SELF-SIMILAR SOLUTIONS OF σ_k^α-CURVATURE FLOW

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ABSTRACT. In this paper, employing a new inequality, we show that under certain curvature pinching condition, the strictly convex closed smooth self-similar solution of σ_k^α-flow must be a round sphere. We also obtain a similar result for the solutions of $F = -\langle X, e_{n+1} \rangle$ $(*)$ with a non-homogeneous function F. At last, we prove that if F can be compared with $\frac{(n-k+1)\sigma_{k-1}}{k\sigma_k}$, then a closed strictly k-convex solution of $(*)$ must be a round sphere.

1. INTRODUCTION

Let $X : M \to \mathbb{R}^{n+1}$ be a smooth embedding of a closed, orientable n-dimensional manifold with $n \geq 2$. Choose an orthonormal frame in $\mathbb{R}^{n+1}$ along M such that $\{e_1, e_2, \cdots, e_n\}$ are tangent to M and e_{n+1} is the inward-pointing unit normal vector of M. Under such a frame, let $A = \{h_{ij}\}$ denote the components of the second fundamental form of X, then the principal curvatures $\lambda_1, \cdots, \lambda_n$ of M are eigenvalues of the second fundamental form A. Define

$$\sigma_k(A) = \frac{1}{k!} \delta \begin{pmatrix} i_1 i_2 \cdots i_n \\ j_1 j_2 \cdots j_n \end{pmatrix} h_{i_1 j_1} h_{i_2 j_2} \cdots h_{i_n j_n},$$

where $\delta \begin{pmatrix} i_1 i_2 \cdots i_n \\ j_1 j_2 \cdots j_n \end{pmatrix}$ is the generalized Kronecker symbol. We use the summation convention throughout this paper unless otherwise stated. For convenience, we set $\sigma_0 = 1$ and $\sigma_k = 0$ for $k > n$.

In this paper, we consider a hypersurface M which satisfies the following equation

$$(1.1) \qquad F(A(x)) = -\langle X(x), e_{n+1}(x) \rangle, \text{ for all } x \in M,$$

where $F(A) = f(\lambda)$ is a smooth function of principal curvatures and $\langle \, , \, \rangle$ denotes the standard Euclidean metric in $\mathbb{R}^{n+1}$.

This type of equation is important for curvature flow of the following type

$$(1.2) \qquad \tilde{X}_t = F(A)e_{n+1}.$$

Actually, if X is a solution of (1.1) and F is homogeneous of degree β, then

$$\tilde{X}(x,t) = ((\beta+1)(T-t))^{\frac{1}{1+\beta}} X(x)$$

gives rise to the solution of (1.2) up to a tangential diffeomorphism [17]. So in the same spirit, we call the solutions of (1.1) self-similar solutions of (1.2). Moreover, for $F = H$, the solution of (1.1) is usually called self-shrinker which describes the

asymptotic behavior of mean curvature flow (see [15, 11]). Huisken proved the following theorem.

Theorem 1.1 ([15]). *If M^n, $n \geq 2$, is a closed hypersurface in $\mathbb{R}^{n+1}$, with non-negative mean curvature H and satisfies the equation*

$$H = -\langle X, e_{n+1} \rangle,$$

then M^n is a round sphere of radius $\sqrt{n}$.

Similar to the case of mean curvature, the solution of (1.1) with $F = \sigma_n^\alpha$ also describes the asymptotic behavior of α-Gauss curvature flow (see [4, 5, 14, 16]). Very recently, for $F = \sigma_n^\alpha$, Brendle, Choi and Daskalopoulos proved that the solution of (1.1) is either a round sphere for $\alpha > \frac{1}{n+2}$ or an ellipsoid for $\alpha = \frac{1}{n+2}$ (see [7, 8]).

In [17], McCoy considered the case which F is a class of concave or convex homogeneous functions of principal curvature with degree 1. Under certain pinching condition, he also obtained a result for higher degree homogeneous functions.

For homogeneous functions with degree greater than 1, the convergence of α-Gauss curvature flow is well-studied. For the flows (1.2) of convex hypersurfaces by speeds $F = H^\alpha$ [18], $F = \sigma_2^\alpha$ [1, 2] and more general F with certain properties [6], similar results are obtained under certain curvature pinching conditions.

In this paper we first consider the self-similar solutions of σ_k^α-flow for strictly convex hypersurfaces when $k \geq 2$ and $\alpha > \frac{1}{k}$ under a different type of curvature pinching condition.

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ denote the principal curvatures of M. M is said to be strictly convex if $\lambda \in \Gamma_+ = \{\mu \in \mathbb{R}^n | \mu_1 > 0, \mu_2 > 0, \cdots, \mu_n > 0\}$ for any point in M. Denote

$$\sigma_k(\lambda) = \sigma_k(\lambda(A)) = \sum_{1 \leq i_1 \leq i_2 \cdots \leq i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

Let $\sigma_k(\lambda|i)$ denote the symmetric function $\sigma_k(\lambda)$ with $\lambda_i = 0$ and $\sigma_k(\lambda|ij)$, with $i \neq j$, denote the symmetric function $\sigma_k(\lambda)$ with $\lambda_i = \lambda_j = 0$. The following two basic equalities are needed in our investigation of the σ_k^α self-similar solutions.

$$\sigma_{k-1}(\lambda|i) = \frac{\partial \sigma_k(\lambda)}{\partial \lambda_i}, \quad \sigma_{k-2}(\lambda|ij) = \frac{\partial^2 \sigma_k(\lambda)}{\partial \lambda_i \partial \lambda_j}.$$

Condition 1.2. Assume

$$\frac{\sigma_1^2(\lambda)\sigma_{k-2}(\lambda|ij)}{(\alpha k - 1)\sigma_1(\lambda)\sigma_{k-1}(\lambda|p) - (\alpha - 1)k^2\sigma_k(\lambda)} \in [1, 1+\delta]$$

holds for all $1 \leq p \leq n, 1 \leq i < j \leq n$, where

$$(1.3) \qquad \delta = \begin{cases} \dfrac{3}{n-1}, & \text{if } k = 2, \\[2mm] \dfrac{\sqrt{n^2 + 8n - 8} + 2 - n}{2(n-1)}, & \text{if } 3 \leq k \leq n-1. \end{cases}$$

Our main result can be stated as follows.

Theorem 1.3. *Let M be a closed strictly convex hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 2$. If $F = \sigma_k^\alpha$ with $2 \leq k \leq n-1$, $\alpha > \frac{1}{k}$ and Condition 1.2 holds, then the solution of (1.1) is a round sphere.*

Remark 1.4. If M is totally umbilical, then Condition 1.2 leads to

$$\frac{\sigma_1^2(\lambda)\sigma_{k-2}(\lambda|ij)}{(\alpha k - 1)\sigma_1(\lambda)\sigma_{k-1}(\lambda|p) - (\alpha - 1)k^2\sigma_k(\lambda)} = \frac{n}{n-1} \in [1, 1+\delta],$$

which implies that Condition 1.2 is a pinching condition for principal curvatures.

Remark 1.5. In the case of $F = \sigma_2$, Condition 1.2 satisfies if $3\lambda_{\min} \geq \lambda_{\max}$, where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum principal curvature and maximum principal curvature, respectively.

Remark 1.6. Brendle, Choi and Daskalopoulos obtained better result for the case of $F = \sigma_n^\alpha$ in [7, 8], so we omit this case in the statement of Theorem 1.3.

Somewhat surprisingly, the above argument enables us to discuss the solution of (1.1) with a non-homogeneous function $F = \sum_{l=1}^n a_l\sigma_l$, where a_l are nonnegative constants with $\sum_{l=2}^n a_l > 0$. Thus we obtain the following result.

Theorem 1.7. *Let M be a closed strictly convex hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 2$. If $F = \sum_{l=1}^n a_l\sigma_l$ and the condition $\lambda_{min} \geq \Theta\lambda_{max}$ holds, where $0 < \Theta \leq 1$ is a constant depending on n, then, the solution of (1.1) is a round sphere.*

Remark 1.8. For $F = \sum_{l=1}^n a_l\sigma_l^{\alpha_l}$ with $\alpha_l > \frac{1}{l}$, under suitable pinching condition, the solution of (1.1) is also a round sphere. The proof is similar to the proof of Theorem 1.7.

For $F = \frac{\sigma_{k-1}}{\sigma_k}$ (which is used in [13]), the solution of (1.1) can be characterized as follows when M is strictly k-convex. A hypersurface M in $\mathbb{R}^{n+1}$ is strictly k-convex, if $\lambda(x) \in \Gamma_k = \{\mu \in \mathbb{R}^n | \sigma_1(\mu) > 0, \cdots, \sigma_k(\mu) > 0\}$ for all $x \in M$. Obviously, $\Gamma_+ = \Gamma_n \subset \Gamma_{n-1} \subset \cdots \subset \Gamma_1$.

Theorem 1.9. *Let M be a closed strictly k-convex hypersurface in $\mathbb{R}^{n+1}$. Assuming $F \geq \frac{(n-k+1)\sigma_{k-1}}{k\sigma_k}$ or $F \leq \frac{(n-k+1)\sigma_{k-1}}{k\sigma_k}$, if there exists a solution of (1.1), then $F = \frac{(n-k+1)\sigma_{k-1}}{k\sigma_k}$ and the solution must be a round sphere.*

The paper is organized as follows. In Section 2, we show a new inequality of symmetric functions, which plays an important role in the proof of our main result. Some basic equations are derived in Section 3. In Section 4, we use the maximum principle to establish our main result (Theorem 1.3). We devote Section 5 to a discussion on the solution of (1.1) with a non-homogeneous function F. Finally the proof of Theorem 1.9 is presented in Section 6.

2. A NEW INEQUALITY OF SYMMETRIC FUNCTIONS

In this section we show a new inequality of symmetric functions, which may have its own interest.

Lemma 2.1. *For any $2 \leq k \leq n$ and $\lambda \in \Gamma_+$, we have*

$$\frac{1}{k(k-1)}\sigma_1(\lambda) - \frac{k\sigma_k(\lambda)}{(k-1)\sigma_{k-1}(\lambda)} + \frac{(k+1)\sigma_{k+1}(\lambda)}{k\sigma_k(\lambda)} \geq 0.$$

Equality occurs if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$.

Proof. Let $S_k(\lambda)$ denote the power sum of λ defined by $S_k(\lambda) = \sum_{i=1}^{n} \lambda_i^k$. Then $2\sigma_2 = S_1^2 - S_2$ and $3\sigma_3 = \frac{1}{2}S_1^3 - \frac{3}{2}S_1 S_2 + S_3$. Thus for $k = 2$, we have

$$\sigma_1 - 4\frac{\sigma_2}{\sigma_1} + 3\frac{\sigma_3}{\sigma_2}$$

$$= \frac{1}{\sigma_1 \sigma_2}(\sigma_1^2 \sigma_2 - 4\sigma_2^2 + 3\sigma_1 \sigma_3)$$

$$= \frac{1}{\sigma_1 \sigma_2}\left(\frac{1}{2}S_1^2(S_1^2 - S_2) - (S_1^2 - S_2)^2 + S_1(\frac{1}{2}S_1^3 - \frac{3}{2}S_1 S_2 + S_3)\right)$$

$$= \frac{1}{\sigma_1 \sigma_2}(S_1 S_3 - S_2^2)$$

$$= \frac{1}{\sigma_1 \sigma_2}\left(\sum_{i<j} \lambda_i \lambda_j (\lambda_i - \lambda_j)^2\right) \geq 0$$

and equality occurs if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$.

We complete the proof by induction for k and assume the lemma is true for $\{2, 3, \cdots, k-1\}$. Let

$$f(x) = \prod_{i=1}^{n}(1 - \lambda_i x) = \sum_{m=0}^{n}(-1)^m \sigma_m(\lambda) x^m.$$

Then

$$\frac{d}{dx}f(x) = \sum_{m=1}^{n}(-1)^m m\sigma_m(\lambda) x^{m-1}.$$

On the other hand, since $\frac{d}{dx}f(x)$ is a polynomial of degree $n-1$, by Rolle's theorem, if all roots of a polynomial $f(x)$ are real and positive, then the same is true for its derivative. This leads to

$$\frac{d}{dx}f(x) = -\sigma_1(\lambda)\prod_{i=1}^{n-1}(1 - \mu_i x) = -\sigma_1(\lambda)\sum_{l=0}^{n-1}(-1)^l \sigma_l(\mu) x^l.$$

By comparing the above two expressions, we conclude that

(2.1) $(m+1)\sigma_{m+1}(\lambda) = \sigma_1(\lambda)\sigma_m(\mu)$ for $0 \leq m \leq n-1.$

Thus, for $2 \leq k \leq n-1$, we obtain

$$\frac{(k+1)\sigma_{k+1}(\lambda)}{k\sigma_k(\lambda)} = \frac{\sigma_k(\mu)}{\sigma_{k-1}(\mu)}$$

$$\geq \frac{k-1}{k}\left(\frac{(k-1)\sigma_{k-1}(\mu)}{(k-2)\sigma_{k-2}(\mu)} - \frac{\sigma_1(\mu)}{(k-1)(k-2)}\right)$$

$$= \frac{k-1}{k}\left(\frac{k\sigma_k(\lambda)}{(k-2)\sigma_{k-1}(\lambda)} - \frac{2\sigma_2(\lambda)}{(k-1)(k-2)\sigma_1(\lambda)}\right)$$

$$= \frac{(k-1)^2}{k(k-2)}\frac{k\sigma_k(\lambda)}{(k-1)\sigma_{k-1}(\lambda)} - \frac{2\sigma_2(\lambda)}{k(k-2)\sigma_1(\lambda)}$$

$$= \frac{k\sigma_k(\lambda)}{(k-1)\sigma_{k-1}(\lambda)} + \frac{1}{k(k-2)}\left(\frac{k\sigma_k(\lambda)}{(k-1)\sigma_{k-1}(\lambda)} - \frac{2\sigma_2(\lambda)}{\sigma_1(\lambda)}\right)$$

$$= \frac{k\sigma_k(\lambda)}{(k-1)\sigma_{k-1}(\lambda)} + \frac{1}{k(k-2)}\sum_{i=2}^{k-1}\left(\frac{(i+1)\sigma_{i+1}(\lambda)}{i\sigma_i(\lambda)} - \frac{i\sigma_i(\lambda)}{(i-1)\sigma_{i-1}(\lambda)}\right)$$

$$\geq \frac{k\sigma_k(\lambda)}{(k-1)\sigma_{k-1}(\lambda)} - \frac{1}{k(k-2)} \sum_{i=2}^{k-1} \frac{\sigma_1(\lambda)}{i(i-1)}$$

$$= \frac{k\sigma_k(\lambda)}{(k-1)\sigma_{k-1}(\lambda)} - \frac{1}{k(k-1)}\sigma_1(\lambda).$$

For $k = n$, since $\sigma_{n+1} = 0$, it is confirmed by the Newton-MacLaurin inequalities. We finish the proof by noticing all equalities occur if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. $\qquad\square$

Remark 2.2. The condition $\lambda \in \Gamma_+$ seems necessary because for $k = 2, n = 3$ and $\lambda = (-1, 3, 3)$, we have $\sigma_1(\lambda) > 0$ and $\sigma_2(\lambda) > 0$ but

$$\sigma_1 - 4\frac{\sigma_2}{\sigma_1} + 3\frac{\sigma_3}{\sigma_2} = \frac{1}{\sigma_1\sigma_2}\Big(\sum_{i<j} \lambda_i\lambda_j(\lambda_i - \lambda_j)^2\Big) < 0.$$

The following corollary of Lemma 2.1 will be used in Section 4.

Corollary 2.3. *For any $2 \leq k \leq n$ and $\lambda \in \Gamma_+$, we have*

$$(k-1)\sigma_1(\lambda) - 2k\frac{\sigma_2(\lambda)}{\sigma_1(\lambda)} + (k+1)\frac{\sigma_{k+1}(\lambda)}{\sigma_k(\lambda)} \geq 0.$$

Equality occurs if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$.

Proof. Notice

$$\frac{(k+1)\sigma_{k+1}(\lambda)}{k\sigma_k(\lambda)} - \frac{2\sigma_2(\lambda)}{\sigma_1(\lambda)} = \sum_{i=2}^{k} \left(\frac{(i+1)\sigma_{i+1}(\lambda)}{i\sigma_i(\lambda)} - \frac{i\sigma_i(\lambda)}{(i-1)\sigma_{i-1}(\lambda)} \right)$$

$$\geq -\sum_{i=2}^{k} \frac{\sigma_1(\lambda)}{i(i-1)}$$

$$= -(1 - \frac{1}{k})\sigma_1(\lambda),$$

where the inequality follows from Lemma 2.1. $\qquad\square$

To finish this section, we list one well-known result (See for example [3] and [12]).

Lemma 2.4. *If $W = (w_{ij})$ is a symmetric real matrix and $\lambda_m = \lambda_m(W)$ is one of its eigenvalues $(m = 1, \cdots, n)$. If $f = f(\lambda)$ is a function on $\mathbb{R}^n$ and $F = F(W) = f(\lambda(W))$, then for any real symmetric matrix $B = (b_{ij})$, we have the following formulae:*

(i) $\dfrac{\partial F}{\partial w_{ij}} b_{ij} = \dfrac{\partial f}{\partial \lambda_p} b_{pp},$

(ii) $\dfrac{\partial^2 F}{\partial w_{ij} \partial w_{st}} b_{ij} b_{st} = \dfrac{\partial^2 f}{\partial \lambda_p \partial \lambda_q} b_{pp} b_{qq} + 2 \sum_{p<q} \dfrac{\frac{\partial f}{\partial \lambda_p} - \frac{\partial f}{\partial \lambda_q}}{\lambda_p - \lambda_q} b_{pq} b_{qp}.$

Remark 2.5. In the above lemma, $\dfrac{\frac{\partial f}{\partial \lambda_p} - \frac{\partial f}{\partial \lambda_q}}{\lambda_p - \lambda_q}$ is interpreted as a limit if $\lambda_p = \lambda_q$.

3. Equations of the test function

In this section, we will obtain some useful equations by direct computation from the following equation

$$(3.1) \qquad\qquad\qquad F = -\langle X, e_{n+1}\rangle.$$

Differentiating (3.1) gives

$$\nabla_j F = h_{jl}\langle X, e_l\rangle,$$

and

$$(3.2) \qquad\qquad \nabla_i\nabla_j F = h_{jli}\langle X, e_l\rangle + h_{ij} - h_{jl}h_{li}F.$$

Then, we obtain

$$(3.3) \qquad \frac{\partial F}{\partial h_{ij}}\nabla_i\nabla_j F = \nabla_l F\langle X, e_l\rangle + \frac{\partial F}{\partial h_{ij}}h_{ij} - \frac{\partial F}{\partial h_{ij}}h_{jl}h_{li}F.$$

Let $G = G(A) = g(\lambda(A))$ be a homogeneous function, called the test function in this paper. By directly calculation, we obtain

$$\nabla_i\nabla_j G = \nabla_i\left(\frac{\partial G}{\partial h_{pq}}h_{pqj}\right) = \frac{\partial^2 G}{\partial h_{pq}\partial h_{st}}h_{pqj}h_{sti} + \frac{\partial G}{\partial h_{pq}}h_{pqji}.$$

By Codazzi equation and Ricci identity, we obtain

$$h_{pqji} = h_{pjqi} = h_{pjiq} + h_{mj}R_{mpqi} + h_{pm}R_{mjqi}.$$

Furthermore, using Gauss equation gives rise to

$$h_{pqji} = h_{ijpq} + h_{mj}(h_{mq}h_{pi} - h_{mi}h_{pq}) + h_{pm}(h_{mq}h_{ji} - h_{mi}h_{jq}).$$

Then, we have

$$\begin{aligned}
\frac{\partial F}{\partial h_{ij}}\nabla_i\nabla_j G &= \frac{\partial F}{\partial h_{ij}}\frac{\partial^2 G}{\partial h_{pq}\partial h_{st}}h_{pqj}h_{sti} + \frac{\partial F}{\partial h_{ij}}\frac{\partial G}{\partial h_{pq}}h_{ijpq} \\
&\quad + \frac{\partial F}{\partial h_{ij}}\frac{\partial G}{\partial h_{pq}}(h_{mj}(h_{mq}h_{pi} - h_{mi}h_{pq}) + h_{pm}(h_{mq}h_{ji} - h_{mi}h_{jq})) \\
&= \frac{\partial F}{\partial h_{ij}}\frac{\partial^2 G}{\partial h_{pq}\partial h_{st}}h_{pqj}h_{sti} + \frac{\partial G}{\partial h_{pq}}\left(\nabla_p\nabla_q F - \frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ijp}h_{stq}\right) \\
&\quad + \frac{\partial F}{\partial h_{ij}}\frac{\partial G}{\partial h_{pq}}(-h_{mj}h_{mi}h_{pq} + h_{pm}h_{mq}h_{ji}).
\end{aligned}$$

Moreover, using (3.2), we obtain

$$(3.4)\qquad
\begin{aligned}
&\frac{\partial F}{\partial h_{ij}}\nabla_i\nabla_j G - \nabla_l G\langle X, e_l\rangle \\
&= \frac{\partial G}{\partial h_{ij}}h_{ij}\left(1 - \frac{\partial F}{\partial h_{pq}}h_{pm}h_{mq}\right) + \frac{\partial G}{\partial h_{ij}}h_{jl}h_{li}\left(\frac{\partial F}{\partial h_{pq}}h_{pq} - F\right) \\
&\quad + \left(\frac{\partial F}{\partial h_{ij}}\frac{\partial^2 G}{\partial h_{pq}\partial h_{st}} - \frac{\partial G}{\partial h_{ij}}\frac{\partial^2 F}{\partial h_{pq}\partial h_{st}}\right)h_{pqj}h_{sti}.
\end{aligned}$$

The above equation is elliptic if $\frac{\partial F}{\partial h_{ij}}$ is positive definite. In fact, for $\lambda \in \Gamma_+$, the cases that we discuss are all elliptic.

For convenience, we denote the first two terms on the right hand of the above equality by TERM I and the last term by TERM II. In fact, by Lemma 2.4, we have

$$\text{TERM I} = \frac{\partial g}{\partial \lambda_i} \lambda_i \left(1 - \frac{\partial f}{\partial \lambda_p} \lambda_p^2 \right) + \frac{\partial g}{\partial \lambda_i} \lambda_i^2 \left(\frac{\partial f}{\partial \lambda_p} \lambda_p - f \right)$$

and

$$\text{TERM II} = \left(\frac{\partial f}{\partial \lambda_i} \frac{\partial^2 g}{\partial \lambda_p \partial \lambda_q} - \frac{\partial g}{\partial \lambda_i} \frac{\partial^2 f}{\partial \lambda_p \partial \lambda_q} \right) h_{ppi} h_{qqi}$$
$$+ 2 \sum_{p<q} \left(\frac{\frac{\partial g}{\partial \lambda_p} - \frac{\partial g}{\partial \lambda_q}}{\lambda_p - \lambda_q} \frac{\partial f}{\partial \lambda_i} - \frac{\frac{\partial f}{\partial \lambda_p} - \frac{\partial f}{\partial \lambda_q}}{\lambda_p - \lambda_q} \frac{\partial g}{\partial \lambda_i} \right) h_{pqi}^2.$$

4. THE CASE OF $F = \sigma_k^\alpha$ ($k \geq 2$)

In this section, we consider the case of $F = \sigma_k^\alpha$ ($k \geq 2$) and we choose $G = \frac{\sigma_1^k}{\sigma_k}$ as the test function. By straightforward calculation, we obtain the following expressions which will be used later:

$$(4.1) \qquad \frac{\partial^2 f}{\partial \lambda_p \partial \lambda_q} = \alpha f \left(\frac{(\alpha-1)\sigma_{k-1}(\lambda|p)\sigma_{k-1}(\lambda|q)}{\sigma_k^2} + \frac{\sigma_{k-2}(\lambda|pq)}{\sigma_k} \right),$$

$$(4.2) \qquad \frac{\frac{\partial f}{\partial \lambda_p} - \frac{\partial f}{\partial \lambda_q}}{\lambda_p - \lambda_q} = -\alpha f \frac{\sigma_{k-2}(\lambda|pq)}{\sigma_k},$$

$$(4.3) \qquad \frac{\partial g}{\partial \lambda_p} = g \left(\frac{k}{\sigma_1} - \frac{\sigma_{k-1}(\lambda|p)}{\sigma_k} \right),$$

$$(4.4) \qquad \begin{aligned} \frac{\partial^2 g}{\partial \lambda_p \partial \lambda_q} = g \Big(& \frac{k(k-1)}{\sigma_1^2} - \frac{k(\sigma_{k-1}(\lambda|p) + \sigma_{k-1}(\lambda|q))}{\sigma_1 \sigma_k} \\ & - \frac{\sigma_{k-2}(\lambda|pq)}{\sigma_k} + \frac{2\sigma_{k-1}(\lambda|p)\sigma_{k-1}(\lambda|q)}{\sigma_k^2} \Big), \end{aligned}$$

$$(4.5) \qquad \frac{\frac{\partial g}{\partial \lambda_p} - \frac{\partial g}{\partial \lambda_q}}{\lambda_p - \lambda_q} = g \frac{\sigma_{k-2}(\lambda|pq)}{\sigma_k}.$$

We will see TERM I is non-negative and TERM II is non-negative under Condition 1.2.

Lemma 4.1. *For $F = \sigma_k^\alpha$ ($k \geq 2$) and $G = \frac{\sigma_1^k}{\sigma_k}$, TERM I is non-negative for $\alpha \geq \frac{1}{k}$. Moreover, it vanishes for $\alpha > \frac{1}{k}$ if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$.*

Proof. Noting that $G = \frac{\sigma_1^k}{\sigma_k}$ is homogeneous of degree 0, we have $\frac{\partial g}{\partial \lambda_i} \lambda_i = 0$. And, (4.3) and $\sum_{i=1}^n \sigma_{k-1}(\lambda|i)\lambda_i^2 = \sigma_1(\lambda)\sigma_k(\lambda) - (k+1)\sigma_{k+1}(\lambda)$ yield

$$\frac{\partial g}{\partial \lambda_p} \lambda_p^2 = g \left(\frac{k(\sigma_1^2 - 2\sigma_2)}{\sigma_1} - \frac{\sigma_{k-1}(\lambda|p)\lambda_p^2}{\sigma_k} \right)$$
$$= g \left((k-1)\sigma_1 - 2k\frac{\sigma_2}{\sigma_1} + (k+1)\frac{\sigma_{k+1}}{\sigma_k} \right).$$

Thus,

$$\text{TERM I} = (k\alpha - 1)fg\left((k-1)\sigma_1 - 2k\frac{\sigma_2}{\sigma_1} + (k+1)\frac{\sigma_{k+1}}{\sigma_k}\right).$$

For $f > 0$ and $g > 0$, the proof is finished by Corollary 2.3. □

Now, we consider TERM II and complete the proof of Theorem 1.1.

Lemma 4.2. *For $F = \sigma_k^\alpha$ $(k \geq 2)$ and $G = \frac{\sigma_1^k}{\sigma_k}$, if G attains its maximum at x_0, then, at x_0,*

$$\text{TERM II} = \frac{\alpha k f g}{\sigma_1^3 \sigma_k}\Big(\sum_i \sum_{p \neq q} \sigma_1^2 \sigma_{k-2}(\lambda|pq)(h_{pqi}^2 - h_{ppi}h_{qqi})$$
$$+ \sum_i \big(-(\alpha-1)k^2\sigma_k + (\alpha k - 1)\sigma_1 \sigma_{k-1}(\lambda|i)\big)(\nabla_i \sigma_1)^2\big)\Big).$$

Proof. By Lemma 2.4, we have $\sigma_{k-1}(\lambda|p)h_{ppi} = \nabla_i \sigma_k$. Then, by (4.1) and (4.4),

$$\frac{\partial^2 f}{\partial \lambda_p \partial \lambda_q}h_{ppi}h_{qqi} = \alpha f\left(\sum_i \frac{(\alpha-1)(\nabla_i \sigma_k)^2}{\sigma_k^2} + \sum_i \sum_{p \neq q} \frac{\sigma_{k-2}(\lambda|pq)}{\sigma_k}h_{ppi}h_{qqi}\right)$$

and

$$\frac{\partial^2 g}{\partial \lambda_p \partial \lambda_q}h_{ppi}h_{qqi} = g\sum_i\Big(\frac{k(k-1)(\nabla_i \sigma_1)^2}{\sigma_1^2} - \frac{2k\nabla_i \sigma_1 \nabla_i \sigma_k}{\sigma_1 \sigma_k}$$
$$- \sum_{p \neq q} \frac{\sigma_{k-2}(\lambda|pq)}{\sigma_k}h_{ppi}h_{qqi} + \frac{2(\nabla_i \sigma_k)^2}{\sigma_k^2}\Big).$$

Since G attains its maximum at x_0, then $\nabla_l G = 0$ at x_0 which implies $\frac{k\nabla_l \sigma_1}{\sigma_1} = \frac{\nabla_l \sigma_k}{\sigma_k}$ at x_0. Thus, at x_0,

$$\frac{\partial^2 f}{\partial \lambda_p \partial \lambda_q}h_{ppi}h_{qqi} = \alpha f\left(\sum_i \frac{(\alpha-1)k^2(\nabla_i \sigma_1)^2}{\sigma_1^2} + \sum_i \sum_{p \neq q} \frac{\sigma_{k-2}(\lambda|pq)}{\sigma_k}h_{ppi}h_{qqi}\right)$$

and

$$\frac{\partial^2 g}{\partial \lambda_p \partial \lambda_q}h_{ppi}h_{qqi} = g\Big(\sum_i \frac{k(k-1)(\nabla_i \sigma_1)^2}{\sigma_1^2} - \sum_i \sum_{p \neq q} \frac{\sigma_{k-2}(\lambda|pq)}{\sigma_k}h_{ppi}h_{qqi}\Big).$$

Furthermore, using (4.2), (4.3), (4.5), at x_0, we get

$$\text{TERM II} = \frac{\alpha k f g}{\sigma_1^3 \sigma_k}\Big(\sum_i \sum_{p \neq q} \sigma_1^2 \sigma_{k-2}(\lambda|pq)(h_{pqi}^2 - h_{ppi}h_{qqi})$$
$$+ \sum_i \big(-(\alpha-1)k^2\sigma_k + (\alpha k - 1)\sigma_1 \sigma_{k-1}(\lambda|i)\big)(\nabla_i \sigma_1)^2\big)\Big).$$

□

For convenience, we denote

$$A_{ij} = \sigma_1^2 \sigma_{k-2}(\lambda|ij)$$

and

$$B_p = -(\alpha - 1)k^2\sigma_k + (\alpha k - 1)\sigma_1\sigma_{k-1}(\lambda|p).$$

Then,

$$\text{TERM II} = \frac{\alpha k f g}{\sigma_1^3 \sigma_k}\Big(\sum_{i \neq j}\sum_p A_{ij}(h_{ijp}^2 - h_{iip}h_{jjp}) + \sum_p B_p(\nabla_p\sigma_1)^2)\Big).$$

Lemma 4.3. *Let M be a closed strictly convex hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 2$ satisfying Condition 1.2. For $F = \sigma_k^\alpha$ ($k \geq 2$) and $G = \frac{\sigma_1^k}{\sigma_k}$, if G attains its maximum at x_0, then, at x_0, TERM II is non-negative.*

Proof. It suffices to check if $\sum_{i \neq j}\sum_p A_{ij}(h_{ijp}^2 - h_{iip}h_{jjp}) + \sum_p B_p(\nabla_p\sigma_1)^2$ is non-negative. Firstly, we notice

$$\sum_{i \neq j}\sum_p A_{ij}(h_{ijp}^2 - h_{iip}h_{jjp}) + \sum_p B_p(\nabla_p\sigma_1)^2$$

$$= \sum_{i \neq j} A_{ij}(h_{iji}^2 + h_{ijj}^2) + \sum_{\neq} A_{ij}h_{ijp}^2 - \sum_{i \neq j} A_{ij}(h_{iii}h_{jji} + h_{iij}h_{jjj})$$

$$- \sum_{\neq} A_{ij}h_{iip}h_{jjp} + \sum_p B_p\Big(\sum_i h_{iip}^2 + \sum_{i \neq j} h_{iip}h_{jjp}\Big)$$

$$= 2\sum_{i \neq j} A_{ij}h_{iij}^2 + \sum_{\neq} A_{ij}h_{ijp}^2 - 2\sum_{i \neq j} A_{ij}h_{iii}h_{jji} - \sum_{\neq} A_{ij}h_{iip}h_{jjp} + \sum_i B_i h_{iii}^2$$

$$+ \sum_{i \neq p} B_p h_{iip}^2 + \sum_{i \neq j}(B_i h_{iii}h_{jji} + B_j h_{iij}h_{jjj}) + \sum_{\neq} B_p h_{iip}h_{jjp}$$

$$= \sum_i B_i h_{iii}^2 + \sum_{i \neq j}(2A_{ij} + B_j)\, h_{iij}^2 + \sum_{\neq} A_{ij}h_{ijp}^2 + 2\sum_{i \neq j}(-A_{ij} + B_i)\, h_{iii}h_{jji}$$

$$+ \sum_{\neq}(-A_{ij} + B_p)\, h_{iip}h_{jjp},$$

where $\neq$ represents i, j, p are pairwise distinct.

Now, we estimate the lower bounds of the last two terms. For fixed i, j and p, we have

$$2\left(-A_{ij} + B_i\right) h_{iii}h_{jji} \geq -a_{ij}h_{iii}^2 - b_{ij}h_{jji}^2\,,$$

where $a_{ij} > 0, b_{ij} > 0$ are constants satisfying

$$(4.6) \qquad\qquad a_{ij}b_{ij} = \left(-A_{ij} + B_i\right)^2.$$

And

$$\left(-A_{ij} + B_p\right) h_{iip}h_{jjp} \geq -c_{ijp}h_{iip}^2 - d_{ijp}h_{jjp}^2\,,$$

where $c_{ijp} > 0$, $d_{ijp} > 0$ are constants satisfying

$$(4.7) \qquad\qquad 4c_{ijp}d_{ijp} = \left(-A_{ij} + B_p\right)^2$$

and $c_{ijp} = c_{jip}$, $d_{ijp} = d_{jip}$.

Thus we obtain

$$\sum_{i \neq j} \sum_p A_{ij}(h_{ijp}^2 - h_{iip}h_{jjp}) + \sum_p B_p(\nabla_p \sigma_1)^2$$

$$\geq \sum_i \left(B_i - \sum_{j \neq i} a_{ij} \right) h_{iii}^2 + \sum_{i \neq j} \left(2A_{ij} + B_j - b_{ji} - \sum_{p \neq i, p \neq j} (c_{ipj} + d_{pij}) \right) h_{iij}^2$$

$$+ \sum_{\neq} A_{ij} h_{ijp}^2.$$

Condition 1.2 implies $B_i > 0$, then we can choose $a_{ij} = \frac{1}{n-1} B_i$. Then, from (4.6), we have

$$b_{ij} = (-A_{ij} + B_i)^2 \, a_{ij}^{-1} = \frac{(n-1)(-A_{ij} + B_i)^2}{B_i}.$$

And, we can choose $c_{ijp} = d_{ijp}$, because h_{iip} and h_{jjp} are the same type of terms. Furthermore, from (4.7), we obtain

$$c_{ipj} = d_{pij} = \frac{1}{2} | - A_{ip} + B_j|.$$

Then, we just need

$$2A_{ij} + B_j \geq \frac{(n-1)(-A_{ij} + B_j)^2}{B_j} + \sum_{p \neq i, p \neq j} | - A_{ip} + B_j|.$$

For $B_j > 0$, the above inequality is equivalent to

(4.8) $$\frac{2A_{ij}}{B_j} + 1 \geq (n-1) \left(-\frac{A_{ij}}{B_j} + 1 \right)^2 + \sum_{p \neq i, p \neq j} | -\frac{A_{ip}}{B_j} + 1|.$$

It is easy to check this inequality holds if $1 \leq \frac{A_{ij}}{B_p} \leq 1 + \delta$ with δ satisfies (1.3) for all $1 \leq p \leq n$ and $1 \leq i < j \leq n$.   □

Proof of Theorem 1.4. The proof is completed by the maximum principle. The equation (3.4) is elliptic and at the maximum point of G, the left hand side of (3.4) is non-positive. But, under Condition 1.2, we know the right hand side of (3.4) is non-negative. This means TERM I must be zero. By Lemma 2.1, we obtain $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. By Newton-Maclaurin inequality, we know $G = \frac{\sigma_1^k}{\sigma_k}$ also reaches its minimum, therefore is a constant. So, $\lambda_1 = \lambda_2 = \cdots = \lambda_n$ is established everywhere on M which implies M is a round sphere.   □

5. For $F = \sum_{l=1}^n a_l \sigma_l$

In this section, for a non-homogeneous function $F = \sum_{l=1}^n a_l \sigma_l$, where a_l is a nonnegative constant and $\sum_{l=2}^n a_l > 0$, we choose $G = \frac{\sigma_1^n}{\sigma_n}$ as the test function. We will analyze TERM I and TERM II as in the previous section.

Lemma 5.1. *For $F = \sum_{l=1}^n a_l \sigma_l$ and $G = \frac{\sigma_1^n}{\sigma_n}$, TERM I is non-negative. Moreover, it vanishes if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$.*

Proof. Just need to notice that $G = \frac{\sigma_1^n}{\sigma_n}$ is homogeneous of degree 0 and $\frac{\partial f}{\partial \lambda_i} \lambda_i - f = \sum_{l=2}^n (l-1) a_l \sigma_l > 0$, the rest of the proof is similar to Lemma 4.1.   □

Now, we regard TERM II as an operator on $C^\infty(M)$, i.e.,

$$\text{TERM II} = \Phi(f, g),$$

where $\Phi : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$. Obviously, $\Phi(\sum_{l=1}^n a_l \sigma_l, g) = \sum_{l=1}^n a_l \Phi(\sigma_l, g)$. Now, we consider $\Phi(\sigma_l, g)$.

Lemma 5.2. *For $G = \frac{\sigma_1^n}{\sigma_n}$, if G attains its maximum at x_0, then, at x_0,*

$$\Phi(\sigma_l, g) = g\Bigg(\sum_i \sum_{p \neq q} \Big(\frac{\sigma_{l-1}(\lambda|i)}{\lambda_p \lambda_q} + \frac{n\sigma_{l-2}(\lambda|pq)}{\sigma_1} - \frac{\sigma_{l-2}(\lambda|pq)}{\lambda_i} \Big)(h_{pqi}^2 - h_{ppi}h_{qqi})$$

$$+ \sum_i \frac{n(n-1)\sigma_{l-1}(\lambda|i)}{\sigma_1^2}(\nabla_i \sigma_1)^2 \Bigg).$$

Proof. By (4.3) and (4.5), we have

$$\frac{\partial g}{\partial \lambda_p} = g\left(\frac{n}{\sigma_1} - \frac{1}{\lambda_p} \right)$$

and

$$\frac{\frac{\partial g}{\partial \lambda_p} - \frac{\partial g}{\partial \lambda_q}}{\lambda_p - \lambda_q} = \frac{g}{\lambda_p \lambda_q}.$$

Since G attains its maximum at x_0, $\nabla_l G = 0$, which implies $\frac{n\nabla_l \sigma_1}{\sigma_1} = \frac{\nabla_l \sigma_n}{\sigma_n}$ at x_0. Thus, at x_0,

$$\frac{\partial^2 g}{\partial \lambda_p \partial \lambda_q} h_{ppi}h_{qqi} = g\Big(\sum_i \frac{n(n-1)(\nabla_i \sigma_1)^2}{\sigma_1^2} - \sum_i \sum_{p \neq q} \frac{1}{\lambda_p \lambda_q} h_{ppi}h_{qqi} \Big).$$

Furthermore, we obtain

$$\Phi(\sigma_l, g) = g\Bigg(\sum_i \sum_{p \neq q} \Big(\frac{\sigma_{l-1}(\lambda|i)}{\lambda_p \lambda_q} + \frac{n\sigma_{l-2}(\lambda|pq)}{\sigma_1} - \frac{\sigma_{l-2}(\lambda|pq)}{\lambda_i} \Big)(h_{pqi}^2 - h_{ppi}h_{qqi})$$

$$+ \sum_i \frac{n(n-1)\sigma_{l-1}(\lambda|i)}{\sigma_1^2}(\nabla_i \sigma_1)^2 \Bigg).$$

$\square$

For convenience, let

$$A_{ijp} = \frac{\sigma_{l-1}(\lambda|i)}{\lambda_p \lambda_q} + \frac{n\sigma_{l-2}(\lambda|pq)}{\sigma_1} - \frac{\sigma_{l-2}(\lambda|pq)}{\lambda_i}$$

and

$$B_p = \frac{n(n-1)}{\sigma_1^2}\sigma_{l-1}(\lambda|p).$$

Then,

$$\Phi(\sigma_l, g) = g\Big(\sum_{i \neq j} \sum_p A_{ijp}(h_{ijp}^2 - h_{iip}h_{jjp}) + \sum_p B_p(\nabla_p \sigma_1)^2 \Big).$$

Lemma 5.3. *Let M be a strictly convex hypersurface in $\mathbb{R}^{n+1}$ satisfying the condition $\lambda_{min} \geq \theta(l, n)\lambda_{max}$, where $0 < \theta(l, n) \leq 1$ is a constant depending on l and n. For $G = \frac{\sigma_1^n}{\sigma_n}$, if G attains its maximum at x_0, then, at x_0, $\Phi(\sigma_l, g)$ is non-negative.*

Proof. Similar to the proof of Lemma 4.3, we obtain

$$\sum_{i\neq j}\sum_p A_{ijp}(h_{ijp}^2 - h_{iip}h_{jjp}) + \sum_p B_p(\nabla_p\sigma_1)^2$$

$$= \sum_i B_i h_{iii}^2 + \sum_{i\neq j}(2A_{iji} + B_j)h_{iij}^2 + \sum_{\neq} A_{ijp}h_{ijp}^2 + 2\sum_{i\neq j}(B_i - A_{iji})h_{iii}h_{jji}$$

$$+ \sum_{\neq}(B_p - A_{ijp})h_{iip}h_{jjp}.$$

We first estimate the lower bounds of the last two terms. As the previous section,

$$2\left(B_i - A_{iji}\right)h_{iii}h_{jji} \geq -a_{ij}h_{iii}^2 - b_{ij}h_{jji}^2,$$

where $a_{ij} > 0, b_{ij} > 0$ are constants satisfying

$$(5.1) \qquad\qquad a_{ij}b_{ij} = \left(B_i - A_{iji}\right)^2.$$

And

$$\left(B_p - A_{ijp}\right)h_{iip}h_{jjp} \geq -c_{ijp}h_{iip}^2 - d_{ijp}h_{jjp}^2,$$

where $c_{ijp} > 0$, $d_{ijp} > 0$ are constants satisfying

$$(5.2) \qquad\qquad 4c_{ijp}d_{ijp} = \left(B_p - A_{ijp}\right)^2,$$

and $c_{ijp} = c_{jip}$, $d_{ijp} = d_{jip}$.

Then, we obtain

$$\sum_{i\neq j}\sum_p A_{ijp}(h_{ijp}^2 - h_{iip}h_{jjp}) + \sum_p B_p(\nabla_p\sigma_1)^2$$

$$\geq \sum_i \left(B_i - \sum_{j\neq i}a_{ij}\right)h_{iii}^2 + \sum_{i\neq j}\left(2A_{iji} + B_j - b_{ji} - \sum_{p\neq i, p\neq j}(c_{ipj} + d_{pij})\right)h_{iij}^2$$

$$+ \sum_{\neq} A_{ijp}h_{ijp}^2.$$

We can choose $a_{ij} = \frac{1}{n-1}B_i$. Then, From (5.1), we have

$$b_{ij} = \left(-A_{iji} + B_i\right)^2 a_{ij}^{-1} = \frac{(n-1)\left(-A_{iji} + B_i\right)^2}{B_i}.$$

And, we can choose $c_{ijp} = d_{ijp}$, because h_{iip} and h_{jjp} are the same type of terms. Furthermore, from (5.2), we can take

$$c_{ipj} = d_{pij} = \frac{1}{2}|-A_{ipj} + B_j|.$$

Then, we just need

$$2A_{iji} + B_j \geq \frac{(n-1)\left(-A_{iji} + B_j\right)^2}{B_j} + \sum_{p\neq i, p\neq j}|-A_{ipj} + B_j|.$$

For $B_j > 0$, the above inequality is equivalent to

$$(5.3) \qquad \frac{2A_{iji}}{B_j} + 1 \geq (n-1)\left(-\frac{A_{iji}}{B_j} + 1\right)^2 + \sum_{p\neq i, p\neq j}|-\frac{A_{ipj}}{B_j} + 1|.$$

Notice that $\frac{A_{ijp}}{B_q} = \frac{n}{n-1}$ at umbilical points of M for any $1 \le i, j, p, q \le n$. Thus we can assume

$$(5.4) \qquad\qquad 1 < \frac{A_{ijp}}{B_q} < 1 + \delta.$$

Then, by solving

$$3 \ge (n-1)\delta^2 + (n-2)\delta,$$

we can choose $\delta = \frac{\sqrt{n^2+8n-8}+2-n}{2(n-1)}$ such that (5.3) holds. By direct calculation, we can choose

$$\theta(l,n) = \begin{cases} \max(\sqrt{\dfrac{n-1}{n}}, \sqrt{\dfrac{n}{(n-1)(1+\delta)}}), & \text{for } l = 1, \\[3ex] \max\left(\left(\dfrac{\frac{n-1}{n}C_{n-1}^{l-1} + C_{n-2}^{l-2}}{C_{n-1}^{l-1} + C_{n-2}^{l-2}}\right)^{\frac{1}{l+1}}, \left(\dfrac{C_{n-1}^{l-1} + C_{n-2}^{l-2}}{\frac{(1+\delta)(n-1)}{n}C_{n-1}^{l-1} + C_{n-2}^{l-2}}\right)^{\frac{1}{l+1}}\right), & \text{for } l = 2, ..., n, \end{cases}$$

such that under the condition $\lambda_{min} > \theta(l,n)\lambda_{max}$, (5.4) holds, where $C_n^k = \frac{n!}{k!(n-k)!}$. Thus the proof is completed. $\qquad\square$

Proof of Theorem 1.7. Now, let $\Theta(n) = \max\limits_{l=1,...,n} \theta(n,l)$. Then, under condition $\lambda_{min} > \Theta\lambda_{max}$, $\Phi(\sigma_l, g)$ is non-negative for all l. Therefore, TERM II is non-negative under the condition. Similarly, by the maximum principle we complete the proof. $\qquad\square$

6. Proof of Theorem 1.9

Proof of Theorem 1.9. By Minkowski identity, we have

$$k \int_M \sigma_k \langle X, e_{n+1}\rangle d\mu + (n-k+1) \int_M \sigma_{k-1} d\mu = 0.$$

By (1.1), we have

$$(6.1) \qquad\qquad 0 = \int_M k\sigma_k \left(-F + \frac{(n-k+1)\sigma_{k-1}}{k\sigma_k}\right) d\mu.$$

Since $\sigma_k > 0$ and $-F + \frac{(n-k+1)\sigma_{k-1}}{k\sigma_k}$ is non-negative or non-positive, we know $F = \frac{(n-k+1)\sigma_{k-1}}{k\sigma_k}$. Notice that (6.1) also holds for $k-1$. Combining Newton-MacLaurin inequalities, we have

$$0 = \int_M (k-1)\sigma_{k-1}\left(-F + \frac{(n-k+2)\sigma_{k-2}}{(k-1)\sigma_{k-1}}\right) d\mu$$
$$= \int_M (k-1)\sigma_{k-1}\left(-\frac{(n-k+1)\sigma_{k-1}}{k\sigma_k} + \frac{(n-k+2)\sigma_{k-2}}{(k-1)\sigma_{k-1}}\right) d\mu \le 0.$$

This implies

$$\frac{(n-k+1)\sigma_{k-1}}{k\sigma_k} = \frac{(n-k+2)\sigma_{k-2}}{(k-1)\sigma_{k-1}}$$

on M. Thus, we have $\lambda_1 = \lambda_2 = \cdots = \lambda_n$ for every point of M which means M is a round sphere. $\qquad\square$

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