}(\Omega_1) = S_{e_1e_3}(\Omega_1) \cup S_{e_2e_2}(\Omega_1)$. Thus

$$\begin{aligned}\mu_1|_{S^{-1}_{e_2^{n_t-2}}(A)} &= p_{e_1}\mu_1|_{B_{1,2}} \circ S_{e_1}^{-1}(S^{-1}_{e_2^{n_t-2}}(A)) + p_{e_2}\mu_1|_{B_{1,1}} \circ S_{e_2}^{-1}(S^{-1}_{e_2^{n_t-2}}(A))\\ &= p_{e_1}\mu_1|_{B_{1,2}} \circ S^{-1}_{e_2^{n_t-2}e_1}(A) + p_{e_2}\mu_1|_{B_{1,1}} \circ S^{-1}_{e_2^{n_t-1}}(A). \end{aligned} \tag{6.56}$$

Multiplying both sides of (6.56) by $w(n_t-2)p_{e_1}^{-1}$, using (1.13) and Lemma 6.3(e), we have

$$\begin{aligned}&w(n_t-2)p_{e_1}^{-1}\mu_1(S^{-1}_{e_2^{n_t-2}}(A))\\ =\;& w(n_t-2)\mu_1|_{B_{1,2}} \circ S^{-1}_{e_2^{n_t-2}e_1}(A) + w(n_t-2)p_{e_2}p_{e_1}^{-1}\mu_1|_{B_{1,1}} \circ S^{-1}_{e_2^{n_t-1}}(A)\\ \ge\;& w(n_t-2)\mu_1|_{B_{1,2}} \circ S^{-1}_{e_2^{n_t-2}e_1}(A) + p_{e_2}^{n_t-1}\mu_1|_{B_{1,1}} \circ S^{-1}_{e_2^{n_t-1}}(A)\\ =\;& \mu_1(A).\end{aligned}$$

Thus $\mu_1|_{S_{e_2}^{n_t-1}}(B_{1,1}) \le w(n_t-2)p_{e_1}^{-1}\mu_1 \circ S^{-1}_{e_2^{n_t-2}}$ on $S_{e_2^{n_t-1}}(B_{1,1})$. Combining this with Proposition 3.4, we have

$$N(e^t, -\Delta_{\mu_1|_{S_{e_2^{n_t-1}}(B_{1,1})}}) \le N(w(n_t-2)p_{e_1}^{-1}r^{n_t-2}e^t, -\Delta_{\mu_1|_{S_{e_2}(B_{1,1})}}) \tag{6.57}$$

It follows from (4.39) that $t + \ln(w(n_t-1)\rho r^{n_t-1}) < t_0$. Hence we have $w(n_t-2)\rho r^{n_t-1}e^t \le w(n_t-1)\rho r^{n_t-1}e^t < e^{t_0}$, and thus $w(n_t-2)p_{e_1}^{-1}r^{n_t-2}e^t \le (\rho r p_{e_1})^{-1}e^{t_0}$. Combining this with (6.57), we get

$$N(e^t, -\Delta_{\mu_1|_{S_{e_2^{n_t-1}}(B_{1,1})}}) \le N((\rho r p_{e_1})^{-1}e^{t_0}, -\Delta_{\mu_1|_{S_{e_2}(B_{1,1})}}) := C.$$

This completes the proof. ☐

Proposition 6.7. *Let α be defined as in* (1.14). *Then there exists some $\sigma > 0$ such that $z_\ell^{(\alpha)}(t) = o(e^{-\sigma t})$ as $t \to \infty$ for $\ell = 1, 2, 3, 4$.*

Proof. It follows from Proposition 6.6 that there exists some constant $C > 0$ such that $z_1^{(\alpha)}(t) \le (C + 4n_t - 4)e^{-\alpha t}$. Moreover, since $z_\ell^{(\alpha)}(t) \le 2e^{-\alpha t}$ for $\ell = 2, 3, 4$, it suffices to show that for any $0 < \sigma < \alpha$, $n_t e^{-\alpha t} = o(e^{-\sigma t})$ as $t \to \infty$. It follows from (4.39) that $t + \ln(p_{e_2} r)^{n_t} < t_0$, and hence $n_t \le (\ln(p_{e_2} r))^{-1}(t_0 - t)$. Consequently for any $0 < \sigma < \alpha$,

$$\frac{n_t e^{-\alpha t}}{e^{-\sigma t}} \le \frac{t_0 - t}{\ln(p_{e_2} r) \cdot e^{(\alpha-\sigma)t}} \to 0 \qquad \text{as } t \to \infty.$$

This complete the proof. ☐

Proof of Corollary 1.4. We can apply Propositions 6.5 and 6.7, and Theorem1.3 to get the result. ☐

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