ORTHOGONAL POLYNOMIALS DEFINED BY SELF-SIMILAR MEASURES WITH OVERLAPS

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Abstract. We study orthogonal polynomials with respect to self-similar measures, focusing on the class of infinite Bernoulli convolutions, which are defined by iterated function systems with overlaps, especially those defined by the Pisot, Garsia, and Salem numbers. By using an algorithm of Mantica, we obtain graphs of the coefficients of the 3-term recursion relation defining the orthogonal polynomials. We use these graphs to predict whether the singular infinite Bernoulli convolutions belong to the Nevai class. Based on our numerical results, we conjecture that all infinite Bernoulli Convolutions with contraction ratios greater than or equal to 1/2 belong to Nevai's class, regardless of the probability weights assigned to the self-similar measures.

1. Introduction

Orthogonal polynomials with respect to fractal measures have been studied by authors including Mantica [9,10], Heilman *et al* [7], Krüger and Simon [8], and Bandt and Peña [1]. In [7], the self-similar measures studied satisfy the open set condition. As the classical theory [6,22] to compute the orthogonal polynomials can be applied to a large class of general measures and algorithms formulated by Mantica to compute the coefficients of the 3-term recurrence relation (see (2.2)) can be applied to general self-similar measures, we use them to study self-similar measures defined by iterated function systems with overlaps. This paper can be consider a continuation of the explorations in [7].

Let μ be a positive measure on $[0,1]$. We say that a measure μ is in Nevai's class (see [12,14,18]) if the recurrence coefficients A_n and r_n in (2.2) satisfy

$$\lim_{n\to\infty} A_n = \frac{1}{2} \qquad \text{and} \qquad \lim_{n\to\infty} r_n = \frac{1}{4}.$$

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Rahmanov [19, 20] proved that if $\mu' > 0$ for Lebesgue a.e. on $[0, 1]$, then μ belongs to Nevai's class. On the other hand, Lubinsky [13] constructed singular continuous measures that belong to Nevai's class. These results seem to suggest the relationship between absolute continuity being in Nevai's class, and motivated our study of infinite Bernoulli convolutions with overlaps. They are self-similar measures defined by an iterated function system (IFS) with two similitudes $\{S_1, S_2\}$ together with probability weights $w_1 = w_2 = 1/2$ as follows:

$$\begin{aligned} &\mu = \frac{1}{2}\mu \circ S_1^{-1} + \frac{1}{2}\mu \circ S_2^{-1}, \qquad \text{where} \\ &S_1(x) = \rho x, \qquad S_2(x) = \rho x + (1-\rho), \qquad 1/2 \le \rho < 1. \end{aligned} \tag{1.1}$$

We are particularly interested in several families of algebraic integers. If ρ^{-1} is a Pisot number (i.e., an algebraic integer > 1 whose algebraic conjugates lie inside the unit circle), then μ is singular [3]. This is the only class of numbers for which μ is known to be singular. Wintner [23] proved that if $\rho = 1/\sqrt[n]{2}$ for some $n \in \mathbb{N}$, then μ is absolutely continuous. Garsia [5] constructed a class of algebraic integers for which μ is absolutely continuous. These are the only numbers for which the self-similar measures are explicitly known to be absolute continuous, and by Rahmanov's theorem, they are in Nevai's class. Solomyak [21] proved that for Lebesgue a.e. $\rho \in (1/2, 1)$, μ is absolutely continuous. Salem numbers are algebraic integers greater than 1 whose algebraic conjugates have moduli less than or equal to 1, with at least one of them being 1. They are of interest in dynamical systems and number theory (see [2, 15] and references therein) as well as fractals (see, e.g., [4]). Although the definition of a Salem number is close to that of a Pisot numbers, Salem numbers, as well as the associated self-similar measures, are much less understood. It is not known whether the self-similar measure associated with a Salem number is absolutely continuous or singular.

We also study weighted infinite Bernoulli convolutions, namely,

$$\begin{aligned} &\mu = w_1\mu \circ S_1^{-1} + w_2\mu \circ S_2^{-1}, \qquad \text{where } w_1 > 0,\ w_2 > 0,\ w_1 + w_2 = 1, \\ &S_1(x) = \rho x, \qquad S_2(x) = \rho x + (1-\rho), \qquad \text{and} \qquad 1/2 \le \rho < 1. \end{aligned} \tag{1.2}$$

As the weights become more biased, the corresponding measure μ tend to be more singular (in the sense that the Hausdorff dimension of μ becomes smaller). However, our numerical experiments suggest, rather unexpectedly, that μ continues to belong to Nevai's class. Unfortunately, we are not able to prove this.

This paper is organized as follows. In Section 2, we establish some properties of the recurrence coefficients for general measures. In Section 3, we graph the orthogonal polynomials defined by the infinite Bernoulli convolution associated with the golden ratio and the 3-fold convolution of the Cantor measure, both being self-similar measures with overlaps that have been studied extensively. In Section 4 we derive explicit formulas for computing the recurrence coefficients by using both Mantica's and Chebyshev's algorithms. Numerical solutions are displayed in Section 5. Finally, we state some open problems and conjectures in Section 6.

2. 3-term recursive relation and symmetric measures

Let μ be a positive measure on $\mathbb{R}$. Let $\mathbb{P}$ be the vector space of all real polynomials and $\mathbb{P}_n$ be the subspace of polynomials of degree $\leq n$. Define an inner product and norm on $\mathbb{P}$ respectively as

$$\langle u, v\rangle_\mu := \int_{\mathbb{R}} uv\, d\mu, \qquad \|u\|_\mu := \sqrt{\langle u, u\rangle_\mu}.$$

The inner product is said to be *positive definite* on $\mathbb{P}$ (resp. $\mathbb{P}_n$) if $\|u\|_\mu > 0$ for all $u \in \mathbb{P}$ (resp. $\mathbb{P}_n$) that is not identically 0. The *orthogonal polynomials* $\{P_n(x)\}_{n\geq 0}$ with respect to μ are polynomials of the form

$$P_n(x) = c_n x^n + \cdots + c_1 x + c_0 \qquad \text{with } c_n > 0 \tag{2.1}$$

that satisfy $\langle P_i, P_j\rangle_\mu = \delta_{ij}$ for all $i, j = 0, 1, 2, \ldots$, where δ_{ij} is the Kronecker delta. $\{P_i(x)\}_{i=0}^n$ form a basis of $\mathbb{P}_n$. The existence and uniqueness of orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ with respect to μ are well-known (see [6, Theorem 1.6]). The orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ satisfy a 3-term recurrence relation (see [6, Theorem 1.29])

$$xP_n(x) = r_{n+1}P_{n+1}(x) + A_nP_n(x) + r_nP_{n-1}(x) \qquad \text{for } n = 0, 1, 2 \ldots, \tag{2.2}$$

where $P_{-1}(x) = 0$, $P_0(x) = 1$, and $r_0 = 1$. Orthogonality implies that for all $n \geq 0$,

$$r_{n+1} = \langle xP_{n+1}, P_n\rangle_\mu \quad \text{and} \quad A_n = \langle xP_n, P_n\rangle_\mu .$$

Monic real polynomials $\widetilde{P}_n(x) = x^n + \cdots, n = 0, 1, 2, \ldots$, are called the *monic orthogonal polynomials* with respect to the measure μ if $\langle P_i, P_j\rangle_\mu = 0$ for all $i \neq j \in \{0, 1, 2, \ldots\}$ and $\|\widetilde{P}_i\|_\mu > 0$ for all $i = 0, 1, 2, \ldots$. Note that $P_n(x) = \widetilde{P}_n(x)/\|\widetilde{P}_n\|_\mu$. Combining [6, Theorem 1.29] and (2.2), we see that monic orthogonal polynomials $\{\widetilde{P}_n(x)\}_{n\geq 0}$ satisfy the following 3-term recurrence relation

$$x\widetilde{P}_n(x) = \widetilde{P}_{n+1}(x) + A_n\widetilde{P}_n(x) + r_n^2\widetilde{P}_{n-1}(x) \qquad \text{for } n = 0, 1, 2 \ldots, \tag{2.3}$$

where $\widetilde{P}_{-1}(x) = 0$ and $\widetilde{P}_0(x) = 1$.

Let $x_0 \in \mathbb{R}$ and $T(x) := 2x_0 - x$ for $x \in \mathbb{R}$. Note that $T = T^{-1}$. We say that μ is *symmetric* with respect to $x = x_0$ if $\mu(E) = \mu \circ T^{-1}(E)$ for all μ-measurable $E \subseteq \mathbb{R}$.

Proposition 2.1. *Let μ be a positive measure on $\mathbb{R}$ and let $\{P_n(x)\}$ be orthogonal polynomials with respect to μ as in 2.1. Assume that μ is symmetric with respect to $x = x_0$ for some $x_0 \in \mathbb{R}$ and let $T(x) := 2x_0 - x$. Then for each $n \geq 0$,*

(a) $P_n(x) = (-1)^n P_n(T(x))$ for all $x \in \mathbb{R}$;

(b) $A_n = x_0$.

Proof. (a) For each $n \geq 0$, define $\hat{P}_n(x) := (-1)^n P_n(T(x))$ for $x \in \mathbb{R}$. It follows from the definitions of $T(x)$ and $P_n(x)$ that $\hat{P}_n(x) = c_n x^n + \cdots$. Moreover, for $0 \leq k \neq \ell$, we have

$$\begin{aligned}\langle \hat{P}_k, \hat{P}_\ell \rangle_\mu &= (-1)^{k+\ell} \int_{\mathbb{R}} P_k(T(x)) P_\ell(T(x))\, d\mu(x) \\ &= (-1)^{k+\ell} \int_{T(\mathbb{R})} P_k(y) P_\ell(y)\, d\mu \circ T^{-1}(y) \\ &= (-1)^{k+\ell} \int_{\mathbb{R}} P_k(y) P_\ell(y)\, d\mu(y) = (-1)^{k+\ell} \langle P_k, P_\ell \rangle_\mu = \delta_{k\ell}.\end{aligned} \tag{2.4}$$

Thus $\{\hat{P}_n(x)\}_{n\geq 0}$ is also a sequence of orthogonal polynomials with respect to μ. By the uniqueness of orthogonal polynomials, $P_n(x) = \hat{P}_n(x)$, i.e., (a) holds.

(b) Combining $T = T^{-1}$, $\mu = \mu \circ T^{-1}$, and (a), we have

$$\begin{aligned}A_n &= \int_{\mathbb{R}} x P_n^2(x)\, d\mu(x) = \int_{T(\mathbb{R})} (T^{-1}(y)) P_n^2(T^{-1}(y))\, d\mu \circ T^{-1}(y) \\ &= \int_{\mathbb{R}} (2x_0 - y) P_n^2(T(y))\, d\mu(y) = \int_{\mathbb{R}} (2x_0 - y) P_n^2(y)\, d\mu(y) \\ &= 2x_0 - A_n \qquad \text{for } n \geq 0.\end{aligned} \tag{2.5}$$

Thus $A_n = x_0$ for all $n \geq 0$. $\square$

3. Graphing orthogonal polynomials with respect to self-similar measures

Let μ be the self-similar measure defined by an IFS $\{S_i\}_{i=1}^N$ on $\mathbb{R}$ of the form

$$S_i(x) = \rho_i x + b_i, \qquad i = 1, \ldots, N, \tag{3.1}$$

together with a probability weight $\{w_i\}_{i=1}^N$. That is, μ is the unique probability measure satisfying the following self-similar identity

$$\mu = \sum_{i=1}^N w_i \mu \circ S_i^{-1}. \tag{3.2}$$

Thus for any continuous function f on $\mathbb{R}$,

$$\int_K f\, d\mu = \sum_{i=1}^N w_i \int_K f \circ S_i\, d\mu, \tag{3.3}$$

where $K := \operatorname{supp}(\mu)$ is called the *self-similar set* defined by $\{S_i\}_{i=1}^N$. It is known that if $N \geq 2$, then μ is continuous. For $n \geq 0$, let m_n be the *n-th moment* with respect to μ, defined as

$$m_n = m_n(\mu) := \int_K x^n \, d\mu. \tag{3.4}$$

Let $\{\widetilde{P}_n(x)\}_{n\geq 0}$ be the monic orthogonal polynomials with respect to μ. In this section, we will draw the figures of $\widetilde{P}_n(x)$. Since the monic orthogonal polynomials $\widetilde{P}_k(x)$ can be expressed in terms of the moments in the following well-known formula (see, e.g., [6]):

$$\widetilde{P}_n(x) = \frac{1}{m_n} \begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}, \tag{3.5}$$

we first need to compute the moments $\{m_i\}_{i=0}^{2n-1}$ with respect to μ.

The following proposition gives a straight-forward derivation of a formula for the moments corresponding to any one-dimensional self-similar measure μ.

Proposition 3.1. *Let μ be defined by* (3.1) *and* (3.2)*, and let m_n be defined as in* (3.4)*. Then*

$$m_n = \frac{1}{1 - \sum_{i=1}^N w_i \rho_i^n} \sum_{i=1}^N w_i \sum_{k=0}^{n-1} \binom{n}{k} \rho_i^k b_i^{n-k} m_k.$$

Proof. Letting $f(x) = x^n$ in (3.3) yields

$$\begin{aligned} m_n &= \sum_{i=1}^N w_i \int_K (\rho_i x + b_i)^n \, d\mu = \sum_{i=1}^N w_i \int_K \left(\rho_i^n x^n + \sum_{k=0}^{n-1} \binom{n}{k} \rho_i^k b_i^{n-k} x^k \right) d\mu \\ &= \left(\sum_{i=1}^N w_i \rho_i^n \right) m_n + \sum_{i=1}^N w_i \sum_{k=0}^{n-1} \binom{n}{k} \rho_i^k b_i^{n-k} m_k, \end{aligned}$$

which gives the desired formula. □

3.1. **Infinite Bernoulli convolution associated with the golden ratio.** The infinite Bernoulli convolution associated with golden ratio μ is the self-similar measure on $[0, 1]$ defined by the IFS

$$S_1(x) = \rho x, \qquad S_2(x) = \rho x + (1 - \rho), \qquad \rho = (\sqrt{5} - 1)/2 \approx 0.618033988\ldots, \tag{3.6}$$

together with probability weights $w_1 = w_2 = 1/2$. That is,

$$\mu = \frac{1}{2} \mu \circ S_1^{-1} + \frac{1}{2} \mu \circ S_2^{-1}. \tag{3.7}$$

It is known that μ is singular (see [3]) with $\mathrm{supp}(\mu) = [0,1]$. By using Proposition 3.1, we have

$$m_n = \frac{1}{2}\frac{\rho^{2n}}{(1-\rho^n)}\sum_{i=0}^{n-1}\binom{n}{i}\rho^{-i}m_i. \tag{3.8}$$

Now the point is how to compute the moments. Using formula (3.8), would takes too much time to compute m_n when $n > 18$. The following are some equalities which may be used to reduce computing time. We first introduce the Lucas and Fibonacci sequences, which are closely related to the golden ratio ρ. The Lucas sequence $\{L_n\}_{n\geq 0}$ is given by the recurrence relation:

$$L_0 = 2, \qquad L_1 = 1, \qquad L_n = L_{n-1} + L_{n-2}, \qquad n \in \mathbb{Z}, \tag{3.9}$$

while the Fibonacci sequence $\{F_n\}_{n\geq 0}$ is given by

$$F_0 = 0, \qquad F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \qquad n \in \mathbb{Z}. \tag{3.10}$$

Let $\phi := \rho^{-1}$. We note that for $n \geq 0$,

$$L_n = \phi^n + (1-\phi)^n \quad \text{and} \quad F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}.$$

Thus

$$\phi^n = (L_n + \sqrt{5}F_n)/2 \qquad \text{for } n \geq 0. \tag{3.11}$$

It follows that

$$\frac{\rho^{2n}}{1-\rho^n} = \frac{1}{\rho^{-2n}-\rho^{-n}} = \frac{1}{\phi^{2n}-\phi^n} = \frac{2}{(L_{2n}-L_n)+\sqrt{5}(F_{2n}-F_n)}. \tag{3.12}$$

Combining (3.8), (3.11) and (3.12), we have

$$m_n = \frac{1}{2}\cdot\frac{1}{(L_{2n}-L_n)+\sqrt{5}(F_{2n}-F_n)}\sum_{i=0}^{n-1}\binom{n}{i}(L_i+\sqrt{5}F_i)m_i.$$

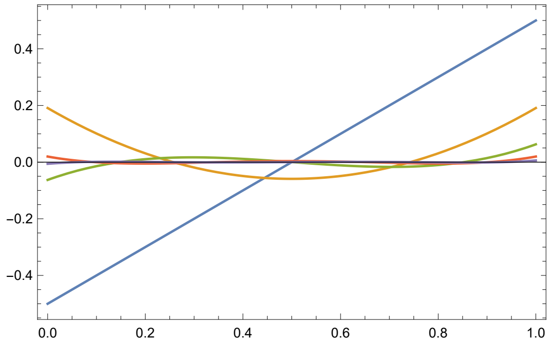
Monic orthogonal polynomials $\widetilde{P}_1(x), \ldots, \widetilde{P}_5(x)$ corresponding to the golden ratio are plotted in Figure 1(a).

3.2. **Three-fold convolution of the Cantor measure.** The 3-fold convolution of the Cantor measure is the self-similar measure defined by the IFS

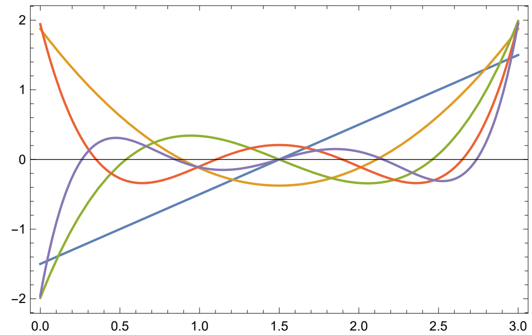
$$S_i(x) = \frac{1}{3}x + \frac{2}{3}(i-1), \qquad i = 1,2,3,4,$$

together with the probability weight $\{1/8, 3/8, 3/8, 1/8\}$, *i.e.*,

$$\mu = \frac{1}{8}\mu\circ S_1^{-1} + \frac{3}{8}\mu\circ S_2^{-1} + \frac{3}{8}\mu\circ S_3^{-1} + \frac{1}{8}\mu\circ S_4^{-1}.$$



(a)



(b)

FIGURE 1. Monic orthogonal polynomials $\widetilde{P}_1(x), \ldots, \widetilde{P}_5(x)$ corresponding to (a) the infinite Bernoulli convolution associated with the golden ratio, and (b) the three-fold convolution of the Cantor measure.

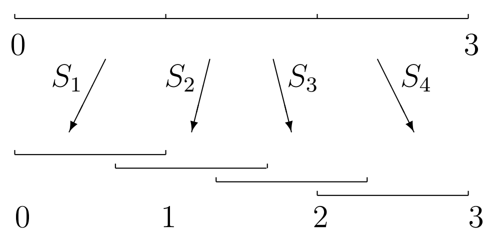


FIGURE 2. Figure showing the IFS defining the 3-fold convolution.

(See Figure 3.2.) We note that $\mu = \mu_c^{*3} := \mu_c * \mu_c * \mu_c$, where μ_c is the standard Cantor measure.

Proposition 3.1 implies that for the 3-fold convolution of the Cantor measure, the moments m_n satisfy

$$m_n = \frac{1}{8(3^n - 1)} \sum_{k=0}^{n-1} \binom{n}{k} (3 \cdot 2^{n-k} + 3 \cdot 4^{n-k} + 6^{n-k}) m_k.$$

Monic orthogonal polynomials $\widetilde{P}_1(x), \ldots, \widetilde{P}_5(x)$ corresponding to the 3-fold convolution are plotted in Figure 1(b).

4. ALGORITHMS FOR COMPUTING A_n AND r_n

In this section, we introduce two methods, namely Mantica's algorithm [9] and modified Chebyshev's algorithm [6], to compute the A_n's and r_n's in (2.2).

4.1. **Mantica's algorithm.** In this subsection, we use a method of Mantica [9] to compute the A_n's and r_n's in (2.2). We first describe the method for an IFS $\{S_i\}_{i=1}^N$ on $\mathbb{R}$, where

$$S_i(x) = \rho_i x + b_i, \qquad i = 1, \dots, N,$$

and let μ be the self-similar measure defined by the IFS together with a set of probability weights $\{w_i\}_{i=1}^N$.

Let $P_n(x)$ be orthogonal polynomials with respect to μ. For any $n \geq 0$ and $i \in \{1, \dots, N\}$, and $0 \leq \ell \leq n$, let $\Gamma_{i,\ell}^n$ be defined by the relation:

$$P_n(S_i(x)) = \sum_{\ell=0}^{n} \Gamma_{i,\ell}^n P_\ell(x). \tag{4.1}$$

According to [9, Lemma 1], the coefficients $\Gamma_{i,\ell}^n$ can be determined recursively from $\{\rho_i\}_{i=1}^N$, $\{b_i\}_{i=1}^N$, $\{A_k\}_{k=0}^{n-1}$, and $\{r_k\}_{k=0}^n$. We state the precise result below.

Proposition 4.1. *Let* $(\Gamma_{i,\ell}^n)$ *be the the coefficients in* (4.1). *For each* $i \in \{1, \dots, N\}$, *we have*

(a) $\Gamma_{i,0}^1 = \big(\rho_i A_0 \Gamma_{i,0}^0 + b_i \Gamma_{i,0}^0 - A_0 \Gamma_{i,0}^0\big)/r_1$ *and* $\Gamma_{i,1}^1 = \rho_i \Gamma_{i,0}^0$.

(b)

$$\begin{aligned}
\Gamma_{i,0}^2 &= \big(\rho_i r_1 \Gamma_{i,1}^1 + \rho_i A_0 \Gamma_{i,0}^1 + b_i \Gamma_{i,0}^1 - A_1 \Gamma_{i,0}^1 - r_1 \Gamma_{i,0}^0\big)/r_2 \\
\Gamma_{i,1}^2 &= \big(\rho_i r_1 \Gamma_{i,0}^1 + \rho_i A_1 \Gamma_{i,1}^1 + b_i \Gamma_{i,1}^1 - A_1 \Gamma_{i,1}^1\big)/r_2 \\
\Gamma_{i,2}^2 &= \rho_i \Gamma_{i,1}^1.
\end{aligned} \tag{4.2}$$

(c) for $n \geq 3$ *and* $1 \leq \ell \leq n-2$,

$$\begin{aligned}
\Gamma_{i,0}^n &= \big(\rho_i A_0 \Gamma_{i,0}^{n-1} + \rho_i r_1 \Gamma_{i,1}^{n-1} + b_i \Gamma_{i,0}^{n-1} - A_{n-1} \Gamma_{i,0}^{n-1} - r_{n-1} \Gamma_{i,0}^{n-2}\big)/r_n; \\
\Gamma_{i,\ell}^n &= \big(\rho_i r_\ell \Gamma_{i,\ell-1}^{n-1} + \rho_i A_\ell \Gamma_{i,\ell}^{n-1} + \rho_i r_{\ell+1} \Gamma_{i,\ell+1}^{n-1} + b_i \Gamma_{i,\ell}^{n-1} - A_{n-1} \Gamma_{i,\ell}^{n-1} - r_{n-1} \Gamma_{i,\ell}^{n-2}\big)/r_n; \\
\Gamma_{i,n-1}^n &= \big(\rho_i r_{n-1} \Gamma_{i,n-2}^{n-1} + \rho_i A_{n-1} \Gamma_{i,n-1}^{n-1} + b_i \Gamma_{i,n-1}^{n-1} - A_{n-1} \Gamma_{i,n-1}^{n-1}\big)/r_n; \\
\Gamma_{i,n}^n &= \rho_i \Gamma_{i,n-1}^{n-1}.
\end{aligned}$$

Proof. We first note that $P_{-1}(x) = 0$. Fix any $i \in \{1, \dots, N\}$.

(a) Using 3-term recurrence relation (2.2), we have

$$xP_0(x) = r_1 P_1(x) + A_0 P_0(x) + r_0 P_{-1}(x) = r_1 P_1(x) + A_0 P_0(x). \tag{4.3}$$

Substituting x by $S_i(x) = \rho_i x + b_i$, we have

$$(\rho_i x + b_i) P_0(S_i(x)) = r_1 P_1(S_i(x)) + A_0 P_0(S_i(x)).$$

It follows from (4.1) that $(\rho_i x+b_i)\Gamma^0_{i,0}P_0(x) = r_1\big(\Gamma^1_{i,0}P_0(x)+\Gamma^1_{i,1}P_1(x)\big)+A_0\Gamma^0_{i,0}P_0(x)$, i.e.,

$$\rho_i\Gamma^0_{i,0}xP_0(x) = \big(r_1\Gamma^1_{i,0} + A_0\Gamma^0_{i,0} - b_i\Gamma^0_{i,0}\big)P_0(x) + r_1\Gamma^1_{i,1}P_1(x). \tag{4.4}$$

Combining (4.4) with (4.3), we have

$$\rho_i\Gamma^0_{i,0}\big(r_1P_1(x) + A_0P_0(x)\big) = \big(r_1\Gamma^1_{i,0} + A_0\Gamma^0_{i,0} - b_i\Gamma^0_{i,0}\big)P_0(x) + r_1\Gamma^1_{i,1}P_1(x). \tag{4.5}$$

Thus $\rho_i r_1\Gamma^0_{i,0} = r_1\Gamma^1_{i,1}$ and $\rho_i A_0\Gamma^0_{i,0} = r_1\Gamma^1_{i,0} + A_0\Gamma^0_{i,0} - b_i\Gamma^0_{i,0}$. Consequently,

$$\Gamma^1_{i,1} = \rho_i\Gamma^0_{i,0}, \qquad \Gamma^1_{i,0} = \big(\rho_iA_0\Gamma^0_{i,0} + b_i\Gamma^0_{i,0} - A_0\Gamma^0_{i,0}\big)/r_1.$$

(b) Similarly, using 3-term recurrence relation (2.2), we have

$$xP_1(x) = r_2P_2(x) + A_1P_1(x) + r_1P_0(x). \tag{4.6}$$

Substituting x by $S_i(x) = \rho_i x + b_i$, we have

$$(\rho_ix + b_i)P_1(S_i(x)) = r_2P_2(S_i(x)) + A_1P_1(S_i(x)) + r_1P_0(S_i(x)). \tag{4.7}$$

Using (4.1), (4.3), and (4.6), we have the left side of (4.7)

$$\begin{aligned}(\rho_ix + b_i)P_1(S_i(x)) &= (\rho_ix + b_i)\big(\Gamma^1_{i,0}P_0(x) + \Gamma^1_{i,1}P_1(x)\big)\\ &= \rho_i\Gamma^1_{i,0}xP_0(x) + \rho_i\Gamma^1_{i,1}xP_1(x) + b_i\Gamma^1_{i,0}P_0(x) + b_i\Gamma^1_{i,1}P_1(x)\\ &= \rho_i\Gamma^1_{i,0}\big(r_1P_1(x) + A_0P_0(x)\big) + \rho_i\Gamma^1_{i,1}\big(r_2P_2(x) + A_1P_1(x) + r_1P_0(x)\big)\\ &\quad + b_i\Gamma^1_{i,0}P_0(x) + b_i\Gamma^1_{i,1}P_1(x)\\ &= \rho_ir_2\Gamma^1_{i,1}P_2(x) + \big(\rho_ir_1\Gamma^1_{i,0} + \rho_iA_1\Gamma^1_{i,1} + b_i\Gamma^1_{i,1}\big)P_1(x)\\ &\quad + \big(\rho_iA_0\Gamma^1_{i,0} + \rho_ir_1\Gamma^1_{i,1} + b_i\Gamma^1_{i,0}\big)P_0(x).\end{aligned} \tag{4.8}$$

On the other hand, (4.1) implies that the right side of (4.7)

$$\begin{aligned}&r_2P_2(S_i(x)) + A_1P_1(S_i(x)) + r_1P_0(S_i(x))\\ &= r_2\big(\Gamma^2_{i,0}P_0(x) + \Gamma^2_{i,1}P_1(x) + \Gamma^2_{i,2}P_2(x)\big) + A_1(\Gamma^1_{i,0}P_0(x) + \Gamma^1_{i,1}P_1(x)\big)\\ &\quad + r_1\Gamma^0_{i,0}P_0(x)\\ &= r_2\Gamma^2_{i,2}P_2(x) + \big(r_2\Gamma^2_{i,0} + A_1\Gamma^1_{i,0} + r_1\Gamma^0_{i,0}\big)P_0(x) + \big(r_2\Gamma^2_{i,1} + A_1\Gamma^1_{i,1}\big)P_1(x).\end{aligned} \tag{4.9}$$

Combining (4.7), (4.7), and (4.7), we obtain

$$\begin{aligned}\Gamma^2_{i,0} &= \big(\rho_ir_1\Gamma^1_{i,1} + \rho_iA_0\Gamma^1_{i,0} + b_i\Gamma^1_{i,0} - A_1\Gamma^1_{i,0} - r_1\Gamma^0_{i,0}\big)/r_2\\ \Gamma^2_{i,1} &= \big(\rho_ir_1\Gamma^1_{i,0} + \rho_iA_1\Gamma^1_{i,1} + b_i\Gamma^1_{i,1} - A_1\Gamma^1_{i,1}\big)/r_2\\ \Gamma^2_{i,2} &= \rho_i\Gamma^1_{i,1}.\end{aligned} \tag{4.10}$$

(c) Fix any $n \geq 3$. Replacing x by $S_i(x) = \rho_ix + b_i$ in (2.2) gives

$$(\rho_ix + b_i - A_{n-1})P_{n-1}(S_i(x)) = r_nP_n(S_i(x)) + r_{n-1}P_{n-2}(S_i(x)). \tag{4.11}$$

Using (4.1), we can write this as

$$\begin{aligned}(\rho_i x + b_i - A_{n-1})\sum_{\ell=0}^{n-1}\Gamma_{i,\ell}^{n-1}P_\ell(x) &= r_n\sum_{\ell=0}^{n}\Gamma_{i,\ell}^{n}P_\ell(x) + r_{n-1}\sum_{\ell=0}^{n-2}\Gamma_{i,\ell}^{n-2}P_\ell(x)\\ &= \sum_{\ell=0}^{n-2}\big(r_n\Gamma_{i,\ell}^{n} + r_{n-1}\Gamma_{i,\ell}^{n-2}\big)P_\ell(x)\\ &\quad + r_n\Gamma_{i,n-1}^{n}P_{n-1}(x) + r_n\Gamma_{i,n}^{n}P_n(x).\end{aligned} \tag{4.12}$$

Moreover, (2.2) implies

$$\begin{aligned}&(\rho_i x + b_i - A_{n-1})\sum_{\ell=0}^{n-1}\Gamma_{i,\ell}^{n-1}P_\ell(x)\\ &= \rho_i\sum_{\ell=0}^{n-1}\Gamma_{i,\ell}^{n-1}xP_\ell(x) + (b_i - A_{n-1})\sum_{\ell=0}^{n-1}\Gamma_{i,\ell}^{n-1}P_\ell(x)\\ &= \rho_i\sum_{\ell=0}^{n-1}\Gamma_{i,\ell}^{n-1}\big(r_{\ell+1}P_{\ell+1}(x) + A_\ell P_\ell(x) + r_\ell P_{\ell-1}(x)\big) + (b_i - A_{n-1})\sum_{\ell=0}^{n-1}\Gamma_{i,\ell}^{n-1}P_\ell(x)\\ &= \big(\rho_i A_0\Gamma_{i,0}^{n-1} + \rho_i r_1\Gamma_{i,1}^{n-1} + b_i\Gamma_{i,0}^{n-1} - A_{n-1}\Gamma_{i,0}^{n-1}\big)P_0(x)\\ &+ \sum_{\ell=1}^{n-2}\big(\rho_i r_\ell\Gamma_{i,\ell-1}^{n-1} + \rho_i A_\ell\Gamma_{i,\ell}^{n-1} + \rho_i r_{\ell+1}\Gamma_{i,\ell+1}^{n-1} + b_i\Gamma_{i,\ell}^{n-1} - A_{n-1}\Gamma_{i,\ell}^{n-1}\big)P_\ell(x)\\ &+ \big(\rho_i r_{n-1}\Gamma_{i,n-2}^{n-1} + \rho_i A_{n-1}\Gamma_{i,n-1}^{n-1} + b_i\Gamma_{i,n-1}^{n-1} - A_{n-1}\Gamma_{i,n-1}^{n-1}\big)P_{n-1}(x) + \rho_i r_n\Gamma_{i,n-1}^{n-1}P_n(x).\end{aligned} \tag{4.13}$$

Equating (4.12) and (4.12) gives, for $1 \le \ell \le n-2$,

$$\begin{aligned}\Gamma_{i,0}^{n} &= \big(\rho_i A_0\Gamma_{i,0}^{n-1} + \rho_i r_1\Gamma_{i,1}^{n-1} + b_i\Gamma_{i,0}^{n-1} - A_{n-1}\Gamma_{i,0}^{n-1} - r_{n-1}\Gamma_{i,0}^{n-2}\big)/r_n;\\ \Gamma_{i,\ell}^{n} &= \big(\rho_i r_\ell\Gamma_{i,\ell-1}^{n-1} + \rho_i A_\ell\Gamma_{i,\ell}^{n-1} + \rho_i r_{\ell+1}\Gamma_{i,\ell+1}^{n-1} + b_i\Gamma_{i,\ell}^{n-1} - A_{n-1}\Gamma_{i,\ell}^{n-1} - r_{n-1}\Gamma_{i,\ell}^{n-2}\big)/r_n;\\ \Gamma_{i,n-1}^{n} &= \big(\rho_i r_{n-1}\Gamma_{i,n-2}^{n-1} + \rho_i A_{n-1}\Gamma_{i,n-1}^{n-1} + b_i\Gamma_{i,n-1}^{n-1} - A_{n-1}\Gamma_{i,n-1}^{n-1}\big)/r_n;\\ \Gamma_{i,n}^{n} &= \rho_i\Gamma_{i,n-1}^{n-1}.\end{aligned}$$

The proof is complete. □

Let $\widetilde{P}_n(x) = r_nP_n(x)$. Hence, we can write a second decomposition

$$\widetilde{P}_n(S_i(x)) = \widetilde{\Gamma}_{i,\ell}^{n}\widetilde{P}_n(x) + \sum_{\ell=0}^{n-1}\widetilde{\Gamma}_{i,\ell}^{n}P_\ell(x). \tag{4.14}$$

It is easy to see that $\widetilde{\Gamma}_{i,\ell}^{n} = r_n\Gamma_{i,\ell}^{n}$ for $0 \le \ell \le n-1$ and $\widetilde{\Gamma}_{i,n}^{n} = \Gamma_{i,n}^{n}$. Thus the coefficients $\Gamma_{i,n}^{n}$ can be computed recursively on the basis of the knowledge of only $(r_k)_{k=1}^{n-1}$ and $(A_k)_{k=1}^{n-1}$.

The following proposition gives formulas for A_n and r_n. The proof is given in [9, Lemmas 2 and 3].

Proposition 4.2. *For $n \geq 0$,*

(a) $r_n^2 = \Big(\sum_{i=1}^N w_i(B_i + C_i)\Big)/\big(1 - \sum_{i=1}^N w_i\rho_i\widetilde{\Gamma}_{i,n}^n\Gamma_{i,n-1}^{n-1}\big)$, *where*

$$B_i = \sum_{\ell=0}^{n-1}(b_i + \rho_i A_\ell)\widetilde{\Gamma}_{i,\ell}^n\Gamma_{i,\ell}^{n-1}, \quad \text{and} \quad C_i = \rho_i\sum_{\ell=0}^{n-2} r_{\ell+1}\big(\widetilde{\Gamma}_{i,\ell}^n\Gamma_{i,\ell+1}^{n-1} + \widetilde{\Gamma}_{i,\ell+1}^n\Gamma_{i,\ell}^{n-1}\big). \tag{4.15}$$

(b) $A_n = \sum_{i=1}^N w_i\Big(\sum_{\ell=0}^{n-1}(\Gamma_{i,\ell}^n)^2(b_i+\rho_i A_\ell)+2\rho_i\sum_{\ell=0}^{n-1} r_{\ell+1}\Gamma_{i,\ell}^n\Gamma_{i,\ell+1}^n+b_i\big(\Gamma_{i,n}^n\big)^2\Big)/(1-\sum_{i=1}^N w_i\rho_i\big(\Gamma_{i,n}^n\big)^2)$.

4.2. **Modified Chebyshev's algorithm.** Let μ be a positive measure on $\mathbb{R}$ with compact or infinite support, for which all moments m_n, $n = 0, 1, \dots,$ exist and are finite. Let $\{\widetilde{P}_n(x)\}_{n\geq 0}$ be the monic orthogonal polynomials with respect to μ. We can also use the *Modified Chebyshev algorithm* [6] to compute the A_n's and r_n's in (2.2), as follows. Let $\{\pi_n(x)\}_{n\geq 0}$ denote a system of monic polynomials satisfying a recurrence relation

$$\begin{aligned} &x\pi_n(x) = \pi_{n+1}(x) + \alpha_n\pi_n(x) + \beta_n\pi_{n-1}(x), \qquad n = 0, 1, 2, \dots, \\ &\pi_{-1}(x) = 0, \quad \pi_0(x) = 1, \end{aligned} \tag{4.16}$$

where $\alpha_n \in \mathbb{R}$ and $\beta_n \geq 0$ are assumed known. In the case $\alpha_n = \beta_n = 0$, it reduces to Chebyshev's original algorithm. We then call

$$v_n = v_n(\mu) := \int_{\mathbb{R}} \pi_n(x)\, d\mu(x), \qquad n = 0, 1, 2, \dots, \tag{4.17}$$

the *modified moments* of the measure μ relative to the polynomial system $\{\pi_n(x)\}$. Combining (2.3) and (4.16), we see that if $\alpha_n = A_n$ and $\beta_n = r_n^2$, then $\pi_n(x) = \widetilde{P}_n(x)$ and the modified moments reduce to the ordinary moments (3.4).

To describe the algorithm, we introduce *mixed moments*, which are defined for $k, \ell \geq -1$, as follows

$$\sigma_{k,\ell} := \int_{\mathbb{R}} \widetilde{P}_k(x)\pi_\ell(x)\, d\mu \qquad \textit{with} \qquad \sigma_{-1,\ell} := 0. \tag{4.18}$$

One obtains the first n coefficients α_k, β_k, $k = 0, 1, \dots, n-1$, from the first $2n$ modified moments v_i, $i = 0, 1, \dots, 2n-1$, by the following modified Chebyshev's algorithm:

Initialization:

$$\begin{aligned} &A_0 = \alpha_0 + v_1/v_0, \\ &r_0 = \sqrt{v_0}, \\ &\sigma_{-1,\ell} = 0, \quad \ell = 1, 2, \dots, 2n-2. \\ &\sigma_{0,\ell} = v_\ell, \quad \ell = 0, 1, 2, \dots, 2n-1. \end{aligned} \tag{4.19}$$

Continuation: For $k = 1, 2, \dots, n-1$,

$$\begin{aligned} \sigma_{k,\ell} &= \sigma_{k-1,\ell+1} - (A_{k-1} - \alpha_\ell)\sigma_{k-1,\ell} - r_{k-1}^2 \sigma_{k-2,\ell} + \beta_\ell \sigma_{k-1,\ell-1}, \\ &\qquad \ell = k, k+1, \dots, 2n-k-1, \\ A_k &= \alpha_k + \frac{\sigma_{k,k+1}}{\sigma_{k,k}} - \frac{\sigma_{k-1,k}}{\sigma_{k-1,k-1}}, \\ r_k &= \sqrt{\frac{\sigma_{k,k}}{\sigma_{k-1,k-1}}}. \end{aligned} \tag{4.20}$$

The algorithm requires $\{v_\ell\}_{\ell=0}^{2n-1}$ and $\{\alpha_k, \beta_k\}_{k=0}^{2n-2}$ as input, and produces $\{A_k, r_k\}_{k=0}^{n-1}$. The complexity in terms of arithmetic operations is $O(n^2)$. We apply the Chebyshev algorithm to infinite Bernoulli convolutions with overlaps. The results are the same as those obtained by Mantica's algorithm.

5. Numerical results for infinite Bernoulli convolutions

In this section, we display numerical results for the recurrence coefficients A_n and r_n in (2.2) for the symmetric and weighted infinite Bernoulli convolutions in (1.1) and (1.2). The value of ρ in (1.1) and (1.2) are taken from the following Table 1 through Table 4. For symmetric Bernoulli convolutions, we have $A_n = 1/2$ for all $n = 0, 1, 2, \dots$ (see Proposition 2.1), and so in this case, we only display numerical results for r_n. For the weighted infinite Bernoulli convolutions in (1.2) with weights $w_1 = 1/4$ and $w_2 = 3/4$, we show numerical results for both A_n and r_n.

We also remark that for the symmetric Bernoulli convolutions, $A_n = 1/2$ is known, and so we are able to compute r_n quite efficiently for n up to 10000. For the weighted Bernoulli convolutions, computations of A_n and r_n take much more machine time and memory and so we only show results up to $n = 1000$.

Label	ρ^{-1}	Appro. value ρ	Label	ρ^{-1}	Appro. value ρ
(a)	$\sqrt[2]{2}$	0.7071067811	(d)	$\sqrt[8]{2}$	0.9170040432
(b)	$\sqrt[4]{2}$	0.8408964152	(e)	$\sqrt[20]{2}$	0.9659363289
(c)	$\sqrt[6]{2}$	0.8908987181	(h)	$\sqrt[100]{2}$	0.9930924954

Table 1. The sequence $\sqrt[n]{2}$.

Label	Minimal polynomial	Appro. value of ρ^{-1}	ρ
(a)	$x^3 - 2x - 2$	1.7692923542	0.5651977173
(b)	$x^3 - x^2 - 2$	1.6956207695	0.5897545123
(c)	$x^3 - 2x^2 + 2x - 2$	1.5436890126	0.6477988712
(d)	$x^3 - x^2 + x - 2$	1.3532099641	0.7389836215
(e)	$x^5 - x^2 - 2$	1.2980299423	0.7703982530
(f)	$x^3 + x^2 - x - 2$	1.2055694304	0.8294835409

TABLE 2. Selected Garsia numbers.

Label	Minimal polynomial	Appro. value of ρ^{-1}	ρ
(a)	$x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$	1.9835828434	0.5041382583
(b)	$x^3 - x^2 - x - 1$	1.8392867552	0.5436890126
(c)	$x^3 - 2x^2 + x - 1$	1.7548776662	0.5698402909
(d)	$x^2 - x - 1$	1.6180339887	0.6180339887
(e)	$x^3 - x^2 - 1$	1.4655712318	0.6823278038
(f)	$x^3 - x - 1$	1.3247179572	0.7548776662

TABLE 3. (f) is the smallest Pisot number. (d) is the golden ratio. We also include the first few Pisot number from the family $x^n - x^{n-1} - \cdots - x - 1 = 0$. These family of Pisot numbers are decreasing and tend to 1.

Label	Minimal polynomial	Appro. value of ρ^{-1}	ρ
(a)	$x^9 - x^8 - x^7 - x^6 - x^5 - x^4 - x^3 - x^2 - x + 1$	1.9940041991	0.5015034574
(b)	$x^7 - x^6 - x^5 - x^4 - x^3 - x^2 - x + 1$	1.9748187082	0.5063755957
(c)	$x^6 - x^5 - x^4 - x^3 - x^2 - x + 1$	1.9468562682	0.5136486017
(d)	$x^4 - x^3 - x^2 - x + 1$	1.7220838057	0.5806918319
(e)	$x^{10} - x^8 - x^7 + x^5 - x^3 - x^2 + 1$	1.2934859531	0.7731046460
(f)	$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$	1.1762808182	0.8501371309

TABLE 4. Selected Salem numbers.

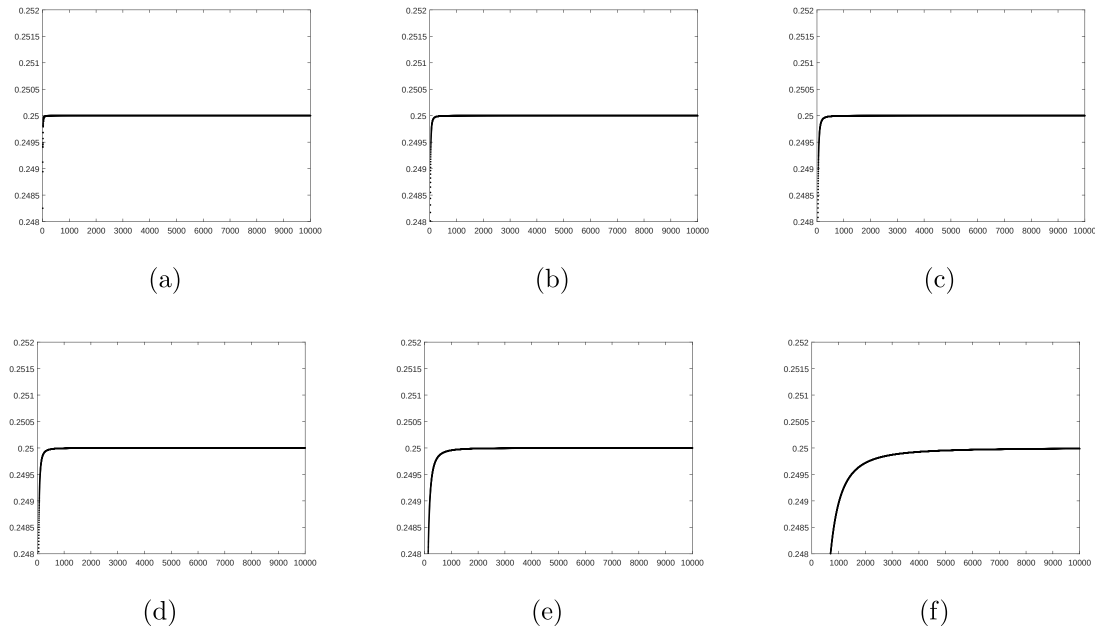


FIGURE 3. Graphs of r_n for the infinite Bernoulli convolutions in (1.1), where ρ comes from the sequence $\{\sqrt[n]{2}\}$ in Table 1.

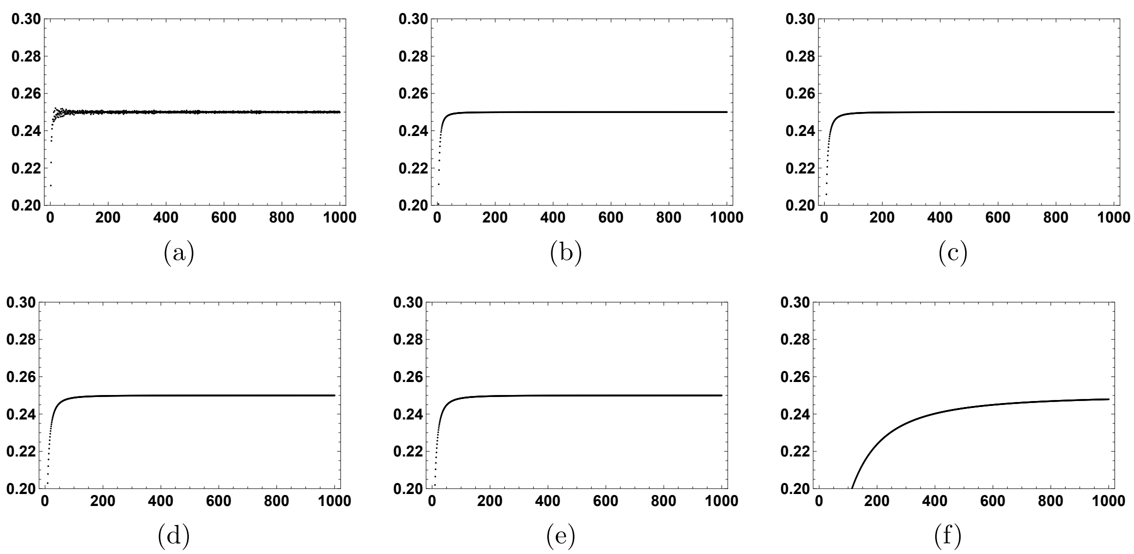


FIGURE 4. Graphs of r_n for the weighted infinite Bernoulli convolutions in (1.2) with weights $w_1 = 1/4$ and $w_2 = 3/4$, where ρ comes from the sequence in $\{\sqrt[n]{2}\}$ in Table 1.

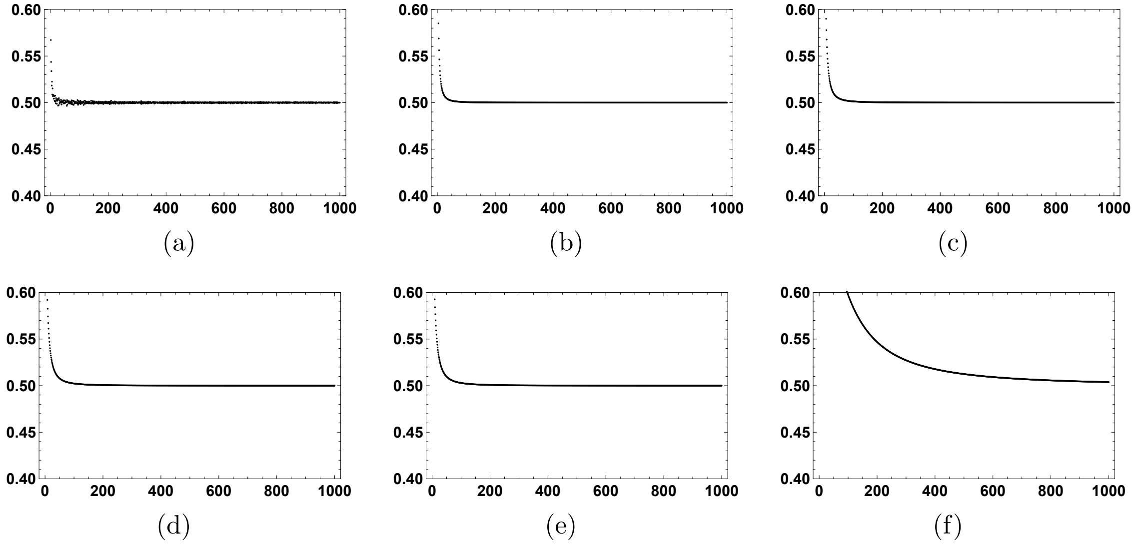


FIGURE 5. Graphs of A_n for the weighted infinite Bernoulli convolutions in (1.2) with weights $w_1 = 1/4$ and $w_2 = 3/4$, where ρ comes from the sequence $\{\sqrt[n]{2}\}$ in Table 1.

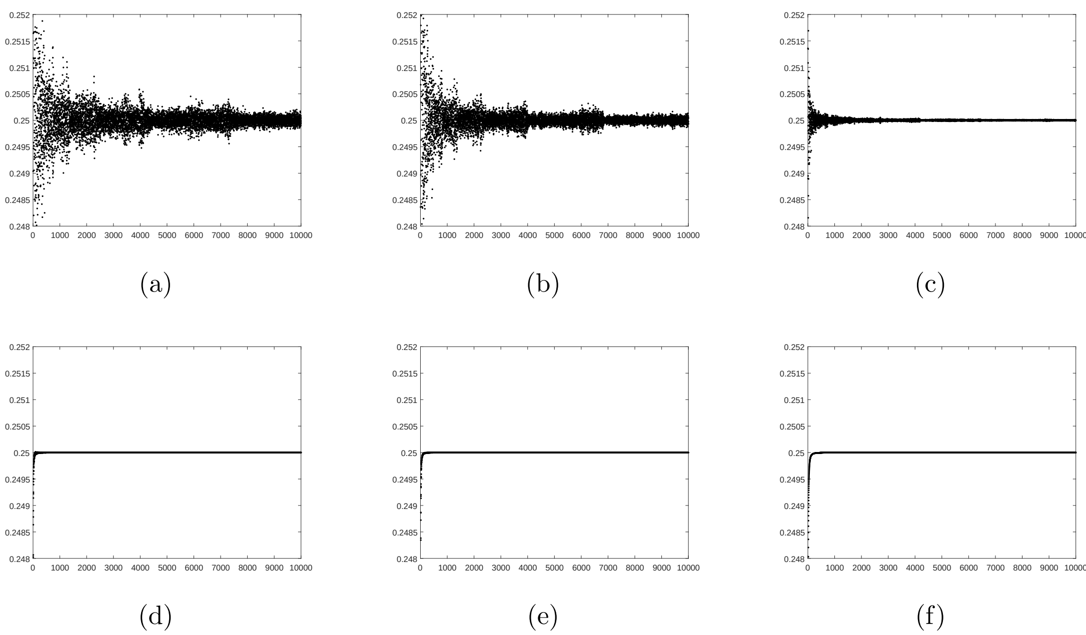


FIGURE 6. Graphs of r_n for the infinite Bernoulli convolutions in (1.1), where ρ are selected Garsia numbers in Table 2.

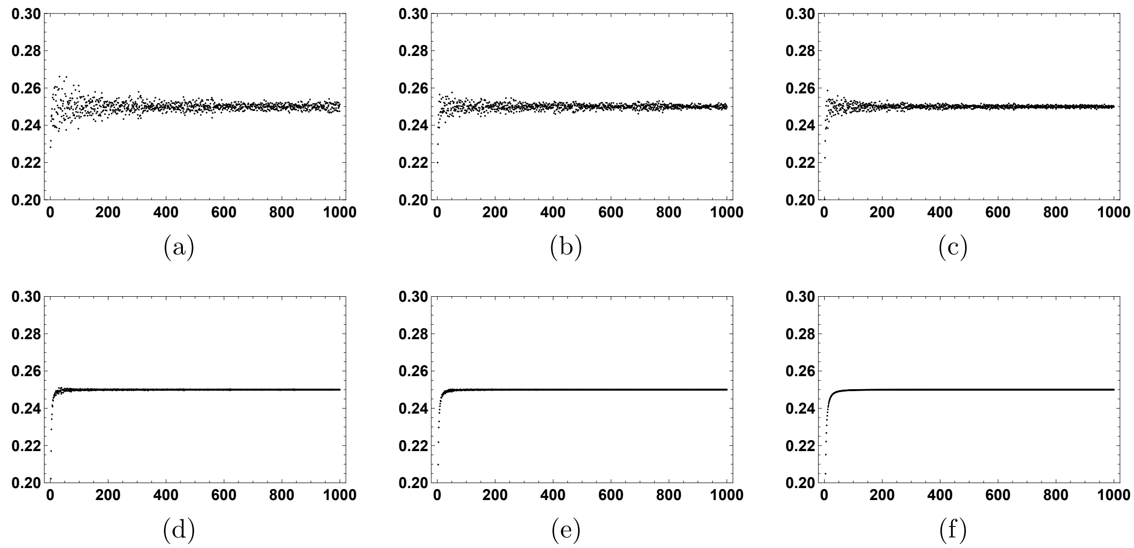


FIGURE 7. Graphs of r_n for the weighted infinite Bernoulli convolutions in (1.2) with weights $w_1 = 1/4$ and $w_2 = 3/4$, where ρ are the Garsia numbers in Table 2.

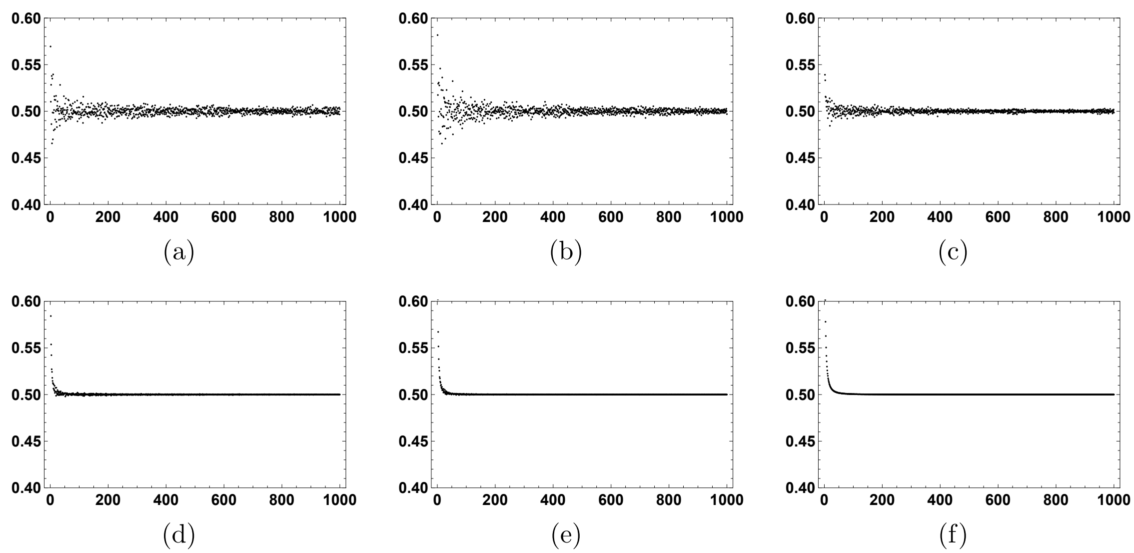


FIGURE 8. Graphs of A_n for the weighted infinite Bernoulli convolutions in (1.2) with weights $w_1 = 1/4$ and $w_2 = 3/4$, where ρ are the Garsia numbers in Table 2.

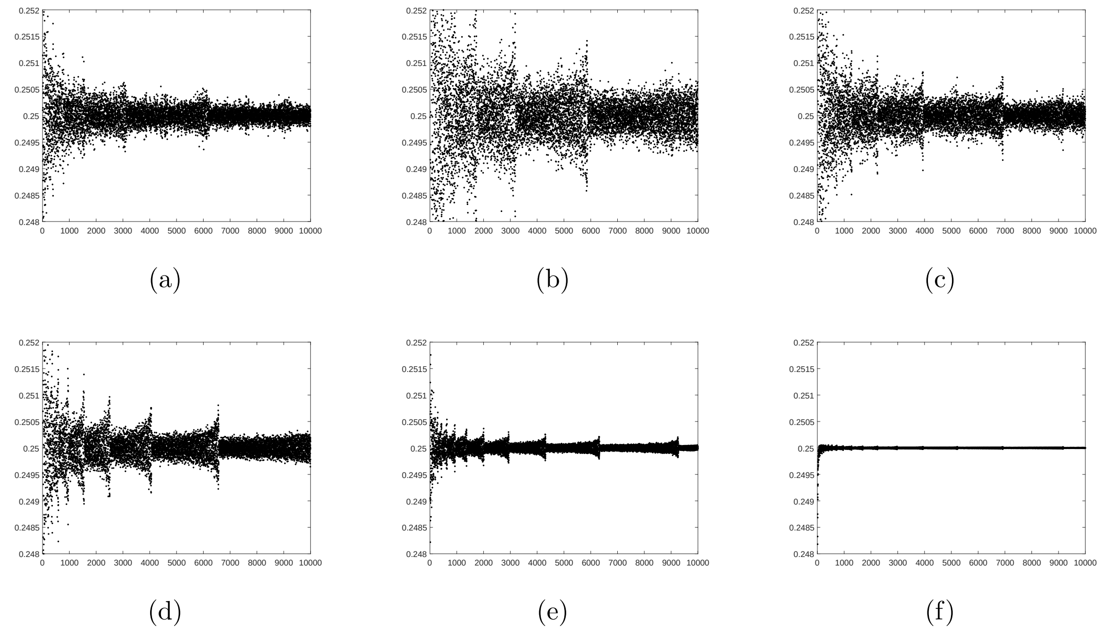


FIGURE 9. Graphs of r_n for the infinite Bernoulli convolutions in (1.1), where ρ are the selected Pisot numbers in Table 3

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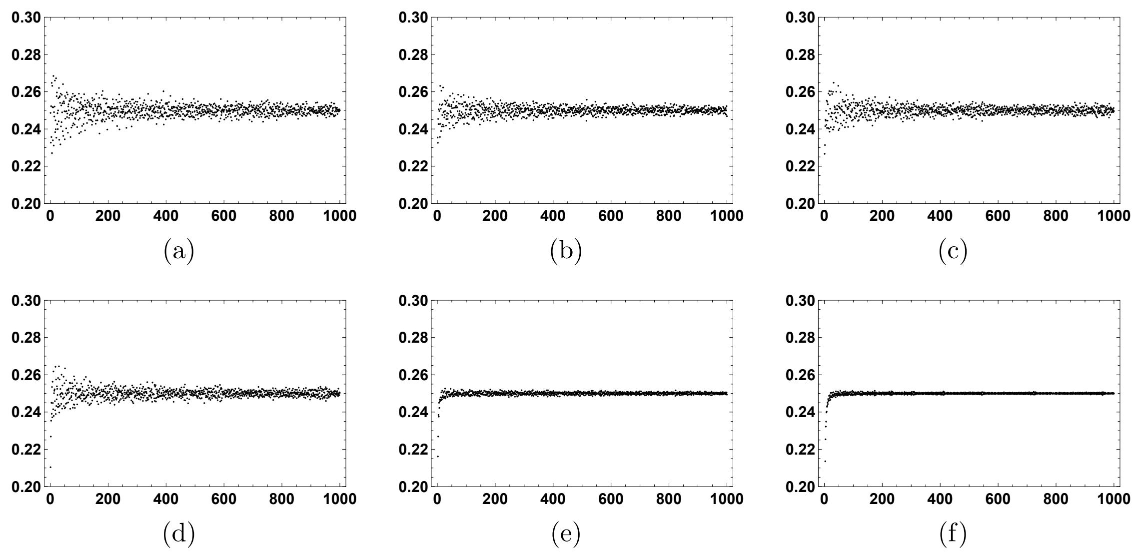

FIGURE 10. Graphs of r_n for the weighted infinite Bernoulli convolutions in (1.2) with weights $w_1 = 1/4$ and $w_2 = 3/4$, where ρ are the selected Pisot numbers in Table 3

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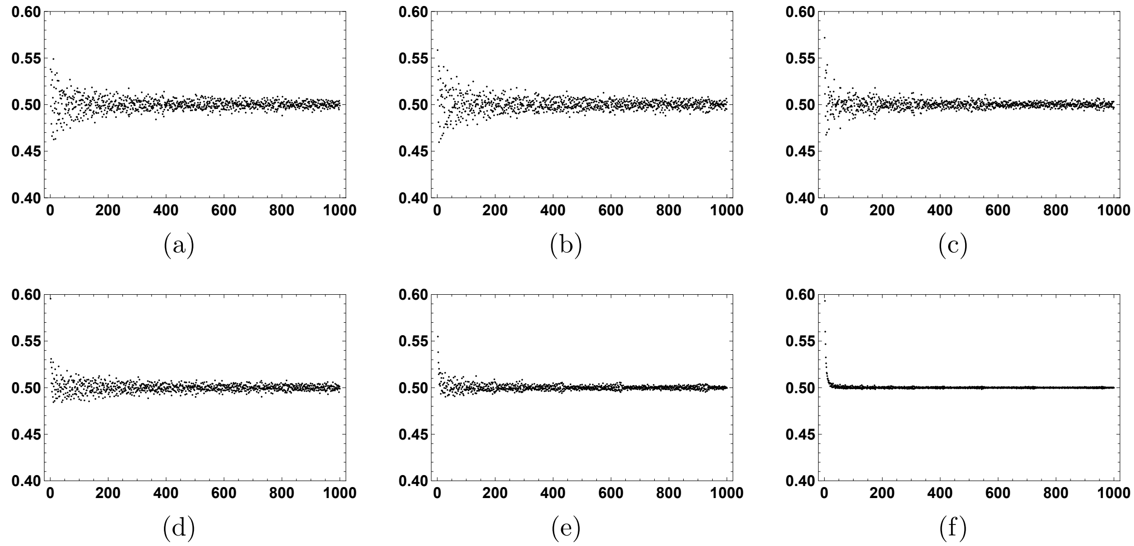

FIGURE 11. Graphs of A_n for the weighted infinite Bernoulli convolutions in (1.2) with weights $w_1 = 1/4$ and $w_2 = 3/4$, where ρ are the selected Pisot numbers in Table 3.

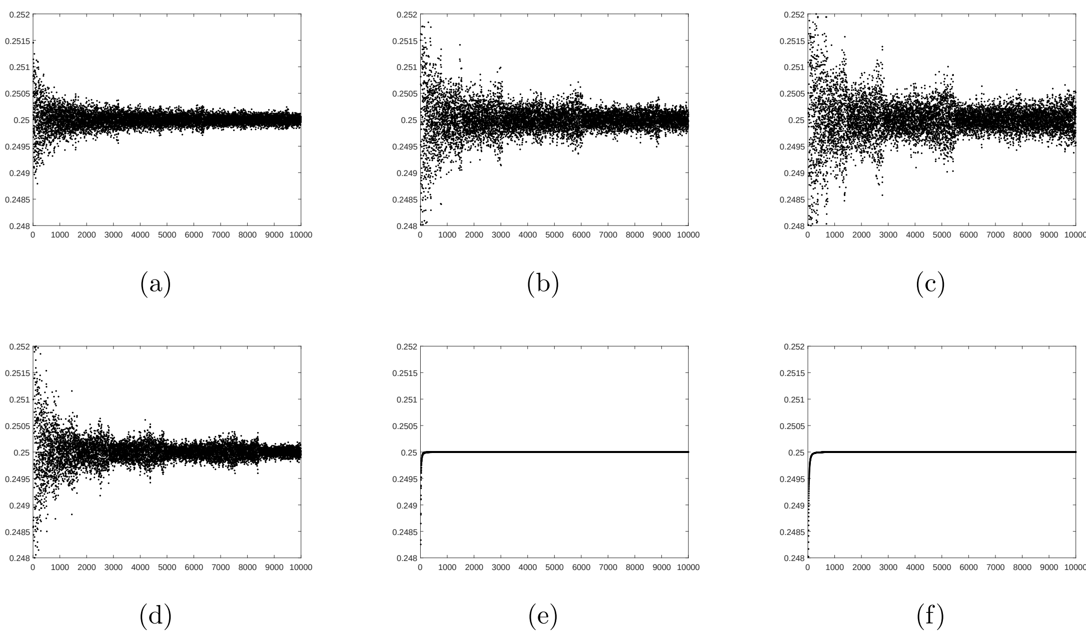

FIGURE 12. Graphs of r_n for the infinite Bernoulli convolutions in (1.1), where ρ are the selected Salem numbers in Table4.

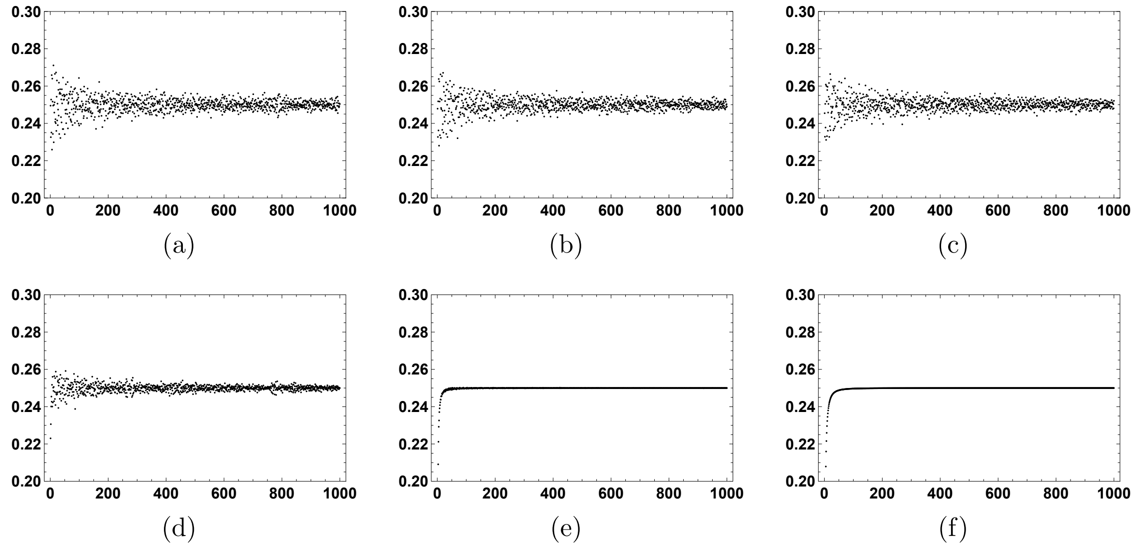

FIGURE 13. Graphs of r_n for the weighted infinite Bernoulli convolutions in (1.2) with weights $w_1 = 1/4$ and $w_2 = 3/4$, where ρ are the selected Salem numbers in Table 4.

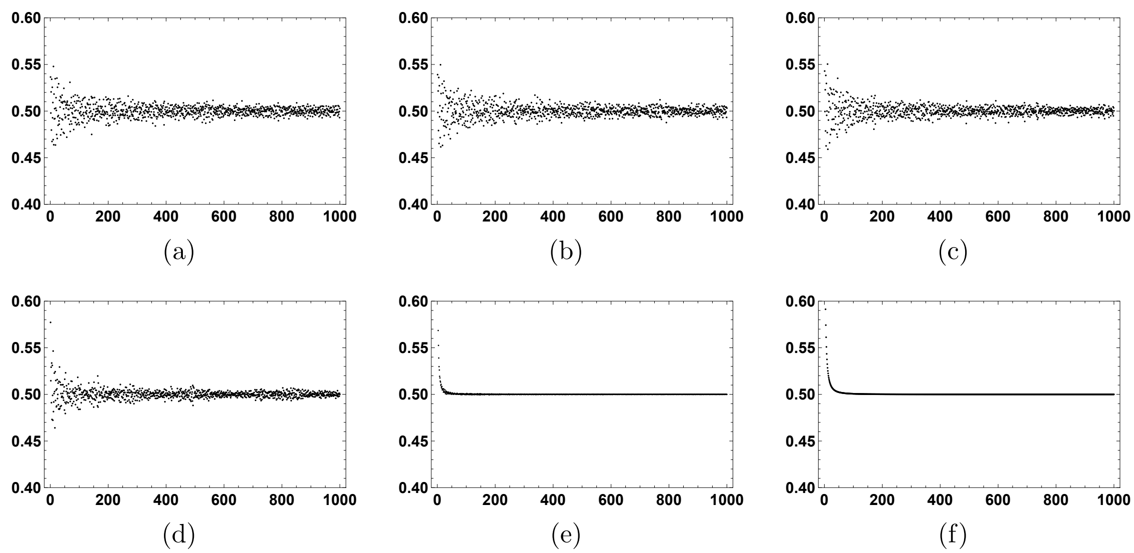

FIGURE 14. Graphs of A_n for the weighted infinite Bernoulli convolutions in (1.2) with weights $w_1 = 1/4$ and $w_2 = 3/4$, where ρ are the selected Salem numbers in Table 4.

6. Comments and open questions

In view of numerical experiments in Section 5, we formulate the following conjecture.

Conjecture 6.1. *The family of infinite Bernoulli convolutions in* (1.2) *belong to Nevai's class, regardless of the probability weights* w_1 *and* w_2*.*

In fact, it is of interest to study a weaker version of this conjecture: the average of A_n tends to $1/2$ and the average of r_n tends to $1/4$.

It is also of interest to study the behaviors of the recurrence coefficients A_n and r_n for other one-dimensional self-similar measures with overlaps, such convolutions of Cantor-type measures (see, e.g., [11, 16]) and measures that are so-called essentially of finite type (see, e.g., [17]).

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