HAMILTONIAN STABILITY OF THE GAUSS IMAGES OF HOMOGENEOUS ISOPARAMETRIC HYPERSURFACES

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Abstract. The image of the Gauss map of any oriented isoparametric hypersurface of the unit standard sphere $S^{n+1}(1)$ is a minimal Lagrangian submanifold in the complex hyperquadric $Q_n(\mathbf{C})$. In this paper we show that the Gauss image of a compact oriented isoparametric hypersurface with g distinct constant principal curvatures in $S^{n+1}(1)$ is a compact monotone and cyclic embedded Lagrangian submanifold with minimal Maslov number $2n/g$. The main result of this paper is to determine completely the Hamiltonian stability of all compact minimal Lagrangian submanifolds embedded in complex hyperquadrics which are obtained as the images of the Gauss map of homogeneous isoparametric hypersurfaces in the unit spheres, by harmonic analysis on homogeneous spaces and fibrations on homogeneous isoparametric hypersurfaces. In addition, the discussions on the exceptional Riemannian symmetric space $(E_6, U(1)\cdot Spin(10))$ and the corresponding Gauss image have their own interest.

Introduction

In 1990's Oh initialized the study of Hamiltonian minimality and Hamiltonian stability of Lagrangian submanifolds in Kähler manifolds ([33], [34], [35]). It provides a constrained volume variational problem of Lagrangian submanifolds in Kähler manifolds under Hamiltonian deformations. Thus it is natural to study what Lagrangian submanifolds in specific Kähler manifolds are Hamiltonian stable. After Oh's pioneer papers, there has been extensive research done on Hamiltonian stabilities of minimal or Hamiltonian minimal Lagrangian submanifolds in various Kähler manifolds, such as complex Euclidean spaces, complex projective spaces, compact Hermitian symmetric spaces, certain toric Kähler manifolds and so on. (See e.g., [2, 9, 38, 40, 43, 49] and references therein.) In particular, a compact minimal Lagrangian submanifold L in a compact homogeneous Einstein-Kähler manifold with positive Einstein constant κ is Hamiltonian stable if and only if the first (positive) eigenvalue λ_1 of the Laplacian of L with respect to the induced metric satisfies $\lambda_1 = \kappa$. Hence in this case, to determine the Hamiltonian stability reduces to calculating the first eigenvalue of the Laplacian, which is an important problem in differential geometry.

On the other hand, isoparametric hypersurfaces are next simple hypersurfaces in spheres after geodesic spheres. The theory of isoparametric hypersurfaces in spheres was originated by Élie Cartan and well developed afterward. Particularly great progress on the classification problem of isoparametric hypersurfaces in spheres were made by the recent work of Cecil-Chi-Jensen ([10]), Immervoll ([21]), Chi ([12, 13]) and Miyaoka ([30]). Among all important

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results of isoparametric hypersurfaces in spheres, Münzner ([31], [32]) showed that the number g of distinct principal curvatures of an isoparametric hypersurface N^n in $S^{n+1}(1)$ must be $g = 1, 2, 3, 4, 6$ and N^n is always real algebraic in the sense that N^n is defined by a certain real homogeneous polynomial of degree g which is called the "Cartan-Münzner polynomial".

It is known that the Gauss image of any compact oriented isoparametric hypersurface in the unit standard sphere is a smooth compact embedded minimal Lagrangian submanifold in the complex hyperquadric and the Gauss map is a covering map with covering transformation group $\mathbf{Z}_g$ ([43, 26]). Thus it can be expected that the Gauss images of isoparametric hypersurfaces in spheres provide a nice class of compact Lagrangian submanifolds embedded in complex hyperquadrics and moreover they should play certain roles in symplectic geometry. Besides properties of Gauss images discussed in our previous paper [26], in this paper we show (see Theorem 2.1)

Theorem. *The Gauss image of a compact oriented isoparametric hypersurface with g distinct constant principal curvature in $S^{n+1}(1)$ is a compact monotone and cyclic embedded Lagrangian submanifold with minimal Maslov number $2n/g$.*

Recall that all isoparametric hypersurfaces in the unit standard sphere are classified into homogeneous ones and non-homogeneous ones. An isoparametric hypersurface N^n in the unit standard sphere $S^{n+1}(1)$ is called *homogeneous* if N^n can be obtained as an orbit of a compact Lie subgroup of $SO(n+2)$. Every homogeneous isoparametric hypersurface in a sphere can be obtained as a principal orbit of a linear isotropy representation of a compact Riemannian symmetric pair (U, K) of rank 2, due to Hsiang-Lawson ([19]) and Takagi-Takahashi ([45]). Only in the case of $g = 4$ there are known to exist non-homogeneous isoparametric hypersurfaces, which were discovered first by Ozeki-Takeuchi ([41], [42]) and extensively generalized by Ferus-Karcher-Münzner ([14]). The purpose of this paper is to determine completely the Hamiltonian stability of all compact minimal Lagrangian embedded submanifolds in $Q_n(\mathbf{C})$ which are obtained as the Gauss images of homogeneous isoparametric hypersurfaces in $S^{n+1}(1)$. This paper is a continuation of [26], where we have already treated the cases of $g = 1, 2, 3$.

The main result of this paper is as follows :

Theorem. *Suppose that (U, K) is not of type* EIII*, that is, $(U, K) \neq (E_6, U(1) \cdot Spin(10))$. Then the Gauss image $L = \mathcal{G}(N)$ is not Hamiltonian stable if and only if $m_2 - m_1 \geq 3$. Moreover if (U, K) is of type* EIII*, then $(m_1, m_2) = (6, 9)$ but $L = \mathcal{G}(N)$ is strictly Hamiltonian stable.*

This paper is organized as follows: In Section 1 we recall the notions and fundamental properties on Hamiltonian minimality, Hamiltonian stability and strictly Hamiltonian stability of Lagrangian submanifolds in Kähler manifolds. In Section 2 we briefly explain properties of minimal Lagrangian submanifolds in complex hyperquadrics as the Gauss images of isoparametric hypersurfaces in spheres. In Section 3 we explain the method of eigenvalue computations of our compact homogeneous spaces which are the Gauss images of compact homogeneous isoparametric hypersurfaces in spheres, and the fibrations on homogeneous isoparametric hypersurfaces by homogeneous isoparametric hypersurfaces. The fibrations are very useful for our computation. In Sections 4 and 5, we determine the strictly Hamiltonian stability of the Gauss images of compact homogeneous isoparametric hypersurfaces with $g = 6$. In Sections 6-11, we determine the strictly Hamiltonian stability of the Gauss images of compact homogeneous isoparametric hypersurfaces with $g = 4$. In particular, the discussions on the exceptional

Riemannian symmetric space $(E_6, U(1) \cdot Spin(10))$ and the corresponding Gauss image have their own interest.

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1. Hamiltonian minimality and Hamiltonian stability

Assume that (M, ω, J, g) is a Kähler manifold with the compatible complex structure J and Kähler metric g. Let $\varphi : L \to M$ be a Lagrangian immersion and H denote the mean curvature vector field of φ. The corresponding 1-form $\alpha_{\mathrm{H}} := \omega(\mathrm{H}, \cdot) \in \Omega^1(L)$ is called the *mean curvature form* of φ. For simplicity, throughout this paper we assume that L is compact without boundary.

Definition 1.1. Let M be a Kähler manifold. A Lagrangian immersion $\varphi : L \to M$ is called *Hamiltonian minimal* (shortly, H-minimal) or *Hamiltonian stationary*, if it is the critical point of the volume functional for all Hamiltonian deformations $\{\varphi_t\}$.

The corresponding Euler-Lagrange equation is $\delta\alpha_{\mathrm{H}} = 0$, where δ is the co-differential operator with the respect to the induced metric on L.

Definition 1.2. An H-minimal Lagrangian immersion φ is called *Hamiltonian stable* (shortly, *H-stable*) if the second variation of the volume is nonnegative under every Hamiltonian deformation $\{\varphi_t\}$.

The second variational formula is given as follows ([35]):

$$
\begin{aligned}
&\frac{d^2}{dt^2}\mathrm{Vol}\,(L, \varphi_t^* g)|_{t=0} \\
&= \int_L \left(\langle \Delta_L^1 \alpha, \alpha \rangle - \langle \overline{R}(\alpha), \alpha \rangle - 2\langle \alpha \otimes \alpha \otimes \alpha_{\mathrm{H}}, S \rangle + \langle \alpha_{\mathrm{H}}, \alpha \rangle^2 \right) dv,
\end{aligned}
$$

where Δ_L^1 denotes the Laplace operator of $(L, \varphi^* g)$ acting on the vector space $\Omega^1(L)$ of smooth 1-forms on L and $\alpha := \omega(V, \cdot) \in B(L)$ is the exact 1-form corresponding to an infinitesimal Hamiltonian deformation V. Here

$$\langle \overline{R}(\alpha), \alpha \rangle := \sum_{i,j=1}^{n} \mathrm{Ric}^M(e_i, e_j)\alpha(e_i)\alpha(e_j)$$

for a local orthonormal frame $\{e_i\}$ on L and

$$S(X, Y, Z) := \omega(B(X, Y), Z)$$

for each $X, Y, Z \in C^\infty(TL)$, which is a symmetric 3-tensor field on L defined by the second fundamental form B of L in M.

For an H-minimal Lagrangian immersion $\varphi : L \to M$, we denote by $E_0(\varphi)$ the null space of the second variation on $B^1(L)$, or equivalently the solution space to the linearized H-minimal Lagrangian submanifold equation, and we call $n(\varphi) := \dim E_0(\varphi)$ the *nullity* of φ.

If $H^1(M, \mathbf{R}) = \{0\}$, then any holomorphic Killing vector field on M is a Hamiltonian vector field, and thus it generates a volume-preserving Hamiltonian deformation of φ. Namely,

$$\{\varphi^*\alpha_X \mid X \text{ is a holomorphic Killing vector field on } M\} \subset E_0(\varphi) \subset B^1(L).$$

Set $n_{hk}(\varphi) := \dim\{\varphi^*\alpha_X \mid X \text{ is a holomorphic Killing vector field on } M\}$, which is called the *holomorphic Killing nullity* of φ.

Definition 1.3. An H-minimal Lagrangian immersion φ is called *strictly Hamiltonian stable* (shortly, strictly H-stable) if φ is Hamiltonian stable and $n_{hk}(\varphi) = n(\varphi)$.

Note that if L is strictly Hamiltonian stable, then L has local minimum volume under each Hamiltonian deformation.

In the case when L is a compact minimal Lagrangian submanifold in an Einstein-Käher manifold M with Einstein constant κ, the second variational formula becomes much simpler. we see that L is H-stable if and only if the first (positive) eigenvalue λ_1 of the Laplacian of L acting on smooth functions satisfies $\lambda_1 \geq \kappa$ ([33]). On the other hand, it is known that the first eigenvalue λ_1 of the Laplacian of any compact minimal Lagrangian submanifold L in a compact homogeneous Einstein-Kähler manifold with positive Einstein constant κ has the upper bound $\lambda_1 \leq \kappa$ ([37], [38]). In this case, L is H-stable if and only if $\lambda_1 = \kappa$.

Assume that (M, ω, J, g) is a Kähler manifold and G is an analytic subgroup of its automorphism group $\mathrm{Aut}(M, \omega, J, g)$. A Lagrangian orbit $L = G \cdot x \subset M$ of G is called a *homogeneous Lagrangian submanifold* of M. An easy but useful observation can be given as follows.

Proposition 1.1. *Any compact homogeneous Lagrangian submanifold in a Kähler manifold is Hamiltonian minimal.*

Proof. Since α_{H} is an invariant 1-form on L, $\delta\alpha_{\mathrm{H}}$ is a constant function on L. Hence by the divergence theorem we obtain $\delta\alpha_{\mathrm{H}} = 0$. □

Set

$$\tilde{G} := \{a \in \mathrm{Aut}(M, \omega, J, g) \mid a(L) = L\}.$$

Then $G \subset \tilde{G}$ and $\tilde{G}$ is the maximal subgroup of $\mathrm{Aut}(M, \omega, J, g)$ preserving L. Moreover we have $n_{hk}(\varphi) = \dim(\mathrm{Aut}(M, \omega, J, g)) - \dim(\tilde{G})$.

2. Gauss maps of isoparametric hypersurfaces in a sphere

2.1. **Gauss maps of oriented hypersurfaces in spheres.** Let N^n be an oriented hypersurface immersed in the unit standard sphere $S^{n+1}(1) \subset \mathbf{R}^{n+2}$. Denote by $\mathbf{x}$ its position vector of a point p of N and $\mathbf{n}$ the unit normal vector field of N in $S^{n+1}(1)$. It is a fundamental fact in symplectic geometry that the *Gauss map* defined by

$$\mathcal{G} : N^n \ni p \longmapsto \mathbf{x}(p) \wedge \mathbf{n}(p) \cong [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] \in \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cong Q_n(\mathbf{C})$$

is always a Lagrangian immersion in the complex hyperquadric $Q_n(\mathbf{C})$. Here the complex hyperquadric $Q_n(\mathbf{C})$ is identified with the real Grassmann manifold $\widetilde{Gr}_2(\mathbf{R}^{n+2})$ of oriented 2-dimensional vector subspaces of $\mathbf{R}^{n+2}$, which has a symmetric space expression $SO(n+2)/(SO(2) \times SO(n))$.

Let $g^{std}_{Q_n(\mathbf{C})}$ be the standard Kähler metric of $Q_n(\mathbf{C})$ induced from the standard inner product of $\mathbf{R}^{n+2}$. Note that the Einstein constant of $g^{std}_{Q_n(\mathbf{C})}$ is equal to n. Let κ_i $(i = 1, \cdots, n)$ denote the principal curvatures of $N^n \subset S^{n+1}(1)$ and H denote the mean curvature vector field of the Gauss map $\mathcal{G}$. Palmer showed the following mean curvature form formula ([43]):

$$\alpha_{\mathrm{H}} = -d\left(\sum_{i=1}^{n} \operatorname{arc}\cot\kappa_i\right) = d\left(\operatorname{Im}\left(\log\prod_{i=1}^{n}(1+\sqrt{-1}\kappa_i)\right)\right). \tag{2.1}$$

Hence, if N^n is an oriented austere hypersurface in $S^{n+1}(1)$, introduced by Harvey-Lawson ([18]), then its Gauss map $\mathcal{G} : N^n \to Q_n(\mathbf{C})$ is a minimal Lagrangian immersion. In particular, since any minimal surface in $S^3(1)$ is austere, its Gauss map is a minimal Lagrangian immersion in $Q_2(\mathbf{C}) \cong S^2 \times S^2$ ([9]). Note that more minimal Lagrangian submanifolds of complex hyperquadrics can be obtained from Gauss maps of certain oriented hypersurfaces in spheres through Palmer's formula ([23]).

2.2. **Gauss maps of isoparametric hypersurfaces in spheres.** Now suppose that N^n is a compact oriented hypersurface in $S^{n+1}(1)$ with constant principal curvatures, i.e., *isoparametric hypersurface*. By Müzner's result ([31, 32]), the number g of distinct principal curvatures must be $1, 2, 3, 4$ or 6, and the distinct principal curvatures have the multiplicities $m_1 = m_3 = \cdots$, $m_2 = m_4 = \cdots$. We may assume that $m_1 \leq m_2$. It follows from (2.1) that its Gauss map $\mathcal{G} : N^n \to Q_n(\mathbf{C})$ is a minimal Lagrangian immersion. Moreover, the "Gauss image"of $\mathcal{G}$ is a compact minimal Lagrangian submanifold $L^n = \mathcal{G}(N^n) \cong N^n/\mathbf{Z}_g$ *embedded* in $Q_n(\mathbf{C})$ so that $\mathcal{G} : N^n \to \mathcal{G}(N^n) = L^n$ is a covering map with the Deck transformation group $\mathbf{Z}_g$ ([26], [27]).

Here we mention the following symplectic topological properties of the Gauss images of isoparametric hypersurfaces.

Theorem 2.1. *The Gauss image $L = \mathcal{G}(N^n)$ is a compact monotone and cyclic Lagrangian submanifold embedded in $Q_n(\mathbf{C})$ and its minimal Maslov number Σ_L is given by*

$$\Sigma_L = \frac{2n}{g} = \begin{cases} m_1 + m_2, & \text{if } g \text{ is even;} \\ 2m_1, & \text{if } g \text{ is odd.} \end{cases}$$

We need to use the following H. Ono's result which generalizes Oh's work [36].

Lemma 2.1 ([37])**.** *Let M be a simply connected Kähler-Einstein manifold with positive scalar curvature with a prequantization complex line bundle E. Then any compact minimal Lagrangian submanifold L in M is monotone and cyclic. Moreover the minimal Maslov number Σ_L of L satisfies the following relation:*

$$n_L\,\Sigma_L \,=\, 2\,\gamma_{c_1}, \tag{2.2}$$

where

$$\gamma_{c_1} := \min\{c_1(M)(A) \mid A \in H_2(M;\mathbb{Z}), c_1(M)(A) > 0\} \in \mathbf{Z}$$

is called the index of a Kähler manifold M and

$$n_L := \min\{k \in \mathbf{Z} \mid k \geq 1, \otimes^k (E, \nabla)_{|L} \textit{ is trivial}\}.$$

Using this lemma and the properties of isoparametric hypersurfaces in a sphere, we shall prove Theorem 2.1.

Proof. It follows from Lemma 2.1 and the minimality of the Gauss image $L = \mathcal{G}(N^n)$ that L is a monotone and cyclic Lagrangian submanifold in $Q_n(\mathbf{C})$. Remark that the index of $Q_n(\mathbf{C})$ is known as follows ([6]): $\gamma_{c_1} = n$ if $n \geq 2$ and $\gamma_{c_1} = 2$ if $n = 1$. So in order to find the minimal

Maslov number Σ_L of L, we only need to compute n_L. Let $\tilde{N}^n$ be the Legendrian lift of N^n to the unit tangent sphere bundle $UTS^{n+1}(1) = V_2(\mathbf{R}^{n+2})$. Then $\pi : V_2(\mathbf{R}^{n+2})|_L \to L = \mathcal{G}(N^n)$ is a flat principal fiber bundle with structure group $SO(2)$ and the covering map $\pi : \tilde{N}^n \to \mathcal{G}(N^n)$ with Deck transformation group $\mathbf{Z}_g$ coincides with its holonomy subbundle with the holonomy group $\mathbf{Z}_g$. Let E be a complex line bundle over $Q_n(\mathbf{C})$ associated with the principal fiber bundle $\pi : V_2(\mathbf{R}^{n+2}) \to \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cong Q_n(\mathbf{C})$ by the standard action of $SO(2) \cong U(1)$ on $\mathbf{C}$. Then $E|_L$ is a flat complex line bundle over $\mathcal{G}(N^n)$ associated with the principal fiber bundle $\pi : V_2(\mathbf{R}^{n+2})|_L \to \mathcal{G}(N^n)$ by the standard action of $SO(2) \cong U(1)$ on $\mathbf{C}$. The tautological complex line bundle $\mathcal{W}$ over $Q_n(\mathbf{C}) \subset \mathbf{C}P^{n+1}$ is defined by $\mathcal{W}_x := \mathbf{C}(\mathbf{a} + \sqrt{-1}\mathbf{b})$ for each $[\mathbf{a}+\sqrt{-1}\mathbf{b}] \in Q_n(\mathbf{C})$. Then $E = \mathcal{W}$ if $n \geq 2$ and $\otimes^2 E = \mathcal{W}$ if $n = 1$. Indeed, $c_1(\mathcal{W})(\mathbf{C}P^1) = 1$ if $n \geq 2$. Here, $\mathbf{C}P^1$ denotes the set of one-dimensional complex vector subspaces in a 2-dimensional isotropic vector subspace of $\mathbf{C}^{n+2}$. For $k = 1, \cdots, g$, the generator $e^{\sqrt{-1}\frac{2\pi}{g}}$ of the holonomy group $\mathbf{Z}_g$ on $E|_L$ induces the multiplication by $e^{\sqrt{-1}\frac{2\pi k}{g}}$ on $\otimes^k E|_L$. Thus the holonomy group of $\otimes^k E|_L$ is generated by $e^{\sqrt{-1}\frac{2\pi k}{g}}$ of $\mathbf{Z}_g$. Hence, $\otimes^k E|_L$ has non-trivial holonomy for $k = 1, \cdots, g-1$ and $\otimes^g E|_L$ has trivial holonomy. Therefore, $n_L = g$ if $n \geq 2$ and $n_L = 2$ if $n = 1$. Thus the conclusion follows from (2.2). $\square$

A hypersurface N^n in $S^{n+1}(1)$ is *homogeneous* if it is obtained as an orbit of a compact connected subgroup G of $SO(n+2)$. Obviously any homogeneous hypersurface in $S^{n+1}(1)$ is an isoparametric hypersurface. It turns out that N^n is homogeneous if and only if its Gauss image $\mathcal{G}(N^n)$ is homogeneous ([26]).

Consider

$$\mathcal{G} : N^n \ni p \longmapsto \mathbf{x}(p) \wedge \mathbf{n}(p) \in \widetilde{Gr}_2(\mathbf{R}^{n+2}) \subset \bigwedge^2 \mathbf{R}^{n+2}.$$

Here $\bigwedge^2 \mathbf{R}^{n+2} \cong \mathfrak{o}(n+2)$ can be identified with the Lie algebra of all (holomorphic) Killing vector fields on $S^{n+1}(1)$ or $\widetilde{Gr}_2(\mathbf{R}^{n+2})$. Let $\tilde{\mathfrak{k}}$ be the Lie subalgebra of $\mathfrak{o}(n+2)$ consisting of all Killing vector fields tangent to N^n or $\mathcal{G}(N^n)$ and $\tilde{K}$ be a compact connected Lie subgroup of $SO(n+2)$ generated by $\tilde{\mathfrak{k}}$. Take the orthogonal direct sum

$$\bigwedge^2 \mathbf{R}^{n+2} = \tilde{\mathfrak{k}} + \mathcal{V},$$

where $\mathcal{V}$ is a vector subspace of $\mathfrak{o}(n+2)$. The linear map

$$\mathcal{V} \ni X \longmapsto \alpha_X|_{\mathcal{G}(N^n)} \in E_0(\mathcal{G}) \subset B^1(\mathcal{G}(N^n))$$

is injective and $n_{hk}(\mathcal{G}) = \dim \mathcal{V}$. Then $\mathcal{G}(N^n) \subset \mathcal{V}$ and thus

$$\mathcal{G}(N^n) \subset \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V}.$$

Indeed, for each $X \in \tilde{\mathfrak{k}}$ and each $p \in N^n$, $\langle X, \mathbf{x}(p) \wedge \mathbf{n}(p)\rangle = \langle X\mathbf{x}(p), \mathbf{n}(p)\rangle - \langle \mathbf{x}(p), X\mathbf{n}(p)\rangle = 2\langle X\mathbf{x}(p), \mathbf{n}(p)\rangle = 0$.

Note that $\mathcal{G}(N^n)$ is a compact minimal submanifold embedded in the unit hypersphere of $\mathcal{V}$ and by the theorem of Tsunero Takahashi each coordinate function of $\mathcal{V}$ restricted to $\mathcal{G}(N^n)$ is an eigenfunction of the Laplace operator with eigenvalue n. Then we observe

Lemma 2.2. *n is just the first (positive) eigenvalue of $\mathcal{G}(N^n)$ if and only if $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is Hamiltonian stable. Moreover the dimension of the vector space $\mathcal{V}$ is equal to the multiplicity* *of the (resp. first) eigenvalue n if and only if $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is Hamiltonian rigid (resp. strictly Hamiltonian stable).*

Next we mention a relationship between the Gauss images $\mathcal{G}(N^n)$ of isoparametric hypersurfaces and the intersection $\widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V}$. In [27] we showed that if N^n is homogeneous, then $\mathcal{G}(N^n) = \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V}$.

Define a map $\mu : \widetilde{Gr}_2(\mathbf{R}^{n+2}) \to \bigwedge^2 \mathbf{R}^{n+2}$ by

$$\mu : \widetilde{Gr}_2(\mathbf{R}^{n+2}) \ni [W] \longmapsto \mathbf{a} \wedge \mathbf{b} \in \bigwedge^2 \mathbf{R}^{n+2} \cong \mathfrak{o}(n+2) = \tilde{\mathfrak{k}} + \mathcal{V}.$$

The moment map of the action $\tilde{K}$ on $\widetilde{Gr}_2(\mathbf{R}^{n+2})$ is given by $\mu_{\tilde{\mathfrak{k}}} := \pi_{\tilde{\mathfrak{k}}} \circ \mu : \widetilde{Gr}_2(\mathbf{R}^{n+2}) \to \tilde{\mathfrak{k}}$, where $\pi_{\tilde{\mathfrak{k}}} : \mathfrak{o}(n+2) \to \tilde{\mathfrak{k}}$ denotes the orthogonal projection onto $\tilde{\mathfrak{k}}$. For any $p \in N^n$, we have

$$\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) \subset \mathcal{G}(N^n) \subset \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V} = \mu_{\tilde{\mathfrak{k}}}^{-1}(0).$$

It is obvious that N^n is homogeneous if and only if $\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) = \mathcal{G}(N^n)$. In [27] we proved its inverse as follows: Assume that $\mathcal{G}(N^n) = \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V}$. Then $\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) = \mathcal{G}(N^n)$, that is, N^n is homogeneous. Therefore we obtain ([27]) that N^n is not homogeneous if and only if

$$\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) \subsetneqq \mathcal{G}(N^n) \subsetneqq \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V} = \mu_{\tilde{\mathfrak{k}}}^{-1}(0).$$

All isoparametric hypersurfaces in spheres are classified into homogeneous one and non-homogeneous one. Due to Hsiang-Lawson ([18]) and Takagi-Takahashi ([45]), any homogeneous isoparametric hypersurface in a sphere can be obtained as a principal orbit of the isotropy representation of a compact Riemannian symmetric pair (U, K) of rank 2 (see Table 1).

Compact homogeneous minimal Lagrangian submanifolds obtained as the Gauss images of homogeneous isoparametric hypersurfaces are constructed in the following way (cf. [26]). Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{u}$ as a symmetric Lie algebra of a symmetric pair (U, K) of rank 2 and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Define an $\mathrm{Ad}U$-invariant inner product $\langle\ ,\ \rangle_{\mathfrak{u}}$ of $\mathfrak{u}$ from the Killing-Cartan form of $\mathfrak{u}$. Then the vector space $\mathfrak{p}$ equipped with the inner product $\langle\ ,\ \rangle_{\mathfrak{u}}$ can be identified with the Euclidean space $\mathbf{R}^{n+2}$ and $S^{n+1}(1)$ denotes the $(n+1)$-dimensional unit standard sphere in $\mathfrak{p}$. The linear isotropy action $\mathrm{Ad}_{\mathfrak{p}}$ of K on $\mathfrak{p}$ and thus on $S^{n+1}(1)$ induces the group action of K on $\widetilde{\mathrm{Gr}}_2(\mathfrak{p}) \cong Q_n(\mathbf{C})$. For each *regular* element H of $\mathfrak{a} \cap S^{n+1}(1)$, we get a homogeneous isoparametric hypersurface in the unit sphere

$$N^n = (\mathrm{Ad}_{\mathfrak{p}} K)H \subset S^{n+1}(1) \subset \mathfrak{p} \cong \mathbf{R}^{n+2}.$$

Its Gauss image is

$$L^n = \mathcal{G}(N^n) = K \cdot [\mathfrak{a}] = [(\mathrm{Ad}_{\mathfrak{p}} K)\mathfrak{a}] \subset \widetilde{Gr}_2(\mathfrak{p}) \cong Q_n(\mathbf{C}).$$

Here N and $\mathcal{G}(N^n)$ have homogeneous space expressions $N \cong K/K_0$ and $\mathcal{G}(N^n) \cong K/K_{[\mathfrak{a}]}$, where we define

$$\begin{aligned}
K_0 &:= \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(H) = H\} \\
&= \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(H) = H \text{ for each } H \in \mathfrak{a}\}, \\
K_{\mathfrak{a}} &:= \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(\mathfrak{a}) = \mathfrak{a}\}, \\
K_{[\mathfrak{a}]} &:= \{k \in K_{\mathfrak{a}} \mid \mathrm{Ad}_{\mathfrak{p}}(k) : \mathfrak{a} \longrightarrow \mathfrak{a} \text{ preserves the orientation of } \mathfrak{a}\}.
\end{aligned}$$

The deck transformation group of the covering map $\mathcal{G} : N \to \mathcal{G}(N^n)$ is equal to $K_{[\mathfrak{a}]}/K_0 = W(U,K)/\mathbf{Z}_2 \cong \mathbf{Z}_g$, where $W(U,K) = K_{\mathfrak{a}}/K_0$ is the Weyl group of (U,K).

Since we know that $\mathrm{Ad}_{\mathfrak{p}}K$ is the maximal compact subgroup of $SO(n+2)$ preserving N and/or $\mathcal{G}(N^n)$ ([19], [26]), in this case its nullity is given as

$$n_{hk}(\mathcal{G}) = n_{hk}(\mathcal{G}(N^n)) = \dim SO(n+2) - \dim K.$$

TABLE 1. Homogeneous isoparametric hypersurfaces in spheres

g	*Type*	(U,K)	dimN	m_1, m_2	K/K_0
1	$S^1\times$ BDII	$(S^1 \times SO(n+2), SO(n+1))$ $(n \geq 1)$ $[\mathbf{R} \oplus A_1]$	n	n	S^n
2	BDII$\times$ BDII	$(SO(p+2) \times SO(n+2-p),$ $SO(p+1) \times SO(n+1-p))$ $(1 \leq p \leq n-1)$ $[A_1 \oplus A_1]$	n	$p, n-p$	$S^p \times S^{n-p}$
3	AI_2	$(SU(3), SO(3))$ $[A_2]$	3	$1,1$	$\frac{SO(3)}{\mathbf{Z}_2+\mathbf{Z}_2}$
3	$\mathfrak{a}_2$	$(SU(3) \times SU(3), SU(3))$ $[A_2]$	6	$2,2$	$\frac{SU(3)}{T^2}$
3	AII_2	$(SU(6), Sp(3))$ $[A_2]$	12	$4,4$	$\frac{Sp(3)}{Sp(1)^3}$
3	EIV	(E_6, F_4) $[A_2]$	24	$8,8$	$\frac{F_4}{Spin(8)}$
4	$\mathfrak{b}_2$	$(SO(5) \times SO(5), SO(5))$ $[B_2]$	8	$2,2$	$\frac{SO(5)}{T^2}$
4	AIII_2	$(SU(m+2), S(U(2) \times U(m)))$ $(m \geq 2)$ $[BC_2] (m \geq 3)$, $[B_2] (m=2)$	$4m-2$	$2,$ $2m-3$	$\frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))}$
4	BDI_2	$(SO(m+2), SO(2) \times SO(m))$ $(m \geq 3)$ $[B_2]$	$2m-2$	$1,$ $m-2$	$\frac{SO(2)\times SO(m)}{\mathbf{Z}_2 \times SO(m-2)}$
4	CII_2	$(Sp(m+2), Sp(2) \times Sp(m))$ $(m \geq 2)$ $[BC_2] (m \geq 3)$, $[B_2] (m=2)$	$8m-2$	$4,$ $4m-5$	$\frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)}$
4	DIII_2	$(SO(10), U(5))$ $[BC_2]$	18	$4,5$	$\frac{U(5)}{SU(2)\times SU(2)\times U(1)}$
4	EIII	$(E_6, U(1) \cdot Spin(10))$ $[BC_2]$	30	$6,9$	$\frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)}$
6	$\mathfrak{g}_2$	$(G_2 \times G_2, G_2)$ $[G_2]$	12	$2,2$	$\frac{G_2}{T^2}$
6	G	$(G_2, SO(4))$ $[G_2]$	6	$1,1$	$\frac{SO(4)}{\mathbf{Z}_2+\mathbf{Z}_2}$

3. THE METHOD OF EIGENVALUE COMPUTATIONS FOR OUR COMPACT HOMOGENEOUS SPACES

3.1. **Basic results from harmonic analysis on compact homogeneous spaces.** First we recall the basic theory of harmonic analysis on general compact homogeneous spaces (cf. [46]). Let $\mathcal{D}(G)$ be the complete set of all inequivalent irreducible unitary representations of a compact connected Lie group G. For a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, let $\Sigma(G)$ be the set of all roots of $\mathfrak{k}$ and $\Sigma^+(G)$ be its subset of all positive root $\alpha \in \Sigma(G)$ relative to a linear order fixed on $\mathfrak{t}$. Set

$$\Gamma(G) := \{\xi \in \mathfrak{t} \mid \exp(\xi) = e\},$$

$$Z(G) := \{\Lambda \in \mathfrak{t}^* \mid \Lambda(\xi) \in 2\pi\mathbf{Z} \text{ for each } \xi \in \Gamma(G)\},$$

$$D(G) := \{\Lambda \in Z(G) \mid \langle \Lambda, \alpha\rangle \geq 0 \text{ for each } \alpha \in \Sigma^+(G)\}.$$

Then there is a bijective correspondence between $D(G) \ni \Lambda \longmapsto (V_\Lambda, \rho_\Lambda) \in \mathcal{D}(G)$, where $(V_\Lambda, \rho_\Lambda)$ denotes an irreducible unitary representation of G with the highest weight Λ equipped with a $\rho_\Lambda(K)$-invariant Hermitian inner product $\langle\ ,\ \rangle_{V_\Lambda}$. Let $\langle\ ,\ \rangle_{\mathfrak{g}}$ be an AdG-invariant inner product of $\mathfrak{g}$. For a compact Lie subgroup H of G with Lie subalgebra $\mathfrak{h}$, we take the orthogonal direct sum decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ relative to $\langle\ ,\ \rangle_{\mathfrak{g}}$. Set

$$D(G,H) := \{\Lambda \in D(G) \mid (V_\Lambda)_H \neq \{0\}\}, \tag{3.1}$$

where

$$(V_\Lambda)_H := \{w \in V_\Lambda \mid \rho_\Lambda(a)w = w \ (\forall a \in H)\}. \tag{3.2}$$

Let $\Lambda \in D(G,H)$. For each $\bar{w} \otimes v \in (V_\Lambda)^*_H \otimes V_\Lambda$, we define a real analytic function $f_{\bar{w}\otimes v}$ on G/H by

$$(f_{\bar{w}\otimes v})(aH) := \langle v, \rho_\Lambda(a)w\rangle_{V_\Lambda} \tag{3.3}$$

for all $aH \in G/H$. By virtue of the Peter-Weyl's theorem and the Frobenius reciprocity law, we have a linear injection

$$(V_\Lambda)^*_H \otimes V_\Lambda \ni \bar{w} \otimes v \longmapsto f_{\bar{w}\otimes v} \in C^\infty(G/H, \mathbf{C}) \tag{3.4}$$

and the decomposition

$$C^\infty(G/H, \mathbf{C}) = \bigoplus_{\Lambda \in D(G,H)} (V_\Lambda)^*_H \otimes V_\Lambda. \tag{3.5}$$

in the sense of C^∞-topology. Via the natural homogeneous projection $\pi : G \to G/H$, the vector space $C^\infty(G/H, \mathbf{C})$ of all complex valued smooth functions on G/H can be identified with the vector space $C^\infty(G, \mathbf{C})_H$ of all complex valued smooth functions on G invariant under the right action of H. Let $\mathrm{U}(\mathfrak{g})$ be the universal enveloping algebra of Lie algebra $\mathfrak{g}$, which is identified to the algebra of all left-invariant linear differential operators on $C^\infty(G, \mathbf{C})$. Let

$$\mathrm{U}(\mathfrak{g})_H := \{D \in \mathrm{U}(\mathfrak{g}) \mid \mathrm{Ad}(h)D = R_h \circ D \circ R_{h^{-1}} = D \text{ for each } h \in H\}$$

be a subalgebra of $\mathrm{U}(\mathfrak{g})$ consisting of elements fixed by the adjoint action of H. Here define $(R_h\tilde{f})(u) := \tilde{f}(uh)$ for $\tilde{f} \in C^\infty(G, \mathbf{C})$. For each $D \in \mathrm{U}(\mathfrak{g})_H$, we have $D(C^\infty(G, \mathbf{C})_H) \subset C^\infty(G, \mathbf{C})_H$. The *Casimir operator* $\mathcal{C}_{G/H, \langle\ ,\ \rangle_{\mathfrak{g}}}$ of (G,H) relative to $\langle\ ,\ \rangle_{\mathfrak{g}}$ is defined by $\mathcal{C} = \mathcal{C}_{G/H, \langle\ ,\ \rangle_{\mathfrak{g}}} := \sum_{i=1}^n (X_i)^2$, where $\{X_i \mid i = 1, \cdots, n\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle_{\mathfrak{g}}$. Then $\mathcal{C}_{G/H, \langle\ ,\ \rangle_{\mathfrak{g}}} \in \mathrm{U}(\mathfrak{g})_H$ and by the AdG-invariance of $\langle\ ,\ \rangle_{\mathfrak{g}}$ and Schur's Lemma there is a non-positive real constant $c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}})$ such that

$$\mathcal{C}_{G/H, \langle\ ,\ \rangle_{\mathfrak{g}}}(f_{\bar{w}\otimes v}) = c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}}) f_{\bar{w}\otimes v} \tag{3.6}$$

for each $\bar{w} \otimes v \in (V_\Lambda)^*_H \otimes V_\Lambda$. The eigenvalue $c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}})$ is given by the Freudenthal's formula

$$c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}}) = -\langle \Lambda, \Lambda + 2\delta\rangle_{\mathfrak{g}}, \tag{3.7}$$

where $2\delta = \sum_{\alpha \in \Sigma^+(G)} \alpha$.

Now we shall consider our compact homogeneous spaces $N^n = K/K_0$ and $L^n = \mathcal{G}(N^n) = K/K_{[\mathfrak{a}]}$ ([26]). Let $\Sigma(U,K)$ be the set of (restricted) roots of $(\mathfrak{u}, \mathfrak{k})$ and $\Sigma^+(U,K)$ be its subset of positive roots. We have the following root decomposition of $\mathfrak{k}$:

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\gamma \in \Sigma^+(U,K)} \mathfrak{k}_\gamma,$$

where

$$
\begin{aligned}
\mathfrak{k}_0 :=&\{X \in \mathfrak{k} \mid [X, \mathfrak{a}] \subset \mathfrak{a}\}\\
=&\{X \in \mathfrak{k} \mid [X, H] = 0 \quad \text{for each } H \in \mathfrak{a}\},\\
\mathfrak{k}_\gamma :=&\{X \in \mathfrak{k} \mid (\mathrm{ad}H)^2 X = (\gamma(H))^2 X \text{ for each } H \in \mathfrak{a}\}.
\end{aligned}
$$

For each $\gamma \in \Sigma^+(U, K)$, set $m(\gamma) := \dim \mathfrak{k}_\gamma$. Define

$$
\mathfrak{m} := \sum_{\gamma \in \Sigma^+(U,K)} \mathfrak{k}_\gamma. \tag{3.8}
$$

Then the tangent vector spaces $T_{eK_0}(K/K_0)$ and $T_{eK_{[\mathfrak{a}]}}(K/K_{[\mathfrak{a}]})$ can be identified with the vector subspace $\mathfrak{m}$ of $\mathfrak{k}$. We can choose an orthonormal basis of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle_{\mathfrak{u}}$

$$
\{X_{\gamma,i} \mid \gamma \in \Sigma^+(U, K), i = 1, 2, \cdots, m(\gamma)\}.
$$

Let $\langle\ ,\ \rangle$ denote the $\mathrm{Ad}_{\mathfrak{m}}(K_0)$-invariant inner product of $\mathfrak{m}$ corresponding to the induced metric $\mathcal{G}^* g^{\mathrm{std}}_{Q_n(\mathbf{C})}$ on K/K_0. Thus we know ([26]) that

$$
\left\{ \frac{1}{\|\gamma\|_{\mathfrak{u}}} X_{\gamma,i} \mid \gamma \in \Sigma^+(U, K), i = 1, 2, \cdots, m(\gamma) \right\}
$$

is an orthonormal basis of $\mathfrak{m}$ relative to $\langle\ ,\ \rangle$.

The Laplace operator $\Delta^0_{L^n} = \delta d$ acting on $C^\infty(K/K_0, \mathbf{C})$ with respect to the induced metric $\mathcal{G}^* g^{\mathrm{std}}_{Q_n(\mathbf{C})}$ corresponds to the linear differential operator $-\mathcal{C}_{L^n}$ on $C^\infty(K, \mathbf{C})_{K_0}$, where $\mathcal{C}_{L^n} \in \mathrm{U}(\mathfrak{k})$ is the Casimir operator relative to the $\mathrm{Ad}_{\mathfrak{m}}(K_0)$-invariant inner product $\langle\ ,\ \rangle$ of $\mathfrak{m}$ defined by

$$
\mathcal{C}_{L^n} := \sum_{\gamma \in \Sigma^+(U,K)} \sum_{i=1}^{m(\gamma)} \frac{1}{\|\gamma\|^2_{\mathfrak{u}}} (X_{\gamma,i})^2. \tag{3.9}
$$

Note that $\mathcal{C}_{L^n} \in \mathrm{U}(\mathfrak{k})_{K_0}$ because of the $\mathrm{Ad}_{\mathfrak{m}}(K_0)$-invariance of $\langle\ ,\ \rangle$.

Suppose that $\Sigma(U, K)$ is irreducible. Let γ_0 denote the highest root of $\Sigma(U, K)$. For $g = 3, 4,$ or 6, the restricted root system $\Sigma(U, K)$ is of type A_2, B_2, BC_2 or G_2. Then we know that for each $\gamma \in \Sigma^+(U, K)$,

$$
\frac{\|\gamma\|^2_{\mathfrak{u}}}{\|\gamma_0\|^2_{\mathfrak{u}}} =
\begin{cases}
1 & \text{if } \Sigma(U, K) \text{ is of type } A_2,\\
1 \text{ or } 1/3 & \text{if } \Sigma(U, K) \text{ is of type } G_2,\\
1 \text{ or } 1/2 & \text{if } \Sigma(U, K) \text{ is of type } B_2,\\
1, 1/2 \text{ or } 1/4 & \text{if } \Sigma(U, K) \text{ is of type } BC_2.
\end{cases}
$$

Set

$$
\Sigma^+_1(U, K) := \{\gamma \in \Sigma^+(U, K) \mid \|\gamma\|^2_{\mathfrak{u}} = \|\gamma_0\|^2_{\mathfrak{u}}\}. \tag{3.10}
$$

Define a symmetric Lie subalgebra $(\mathfrak{u}_1, \mathfrak{k}_1)$ by

$$
\begin{aligned}
&\mathfrak{k}_1 := \mathfrak{k}_0 + \sum_{\gamma \in \Sigma^+_1(U,K)} \mathfrak{k}_\gamma, \quad \mathfrak{p}_1 := \mathfrak{a} + \sum_{\gamma \in \Sigma^+_1(U,K)} \mathfrak{p}_\gamma,\\
&\mathfrak{u}_1 := \mathfrak{k}_1 + \mathfrak{p}_1.
\end{aligned}
$$

Let K_1 and U_1 denote connected compact Lie subgroups of K and U generated by $\mathfrak{k}_1$ and $\mathfrak{u}_1$.

Suppose that $\Sigma^+(U,K)$ is of type BC_2. Define

$$\Sigma_2^+(U,K) := \{\gamma \in \Sigma^+(U,K) \mid \|\gamma\|_{\mathfrak{u}}^2 = \|\gamma_0\|_{\mathfrak{u}}^2 \text{ or } \|\gamma_0\|_{\mathfrak{u}}^2/2\}. \tag{3.11}$$

Define a symmetric Lie subalgebra $(\mathfrak{u}_2, \mathfrak{k}_2)$ by

$$\begin{aligned} &\mathfrak{k}_2 := \mathfrak{k}_0 + \sum_{\gamma \in \Sigma_2^+(U,K)} \mathfrak{k}_\gamma, \quad \mathfrak{p}_2 := \mathfrak{a} + \sum_{\gamma \in \Sigma_2^+(U,K)} \mathfrak{p}_\gamma, \\ &\mathfrak{u}_2 := \mathfrak{k}_2 + \mathfrak{p}_2. \end{aligned}$$

Let K_2 and U_2 denote connected compact Lie subgroups of K and U generated by $\mathfrak{k}_2$ and $\mathfrak{u}_2$. We have the following subgroups of K in each case:

$$\begin{array}{ll} K_0 \subset K, & \text{if } \Sigma(U,K) \text{ is of type } A_2; \\ K_0 \subset K_1 \subset K, & \text{if } \Sigma(U,K) \text{ is of type } B_2 \text{ or } G_2; \\ K_0 \subset K_1 \subset K_2 \subset K, & \text{if } \Sigma(U,K) \text{ is of type } BC_2. \end{array}$$

Set

$$\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} := \sum_{\gamma \in \Sigma^+(U,K)} \sum_{i=1}^{m(\gamma)} (X_{\gamma,i})^2,$$

$$\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} := \sum_{\gamma \in \Sigma_1^+(U,K)} \sum_{i=1}^{m(\gamma)} (X_{\gamma,i})^2, \tag{3.12}$$

$$\mathcal{C}_{K_2/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} := \sum_{\gamma \in \Sigma_2^+(U,K)} \sum_{i=1}^{m(\gamma)} (X_{\gamma,i})^2.$$

Then $\mathcal{C}_{K/K_0}, \mathcal{C}_{K_1/K_0}, \mathcal{C}_{K_2/K_0} \in \mathrm{U}(\mathfrak{k})_{K_0}$ and the Casimir operator $\mathcal{C}_{L^n}$ can be decomposed as follows:

Lemma 3.1.

$$\mathcal{C}_{L^n} = \begin{cases} \dfrac{1}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} \quad \textit{if } \Sigma(U,K) \textit{ is of type } A_2; \\[2ex] \dfrac{3}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \dfrac{2}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} \quad \textit{if } \Sigma(U,K) \textit{ is of type } G_2; \\[2ex] \dfrac{2}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \dfrac{1}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} \quad \textit{if } \Sigma(U,K) \textit{ is of type } B_2; \\[2ex] \dfrac{4}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \dfrac{1}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \dfrac{2}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_2/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} \\ \qquad\qquad \textit{if } \Sigma(U,K) \textit{ is of type } BC_2. \end{cases}$$

3.2. **Fibrations on homogeneous isoparametric hypersurfaces by homogeneous isoparametric hypersurfaces.** For $g = 4$ or 6, (U, K) is of type G_2, B_2 or BC_2 as indicated at the 3rd column of Table 1.

In the case when (U, K) is of type B_2 or G_2, we have one fibration as follows:

$$
\begin{tikzcd}
N^n = K/K_0 \arrow[d, "K_1/K_0"] \\
K/K_1
\end{tikzcd}
$$

In the case when (U, K) is of type BC_2, we have the following two fibrations:

$$
\begin{tikzcd}
N^n = K/K_0 \arrow[r, "="] \arrow[d, "K_1/K_0"] & K/K_0 \arrow[d, "K_2/K_0"] \\
K/K_1 \arrow[r, "K_2/K_1"] & K/K_2
\end{tikzcd}
$$

3.2.1. *In case $g = 6$ and $(U, K) = (G_2, SO(4))$, $(m_1, m_2) = (1, 1)$.*

$$
\begin{tikzcd}
N^6 = K/K_0 = SO(4)/(\mathbf{Z}_2 + \mathbf{Z}_2) \arrow[d, "K_1/K_0 = SO(3)/(\mathbf{Z}_2 + \mathbf{Z}_2)"] \\
K/K_1 = SO(4)/SO(3) \cong S^3
\end{tikzcd}
$$

Here $U_1/K_1 = SU(3)/SO(3)$ is a maximal totally geodesic submanifold of $U/K = G_2/SO(4)$. $K/K_0 = SO(4)/(\mathbf{Z}_2 + \mathbf{Z}_2)$ is a homogeneous isoparametric hypersurface with $g = 6$, $m_1 = m_2 = 1$ and $K_1/K_0 = SO(3)/(\mathbf{Z}_2 + \mathbf{Z}_2)$ is a homogenous isoparametric hypersurface with $g = 3$, $m_1 = m_2 = 1$.

Remark ([25]). Maximal totally geodesic submanifolds embedded in $G_2/SO(4)$ are classified as $SU(3)/SO(3)$, $\mathbf{C}P^2$, $S^2 \cdot S^2$.

3.2.2. *In case $g = 6$ and $(U, K) = (G_2 \times G_2, G_2)$, $(m_1, m_2) = (2, 2)$.*

$$\begin{array}{c} N^{12} = K/K_0 = G_2/T^2 \\ \Big\downarrow {\scriptstyle K_1/K_0 = SU(3)/T^2} \\ K/K_1 = G_2/SU(3) \cong S^6 \end{array}$$

Here $U_1/K_1 = (SU(3) \times SU(3))/SU(3)$ is a maximal totally geodesic submanifold of $U/K = (G_2 \times G_2)/G_2$. $K/K_0 = G_2/T^2$ is a homogenous isoparametric hypersurface with $g = 6$, $m_1 = m_2 = 2$ and $K_1/K_0 = SU(3)/T^2$ is a homogenous isoparametric hypersurface with $g = 3$, $m_1 = m_2 = 2$.

Remark ([25]). Maximal totally geodesic submanifolds embedded in G_2 are classified as $G_2/SO(4)$, $SU(3)$, $S^3 \cdot S^3$.

3.2.3. *In case $g = 4$ and $(U, K) = (SO(5) \times SO(5), SO(5))$, $(m_1, m_2) = (2, 2)$.*

$$\begin{array}{c} N^{8} = K/K_0 = SO(5)/T^2 \\ \Big\downarrow {\scriptstyle K_1/K_0 = SO(4)/T^2} \\ K/K_1 = SO(5)/SO(4) \cong S^4 \end{array}$$

Here $U_1/K_1 = (SO(4) \times SO(4))/SO(4) \cong SO(4) \cong S^3 \cdot S^3$ is a maximal totally geodesic submanifold of $U/K = (SO(5) \times SO(5))/SO(5) \cong SO(5)$. $K/K_0 = SO(5)/T^2$ is a homogeneous isoparametric hypersurface with $g = 4$, $m_1 = m_2 = 2$ and $K_1/K_0 = SO(4)/T^2 \cong S^2 \times S^2$ is a homogeneous isoparametric hypersurface with $g = 2$, $m_1 = m_2 = 2$.

Remark ([25]). Maximal totally geodesic submanifolds embedded in $Sp(2) \cong Spin(5)$ are classified as $\widetilde{Gr}_2(\mathbf{R}^5)$, $S^1 \cdot S^3$, $S^3 \times S^3$, S^4.

3.2.4. *In case $g = 4$ and $(U, K) = (SO(10), U(5))$, $(m_1, m_2) = (4, 5)$.*

$$\begin{tikzcd}
N^{18} = K/K_0 = \dfrac{U(5)}{SU(2) \times SU(2) \times U(1)} \arrow[r, "="] \arrow[d, "K_1/K_0 = \frac{U(2)\times U(2)\times U(1)}{SU(2)\times SU(2)\times U(1)}"] & K/K_0 = \dfrac{U(5)}{SU(2) \times SU(2) \times U(1)} \arrow[d, "K_2/K_0 = \frac{U(4)\times U(1)}{SU(2)\times SU(2)\times U(1)}"] \\
K/K_1 = \dfrac{U(5)}{U(2) \times U(2) \times U(1)} \arrow[r, "K_2/K_1 = \frac{U(4)\times U(1)}{U(2)\times U(2)\times U(1)}"] & K/K_2 = \dfrac{U(5)}{U(4) \times U(1)}
\end{tikzcd}$$

Here $U_2/K_2 = \frac{SO(8)\times SO(2)}{U(4)\times U(1)} \cong \frac{SO(8)}{U(4)} \cong \frac{SO(8)}{SO(2)\times SO(6)} \cong \widetilde{Gr}_2(\mathbf{R}^8)$ is a maximal totally geodesic submanifold of $U/K = SO(10)/U(5)$, but $U_1/K_1 = \frac{SO(4)\times SO(4)\times SO(2)}{U(2)\times U(2)\times U(1)} \cong \widetilde{Gr}_2(\mathbf{R}^4)$ is not a maximal totally geodesic submanifold of U_2/K_2. Notice that $K/K_0 = \frac{U(5)}{SU(2)\times SU(2)\times U(1)}$ is a homogeneous isoparametric hypersurface with $g=4$, $(m_1,m_2)=(4,5)$, $K_2/K_0 = \frac{U(4)\times U(1)}{SU(2)\times SU(2)\times U(1)} \cong \frac{SO(2)\times SO(6)}{\mathbf{Z}_2\times SO(4)}$ is a homogeneous isoparametric hypersurface with $g=4$, $(m_1,m_2)=(1,4)$ and $K_1/K_0 = \frac{U(2)\times U(2)\times U(1)}{SU(2)\times SU(2)\times U(1)} \cong \frac{U(2)}{SU(2)} \times \frac{U(2)}{SU(2)} \cong S^1\times S^1$ is a homogeneous isoparametric hypersurface with $g=2$, $(m_1,m_2)=(1,1)$.

Remark ([25]). Maximal totally geodesic submanifolds embedded in $\dfrac{SO(10)}{U(5)}$ are classified as $\widetilde{Gr}_2(\mathbf{R}^8)$, $Gr_2(\mathbf{C}^5)$, $SO(5)$, $S^2\times\mathbf{C}P^3$, $\mathbf{C}P^4$.

Remark ([25]). Maximal totally geodesic submanifolds embedded in $\widetilde{Gr}_2(\mathbf{R}^8)$ are classifed as $\widetilde{Gr}_2(\mathbf{R}^7)$, $S^p\cdot S^q$ $(p+q=6)$, $\mathbf{C}P^3$.

3.2.5. *In case $g=4$ and $(U,K)=(SO(m+2),SO(2)\times SO(m))\,(m\geq 3)$, $(m_1,m_2)=(1,m-2)$.*

$$\begin{array}{l}
N^{2m-2} = K/K_0 = \frac{SO(2)\times SO(m)}{\mathbf{Z}_2\times SO(m-2)} \\
\Big\downarrow \quad K_1/K_0 = \frac{SO(2)\times SO(2)\times SO(m-2)}{\mathbf{Z}_2\times SO(m-2)} \cong \frac{SO(2)\times SO(2)}{\mathbf{Z}_2} \cong S^1\times S^1 \\
K/K_1 = \frac{SO(2)\times SO(m)}{SO(2)\times SO(2)\times SO(m-2)} \cong \frac{SO(m)}{SO(2)\times SO(m-2)} \cong \widetilde{Gr}_2(\mathbf{R}^m)
\end{array}$$

Here $U_1/K_1 = \frac{SO(4)\times SO(m-2)}{SO(2)\times SO(2)\times SO(m-2)} \cong \widetilde{Gr}_2(\mathbf{R}^4) \cong S^2\times S^2$ is not maximal totally geodesic submanifold of $U/K = \frac{SO(m+2)}{SO(2)\times SO(m)} \cong \widetilde{Gr}_2(\mathbf{R}^{m+2})$. Notice that $K/K_0 = \frac{SO(2)\times SO(m)}{\mathbf{Z}_2\times SO(m-2)}$ is a homogeneous isoparametric hypersurface with $g=4$, $(m_1,m_2)=(1,m-2)$ and $K_1/K_0 = \frac{SO(2)\times SO(2)\times SO(m-2)}{\mathbf{Z}_2\times SO(m-2)} \cong \frac{SO(2)\times SO(2)}{\mathbf{Z}_2} \cong S^1\times S^1$ is a homogeneous isoparametric hypersurface with $g=2$, $(m_1,m_2)=(1,1)$.

Remark ([25]). Maximal totally geodesic submanifolds embedded in $\widetilde{Gr}_2(\mathbf{R}^{m+2})$ $(m\geq 3)$ are classified as $\widetilde{Gr}_2(\mathbf{R}^{m+1})$, $S^p\cdot S^q(p+q=m)$, $\mathbf{C}P^{[\frac{m}{2}]}$.

3.2.6. *In case $g=4$ and $(U,K)=(SU(m+2),S(U(2)\times U(m))\,(m\geq 2)$, $(m_1,m_2)=(2,2m-3)$.*

(i) $m=2$, $(U,K)=(SU(4),S(U(2)\times U(2)))$, $(m_1,m_2)=(2,1)$

$$\begin{array}{l}
N^6 = K/K_0 = \frac{S(U(2)\times U(2))}{S(U(1)\times U(1))} \\
\Big\downarrow \quad K_1/K_0 = \frac{S(U(1)\times U(1)\times U(1)\times U(1))}{S(U(1)\times U(1))} \cong S^1\times S^1 \\
K/K_1 = \frac{S(U(2)\times U(2))}{S(U(1)\times U(1)\times U(1)\times U(1))} \cong S^2\times S^2
\end{array}$$

Here $U_1/K_1 = \frac{S(U(2)\times U(2))}{S(U(1)\times U(1)\times U(1)\times U(1))} \cong S^2\times S^2$ is not a maximal totally geodesic submanifold in $U/K = \frac{SU(4)}{S(U(2)\times U(2))} \cong Gr_2(\mathbf{C}^4) \cong \widetilde{Gr}_2(\mathbf{R}^6)$. Notice that $K/K_0 = \frac{S(U(2)\times U(2))}{S(U(1)\times U(1))}$ is a homogeneous isoparametric hypersurface with $g=4$, $(m_1,m_2)=(2,1)$

and $K_1/K_0 \cong S^1 \times S^1$ is a homogeneous isoparametric hypersurface with $g = 2$, $(m_1, m_2) = (1, 1)$.

(ii) $m \geq 3$

$$\begin{array}{ccc}
N^{4m-2} = \frac{K}{K_0} = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))} & \xrightarrow{=} & \frac{K}{K_0} = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))} \\
\Big\downarrow \frac{K_1}{K_0} = \frac{S(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2))}{S(U(1)\times U(1)\times U(m-2))} & & \Big\downarrow \frac{K_2}{K_0} = \frac{S(U(2)\times U(2)\times U(m-2))}{S(U(1)\times U(1)\times U(m-2))} \\
\frac{K}{K_1} = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2))} & \xrightarrow{\frac{K_2}{K_1} \cong \mathbf{C}P^1 \times \mathbf{C}P^1} & \frac{K}{K_2} = \frac{S(U(2)\times U(m))}{S(U(2)\times U(2)\times U(m-2))}
\end{array}$$

Here $U_2/K_2 \cong Gr_2(\mathbf{C}^4)$ is not a maximal totally geodesic submanifold of $U/K = \frac{SU(m+2)}{S(U(2)\times U(m))} \cong Gr_2(\mathbf{C}^{m+2})$ and $U_1/K_1 = \frac{S(U(2)\times U(2)\times U(m-2))}{S(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2))} \cong \mathbf{C}P^1 \times \mathbf{C}P^1$ is not a maximal totally geodesic submanifold of U_2/K_2. Notice that $K/K_0 = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (2, 2m-3)$, $K_2/K_0 = \frac{S(U(2)\times U(2)\times U(m-2))}{S(U(1)\times U(1)\times U(m-2))}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (2, 1)$ and $K_1/K_0 \cong S^1 \times S^1$ is a homogeneous isoparametric hypersurface with $g = 2$, $(m_1, m_2) = (1, 1)$.

Remark. ([25]) Maximal totally geodesic submanifolds embedded in $Gr_2(\mathbf{C}^{m+2})$ $(m \geq 3)$ are classified as $Gr_2(\mathbf{C}^{m+1})$, $Gr_2(\mathbf{R}^{m+2})$, $\mathbf{C}P^p \times \mathbf{C}P^q$ $(p + q = m)$, $\mathbf{H}P^{[\frac{m}{2}]}$.

3.2.7. *In case $g = 4$ and $(U, K) = (Sp(m+2), Sp(2)\times Sp(m))$ $(m \geq 2)$, $(m_1, m_2) = (4, 4m-5)$.*

(i) In case $g = 4$ and $(U, K) = (Sp(4), Sp(2) \times Sp(2))$ $(m = 2)$, $(m_1, m_2) = (4, 3)$

$$\begin{array}{l}
N^{14} = K/K_0 = \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)} \\
\Big\downarrow K_1/K_0 = \frac{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)}{Sp(1)\times Sp(1)} \cong S^3 \times S^3 \\
K/K_1 = \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)} \cong \mathbf{H}P^1 \times \mathbf{H}P^1 \cong S^4 \times S^4
\end{array}$$

Here $U_1/K_1 = \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)} \cong \mathbf{H}P^1 \times \mathbf{H}P^1$ is a maximal totally geodesic submanifold of $U/K = \frac{Sp(4)}{Sp(2)\times Sp(2)} \cong Gr_2(\mathbf{H}^4)$. Notice that $K/K_0 = \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (4, 3)$ and $K_1/K_0 = \frac{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)}{Sp(1)\times Sp(1)} \cong S^3 \times S^3$ is a homogeneous isoparametric hypersurface with $g = 2$, $(m_1, m_2) = (3, 3)$.

(ii) $m \geq 3$

$$\begin{tikzcd}
N^{8m-2} = \frac{K}{K_0} = \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)} \arrow[r, "="] \arrow[d, "\frac{K_1}{K_0} = \frac{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\times Sp(m-2)}{Sp(1)\times Sp(1)\times Sp(m-2)}"] & \frac{K}{K_0} = \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)} \arrow[d, "\frac{K_2}{K_0} = \frac{Sp(2)\times Sp(2)\times Sp(m-2)}{Sp(1)\times Sp(1)\times Sp(m-2)}"] \\
\frac{K}{K_1} = \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\times Sp(m-2)} \arrow[r, "\frac{K_2}{K_1} \cong \mathbf{H}P^1 \times \mathbf{H}P^1"] & \frac{K}{K_2} = \frac{Sp(2)\times Sp(m))}{Sp(2)\times Sp(2)\times Sp(m-2)}
\end{tikzcd}$$

Here $U_2/K_2 = \frac{Sp(4)\times Sp(m-2)}{Sp(2)\times Sp(2)\times Sp(m-2)} \cong Gr_2(\mathbf{H}^4)$ is not a maximal totally geodesic submanifold of $U/K = \frac{Sp(m+2)}{Sp(2)\times Sp(m)} \cong Gr_2(\mathbf{H}^{m+2})$ but $U_1/K_1 = \frac{Sp(2)\times Sp(2)\times Sp(m-2)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\times Sp(m-2)} \cong \mathbf{H}P^1 \times \mathbf{H}P^1$ is a maximal totally geodesic submanifold of U_2/K_2. Notice that $K/K_0 = \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (4, 4m-5)$, $K_2/K_0 \cong \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (4, 3)$ and $K_1/K_0 \cong S^3 \times S^3$ is a homogeneous isoparametric hypersurface with $g = 2$, $(m_1, m_2) = (3, 3)$.

Remark. ([25]) Maximal totally geodesic submanifolds embedded in $Gr_2(\mathbf{H}^4)$ are classified as $Sp(2)$, $\mathbf{H}P^2$, $S^1 \cdot S^5$, $S^4 \times S^4$, $Gr_2(\mathbf{C}^4)$.

Maximal totally geodesic submanifolds embedded in $Gr_2(\mathbf{H}^{m+2})$ $(m \geq 3)$ are classified as $Gr_2(\mathbf{H}^{m+1})$, $Gr_2(\mathbf{C}^{m+2})$, $\mathbf{H}P^p \times \mathbf{H}P^q$ $(p + q = m)$.

3.2.8. *In case $g = 4$ and $(U, K) = (E_6, U(1)\cdot Spin(10))$, $(m_1, m_2) = (6, 9)$.*

$$\begin{tikzcd}
N^{30} = \frac{K}{K_0} = \frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)} \arrow[r, "="] \arrow[d, "\frac{K_1}{K_0} = \frac{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6)))}{S^1\cdot Spin(6)}"] & \frac{K}{K_0} = \frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)} \arrow[d, "\frac{K_2}{K_0} = \frac{U(1)\cdot(Spin(2)\cdot Spin(8))}{S^1\cdot Spin(6)}"] \\
\frac{K}{K_1} = \frac{U(1)\cdot Spin(10)}{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6)))} \arrow[r, "\frac{K_2}{K_1} = \frac{U(1)\cdot(Spin(2)\cdot Spin(8))}{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6))}"] & \frac{K}{K_2} = \frac{U(1)\cdot Spin(10)}{U(1)\cdot(Spin(2)\cdot Spin(8))}
\end{tikzcd}$$

Here $U_2/K_2 = \frac{U(1)\cdot Spin(10)}{U(1)\cdot(Spin(2)\cdot Spin(8))} \cong \widetilde{Gr}_2(\mathbf{R}^{10})$ is a maximal totally geodesic submanifold of $U/K = \frac{E_6}{U(1)\cdot Spin(10)}$ but $U_1/K_1 = \frac{S^1\cdot Spin(4)\cdot Spin(6)}{S^1\cdot(Spin(2)\cdot Spin(2)\cdot Spin(6))} \cong S^2 \times S^2$ is not a maximal totally geodesic submanifold in U_2/K_2. Notice that $K/K_0 = \frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (6, 9)$, $K_2/K_0 = \frac{U(1)\cdot(Spin(2)\cdot Spin(8))}{S^1\cdot Spin(6)} \cong \frac{Spin(2)\cdot Spin(8)}{Spin(6)} \cong \frac{SO(2)\times SO(8)}{\mathbf{Z}_2\times SO(6)}$ is a homogeneous isoparametric hypersurface with $g = 4, (m_1, m_2) = (1, 6)$ and $K_1/K_0 = \frac{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6)))}{S^1\cdot Spin(6)} \cong S^1 \times S^1$ is a homogeneous isoparametric hypersurface with $g = 2$, $(m_1, m_2) = (1, 1)$.

Remark ([25]). Maximal totally geodesic submanifolds embedded in $E_6/U(1)\cdot Spin(10)$ are classified as $Gr_2(\mathbf{H}^4)/\mathbf{Z}_2$, $\mathbf{O}P^2$, $S^2 \times \mathbf{C}P^2$, $SO(10)/U(5)$, $Gr_2(\mathbf{C}^6)$, $\widetilde{Gr}_2(\mathbf{R}^{10})$.

4. The case $(U, K) = (G_2 \times G_2, G_2)$

Let $U = G_2 \times G_2$, $K = \{(x, x) \in U \mid x \in G_2\}$ and (U, K) is of type G_2. Then $K_0 = \{k \in K \mid \mathrm{Ad}(k)H = H$ for each $H \in \mathfrak{a}\} \cong T^2$ is a maximal torus of G_2 and $N^{12} = K/K_0 \cong G_2/T^2$ is a maximal flag manifold of dimension $n = 12$. Thus its Gauss image is $L^{12} = \mathcal{G}(N^{12})(\cong N^{12}/\mathbf{Z}_6) = K \cdot [\mathfrak{a}] \cong (K/K_{[\mathfrak{a}]}) \subset Q_{12}(\mathbf{C})$.

Set $\langle\ ,\ \rangle_{\mathfrak{u}} = -B_{\mathfrak{u}}(\ ,\ )$, where $B_{\mathfrak{u}}(\ ,\ )$ denotes the Killing-Cartan form of $\mathfrak{u}$. Let $\langle\ ,\ \rangle$ be the inner product of $\mathfrak{m}$ corresponding to the invariant induced metric on L^n from $(Q_n(\mathbf{C}), g^{\mathrm{std}}_{Q_n(\mathbf{C})})$.

The restricted root system $\Sigma(U, K)$ of type G_2, can be given as follows ([7]):

$$\begin{aligned}\Sigma(U,K) =&\{\pm(\varepsilon_1-\varepsilon_2)=\pm\alpha_1, \pm(\varepsilon_3-\varepsilon_1)=\pm(\alpha_1+\alpha_2),\\ &\pm(\varepsilon_3-\varepsilon_2)=\pm(2\alpha_1+\alpha_2), \pm(-2\varepsilon_1+\varepsilon_2+\varepsilon_3)=\pm\alpha_2,\\ &\pm(\varepsilon_1-2\varepsilon_2+\varepsilon_3)=\pm(3\alpha_1+\alpha_2),\\ &\pm(2\varepsilon_3-\varepsilon_1-\varepsilon_2)=\pm(3\alpha_1+2\alpha_2)=\tilde{\alpha}\},\end{aligned}$$

where $\Pi(U, K) = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$ is its fundamental root system. Here

$$\|\gamma\|^2_{\mathfrak{u}} = \begin{cases} \dfrac{1}{24} & \text{if } \gamma \text{ is short,} \\ \\ \dfrac{1}{8} & \text{if } \gamma \text{ is long.}\end{cases}$$

Now $K_1 = SU(3)$ and $K_0 = T^2 \subset K_1 = SU(3) \subset K = G_2$.

In Lemma 3.1 the Casimir operator

$$\mathcal{C}_L = \frac{3}{\|\gamma_0\|^2_{\mathfrak{u}}}\mathcal{C}_{K/K_0,\langle\,,\,\rangle_{\mathfrak{u}}} - \frac{2}{\|\gamma_0\|^2_{\mathfrak{u}}}\mathcal{C}_{K_1/K_0,\langle\,,\,\rangle_{\mathfrak{u}}},$$

of L^n with respect to $\langle\ ,\ \rangle$ corresponding to $-\Delta_{L^{12}}$ becomes

$$\begin{aligned}\mathcal{C}_L =&24\,\mathcal{C}_{K/K_0,\langle\,,\,\rangle_{\mathfrak{u}}} - 16\,\mathcal{C}_{K_1/K_0,\langle\,,\,\rangle_{\mathfrak{u}}}\\ =&12\,\mathcal{C}^{\mathfrak{k}}_{K/K_0} - 8\,\mathcal{C}^{\mathfrak{k}}_{K_1/K_0}\\ =&12\,\mathcal{C}^{\mathfrak{k}}_{K/K_0} - 6\,\mathcal{C}^{\mathfrak{k}_1}_{K_1/K_0},\end{aligned}$$

where $\mathcal{C}^{\mathfrak{k}}_{K/K_0}$ and $\mathcal{C}^{\mathfrak{k}}_{K_1/K_0}$ denote the Casimir operators of K/K_0 and K_1/K_0 relative to the K_0-invariant inner product induced from the Killing-Cartan form of $\mathfrak{k}$, respectively, and $\mathcal{C}^{\mathfrak{k}_1}_{K_1/K_0}$ denotes the Casimir operator of K_1/K_0 relative to the K_0-invariant inner product induced from the Killing-Cartan form of $\mathfrak{k}_1$.

Let $\{\alpha_1, \alpha_2\}$ be the fundamental root system of G_2 and $\{\Lambda_1, \Lambda_2\}$ be the fundamental weight system of G_2. In our work we frequently use Satoru Yamaguchi's results ([50]) on the spectra of maximal flag manifolds.

Lemma 4.1 (Branching law of (G_2, T^2) [50])**.**

$$\begin{aligned}D(K,K_0) =& D(G_2,T^2) = D(G_2)\\ =&\{\Lambda = m_1\Lambda_1 + m_2\Lambda_2 \mid m_1, m_2 \in \mathbf{Z}, m_1 \geq 0, m_2 \geq 0\} \qquad (4.1)\\ =&\{\Lambda = p_1\alpha_1 + p_1\alpha_2 \mid p_1, p_2 \in \mathbf{Z}, p_1 \geq 1, p_2 \geq 1\}\end{aligned}$$

The eigenvalue formula of the Casimir operator $\mathcal{C}_{K/K_0}$ relative to the inner product induced from the Killing-Cartan form of G_2 is

$$-c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}_2}) = \frac{1}{24}(m_1p_1 + 3m_2p_2 + 2p_1 + 6p_2) \tag{4.2}$$

for each $\Lambda \in D(G_2, T^2) = D(G_2)$.

Since

$$-\mathcal{C}_L = -\left(4\,\mathcal{C}^{\mathfrak{g}_2}_{K/K_0} + \sum_{\gamma:\text{short}} 16\,(X_{\gamma,i})^2\right) \geq -4\,\mathcal{C}^{\mathfrak{g}_2}_{K/K_0}\,,$$

if the eigenvalue $-c_L$ of $-\mathcal{C}_L$ satisfies $-c_L \leq n = 12$, then $-c_\Lambda \leq 3$.

By using the formula (4.2), we get

$$\begin{aligned}
&\{\Lambda \in D(G_2, T^2) \mid -c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}_2}) \leq 3\}\\
=&\{0, \Lambda_1((p_1,p_2)=(2,1)), 2\Lambda_1((p_1,p_2)=(4,2)), 3\Lambda_1((p_1,p_2)=(6,3)),\\
&\Lambda_2((p_1,p_2)=(3,2)), 2\Lambda_2((p_1,p_2)=(6,4)), \Lambda_1+\Lambda_2((p_1,p_2)=(5,3)),\\
&2\Lambda_1+\Lambda_2((p_1,p_2)=(7,4))\}.
\end{aligned}$$

Let $\{\alpha_1', \alpha_2'\}$ be the fundamental root system of $SU(3)$ and $\{\Lambda_1', \Lambda_2'\}$ be the fundamental weight system of $SU(3)$. For each $\Lambda \in D(G_2, T^2)$ with $-c(\Lambda, \langle\,,\rangle_{\mathfrak{g}_2}) \leq 3$, by using the branching law of $(G_2, SU(3))$ in [28], we can determine all irreducible $SU(3)$-submodule $V_{\Lambda'}$ with the highest weight $\Lambda' = m_1'\Lambda_1' + m_2\Lambda_2'$ contained in an irreducible G_2-module V_Λ as in the following table:

(m_1, m_2)	(p_1, p_2)	$-c$	$\dim_{\mathbf{C}} V_\Lambda$	irred. $SU(3)$-submodules (m_1', m_2')
$(1,0)$	$(2,1)$	$\frac{1}{2}$	7	$(1,0), (0,1), (0,0)$
$(2,0)$	$(4,2)$	$\frac{7}{6}$	27	$(2,0), (1,1), (0,2), (1,0), (0,1), (0,0)$
$(3,0)$	$(6,3)$	2	77	$(3,0), (2,1), (1,2), (0,3), (2,0), (1,1), (0,2), (1,0), (0,1), (0,0)$
$(0,1)$	$(3,2)$	1	14	$(1,1), (1,0), (0,1)$
$(0,2)$	$(6,4)$	$\frac{5}{2}$	77	$(2,2), (2,1), (1,2), (2,0), (1,1), (0,2)$
$(1,1)$	$(5,3)$	$\frac{7}{4}$	64	$(2,1), (1,2), (2,0), 2(1,1), (0,2), (1,0), (0,1)$
$(2,1)$	$(7,4)$	$\frac{8}{3}$	189	$(3,1), (2,2), (1,3), (3,0), 2(2,1), 2(1,2), (0,3), (2,0), 2(1,1), (0,2), (1,0), (0,1)$

Since

$$\begin{aligned}
\mathfrak{g}_2^{\mathbf{C}} &= \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha\in\Sigma(G_2)} \mathfrak{g}^\alpha = \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha:\text{short}} \mathfrak{g}^\alpha + \sum_{\alpha:\text{long}} \mathfrak{g}^\alpha,\\
\mathfrak{su}(3)^{\mathbf{C}} &= \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha:\text{long}} \mathfrak{g}^\alpha,
\end{aligned}$$

we know that

$$\begin{aligned}T^2\cdot\mathbf{Z}_6 =&\{a\in G_2 \mid \mathrm{Ad}(a)(\mathfrak{t})=\mathfrak{t} \text{ preserving the orientation of } \mathfrak{t}\}\\ \supset&\{a\in SU(3) \mid \mathrm{Ad}(a)(\mathfrak{t})=\mathfrak{t} \text{ preserving the orientation of } \mathfrak{t}\}\\ =&T^2\cdot\mathbf{Z}_3.\end{aligned}$$

Now we use results in $SU(3)/T^2$, which were already treated in the case of $g=3$ and $m=2$ ([26]).

Lemma 4.2 (Branching law of $(SU(3),T^2)$ [50])**.**

$$\begin{aligned}D(K_1,K_0) =&D(SU(3),T^2)\\ =&\{\Lambda'=m_1'\Lambda_1'+m_2'\Lambda_2' \mid m_i'\in\mathbf{Z}, m_i'\geq 0\}\\ =&\{\Lambda'=p_1'\alpha_1'+p_2'\alpha_2' \mid p_i'\in\mathbf{Z}, p_i'\geq 1\},\end{aligned}\tag{4.3}$$

where

$$m_1'=2p_1'-p_2'\geq 0,\quad m_2'=-p_1'+2p_2'\geq 0.$$

The eigenvalue formula is

$$-c(\Lambda',\langle\ ,\ \rangle_{\mathfrak{su}(3)})=\frac{1}{6}(m_1'p_1'+m_2'p_2'+2p_1'+2p_2')\tag{4.4}$$

for each $\Lambda'\in D(SU(3),T^2)$.

Using Lemma 4.2, we get that $\Lambda'=m_1'\Lambda_1'+m_2'\Lambda_2'\in D(SU(3),T^2)$ such that $V_{\Lambda'}\subset V_\Lambda$ for some $\Lambda\in D(G_2,T^2)$ with $-c(\Lambda,\langle\ ,\ \rangle_{\mathfrak{g}_2})\leq 3$ satisfies

$$(m_1',m_2')\in\{(1,1),(3,0),(0,3),(2,2)\}.$$

By using the formula (4.4), we compute the corresponding eigenvalues of $\mathcal{C}_{K_1/K_0}$ as follows:

(p_1',p_2')	(m_1',m_2')	$-c'=-c(\Lambda',\langle\ ,\ \rangle_{\mathfrak{su}(3)})$
$(1,1)$	$(1,1)$	1
$(2,1)$	$(3,0)$	2
$(1,2)$	$(0,3)$	2
$(2,2)$	$(2,2)$	$\frac{8}{3}$

Therefore, for all $\Lambda\in D(G_2,T^2)$ and all $\Lambda'\in D(SU(3),T^2)$ such that $V_{\Lambda'}\subset V_\Lambda$ and $-c(\Lambda,\langle\ ,\ \rangle_{\mathfrak{g}_2})\leq 3$, the corresponding eigenvalues of $-\mathcal{C}_L=-12\,\mathcal{C}^{\mathfrak{k}}_{K/K_0}+6\,\mathcal{C}^{\mathfrak{k}_1}_{K_1/K_0}=-12c+6c'$ are given in the following table:

(m_1,m_2)	(p_1,p_2)	$\dim_{\mathbf{C}} V_\Lambda$	$-c$	(m'_1,m'_2)	$-c'$	$-12c+6c'$
$(2,0)$	$(4,2)$	27	$\frac{7}{6}$	$(1,1)$	1	8
$(3,0)$	$(6,3)$	77	2	$(1,1)$	1	18
$(3,0)$	$(6,3)$	77	2	$(3,0)$	2	12
$(3,0)$	$(6,3)$	77	2	$(0,3)$	2	12
$(0,1)$	$(3,2)$	14	1	$(1,1)$	1	6
$(0,2)$	$(6,4)$	77	$\frac{5}{2}$	$(1,1)$	1	24
$(0,2)$	$(6,4)$	77	$\frac{5}{2}$	$(2,2)$	$\frac{8}{3}$	14
$(1,1)$	$(5,3)$	64	$\frac{7}{4}$	$2(1,1)$	1	15
$(2,1)$	$(7,4)$	189	$\frac{8}{3}$	$2(1,1)$	1	26
$(2,1)$	$(7,4)$	189	$\frac{8}{3}$	$(3,0)$	2	20
$(2,1)$	$(7,4)$	189	$\frac{8}{3}$	$(0,3)$	2	20
$(2,1)$	$(7,4)$	189	$\frac{8}{3}$	$(2,2)$	$\frac{8}{3}$	16

Since $\Lambda'_1+\Lambda'_2$ ($(m'_1,m'_2)=(1,1)$) corresponds to the complexified adjoint representation of $SU(3)$, we see that $(V'_{\Lambda'_1+\Lambda'_2})_{T^2}\cong\mathfrak{t}^2$ and $(V'_{\Lambda'_1+\Lambda'_2})_{T^2\cdot\mathbf{Z}_3}=\{0\}$. Then

$$\Lambda'_1+\Lambda'_2\notin D(SU(3),T^2\cdot\mathbf{Z}_3).$$

and thus

$$2\Lambda_1,\Lambda_2\notin D(G_2,T^2\cdot\mathbf{Z}_6).$$

Now we obtain that $\mathcal{G}(G_2/T^2)\subset Q_{12}(\mathbf{C})$ is Hamiltonian stable.

We need to examine whether $3\Lambda_1\in D(K,K_{[\mathfrak{a}]})=D(G_2,T^2\cdot\mathbf{Z}_6)$ or not. Consider

$$(V_{3\Lambda_1})_{T^2}=(V'_{3\Lambda'_1})_{T^2}\oplus(V'_{3\Lambda'_2})_{T^2}\oplus(V'_{\Lambda'_1+\Lambda'_2})_{T^2}.$$

Since

$$V'_{3\Lambda'_1}\cong\mathrm{Sym}^3(\mathbf{C}^3)=\mathrm{span}_{\mathbf{C}}\{e_{i_1}\cdot e_{i_2}\cdot e_{i_3}\mid 1\le i_1\le i_2\le i_3\le 3\},$$

where $\{e_1,e_2,e_3\}$ is the standard basis of $\mathbf{C}^3$, we get

$$(V'_{3\Lambda'_1})_{T^2}=(V'_{3\Lambda'_1})_{T^2\cdot\mathbf{Z}_3}=\mathrm{span}_{\mathbf{C}}\{e_1\cdot e_2\cdot e_3\}.$$

Similarly, we get $V'_{3\Lambda'_2}\cong\mathrm{Sym}^3(\bar{\mathbf{C}}^3)$ and $(V'_{3\Lambda'_2})_{T^2}=(V'_{3\Lambda'_2})_{T^2\cdot\mathbf{Z}_3}$ with dimension 1. On the other hand we know that $(V'_{\Lambda'_1+\Lambda'_2})_{T^2}\cong\mathfrak{t}$ and $(V'_{\Lambda'_1+\Lambda'_2})_{T^2\cdot\mathbf{Z}_3}=\{0\}$. Hence we get $\dim_{\mathbf{C}}(V_{3\Lambda_1})_{T^2}=4$ and $\dim_{\mathbf{C}}(V_{3\Lambda_1})_{T^2\cdot\mathbf{Z}_3}=2$. However $\dim_{\mathbf{C}}(V_{3\Lambda_1})_{T^2\cdot\mathbf{Z}_6}=1$. In fact, $T^2\cdot\mathbf{Z}_6\subset G_2$, $T^2\cdot\mathbf{Z}_6\not\subset SU(3)$, $T^2\cdot\mathbf{Z}_3\subset SU(3)$ and $(T^2\cdot\mathbf{Z}_6)/(T^2\cdot\mathbf{Z}_3)\cong\mathbf{Z}_2$. Thus there exists an element $u\in T^2\cdot\mathbf{Z}_6\subset G_2$ with $u\notin SU(3)$ which satisfies $\mathrm{Ad}(u)(SU(3))\subset SU(3)$ and provides the generators of both $(T^2\cdot\mathbf{Z}_6)/T^2\cong\mathbf{Z}_6$ and $(T^2\cdot\mathbf{Z}_6)/(T^2\cdot\mathbf{Z}_3)\cong\mathbf{Z}_2$. Then we observe that $\rho_{3\Lambda'_1}\circ\mathrm{Ad}(u)|_{SU(3)}\cong\rho_{3\Lambda'_2}$ and $\rho_{3\Lambda_1}(u)(V'_{3\Lambda'_1})=V'_{3\Lambda'_2}$. Thus $\rho_{3\Lambda_1}(u)(V'_{3\Lambda'_1})_{T^2\cdot\mathbf{Z}_3}=(V'_{3\Lambda'_2})_{T^2\cdot\mathbf{Z}_3}$ and $(\rho_{3\Lambda_1}(u))^2|_{(V'_{3\Lambda'_1})_{T^2\cdot\mathbf{Z}_3}}=(\rho_{3\Lambda_1}(u^2))|_{(V'_{3\Lambda'_1})_{T^2\cdot\mathbf{Z}_3}}=\mathrm{Id}$, because $u^2\in T^2\cdot\mathbf{Z}_3$. Hence we have $(V_{3\Lambda_1})_{T^2\cdot\mathbf{Z}_6}\subset(V'_{3\Lambda'_1})_{T^2\cdot\mathbf{Z}_3}\oplus(V'_{3\Lambda'_2})_{T^2\cdot\mathbf{Z}_3}$ and $\dim(V_{3\Lambda_1})_{T^2\cdot\mathbf{Z}_6}=1$. Therefore $3\Lambda_1\in D(G_2,T^2\cdot\mathbf{Z}_6)$ and its multiplicity is equal to 1. It follows that

$$n(L^{12})=\dim_{\mathbf{C}}(V_{3\Lambda_1})=77=91-14=\dim(SO(14))-\dim(G_2)=n_{hk}(L^{12}),$$

that is, $\mathcal{G}(G_2/T^2)\subset Q_{12}(\mathbf{C})$ is Hamiltonian rigid.

Let $\bigwedge^2 \mathbf{R}^{14} = \mathfrak{o}(n+2) = \mathrm{ad}_{\mathfrak{p}}(\mathfrak{g}_2) + \mathcal{V} \cong \mathfrak{g}_2 + \mathcal{V}$. Then

$$\bigwedge^2 \mathbf{C}^{14} = (\bigwedge^2 \mathbf{R}^{14})^{\mathbf{C}} = \mathfrak{o}(n+2)^{\mathbf{C}} = \mathfrak{o}(n+2,\mathbf{C}) = \mathrm{ad}_{\mathfrak{p}}(\mathfrak{g}_2^{\mathbf{C}}) + \mathcal{V}^{\mathbf{C}} \cong \mathfrak{g}_2^{\mathbf{C}} + \mathcal{V}^{\mathbf{C}},$$

where $\dim \mathcal{V} = 77$ and $\dim_{\mathbf{C}} \mathcal{V}^{\mathbf{C}} = 77$. More precisely, we observe that $\mathcal{V}$ is a real 77-dimensional irreducible G_2-module with $(\mathcal{V})_{T^2\cdot\mathbf{Z}_6} \neq \{0\}$, and $\mathcal{V}^{\mathbf{C}}$ is a complex 77-dimensional G_2-module with $(\mathcal{V})^{\mathbf{C}}_{T^2\cdot\mathbf{Z}_6} \neq \{0\}$. Moreover, we have $\mathcal{V}^{\mathbf{C}} \cong V_{3\Lambda_1}$ with $\dim_{\mathbf{C}}(\mathcal{V})^{\mathbf{C}}_{T^2\cdot\mathbf{Z}_6} = 1$.

From these arguments we conclude that

Theorem 4.1. *The Gauss image $L^{12} = \mathcal{G}(G_2/T^2) = \frac{G_2}{T^2\cdot\mathbf{Z}_6} \subset Q_{12}(\mathbf{C})$ is strictly Hamiltonian stable.*

5. The case $(U,K) = (G_2, SO(4))$

Let $U = G_2$, $K = SO(4)$ and (U,K) is of type G_2. Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the orthogonal symmetric Lie algebra of $(G_2, SO(4))$ and $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{p}$. Here $\mathfrak{u} = \mathfrak{g}_2$, $\mathfrak{k} = \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Let

$$p : \widetilde{K} = Spin(4) = SU(2) \times SU(2) \longrightarrow K = SO(4)$$

be the universal covering Lie group homomorphism with kernel $\mathbf{Z}_2$.

Recall that the complete set of all inequivalent irreducible unitary representations of $SU(2)$ is given by

$$\mathcal{D}(SU(2)) = \{(V_m, \rho_m) \mid m \in \mathbf{Z}, m \geq 0\},$$

where V_m denotes the complex vector space of complex homogeneous polynomials of degree m with two variables z_0, z_1 and the representation ρ_m of $SU(2)$ on V_m is defined by $(\rho_m(g)f)(z_0,z_1) = f((z_0,z_1)g)$ for each $g \in SU(2)$. Set

$$v_k^{(m)}(z_0,z_1) := \frac{1}{\sqrt{k!(m-k)!}} z_0^{m-k} z_1^k \in V_m \quad (k = 0,1,\ldots,m) \tag{5.1}$$

and define the standard Hermitian inner product of V_m invariant under $\rho_m(SU(2))$ such that $\{v_0^{(m)}, \ldots, v_m^{(m)}\}$ is a unitary basis of V_m. Let $(V_l \otimes V_m, \rho_l \boxtimes \rho_m)$ denote an irreducible unitary representation of $SU(2) \times SU(2)$ of complex dimension $(l+1)(m+1)$ obtained by taking the exterior tensor product of V_l and V_m and then

$$\{(V_l \otimes V_m, \rho_l \boxtimes \rho_m) \mid l, m \in \mathbf{Z},\ l, m \geq 0\}$$

is the complete set of all inequivalent irreducible unitary representations of $SU(2) \times SU(2)$.

The isotropy representation of $(G_2, SO(4))$ is explicitly described as follows (cf. [17]): Suppose that $(l,m) = (3,1)$. The real 8-dimensional vector subspace W of $V_3 \otimes V_1$ spanned over $\mathbf{R}$ by

$$\begin{aligned}
\{\ & v_0^{(3)} \otimes v_0^{(1)} + v_3^{(3)} \otimes v_1^{(1)},\ \sqrt{-1}\,(v_0^{(3)} \otimes v_0^{(1)} - v_3^{(3)} \otimes v_1^{(1)}),\\
& v_1^{(3)} \otimes v_0^{(1)} - v_2^{(3)} \otimes v_1^{(1)},\ \sqrt{-1}\,(v_1^{(3)} \otimes v_0^{(1)} + v_2^{(3)} \otimes v_1^{(1)}),\\
& v_0^{(3)} \otimes v_1^{(1)} - v_3^{(3)} \otimes v_0^{(1)},\ \sqrt{-1}\,(v_0^{(3)} \otimes v_1^{(1)} + v_3^{(3)} \otimes v_0^{(1)}),\\
& v_2^{(3)} \otimes v_0^{(1)} + v_1^{(3)} \otimes v_1^{(1)},\ \sqrt{-1}\,(v_2^{(3)} \otimes v_0^{(1)} - v_1^{(3)} \otimes v_1^{(1)})\ \}
\end{aligned}$$

gives an irreducible orthogonal representation of $SU(2) \times SU(2)$ whose complexification is $V_3 \otimes V_1$, i.e. W is a *real form* of $V_3 \otimes V_1$. Then the isotropy representation $\mathrm{Ad}_{\mathfrak{p}}$ of $(G_2, SO(4))$

is given by $\mathrm{Ad}_{\mathfrak{p}}^{\mathbf{C}} \circ p \cong \rho_3 \boxtimes \rho_1$ and the vector space $\mathfrak{p}$ is linearly isomorphic to W. Moreover $\mathfrak{a}$ corresponds to a vector subspace

$$\mathbf{R}(v_0^{(3)} \otimes v_0^{(1)} + v_3^{(3)} \otimes v_1^{(1)}) + \mathbf{R}(v_2^{(3)} \otimes v_0^{(1)} + v_1^{(3)} \otimes v_1^{(1)}) .$$

For each $X = \begin{pmatrix} \sqrt{-1}x & u \\ -\bar{u} & -\sqrt{-1}x \end{pmatrix}$, $Y = \begin{pmatrix} \sqrt{-1}y & w \\ -\bar{w} & -\sqrt{-1}y \end{pmatrix} \in \mathfrak{su}(2)$, the following useful formula holds:

Lemma 5.1.

$$\begin{aligned}
&[d(\rho_l \boxtimes \rho_m)(X,Y)] \left(v_i^{(l)} \otimes v_j^{(m)} \pm v_{l-i}^{(l)} \otimes v_{m-j}^{(m)}\right) \\
=&\{(2i-l)x + (2j-m)y\} \sqrt{-1}(v_i^{(l)} \otimes v_j^{(m)} \mp v_{l-i}^{(l)} \otimes v_{m-j}^{(m)}) \\
&- \sqrt{i(l-i+1)}\ \mathrm{Re}(u)\ (v_{i-1}^{(l)} \otimes v_j^{(m)} \mp v_{l-i+1}^{(l)} \otimes v_{m-j}^{(m)}) \\
&+ \sqrt{i(l-i+1)}\ \mathrm{Im}(u)\ \sqrt{-1}(v_{i-1}^{(l)} \otimes v_j^{(m)} \pm v_{l-i+1}^{(l)} \otimes v_{m-j}^{(m)}) \\
&- \sqrt{j(m-j+1)}\ \mathrm{Re}(w)\ (v_i^{(l)} \otimes v_{j-1}^{(m)} \mp v_{l-i}^{(l)} \otimes v_{m-j+1}^{(m)}) \\
&+ \sqrt{j(m-j+1)}\ \mathrm{Im}(w)\ \sqrt{-1}(v_i^{(l)} \otimes v_{j-1}^{(m)} \pm v_{l-i}^{(l)} \otimes v_{m-j+1}^{(m)}) \\
&+ \sqrt{(l-i)(i+1)}\ \mathrm{Re}(u)\ (v_{i+1}^{(l)} \otimes v_j^{(m)} \mp v_{l-i-1}^{(l)} \otimes v_{m-j}^{(m)}) \\
&+ \sqrt{(l-i)(i+1)}\ \mathrm{Im}(u)\ \sqrt{-1}(v_{i+1}^{(l)} \otimes v_j^{(m)} \pm v_{l-i-1}^{(l)} \otimes v_{m-j}^{(m)}) \\
&+ \sqrt{(m-j)(j+1)}\ \mathrm{Re}(w)\ (v_i^{(l)} \otimes v_{j+1}^{(m)} \mp v_{l-i}^{(l)} \otimes v_{m-j-1}^{(m)}) \\
&+ \sqrt{(m-j)(j+1)}\ \mathrm{Im}(w)\ \sqrt{-1}(v_i^{(l)} \otimes v_{j+1}^{(m)} \pm v_{l-i}^{(l)} \otimes v_{m-j-1}^{(m)}) .
\end{aligned} \tag{5.2}$$

Remark. By using the formula (5.2) we can check that the real vector subspace W is invariant under the action of $SU(2) \times SU(2)$ via $\rho_3 \boxtimes \rho_1$.

Define an orthonormal basis of the real vector space $W \cong \mathfrak{p}$ by

$$\begin{aligned}
H_1 :&= \frac{1}{\sqrt{2}}(v_0^{(3)} \otimes v_0^{(1)} + v_3^{(3)} \otimes v_1^{(1)}), \\
H_2 :&= \frac{1}{\sqrt{2}}(v_2^{(3)} \otimes v_0^{(1)} + v_1^{(3)} \otimes v_1^{(1)}), \\
E_1 :&= \frac{1}{\sqrt{2}}\sqrt{-1}\ (v_0^{(3)} \otimes v_0^{(1)} - v_3^{(3)} \otimes v_1^{(1)}), \\
E_2 :&= \frac{1}{\sqrt{2}}(v_1^{(3)} \otimes v_0^{(1)} - v_2^{(3)} \otimes v_1^{(1)}), \\
E_3 :&= \frac{1}{\sqrt{2}}\sqrt{-1}\ (v_1^{(3)} \otimes v_0^{(1)} + v_2^{(3)} \otimes v_1^{(1)}), \\
E_4 :&= \frac{1}{\sqrt{2}}(v_0^{(3)} \otimes v_1^{(1)} - v_3^{(3)} \otimes v_0^{(1)}), \\
E_5 :&= \frac{1}{\sqrt{2}}\sqrt{-1}\ (v_0^{(3)} \otimes v_1^{(1)} + v_3^{(3)} \otimes v_0^{(1)}), \\
E_6 :&= \frac{1}{\sqrt{2}}\sqrt{-1}\ (v_2^{(3)} \otimes v_0^{(1)} - v_1^{(3)} \otimes v_1^{(1)}) .
\end{aligned}$$

Then we have the matrix expression as follows:

$$
\begin{aligned}
&[d(\rho_3 \boxtimes \rho_1)(X,Y)]\,(H_1,H_2)\\
&= (E_1,E_2,E_3,E_4,E_5,E_6,) \begin{pmatrix} -(3x+y) & 0 \\ \sqrt{3}\ \mathrm{Re}(u) & -(2\ \mathrm{Re}(u)+\mathrm{Re}(w)) \\ \sqrt{3}\ \mathrm{Im}(u) & 2\ \mathrm{Im}(u)+\mathrm{Im}(w) \\ \mathrm{Re}(w) & -\sqrt{3}\ \mathrm{Re}(u) \\ \mathrm{Im}(w) & \sqrt{3}\ \mathrm{Im}(u) \\ 0 & x-y \end{pmatrix}.
\end{aligned}
$$

The inner product $\langle\ ,\ \rangle$ corresponding to the metric induced from $g^{std}_{Q_6(\mathbf{C})}$ is given as follows: For $(X,X'),(Y,Y') \in \mathfrak{su}(2)\oplus\mathfrak{su}(2)$,

$$
\begin{aligned}
&\langle (X,X'),(Y,Y')\rangle\\
:=\ & (3x+x')(3y+y')\\
&+ 3\ \mathrm{Re}(u)\mathrm{Re}(w) + (2\ \mathrm{Re}(u)+\mathrm{Re}(u'))(2\ \mathrm{Re}(w)+\mathrm{Re}(w'))\\
&+ 3\ \mathrm{Im}(u)\mathrm{Im}(w) + (2\ \mathrm{Im}(u)+\mathrm{Im}(u'))(2\ \mathrm{Im}(w)+\mathrm{Im}(w'))\\
&+ \mathrm{Re}(u')\mathrm{Re}(w') + 3\mathrm{Re}(u)\mathrm{Re}(w)\\
&+ \mathrm{Im}(u')\mathrm{Im}(w') + 3\mathrm{Im}(u)\mathrm{Im}(w)\\
&+ (x-x')(y-y')\\
=\ & 10xy + 2x'y + 2xy' + 2x'y'\\
&+ 10\ \mathrm{Re}(u)\mathrm{Re}(w) + 2\ \mathrm{Re}(u')\mathrm{Re}(w) + 2\ \mathrm{Re}(u)\mathrm{Re}(w') + 2\mathrm{Re}(u')\mathrm{Re}(w')\\
&+ 10\ \mathrm{Im}(u)\mathrm{Im}(w) + 2\ \mathrm{Im}(u')\mathrm{Im}(w) + 2\ \mathrm{Im}(u)\mathrm{Im}(w') + 2\mathrm{Im}(u')\mathrm{Im}(w').
\end{aligned}
$$

Thus the Casimir operator of $(\widetilde{K}, \widetilde{K}_{[\mathfrak{a}]})$ relative to the inner product $\langle\ ,\ \rangle$ is given as follows:

$$
\begin{aligned}
\mathcal{C}_L =\ & \frac{1}{2}\ (X_1,0)\cdot(X_1,0) + \frac{1}{2}\ (X_2,0)\cdot(X_2,0) + \frac{1}{2}\ (X_3,0)\cdot(X_3,0)\\
&+\frac{5}{2}\ (0,X_1)\cdot(0,X_1) + \frac{5}{2}\ (0,X_2)\cdot(0,X_2) + \frac{5}{2}\ (0,X_3)\cdot(0,X_3)\\
&-\ (X_1,0)\cdot(0,X_1) -\ (X_2,0)\cdot(0,X_2) -\ (X_3,0)\cdot(0,X_3),
\end{aligned}
$$

where

$$
X_1 := \frac{1}{2}\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix},\ X_2 := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ X_3 := \frac{1}{2}\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}
$$

is a basis of $\mathfrak{su}(2)$ and $\{(X_1,0),(X_2,0),(X_3,0),(0,X_1),(0,X_2),(0,X_3)\}$ is a basis of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$. Hence, we have the following formula for the Casimir operator:

Lemma 5.2.

$$
\begin{aligned}
&[d(\rho_l \boxtimes \rho_m)(\mathcal{C}_L)](v_i^{(l)} \otimes v_a^{(m)})\\
&= -\Big\{\frac{l(l+2)}{8} + \frac{5m(m+2)}{8} - \frac{(2i-l)(4a-m)}{4}\Big\}(v_i^{(l)} \otimes v_a^{(m)})\\
&\quad+ \frac{1}{2}\sqrt{(i+1)(l-i)a(m-a+1)}(v_{i+1}^{(l)} \otimes v_{a-1}^{(m)})\\
&\quad+ \frac{1}{2}\sqrt{i(l-i+1)(a+1)(m-a)}(v_{i-1}^{(l)} \otimes v_{a+1}^{(m)}).
\end{aligned}
$$

Set

$$\widetilde{K}_0 := \{(A,B) \in \widetilde{K} \mid \mathrm{Ad}(p(A,B))H = H \text{ for each } H \in \mathfrak{a}\ \}.$$

Then using this description of the isotropy representation we can compute directly

$$\begin{aligned}
&\widetilde{K}_0 \\
=&\Big\{\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right),\ \left(\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix},\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}\right),\\
&\left(\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right),\ \left(\begin{pmatrix}-\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix},\begin{pmatrix}-\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix}\right),\\
&\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right),\ \left(\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix},\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}\right),\\
&\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right),\ \left(\begin{pmatrix}0 & -\sqrt{-1}\\ -\sqrt{-1} & 0\end{pmatrix},\begin{pmatrix}0 & -\sqrt{-1}\\ -\sqrt{-1} & 0\end{pmatrix}\right)\Big\}.
\end{aligned}$$

In particular, the order of $\widetilde{K}_0$ is 8. This result is consistent with ones of [4, p.611], [5, p.651] and [48, p.573] in topology of transformation group theory. Moreover we obtain

$$\begin{aligned}
&K_0\\
=&\Big\{p\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) = p\left(\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right),\\
&p\left(\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix},\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}\right) = p\left(\begin{pmatrix}-\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix},\begin{pmatrix}-\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix}\right),\\
&p\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right) = p\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right),\\
&p\left(\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix},\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}\right) = p\left(\begin{pmatrix}0 & -\sqrt{-1}\\ -\sqrt{-1} & 0\end{pmatrix},\begin{pmatrix}0 & -\sqrt{-1}\\ -\sqrt{-1} & 0\end{pmatrix}\right)\Big\}\\
\cong&\ \mathbf{Z}_2 + \mathbf{Z}_2.
\end{aligned}$$

Hence the order of group K_0 is equal to 4 and

$$\widetilde{K}/\widetilde{K}_0 \cong K/K_0 = SO(4)/\mathbf{Z}_2 + \mathbf{Z}_2.$$

For each $l, m \in \mathbf{Z}$ with $l, m \geq 0$, the vector subspace of $V_l \otimes V_m$

$$\begin{aligned}
&(V_l \otimes V_m)_{\widetilde{K}_0}\\
:=&\{\xi \in V_l \otimes V_m \mid [(\rho_l \boxtimes \rho_m)(A,B)](\xi) = \xi \text{ for any } (A,B) \in \widetilde{K}_0\ \}
\end{aligned}$$

can be described explicitly as follows:

Lemma 5.3. *When $(l+m)/2$ is even,*

$$\begin{aligned}
&(V_l \otimes V_m)_{\widetilde{K}_0}\\
=&\{\ \xi = \sum_{i+a:even} \xi_{i,a}(v_i^{(l)} \otimes v_a^{(m)} + v_{l-i}^{(l)} \otimes v_{m-a}^{(m)}) \mid \xi_{i,a} \in \mathbf{C}\ \}
\end{aligned}$$

and when $(l+m)/2$ is odd,

$$\begin{aligned}&(V_l \otimes V_m)_{\widetilde{K}_0}\\ =&\{\ \xi = \sum_{i+a:odd} \xi_{i,a}(v_i^{(l)} \otimes v_a^{(m)} - v_{l-i}^{(l)} \otimes v_{m-a}^{(m)}) \mid \xi_{i,a} \in \mathbf{C}\ \}.\end{aligned}$$

Next we describe the subgroups of $\widetilde{K}$ defined as

$$\begin{aligned}\widetilde{K}_{\mathfrak{a}}\ :=&\{(A,B) \in \widetilde{K} \mid [(\rho_3 \boxtimes \rho_1)(A,B)](\mathfrak{a}) = \mathfrak{a}\},\\ \widetilde{K}_{[\mathfrak{a}]} :=&\{(A,B) \in \widetilde{K} \mid [(\rho_3 \boxtimes \rho_1)(A,B)](\mathfrak{a}) = \mathfrak{a}\\ &\qquad\qquad\qquad\qquad \text{preserving the orientation of } \mathfrak{a}\} \subset \widetilde{K}_{\mathfrak{a}}.\end{aligned}$$

For $(A,B) \in \widetilde{K} = SU(2) \times SU(2)$, we compute that $(A,B) \in \widetilde{K}_{\mathfrak{a}}$ if and only if (A,B) is one of the following elements:

$$\left(\begin{pmatrix} e^{\sqrt{-1}\theta_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} e^{\sqrt{-1}\theta'_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta'_1} \end{pmatrix} \right),$$

where $\theta_1 = \frac{\pi}{4}k_1$, $\theta'_1 = \frac{\pi}{4}k'_1$, $k_1, k'_1 \in \mathbf{Z}$, $k_1 - k'_1 \in 4\mathbf{Z}$,

$$\left(\begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta_2} \\ e^{\sqrt{-1}\theta_2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta'_2} \\ e^{\sqrt{-1}\theta'_2} & 0 \end{pmatrix} \right),$$

where $\theta_2 = \frac{\pi}{4}k_2$, $\theta'_2 = \frac{\pi}{4}k'_2$, $k_2, k'_2 \in \mathbf{Z}$, $k_2 - k'_2 \in 4\mathbf{Z}$,

$$\left(\begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta'_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta'_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta'_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta'_1} \end{pmatrix} \right)$$

where $\theta_1 = \frac{\pi}{4}k_1$, $\theta_2 = \frac{\pi}{4}k_2$, $\theta'_1 = \frac{\pi}{4}k'_1$, $\theta'_2 = \frac{\pi}{4}k'_2$ and k_1, k_2, k'_1, $k'_2 \in \mathbf{Z}$, $k_1 + k_2$, $k_1 - k_2$, $k'_1 + k'_2$, $k'_1 - k'_2 \in 2\mathbf{Z}$, $k_1 - k'_1$, $k_2 - k'_2 \in 4\mathbf{Z}$, $k_1 + k_2 - k'_1 - k'_2$, $k_1 - k_2 - k'_1 + k'_2 \in 8\mathbf{Z}$.

In particular, the order of $\widetilde{K}_{\mathfrak{a}}$ is equal to $16 + 16 + 32 + 32 = 96$.

Moreover, for $(A,B) \in \widetilde{K} = SU(2) \times SU(2)$, we have that $(A,B) \in \widetilde{K}_{[\mathfrak{a}]}$ if and only if (A,B) is one of the following elements:

$$\left(\begin{pmatrix} e^{\sqrt{-1}\theta_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} e^{\sqrt{-1}\theta'_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta'_1} \end{pmatrix} \right)$$

where $\theta_1 = \frac{\pi}{4}k_1, \theta'_1 = \frac{\pi}{4}k'_1$, $k_1, k'_1 \in 2\mathbf{Z}$, $k_1 - k'_1 \in 4\mathbf{Z}$,

$$\left(\begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta_2} \\ e^{\sqrt{-1}\theta_2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta'_2} \\ e^{\sqrt{-1}\theta'_2} & 0 \end{pmatrix} \right)$$

where $\theta_2 = \frac{\pi}{4}k_2, \theta'_2 = \frac{\pi}{4}k'_2$, $k_2, k'_2 \in 2\mathbf{Z}$, $k_2 - k'_2 \in 4\mathbf{Z}$,

$$\left(\begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta'_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta'_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta'_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta'_1} \end{pmatrix} \right)$$

where $\theta_1 = \frac{\pi}{4}k_1$, $\theta_2 = \frac{\pi}{4}k_2$, $\theta_1' = \frac{\pi}{4}k_1'$, $\theta_2' = \frac{\pi}{4}k_2'$ and $k_1, k_2, k_1', k_2' \in 2\mathbf{Z}+1$, $k_1+k_2, k_1-k_2, k_1'+k_2', k_1'-k_2' \in 2\mathbf{Z}$, $k_1-k_1', k_2-k_2' \in 4\mathbf{Z}$, $k_1+k_2-(k_1'+k_2'), k_1-k_2-(k_1'-k_2') \in 8\mathbf{Z}$. In other words, $(A,B) \in \widetilde{K}_{[\mathfrak{a}]}$ if and only if (A,B) is one of the following elements:

$$\left(\begin{pmatrix} e^{\sqrt{-1}\theta_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} e^{\sqrt{-1}\theta_1'} & 0 \\ 0 & e^{-\sqrt{-1}\theta_1'} \end{pmatrix} \right),$$

where $\theta_1 = \frac{\pi}{2}l_1, \theta_1' = \frac{\pi}{2}l_1'$, $l_1, l_1' \in \mathbf{Z}$, $l_1 - l_1' \in 2\mathbf{Z}$,

$$\left(\begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta_2} \\ e^{\sqrt{-1}\theta_2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta_2'} \\ e^{\sqrt{-1}\theta_2'} & 0 \end{pmatrix} \right),$$

where $\theta_2 = \frac{\pi}{2}l_2$, $\theta_2' = \frac{\pi}{2}l_2'$, $l_2, l_2' \in \mathbf{Z}$, $l_2 - l_2' \in 2\mathbf{Z}$,

$$\left(\begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_1'} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_2'} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_2'} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_1'} \end{pmatrix} \right)$$

where $\theta_1 = \frac{\pi}{2}l_1 + \frac{\pi}{4}$, $\theta_2 = \frac{\pi}{2}l_2 + \frac{\pi}{4}$, $\theta_1' = \frac{\pi}{2}l_1' + \frac{\pi}{4}$, $\theta_2' = \frac{\pi}{2}l_2' + \frac{\pi}{4}$, $l_1, l_2, l_1', l_2' \in \mathbf{Z}$, $l_1 - l_1', l_2 - l_2' \in 2\mathbf{Z}$, $l_1 + l_2 - (l_1' + l_2'), l_1 - l_2 - (l_1' - l_2') \in 4\mathbf{Z}$. In particular, the order of $\widetilde{K}_{[\mathfrak{a}]}$ is equal to $8+8+16+16 = 48 = 8 \times 6 = \sharp\widetilde{K}_0 \times \sharp\mathbf{Z}_6$. Then we obtain

Lemma 5.4. *$\widetilde{K}_{[\mathfrak{a}]}/\widetilde{K}_0 \cong \mathbf{Z}_6$.*

Proof. We compute

$$\begin{aligned} A &= \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}(\frac{\pi}{2}l_1+\frac{\pi}{4})} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}(\frac{\pi}{2}l_2+\frac{\pi}{4})} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}(\frac{\pi}{2}l_2+\frac{\pi}{4})} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}(\frac{\pi}{2}l_1+\frac{\pi}{4})} \end{pmatrix}, \\ A^3 &= \begin{pmatrix} -\sqrt{2}\cos(\frac{\pi}{2}l_1+\frac{\pi}{4}) & 0 \\ 0 & -\sqrt{2}\cos(\frac{\pi}{2}l_1+\frac{\pi}{4}) \end{pmatrix} \\ &= \begin{cases} -\mathrm{I}_2 & \text{if } l_1 \equiv 0 \text{ or } 3 \pmod 4 \\ \mathrm{I}_2 & \text{if } l_1 \equiv 1 \text{ or } 2 \pmod 4 \end{cases}, \\ A^6 &= \mathrm{I}_2 \ . \end{aligned}$$

The generator of $\widetilde{K}_{[\mathfrak{a}]}/\widetilde{K}_0 \cong \mathbf{Z}_6$ is represented by the element

$$\left(\begin{pmatrix} \frac{1+\sqrt{-1}}{2} & -\frac{1-\sqrt{-1}}{2} \\ \frac{1+\sqrt{-1}}{2} & \frac{1-\sqrt{-1}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1+\sqrt{-1}}{2} & \frac{1-\sqrt{-1}}{2} \\ -\frac{1+\sqrt{-1}}{2} & -\frac{1-\sqrt{-1}}{2} \end{pmatrix} \right) . \tag{5.3}$$

□

Then using Lemmas 5.2 and 5.3 we can determine directly all eigenvalues of $\mathcal{C}_L$ on $\tilde{K}/\tilde{K}_0$ less than or equal to $\dim L = 6$ and corresponding representations of $\tilde{K}$ as in the following table:

(l,m)	$\dim(V_l \otimes V_m)_{\tilde{K}_0}$	eigenvalues of $\mathcal{C}_L$	$-\lambda \leq 6$
$(1,1)$	1	-3	*
$(2,0)$	0		
$(0,2)$	0		
$(3,1)$	2	$-3, -3$	*
$(1,3)$	2	$-9, -9$	
$(4,0)$	2	$-3, -3$	*
$(0,4)$	2	$-15, -15$	*
$(2,2)$	3	$-5, -5, -8$	*
$(5,1)$	3	$-8, -5, -8$	*
$(6,0)$	1	-6	*
$(4,2)$	3	$-6, -9, -9$	*
$(3,3)$	4	$-9, -12, -12, -15$	
$(8,0)$	2	$-10, -10$	
$(7,1)$	4	$-12, -12, -8, -8$	
$(6,2)$	5	$-15, -12, -8, -8, -12$	

Hence we get

$$\begin{aligned}&\{(l,m) \mid -c_L \leq 6 \text{ and } (V_l \otimes V_m)_{\tilde{K}_0} \neq \{0\}\}\\ =&\{(1,1),(4,0),(2,2),(3,1),(6,0),(5,1),(4,2)\}.\end{aligned}$$

Using the generator (5.3) of $\widetilde{K}_{[\mathfrak{a}]}/\widetilde{K}_0 \cong \mathbf{Z}_6$, we compute that $(V_l \otimes V_m)_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ for $(l,m) = (1,1)$, $(4,0)$, $(3,1)$, $(5,1)$ and $\dim_{\mathbf{C}}(V_l \otimes V_m)_{\tilde{K}_{[\mathfrak{a}]}} = 1$ for $(l,m) = (2,2), (6,0), (4,2)$. But we observe that the fixed vector in $(V_2 \otimes V_2)_{\tilde{K}_{[\mathfrak{a}]}} \neq \{0\}$ corresponds to the larger eigenvalue $8 > 6$. Hence we obtain that the Gauss image $L^6 = \mathcal{G}(\frac{SO(4)}{\mathbf{Z}_2+\mathbf{Z}_2}) = \frac{SO(4)}{(\mathbf{Z}_2+\mathbf{Z}_2)\cdot\mathbf{Z}_6} \subset Q_6(\mathbf{C})$ is Hamiltonian stable.

Moreover from the above result of dimension computation we have

$$\begin{aligned}n(L^6) &= \dim_{\mathbf{C}} V_6 \boxtimes V_0 + \dim_{\mathbf{C}} V_4 \boxtimes V_2 = 7 \times 1 + 5 \times 3 = 7 + 15 = 22\\ &= \dim SO(8) - \dim SO(4) = n_{hk}(L).\end{aligned}$$

Thus the Gauss image $L^6 = \mathcal{G}(\frac{SO(4)}{\mathbf{Z}_2+\mathbf{Z}_2}) = \frac{SO(4)}{(\mathbf{Z}_2+\mathbf{Z}_2)\cdot\mathbf{Z}_6} \subset Q_6(\mathbf{C})$ is Hamiltonian rigid. From these results we conclude

Theorem 5.1. *The Gauss image* $L^6 = \mathcal{G}\left(\frac{SO(4)}{\mathbf{Z}_2+\mathbf{Z}_2}\right) = \frac{SO(4)}{(\mathbf{Z}_2+\mathbf{Z}_2)\cdot\mathbf{Z}_6} \subset Q_6(\mathbf{C})$ *is strictly Hamiltonian stable.*

6. The case $(U,K) = (SO(5) \times SO(5), SO(5))$

Now (U,K) is of type B_2 and $U = SO(5) \times SO(5)$, $K = \{(x,x) \in U \mid x \in SO(5)\}$. Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition, where $\mathfrak{u} = \mathfrak{o}(5) \oplus \mathfrak{o}(5)$, $\mathfrak{k} = \{(X,X) \mid X \in \mathfrak{o}(5)\} \cong \mathfrak{o}(5)$

and $\mathfrak{p} = \{(X, -X) \mid X \in \mathfrak{o}(5)\}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ given by

$$\begin{aligned}\mathfrak{a} = &\left\{ (H, -H) \mid H = H(\xi_1, \xi_2) = \begin{pmatrix} 0 & -\xi_1 & 0 & 0 & 0 \\ \xi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\xi_2 & 0 \\ 0 & 0 & \xi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \xi_1, \xi_2 \in \mathbf{R} \right\} \\ &\cong \mathfrak{t} = \{H(\xi_1, \xi_2) \mid \xi_1, \xi_2 \in \mathbf{R}\} \subset \mathfrak{o}(5).\end{aligned}$$

Then the centralizer K_0 of $\mathfrak{a}$ in K is given by

$$K_0 = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid A, B \in SO(2) \right\} \cong T^2,$$

which is a maximal torus of $SO(5)$ and $N = K/K_0 \cong SO(5)/T^2$ is a maximal flag manifold of dimension $n = 8$. Moreover $K_{[\mathfrak{a}]}$ is described as

$$\begin{aligned} K_{[\mathfrak{a}]} = &\begin{pmatrix} \mathrm{I}_2 & 0 & 0 \\ 0 & \mathrm{I}_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot T^2 \cup \begin{pmatrix} & & 1 & 0 & \\ & & 0 & 1 & \\ 1 & 0 & & & \\ 0 & -1 & & & \\ & & & & -1 \end{pmatrix} \cdot T^2 \\ &\cup \begin{pmatrix} 1 & 0 & & & \\ 0 & -1 & & & \\ & & 1 & 0 & \\ & & 0 & -1 & \\ & & & & 1 \end{pmatrix} \cdot T^2 \cup \begin{pmatrix} & & 1 & 0 & \\ & & 0 & -1 & \\ 1 & 0 & & & \\ 0 & 1 & & & \\ & & & & -1 \end{pmatrix} \cdot T^2. \end{aligned}$$

The deck transformation group of the covering map $\mathcal{G} : N^8 \to \mathcal{G}(N^8)$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

6.1. **Description of the Casimir operator.** Choose $\langle X, Y \rangle_{\mathfrak{k}} := -\mathrm{tr}(XY)$ for each $X, Y \in \mathfrak{k} = \mathfrak{so}(5)$. The restricted root system $\Sigma(U, K)$ of type B_2, can be described as follows (cf. [7]):

$$\begin{aligned}\Sigma(U, K) =&\{\pm(\epsilon_1 - \epsilon_2) = \pm\alpha_1, \pm\epsilon_2 = \pm\alpha_2, \pm(\epsilon_1 + \epsilon_2) = \pm(\alpha_1 + 2\alpha_2), \\ &\pm \epsilon_1 = \pm(\alpha_1 + \alpha_2)\}.\end{aligned}$$

Then the square length of each $\gamma \in \Sigma(U, K)$ relative to $\langle\ ,\ \rangle_{\mathfrak{k}}$ is

$$\|\gamma\|_{\mathfrak{u}}^2 = \begin{cases} \dfrac{1}{4} & \text{if } \gamma \text{ is short,} \\ \dfrac{1}{2} & \text{if } \gamma \text{ is long.} \end{cases}$$

In this case $K = SO(5) \supset K_1 = SO(4) \supset K_0 = T^2$. The Casimir operator $\mathcal{C}_L$ of L^n relative to the induced metric from $g^{\mathrm{std}}_{Q_n(\mathbf{C})}$ becomes

$$\begin{aligned}
\mathcal{C}_L &= \frac{2}{\|\gamma_0\|^2_{\mathfrak{u}}}\mathcal{C}_{K/K_0,\langle\,,\,\rangle_{\mathfrak{u}}} - \frac{1}{\|\gamma_0\|^2_{\mathfrak{u}}}\mathcal{C}_{K_1/K_0,\langle\,,\,\rangle_{\mathfrak{u}}} \\
&= 4\,\mathcal{C}_{K/K_0,\langle\,,\,\rangle_{\mathfrak{u}}} - 2\,\mathcal{C}_{K_1/K_0,\langle\,,\,\rangle_{\mathfrak{u}}} \\
&= 2\,\mathcal{C}_{K/K_0} - \mathcal{C}_{K_1/K_0} \\
&= \ \mathcal{C}_{K/K_0} + \mathcal{C}_{K/K_1},
\end{aligned}$$

where $\mathcal{C}_{K/K_0}$ and $\mathcal{C}_{K_1/K_0}$ denote the Casimir operators of K/K_0 and K_1/K_0 relative to $\langle\ ,\ \rangle_{\mathfrak{k}}$ and $\langle\ ,\ \rangle_{\mathfrak{k}}|_{\mathfrak{k}_1}$, respectively.

6.2. **Descriptions of $D(K)$ and $D(K_1)$.** Since the maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ can be given by

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & -\xi_1 & & & \\ \xi_1 & 0 & & & \\ & & 0 & -\xi_2 & \\ & & \xi_2 & 0 & \\ & & & & 0 \end{pmatrix} \mid \xi_1, \xi_2 \in \mathbf{R} \right\} \subset \mathfrak{k}_1 \subset \mathfrak{k},$$

we have

$$\begin{aligned}
&\Gamma(K) = \Gamma(K_1) \\
&= \left\{ \xi = \begin{pmatrix} 0 & -\xi_1 & & & \\ \xi_1 & 0 & & & \\ & & 0 & -\xi_2 & \\ & & \xi_2 & 0 & \\ & & & & 0 \end{pmatrix} \mid \xi_1, \xi_2 \in 2\pi\mathbf{Z} \right\}.
\end{aligned}$$

Denote by ε_i $(i = 1, 2)$ a linear function $\epsilon_i : \mathfrak{t} \ni \xi \mapsto \xi_i \in \mathbf{R}$. Then

$$\begin{aligned}
D(K) = D(SO(5)) &= \{\Lambda = k_1\epsilon_1 + k_2\epsilon_2 \mid k_1, k_2 \in \mathbf{Z}, k_1 \geq k_2 \geq 0\}, \\
D(K_1) = D(SO(4)) &= \{\Lambda = k_1\epsilon_1 + k_2\epsilon_2 \mid k_1, k_2 \in \mathbf{Z}, k_1 \geq |k_2|\}.
\end{aligned}$$

6.3. **Branching law of $(SO(5), SO(4))$.**

Lemma 6.1 (Branching law of $(SO(5), SO(4))$ [20])**.** *Let $\Lambda = k_1\epsilon_1 + k_2\epsilon_2 \in D(SO(5))$ be the highest weight of an irreducible $SO(5)$-module V_Λ, where $k_1, k_2 \in \mathbf{Z}$ and $k_1 \geq k_2 \geq 0$. Then V_Λ contains an irreducible $SO(4)$-module $W_{\Lambda'}$ with the highest weight $\Lambda' = k'_1\epsilon_1 + k'_2\epsilon_2 \in D(SO(4))$, where $k'_1, k'_2 \in \mathbf{Z}$, $k'_1 \geq |k'_2|$, if and only if*

$$k_1 \geq k'_1 \geq k_2 \geq |k'_2|. \tag{6.1}$$

6.4. **Descriptions of $D(K, K_0)$ and $D(K_1, K_0)$.** Define an $\mathrm{Ad}(K)$-invariant inner product of $\mathfrak{k}$ by $\langle X, Y\rangle_{\mathfrak{k}} := -\mathrm{tr}(XY)$ $(X, Y \in \mathfrak{k} = \mathfrak{o}(5))$.

Let $\{\alpha'_1 = \epsilon_1 - \epsilon_2, \alpha'_2 = \epsilon_1 + \epsilon_2\}$ be the fundamental root system of $SO(4)$ and $\{\Lambda'_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2), \Lambda'_2 = \frac{1}{2}(\epsilon_1 + \epsilon_2)\}$ be the fundamental weight system of $SO(4)$. Then

Lemma 6.2 ([50]).

$$
\begin{aligned}
D(K_1, K_0) = & \ D(SO(4), T^2) \\
= & \Big\{ \Lambda' = k'_1\epsilon_1 + k'_2\epsilon_2 = m'_1\Lambda'_1 + m'_2\Lambda'_2 = p'_1\alpha'_1 + p'_2\alpha'_2 \mid \\
& k'_i \in \mathbf{Z},\ k'_1 \geq |k'_2|, m'_i \in \mathbf{Z}, m'_i \geq 0, p'_i \in \mathbf{Z}, p'_i \geq 1, \\
& m'_1 = k'_1 - k'_2 = 2p'_1 \geq 0, m'_2 = k'_1 + k'_2 = 2p'_2 \geq 0 \Big\}.
\end{aligned} \tag{6.2}
$$

The eigenvalue formula the Casimir operator $\mathcal{C}_{K_1/K_0}$ relative to $\langle X, Y\rangle_{\mathfrak{k}}|_{\mathfrak{k}_1}$ is

$$
-c_{\Lambda'} = \frac{1}{2}((k'_1)^2 + (k'_2)^2 + 2k'_1).
$$

for each $\Lambda' = k'_1\epsilon_1 + k'_2\epsilon_2 \in D(K_1, K_0)$.

Let $\{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2\}$ be the fundamental root system of $SO(5)$ and $\{\Lambda_1 = \epsilon_1, \Lambda_2 = \frac{1}{2}(\epsilon_1 + \epsilon_2)\}$ be the fundamental weight system of $SO(5)$. Then

Lemma 6.3 ([50]).

$$
\begin{aligned}
D(K, K_0) = & \ D(SO(5), T^2) \\
= & \Big\{ \Lambda = k_1\epsilon_1 + k_2\epsilon_2 = m_1\Lambda_1 + m_2\Lambda_2 = p_1\alpha_1 + p_2\alpha_2 \mid \\
& k_i \in \mathbf{Z},\ k_1 \geq k_2 \geq 0,\ m_i \in \mathbf{Z},\ m_i \geq 0,\ p_i \in \mathbf{Z},\ p_i \geq 1, \\
& m_1 = 2p_1 - p_2 \geq 0,\ m_2 = -2p_1 + 2p_2 \geq 0,\ p_1 = k_1,\ p_2 = k_1 + k_2 \Big\}
\end{aligned} \tag{6.3}
$$

The eigenvalue formula of the Casimir operator $\mathcal{C}_{K/K_0}$ with respect to the inner product $\langle X, Y\rangle_{\mathfrak{k}}$ is

$$
-c_\Lambda = \frac{1}{2}(k_1^2 + k_2^2 + 3k_1 + k_2).
$$

for each $\Lambda = k_1\epsilon_1 + k_2\epsilon_2 \in D(K, K_0)$.

6.5. **Eigenvalue computation.** By Lemmas 6.3 and 6.2 we have the following eigenvalue formula for $\mathcal{C}_L$.

$$
\begin{aligned}
-c_L = & \ -2c_{K/K_0} + c_{K_1/K_0} \\
& = (k_1^2 + k_2^2 + 3k_1 + k_2) - \frac{1}{2}((k'_1)^2 + (k'_2)^2 + 2k'_1).
\end{aligned}
$$

Since

$$
-\mathcal{C}_L = -\mathcal{C}_{K/K_0} - \mathcal{C}_{S^4} \geq -\mathcal{C}_{K/K_0},
$$

the eigenvalue of $\mathcal{C}_L$, $-c_L \leq n = 8$ implies $-c_\Lambda \leq 8$. Using Lemma 6.3 we compute

$$
\begin{aligned}
& \{\Lambda \in D(SO(5), T^2) \mid -c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{k}}) \leq 8\} \\
= & \{\epsilon_1\ ((k_1, k_2) = (1, 0)),\ \epsilon_1 + \epsilon_2\ ((k_1, k_2) = (1, 1)),\ 2\epsilon_1\ ((k_1, k_2) = (2, 0)), \\
& 2\epsilon_1 + \epsilon_2\ ((k_1, k_2) = (2, 1)),\ 2\epsilon_1 + 2\epsilon_2\ ((k_1, k_2) = (2, 2))\}.
\end{aligned}
$$

Suppose that $(k_1, k_2) = (1, 0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 5$. It follows from Lemma 6.1 that $(k'_1, k'_2) = (0, 0)$ or $(1, 0)$. By Lemma 6.2, we have $(p'_1, p'_2) = (0, 0)$ or $(\frac{1}{2}, \frac{1}{2})$, but $\Lambda'|_{(p'_1, p'_2) = (0,0)}$, $\Lambda'|_{(p'_1, p'_2) = (\frac{1}{2}, \frac{1}{2})} \notin D(SO(4), T^2)$. Hence $\Lambda = (1, 0) \notin D(SO(5), T^2) = D(K, K_0)$.

Suppose that $(k_1,k_2)=(1,1)$. Then $\dim_{\mathbf{C}} V_\Lambda = 10$, $V_\Lambda \cong \mathfrak{o}(5,\mathbf{C})$ and $K_{[\mathfrak{a}]}/K_0$ acts on $(V_\Lambda)_{K_0} \cong (\mathfrak{t}^2)^{\mathbf{C}} \cong \mathfrak{a}^{\mathbf{C}}$ via the action of Weyl group $W(U,K)$. Thus it must be $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}$. Hence, $\Lambda|_{(k_1,k_2)=(1,1)} \notin D(K,K_{[\mathfrak{a}]})$.

Suppose that $(k_1,k_2)=(2,0)$. Then $(m_1,m_2)=(2,0)$ and $\dim_{\mathbf{C}} V_{2\Lambda_1} = 14$. It follows from Lemma 6.1 that $(k_1',k_2') = (0,0),(1,0)$ or $(2,0)$. By Lemma 6.2, we have $(p_1',p_2') = (0,0),(\frac{1}{2},\frac{1}{2})$ or $(1,1)$. Note that $\Lambda'|_{(p_1',p_2')=(0,0)}$, $\Lambda'|_{(p_1',p_2')=(\frac{1}{2},\frac{1}{2})} \notin D(SO(4),T^2)$. If $(p_1',p_2')=(1,1)$, then $(m_1',m_2')=(2,2)$ and $-c_\Lambda=5$, $-c_{\Lambda'}=4$, thus

$$-c_L = -2c_\Lambda + c_{\Lambda'} = 10-4 = 6 < 8.$$

On the other hand, we observe that

$$\begin{aligned}
V_{2\Lambda_1} \cong & \mathrm{Sym}_0(\mathbf{C}^5)\\
=&\mathbf{C}\cdot\begin{pmatrix} -\frac{1}{4}I_4 & 0\\ 0 & 1\end{pmatrix} \oplus \left\{ \begin{pmatrix} X & 0\\ 0 & 0\end{pmatrix} \mid X\in \mathrm{Sym}_0(\mathbf{C}^4)\right\}\\
&\oplus \left\{ \begin{pmatrix} 0 & Z\\ {}^tZ & 0\end{pmatrix} \mid Z \in M(4,1;\mathbf{C})\right\}\\
=&W_{|\Lambda'=0} \oplus W_{2\Lambda_1'+2\Lambda_2'} \oplus W_{\Lambda_1'+\Lambda_2'},
\end{aligned}$$

and

$$(V_{2\Lambda_1})_{K_0} = \left\{ \begin{pmatrix} c_1 I_2 & & \\ & c_2 I_2 & \\ & & c_3\end{pmatrix} \mid c_1,c_2,c_3\in\mathbf{C}, 2c_1+2c_2+c_3=0\right\}.$$

As

$$\begin{pmatrix} 0&0&1&0&0\\ 0&0&0&-1&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&-1\end{pmatrix} \begin{pmatrix} c_1 I_2 & & \\ & c_2 I_2 & \\ & & c_3\end{pmatrix} \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&-1&0&0&0\\ 0&0&0&0&-1\end{pmatrix}$$
$$= \begin{pmatrix} c_2 I_2 & & \\ & c_1 I_2 & \\ & & c_3\end{pmatrix},$$

we get

$$(V_{2\Lambda_1})_{K_{[\mathfrak{a}]}} = \left\{ \begin{pmatrix} -\frac{c}{4}\mathrm{I}_4 & \\ & c\end{pmatrix} \mid c\in\mathbf{C}\right\} = W_{|\Lambda'=0}.$$

Thus

$$W'_{2\Lambda_1'+2\Lambda_2'} \cap (V_{2\Lambda_1})_{K_{[\mathfrak{a}]}} = \{0\}.$$

Suppose that $(k_1,k_2)=(2,1)$. Then $(m_1,m_2)=(2,1)$ and $\dim_{\mathbf{C}} V_{2\Lambda_1+\Lambda_2} = 35$. It follows from Lemma 6.1 that $(k_1',k_2') = (1,0)$, $(1,-1)$, $(1,1)$, $(2,0)$, $(2,-1)$ or $(2,1)$, that is, $(m_1',m_2') = (1,1)$, $(2,0)$, $(0,2)$, $(2,2)$, $(3,1)$ or $(1,3)$, and thus

$$V_{2\Lambda_1+\Lambda_1} = W_{\Lambda_1'+\Lambda_2'} \oplus W_{2\Lambda_1'} \oplus W_{2\Lambda_2'} \oplus W_{2\Lambda_1'+2\Lambda_2'} \oplus W_{3\Lambda_1'+\Lambda_2'} \oplus W_{\Lambda_1'+3\Lambda_2'}.$$

By Lemma 6.3, we have $(p_1',p_2') = (\frac{1}{2},\frac{1}{2})$, $(1,0)$, $(0,1)$, $(1,1)$, $(\frac{3}{2},\frac{1}{2})$ or $(\frac{1}{2},\frac{3}{2})$. Then by Lemma 6.2 we see that $\Lambda'|_{(p_1',p_2')=(\frac{1}{2},\frac{1}{2})}$, $\Lambda'|_{(p_1',p_2')=(1,0)}$, $\Lambda'|_{(p_1',p_2')=(0,1)}$, $\Lambda'|_{(p_1',p_2')=(\frac{3}{2},\frac{1}{2})}$, $\Lambda'|_{(p_1',p_2')=(\frac{1}{2},\frac{3}{2})}$

$\notin D(SO(4),T^2)$. If $(p'_1,p'_2)=(1,1)$, i.e. $(m'_1,m'_2)=(2,2)$, then $-c_\Lambda=6$, $-c_{\Lambda'}=4$ and thus

$$-c_L=-2\,c_\Lambda+c_{\Lambda'}=12-4=8.$$

So we need to determine the dimension of $(W_{2\Lambda'_1+2\Lambda'_2})_{K_{[\mathfrak{a}]}}\neq\{0\}$.

Since $W_{2\Lambda'_1+2\Lambda'_2}\cong\mathfrak{sl}(2,\mathbf{C})\boxtimes\mathfrak{sl}(2,\mathbf{C})$ and

$$(W_{2\Lambda'_1+2\Lambda'_2})_{K_0}\cong(\mathfrak{sl}(2,\mathbf{C})\boxtimes\mathfrak{sl}(2,\mathbf{C}))_{K_0}=\mathbf{C}\boxtimes\mathbf{C},$$

we have $\dim_{\mathbf{C}}(W_{2\Lambda'_1+2\Lambda'_2})_{K_0}=1$. Let $\wedge^2\mathbf{R}^{10}=\mathfrak{so}(10)=\mathrm{ad}_{\mathfrak{p}}(\mathfrak{so}(5))+\mathcal{V}$. Then $\wedge^2\mathbf{C}^{10}=(\wedge^2\mathbf{R}^{10})^{\mathbf{C}}=\mathfrak{so}(10,\mathbf{C})=\mathrm{ad}(\mathfrak{so}(5))^{\mathbf{C}}+\mathcal{V}^{\mathbf{C}}\cong\mathfrak{so}(5,\mathbf{C})+\mathcal{V}^{\mathbf{C}}$, where $\{0\}\neq\mathcal{V}^{\mathbf{C}}\subset V_{2\Lambda_1+\Lambda_2}$. By the irreducibility of $V_{2\Lambda_1+\Lambda_2}$, we see $\mathcal{V}^{\mathbf{C}}=V_{2\Lambda_1+\Lambda_2}$. Since

$$\{0\}\neq(\mathcal{V}^{\mathbf{C}})_{K_{[\mathfrak{a}]}}=(W_{2\Lambda'_1+2\Lambda'_2})_{K_{[\mathfrak{a}]}}\subset(W_{2\Lambda'_1+2\Lambda'_2})_{K_0}$$

and $\dim_{\mathbf{C}}(W_{2\Lambda'_1+2\Lambda'_2})_{K_0}=1$, we get

$$\{0\}\neq(\mathcal{V}^{\mathbf{C}})_{K_{[\mathfrak{a}]}}=(W_{2\Lambda'_1+2\Lambda'_2})_{K_{[\mathfrak{a}]}}=(W_{2\Lambda'_1+2\Lambda'_2})_{K_0}$$

and $\dim_{\mathbf{C}}(W_{2\Lambda'_1+2\Lambda'_2})_{K_{[\mathfrak{a}]}}=1$. Hence $2\Lambda_1+\Lambda_2\in D(K,K_{[\mathfrak{a}]})$ and its multiplicity is equal to 1.

Suppose that $(k_1,k_2)=(2,2)$. It follows from Lemma 6.1 that $(k'_1,k'_2)=(2,0)$, $(2,1)$, $(2,2)$, $(2,-1)$ or $(2,-2)$. By Lemma 6.2, we have $(p'_1,p'_2)=(1,1)$, $(\frac{1}{2},\frac{3}{2})$, $(0,2)$, $(\frac{3}{2},\frac{1}{2})$ or $(2,0)$ and thus $\Lambda'|_{(p'_1,p'_2)=(\frac{1}{2},\frac{3}{2})}$, $\Lambda'|_{(p'_1,p'_2)=(0,2)}$, $\Lambda'|_{(p'_1,p'_2)=(\frac{3}{2},\frac{1}{2})}$, $\Lambda'|_{(p'_1,p'_2)=(2,0)}\notin D(SO(4),T^2)$. If $(p'_1,p'_2)=(1,1)$, then $-c_\Lambda=8$, $-c_{\Lambda'}=4$ and hence

$$-c_L=-2c_\Lambda+c_{\Lambda'}=16-4=12>8.$$

Now we obtain that the Gauss image $L^8=\mathcal{G}(SO(5)/T^2)\subset Q_8(\mathbf{C})$ is Hamiltonian stable. Moreover it also follows that

$$n(L^8)=\dim_{\mathbf{C}}(V_{2\Lambda_1+\Lambda_2})=35=\dim(SO(10))-\dim(SO(5))=n_{hk}(L^8).$$

Hence the Gauss image $L^8=\mathcal{G}(SO(5)/T^2)\subset Q_8(\mathbf{C})$ is Hamiltonian rigid.

From theses results we conclude that

Theorem 6.1. *The Gauss image $L^8=\mathcal{G}(SO(5)/T^2)=\frac{SO(5)}{T^2\cdot\mathbf{Z}_2}\subset Q_8(\mathbf{C})$ is strictly Hamiltonian stable.*

7. The case $(U,K)=(SO(10),U(5))$

In this case, (U,K) is of BC_2 type and $K=U(5)\subset U=SO(10)$. Here each $A+\sqrt{-1}B\in U(5)$ can be identified with an element $\begin{pmatrix}A & -B\\ B & A\end{pmatrix}\in SO(10)$ with $A,B\in\mathfrak{g}l(5,\mathbf{R})$. The canonical decomposition $\mathfrak{u}=\mathfrak{k}+\mathfrak{p}$ of $\mathfrak{u}$ and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ are given by $\mathfrak{u}=\mathfrak{so}(10)$,

$$\begin{aligned}\mathfrak{k}&=\left\{\begin{pmatrix}X & -Y\\ Y & X\end{pmatrix}\in\mathfrak{so}(10)\mid -X^t=X,Y^t=Y\right\}\\ &\cong\mathfrak{u}(5)=\left\{T=X+\sqrt{-1}Y\in\mathfrak{gl}(5,\mathbf{C})\mid T^*=-T\right\},\\ \mathfrak{p}&=\left\{\begin{pmatrix}X & Y\\ Y & -X\end{pmatrix}\in\mathfrak{so}(10)\mid X,Y\in\mathfrak{so}(5)\right\}\end{aligned}$$

and

$$\mathfrak{a} = \left\{ \begin{pmatrix} H_1 & 0 \\ 0 & -H_1 \end{pmatrix} \mid H_1 = \begin{pmatrix} 0 & -\xi_1 & & & \\ \xi_1 & 0 & & & \\ & & 0 & -\xi_2 & \\ & & \xi_2 & 0 & \\ & & & & 0 \end{pmatrix} \xi_1, \xi_2 \in \mathbf{R} \right\}.$$

Then the centralizer K_0 of $\mathfrak{a}$ in K is as follows:

$$\begin{aligned} & K_0 \\ =& \left\{ \begin{pmatrix} a_{11}+\mathbf{i}b_{11} & a_{12}+\mathbf{i}b_{12} & 0 & 0 & 0 \\ -a_{12}+\mathbf{i}b_{12} & a_{11}-\mathbf{i}b_{11} & 0 & 0 & 0 \\ 0 & 0 & a_{22}+\mathbf{i}b_{22} & a_{21}+\mathbf{i}b_{21} & 0 \\ 0 & 0 & -a_{21}+\mathbf{i}b_{21} & a_{22}-\mathbf{i}b_{22} & 0 \\ 0 & 0 & 0 & 0 & a_{33}+\mathbf{i}b_{33} \end{pmatrix} \in U(5) \right\} \\ \cong & SU(2) \times SU(2) \times U(1) \end{aligned}$$

and $N = K/K_0 \cong U(5)/SU(2) \times SU(2) \times U(1)$ is of dimension 18. Moreover,

$$\begin{aligned} K_{[\mathfrak{a}]} =& K_0 \cup \begin{pmatrix} & & 1 & 0 & \\ & & 0 & 1 & \\ 1 & 0 & & & \\ 0 & -1 & & & \\ & & & & 1 \end{pmatrix} \cdot K_0 \cup \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix} \cdot K_0 \\ & \cup \begin{pmatrix} & & 1 & 0 & \\ & & 0 & -1 & \\ 1 & 0 & & & \\ 0 & 1 & & & \\ & & & & 1 \end{pmatrix} \cdot K_0. \end{aligned}$$

It means that the deck transformation group of the covering map $\mathcal{G} : N \to \mathcal{G}(N^{18})$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

7.1. **Description of the Casimir operator.** Choose $\langle X, Y \rangle_{\mathfrak{u}} := -\mathrm{tr}(XY)$ for each $X, Y \in \mathfrak{u} = \mathfrak{so}(10)$. The restricted root system $\Sigma(U, K)$ of type BC_2 can be given as follows ([7]):

$$\begin{aligned} & \Sigma(U,K) \\ =& \{\pm\epsilon_2 = \pm\alpha_1, \pm(\epsilon_1 - \epsilon_2) = \pm\alpha_2, \pm\epsilon_1 = \pm(\alpha_1+\alpha_2), \\ & \pm(\epsilon_1+\epsilon_2) = \pm(2\alpha_1+\alpha_2), \pm 2\epsilon_1 = \pm(2\alpha_1 + 2\alpha_2), \pm 2\epsilon_2 = \pm 2\alpha_1\}. \end{aligned}$$

Then the square length of each $\gamma \in \Sigma(U, K)$ relative to $\langle \, , \, \rangle_{\mathfrak{u}}$ is

$$\|\gamma\|_{\mathfrak{u}}^2 = \frac{1}{4}, \frac{1}{2} \text{ or } 1.$$

Hence the Casimir operator $\mathcal{C}_L$ of L^n with respect to the induced metric from $g^{\mathrm{std}}_{Q_n(\mathbf{C})}$ can be expressed as follows:

$$\begin{aligned}\mathcal{C}_L &= \frac{4}{\|\gamma_0\|^2_{\mathfrak{u}}}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \frac{1}{\|\gamma_0\|^2_{\mathfrak{u}}}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \frac{2}{\|\gamma_0\|^2_{\mathfrak{u}}}\mathcal{C}_{K_2/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} \\ &= 4\,\mathcal{C}_{K/K_0,\langle,\rangle_{\mathfrak{u}}} - \mathcal{C}_{K_1/K_0,\langle,\rangle_{\mathfrak{u}}} - 2\,\mathcal{C}_{K_2/K_0,\langle,\rangle_{\mathfrak{u}}} \\ &= 2\,\mathcal{C}_{K/K_0} - \mathcal{C}_{K_2/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0} \\ &= \mathcal{C}_{K/K_0} + \mathcal{C}_{K/K_2} + \frac{1}{2}\mathcal{C}_{K_1/K_0},\end{aligned}$$

where $\mathcal{C}_{K/K_0}$, $\mathcal{C}_{K_2/K_0}$ and $\mathcal{C}_{K_1/K_0}$ denote the Casimir operator of K/K_0, K_2/K_0 and K_1/K_0 relative to $\langle\ ,\ \rangle|_{\mathfrak{k}}$, $\langle\ ,\ \rangle|_{\mathfrak{k}_2}$ and $\langle\ ,\ \rangle|_{\mathfrak{k}_1}$, respectively. Here, $\langle X, Y\rangle := -\mathrm{tr}(\mathrm{Re}(XY))$ for all $X, Y \in \mathfrak{k} = u(5)$.

7.2. **Descriptions of** $D(K)$**,**$D(K_1)$ **and** $D(K_2)$**.** Using a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ given by

$$\mathfrak{t} = \left\{ \sqrt{-1}\begin{pmatrix} y_1 & 0 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 & 0 \\ 0 & 0 & y_3 & 0 & 0 \\ 0 & 0 & 0 & y_4 & 0 \\ 0 & 0 & 0 & 0 & y_5 \end{pmatrix} \mid y_1, y_2, y_3, y_4, y_5 \in \mathbf{R} \right\} \subset \mathfrak{k},$$

we have

$$\begin{aligned}\Gamma(K) &= \Gamma(K_2) = \Gamma(K_1) = \Gamma(K_0) \\ &= \left\{ \xi = \sqrt{-1}\begin{pmatrix} \xi_1 & 0 & 0 & 0 & 0 \\ 0 & \xi_2 & 0 & 0 & 0 \\ 0 & 0 & \xi_3 & 0 & 0 \\ 0 & 0 & 0 & \xi_4 & 0 \\ 0 & 0 & 0 & 0 & \xi_5 \end{pmatrix} \mid \xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \in 2\pi\mathbf{Z} \right\}, \\ \Gamma(C(K)) &= 2\pi\mathbf{Z}\mathrm{I}_5.\end{aligned}$$

Then $D(K)$, $D(K_1)$ and $D(K_2)$ are given as follows:

$$\begin{aligned} D(K) =& D(U(5)) \\ =& \{\Lambda = p_1y_1 + \cdots + p_5y_5 \mid p_1, \cdots, p_5 \in \mathbf{Z}, p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5\}, \\ D(K_2) =& D(U(4)\times U(1)) \\ =& \{\Lambda = p_1y_1 + \cdots + p_5y_5 \mid p_1, \cdots, p_5 \in \mathbf{Z}, p_1 \geq p_2 \geq p_3 \geq p_4\}, \\ D(K_1) =& D(U(2)\times U(2)\times U(1)) \\ =& \{\Lambda = p_1y_1 + \cdots + p_5y_5 \mid p_1, \cdots, p_5 \in \mathbf{Z}, p_1 \geq p_2, p_3 \geq p_4\}.\end{aligned}$$

7.3. **Branching laws of** $(U(m+1), U(m)\times U(1))$**.**

The branching laws for $(SU(m+1), S(U(1)\times U(m)))$ was shown by Ikeda and Taniguchi [20]. It can be reformulated to the branching laws for $(U(m+1), U(m)\times U(1))$ as follows:

Lemma 7.1 (Branching laws for $(U(m+1), U(m)\times U(1))$)**.** *Let* $\Lambda = p_1y_1 + \cdots + p_my_m \in D(U(m))$ *be the highest weight of an irreducible* $U(m)$*-module* V_Λ*, where* $p_i \in \mathbf{Z}$ $(i = 1, \cdots, m)$

and $p_1 \geq p_2 \geq \cdots \geq p_m$. *Then the irreducible decomposition of* V_Λ *as a* $U(m) \times U(1)$*-module contains an irreducible* $U(m)\times U(1)$*-module* $V_{\Lambda'}$ *with the highest weight* $V_{\Lambda'} = q_1y_1+\cdots+q_my_m \in D(U(m) \times U(1))$, *where* $q_i \in \mathbf{Z}$ *and* $q_1 \geq q_2 \geq \cdots \geq q_m$, *if and only if*

$$p_1 \geq q_1 \geq p_2 \geq q_2 \geq p_3 \geq q_3 \geq \cdots \geq p_{m-1} \geq q_{m-1} \geq p_m,$$
$$\sum_{i=1}^{m} p_i = \sum_{i=1}^{m} q_i.$$

In particluar the multiplicity of $V_{\Lambda'}$ *is* 1.

In the next subsection we use the branching laws of $(U(m+1), U(m) \times U(1))$ and $(U(m), U(2)\times U(m-2))$ in the case of $m = 4$. The branching laws of $(U(m), U(2)\times U(m-2))$ are described in Lemma 9.1 of Section 9.

7.4. **Descriptions of** $D(K, K_0)$**,** $D(K_2, K_0)$ **and** $D(K_1, K_0)$**.**

Each $\Lambda \in D(K) = D(U(5))$ is expressed as

$$\Lambda = p_1y_1 + \cdots p_5y_5,$$

where $p_i \in \mathbf{Z}$, $p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5$. Then by Lemma 7.1 in the case of $m = 4$, V_Λ can be decomposed into irreducible $U(4) \times U(1)$-modules as

$$V_\Lambda = \bigoplus_{i=1}^{s} V'_{\Lambda'_i} = \bigoplus_{i=1}^{s} W'_{\Lambda'_{1i}} \boxtimes U_{q_5y_5},$$

where $\Lambda'_i = q_1y_1 + q_2y_2 + q_3y_3 + q_4y_4 + q_5y_5 \in D(K_2) = D(U(4) \times U(1))$, $\Lambda'_{1i} = q_1y_1 + q_2y_2 + q_3y_3 + q_4y_4 \in D(U(4))$, $q_5y_5 \in D(U(1))$ and $q_i \in \mathbf{Z}$ $(i = 1, 2, 3, 4, 5)$ satisfy

$$p_1 \geq q_1 \geq p_2 \geq q_2 \geq p_3 \geq q_3 \geq p_4 \geq q_4 \geq p_5,$$
$$\sum_{i=1}^{5} p_i = \sum_{j=1}^{5} q_j.$$

By the branching law for $(U(4), U(2) \times U(2))$ in Lemma 9.1, each $W'_{\Lambda'_{1i}}$ can be decomposed as

$$W'_{\Lambda'_{1i}} = \bigoplus W''_{\Lambda''} = \bigoplus W''_{\tilde{\Lambda}_\sigma} \boxtimes W''_{\tilde{\Lambda}_\rho},$$

where $\Lambda'' = k_1y_1 + k_2y_2 + k_3y_3 + k_4y_4 \in D(U(2) \times U(2))$, $\tilde{\Lambda}_\sigma = k_1y_1 + k_2y_2 \in D(U(2))$, $\tilde{\Lambda}_\rho = k_3y_3 + k_4y_4 \in D(U(2))$ and $k_i \in \mathbf{Z}$ $(i = 1, 2, 3, 4)$ satisfy

(i) $\sum_{i=1}^{4} k_i = \sum_{i=1}^{4} q_i$;

(ii) $q_1 \geq k_1 \geq q_3$, $q_2 \geq k_2 \geq q_4$;

(iii) in the finite power series expansion in X of $\dfrac{\prod_{i=1}^{3}(X^{r_i+1} - X^{-(r_i+1)})}{(X - X^{-1})^2}$, where $r_i (i = 1, 2, 3)$ are defined by

$$r_1 := q_1 - \max(k_1, q_2),$$
$$r_2 := \min(k_1, q_2) - \max(k_2, q_3),$$
$$r_3 := \min(k_2, q_3) - q_4,$$

the coefficient of $X^{k_3-k_4+1}$ does not vanish. Moreover the value of this coefficient is the multiplicity of the $U(2) \times U(2)$-module $W''_{\Lambda''}$.

By the branching law of $(U(2), SU(2))$ (see Section 9), as $SU(2)$-modules they become

$$W''_{\tilde{\Lambda}_\sigma} = W''_{\Lambda_\sigma}, \quad W''_{\tilde{\Lambda}_\rho} = W''_{\Lambda_\rho},$$

where $\Lambda_\sigma = \dfrac{k_1 - k_2}{2}(y_1 - y_2) \in D(SU(2))$, $\Lambda_\rho = \dfrac{k_3 - k_4}{2}(y_3 - y_4) \in D(SU(2))$.

Hence, one can decompose a K-module V_Λ into the following irreducible K_0-modules

$$V_\Lambda = \bigoplus \bigoplus W''_{\Lambda_\sigma} \boxtimes W''_{\Lambda_\rho} \boxtimes U_{q_5 y_5}.$$

Now assume that $\Lambda \in D(K, K_0)$. Then there exists at least one nonzero trivial irreducible K_0-module in the above decomposition for some σ and ρ. So in this case, we have

$$k_1 - k_2 = 0, \quad k_3 - k_4 = 0, \quad q_5 = 0.$$

So we know that

$$\begin{aligned}
&2k_1 + 2k_3 = \sum_{i=1}^{4} q_i = \sum_{j=1}^{5} p_j, \\
&q_2 \geq k_1 = k_2 \geq q_3, \\
&r_1 = q_1 - q_2, \\
&r_2 = k_1 - k_2 = 0, \\
&r_3 = q_3 - q_4
\end{aligned}$$

and in the finite power series expansion in X of

$$\frac{(X^{q_1 - q_2 + 1} - X^{-(q_1 - q_2 + 1)})(X^{q_3 - q_4 + 1} - X^{-(q_3 - q_4 + 1)})}{X - X^{-1}},$$

the coefficient of X does not vanish. Moreover, the value of this coefficient is the multiplicity of the $U(2) \times U(2)$-module.

Therefore, in the above notations, for each $\Lambda \in D(K, K_0)$ given by $\Lambda = p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + p_5 y_5$, where $p_1, \cdots, p_5 \in \mathbf{Z}$, $p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5$, each $\Lambda' \in D(K_2, K_0)$ is given by $\Lambda' = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4$, where $q_1, \cdots, q_4 \in \mathbf{Z}$, $q_1 \geq q_2 \geq q_3 \geq q_4$, $\sum_{i=1}^{5} p_i = \sum_{j=1}^{4} q_j$. Moreover, each $\Lambda'' \in D(K_1, K_0)$ is given by $\Lambda'' = k_1 y_1 + k_2 y_2 + k_3 y_3 + k_4 y_4$, where $k_1, \cdots, k_4 \in \mathbf{Z}$, $k_1 = k_2$, $k_3 = k_4$, $2k_1 + 2k_3 = \sum_{j=1}^{4} q_j$.

7.5. **Eigenvalue computation.** For each $\Lambda = p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + p_5 y_5 \in D(K, K_0)$, with $p_i \in \mathbf{Z}$, $p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5$, the eigenvalue formula of the Casimir operator $\mathcal{C}_{K/K_0}$ with respect to the inner product $\langle X, Y \rangle_{\mathfrak{k}} = -\mathrm{Tr}(\mathrm{Re}(\mathrm{XY}))$ for any $X, Y \in \mathfrak{k} = \mathfrak{u}(5)$ is given by

$$-c_\Lambda = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + 4p_1 + 2p_2 - 2p_4 - 4p_5.$$

For each $\Lambda' = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 \in D(K_2, K_0)$ with $q_i \in \mathbf{Z}$ and $q_1 \geq q_2 \geq q_3 \geq q_4$, the eigenvalue formula of the Casimir operator $\mathcal{C}_{K_2/K_0}$ with respect to the inner product $\langle\, ,\, \rangle_{\mathfrak{k}}|_{\mathfrak{k}_2}$ is given by

$$-c_{\Lambda'} = q_1^2 + q_2^2 + q_3^2 + q_4^2 + 3q_1 + q_2 - q_3 - 3q_4.$$

For each $\Lambda'' = k_1y_1+k_2y_2+k_3y_3+k_4y_4 \in D(K_1, K_0)$ with $k_1 = k_2$ and $k_3 = k_4$, the eigenvalue formula of the Casimir operator $\mathcal{C}_{K_1/K_0}$ with respect to the inner product $\langle\ ,\ \rangle_{\mathfrak{k}}|_{\mathfrak{k}_1}$ is given by

$$\begin{aligned}-c_{\Lambda''} =& k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_1 - k_2 + k_3 - k_4 \\ =& k_1^2 + k_2^2 + k_3^2 + k_4^2.\end{aligned}$$

Hence, we have the following eigenvalue formula

$$\begin{aligned}-c_L =& -2c_\Lambda + c_{\Lambda'} + \frac{1}{2}c_{\Lambda''} \\ =& 2(p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + 4p_1 + 2p_2 - 2p_4 - 4p_5) \\ & - (q_1^2 + q_2^2 + q_3^2 + q_4^2 + 3q_1 + q_2 - q_3 - 3q_4) \\ & - \frac{1}{2}(k_1^2 + k_2^2 + k_3^2 + k_4^2).\end{aligned} \tag{7.1}$$

By using and estimating the formula from above by 18, we get that

Lemma 7.2. *$\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in D(K)$ belongs to $\Lambda \in D(K, K_0)$ with eigenvalue $-c_L \leq 18$ if and only if $(p_1, p_2, p_3, p_4, p_5)$ is one of*

$$\Big\{ (0, -1, -1, -1, -1), (1, 1, 1, 1, 0), (1, 1, 0, 0, 0), (0, 0, 0, -1, -1), \\ (1, 0, 0, 0, -1), (2, 1, 1, 0, 0), (0, 0, -1, -1, -2), (1, 1, 0, -1, -1) \Big\}.$$

Denote by $\omega_1, \omega_2, \omega_3, \omega_4$ the fundamental weight system of $SU(5)$.

Suppose that $\Lambda = (1, 1, 1, 1, 0)$. Then $\dim V_\Lambda = 5$. By the branching law of $(U(5), U(4) \times U(1))$, $\Lambda' = (1, 1, 1, 1, 0)$ or $(1, 1, 1, 0, 1)$, where $\Lambda' = (1, 1, 1, 1, 0) \in D(K_2, K_0)$. By the branching law of $(U(4), U(2) \times U(2))$, $\Lambda'' = (1, 1, 1, 1) \in D(K_1, K_0)$. Thus $-c_\Lambda = 8$, $-c_{\Lambda'} = 4$, $-c_{\Lambda''} = 4$ and $-c_L = -2c_\Lambda + c_{\Lambda'} + \frac{1}{2}c_{\Lambda''} = 10 < 18$.

On the other hand, $\Lambda = \Lambda_0 + \omega_4$, where $\Lambda_0 = \frac{4}{5}\sum_{i=1}^5 y_i$. The group $K = U(5) = C(U(5)) \cdot SU(5)$ acts on $\dim V_\Lambda = 5$ and $V_\Lambda \cong \mathbf{C} \otimes \bar{\mathbf{C}}^5$ by $\rho_{\Lambda_0} \boxtimes \bar{\mu}_5$, where $\bar{\mu}_5$ denotes the conjugate representation of the standard representation of $SU(5)$ on $\mathbf{C}^5$. For each element $g_0 = \begin{pmatrix} A & & \\ & B & \\ & & e^{\sqrt{-1}\theta} \end{pmatrix} \in K_0$ and each element $u \otimes \mathbf{w} \in \mathbf{C} \otimes \bar{\mathbf{C}}^5$, where $A, B \in SU(2)$ and $\theta \in \mathbf{R}$,

$$\begin{aligned}\rho_\Lambda(g_0)(u \otimes \mathbf{w}) =& \rho_{\Lambda_0}(e^{\frac{\sqrt{-1}}{5}\theta} I_5)(u) \otimes \rho_{\omega_4}(e^{-\frac{\sqrt{-1}}{5}\theta} g_0)\mathbf{w} \\ =& e^{\frac{4\sqrt{-1}}{5}\theta} u \otimes \begin{pmatrix} e^{\frac{\sqrt{-1}}{5}\theta} \bar{A} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ e^{\frac{\sqrt{-1}}{5}\theta} \bar{B} \begin{pmatrix} w_3 \\ w_4 \end{pmatrix} \\ e^{-\frac{4\sqrt{-1}}{5}\theta} w_5 \end{pmatrix}.\end{aligned}$$

Hence $(V_\Lambda)_{K_0} = \mathrm{span}_{\mathbf{C}}\{1 \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$.

For a generator $g = \begin{pmatrix} & & 1 & 0 & \\ & & 0 & 1 & \\ 1 & 0 & & & \\ 0 & -1 & & & \\ & & & & 1 \end{pmatrix} \in K_{[\mathfrak{a}]} \subset K_2$ in $\mathbf{Z}_4$,

$$\begin{aligned}\rho_\Lambda(g)(u \otimes \mathbf{e}_5) &= \rho_{\Lambda_0}(e^{\sqrt{-1}\frac{\pi}{5}} I_5)(u) \otimes \rho_{\omega_4}(e^{-\sqrt{-1}\frac{\pi}{5}} g)(\mathbf{e}_5) \\ &= e^{\sqrt{-1}\frac{4\pi}{5}} u \otimes e^{\sqrt{-1}\frac{\pi}{5}} \mathbf{e}_5 = -u \otimes \mathbf{e}_5.\end{aligned}$$

So $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}$, i.e., $\Lambda = (1,1,1,1,0) \notin D(K, K_{[\mathfrak{a}]})$. Similarly, we get $\Lambda = (0,-1,-1,-1,-1) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (1,1,0,0,0)$. Then $\dim V_\Lambda = 10$. By the branching law of $(U(5), U(4) \times U(1))$, $\Lambda' = (1,1,0,0,0)$ or $(1,0,0,0,1)$, where $\Lambda' = (1,1,0,0,0) \in D(K_2, K_0)$. By the branching law of $(U(4), U(2) \times U(2))$, $\Lambda'' = (1,1,0,0), (0,0,1,1)$ or $(1,0,1,0)$, where $\Lambda'' = (1,1,0,0)$ or $(0,0,1,1) \in D(K_1, K_0)$. Thus $-c_\Lambda = 8$, $-c_{\Lambda'} = 6$, $-c_{\Lambda''} = 2$ and $-c_L = -2c_\Lambda + c_{\Lambda'} + \frac{1}{2} c_{\Lambda''} = 9 < 18$.

On the other hand, $\Lambda = \Lambda_0 + \omega_2$, where $\Lambda_0 = \frac{2}{5}\sum_{i=1}^5 y_i$. $V_\Lambda \cong \mathbf{C} \oplus \wedge^2 \mathbf{C}^5$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ be the standard basis of $\mathbf{C}^5$. For each element $g_0 \in K_0$ expressed as above and each element $u \otimes \mathbf{e}_i \wedge \mathbf{e}_j \in V_\Lambda$ $(1 \le i < j \le 5)$,

$$\begin{aligned}\rho_\Lambda(g_0)(u \otimes \mathbf{e}_i \wedge \mathbf{e}_j) &= \rho_{\Lambda_0}(e^{\frac{\sqrt{-1}}{5}\theta} I_5)(u) \otimes \rho_{\omega_2}(e^{-\frac{\sqrt{-1}}{5}\theta} g_0)(\mathbf{e}_i \wedge \mathbf{e}_j) \\ &= e^{\sqrt{-1}\frac{2}{5}\theta} u \otimes (e^{-\frac{\sqrt{-1}}{5}\theta} g_0 \mathbf{e}_i \wedge e^{-\frac{\sqrt{-1}}{5}\theta} g_0 \mathbf{e}_j).\end{aligned}$$

It follows from this that $(V_\Lambda)_{K_0} = \mathrm{span}_{\mathbf{C}}\{1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2), 1 \otimes (\mathbf{e}_3 \wedge \mathbf{e}_4)\}$. For the generator $g \in K_{[\mathfrak{a}]}$ of $\mathbf{Z}_4$ given above, we have

$$\begin{aligned}\rho_\Lambda(g)(1 \otimes \mathbf{e}_1 \wedge \mathbf{e}_2) &= -1 \otimes \mathbf{e}_3 \wedge \mathbf{e}_4, \\ \rho_\Lambda(g)(1 \otimes \mathbf{e}_3 \wedge \mathbf{e}_4) &= 1 \otimes \mathbf{e}_1 \wedge \mathbf{e}_2.\end{aligned}$$

Hence $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}$, i.e., $\Lambda = (1,1,0,0,0) \notin D(K, K_{[\mathfrak{a}]})$. Similarly, we get $\Lambda = (0,0,0,-1,-1) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (1,0,0,0,-1)$. Then $\dim V_\Lambda = 24$. By the branching law of $(U(5), U(4) \times U(1))$, $\Lambda' = (1,0,0,0,-1)$, $(1,0,0,-1,0)$, $(0,0,0,0,0)$ or $(0,0,0,-1,1)$, where $\Lambda'_1 = (1,0,0,-1,0)$, $\Lambda'_2 = (0,0,0,0,0) \in D(K_2, K_0)$. By the branching law of $(U(4), U(2) \times U(2))$, $\Lambda''_1 = (1,0,0,-1)$, $(1,-1,0,0)$, $(0,0,0,0)$, $(0,0,1,-1)$ or $(0,-1,1,0)$, where $\Lambda''_1 = (0,0,0,0) \in D(K_1, K_0)$. Also, $\Lambda''_2 = (0,0,0,0) \in D(K_1, K_0)$. Thus $-c_\Lambda = 10$, $-c_{\Lambda'_1} = 8$, $-c_{\Lambda''_1} = 0$, $-c_L = -2c_\Lambda + c_{\Lambda'} + \frac{1}{2}c_{\Lambda''} = 12 < 18$ and $-c_{\Lambda'_2} = 0$, $-c_{\Lambda''_2} = 0$, $-c_L = 20 > 18$.

On the other hand, $\Lambda = \omega_1 + \omega_4$ corresponds to the adjoint representation of $SU(5)$.

$$\begin{aligned}V_\Lambda = \mathbf{C} \otimes (\mathbf{C} \cdot \begin{pmatrix} -\frac{1}{4} I_4 & 0 \\ 0 & 1 \end{pmatrix} &\oplus \mathbf{C} \cdot \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \\ \oplus \mathbf{C} \cdot \begin{pmatrix} & & * \\ & & 0 \\ * & 0 & 0 \end{pmatrix} &\oplus \mathbf{C} \cdot \begin{pmatrix} & & 0 \\ & & * \\ 0 & * & 0 \end{pmatrix}) \\ = V'_{(0,0,0,0,0)} \oplus V'_{(1,0,0,-1,0)} &\oplus V'_{(1,0,0,0,-1)} \oplus V'_{(0,0,0,-1,1)}.\end{aligned}$$

$$(V_\Lambda)_{K_0} = \left\{ \begin{pmatrix} c_1 I_2 & & \\ & c_2 I_2 & \\ & & c_3 \end{pmatrix} \mid c_1, c_2, c_3 \in \mathbf{C}, 2c_1 + 2c_2 + c_3 = 0 \right\}$$
$$\subset V'_{(0,0,0,0,0)} \oplus V'_{(1,0,0,-1,0)}.$$

By direct calculations, we get that for a generator $g \in K_{[\mathfrak{a}]} \subset K_2$ in $\mathbf{Z}_4$ as above,

$$\mathrm{Ad}(g) \begin{pmatrix} c_1 I_2 & & \\ & c_2 I_2 & \\ & & c_3 \end{pmatrix} = \begin{pmatrix} c_2 I_2 & & \\ & c_1 I_2 & \\ & & c_3 \end{pmatrix}.$$

Hence,

$$(V_\Lambda)_{K_{[\mathfrak{a}]}} = \left\{ \begin{pmatrix} -\frac{c}{4} \mathrm{I}_4 & \\ & c \end{pmatrix} \mid c \in \mathbf{C} \right\} = V'_{(0,0,0,0,0)}.$$

But this 1-dimensional fixed vector space corresponds to the larger eigenvalue 20.

Suppose that $\Lambda = (2,1,1,0,0)$. Then $\dim V_\Lambda = 45$. By the branching law of $(U(5), U(4) \times U(1))$ that V_Λ can be decomposed into the following irreducible $K_2 = U(4) \times U(1)$-submodules:

$$V_\Lambda = V'_{(2,1,1,0,0)} \oplus V'_{(1,1,1,0,1)} \oplus V'_{(2,1,0,0,1)} \oplus V'_{(1,1,0,0,2)},$$

where $\Lambda' = (2,1,1,0,0) \in D(K_2, K_0)$. By the branching law of $(U(4), U(2) \times U(2))$, $\Lambda'' = (2,1,1,0)$, $(2,0,1,1)$, $(1,1,2,0)$, $(1,1,1,1)$ or $(1,0,2,1)$, where $\Lambda'' = (1,1,1,1) \in D(K_1, K_0)$. Thus $-c_\Lambda = 16$, $-c_{\Lambda'} = 12$, $-c_{\Lambda''} = 4$, $-c_L = -2c_\Lambda + c_{\Lambda'} + \frac{1}{2} c_{\Lambda''} = 18$.

On the other hand, since $V'_{(1,1,1,0,1)} \oplus V'_{(2,1,0,0,1)} \oplus V'_{(1,1,0,0,2)}$ has no nonzero vectors fixed by K_0, we see that $(V_\Lambda)_{K_0} \subset V'_{(2,1,1,0,0)}$. Note that $\Lambda' = 2y_1 + y_2 + y_3 = \sum_{i=1}^4 y_i + y_1 - y_4 \in D(K_2, K_0)$ corresponds to the tensor product of $C(U(4))$ representation with the highest weight $\sum_{i=1}^4 y_i$, the adjoint representation of $SU(4)$ with the highest weight $y_1 - y_4$ and the trivial representation of $U(1)$. Then for each element $g_0 \in K_0$ and each element $u \otimes X \otimes v \in \mathbf{C} \otimes \mathfrak{su}(4) \otimes \mathbf{C} \cong V_{\Lambda'}$,

$$\rho_{\Lambda'}(g_0)(u \otimes X \otimes v) = u \otimes \mathrm{Ad} \begin{pmatrix} A & \\ & B \end{pmatrix} (X) \otimes v.$$

Thus $(V_\Lambda)_{K_0} = \mathrm{span}\{1 \otimes \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix} \otimes 1\}$. For the element $g \in K_{[\mathfrak{a}]} \subset K_2$,

$$\rho_{\Lambda'}(g)(u \otimes \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix} \otimes v) = e^{\sqrt{-1}\pi} u \otimes \begin{pmatrix} -I_2 & \\ & I_2 \end{pmatrix} \otimes v.$$

It follows that $(V_\Lambda)_{K_{[\mathfrak{a}]}} = (V_\Lambda)_{K_0}$, i.e., $\Lambda = (2,1,1,0,0) \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 1. Similarly, $\Lambda = (0,0,-1,-1,-2) \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 1 and it also gives the eigenvalue 18.

Suppose that $\Lambda = (1,1,0,-1,-1)$. Then $\dim V_\Lambda = 75$. By the branching law of $(U(4), U(2) \times U(2))$, V_Λ can be decomposed the following irreducible $K_1 = U(4) \times U(1)$-submodules:

$$V_\Lambda = V'_{(1,1,0,-1,-1)} \oplus V'_{(1,1,-1,-1,0)} \oplus V'_{(1,0,0,-1,0)} \oplus V'_{(1,0,-1,-1,1)},$$

where $\Lambda'_1 = (1,1,-1,-1,0)$ and $\Lambda'_2 = (1,0,0,-1,0) \in D(K_2, K_0)$. For Λ'_2, by the branching law of $(U(4), U(2) \times U(2))$, $\Lambda''_2 = (1,0,0,-1)$, $(1,-1,0,0)$, $(0,0,-1,-1)$, $(0,0,0,0)$ or $(0,-1,1,0)$, where $\Lambda''_2 = (0,0,0,0) \in D(K_1, K_0)$. Therefore, $-c_\Lambda = 16$, $-c_{\Lambda'_2} = 8$, $-c_{\Lambda''_2} = 0$, and $-c_L = -2c_\Lambda + c_{\Lambda'_2} + \frac{1}{2} c_{\Lambda''_2} = 24 > 18$. For Λ'_1, by the branching law of $(U(4), U(2) \times U(2))$,

$\Lambda'' = (1,1,-1,-1), (1,0,0,-1), (1,-1,1,-1), (0,0,0,0), (0,-1,1,0)$ or $(-1,-1,1,1)$, where $\Lambda''_{11} = (1,1,-1,-1)$, $\Lambda''_{12} = (-1,-1,1,1)$, $\Lambda''_{13} = (0,0,0,0) \in D(K_1, K_0)$. Thus $-c_\Lambda = 16$, $-c_{\Lambda'} = 12$, $-c_{\Lambda''_{11}} = -c_{\Lambda''_{12}} = 4$, $-c_{\Lambda''_{13}} = 0$, $-c_L = -2c_\Lambda + c_{\Lambda'} + \frac{1}{2}c_{\Lambda''} = 18, 18$ or 20. Moreover, from the above irreducible K_2-decomposition of V_Λ and eigenvalue calculations, we only need to determine $\dim(V_\Lambda)_{K_{[\mathfrak{a}]}} \cap (V''_{11} \oplus V''_{12})$ since the fixed vectors in this subspace by $K_{[\mathfrak{a}]}$ give the eigenvalue 18. Here we set $V''_{11} := V''_{\Lambda''_{11}}$ and $V''_{12} := V''_{\Lambda''_{12}}$.

Recall that the irreducible representation of $SU(4)$ with the highest weight $\Lambda'_1 = y_1 + y_2 - y_3 - y_4 = 2\omega_2$ can be described as follows ([16]):

$$\mathrm{Sym}^2(\wedge^2 \mathbf{C}^4) = I(Gr_2(\mathbf{C}^4))_2 \oplus V'_{\Lambda'_1},$$

where $I(Gr_2(\mathbf{C}^4))_2$, the ideal of the Grassmannian $Gr_2(\mathbf{C}^4)$, denotes the space of all homogeneous polynomials of degree 2 on $\mathbf{P}(\wedge^2 \mathbf{C}^{4*})$ that vanish on $Gr_2(\mathbf{C}^4)$. Here $I(Gr_2(\mathbf{C}^4))_2 \cong \wedge^4 \mathbf{C}^4 \cong \mathbf{C}$ can be written down explicitly in terms of a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ of $\mathbf{C}^4$:

$$\begin{aligned} I(Gr_2(\mathbf{C}^4))_2 = \mathrm{span}\{&(\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) + (\mathbf{e}_1 \wedge \mathbf{e}_4) \cdot (\mathbf{e}_2 \wedge \mathbf{e}_3) \\ &- (\mathbf{e}_1 \wedge \mathbf{e}_3) \cdot (\mathbf{e}_2 \wedge \mathbf{e}_4)\}. \end{aligned}$$

Thus a basis for $V'_{\Lambda'_1}$ can be given explicitly. For any element $g_0 \in K_0$, denote $g'_0 = \begin{pmatrix} A & \\ & B \end{pmatrix} \in SU(2) \times SU(2) \subset U(4)$. The representation of K_0 on any element $u \otimes X \otimes w \in \mathbf{C} \otimes V'_{\Lambda'_1} \otimes \mathbf{C}$ is

$$\rho_\Lambda(g)(u \otimes X \otimes w) = \rho_0(1)(u) \otimes \rho_{\Lambda'_1}(g'_0)(X) \otimes \rho_0(e^{\sqrt{-1}\theta})(w).$$

By direct computations, we obtain

$$\begin{aligned} (V_\Lambda)_{K_0} \cap V'_{\Lambda'_1} = \mathrm{span}_{\mathbf{C}}\{\ &1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2) \otimes 1, \\ &1 \otimes (\mathbf{e}_3 \wedge \mathbf{e}_4) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \otimes 1, \\ &1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \otimes 1\}, \end{aligned}$$

where $(\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2) \in V''_{11}$, $(\mathbf{e}_3 \wedge \mathbf{e}_4) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \in V''_{12}$ and $(\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \in V''_{13}$. For the generator $g \in K_{[\mathfrak{a}]} \subset K_2$, denote $g' = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}$. The representation of g on $u \otimes X \otimes w$ is

$$\rho_\Lambda(g)(u \otimes X \otimes w) = \rho_0(e^{\frac{\sqrt{-1}}{4}\pi} I_4)(u) \otimes \rho_{\Lambda'_1}(e^{-\frac{\sqrt{-1}}{4}\pi} g')(X) \otimes \rho_0(1)(w).$$

It follows that

$$\begin{aligned} &(V_\Lambda)_{K_{[\mathfrak{a}]}} \cap V'_{\Lambda'_1} = \mathrm{span}_{\mathbf{C}}\{1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \otimes 1, \\ &1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2) \otimes 1 - 1 \otimes (\mathbf{e}_3 \wedge \mathbf{e}_4) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \otimes 1\}. \end{aligned}$$

In particular, $\Lambda = (1,1,0,-1,-1) \in D(K, K_{[\mathfrak{a}]})$ and

$$\begin{aligned} &(V_\Lambda)_{K_{[\mathfrak{a}]}} \cap (V''_{11} \oplus V''_{12}) \\ &\quad = \mathrm{span}_{\mathbf{C}}\{1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2) \otimes 1 - 1 \otimes (\mathbf{e}_3 \wedge \mathbf{e}_4) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \otimes 1\} \end{aligned}$$

with dimension 1, which corresponds to the eigenvalue 18.

Now we obtain that the Gauss image L^{18} is Hamiltonian stable. Moreover,

$$\begin{aligned} n(L^{18}) =& \dim V_{(0,0,-1,-1,-2)} + \dim V_{(2,1,1,0,0)} + \dim V_{(1,1,0,-1,-1)} \\ =& 45 + 45 + 75 = 165 = \dim SO(20) - \dim U(5) = n_{hk}(L^{18}). \end{aligned}$$

Hence the Gauss image L^{18} is Hamiltonian rigid.

Therefore, we conclude that the Gauss image L^{18} is Hamiltonian stable.

Theorem 7.1. *The Gauss image* $L^{18} = \mathcal{G}\left(\frac{U(5)}{(SU(2)\times SU(2)\times U(1))}\right) = \frac{U(5)}{(SU(2)\times SU(2)\times U(1))\cdot \mathbf{Z}_4} \subset Q_{18}(\mathbf{C})$ *is strictly Hamiltonian stable.*

8. The case $(U, K) = (SO(m+2), SO(2)\times SO(m))$ $(m \geq 3)$

In this case (U, K) is of type B_2. The canonical decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{u} = \mathfrak{o}(m+2)$ and a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ are given as

$$\begin{aligned} \mathfrak{k} &= \left\{ \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \mid T_1 \in \mathfrak{o}(2), T_2 \in \mathfrak{o}(m) \right\} = \mathfrak{o}(2) + \mathfrak{o}(m), \\ \mathfrak{p} &= \left\{ \begin{pmatrix} 0 & -{}^tX \\ X & 0 \end{pmatrix} \mid X \in M(m, 2; \mathbf{R}) \right\}, \\ \mathfrak{a} &= \left\{ H = H(\xi_1, \xi_2) = \begin{pmatrix} 0 & -{}^t\xi & 0 \\ \xi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \xi = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \xi_1, \xi_2 \in \mathbf{R} \right\}. \end{aligned}$$

Then

$$\begin{aligned} K_0 &= \left\{ \begin{pmatrix} \pm \mathrm{I}_4 & 0 \\ 0 & T \end{pmatrix} \mid T \in SO(m-2) \right\} \\ &\cong \mathbf{Z}_2 \times SO(m-2). \end{aligned}$$

Moreover

$$K_{[\mathfrak{a}]} \cong (\mathbf{Z}_2 \times SO(m-2)) \cdot \mathbf{Z}_4$$

consists of all elements

$$a = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B' \end{pmatrix} \in K = SO(2) \times SO(m),$$

where

$$\begin{aligned} (A, B) =& \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \\ & \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right), \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right), \\ & \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \\ & \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right), \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right). \end{aligned}$$

Here note that $K_{[\mathfrak{a}]} \not\subset K_1 = SO(2)\times SO(2)\times SO(m-2)$. Thus the deck transformation group of the covering map $\mathcal{G} : N^{2m-2} \to \mathcal{G}(N^{2m-2})$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

8.1. **Description of the Casimir operator.** Denote $\langle X,Y\rangle_{\mathfrak{u}} := -\frac{1}{2}\mathrm{tr}XY$ for each $X,Y \in \mathfrak{u} = \mathfrak{o}(m+2)$. The restricted root system $\Sigma(U,K)$ of type B_2, can be given as follows ([7]):

$$\Sigma^+(U,K) = \{\varepsilon_1 - \epsilon_2 = \alpha_1,\, \varepsilon_2 = \alpha_2,\, \varepsilon_1 + \epsilon_2 = \alpha_1 + 2\alpha_2,\, \varepsilon_1 = \alpha_1 + \alpha_2\}.$$

Then, relative to the above inner product $\langle\, ,\rangle_{\mathfrak{u}}$, the square length of any restrict root $\gamma \in \Sigma(U,K)$ is $\|\gamma\|^2_{\mathfrak{u}} = 1$ or 2. Hence the Casimir operator $\mathcal{C}_L$ of L with respect to the induced metric from $Q_{2m-2}(\mathbf{C})$ is given as follows:

$$\begin{aligned}\mathcal{C}_L &= \frac{2}{\|\gamma_0\|^2_{\mathfrak{u}}}\mathcal{C}_{K/K_0,\langle\,,\rangle_{\mathfrak{u}}} - \frac{1}{\|\gamma_0\|^2_{\mathfrak{u}}}\mathcal{C}_{K_1/K_0,\langle\,,\rangle_{\mathfrak{u}}} \\ &= \mathcal{C}_{K/K_0} - \frac{1}{2}\,\mathcal{C}_{K_1/K_0},\end{aligned} \tag{8.1}$$

where $K = SO(2)\times SO(m) \supset K_1 = SO(2)\times SO(2)\times SO(m-2) \supset K_0 = \mathbf{Z}_2 \times SO(m-2)$ and $\mathcal{C}_{K/K_0}$, $\mathcal{C}_{K_1/K_0}$ denote the Casimir operators of K/K_0 and K_1/K_0 relative to $\langle\, ,\, \rangle_{\mathfrak{u}}|_{\mathfrak{k}}$ and $\langle\, ,\, \rangle_{\mathfrak{u}}|_{\mathfrak{k}_1}$, respectively.

8.2. **Branching laws for** $(SO(n+2), SO(2)\times SO(n))$**.** We need the branching laws for $(SO(n+2), SO(2)\times SO(n))$ by Tsukamoto ([47]).

Lemma 8.1 (Branching laws for $(SO(2p+2), SO(2)\times SO(2p)), p \geq 1$)**.** *Let $\Lambda = h_0\varepsilon_0 + h_1\varepsilon_1 + \cdots + h_{p-1}\varepsilon_{p-1} + \epsilon h_p\varepsilon_p \in D(SO(2p+2))$, where $\epsilon = 1$ or -1 and $h_0, h_1, \cdots, h_p$ are integers satisfying*

$$h_0 \geq h_1 \geq \cdots \geq h_p \geq 0 \tag{8.2}$$

and $\Lambda' = k_0\varepsilon_0 + k_1\varepsilon_1 + \cdots + k_{p-1}\varepsilon_{p-1} + \epsilon' k_p\varepsilon_p \in D(SO(2)\times SO(2p))$, where $\epsilon' = 1$ or -1 and $k_0, k_1, \cdots, k_p$ are integers satisfying

$$k_1 \geq \cdots \geq k_p \geq 0. \tag{8.3}$$

The irreducible decomposition of V_Λ as a $SO(2)\times SO(2p)$-module contains an irreducible $SO(2)\times SO(2p)$-module $V'_{\Lambda'}$ if and only if

$$\begin{aligned}&h_{i-1} \geq k_i \geq h_{i+1} \quad (1 \leq i \leq p-1),\\ &h_{p-1} \geq k_p \geq 0,\end{aligned}$$

and the coefficient of X^{k_0} in the finite power series

$$X^{\epsilon\epsilon' l_p}\prod_{i=0}^{p-1}\frac{X^{l_i+1} - X^{-l_i-1}}{X - X^{-1}}$$

does not vanish, where

$$\begin{aligned}&l_0 := h_0 - \max\{h_1, k_1\},\\ &l_i := \min\{h_i, k_i\} - \max\{h_{i+1}, k_{i+1}\} \quad (1 \leq i \leq p-1),\\ &l_p := \min\{h_p, k_p\}.\end{aligned} \tag{8.4}$$

Moreover, the coefficient of X^{k_0} is equal to the multiplicity of $V'_{\Lambda'}$ appearing in the irreducible decomposition.

Lemma 8.2 (Branching laws for $(SO(2p+3), SO(2) \times SO(2p+1)), p \geq 1)$**.** *Let* $\Lambda = h_0\varepsilon_0 + h_1\varepsilon_1 + \cdots + h_{p-1}\varepsilon_{p-1} + h_p\varepsilon_p \in D(SO(2p+3))$*, where* $h_0, h_1, \cdots, h_p$ *are integers satisfying* (8.2) *and* $\Lambda' = k_0\varepsilon_0 + k_1\varepsilon_1 + \cdots + k_{p-1}\varepsilon_{p-1} + k_p\varepsilon_p \in D(SO(2) \times SO(2p+1))$*, where* $k_0, k_1, \cdots, k_p$ *are integers satisfying* (8.3)*. The irreducible decomposition of* V_Λ *as a* $SO(2) \times SO(2p+1)$*-module contains an irreducible* $SO(2) \times SO(2p+1)$*-module* $V'_{\Lambda'}$ *if and only if*

$$\begin{aligned} &h_{i-1} \geq k_i \geq h_{i+1}, \quad (1 \leq i \leq p-1) \\ &h_{p-1} \geq k_p \geq 0, \end{aligned}$$

and the coefficient of X^{k_0} *in the finite power series*

$$\left(\prod_{i=0}^{p-1} \frac{X^{l_i+1} - X^{-l_i-1}}{X - X^{-1}}\right) \frac{X^{l_p+\frac{1}{2}} - X^{-l_p-\frac{1}{2}}}{X^{\frac{1}{2}} - X^{-\frac{1}{2}}}$$

does not vanish, where integers $l_0, l_1, \cdots, l_p$ *are defined by* (8.4)*. Moreover, the coefficient of* X^{k_0} *is equal to the multiplicity of* $V'_{\Lambda'}$ *appearing in the irreducible decomposition.*

8.3. **Description of** $D(K, K_0)$ **and eigenvalue computations.**

For $m = 2p$ ($p \geq 2$) or $m = 2p+1$ ($p \geq 1$), each $\tilde{\Lambda} \in D(K) = D(SO(2) \times SO(m))$ can be expressed as

$$\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 + \cdots + k_p\varepsilon_p,$$

where $k_0\varepsilon_0 \in D(SO(2))$, $\Lambda := k_1\varepsilon_1 + \cdots + k_p\varepsilon_p \in D(SO(m))$ and $k_0, k_1, \cdots, k_p \in \mathbf{Z}$ satisfying

$$\begin{aligned} &k_1 \geq k_2 \geq \cdots \geq k_{p-1} \geq |k_p| \quad \text{ if } m = 2p, \\ &k_1 \geq k_2 \geq \cdots \geq k_{p-1} \geq k_p \geq 0 \quad \text{ if } m = 2p+1. \end{aligned}$$

Then we have

$$\tilde{V}_{\tilde{\Lambda}} = U_{k_0\varepsilon_0} \otimes V_\Lambda.$$

Note that

$$\begin{aligned} D(K, K_0) &= D(SO(2) \times SO(m), \mathbf{Z}_2 \times SO(m-2)) \\ &\subset D(SO(2) \times SO(m), SO(m-2)), \\ D(K_1, K_0) &= D(SO(2) \times SO(2) \times SO(m-2), \mathbf{Z}_2 \times SO(m-2)) \\ &\subset D(SO(2) \times SO(2) \times SO(m-2), SO(m-2)). \end{aligned}$$

By applying Lemmas 8.1 and 8.2 to both cases $(SO(2p), SO(2) \times SO(2p-2))$ and $(SO(2p), SO(2) \times SO(2p-1))$, we can describe $D(K, K_0)$ as follows:

Lemma 8.3. *Assume that* $p \geq 2$*. Let* $\tilde{\Lambda} \in D(K)$*. Then an irreducible* K*-module* $\tilde{V}_{\tilde{\Lambda}}$ *with the highest weight* $\tilde{\Lambda}$ *contains an irreducible* K_1*-module* $\tilde{V}'_{\tilde{\Lambda}'}$ *with the highest weight* $\tilde{\Lambda}' \in D(K_1)$ *satisfying* $(\tilde{V}'_{\tilde{\Lambda}'})_{K_0} \neq \{0\}$ *if and only if*

$$\begin{aligned} &\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 + k_2\varepsilon_2 \in D(K), \\ &\tilde{\Lambda}' = k_0\varepsilon_0 + k'_1\varepsilon_1 \in D(K_1), \end{aligned}$$

where $k_0, k_1, k_2, k'_1 \in \mathbf{Z}$*,* $k_1 \geq k_2 \geq 0$ *satisfy the following conditions:*

(i) *The coefficient of* $X^{k'_1}$ *in the finite Laurent series expansion* $\dfrac{X^{k_1-k_2+1} - X^{-(k_1-k_2+1)}}{X - X^{-1}}$ *of* X *does not vanish.*

(ii) $k_0 + k'_1$ *is even.*

In particular $-(k_1-k_2)\leq k_1'\leq (k_1-k_2)$*. Here the coefficient is equal to the multiplicity of* $\tilde{V}'_{\tilde{\Lambda}'}$.

8.3.1. *The case* $m=2p$ $(p\geq 2)$.

Suppose that $m=2p$ $(p\geq 2)$. For each

$$\tilde{\Lambda}=k_0\varepsilon_0+k_1\varepsilon_1+k_2\varepsilon_2\in D(K,K_0)=D(SO(2)\times SO(2p),\mathbf{Z}_2\times SO(2p-2))$$

with $\tilde{\Lambda}'=k_0\varepsilon_0+k_1'\varepsilon_1\in D(K_1,K_0)=D(SO(2)\times SO(2)\times SO(2p-2),\mathbf{Z}_2\times SO(2p-2))$ as in Lemma 8.3, $-\mathcal{C}_{K/K_0}$ and $-\mathcal{C}_{K_1/K_0}$ have eigenvalues

$$\begin{aligned} -c_{\tilde{\Lambda}}&=k_0^2+k_1^2+k_2^2+2(p-1)k_1+2(p-2)k_2,\\ -c_{\tilde{\Lambda}'}&=\frac{1}{2}(k_0^2+k'^2_1). \end{aligned}$$

Hence by the formula (8.1) the corresponding eigenvalue of $-\mathcal{C}_L$ is

$$\begin{aligned} -c_L=&\ -c_{\tilde{\Lambda}}+\frac{1}{2}c_{\tilde{\Lambda}'}\\ =&\ k_0^2+k_1^2+k_2^2+2(p-1)k_1+2(p-2)k_2-\frac{1}{2}(k_0^2+k'^2_1). \end{aligned}\tag{8.5}$$

Denote $\tilde{\Lambda}=k_0\varepsilon_0+k_1\varepsilon_1+k_2\varepsilon_2\in D(K,K_0)$ by $\tilde{\Lambda}=(k_0,k_1,k_2)$.

For each $\tilde{\Lambda}=k_0\varepsilon_0=(k_0,0,0)\in D(K,K_0)$, as $k_1'=0$, $k_0=k_0+k_1'$ is even and $-c_L=\frac{1}{2}k_0^2$, we see that

$$-c_L\leq 2m-2=4p-2 \text{ if and only if } k_0^2\leq 4(2p-1).\tag{8.6}$$

As $\tilde{V}_{\tilde{\Lambda}}\cong U_{k_0\varepsilon_0}\otimes\mathbf{C}\cong U_{k_0\varepsilon_0}$, for a generator $g=\begin{pmatrix}0&1&&&\\-1&0&&&\\&&0&1&\\&&1&0&\\&&&&T'\end{pmatrix}\in K_{[\mathfrak{a}]}$ which will be used throughout this section, we have

$$\rho_{k_0\varepsilon_0}(g)(v\otimes 1)=e^{\sqrt{-1}\frac{\pi}{2}k_0}(v\otimes 1).$$

Hence

$$(k_0,0,0)\in D(K,K_{[\mathfrak{a}]}) \text{ if and only if } k_0\in 4\mathbf{Z}.\tag{8.7}$$

(i) The case $\mathcal{G}(N^6)\cong\frac{SO(2)\times SO(4)}{(\mathbf{Z}_2\times SO(2))\cdot\mathbf{Z}_4}\to Q_6(\mathbf{C})$ with $p=2$.

Since $-c_L=-\frac{1}{2}c_{K/K_0}-\frac{1}{2}c_{K/K_1}\geq-\frac{1}{2}c_{K/K_0}$, note that $-c_L\geq 6$ implies $-c_{\tilde{\lambda}}=-c_{K/K_0}\leq 12$. Using the eigenvalue formula (8.5) we compute that

Lemma 8.4. *$\tilde{\Lambda}=k_0\varepsilon_0+k_1\varepsilon_1+k_2\varepsilon_2\in D(K,K_0)$ has eigenvalue $-c_L\leq 6$ if and only if (k_0,k_1,k_2) is one of*

$$\{0,(\pm 2,0,0),(\pm 1,1,0),(0,1,1),(\pm 2,1,1),(0,2,0),(0,1,-1),(\pm 2,1,-1)\}.$$

Suppose that $\tilde{\Lambda}=(\pm 2,0,0)$. Then by (8.7) $\tilde{\Lambda}=(\pm 2,0,0)\notin D(K,K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda}=(\pm 1,1,0)$. Then $\dim\tilde{V}_{\tilde{\Lambda}}=4$ and $\tilde{V}_{\tilde{\Lambda}}\cong U_{k_0\varepsilon_0}\otimes\mathbf{C}^4$, where $\Lambda=\varepsilon_1\in D(K)$ corresponds to the matrix multiplication of $SO(4)$ on $\mathbf{C}^4$. It follows from the branching law

(Lemma 8.1, p=2) of $(SO(4), SO(2)\times SO(2))$ that $k_1' = \pm 1$. Hence $-c_L = \frac{1}{2}k_0^2 + \frac{5}{2}$. Note that $U_{k_0\varepsilon_0}\otimes \mathbf{C}^4$ can be decomposed into irreducible $SO(2)\times SO(2)\times SO(2)$-modules as

$$U_{k_0\varepsilon_0}\otimes \mathbf{C}^4 = (U_{k_0\varepsilon_0}\otimes (\mathbf{C}^2\oplus\{0\}))\oplus(U_{k_0\varepsilon_0}\otimes(\{0\}\oplus\mathbf{C}^2)).$$

There is no nonzero fixed vector by $\mathbf{Z}_2\times SO(2)$ in $U_{k_0\varepsilon_0}\otimes(\{0\}\oplus\mathbf{C}^2)$. Moreover, since

$$\begin{aligned}
&\rho_{k_0\varepsilon_0+\varepsilon_1}\begin{pmatrix}-I_2 & & \\ & -I_2 & \\ & & T\end{pmatrix}(v\otimes\begin{pmatrix}w_1\\w_2\\0\\0\end{pmatrix})\\
&= e^{\sqrt{-1}\pi k_0}v\otimes\begin{pmatrix}-w_1\\-w_2\\0\\0\end{pmatrix} = e^{\sqrt{-1}\pi(k_0+1)}v\otimes\begin{pmatrix}w_1\\w_2\\0\\0\end{pmatrix},
\end{aligned}$$

it follows that $(\tilde{V}_{\tilde{\Lambda}})_{K_0} = (\tilde{V}_{\tilde{\Lambda}})_{\mathbf{Z}_2\times SO(2)} \neq \{0\}$ if and only if k_0 is odd, and then $(\tilde{V}_{\tilde{\Lambda}})_{\mathbf{Z}_2\times SO(2)} = U_{k_0\varepsilon_0}\otimes(\mathbf{C}^2\oplus\{0\})$. Let k_0 be odd. However since

$$\rho_{k_0\varepsilon_0+\varepsilon_1}(g)(v\otimes\begin{pmatrix}w_1\\w_2\\0\\0\end{pmatrix}) = e^{\sqrt{-1}\frac{\pi}{2}k_0}v\otimes\begin{pmatrix}w_2\\w_1\\0\\0\end{pmatrix},$$

$U_{k_0\varepsilon_0}\otimes(\mathbf{C}^2\oplus\{0\})$ has no nonzero fixed vector by $(\mathbf{Z}_2\times SO(2))\cdot\mathbf{Z}_4$, and hence $(k_0,1,0)\notin D(K,K_{[\mathfrak{a}]})$. In particular $(\pm1,1,0)\notin D(K,K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda}_1 = (k_0,1,1)$ and $\tilde{\Lambda}_2 = (k_0,1,-1)$. Then $\dim\tilde{V}_{\tilde{\Lambda}_1} = \dim\tilde{V}_{\tilde{\Lambda}_2} = 3$ and $\tilde{V}_{\tilde{\Lambda}_1}\oplus\tilde{V}_{\tilde{\Lambda}_2}\cong\mathbf{C}\otimes\wedge^2\mathbf{C}^4$. It follows from the branching law (Lemma 8.1, p=2) $(SO(4), SO(2)\times SO(2))$ that

$$\tilde{V}_{\tilde{\Lambda}_1} = \tilde{V}'_{(k_0,1,1)}\oplus\tilde{V}'_{(k_0,-1,-1)}\oplus\tilde{V}'_{(k_0,0,0)},$$

where $(k_0,0,0)\in D(K_1,K_0)$. Thus $-c_L = \frac{1}{2}k_0^2+4$, which is equals to 4 when $k_0=0$ and 6 when $k_0=\pm2$.

Let $\{e_1,e_2,e_3,e_4\}$ be the standard basis of $\mathbf{C}^4$. Then we have

$$\begin{aligned}
\tilde{V}_{\tilde{\Lambda}_1} &= \mathrm{span}\{e_1\wedge e_2, e_1\wedge e_3 - e_2\wedge e_4, e_1\wedge e_4 + e_2\wedge e_3\},\\
\tilde{V}_{\tilde{\Lambda}_2} &= \mathrm{span}\{e_3\wedge e_4, e_1\wedge e_3 + e_2\wedge e_4, e_1\wedge e_4 - e_2\wedge e_3\}.
\end{aligned}$$

Since $e_1\wedge e_2\in\wedge^2\mathbf{C}^4$ is fixed by the representation of $SO(2)\times SO(2)$ with respect to the highest weight $\tilde{\Lambda}_1$,

$$(\tilde{V}_{\tilde{\Lambda}_1})_{K_0} = \mathrm{span}\{1\otimes(e_1\wedge e_2)\}.$$

Moreover,

$$\rho_{\tilde{\Lambda}_1}(g)(v\otimes(e_1\wedge e_2)) = e^{\sqrt{-1}\frac{\pi}{2}k_0}v\otimes(e_2\wedge e_1).$$

Hence, $\tilde{\Lambda}_1 = (0,1,1)\notin D(K,K_{[\mathfrak{a}]})$ but $\tilde{\Lambda}_1 = (\pm2,1,1)\in D(K,K_{[\mathfrak{a}]})$ and $(\tilde{V}_{\tilde{\Lambda}_1})_{K_{[\mathfrak{a}]}}\cong\mathbf{C}\otimes\mathbf{C}\{e_1\wedge e_2\}$ for $k_0=2$ or -2, both of which give eigenvalue 6. Similarly, $\tilde{\Lambda}_2 = (0,1,-1)\notin D(K,K_{[\mathfrak{a}]})$ but $\tilde{\Lambda}_2 = (\pm2,1,-1)\in D(K,K_{[\mathfrak{a}]})$ and $(\tilde{V}_{\tilde{\Lambda}_2})_{K_{[\mathfrak{a}]}}\cong\mathbf{C}\otimes\mathbf{C}\{e_3\wedge e_4\}$ for $k_0=2$ or -2, both of which give eigenvalue 6.

Suppose that $\tilde{\Lambda} = (0,2,0)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 9$ and $\tilde{V}_{\tilde{\Lambda}} \cong \mathbf{C} \otimes \mathrm{S}^2_0(\mathbf{C}^4)$, where the corresponding representation of $SO(4)$ is just the adjoint representation on $\mathrm{S}^2_0(\mathbf{C}^4)$. It follows from the branching law of $(SO(4), SO(2) \times SO(2))$ that $k'_1 = 0, \pm 2$. Thus $-c_L = 8 - \frac{1}{2}{k'_1}^2$. When $k'_1 = \pm 2$, $-c_L = 6$, otherwise $-c_L = 8 > 6$. On the other hand, $\mathrm{S}^2_0(\mathbf{C}^4)$ can be decomposed into the following $SO(2) \times SO(2)$-modules:

$$V_{2\varepsilon_1} \cong \mathrm{S}^2_0(\mathbf{C}^4) = \mathrm{S}^2_0(\mathbf{C}^2) \oplus \mathrm{S}^2_0(\mathbf{C}^2) \oplus M(2,2;\mathbf{C}) \oplus \mathbf{C} \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}.$$

Thus, $\mathrm{S}^2_0(\mathbf{C}^2) \oplus \mathbf{C} \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}$ is fixed by $\{-I_2\} \times SO(2)$ and $\dim(\tilde{V}_{\tilde{\Lambda}})_{K_0} = 3$. Moreover,

$$\rho_{\tilde{\Lambda}}(g)(v \otimes \begin{pmatrix} a & b & \\ b & -a & \\ & & 0 \end{pmatrix}) = v \otimes \begin{pmatrix} -a & b & \\ b & a & \\ & & 0 \end{pmatrix},$$
$$\rho_{\tilde{\Lambda}}(g)(v \otimes \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}) = v \otimes \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}.$$

Hence,

$$(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}} = \mathbf{C} \otimes \mathbf{C} \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \oplus \mathbf{C} \otimes \mathbf{C} \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}.$$

Notice that the first summand lies in the $SO(2) \times SO(2) \times SO(2)$-module $V'_{2\varepsilon_1} \oplus V'_{-2\varepsilon_1}$, which gives eigenvalue 6 and the second summand lies in the $SO(2) \times SO(2) \times SO(2)$-module with respect to weight $(0,0,0) \in D(K_1, K_0)$, which gives eigenvalue $8 > 6$. Therefore, $\tilde{\Lambda} = (0,2,0) \in D(K, K_{[\mathfrak{a}]})$ and the multiplicity corresponding to eigenvalue 6 is 1.

Now we know that $\mathcal{G}(N^6) \subset Q_6(\mathbf{C})$ is Hamiltonian stable. Since $\tilde{\Lambda} = (2,1,1)$, $(-2,1,1)$, $(2,1,-1)$, $(-2,1,-1)$, $(0,2,0) \in D(K, K_{[\mathfrak{a}]})$ give the smallest eigenvalue 6 with multiplicity 1 and

$$\begin{aligned} & n(L^6) \\ =& \dim \tilde{V}_{(2,1,1)} + \dim \tilde{V}_{(-2,1,1)} + \dim \tilde{V}_{(2,1,-1)} + \dim \tilde{V}_{(-2,1,-1)} + \dim \tilde{V}_{(0,2,0)} \\ =& 3+3+3+3+9 = 21 = \dim SO(8) - \dim(SO(2) \times SO(4)) = n_{hk}(L^6). \end{aligned}$$

Hence we obtain that $\mathcal{G}(N^6) \subset Q_6(\mathbf{C})$ is strictly Hamiltonian stable.

(ii) The case $\mathcal{G}(N^{4p-2}) \cong \frac{SO(2)\times SO(2p)}{(\mathbf{Z}_2 \times SO(2p-2))\cdot \mathbf{Z}_4} \to Q_{4p-2}(\mathbf{C})$ with $p \geq 3$.

Suppose that $\tilde{\Lambda} = (k_0, 0, 0)$ and $k_0 \in 4\mathbf{Z} \setminus \{0\}$. Then $k'_1 = 0$ and by (8.6) $\tilde{\Lambda} \in D(K, K_{[\mathfrak{a}]})$. As $p \geq 3$, we have $16 < 20 \leq 4(2p-1)$. Hence by (8.7) we see that for every $k_0 \in 4\mathbf{Z} \setminus \{0\}$ such that $16 \leq k_0^2 < 4(2p-1)$ we have eigenvalue $-c_L = \frac{1}{2}k_0^2 < 4p-2$. Therefore, $\mathcal{G}(N^{4p-2}) \cong \frac{SO(2)\times SO(2p)}{(\mathbf{Z}_2 \times SO(2p-2))\cdot \mathbf{Z}_4} \to Q_{4p-2}(\mathbf{C})$ is not Hamiltonian stable if $p \geq 3$.

Theorem 8.1.

$$L^{4p-2} = (SO(2) \times SO(2p))/(\mathbf{Z}_2 \times SO(2p-2))\mathbf{Z}_4 \quad (p \geq 2)$$

is not Hamiltonian stable if and only if $(m-2)-1 = 2p-3 \geq 3$. *If* $p = 2$, *then it is strictly Hamiltonian stable.*

Remark. The index $i(L^{4p-2})$ goes to ∞ as $p \to \infty$.

8.3.2. *The case $m=2p+1$ $(p\geq 1)$.*

Assume that $m=2p+1$ $(p\geq 2)$. For each

$$\tilde{\Lambda}=k_0\varepsilon_0+k_1\varepsilon_1+k_2\varepsilon_2\in D(K,K_0)=D(SO(2)\times SO(2p+1),\mathbf{Z}_2\times SO(2p-1))$$

with $\tilde{\Lambda}'=k_0\varepsilon_0+k_1'\varepsilon_1\in D(K_1,K_0)=D(SO(2)\times SO(2)\times SO(2p-1),\mathbf{Z}_2\times SO(2p-1))$ as in Lemma 8.3, $-\mathcal{C}_{K/K_0}$ and $-\mathcal{C}_{K_1/K_0}$ have eigenvalues

$$\begin{aligned}-c_{\tilde{\Lambda}}&=k_0^2+k_1^2+k_2^2+(2p-1)k_1+(2p-3)k_2,\\ -c_{\tilde{\Lambda}'}&=-\frac{1}{2}(k_0^2+{k'}_1^2).\end{aligned}$$

Hence by the formula (8.1) the corresponding eigenvalue of $-\mathcal{C}_L$ is

$$\begin{aligned}-c_L=&\ -c_{\tilde{\Lambda}}+\frac{1}{2}c_{\tilde{\Lambda}'}\\ =&\ k_0^2+k_1^2+k_2^2+(2p-1)k_1+(2p-3)k_2-\frac{1}{2}(k_0^2+{k'}_1^2).\end{aligned}\tag{8.8}$$

Denote $\tilde{\Lambda}=k_0\varepsilon_0+k_1\varepsilon_1+k_2\varepsilon_2\in D(K,K_0)$ by $\tilde{\Lambda}=(k_0,k_1,k_2)$.

For each $\tilde{\Lambda}=k_0\varepsilon_0=(k_0,0,0)\in D(K,K_0)$, as $k_1'=0$, $k_0=k_0+k_1'$ is even and $-c_L=\frac{1}{2}k_0^2$, we see that

$$-c_L\leq 2m-2=4p \text{ if and only if } k_0^2\leq 8p.\tag{8.9}$$

As $\tilde{V}_{\tilde{\Lambda}}\cong U_{k_0\varepsilon_0}\otimes\mathbf{C}\cong U_{k_0\varepsilon_0}$, we have

$$\rho_{k_0\varepsilon_0}(g)(v\otimes 1)=e^{\sqrt{-1}\frac{\pi}{2}k_0}(v\otimes 1).$$

Hence

$$(k_0,0,0)\in D(K,K_{[\mathfrak{a}]}) \text{ if and only if } k_0\in 4\mathbf{Z}.\tag{8.10}$$

(i) The case $\mathcal{G}(N^4)\cong\frac{SO(2)\times SO(3)}{\mathbf{Z}_2\cdot\mathbf{Z}_4}\to Q_4(\mathbf{C})$ with $p=1$.

In this case $K=SO(2)\times SO(3)$, $K_1=SO(2)\times SO(2)$ and $K_0=\mathbf{Z}_2$, where $\mathbf{Z}_2$ is generated by $\begin{pmatrix}-I_4&0\\0&1\end{pmatrix}\in U=SO(5)$. Let $V_{\tilde{\Lambda}}$ be an irreducible $SO(2)\times SO(3)$-module with the highest weight $\tilde{\Lambda}=k_0\varepsilon_0+k_1\varepsilon_1\in D(K)=D(SO(2)\times SO(3))$, where $k_0,k_1\in\mathbf{Z}$ and $k_1\geq 0$. It follows from the branching law of $(SO(3),SO(2))$ that $V_{\tilde{\Lambda}}$ contains an irreducible $SO(2)\times SO(2)$-module $V_{\tilde{\Lambda}'}$ with the highest weight $\tilde{\Lambda}'=k_0\varepsilon_0+k_1'\varepsilon_1\in D(K_1)=D(SO(2)\times SO(2))$, where $k_1'\in\mathbf{Z}$, if and only if $|k_1'|\leq k_1$. Then we see that $\tilde{\Lambda}'\in D(SO(2)\times SO(2),\mathbf{Z}_2)$ if and only if k_0+k_1' is even. By the formula (8.1) the corresponding eigenvalue of the Casimir operator $-\mathcal{C}_L$ is

$$-c_L=k_0^2+k_1^2+k_1-\frac{1}{2}(k_0^2+{k'}_1^2)=\frac{1}{2}k_0^2+k_1^2+k_1-\frac{1}{2}{k'}_1^2.\tag{8.11}$$

Denote $\tilde{\Lambda}=k_0\varepsilon_0+k_1\varepsilon_1\in D(SO(2)\times SO(3),\mathbf{Z}_2)$ by $\tilde{\Lambda}=(k_0,k_1)$. Using the eigenvalue formula 8.11, we compute that $\tilde{\Lambda}=k_0\varepsilon_0+k_1\varepsilon_1\in D(K,K_0)$ has eigenvalue $-c_L\leq 4$ if and only if (k_0,k_1) is one of

$$\Big\{(\pm 2,0),(\pm 2,1),(\pm 1,1),(0,1),(0,2)\Big\}.$$

Suppose that $\tilde{\Lambda} = (\pm 2, 0)$. Notice that for any $v \otimes w \in \tilde{V}_{k_0\varepsilon_0} \cong \mathbf{C} \otimes \mathbf{C}$,

$$\rho_{k_0\varepsilon_0}(g)(v \otimes w) = e^{\sqrt{-1}k_0\frac{\pi}{2}} v \otimes w,$$

$\tilde{\Lambda} = k_0\varepsilon_0 \in D(K, K_{[\mathfrak{a}]})$ if and only if $k_0 \in 4\mathbf{Z}$. Hence $\tilde{\Lambda} = (\pm 2, 0) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (k_0, 1)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 3$. The complex representation of $K = SO(2) \times SO(3)$ with the highest weight $\tilde{\Lambda}$ corresponds to

$$\tilde{V}_{\tilde{\Lambda}} = U_{k_0\varepsilon_0} \otimes V_{\varepsilon_1} \cong U_{k_0\varepsilon_0} \otimes \mathbf{C}^3 = (U_{k_0\varepsilon_0} \otimes \mathbf{C}^2) \oplus (U_{k_0\varepsilon_0} \otimes \mathbf{C}^1).$$

For each $v \otimes w \in U_{k_0\varepsilon_0} \otimes \mathbf{C}^3$ and $\mathrm{diag}(-I_2, -I_2, 1) \in K_0$, where $w = (w_1, w_2, w_2)^t \in \mathbf{C}^3$, the representation of K_0 is given by

$$\rho_{\tilde{\Lambda}}(\mathrm{diag}(-I_2, -I_2, 1))(v \otimes w) = e^{\sqrt{-1}k_0\pi} v \otimes (-w_1, -w_2, w_3)^t.$$

Then $(V_{\tilde{\Lambda}})_{K_0} = \mathbf{C} \otimes \mathbf{C}(0, 0, w_3)^t \cong \mathbf{C} \otimes \mathbf{C}$ if k_0 is even and $(V_{\tilde{\Lambda}})_{K_0} = \mathbf{C} \otimes \mathbf{C}(w_1, w_2, 0)^t \cong \mathbf{C} \otimes \mathbf{C}^2$ if k_0 is odd. Moreover,

$$\rho_{\tilde{\Lambda}}(g)(v \otimes w) = e^{\sqrt{-1}k_0\frac{\pi}{2}} v \otimes \begin{pmatrix} w_2 \\ w_1 \\ -w_3 \end{pmatrix}.$$

Thus $\tilde{\Lambda} \in D(K, K_{[\mathfrak{a}]})$ if and only if $k_0 \equiv 2 \bmod 4$ and its multiplicity is 1. In particular, $\tilde{\Lambda} = (0, 1)$ or $(\pm 1, 1) \notin D(K, K_{[\mathfrak{a}]})$ and $\tilde{\Lambda} = (\pm 2, 1) \in D(K, K_{[\mathfrak{a}]})$. For $\tilde{\Lambda} = (\pm 2, 1)$, it follows from the branching laws of $(SO(3), SO(2))$ that $|k_1'| \leq k_1$ thus $k_1' = 0$ such that $k_0 + k_1'$ is even. Hence, $-c_L = 4$.

Suppose that $\tilde{\Lambda} = (0, 2)$. Then $\dim_{\mathbf{C}} \tilde{V}_{\tilde{\Lambda}} = 5$. It follows from the branching law of $(SO(3), SO(2))$ that $k_1' = 0$ or ± 2. If $k_1' = \pm 2$, then $-c_L = 4$. If $k_1' = 0$, then $-c_L = 6 > 4$. On the other hand, $\Lambda = 2\varepsilon_1 \in D(SO(3))$ corresponds to $V_\Lambda \cong \mathrm{S}^2_0(\mathbf{C}^3)$ and the representation of $SO(3)$ on $\mathrm{S}^2_0(\mathbf{C}^3)$ is just the complexified isotropy representation of a symmetric pair $(SU(3), SO(3))$. Thus $\mathrm{S}^2_0(\mathbf{C}^3)$ can be decomposed into irreducible $SO(2)$-modules as

$$\begin{aligned} V_{2\varepsilon_1} &\cong \mathrm{S}^2_0(\mathbf{C}^3) \\ &= \mathrm{S}^2_0(\mathbf{C}^2) \oplus \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \end{pmatrix} \mid a, b \in \mathbf{C} \right\} \oplus \mathbf{C} \begin{pmatrix} I_2 & \\ & -2 \end{pmatrix} \\ &= \mathbf{C} \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{pmatrix} \oplus \mathbf{C} \begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & -1 \end{pmatrix} \\ &\quad \oplus \mathbf{C} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \sqrt{-1} \\ 1 & \sqrt{-1} & 0 \end{pmatrix} \oplus \mathbf{C} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\sqrt{-1} \\ 1 & -\sqrt{-1} & 0 \end{pmatrix} \oplus \mathbf{C} \begin{pmatrix} I_2 & \\ & -2 \end{pmatrix} \\ &= V'_{2\varepsilon_1} \oplus V'_{-2\varepsilon_1} \oplus V'_{\varepsilon_1} \oplus V'_{-\varepsilon_1} \oplus V'_0. \end{aligned}$$

Using this expression we can directly show that $(\tilde{V}_{\tilde{\Lambda}})_{K_0} \cong (\mathbf{C} \otimes \mathrm{S}^2_0(\mathbf{C}^2)) \oplus (\mathbf{C} \otimes \mathbf{C} \begin{pmatrix} I_2 & \\ & -2 \end{pmatrix})$ and $(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}} \cong \mathbf{C} \otimes \mathbf{C} \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \oplus (\mathbf{C} \otimes \mathbf{C} \begin{pmatrix} I_2 & \\ & -2 \end{pmatrix})$. Hence $\tilde{\Lambda} = (0, 2) \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 2. Note that the first summand of $(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}}$ lies in $\mathbf{C} \otimes (V'_{2\varepsilon_1} \oplus V'_{-2\varepsilon_1})$, which gives eigenvalue 4 with multiplicity 1 and the second summand of $(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}}$ lies in $\mathbf{C}\otimes V_0'$, which gives eigenvalue $6(>4)$ with multiplicity 1.

Now we obtain that $\mathcal{G}(N^4)\subset Q_4(\mathbf{C})$ is Hamiltonian stable. Moreover since

$$\begin{aligned} n(L^4) &= \dim \tilde{V}_{(2,1)} + \dim \tilde{V}_{(-2,1)} + \dim \tilde{V}_{(0,2)} = 3+3+5 \\ &= 11 = \dim SO(6) - \dim(SO(2)\times SO(3)) = n_{hk}(L^4), \end{aligned}$$

$L^4=\mathcal{G}(N^4)\subset Q_4(\mathbf{C})$ is Hamiltonian rigid. Therefore $\mathcal{G}(N^4)\subset Q_4(\mathbf{C})$ is strictly Hamiltonian stable.

(ii) The case $\mathcal{G}(N^8)\cong \frac{SO(2)\times SO(5)}{(\mathbf{Z}_2\times SO(3))\cdot \mathbf{Z}_4}\rightarrow Q_8(\mathbf{C})$ with $p=2$

Denote $\tilde{\Lambda}=k_0\varepsilon_0+k_1\varepsilon_1+k_2\varepsilon_2\in D(K,K_0)=D(SO(2)\times SO(5),\mathbf{Z}_2\times SO(3))$ by $\tilde{\Lambda}=(k_0,k_1,k_2)$. Let $\tilde{\Lambda}'=k_0\varepsilon_0+k_1'\varepsilon_1\in D(K_1,K_0)=D(SO(2)\times SO(2)\times SO(3),\mathbf{Z}_2\times SO(3))$ as in Lemma 8.3. Then using the eigenvalue formula (8.8) we compute

Lemma 8.5. *$\tilde{\Lambda}=k_0\varepsilon_0+k_1\varepsilon_1+k_2\varepsilon_2\in D(K,K_0)$ has eigenvalue $-c_L\leq 8$ if and only if (k_0,k_1,k_2) is one of*

$$\{\,(\pm 4,0,0),(\pm 1,1,0),(\pm 3,1,0),(0,1,1),(\pm 2,1,1),(0,2,0)\,\}.$$

Suppose that $\tilde{\Lambda}=(\pm 4,0,0)$. Then $\dim\tilde{V}_{\tilde{\Lambda}}=1$. It follows from the branching law of $(SO(5),SO(2)\times SO(3))$ that $k_1'=0$. Thus $-c_L=8$. On the other hand, it follows from (8.10) that $\tilde{\Lambda}=(\pm 4,0,0)\in D(K,K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda}=(k_0,1,0)$. Then $\dim\tilde{V}_{\tilde{\Lambda}}=5$ and $\tilde{V}_{\tilde{\Lambda}}\cong U_{k_0\varepsilon_0}\otimes\mathbf{C}^5$, where $\Lambda=\varepsilon_1\in D(K)$ corresponds to the matrix multiplication of $SO(5)$ on $\mathbf{C}^5$. It follows from the branching law of $(SO(5),SO(2)\times SO(3))$ that $k_1'=\pm 1$. Hence $-c_L=\frac{1}{2}k_0^2+\frac{7}{2}$. Notice that $U_{k_0\varepsilon_0}\otimes\mathbf{C}^5$ can be decomposed into the following $SO(2)\times SO(3)$-modules:

$$U_{k_0\varepsilon_0}\otimes\mathbf{C}^5=(U_{k_0\varepsilon_0}\otimes(\mathbf{C}^2\oplus\{0\}))\oplus(U_{k_0\varepsilon_0}\otimes(\{0\}\oplus\mathbf{C}^3)),$$

where $U_{k_0}\varepsilon_0\otimes(\{0\}\oplus\mathbf{C}^3)$ has no nonzero fixed vector by $\mathbf{Z}_2\times SO(3)$. If k_0 is odd, then

$$\rho_{k_0\varepsilon_0+\varepsilon_1}\begin{pmatrix} -I_2 & & \\ & -I_2 & \\ & & T\end{pmatrix}(v\otimes\begin{pmatrix} w_1\\ w_2\\ 0\\ 0\\ 0\end{pmatrix})=e^{\sqrt{-1}\pi k_0}v\otimes\begin{pmatrix} -w_1\\ -w_2\\ 0\\ 0\\ 0\end{pmatrix}=v\otimes\begin{pmatrix} w_1\\ w_2\\ 0\\ 0\\ 0\end{pmatrix},$$

i.e., $(\tilde{V}_{\tilde{\Lambda}})_{\mathbf{Z}_2\otimes SO(3)}=U_{k_0\varepsilon_0}\otimes(\mathbf{C}^2\oplus\{0\})$ if k_0 is odd. But since

$$\rho_{k_0\varepsilon_0+\varepsilon_1}(g)(v\otimes\begin{pmatrix} w_1\\ w_2\\ 0\\ 0\\ 0\end{pmatrix})=e^{\sqrt{-1}\frac{\pi}{2}k_0}v\otimes\begin{pmatrix} w_2\\ w_1\\ 0\\ 0\\ 0\end{pmatrix},$$

$U_{k_0\varepsilon_0}\otimes(\mathbf{C}^2\oplus\{0\})$ has no nonzero fixed vector by $(\mathbf{Z}_2\times SO(3))\cdot\mathbf{Z}_4$, i.e., neither $(\pm 1,1,0)$ and $(\pm 3,1,0)$ is in $D(K,K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda}=(k_0,1,1)$. Then $\dim\tilde{V}_{\tilde{\Lambda}}=10$ and $\tilde{V}_{\tilde{\Lambda}}\cong\mathbf{C}\otimes\wedge^2\mathbf{C}^5$. It follows from the branching law of $(SO(5),SO(2)\times SO(3))$ that $k_1'=0$. Thus $-c_L=\frac{1}{2}k_0^2+6$. On the other hand, since $e_1 \wedge e_2 \in \wedge^2 \mathbf{C}^5$ is fixed by $SO(2) \times SO(3)$, $v \otimes (e_1 \wedge e_2) \in \mathbf{C} \otimes \wedge^2 \mathbf{C}^5$ is fixed by $\mathbf{Z}_2 \times SO(3) \subset SO(2) \times SO(2) \times SO(3)$. Moreover,

$$\rho_{k_0\varepsilon_0+\varepsilon_1+\varepsilon_2}(g)(v \otimes (e_1 \wedge e_2)) = e^{\sqrt{-1}\frac{\pi}{2}k_0} v \otimes (e_2 \wedge e_1).$$

Hence, $\tilde{\Lambda} = (0,1,1) \notin D(K, K_{[\mathfrak{a}]})$ but $\tilde{\Lambda} = (\pm 2, 1, 1) \in D(K, K_{[\mathfrak{a}]})$ and $(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}} \cong \mathbf{C} \otimes \mathbf{C}\{e_1 \wedge e_2\}$ for $k_0 = 2$ or -2, both of which give eigenvalue 8.

Suppose that $\tilde{\Lambda} = (0,2,0)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 14$ and $\tilde{V}_{\tilde{\Lambda}} \cong \mathbf{C} \otimes \mathrm{S}^2_0(\mathbf{C}^5)$, where the representation of $SO(5)$ with highest weight $2\varepsilon_1$ is just the adjoint representation on $\mathrm{S}^2_0(\mathbf{C}^5)$. It follows from the branching law of $(SO(5), SO(2) \times SO(3))$ that $k_1' = 0, \pm 2$. Thus $-c_L = 10 - \frac{1}{2}{k_1'}^2$. When $k_1' = \pm 2$, $-c_L = 8$, otherwise $-c_L = 10 > 8$. On the other hand, $\mathrm{S}^2_0(\mathbf{C}^5)$ can be decomposed into the following $SO(2) \times SO(3)$-modules:

$$\begin{aligned} V_{2\varepsilon_1} &\cong \mathrm{S}^2_0(\mathbf{C}^5) \\ &= \mathrm{S}^2_0(\mathbf{C}^2) \oplus \mathrm{S}^2_0(\mathbf{C}^3) \oplus M(2,3;\mathbf{C}) \oplus \left\{ \begin{pmatrix} zI_2 & \\ 0 & wI_3 \end{pmatrix} \mid z, w \in \mathbf{C}, 2z + 3w = 0 \right\}. \end{aligned}$$

Thus, $\mathrm{S}^2_0(\mathbf{C}^2)$ is fixed by $\{-I_2\} \times SO(3)$ and

$$(\tilde{V}_{\tilde{\Lambda}})_{K_0} \cong \mathbf{C} \otimes \mathrm{S}^2_0(\mathbf{C}^2) \oplus \mathbf{C} \otimes \mathbf{C} \begin{pmatrix} 3I_2 & \\ & -2I_3 \end{pmatrix}.$$

Moreover,

$$\rho_{2\varepsilon_1}(g)(v \otimes \begin{pmatrix} a & b & \\ b & -a & \\ & & 0 \end{pmatrix}) = v \otimes \begin{pmatrix} -a & b & \\ b & a & \\ & & 0 \end{pmatrix}.$$

Hence, $(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}} = \mathbf{C} \otimes \mathbf{C} \cdot \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \oplus \mathbf{C} \otimes \mathbf{C} \begin{pmatrix} 3I_2 & \\ & -2I_3 \end{pmatrix}$. Therefore, $\tilde{\Lambda} = (0,2,0) \in D(K, K_{[\mathfrak{a}]})$. Notice the first summand lies in $\tilde{V}'_{(0,2,0)} \oplus \tilde{V}'_{(0,-2,0)}$ which gives eigenvalue 8 and the second summand lies in $\tilde{V}'_{(0,0,0)}$ which gives eigenvalue 10. Hence the multiplicity corresponding to eigenvalue 8 is 1.

Since $\tilde{\Lambda} = (4,0,0)$, $(-4,0,0)$, $(2,1,1)$, $(-2,1,1)$, $(0,2,0) \in D(K, K_{[\mathfrak{a}]})$ give the smallest eigenvalue 8 with multiplicity 1 and

$$\begin{aligned} n(L^8) &= \dim \tilde{V}_{(4,0,0)} + \dim \tilde{V}_{(-4,0,0)} + \dim \tilde{V}_{(2,1,1)} + \dim \tilde{V}_{(-2,1,1)} + \dim \tilde{V}_{(0,2,0)} \\ &= 1 + 1 + 10 + 10 + 14 = 36 \\ &> 34 = \dim SO(10) - \dim SO(2) \times SO(5) = n_{hk}(L^8), \end{aligned}$$

$\mathcal{G}(N^8) \subset Q_8(\mathbf{C})$ is not Hamiltonian rigid. Therefore $\mathcal{G}(N^8) \subset Q_8(\mathbf{C})$ is Hamiltonian stable but not strictly Hamiltonian stable.

(iii) The case $\mathcal{G}(N^{4p}) \cong \frac{SO(2) \times SO(2p+1)}{(\mathbf{Z}_2 \times SO(2p-1)) \cdot \mathbf{Z}_4} \to Q_{4p}(\mathbf{C})$ with $p \geq 3$.

Suppose that $\tilde{\Lambda} = (k_0, 0, 0)$ and $k_0 \in 4\mathbf{Z} \setminus \{0\}$. Then $k_1' = 0$ and by (8.9) $\tilde{\Lambda} \in D(K, K_{[\mathfrak{a}]})$. As $p \geq 3$, we have $16 < 24 \leq 8p$. Hence by (8.10) we see that for every $k_0 \in 4\mathbf{Z} \setminus \{0\}$ such that $16 \leq k_0^2 < 8p$ we have eigenvalue $-c_L = \frac{1}{2}k_0^2 < 4p$. Therefore, $\mathcal{G}(N^{4p}) \cong \frac{SO(2) \times SO(2p+1)}{(\mathbf{Z}_2 \times SO(2p-1)) \cdot \mathbf{Z}_4} \to Q_{4p-2}(\mathbf{C})$ is not Hamiltonian stable if $p \geq 3$.

Therefore, we obtain

Theorem 8.2. *The Gauss image* $L^{4p} = \frac{SO(2)\times SO(2p+1)}{(\mathbf{Z}_2\times SO(2p-1))\mathbf{Z}_4} \to Q_{4p}(\mathbf{C})$ $(p \geq 1)$ *is not Hamiltonian stable if and only if* $(m-2)-1 = 2p-2 \geq 3$. *If* $p=1$, *it is strictly Hamiltonian stable and if* $p=2$, *it is Hamiltonian stable but not strictly Hamiltonian stable.*

Remark. The index $i(L^{4p})$ goes to ∞ as $p \to \infty$.

9. The case $(U,K) = (SU(m+2), S(U(2)\times U(m)))$ $(m \geq 2)$

In this case, $U = SU(m+2)$ and $K = S(U(2)\times U(m))$ with $m \geq 2$. Then (U,K) is of B_2 type for $m=2$ and BC_2 type for $m \geq 3$.

In this case we use the formulation by the unitary group $U(m)$ rather than one by the special unitary groups $SU(m)$. It seems to work more successfully in our argument of applying the branching laws. Here we will also indicate the relations between both formulations. Let $\tilde{U} := U(m+2)$, $\tilde{K} := U(2)\times U(m)$, $\tilde{K}_2 := U(2)\times U(2)\times U(m-2)$, $\tilde{K}_1 := U(1)\times U(1)\times U(1)\times U(1)\times U(m-2)$ and $\tilde{K}_0 := U(1)\times U(1)\times U(m-2)$. Then $\tilde{U} = C(\tilde{U})\cdot U$, $\tilde{K} = C(\tilde{U})\cdot K$, $\tilde{K}_2 = C(\tilde{U})\cdot K_2$, $\tilde{K}_1 = C(\tilde{U})\cdot K_1$, and $\tilde{K}_0 = C(\tilde{U})\cdot K_0$, where $C(\tilde{U})$ is the center of $\tilde{U}$.

Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ and $\tilde{\mathfrak{u}} = \tilde{\mathfrak{k}} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{u}$ and $\tilde{\mathfrak{u}}$ corresponding to (U,K) and $(\tilde{U},\tilde{K})$, respectively. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, where

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & H_{12} \\ -\bar{H}_{12}^t & 0 \end{pmatrix} \mid H_{12} = \begin{pmatrix} \xi_1 & 0 & 0 & \cdots & 0 \\ 0 & \xi_2 & 0 & \cdots & 0 \end{pmatrix}, \xi_1, \xi_2 \in \mathbf{R} \right\}.$$

Then the centralizer $\tilde{K}_0$ of $\mathfrak{a}$ in $\tilde{K}$ is given as follows:

$$\begin{aligned} \tilde{K}_0 = &\left\{ P = \begin{pmatrix} e^{is} &&&& \\ & e^{it} &&& \\ && e^{is} && \\ &&& e^{it} & \\ &&&& T \end{pmatrix} \mid T \in U(m-2) \right\} \\ \cong &\, U(1)\times U(1)\times U(m-2). \end{aligned}$$

Moreover,

$$\tilde{K}_{[\mathfrak{a}]} = \tilde{K}_0 \cup (Q\cdot \tilde{K}_0) \cup (Q^2\cdot \tilde{K}_0) \cup (Q^3 \cdot \tilde{K}_0),$$

where

$$Q = \begin{pmatrix} 0 & 1 &&&& \\ -1 & 0 &&&& \\ && 0 & -1 && \\ && -1 & 0 && \\ &&&& I_{m-2} \end{pmatrix} \in \tilde{K}_2 \subset \tilde{K}.$$

Thus the deck transformation group of the covering map $\mathcal{G} : N^{8m-2} \to \mathcal{G}(N^{4m-2})$ $(m \geq 2)$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \tilde{K}_{[\mathfrak{a}]}/\tilde{K}_0 \cong \mathbf{Z}_4$. Remark that we will use P and Q to denote the element in $\tilde{K}_0$ and the generator of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ throughout this section.

9.1. Description of the Casimir operator.

Define an inner product $\langle X, Y\rangle_{\mathfrak{u}} := -\mathrm{tr} XY$ for each $X, Y \in \mathfrak{u} = \mathfrak{su}(m+2)$ or for each $X, Y \in \tilde{\mathfrak{u}} = \mathfrak{u}(m+2)$. The restricted root system $\Sigma(U,K)$ is of type B_2 for $m=2$ and type

BC_2 for $m \geq 3$. Then the square length of each restricted roots with respect to $\langle\ ,\ \rangle_{\mathfrak{u}}$, is given by

$$\|\gamma\|_{\mathfrak{u}}^2 = \begin{cases} 1 \text{ or } 2, & m = 2; \\ \frac{1}{2}, 1 \text{ or } 2, & m \geq 3. \end{cases}$$

Hence the Casimir operator $\mathcal{C}_L$ of L with respect to the induced metric from $g^{\mathrm{std}}_{Q_{4m-2}(\mathbf{C})}$ can be expressed as follows:

$$\mathcal{C}_L = \begin{cases} \mathcal{C}_{K/K_0} - \frac{1}{2}\,\mathcal{C}_{K_1/K_0}, & m = 2; \\ 2\mathcal{C}_{K/K_0} - \mathcal{C}_{K_2/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0}, & m \geq 3, \end{cases}$$

where $\mathcal{C}_{K/K_0}$, $\mathcal{C}_{K_2/K_0}$ and $\mathcal{C}_{K_1/K_0}$ denote the Casimir operator of K/K_0, K_2/K_0 and K_1/K_0 relative to $\langle\ ,\ \rangle_{\mathfrak{u}}|_{\mathfrak{k}}$, $\langle\ ,\ \rangle_{\mathfrak{u}}|_{\mathfrak{k}_2}$ and $\langle\ ,\ \rangle_{\mathfrak{u}}|_{\mathfrak{k}_1}$, respectively.

9.2. **Descriptions of $D(\tilde{U})$, $D(U)$ and etc.**

$D(\tilde{U})$, $D(C(\tilde{U}))$ and $D(U)$ are described as follows:

$$\begin{aligned} D(\tilde{U}) = D(U(m+2)) = \Big\{\tilde{\Lambda} = \tilde{p}_1 y_1 + \cdots + \tilde{p}_{m+2} y_{m+2} \mid \tilde{p}_1, \cdots, \tilde{p}_{m+2} \in \mathbf{Z}, \\ \tilde{p}_i - \tilde{p}_{i+1} \geq 0 \ (i = 1, \cdots, m+1)\Big\}, \end{aligned}$$

$$D(C(\tilde{U})) = D(C(U(m+2))) = \Big\{\Lambda = p_0(y_1 + \cdots + y_{m+2}) \mid p_0 \in \frac{1}{m+2}\mathbf{Z}\Big\},$$

$$\begin{aligned} D(U) = D(SU(m+2)) = \Big\{\Lambda = p_1 y_1 + \cdots + p_{m+2} y_{m+2} \mid \sum_{i=1}^{m+2} p_i = 0, \\ p_i - p_{m+2} \in \mathbf{Z}, p_i - p_{i+1} \geq 0 \ (i = 1, \cdots, m+1)\Big\}. \end{aligned}$$

Each $\tilde{\Lambda} = \tilde{p}_1 y_1 + \cdots + \tilde{p}_{m+2} y_{m+2} \in D(U(m+2))$ can be decomposed as $\tilde{\Lambda} = \Lambda^0 + \Lambda$, where

$$\Lambda^0 = \left(\frac{1}{m+2}\sum_{i=1}^{m+2} \tilde{p}_i\right)\left(\sum_{i=1}^{m+2} y_i\right) \in D(C(U(m+2)))$$

and

$$\Lambda = (\tilde{p}_1 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{p}_i) y_1 + \cdots + (\tilde{p}_{m+2} - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{p}_i) y_{m+2} \in D(SU(m+2)).$$

Note that this projection $D(\tilde{U}) \to D(U), \tilde{\Lambda} \mapsto \Lambda$ is surjective.

$$\begin{aligned} & D(\tilde{K}) = D(U(2) \times U(m)) \\ =& \{\tilde{\Lambda} = \tilde{q}_1 y_1 + \tilde{q}_2 y_2 + \tilde{q}_3 y_3 + \cdots + \tilde{q}_{m+2} y_{m+2} \mid \\ & \tilde{q}_i \in \mathbf{Z} \ (i = 1, \cdots, m+2), \tilde{q}_1 - \tilde{q}_2 \geq 0, \tilde{q}_i - \tilde{q}_{i+1} \geq 0 \ (i = 3, \cdots, m+1)\}, \\ & D(K) = D(S(U(2) \times U(m))) \\ =& \{\Lambda = q_1 y_1 + q_2 y_2 + q_3 y_3 + \cdots + q_{m+2} y_{m+2} \mid \sum_{i=1}^{m+2} q_i = 0, q_i - q_j \in \mathbf{Z} \\ & (i, j = 1, 2, \cdots, m+2), q_1 - q_2 \geq 0, q_i - q_{i+1} \geq 0 \ (i = 3, 4, \cdots, m+1)\}, \end{aligned}$$

$$
\begin{aligned}
&D(\tilde{K}_2) = D(U(2) \times U(2) \times U(m-2))\\
=&\{\tilde{\Lambda} = \tilde{q}_1 y_1 + \tilde{q}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4 + \tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2} \mid\\
&\qquad \tilde{q}_i \in \mathbf{Z}\ (i = 1, \cdots, m+2),\\
&\qquad \tilde{q}_1 - \tilde{q}_2, \tilde{q}_3 - \tilde{q}_4, \tilde{q}_i - \tilde{q}_{i+1} \geq 0\ (i = 5, \cdots, m+1)\},\\
&D(K_2) = D(S(U(2) \times U(2) \times U(m-2)))\\
=&\{\Lambda = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 + q_5 y_5 + \cdots + q_{m+2} y_{m+2} \mid \sum_{i=1}^{m+2} q_i = 0,\\
&q_i - q_j \in \mathbf{Z}\ (i, j = 1, 2, \cdots, m+2), q_1 - q_2, q_3 - q_4, q_i - q_{i+1} \geq 0\ (i = 5, \cdots, m+1)\},\\
&D(\tilde{K}_1) = D(U(1) \times U(1) \times U(1) \times U(1) \times U(m-2))\\
=&\{\tilde{\Lambda} = \tilde{q}_1 y_1 + \tilde{q}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4 + \tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2} \mid\\
&\qquad \tilde{q}_i \in \mathbf{Z}\ (i = 1, \cdots, m+2), \tilde{q}_i - \tilde{q}_{i+1} \geq 0\ (i = 5, \cdots, m+1)\ \},\\
&D(K_1) = D(S(U(1) \times U(1) \times U(1) \times U(1) \times U(m-2)))\\
=&\Big\{\Lambda = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 + q_5 y_5 + \cdots + q_{m+2} y_{m+2} \mid \sum_{i=1}^{m+2} q_i = 0,\\
&\qquad q_i - q_j \in \mathbf{Z}\ (i, j = 1, \cdots, m+2), q_i - q_{i+1} \geq 0\ (i = 5, \cdots, m+1)\Big\},\\
&D(\tilde{K}_0) = D(U(1) \times U(1) \times U(m-2))\\
=&\{\tilde{\Lambda} = \tilde{q}_1 y_1 + \tilde{q}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4 + \tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2} \mid\\
&\qquad \tilde{q}_3 = \tilde{q}_1 \in \frac{1}{2}\mathbf{Z}, \tilde{q}_4 = \tilde{q}_2 \in \frac{1}{2}\mathbf{Z}, \tilde{q}_i \in \mathbf{Z}\ (i = 5, \cdots, m+2),\\
&\qquad \tilde{q}_i - \tilde{q}_{i+1} \geq 0\ (i = 5, 6, \cdots, m+1)\},\\
&D(K_0) = D(S(U(1) \times U(1) \times U(m-2)))\\
=&\{\Lambda = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 + q_5 y_5 + \cdots + q_{m+2} y_{m+2} \mid\\
&\qquad \sum_{i=1}^{m+2} q_i = 0, q_i - q_j \in \mathbf{Z}\ (i, j = 1, \cdots, m+2),\\
&\qquad q_3 = q_1, q_4 = q_2, q_i - q_{i+1} \geq 0\ (i = 5, \cdots, m+1)\}.
\end{aligned}
$$

The natural maps $D(\tilde{K}) \longrightarrow D(K)$, $D(\tilde{K}_2) \longrightarrow D(K_2)$, $D(\tilde{K}_1) \longrightarrow D(K_1)$ and $D(\tilde{K}_0) \longrightarrow D(K_0)$ are also surjective.

9.3. **Branching laws of** $(U(m), U(2) \times U(m-2))$**.** The branching laws for $(SU(m), S(U(m) \times U(2)))$ given in [29] can be reformulated to the branching laws for $(U(m), U(2) \times U(m-2))$ as follows:

Lemma 9.1 (Branching law of $(U(m), U(2) \times U(m-2))$)**.** *For each $\tilde{\Lambda} = \tilde{p}_1 y_1 + \cdots + \tilde{p}_m y_m \in D(U(m))$, an irreducible $U(m)$-module $V_{\tilde{\Lambda}}$ with the highest weight $\tilde{\Lambda}$ can be decomposed into the direct sum of irreducible $U(2) \times U(m-2)$-modules as follows:*

$$V_{\tilde{\Lambda}} = \bigoplus_{\tilde{\Lambda}' \in D(U(2) \times U(m-2))} V'_{\tilde{\Lambda}'}.$$

Here $V_{\tilde{\Lambda}}$ contains an irreducible $U(2) \times U(m-2)$-module $V'_{\tilde{\Lambda}'}$ with the highest weight $\tilde{\Lambda}' = \tilde{q}_1 y_1 + \cdots + \tilde{q}_m y_m \in D(U(2) \times U(m-2))$ if and only if the following conditions are satisfied:

(i) $\tilde{q}_1 - \tilde{p}_1 \in \mathbf{Z}$;

(ii) $\tilde{p}_{i-2} \geq \tilde{q}_i \geq \tilde{p}_i \ (i = 3, \cdots, m)$;

(iii) *In the finite power series expansion in X of $\dfrac{\prod_{i=2}^{m}(X^{r_i+1} - X^{-(r_i+1)})}{(X - X^{-1})^{m-2}}$, where*

$$\begin{aligned} r_2 &:= \tilde{p}_1 - \max(\tilde{q}_3, \tilde{p}_2) \\ r_i &:= \min(\tilde{q}_i, \tilde{p}_{i-1}) - \max(\tilde{q}_{i+1}, \tilde{p}_i), \quad (3 \leq i \leq m-1) \\ r_m &:= \min(\tilde{q}_m, \tilde{p}_{m-1}) - \tilde{p}_m, \end{aligned}$$

the coefficient of $X^{\tilde{q}_1 - \tilde{q}_2 + 1}$ does not vanish. Moreover, the value of this coefficient is equal to the multiplicity of the irreducible $U(2) \times U(m-2)$-module $V'_{\tilde{\Lambda}'}$.

9.4. **Branching law of $(U(3), U(2) \times U(1))$.** Now following Lemma 7.1 the branching law of $(U(3), U(2) \times U(1))$ is described as

Lemma 9.2. *Let $\tilde{V}_{\tilde{\Lambda}}$ be an irreducible $U(3)$-module with the highest weight $\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 \in D(U(3))$, where $\tilde{p}_i \in \mathbf{Z}$ $(i = 1, 2, 3)$ and $\tilde{p}_1 \geq \tilde{p}_2 \geq \tilde{p}_3$. Then $\tilde{V}_{\tilde{\Lambda}}$ can be decomposed into irreducible $U(2) \times U(1)$-modules as*

$$\tilde{V}_{\tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3} = \bigoplus_{\alpha=0}^{\tilde{p}_1 - \tilde{p}_2} \bigoplus_{\beta=0}^{\tilde{p}_2 - \tilde{p}_3} \tilde{V}'_{(\tilde{p}_1 - \alpha) y_1 + (\tilde{p}_2 - \beta) y_2 + (\tilde{p}_3 + \alpha + \beta) y_3}.$$

9.5. **Descriptions of $D(\tilde{K}, \tilde{K}_0)$, $D(\tilde{K}_2, \tilde{K}_0)$, $D(\tilde{K}_1, \tilde{K}_0)$.** Let

$$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \cdots + \tilde{p}_{m+2} y_{m+2} \in D(\tilde{K}) = D(U(2) \times U(m)),$$

where $\tilde{p}_1, \cdots, \tilde{p}_{m+2} \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{p}_3 \geq \cdots \geq \tilde{p}_{m+2}$. Thus $\Lambda_\sigma = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 \in D(U(2))$, $\Lambda_\tau = \tilde{p}_3 y_3 + \cdots + \tilde{p}_{m+2} y_{m+2} \in D(U(m))$ and

$$\tilde{\rho}_{\tilde{\Lambda}} = \sigma \boxtimes \tau \in \mathcal{D}(\tilde{K}) = \mathcal{D}(U(2) \times U(m)),$$

where $\sigma \in \mathcal{D}(U(2))$, $\tau \in \mathcal{D}(U(m))$.

By Lemma 9.1, an irreducible $U(m)$-module V_τ with the highest weight Λ_τ can be decomposed into the direct sum of irreducible $U(2) \times U(m-2)$-modules as

$$V_\tau = \bigoplus V'_{\tilde{\Lambda}'_\tau},$$

where $\tilde{\Lambda}'_\tau = \sum_{i=3}^{m+2} \tilde{q}_i y_i \in D(U(2) \times U(m-2))$ with $\tilde{q}_3, \cdots, \tilde{q}_{m+2} \in \mathbf{Z}$, $\tilde{q}_i - \tilde{q}_{i+1} \geq 0$ $(i = 3, 5, \cdots, m+1)$. Note that setting $\Lambda_\varsigma := \tilde{q}_3 y_3 + \tilde{q}_4 y_4 \in D(U(2))$ and $\Lambda_\gamma := \tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2} \in D(U(m-2))$, we get a decomposition into the direct sum of irreducible $\tilde{K}_2$-modules as

$$V_{\tilde{\Lambda}} = \bigoplus_{\varsigma, \gamma} (V_\sigma \boxtimes V_\varsigma \boxtimes V_\gamma).$$

By the branching law of $(U(2), U(1) \times U(1))$ (see Lemma 7.1),

$$V_\sigma = V_{\tilde{p}_1 y_1 + \tilde{p}_2 y_2} = \bigoplus_{\alpha=0}^{\tilde{p}_1 - \tilde{p}_2} V'_{(\tilde{p}_1 - \alpha) y_1 + (\tilde{p}_2 + \alpha) y_2},$$

$$V_\varsigma = V_{\tilde{q}_3 y_3 + \tilde{q}_4 y_4} = \bigoplus_{\beta=0}^{\tilde{q}_3 - \tilde{q}_4} V'_{(\tilde{q}_3 - \beta) y_3 + (\tilde{q}_4 + \beta) y_4}.$$

Thus we have a decomposition into the direct sum of irreducible $\tilde{K}_1$-modules:

$$\begin{aligned} & V_{\tilde{\Lambda}} \\ = & \bigoplus \bigoplus_{\alpha=0}^{\tilde{p}_1 - \tilde{p}_2} \bigoplus_{\beta=0}^{\tilde{q}_3 - \tilde{q}_4} (V'_{(\tilde{p}_1 - \alpha) y_1 + (\tilde{p}_2 + \alpha) y_2} \boxtimes V'_{(\tilde{q}_3 - \beta) y_3 + (\tilde{q}_4 + \beta) y_4} \boxtimes V_{\tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2}}). \end{aligned}$$

Since as a $U(1) \times U(1)$-module

$$V'_{(\tilde{p}_1 - \alpha) y_1 + (\tilde{p}_2 + \alpha) y_2} \boxtimes V'_{(\tilde{q}_3 - \beta) y_3 + (\tilde{q}_4 + \beta) y_4} = V''_{\frac{1}{2}(\tilde{p}_1 + \tilde{q}_3 - \alpha - \beta)(y_1 + y_3) + \frac{1}{2}(\tilde{p}_2 + \tilde{q}_4 + \alpha + \beta)(y_2 + y_4)},$$

we have a decomposition into the direct sum of irreducible $\tilde{K}_0$ -modules:

$$\begin{aligned} & V_{\tilde{\Lambda}} \\ = & \bigoplus_{\varsigma, \gamma} (V_{\tilde{p}_1 y_1 + \tilde{p}_2 y_2} \boxtimes V_{\tilde{q}_3 y_3 + \tilde{q}_4 y_4} \boxtimes V_{\tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2}}) \\ = & \bigoplus \bigoplus_{\alpha=0}^{\tilde{p}_1 - \tilde{p}_2} \bigoplus_{\beta=0}^{\tilde{q}_3 - \tilde{q}_4} (V'_{(\tilde{p}_1 - \alpha) y_1 + (\tilde{p}_2 + \alpha) y_2} \boxtimes V'_{(\tilde{q}_3 - \beta) y_3 + (\tilde{q}_4 + \beta) y_4} \boxtimes V_{\tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2}}) \\ = & \bigoplus \bigoplus_{\alpha=0}^{\tilde{p}_1 - \tilde{p}_2} \bigoplus_{\beta=0}^{\tilde{q}_3 - \tilde{q}_4} V''_{\frac{1}{2}(\tilde{p}_1 + \tilde{q}_3 - \alpha - \beta)(y_1 + y_3) + \frac{1}{2}(\tilde{p}_2 + \tilde{q}_4 + \alpha + \beta)(y_2 + y_4)} \boxtimes V_{\tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2}} . \end{aligned}$$

Thus we obtain that $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ if and only if there exist $\alpha, \beta \in \mathbf{Z}$ with $0 \leq \alpha \leq \tilde{p}_1 - \tilde{p}_2$ and $0 \leq \beta \leq \tilde{q}_3 - \tilde{q}_4$ such that

$$V''_{\frac{1}{2}(\tilde{p}_1 + \tilde{q}_3 - \alpha - \beta)(y_1 + y_3) + \frac{1}{2}(\tilde{p}_2 + \tilde{q}_4 + \alpha + \beta)(y_2 + y_4)} \boxtimes V_{\tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2}}$$

is a trivial $\tilde{K}_0$ -module, that is,

$$\begin{cases} \tilde{p}_1 + \tilde{q}_3 - \alpha - \beta = 0, \\ \tilde{p}_2 + \tilde{q}_4 + \alpha + \beta = 0, \\ \tilde{q}_5 = \cdots = \tilde{q}_{m+2} = 0. \end{cases}$$

Hence $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ must satisfy

$$\begin{aligned} & \tilde{p}_5 = \tilde{p}_6 = \cdots = \tilde{p}_m = 0, \\ & \tilde{p}_3 \geq \tilde{p}_4 \geq 0, \ \tilde{p}_{m+2} \leq \tilde{p}_{m+1} \leq 0, \\ & \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_{m+1} + \tilde{p}_{m+2} = 0. \end{aligned}$$

If $m \geq 4$, then each $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ is expressed as

$$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_{m+1} y_{m+1} + \tilde{p}_{m+2} y_{m+2},$$

where $\tilde{p}_i \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{p}_3 \geq \tilde{p}_4 \geq 0 \geq \tilde{p}_{m+1} \geq \tilde{p}_{m+2}$,

$$\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_{m+1} + \tilde{p}_{m+2} = 0.$$

If $m = 3$, then each $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ is expressed as

$$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_5 y_5,$$

where $\tilde{p}_i \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{p}_3 \geq \tilde{p}_4 \geq \tilde{p}_5$, $\tilde{p}_3 \geq 0$, $\tilde{p}_5 \leq 0$,

$$\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_5 = 0.$$

If $m = 2$, then each $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ is expressed as

$$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4,$$

where $\tilde{p}_i \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{p}_3 \geq \tilde{p}_4$, $\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 = 0$.

Correspondingly, each $\tilde{\Lambda}' \in D(\tilde{K}_2, \tilde{K}_0)$ is expressed as $\tilde{\Lambda}' = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4$, where $\tilde{p}_1, \tilde{p}_2, \tilde{q}_3, \tilde{q}_4 \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{q}_3 \geq \tilde{q}_4$, $\tilde{p}_1 + \tilde{p}_2 + \tilde{q}_3 + \tilde{q}_4 = 0$, in other words, $\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_{m+1} + \tilde{p}_{m+2} = 0$ if $m \geq 4$, $\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_5 = 0$ if $m = 3$. Each $\tilde{\Lambda}'' \in D(\tilde{K}_1, \tilde{K}_0)$ is expressed as $\tilde{\Lambda}'' = \tilde{q}'_1 y_1 + \tilde{q}'_2 y_2 + \tilde{q}'_3 y_3 + \tilde{q}'_4 y_4$, where $\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4 \in \mathbf{Z}$, $\tilde{q}'_1 + \tilde{q}'_3 = 0$, $\tilde{q}'_2 + \tilde{q}'_4 = 0$, $\tilde{q}'_1 = -\alpha + \tilde{p}_1$, $\tilde{q}'_2 = \alpha + \tilde{p}_2$ for some $\alpha = 0, \cdots, \tilde{p}_1 - \tilde{p}_2$, and $\tilde{q}'_3 = -\beta + \tilde{q}_3$, $\tilde{q}'_4 = \beta + \tilde{q}_4$ for some $\beta = 0, \cdots, \tilde{q}_3 - \tilde{q}_4$.

Moreover the coefficient of $X^{\tilde{q}_3 - \tilde{q}_4 + 1}$ in

$$\begin{aligned}
&\frac{1}{X - X^{-1}}(X^{\tilde{p}_3 - \tilde{p}_4 + 1} - X^{-(\tilde{p}_3 - \tilde{p}_4 + 1)})(X^{\tilde{p}_{m+1} - \tilde{p}_{m+2} + 1} - X^{-(\tilde{p}_{m+1} - \tilde{p}_{m+2} + 1)}) \\
&= \sum_{i=0}^{\tilde{p}_3 - \tilde{p}_4} (X^{(\tilde{p}_3 - \tilde{p}_4) + (\tilde{p}_{m+1} - \tilde{p}_{m+2}) - 2i + 1} - X^{(\tilde{p}_3 - \tilde{p}_4) - (\tilde{p}_{m+1} - \tilde{p}_{m+2}) - 2i - 1})
\end{aligned}$$

is equal to the multiplicity of the $\tilde{K}_2$-module with the highest weight $\tilde{\Lambda}' = \Lambda_\sigma + \tilde{\Lambda}'_\tau = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4 \in D(\tilde{K}_2, \tilde{K}_0)$.

9.6. **Eigenvalue computation when** $m = 2$**.** For each $\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 \in D(\tilde{K}, \tilde{K}_0)$ and $\tilde{\Lambda}'' = \tilde{q}'_1 y_1 + \tilde{q}'_2 y_2 + \tilde{q}'_3 y_3 + \tilde{q}'_4 y_4 \in D(\tilde{K}_1, \tilde{K}_0)$ defined as above, the corresponding eigenvalue of $-\mathcal{C}_L$ is

$$\begin{aligned}
-c_L &= -c_{\tilde{\Lambda}} + \frac{1}{2} c_{\tilde{\Lambda}''} \\
&= \tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 + \tilde{p}_4^2 + (\tilde{p}_1 - \tilde{p}_2) + (\tilde{p}_3 - \tilde{p}_4) \qquad (9.1)\\
&\quad - \frac{1}{2}((\tilde{q}'_1)^2 + (\tilde{q}'_2)^2 + (\tilde{q}'_3)^2 + (\tilde{q}'_4)^2).
\end{aligned}$$

Since

$$-\mathcal{C}_L = -\frac{1}{2}\mathcal{C}_{K/K_0} - \mathcal{C}_{K/K_1} \geq -\frac{1}{2}\mathcal{C}_{K/K_0},$$

the first eigenvalue of $-\mathcal{C}_L$, $-c_L \leq n = 6$ implies $-c_{\tilde{\Lambda}} \leq 12$. By estimating the eigenvalue formula (9.1) from above by 6, we compute

Lemma 9.3. *$\tilde\Lambda = \tilde p_1 y_1 + \tilde p_2 y_2 + \tilde p_3 y_3 + \tilde p_4 y_4 \in D(\tilde K, \tilde K_0)$ has eigenvalue $-c_L \le 6$ if and only if $(\tilde p_1, \tilde p_2, \tilde p_3, \tilde p_4)$ is one of*

$$\Big\{(0,0,0,0),(1,1,-1,-1),(1,0,0,-1),(1,-1,0,0),(1,-1,1,-1),\\ (1,1,0,-2),(2,0,-1,-1),(0,-1,1,0),(0,0,1,-1),\\ (0,-2,1,1),(-1,-1,2,0),(-1,-1,1,1)\Big\}.$$

Denote $\tilde\Lambda = \tilde p_1 y_1 + \tilde p_2 y_2 + \tilde p_3 y_3 + \tilde p_4 y_4 \in D(\tilde K, \tilde K_0)$ by $\tilde\Lambda = (\tilde p_1, \tilde p_2, \tilde p_3, \tilde p_4)$.

Suppose that $\tilde\Lambda = (1,1,-1,-1)$. Then $\dim_{\mathbf{C}} V_{\tilde\Lambda} = 1$. By the branching law of $(U(2), U(1)\times U(1))$, $(\tilde q_1', \tilde q_2', \tilde q_3', \tilde q_4') = (1,1,-1,-1) \in D(\tilde K_1, \tilde K_0)$. Then $-c_{\tilde\Lambda} = 4$, $-c_{\tilde\Lambda''} = 4$, $-c_L = -c_{\tilde\Lambda} + \frac{1}{2}c_{\tilde\Lambda''} = 2 < 6$. On the other hand, $V_{\tilde\Lambda} = \mathbf{C} \boxtimes \mathbf{C}$, which is fixed by the $\rho_{\tilde\Lambda}|_{\tilde K_0}$-action. But for a generator Q of $\mathbf{Z}_4$ in $\tilde K_{[\mathfrak{a}]}$, $\rho_{\tilde\Lambda}(Q) = -\mathrm{Id}$ on $V_{\tilde\Lambda}$. Hence, $\tilde\Lambda = (1,1,-1,-1) \notin D(\tilde K, \tilde K_{[\mathfrak{a}]})$. Similarly, $\tilde\Lambda = (-1,-1,1,1) \notin D(\tilde K, \tilde K_{[\mathfrak{a}]})$.

Suppose that $\tilde\Lambda = (1,0,0,-1)$. Then $\dim_{\mathbf{C}} V_{\tilde\Lambda} = 4$. It follows from the branching law of $(U(2), U(1)\times U(1))$ that $(\tilde q_1', \tilde q_2') = (1,0) \oplus (0,1)$ and $(\tilde q_3', \tilde q_4') = (0,-1)$ or $(-1,0)$. Then $(\tilde q_1', \tilde q_2', \tilde q_3', \tilde q_4') = (1,0,-1,0)$ or $(0,1,0,-1) \in D(\tilde K_1, \tilde K_0)$. Hence, $-c_{\tilde\Lambda} = 4$, $-c_{\tilde\Lambda''} = 2$, $-c_L = -c_{\tilde\Lambda} + \frac{1}{2}c_{\tilde\Lambda''} = 3 < 6$.

Let $\mathcal{D}(SU(2)) = \{(V_\ell, \rho_\ell) \mid \ell \in \mathbf{Z}, \ell > 0\}$ be a complete set of inequivalent irreducible unitary representations of $SU(2)$ described in Section 5. Let $\{v_0^{(\ell)}, v_1^{(\ell)}, \cdots, v_\ell^{(\ell)}\}$ be a unitary basis of V_ℓ defined by (5.1). Then

$$V_{\tilde\Lambda} = (W'_{\frac{1}{2}(y_1+y_2)} \otimes V_1) \boxtimes (W'_{-\frac{1}{2}(y_1+y_2)} \otimes V_1).$$

The representation of $\tilde K_0$ on $v_i^{(1)} \otimes v_j^{(1)} \in V_{\tilde\Lambda}$ $(i,j = 0,1)$ is given by

$$\begin{aligned} &\rho_{\tilde\Lambda}(P)(v_i^{(1)} \otimes v_j^{(1)}) \\ =&\left[\rho_1\begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}}\end{pmatrix}\right](v_i^{(1)}) \otimes \left[\rho_1\begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}}\end{pmatrix}\right](v_j^{(1)}) \\ =&e^{\sqrt{-1}(s-t)[1-(i+j)]} v_i^{(1)} \otimes v_j^{(1)}. \end{aligned}$$

Then $(V_{\tilde\Lambda})_{\tilde K_0} = \mathrm{span}_{\mathbf{C}}\{v_1^{(1)} \otimes v_0^{(1)}, v_0^{(1)} \otimes v_1^{(1)}\}$. But for $\mathrm{diag}(1,1,-1,-1) \in \tilde K_{[\mathfrak{a}]}$ and $i,j = 0,1$, $\rho_{\tilde\Lambda}(\mathrm{diag}(1,1,-1,-1))(v_i^{(1)} \otimes v_j^{(1)}) = -v_i^{(1)} \otimes v_j^{(1)}$. So $(V_{\tilde\Lambda})_{\tilde K_{[\mathfrak{a}]}} = \{0\}$ and $\tilde\Lambda = (1,0,0,-1) \notin D(\tilde K, \tilde K_{[\mathfrak{a}]})$. Similarly, $\tilde\Lambda = (0,-1,1,0) \notin D(\tilde K, \tilde K_{[\mathfrak{a}]})$.

Suppose that $\tilde\Lambda = (1,-1,0,0)$. Then $\dim_{\mathbf{C}} V_{\tilde\Lambda} = 3$. It follows from the branching law of $(U(2), U(1)\times U(1))$ that $(\tilde q_1', \tilde q_2') = (1,-1), (0,0)$ or $(-1,1)$ and $(\tilde q_3', \tilde q_4') = (0,0)$. Then $(\tilde q_1', \tilde q_2', \tilde q_3', \tilde q_4') = (0,0,0,0) \in D(\tilde K, \tilde K_0)$. Hence, $-c_{\tilde\Lambda} = 4$, $-c_{\tilde\Lambda''} = 0$, $-c_L = -c_{\tilde\Lambda} + \frac{1}{2}c_{\tilde\Lambda''} = 4 < 6$. On the other hand, $V_{\tilde\Lambda} \cong V_2 \boxtimes \mathbf{C}$. The representation of $\tilde K_0$ on $v_i^{(2)} \otimes w \in V_{\tilde\Lambda}$ is given by

$$\rho_{\tilde\Lambda}(P)(v_i^{(2)} \otimes w) = e^{\sqrt{-1}(s-t)(1-i)} v_i^{(2)} \otimes w.$$

Then $(V_{\tilde\Lambda})_{\tilde K_0} = \mathrm{span}_{\mathbf{C}}\{v_1^{(2)} \otimes w\}$. But for the generator $Q \in \tilde K_{[\mathfrak{a}]}$,

$$\rho_{\tilde\Lambda}(Q)(v_1^{(2)} \otimes w) = -v_1^{(2)} \otimes w.$$

So $(V_{\tilde\Lambda})_{\tilde K_{[\mathfrak{a}]}} = \{0\}$ and $\tilde\Lambda = (1,-1,0,0) \notin D(\tilde K, \tilde K_{[\mathfrak{a}]})$. Similarly, $\tilde\Lambda = (0,0,1,-1) \notin D(\tilde K, \tilde K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (1,-1,1,-1)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 9$. It follows from the branching laws of $(U(2), U(1)\times U(1))$ that $(\tilde{q}'_1,\tilde{q}'_2) = (1,-1)$ or $(0,0)$ and $(\tilde{q}'_3,\tilde{q}'_4) = (1,-1)$ or $(0,0)$. Then $(\tilde{q}'_1,\tilde{q}'_2,\tilde{q}'_3,\tilde{q}'_4) = (1,-1,-1,1)$, $(-1,1,1,-1)$ or $(0,0,0,0) \in D(\tilde{K},\tilde{K}_0)$. When $(\tilde{q}'_1,\tilde{q}'_2,\tilde{q}'_3,\tilde{q}'_4) = (0,0,0,0)$, $-c_{\tilde{\Lambda}} = 8$, $-c_{\tilde{\Lambda}''} = 0$, $-c_L = -c_{\tilde{\Lambda}} + \frac{1}{2}c_{\tilde{\Lambda}''} = 8 > 6$. When $(\tilde{q}'_1,\tilde{q}'_2,\tilde{q}'_3,\tilde{q}'_4) = (1,-1,-1,1)$ or $(-1,1,1,-1)$, $-c_{\tilde{\Lambda}} = 8$, $-c_{\tilde{\Lambda}''} = 4$, $-c_L = -c_{\tilde{\Lambda}} + \frac{1}{2}c_{\tilde{\Lambda}''} = 6$. On the other hand, $V_{\tilde{\Lambda}} \cong V_2 \boxtimes V_2$. The representation of $\tilde{K}_0$ on $v_i^{(2)} \otimes v_j^{(2)} \in V_{\tilde{\Lambda}}$ $(i,j = 0,1,2)$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(P)(v_i^{(2)} \otimes v_j^{(2)})\\
=&\left[\rho_2\begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}}\end{pmatrix}\right](v_i^{(2)}) \otimes \left[\rho_2\begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}}\end{pmatrix}\right](v_j^{(2)})\\
=&e^{\sqrt{-1}(s-t)[2-(i+j)]} v_i^{(2)} \otimes v_j^{(2)}.
\end{aligned}$$

Hence $(V_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_0^{(2)} \otimes v_2^{(2)}, v_1^{(2)} \otimes v_1^{(2)}, v_2^{(2)} \otimes v_0^{(2)}\}$. Moreover, the action of the generator Q of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $v_i^{(2)} \otimes v_j^{(2)}$ is given by

$$\rho_{\tilde{\Lambda}}(Q)(v_i^{(2)} \otimes v_j^{(2)}) = (-1)^{3-i} v_{2-i}^{(2)} \otimes v_{2-j}^{(2)}.$$

Therefore, $(V_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}\{v_0^{(2)} \otimes v_2^{(2)} - v_2^{(2)} \otimes v_0^{(2)}, v_1^{(2)} \otimes v_1^{(2)}\}$ and $\tilde{\Lambda} = (1,-1,1,-1) \in D(\tilde{K},\tilde{K}_{[\mathfrak{a}]})$. Note that the $\tilde{K}_{[\mathfrak{a}]}$-fixed vector $v_1^{(2)} \otimes v_1^{(2)} \in V'_0$, which corresponds eigenvalue 8 and the $\tilde{K}_{[\mathfrak{a}]}$-fixed vector $v_0^{(2)} \otimes v_2^{(2)} - v_2^{(2)} \otimes v_0^{(2)} \in V'_{y_1-y_2-y_3+y_4} \oplus V'_{-y_1+y_2+y_3-y_4}$, which gives eigenvalue 6.

Suppose that $\tilde{\Lambda} = (2,0,-1,-1)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 3$. It follows from the branching law of $(U(2), U(1)\times U(1))$ that $(\tilde{q}'_1,\tilde{q}'_2) = (2,0), (1,1)$ or $(0,2)$ and $(\tilde{q}'_3,\tilde{q}'_4) = (-1,-1)$. Then $(\tilde{q}'_1,\tilde{q}'_2,\tilde{q}'_3,\tilde{q}'_4) = (1,1,-1,-1) \in D(\tilde{K},\tilde{K}_0)$. Hence, $-c_{\tilde{\Lambda}} = 8$, $-c_{\tilde{\Lambda}''} = 4$, $-c_L = -c_{\tilde{\Lambda}} + \frac{1}{2}c_{\tilde{\Lambda}''} = 6$. On the other hand,

$$V_{\tilde{\Lambda}} \cong (V_2 \otimes \mathbf{C}) \boxtimes \mathbf{C}.$$

The representation of $\tilde{K}_0$ on $v_i^{(2)} \otimes w \in V_{\tilde{\Lambda}}$ $(i = 0,1,2)$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(P)(v_i^{(2)} \otimes w)\\
=&\,e^{\sqrt{-1}(s+t)}\left[\rho_2\begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}}\end{pmatrix}\right](v_i^{(2)}) \otimes e^{-\sqrt{-1}(s+t)} w\\
=&\,e^{\sqrt{-1}(s-t)(1-i)} v_i^{(2)} \otimes w.
\end{aligned}$$

Hence, $(V_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_1^{(2)} \otimes 1\}$. Moreover, the action of the generator Q of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $v_i^{(2)} \otimes w$ is given by $\rho_{\tilde{\Lambda}}(Q)(v_i^{(2)} \otimes 1) = (-1)^{1-i} v_{2-i}^{(2)} \otimes 1$. Therefore, $(V_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}\{v_1^{(2)} \otimes 1\}$ and $\tilde{\Lambda} = (2,0,-1,-1) \in D(\tilde{K},\tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 6. Similarly, $\tilde{\Lambda} = (-1,-1,2,0)$, $(1,1,0,-2)$, $(0,-2,1,1) \in D(\tilde{K},\tilde{K}_{[\mathfrak{a}]})$, which give eigenvalue 6 and with multiplicity 1, respectively.

Moreover we observe that

$$\begin{aligned}
n(L^6) =& \dim_{\mathbf{C}} V_{(2,0,-1,-1)} + \dim_{\mathbf{C}} V_{(-1,-1,2,0)} + \dim_{\mathbf{C}} V_{(1,1,0,-2)}\\
&+ \dim_{\mathbf{C}} V_{(0,-2,1,1)} + \dim_{\mathbf{C}} V_{(1,-1,1,-1)} = 3+3+3+3+9\\
=&21 = \dim SO(8) - \dim S(U(2)\times U(2)) = n_{hk}(L^6).
\end{aligned}$$

Therefore we obtain that $L^6 = \mathcal{G}(\frac{S(U(2)\times U(2))}{S(U(1)\times U(1))}) \subset Q_6(\mathbf{C})$ is strictly Hamiltonian stable.

9.7. **Eigenvalue computation when** $m = 3$**.** For each $\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_5 y_5 \in D(\tilde{K}, \tilde{K}_0)$, $\tilde{\Lambda}' = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4 \in D(\tilde{K}_2, \tilde{K}_0)$ and $\tilde{\Lambda}'' = \tilde{q}'_1 y_1 + \tilde{q}'_2 y_2 + \tilde{q}'_3 y_3 + \tilde{q}'_4 y_4 \in D(\tilde{K}_1, \tilde{K}_0)$ given as in Subsection 9.5, the corresponding eigenvalue of $-\mathcal{C}_L$ is

$$\begin{aligned} -c_L =& -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2} c_{\tilde{\Lambda}''} \\ =& \tilde{p}_1^2 + \tilde{p}_2^2 + 2(\tilde{p}_3^2 + \tilde{p}_4^2 + \tilde{p}_5^2) + (\tilde{p}_1 - \tilde{p}_2) + 4(\tilde{p}_3 - \tilde{p}_5) \\ & - (\tilde{q}_3^2 + \tilde{q}_4^2) - (\tilde{q}_3 - \tilde{q}_4) - \frac{1}{2}((\tilde{q}'_1)^2 + (\tilde{q}'_2)^2 + (\tilde{q}'_3)^2 + (\tilde{q}'_4)^2). \end{aligned} \tag{9.2}$$

$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_5 y_5 \in D(\tilde{K}, \tilde{K}_0)$ is denoted by $\tilde{\Lambda} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5)$. Since $-\mathcal{C}_L \geq -\frac{1}{2}\mathcal{C}_{K/K_0}$, the eigenvalue of $-\mathcal{C}_L$, $-c_L \leq n = 10$ implies $-c_{\tilde{\Lambda}} \leq 20$. It then follows that

Lemma 9.4. *$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_5 y_5 \in D(\tilde{K}, \tilde{K}_0)$ has eigenvalue $-c_L \leq 10$ if and only if $(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5)$ is one of*

$$\begin{aligned} \big\{&(0,0,0,0,0), (1,-1,1,0,-1), (2,0,0,-1,-1), (0,-2,1,1,0), \\ &(1,1,0,0,-2), (-1,-1,2,0,0), (1,-1,0,0,0), (1,0,0,0,-1), \\ &(0,-1,1,0,0), (1,1,0,-1,-1), (-1,-1,1,1,0), (0,0,1,0,-1)\big\}. \end{aligned}$$

Suppose that $\tilde{\Lambda} = (1,-1,1,0,-1)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 24$. It follows from Lemma 9.2 that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1,-1,0)$ or $(0,0,0)$. When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,0,0)$, by the branching law of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4, \tilde{q}'_5) = (0,0,0,0,0)$. Hence, $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''} = 16 > 10$. When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1,-1,0)$, by the branching law of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4, \tilde{q}'_5) = (1,-1,-1,1,0)$, $(0,0,0,0,0)$ or $(-1,1,1,-1,0) \in D(\tilde{K}, \tilde{K}_0)$, respectively. Hence, $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''} = 10$, 12 or 10, respectively. On the other hand, now

$$\begin{aligned} (\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} &\subset (W_{y_1-y_2} \boxtimes W_{y_3-y_4} \boxtimes W_0) \oplus (W_{y_1-y_2} \boxtimes W_0 \boxtimes W_0) \\ &\cong (V_2 \boxtimes V_2 \boxtimes \mathbf{C}) \oplus (V_2 \boxtimes \mathbf{C} \boxtimes \mathbf{C}), \end{aligned}$$

where the latter is a $\tilde{K}_2$-module. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u_i \otimes v_j \otimes w \in V_2 \boxtimes V_2 \boxtimes \mathbf{C}$ $(i, j = 0, 1, 2)$ is given by

$$\begin{aligned} &\rho_{\tilde{\Lambda}}(P)(v_i^{(2)} \otimes v_j^{(2)} \otimes w) \\ =& \rho_{y_1-y_2} \begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix} (v_i^{(2)}) \otimes \rho_{y_3-y_4} \begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix} (v_j^{(2)}) \otimes w \\ =& e^{\sqrt{-1}(s-t)(2-i-j)} v_i^{(2)} \otimes v_j^{(2)} \otimes w. \end{aligned}$$

The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $v_i^{(2)} \otimes v \otimes w \in V_2 \boxtimes \mathbf{C} \boxtimes \mathbf{C}$ $(i = 0, 1, 2)$ is given by

$$\begin{aligned} \rho_{\tilde{\Lambda}}(P)(v_i^{(2)} \otimes v \otimes w) =& \rho_{y_1-y_2} \begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix} (v_i^{(2)}) \otimes v \otimes w \\ =& e^{\sqrt{-1}(s-t)(1-i)} v_i^{(2)} \otimes v \otimes w. \end{aligned}$$

Thus, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_2^{(2)} \otimes v_0^{(2)} \otimes w, v_0^{(2)} \otimes v_2^{(2)} \otimes w, v_1^{(2)} \otimes v_1^{(2)} \otimes w, v_1^{(2)} \otimes v \otimes w\}$. Moreover, the action of the generator Q of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $v_i^{(2)} \otimes v_{2-i}^{(2)} \otimes w$ is given by

$$\begin{aligned}\rho_{\tilde{\Lambda}}(Q)(v_i^{(2)} \otimes v_{2-i}^{(2)} \otimes w) &= \rho_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (v_i^{(2)}) \otimes \rho_2 \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} (v_{2-i}^{(2)}) \otimes w \\ &= (-1)^{1-i} u_{2-i} \otimes v_i^{(2)} \otimes w\end{aligned}$$

and the action on $v_i^{(2)} \otimes v \otimes w$ is given by

$$\begin{aligned}\rho_{\tilde{\Lambda}}(Q)(v_i^{(2)} \otimes v \otimes w) &= \rho_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (v_i^{(2)}) \otimes v \otimes w \\ &= (-1)^{2-i} v_{2-i}^{(2)} \otimes v \otimes w.\end{aligned}$$

Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}_{\mathbf{C}}\{v_2^{(2)} \otimes v_0^{(2)} \otimes w - v_0^{(2)} \otimes v_2^{(2)} \otimes w, v_1^{(2)} \otimes v_1^{(2)} \otimes w\}$ and $\tilde{\Lambda} = (1,-1,1,0,-1) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$. Notice that the $\tilde{K}_{[\mathfrak{a}]}$-fixed vector $v_1^{(2)} \otimes v_1^{(2)} \otimes w \in V_{\tilde{\Lambda}''}$, which corresponds eigenvalue 12, where $\tilde{\Lambda}'' = 0$. And the $\tilde{K}_{[\mathfrak{a}]}$-fixed vector $v_2^{(2)} \otimes v_0^{(2)} \otimes w - v_0^{(2)} \otimes v_2^{(2)} \otimes w \in V_{\tilde{\Lambda}_1''} \oplus V_{\tilde{\Lambda}_2''}$, which gives eigenvalue 10, where $\tilde{\Lambda}_1'' = (1,-1,-1,1,0)$ and $\tilde{\Lambda}_2'' = (-1,1,1,-1,0)$.

Suppose that $\tilde{\Lambda} = (2,0,0,-1,-1)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 9$. It follows from the branching law of $(U(3), U(2) \times U(1))$ that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,-1,-1)$ or $(-1,-1,0)$. When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (-1,-1,0)$, by the branching law of $(U(2), U(1) \times U(1))$, $(\tilde{q}_1', \tilde{q}_2', \tilde{q}_3', \tilde{q}_4', \tilde{q}_5') = (1,1,-1,-1,0)$. Hence, $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2} c_{\tilde{\Lambda}''} = 10$. On the other hand,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} \subset (W_{2y_1} \boxtimes W_{-(y_3+y_4)} \boxtimes W_0) \cong V_2 \boxtimes \mathbf{C} \boxtimes \mathbf{C},$$

and the representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $v_i^{(2)} \otimes v \otimes w \in V_2 \boxtimes \mathbf{C} \boxtimes \mathbf{C}$ $(i = 0,1,2)$ is given by

$$\begin{aligned}&\rho_{\tilde{\Lambda}}(P)(v_i^{(2)} \otimes v \otimes w) \\ =&\rho_{2y_1} \begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix} (v_i^{(2)}) \otimes \rho_{-y_3-y_4} \begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix} (v) \otimes w \\ =&e^{\sqrt{-1}(s-t)(1-i)} v_i^{(2)} \otimes v \otimes w.\end{aligned}$$

Thus, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_1^{(2)} \otimes v \otimes w\}$. Moreover, the action of the generator Q of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $v_i^{(2)} \otimes v \otimes w$ is given by

$$\begin{aligned}\rho_{\tilde{\Lambda}}(Q)(v_i^{(2)} \otimes v \otimes w) &= \rho_{2y_1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (v_i^{(2)}) \otimes \rho_{-(y_3+y_4)} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} (v) \otimes w \\ &= (-1)^{1+i}\, v_{2-i}^{(2)} \otimes v \otimes w.\end{aligned}$$

Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}_{\mathbf{C}}\{v_1^{(2)} \otimes v \otimes w\}$, where $\dim_{\mathbf{C}}(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = 1$ and $\tilde{\Lambda} = (2,0,0,-1,-1) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 10. Similarly, $\tilde{\Lambda} = (0,-2,1,1,0) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 10 and with multiplicity 1 and dimension 9.

Suppose that $\tilde{\Lambda} = (1,1,0,0,-2)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 6$. It follows from Lemma 9.2 that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,0,-2), (0,-1,-1)$ or $(0,-2,0)$. When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,-2,0)$, by the branching law of $(U(2), U(1)\times U(1))$, $(\tilde{q}'_1,\tilde{q}'_2,\tilde{q}'_3,\tilde{q}'_4,\tilde{q}'_5)=(1,1,-1,-1,0)$. Hence, $-c_L=-2c_{\tilde{\Lambda}}+c_{\tilde{\Lambda}'}+\frac{1}{2}c_{\tilde{\Lambda}''}=10$. On the other hand,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0}\subset W_0\boxtimes W_{-2y_4}\boxtimes W_0\cong \mathbf{C}\boxtimes V_2\boxtimes \mathbf{C}$$

and the representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u\otimes v_i^{(2)}\otimes w\in \mathbf{C}\boxtimes V_2\boxtimes\mathbf{C}$ $(i=0,1,2)$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(P)(u\otimes v_i^{(2)}\otimes w)\\
=&\rho_{y_1+y_2}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t}\end{pmatrix}(u)\otimes\rho_{-2y_4}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t}\end{pmatrix}(v_i^{(2)})\otimes w\\
=&e^{\sqrt{-1}(s-t)(1-i)}u\otimes v_i^{(2)}\otimes w.
\end{aligned}$$

Thus, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0}=\mathrm{span}_{\mathbf{C}}\{u\otimes v_1^{(2)}\otimes w\}$. Moreover, the action of the generator Q of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u\otimes v_i^{(2)}\otimes w$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(Q)(u\otimes v_i^{(2)}\otimes w)\\
=&\rho_{y_1+y_2}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}(u)\otimes\rho_{-2y_4}\begin{pmatrix} 0 & -1\\ -1 & 0\end{pmatrix}(v_i^{(2)})\otimes w=u\otimes v_{2-i}^{(2)}\otimes w.
\end{aligned}$$

Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}}=\mathrm{span}_{\mathbf{C}}\{u\otimes v_1^{(2)}\otimes w\}$, where $\dim_{\mathbf{C}}(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}}=1$ and $\tilde{\Lambda}=(1,1,0,0,-2)\in D(\tilde{K},\tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 10. Similarly, $\tilde{\Lambda}=(-1,-1,2,0,0)\in D(\tilde{K},\tilde{K}_{[\mathfrak{a}]})$, gives eigenvalue 10 and has multiplicity 1 and dimension 6.

Suppose that $\tilde{\Lambda}=(1,-1,0,0,0)$. Then $(\tilde{q}_1,\tilde{q}_2,\tilde{q}_3,\tilde{q}_4,\tilde{q}_5)=(1,-1,0,0,0)$. It follows from the branching law of $(U(2),U(1)\times U(1))$, $(\tilde{q}'_1,\tilde{q}'_2,\tilde{q}'_3,\tilde{q}'_4,\tilde{q}'_5)=(0,0,0,0,0)\in D(\tilde{K}_1,\tilde{K}_0)$. Hence, $-c_L=-2c_{\tilde{\Lambda}}+c_{\tilde{\Lambda}'}+\frac{1}{2}c_{\tilde{\Lambda}''}=4<10$. On the other hand, $\tilde{V}_{\tilde{\Lambda}}=W_{y_1-y_2}\boxtimes W_0\cong V_2\boxtimes\mathbf{C}$ and the representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u_i\otimes w\in V_2\boxtimes\mathbf{C}$ $(i=0,1,2)$ is given by

$$\begin{aligned}
\rho_{\tilde{\Lambda}}(P)(u_i\otimes w)&=\rho_2\begin{pmatrix} e^{\sqrt{-1}\frac{s-t}{2}} & \\ & e^{-\sqrt{-1}\frac{s-t}{2}}\end{pmatrix}(u_i)\otimes w\\
&=e^{\sqrt{-1}(s-t)(1-i)}u_i\otimes w.
\end{aligned}$$

Thus, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0}=\mathrm{Span}_{\mathbf{C}}\{u_1\otimes w\}$. Moreover, the action of the generator Q of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u_1\otimes w$ is given by $\rho_{\tilde{\Lambda}}(Q)(u_1\otimes w)=-u_1\otimes w$. Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}}=\{0\}$ and $\tilde{\Lambda}=(1,-1,0,0,0)\notin D(\tilde{K},\tilde{K}_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda}=(1,0,0,0,-1)$. It follows from Lemma 9.2 that $(\tilde{q}_3,\tilde{q}_4,\tilde{q}_5)=(0,0,-1)$ or $(0,-1,0)$. When $(\tilde{q}_3,\tilde{q}_4,\tilde{q}_5)=(0,-1,0)$, by the branching law of $(U(2),U(1)\times U(1))$, $(\tilde{q}'_1,\tilde{q}'_2,\tilde{q}'_3,\tilde{q}'_4,\tilde{q}'_5)=(1,0,-1,0,0)$ or $(0,1,0,-1,0)\in D(\tilde{K}_1,\tilde{K}_0)$. Hence, $-c_L=-2c_{\tilde{\Lambda}}+c_{\tilde{\Lambda}'}+\frac{1}{2}c_{\tilde{\Lambda}''}=5,5<10$. On the other hand, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0}\subset W_{y_1}\boxtimes W_{-y_4}\boxtimes W_0\cong V_1\boxtimes V_1\boxtimes\mathbf{C}$, where the latter is the $\tilde{K}_2=U(2)\times U(2)\times U(1)$-module. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $v_i^{(1)}\otimes v_j^{(1)}\otimes w\in V_1\boxtimes V_1\boxtimes\mathbf{C}$ $(i,j=0,1)$ is given by

$$\rho_{\tilde{\Lambda}}(P)(v_i^{(1)}\otimes v_j^{(1)}\otimes w)=e^{\sqrt{-1}(s-t)(1-i-j)}\,v_i^{(1)}\otimes v_j^{(1)}\otimes w.$$

Thus, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_1^{(1)} \otimes v_0^{(1)} \otimes w, u_0 \otimes v_1^{(1)} \otimes w\}$. Moreover, the action of the generator Q of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $v_i^{(1)} \otimes v_{1-i}^{(1)} \otimes w$ $(i = 0, 1)$ is given by

$$\rho_{\tilde{\Lambda}}(Q)(v_i^{(1)} \otimes v_{1-i}^{(1)} \otimes w) = (-1)^{1-i}\, v_{1-i}^{(1)} \otimes v_i^{(1)} \otimes w.$$

Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ and $\tilde{\Lambda} = (1, 0, 0, 0, -1) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$. Similarly, $\tilde{\Lambda} = (0, -1, 1, 0, 0) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (1, 1, 0, -1, -1)$. It follows from Lemma 9.2 that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0, -1, -1)$ or $(-1, -1, 0)$. For the element $(\tilde{p}_1, \tilde{p}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1, 1, -1, -1, 0)$ in $D(\tilde{K}_2, \tilde{K}_0)$, by the branching law of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (1, 1, -1, -1) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence, $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''} = 6 < 10$. On the other hand, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} \subset W_{y_1+y_2} \boxtimes W_{-y_3-y_4} \boxtimes W_0 \cong \mathbf{C} \boxtimes \mathbf{C} \boxtimes \mathbf{C}$, where the latter is the $\tilde{K}_2 = U(2) \times U(2) \times U(1)$-module. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u \otimes v \otimes w \in \mathbf{C} \boxtimes \mathbf{C} \boxtimes \mathbf{C}$ is given by

$$\rho_{\tilde{\Lambda}}(P)(u \otimes v \otimes w) = e^{\sqrt{-1}(s+t)} u \otimes e^{-\sqrt{-1}(s+t)} v \otimes w = u \otimes v \otimes w.$$

It follows that $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{1 \otimes 1 \otimes 1\}$. Moreover, the action of the generator Q of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u \otimes v \otimes w$ is given by

$$\rho_{\tilde{\Lambda}}(Q)(u \otimes v \otimes w) = -u \otimes v \otimes w.$$

Therefore $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ and $\tilde{\Lambda} = (1, 1, 0, -1, -1) \notin D(\tilde{K}, \tilde{K}_0)$. Similarly, $\tilde{\Lambda} = (-1, -1, 1, 1, 0) \notin D(\tilde{K}, \tilde{K}_0)$.

Suppose that $\tilde{\Lambda} = (0, 0, 1, 0, -1)$. It follows from the branching law of $(U(3), U(2) \times U(1))$ that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1, 0, -1), (0, 0, 0), (1, -1, 0)$ or $(0, -1, 1)$. For the element $(\tilde{p}_1, \tilde{p}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0, 0, 0, 0, 0)$ in $D(\tilde{K}_2, \tilde{K}_0)$, by the branching law of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (0, 0, 0, 0) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence, $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''} = 12 > 10$. For the element $(\tilde{p}_1, \tilde{p}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0, 0, 1, -1, 0)$ in $D(\tilde{K}_2, \tilde{K}_0)$, by the branching laws of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (0, 0, 0, 0) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence, $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''} = 8 < 10$. On the other hand, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} \subset \tilde{V}'_{(0,0,0,0,0)} \oplus \tilde{V}'_{(0,0,1,-1,0)}$. We are concerned with only $\tilde{V}'_{(0,0,1,-1,0)}$ since it corresponds to the smaller eigenvalue 8. Note that $\tilde{V}'_{(0,0,1,-1,0)} = W_0 \boxtimes W_{y_3-y_4} \boxtimes W_0 \cong \mathbf{C} \boxtimes V_2 \boxtimes \mathbf{C}$, which is a $\tilde{K}_2$-module. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u \otimes v_i^{(2)} \otimes w \in \tilde{V}'_{(0,0,1,-1,0)}$ $(i = 0, 1, 2)$ is given by

$$\rho_{\tilde{\Lambda}}(P)(u \otimes v_i^{(2)} \otimes w) = e^{\sqrt{-1}(s-t)(1-i)} u \otimes v_i^{(2)} \otimes w.$$

Thus $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{Span}_{\mathbf{C}}\{1 \otimes v_1 \otimes 1\} \oplus \tilde{V}'_{(0,0,0,0,0)}$. Moreover, the action of the generator Q of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u \otimes v_1^{(2)} \otimes w$ is given by

$$\begin{aligned} &\rho_{\tilde{\Lambda}}(Q)(u \otimes v_1^{(2)} \otimes w) \\ =&u \otimes \rho_2(\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}) v_1 \otimes w = -u \otimes v_1^{(2)} \otimes w. \end{aligned}$$

Therefore, $1 \otimes v_1^{(2)} \otimes 1 \notin (\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}}$ and $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \tilde{V}'_{(0,0,0,0,0)}$, which gives a larger eigenvalue 10.

Moreover,

$$\begin{aligned} n(L^{10}) =& \dim_{\mathbf{C}} V_{(1,-1,1,0,-1)} + \dim_{\mathbf{C}} V_{(2,0,0,-1,-1)} + \dim_{\mathbf{C}} V_{(0,-2,1,1,0)} \\ & + \dim_{\mathbf{C}} V_{(1,1,0,0,-2)} + \dim_{\mathbf{C}} V_{(-1,-1,2,0,0)} \\ =& 24+9+9+6+6 = 54 \\ =& \dim SO(12) - \dim S(U(2)\times U(3)) = n_{hk}(L^{10}). \end{aligned}$$

Therefore we obtain that $L^{10} = \mathcal{G}(\frac{S(U(2)\times U(3))}{S(U(1)\times U(1)\times U(1))}) \subset Q_{10}(\mathbf{C})$ is strictly Hamiltonian stable.

9.8. **Eigenvalue computation when** $m \geq 4$**.** For each $\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_{m+1} y_{m+1} + \tilde{p}_{m+2} y_{m+2} \in D(\tilde{K}, \tilde{K}_0)$, $\tilde{\Lambda}' = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4 \in D(\tilde{K}_2, \tilde{K}_0)$ and $\tilde{\Lambda}'' = \tilde{q}'_1 y_1 + \tilde{q}'_2 y_2 + \tilde{q}'_3 y_3 + \tilde{q}'_4 y_4 \in D(\tilde{K}_1, \tilde{K}_0)$, the eigenvalue formula is

$$\begin{aligned} -c_L =& -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2} c_{\tilde{\Lambda}''} \\ =& \tilde{p}_1^2 + \tilde{p}_2^2 + 2(\tilde{p}_3^2 + \tilde{p}_4^2 + \tilde{p}_{m+1}^2 + \tilde{p}_{m+2}^2) \\ & + (\tilde{p}_1 - \tilde{p}_2) + 2(m-1)(\tilde{p}_3 - \tilde{p}_{m+2}) + 2(m-3)(\tilde{p}_4 - \tilde{p}_{m+1}) \\ & - (\tilde{q}_3^2 + \tilde{q}_4^2) - (\tilde{q}_3 - \tilde{q}_4) - \frac{1}{2}((\tilde{q}'_1)^2 + (\tilde{q}'_2)^2 + (\tilde{q}'_3)^2 + (\tilde{q}'_4)^2). \end{aligned}$$

In case $\tilde{\Lambda} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_{m+1}, \tilde{p}_{m+2}) = (\tilde{p}_1, \tilde{p}_2, 0,0,0,0) \in D(\tilde{K}, \tilde{K}_0)$, since $\tilde{p}_3 = \tilde{p}_4 = \tilde{p}_{m+1} = \tilde{p}_{m+2} = 0$, we have $\tilde{q}_3 = \tilde{q}_4 = \tilde{q}_5 = \cdots = \tilde{q}_{m+2} = 0$ and thus $\tilde{q}'_3 = \tilde{q}'_4 = 0$. Since $\tilde{p}_1 + \tilde{p}_2 = 0$, by the branching law of $(U(2), U(1)\times U(1))$ we have $\tilde{q}'_1 = -\alpha + \tilde{p}_1$, $\tilde{q}'_2 = \alpha + \tilde{p}_2 = \alpha - \tilde{p}_1 = -\tilde{q}'_1$ for some $\alpha = 0, 1 \cdots, \tilde{p}_1 - \tilde{p}_2 = 2\tilde{p}_1$. $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ implies that $\tilde{q}'_1 = \tilde{q}'_2 = 0$ since $\tilde{q}'_1 + \tilde{q}'_3 = 0$ and $\tilde{q}'_2 + \tilde{q}'_4 = 0$. Then $-c_L = 2\tilde{p}_1(\tilde{p}_1 + 1)$.

Now $\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 = 2\tilde{p}_1 \frac{1}{2}(y_1 - y_2)$. Set $\ell := 2\tilde{p}_1$. Then $\tilde{V_{\tilde{\Lambda}}} \cong V_\ell \boxtimes \mathbf{C}$. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $v_i^{(\ell)} \otimes w \in \tilde{V}_{\tilde{\Lambda}}$ is given by

$$\begin{aligned} \rho_{\tilde{\Lambda}}(P)(v_i^{(\ell)} \otimes w) =& \left[\rho_\ell \begin{pmatrix} e^{\sqrt{-1}(s-t)/2} & 0 \\ 0 & e^{-\sqrt{-1}(s-t)/2} \end{pmatrix}\right] (v_i^{(\ell)}) \otimes w \\ =& e^{\frac{\sqrt{-1}(s-t)}{2}(\ell - 2i)} v_i^{(\ell)} \otimes w. \end{aligned}$$

Hence, $(\tilde{V_{\tilde{\Lambda}}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_{\tilde{p}_1}^{(\ell)} \otimes w\}$. On the other hand, the action of the generator Q of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ is given by

$$\begin{aligned} \rho_{\tilde{\Lambda}}(Q)(v_{\tilde{p}_1}^{(\ell)} \otimes w) =& \left[\rho_\ell \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] (v_{\tilde{p}_1}^{(\ell)}) \otimes w \\ =& (-1)^{\tilde{p}_1} v_{\tilde{p}_1}^{(\ell)} \otimes w. \end{aligned}$$

Therefore, $(\tilde{V_{\tilde{\Lambda}}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}_{\mathbf{C}}\{v_{\tilde{p}_1}^{(\ell)} \otimes w\}$ for $\tilde{p}_1$ is even. As $m \geq 4$, for every even number $\tilde{p}_1 \geq 2$ such that $12 \leq 2\tilde{p}_1(\tilde{p}_1 + 1) < 4m - 2$, $\tilde{\Lambda} = \tilde{p}_1(y_1 - y_2) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$ has eigenvalue $12 \leq -c_L = 2\tilde{p}_1(\tilde{p}_1 + 1) < 4m - 2$. It means that $L^{4m-2} \subset Q_{4m-2}(\mathbf{C})$ is NOT Hamiltonian stable for $m \geq 4$.

From these results we conclude

Theorem 9.1. *The Gauss image* $L^{4m-2} = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))\cdot \mathbf{Z}_4} \subset Q_{4m-2}(\mathbf{C})$ $(m \geq 2)$ *is not Hamiltonian stable if and only if* $m \geq 4$*. If* $m = 2$ *or* 3*, it is strictly Hamiltonian stable.*

Remark. The index $i(L^{4m-2})$ goes to ∞ as $m \to \infty$.

10. The case $(U, K) = (Sp(m+2), Sp(2) \times Sp(m))$ $(m \geq 2)$

In this case, $K = Sp(2) \times Sp(m) \subset U = Sp(m+2)$, (U, K) is of type B_2 for $m = 2$ and type BC_2 for $m \geq 3$. Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{u}$ and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, where

$$\begin{aligned}\mathfrak{u} =& \mathfrak{sp}(m+2)\\ =& \left\{ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \mid A \in \mathfrak{u}(m+2), B \in M(m+2, \mathbf{C}), B^t = B \right\} \subset \mathfrak{u}(2m+4),\end{aligned}$$

$$\begin{aligned}\mathfrak{k} =& \mathfrak{sp}(2) + \mathfrak{sp}(m)\\ =& \left\{ \begin{pmatrix} A_{11} & 0 & B_{11} & 0 \\ 0 & A_{22} & 0 & B_{22} \\ -\bar{B}_{11} & 0 & \bar{A}_{11} & 0 \\ 0 & -\bar{B}_{22} & 0 & \bar{A}_{22} \end{pmatrix} \mid A_{11} \in \mathfrak{u}(2), B_{11} \in M(2, \mathbf{C}), B_{11}^t = B_{11}, \right.\\ & \left. \qquad A_{22} \in \mathfrak{u}(m), B_{22} \in M(m, \mathbf{C}), B_{22}^t = B_{22} \right\},\end{aligned}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & A_{12} & 0 & B_{12} \\ -\bar{A}_{12}^t & 0 & B_{12}^t & 0 \\ 0 & -\bar{B}_{12} & 0 & \bar{A}_{12} \\ -\bar{B}_{11}^t & 0 & -A_{12}^t & 0 \end{pmatrix} \middle| A_{12} \in M(2, m; \mathbf{C}), B_{12} \in M(2, m; \mathbf{C}) \right\},$$

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & H_{12} & 0 & 0 \\ -\bar{H}_{12}^t & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{H}_{12} \\ 0 & 0 & -H_{12}^t & 0 \end{pmatrix} \mid H_{12} = \begin{pmatrix} \xi_1 & 0 & 0 & \cdots & 0 \\ 0 & \xi_2 & 0 & \cdots & 0 \end{pmatrix}, \xi_1, \xi_2 \in \mathbf{R} \right\}.$$

Then the centralizer K_0 of $\mathfrak{a}$ in K is given as follows:

$$\begin{aligned}K_0 =& Sp(1) \times Sp(1) \times Sp(m-2)\\ =& \left\{ \begin{pmatrix} a_1 & 0 & & & & b_1 & 0 & & & \\ 0 & a_2 & & & & 0 & b_2 & & & \\ & & a_1 & 0 & & & & b_1 & 0 & \\ & & 0 & a_2 & & & & 0 & b_2 & \\ & & & & A_{11} & & & & & A_{12} \\ -\bar{b}_1 & 0 & & & & \bar{a}_1 & 0 & & & \\ 0 & -\bar{b}_2 & & & & 0 & \bar{a}_2 & & & \\ & & -\bar{b}_1 & 0 & & & & \bar{a}_1 & 0 & \\ & & 0 & -\bar{b}_2 & & & & 0 & \bar{a}_2 & \\ & & & & A_{21} & & & & & A_{22} \end{pmatrix} \right. \mid \\ & \left. \begin{pmatrix} a_1 & b_1 \\ -\bar{b}_1 & \bar{a}_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ -\bar{b}_2 & \bar{a}_2 \end{pmatrix} \in Sp(1) = SU(2), \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in Sp(m-2) \right\}.\end{aligned}$$

Moreover,

$$K_{[\mathfrak{a}]} = K_0 \cup (Q \cdot K_0) \cup (Q^2 \cdot K_0) \cup (Q^3 \cdot K_0),$$

where

$$D = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & I_{m-2} \end{pmatrix} \text{ and } Q := \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}.$$

Thus the deck transformation group of the covering map $\mathcal{G} : N^{8m-2} \to \mathcal{G}(N^{8m-2})$ $(m \geq 2)$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

10.1. **Description of the Casimir operator.**

Denote $\langle X, Y \rangle_{\mathfrak{u}} := -\frac{1}{2}\mathrm{tr}XY$ for each $X, Y \in \mathfrak{sp}(m+2) \subset \mathfrak{u}(2m+4)$. Then the square length of each restricted root relative to the above inner product $\langle \ , \ \rangle_{\mathfrak{u}}$, is given by

$$\|\gamma\|_{\mathfrak{u}}^2 = \begin{cases} 1 \text{ or } 2, & m = 2; \\ \frac{1}{2}, 1 \text{ or } 2, & m \geq 3. \end{cases}$$

Hence the Casimir operator $\mathcal{C}_L$ of L, with respect to the induced metric from $g^{\mathrm{std}}_{Q_{8m-2}(\mathbf{C})}$ can be expressed as follows:

$$\mathcal{C}_L = \begin{cases} \mathcal{C}_{K/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0}, & m = 2; \\ 2\,\mathcal{C}_{K/K_0} - \mathcal{C}_{K_2/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0}, & m \geq 3, \end{cases}$$

where $\mathcal{C}_{K/K_0}$, $\mathcal{C}_{K_2/K_0}$ and $\mathcal{C}_{K_1/K_0}$ denote the Casimir operator of K/K_0, K_2/K_0 and K_1/K_0 relative to $\langle \ , \ \rangle_{\mathfrak{u}}|_{\mathfrak{k}}$, $\langle \ , \ \rangle_{\mathfrak{u}}|_{\mathfrak{k}_2}$ and $\langle \ , \ \rangle_{\mathfrak{u}}|_{\mathfrak{k}_1}$, respectively.

10.2. **Descriptions of $D(Sp(m))$ and $D(Sp(2) \times Sp(m))$.**

Let $G = Sp(m)$ and $K = Sp(2) \times Sp(m-2)$ in this subsection. Their Lie algebras are $\mathfrak{g}$ and $\mathfrak{k}$, respectively.

$$\mathfrak{t} = \{\xi = \sqrt{-1}\mathrm{diag}(\xi_1, \cdots, \xi_m, -\xi_1, \cdots, -\xi_m) \mid \xi_1, \cdots, \xi_m \in \mathbf{R}\}.$$

is a maximal abelian subalgebra $\mathfrak{t}$ in both $\mathfrak{g}$ and $\mathfrak{k}$. Let $y_i : \xi \mapsto \xi_i$ be a linear form on $\mathfrak{t}$. Then the fundamental root system of $\mathfrak{g}$ relative to $\mathfrak{t}$ is given by $\{\alpha_1 = y_1 - y_2, \cdots, \alpha_{m-1} = y_{m-1} - y_m, \alpha_m = 2y_m\}$ and the fundamental root system of $\mathfrak{k}$ relative to $\mathfrak{t}$ can be given by $\{\alpha' = y_1 - y_2, \alpha' = 2y_2, \alpha'_3 = y_3 - y_4, \cdots, \alpha'_{m-1} = y_{m-1} - y_m, \alpha'_m = 2y_m\}$. Thus each $\Lambda \in D(G)$ for $G = Sp(m)$ relative to $\mathfrak{t}$ is uniquely expressed as $\Lambda = p_1y_1 + \cdots + p_my_m$ with $p_1, \cdots, p_m \in \mathbf{Z}$ and $p_1 \geq p_2 \geq \cdots \geq p_m \geq 0$. And also each $\Lambda \in D(K)$ for $K = Sp(2) \times Sp(m-2)$ relative to $\mathfrak{t}$ is uniquely expressed as $\Lambda' = q_1y_1 + \cdots + q_my_m$ with $q_1, \cdots, q_m \in \mathbf{Z}$ and $q_1 \geq q_2 \geq 0, q_3 \geq \cdots \geq q_m \geq 0$.

10.3. **Branching law of $(Sp(2), Sp(1) \times Sp(1))$.**

Lemma 10.1 (Branching law of $(Sp(2), Sp(1) \times Sp(1))$ [24], [47]). *Let V_Λ be an irreducible $Sp(2)$-module with the highest weight $\Lambda = p_1y_1 + p_2y_2 \in D(Sp(2))$, where $p_1, p_2 \in \mathbf{Z}$ and $p_1 \geq p_2 \geq 0$. Then V_Λ contains an irreducible $Sp(1) \times Sp(1)$-module $V_{\Lambda'}$ with the highest weight $\Lambda' = q_1y_1 + q_2y_2 \in D(Sp(1) \times Sp(1))$, where $q_1, q_2 \in \mathbf{Z}$ and $q_1 \geq 0, q_2 \geq 0$, if and only if*

(i) *$p_1 \geq q_2 \geq 0$;*

(ii) *in the finite power series expansion in X of $\frac{\prod_{i=0}^{1}(X^{r_i+1}-X^{-(r_i+1)})}{X-X^{-1}}$, where $r_i (i=0,1)$ are defined as*

$$r_0 := p_1 - \max(p_2, q_2), \quad r_1 := \min(p_2, q_2),$$

the coefficient of X^{q_1+1} does not vanish.

Here that coefficient is equal to the multiplicity of a $Sp(1)\times Sp(1)$-module $V_{\Lambda'}$ in V_Λ.

10.4. **Descriptions of $D(K,K_0)$ and $D(K_1,K_0)$ when $m=2$.**

For each $\Lambda = p_1y_1+p_2y_2+p_3y_3+p_4y_4 \in D(K) = D(Sp(2)\times Sp(2))$ with $p_1,\cdots,p_4\in\mathbf{Z}$ and $p_1\geq p_2\geq 0$, $p_3\geq p_4\geq 0$, we know that $p_1y_1+p_2y_2\in D(Sp(2))$, $p_3y_3+p_4y_4\in D(Sp(2))$ and $V_\Lambda = W_{p_1y_1+p_2y_2}\boxtimes W_{p_3y_3+p_4y_4}$. By Lemma 10.1, $W_{p_1y_1+p_2y_2}$ and $W_{p_3y_3+p_4y_4}$ can be decomposed into irreducible $Sp(1)\times Sp(1)$-modules as

$$W_{p_1y_1+p_2y_2} = \bigoplus_{q_1,q_2} W'_{q_1y_1+q_2y_2}, \quad W_{p_3y_3+p_4y_4} = \bigoplus_{q_3,q_4} W'_{q_3y_3+q_4y_4},$$

where q_1,q_2 and q_3,q_4 vary as in Lemma 10.1. Thus we have a decomposition of V_Λ into the direct sum of irreducible $Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)$-modules:

$$V_\Lambda = \bigoplus_{q_1,q_2}\bigoplus_{q_3,q_4}(W'_{q_1y_1+q_2y_2}\boxtimes W'_{q_3y_3+q_4y_4}).$$

Further by the Clebsch-Gordan formula it can be decomposed into into the sum of irreducible $Sp(1)\times Sp(1)$-modules as

$$V_\Lambda = \bigoplus_{q_1,q_2}\bigoplus_{q_3,q_4}\left(\bigoplus_{i=1}^{q_3} U_{q_1+q_3-2i}\right)\boxtimes\left(\bigoplus_{j=0}^{q_4} U_{q_2+q_4-2j}\right).$$

Here we assume that $q_1\geq q_3\geq 0$ and $q_2\geq q_4\geq 0$. Hence

Lemma 10.2. *$\Lambda\in D(K,K_0)$ if and only if there exist $i,j\in\mathbf{Z}$ with $0\leq i\leq q_3$ and $0\leq j\leq q_4$ such that $U_{q_1+q_3-2i}\boxtimes U_{q_2+q_4-2j}$ is a trivial $Sp(1)\times Sp(1)$-module. Then it must be that $(q_1,q_2)=(q_3,q_4)$.*

10.5. **Eigenvalue computation when $m=2$.** For $\Lambda = p_1y_1+p_2y_2+p_3y_3+p_4y_4\in D(K,K_0)$ and $\Lambda' = q_1y_1+q_2y_2+q_3y_3+q_4y_4\in D(K_1,K_0)$ with $q_1=q_3$, $q_2=q_4$ as in Lemma 10.2, the corresponding eigenvalue of $-\mathcal{C}_L$ is

$$\begin{aligned}-c_L &= -c_\Lambda + \frac{1}{2}c_{\Lambda'}\\ &= \Big(\sum_{i=1}^{4} p_i^2 + 4p_1+2p_2+4p_3+2p_4\Big) - \Big(q_1^2+q_2^2+2q_1+2q_2\Big).\end{aligned} \tag{10.1}$$

Denote $\Lambda = p_1y_1+p_2y_2+p_3y_3+p_4y_4\in D(K,K_0)$ by $\Lambda=(p_1,p_2,p_3,p_4)$. Then using the eigenvalue formula (10.1) we compute

Lemma 10.3. *$\Lambda\in D(K,K_0)$ has eigenvalue $-c_L\leq 14$ if and only if (p_1,p_2,p_3,p_4) is one of*

$$\{(0,0,0,0),(1,1,0,0),(0,0,1,1),(1,0,1,0),(1,1,1,1),(1,1,2,0),(2,0,1,1)\}.$$

Suppose that $\Lambda = (1,1,0,0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 5$. It follows from Lemma 10.1 that $(q_1, q_2) = (0,0)$ or $(1,1)$ and $(q_3, q_4) = (0,0)$. Then $(q_1, q_2, q_3, q_4) = (0,0,0,0) \in D(K_1, K_0)$. Hence, $-c_\Lambda = 8$, $-c_{\Lambda'} = 0$, $-c_L = -c_\Lambda + \frac{1}{2} c_{\Lambda'} = 8 < 14$. On the other hand, there is a double covering $\pi : Sp(2) \to SO(5)$, and $\pi(Sp(1) \times Sp(1)) = SO(4)$. Let λ_5 denote the standard representation of $SO(5)$ and 1 the trivial representation of $SO(5)$. Then the complex representation of $K = Sp(2) \times Sp(2)$ with the highest weight $(1,1,0,0)$ is $(\lambda_5 \otimes 1) \otimes \mathbf{C}$ and $V_\Lambda = \mathbf{C}^5$. It is easy to see that $(V_\Lambda)_{K_0} = \mathbf{C}\mathbf{e}_1$, where $\mathbf{e}_1 = (1,0,0,0,0)^t \in \mathbf{C}^5$. However for

$$a = \begin{pmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & 0 & 1 & & \\ & & & & 1 & 0 & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix} \in K_{[\mathfrak{a}]} \subset K, \quad a \notin K_0,$$

$\pi(a) = \mathrm{diag}(-1,1,-1,-1,-1) \notin SO(4)$ and $\pi(a)\mathbf{e}_1 = -\mathbf{e}_1 \neq \mathbf{e}_1$. Therefore $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}$ and $\Lambda = (1,1,0,0) \notin D(K, K_{[\mathfrak{a}]})$. Similarly, $\Lambda = (0,0,1,1) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (1,0,1,0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 16$. The corresponding representation with the highest weight Λ is just the complexified isotropy representation $\mathrm{Ad}_{\mathfrak{p}}(K)^{\mathbf{C}}$. Hence $\Lambda \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (1,1,1,1)$. Then $\dim_{\mathbf{C}} V_\Lambda = 25$. By Lemma 10.1, $(q_1, q_2) = (1,1)$ or $(0,0)$ and $(q_3, q_4) = (1,1)$ or $(0,0)$. Then $(q_1, q_2, q_3, q_4) = (1,1,1,1)$ or $(0,0,0,0) \in D(K_1, K_0)$. If $(q_1, q_2, q_3, q_4) = (1,1,1,1)$, then $-c_L = 10 < 14$. If $(q_1, q_2, q_3, q_4) = (0,0,0,0)$, then $-c_L = 16 > 14$. On the other hand, $V_{(1,1,1,1)}$ is explicitly given as

$$V_{(1,1,1,1)} = \mathbf{C}^5 \boxtimes \mathbf{C}^5 \cong M(5, \mathbf{C}).$$

There are doubly covering homomorphisms

$$\begin{aligned} \pi : K = Sp(2) \times Sp(2) &\longrightarrow SO(5) \times SO(5), \\ \pi|_{K_1} : K_1 = Sp(1) \times Sp(1) \times Sp(1) \times Sp(1) &\longrightarrow SO(4) \times SO(4), \\ \pi|_{K_0} : K_0 = Sp(1) \times Sp(1) &\longrightarrow SO(4). \end{aligned}$$

The representation of K on V_Λ is realized as the action of $\pi(K) = SO(5) \times SO(5)$ on $M(5, \mathbf{C})$ in the following way: For each $(A, B) \in SO(5) \times SO(5)$, $X \in M(5, \mathbf{C})$ is mapped to $AXB^{-1} \in M(5, \mathbf{C})$. Then as a K_1-module,

$$\begin{aligned} M(5, \mathbf{C}) &= \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \right\} \\ &= W_{(1,1,0,0)} \oplus W_{(0,0,1,1)} \oplus W_{(0,0,0,0)} \oplus W_{(1,1,1,1)}. \end{aligned}$$

K_0 acts on $M(5, \mathbf{C})$ by the adjoint action as a diagonal subgroup of K_1. Hence,

$$(M(5, \mathbf{C}))_{K_0} = \left\{ \begin{pmatrix} x & 0 \\ 0 & yI_4 \end{pmatrix} \mid x, y \in \mathbf{C} \right\},$$

$$(M(5, \mathbf{C}))_{K_{[\mathfrak{a}]}} = \mathbf{C} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = W(0,0,0,0).$$

Though $\Lambda = (1,1,1,1) \in D(K, K_{[\mathfrak{a}]})$, by the preceding computation (in case $(q_1, q_2, q_3, q_4) = (0,0,0,0)$) we see that a nonzero element in $(M(5,\mathbf{C}))_{K_{[\mathfrak{a}]}} = W(0,0,0,0)$ gives eigenvalue $-c_L = 16 > 14$.

Suppose that $\Lambda = (1,1,2,0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 50$. It follows from Lemma 10.1 that $(q_1, q_2) = (1,1)$ or $(0,0)$ and $(q_3, q_4) = (0,2), (1,1)$ or $(2,0)$. Thus

$$\begin{aligned} V_\Lambda = &(W_{(1,1)} \boxtimes U_{(0,2)}) \oplus (W_{(1,1)} \boxtimes U_{(1,1)}) \oplus (W_{(1,1)} \boxtimes U_{(2,0)}) \\ &\oplus (W_{(0,0)} \boxtimes U_{(0,2)}) \oplus (W_{(0,0)} \boxtimes U_{(1,1)}) \oplus (W_{(0,0)} \boxtimes U_{(2,0)}). \end{aligned}$$

Here only $(q_1, q_2, q_3, q_4) = (1,1,1,1)$ $(W_{(1,1)} \boxtimes U_{(1,1)})$ belongs to $D(K_1, K_0)$. and the corresponding eigenvalue is $-c_L = 14$. On the other hand, the representation of K with highest weight $\Lambda = (1,1,2,0)$ is $\lambda_5 \boxtimes \mathrm{Ad}^{\mathbf{C}}_{\mathfrak{sp}(2)}$. Set $\Lambda_1 = (p_1, p_2) = (1,1) \in D(Sp(2))$. Then

$$V_{\Lambda_1} \cong \mathbf{C}^5 = \mathbf{C}\mathbf{e}_1 \oplus \mathrm{span}_{\mathbf{C}}\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\} = W_{(0,0)} \oplus W_{(1,1)}.$$

Using the quaternionic representation

$$\mathfrak{sp}(2) = \{X \in M(2,\mathbf{H}) \mid X^* + X = 0\},$$

we chose the following basis of $\mathfrak{sp}(2)$:

$$\begin{aligned} &E_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ E_2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ E_3 := \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \ E_4 := \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \\ &E_5 := \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \ E_6 := \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \ E_7 := \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \\ &E_8 := \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \ E_9 := \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \ E_{10} := \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}, \end{aligned}$$

where $\{i, j, k\}$ denote the unit pure quaternions.

Set $\Lambda_2 = (p_3, p_4) = (2,0) \in D(Sp(2))$. Then

$$\begin{aligned} V_{\Lambda_2} &\cong \mathrm{span}_{\mathbf{C}}\{E_1, E_2, E_3, E_4\} \oplus \mathrm{span}_{\mathbf{C}}\{E_5, E_6, E_7\} \oplus \mathrm{span}_{\mathbf{C}}\{E_8, E_9, E_{10}\} \\ &= W_{(1,1)} \oplus W_{(2,0)} \oplus W_{(0,2)} \end{aligned}$$

By a direct computation, we get that

$$\begin{aligned} (V_\Lambda)_{K_0} =&\mathrm{span}_{\mathbf{C}}\{\mathbf{e}_2 \otimes E_1 + \mathbf{e}_3 \otimes E_2 + \mathbf{e}_4 \otimes E_3 + \mathbf{e}_5 \otimes E_4\} \\ =&(V_\Lambda)_{K_{[\mathfrak{a}]}} \subset W_{(1,1)} \otimes U_{(1,1)}. \end{aligned}$$

Therefore, $\Lambda = (1,1,2,0) \in D(K, K_{[\mathfrak{a}]})$, which gives eigenvalue 14 with multiplicity 1. Similarly, we can show that $\Lambda = (2,0,1,1) \in D(K, K_{[\mathfrak{a}]})$ which gives eigenvalue 14 with multiplicity 1.

Moreover, we observe that

$$\begin{aligned} n(L^{14}) &= \dim_{\mathbf{C}} V_{(1,1,2,0)} + \dim_{\mathbf{C}} V_{(2,0,1,1)} = 100 \\ &= \dim SO(16) - \dim Sp(2) \times Sp(2) = n_{hk}(L^{14}). \end{aligned}$$

From these results we obtain that $L^{14} = \mathcal{G}(\frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)}) \subset Q_{14}(\mathbf{C})$ is strictly Hamiltonian stable.

10.6. **Eigenvalue computation when $m \geq 3$.** For each

$$\Lambda = p_1 y_1 + p_2 y_2 + p_3 y_3 + \cdots + p_{m+2} y_{m+2} \in D(K, K_0)$$

with $p_i \in \mathbf{Z}$, $p_1 \geq p_2$, $p_3 \geq p_4 \geq \cdots \geq p_{m+2} \geq 0$,

$$\Lambda' = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 + q_5 y_5 + \cdots + q_{m+2} y_{m+2} \in D(K_2, K_0),$$

with $q_i \in \mathbf{Z}$, $q_1 \geq q_2 \geq 0$, $q_3 \geq q_4 \geq 0$, $q_5 \geq \cdots \geq q_{m+2} \geq 0$, $q_1 = p_1$, $q_2 = p_2$, and

$$\Lambda'' = k_1 y_1 + k_2 y_2 + k_3 y_3 + k_4 y_4 + k_5 y_5 + \cdots + k_{m+2} y_{m+2} \in D(K_1, K_0)$$

with $k_i \in \mathbf{Z}$, $k_i \geq 0$ for $1 \leq i \leq 4$, $k_5 \geq k_6 \geq \cdots \geq k_{m+2} \geq 0$, $k_j = q_j$ for $5 \leq j \leq m+2$, the corresponding eigenvalue of $-\mathcal{C}_L$ is expressed as follows:

$$\begin{aligned}
-c_L = & -2c_{\Lambda} + c_{\Lambda'} + \frac{1}{2} c_{\Lambda''} \\
= & 2\Big(\sum_{i=1}^{m+2} p_i^2 + 4p_1 + 2p_2 + 2mp_3 + (2m-2)p_4 + \cdots + 2p_{m+2}\Big) \\
& - \Big(\sum_{i=1}^{m+2} q_i^2 + 4q_1 + 2q_2 + 4q_3 + 2q_4 + (2m-4)q_5 + \cdots + 2q_{m+2}\Big) \\
& - \frac{1}{2}\Big(\sum_{i=1}^{m+2} k_i^2 + 2k_1 + 2k_2 + 2k_3 + 2k_4 + (2m-4)k_5 + \cdots + 2k_{m+2}\Big),
\end{aligned} \tag{10.2}$$

where $q_i = k_i$ for $5 \leq i \leq m+2$, $p_1 = q_1$, $p_2 = q_2$ and $k_1 = k_3$, $k_2 = k_4$.

Suppose that $\Lambda = (p_1, p_2, \cdots, p_{m+2}) = (2, 2, 0, \cdots, 0) \in D(K)$. Then by using the branching law of $(Sp(2), Sp(1) \times Sp(1))$ we see that $\Lambda \in D(K, K_0)$, $\Lambda' = (q_1, q_2, \cdots, q_{m+2}) = (2, 2, 0, \cdots, 0) \in D(K_2, K_0)$ and $\Lambda'' = (k_1, k_2, \cdots, k_{m+2}) = (0, 0, 0, \cdots, 0) \in D(K_1, K_0)$. Hence by (10.2) the corresponding eigenvalue is $-c_L = 20 < 8m - 2$ for $m \geq 3$. On the other hand, the representation of K with highest weight $\Lambda = (2, 2, 0, \cdots, 0)$ is a 14-dimensional irreducible representation $\rho_{\mathrm{Sym}_0^2(\mathbf{C}^5)} \boxtimes \mathrm{I}$ of $Sp(2) \times Sp(m)$, where $\rho_{\mathrm{Sym}_0^2(\mathbf{C}^5)}$ is the composition of the natural surjective homomorphism $Sp(2) \to SO(5)$ and the traceless symmetric product representation of $SO(5)$ on $\mathrm{Sym}_0^2(\mathbf{C}^5) := \{X \in M(5; \mathbf{C}) \mid X^t = X, \mathrm{tr} X = 0\}$. Here each $A \in SO(5)$ acts on $\mathrm{Sym}_0^2(\mathbf{C}^5)$ by $\mathrm{Sym}_0^2(\mathbf{C}^5) \ni X \mapsto AXA^t \in \mathrm{Sym}_0^2(\mathbf{C}^5)$. So

$$\begin{aligned}
\mathrm{Sym}_0(\mathbf{C}^5) = & \mathbf{C} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4} I_4 \end{pmatrix} \oplus \Big\{ \begin{pmatrix} 0 & 0 \\ 0 & X' \end{pmatrix} \mid X' \in \mathrm{Sym}_0(\mathbf{C}^4) \Big\} \\
& \oplus \Big\{ \begin{pmatrix} 0 & Z \\ Z^t & 0 \end{pmatrix} \mid Z \in M(1, 4; \mathbf{C}) \Big\} \\
= & \mathbf{C} \oplus \mathrm{Sym}_0(\mathbf{C}^4) \oplus \mathbf{C}^4
\end{aligned}$$

and

$$(\mathrm{Sym}_0(\mathbf{C}^5))_{SO(4)} = \mathbf{C} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4} I_4 \end{pmatrix} \cong \mathbf{C}.$$

Under the natural surjective homomorphism $Sp(2) (\subset SU(4)) \to SO(5)$, the element $\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \in Sp(2)$ corresponds to $\mathrm{diag}(-1, 1, -1, -1, -1) \in SO(5)$, denoted by Q'. By a direct computation, we know that $(\mathrm{Sym}_0(\mathbf{C}^5))_{Q'\cdot SO(4)} \cap (\mathrm{Sym}_0(\mathbf{C}^5))_{SO(4)} = (\mathrm{Sym}_0(\mathbf{C}^5))_{SO(4)}$. Thus,

$$(V_{\Lambda=(2,2,0,\cdots,0)})_{K_0} = \mathbf{C}\cdot\begin{pmatrix}1 & 0\\ 0 & -\frac{1}{4}I_4\end{pmatrix}\boxtimes\mathbf{C}$$

and moreover,

$$(V_{\Lambda=(2,2,0,\cdots,0)})_{K_{[\mathfrak{a}]}} = \mathbf{C}\cdot\begin{pmatrix}1 & 0\\ 0 & -\frac{1}{4}I_4\end{pmatrix}\boxtimes\mathbf{C}.$$

This means that $\Lambda = (2,2,0,\cdots,0) \in D(K, K_{[\mathfrak{a}]})$ has multiplicity 1, which corresponds to eigenvalue $20 < 8m-2$. Therefore, $L^{8m-2} \subset Q_{8m-2}(\mathbf{C})$ is not Hamiltonian stable.

From our results of this section we conclude

Theorem 10.1. *The Gauss image* $L = \frac{Sp(2)\times Sp(m)}{(Sp(1)\times Sp(1)\times Sp(m-2))\cdot \mathbf{Z}_4} \subset Q_{8m-2}(\mathbf{C})$ $(m \geq 2)$ *is not Hamiltonian stable if and only if* $m \geq 3$. *If* $m = 2$, *it is strictly Hamiltonian stable.*

11. The case $(U, K) = (E_6, U(1)\cdot Spin(10))$

In this case, $U = E_6$ and $K = U(1)\cdot Spin(10)$. Then (U, K) is of BC_2 type. For the sake of completeness, we first settle our notations following [42], [51], [22] and the references therein.

11.1. **Cayley algebra.** Let $\mathbf{K}$ be the real Cayley algebra and $\{c_0 = 1, c_1, \cdots, c_7\}$ the standard units of $\mathbf{K}$. They satisfy the following relations ([42]):

$$\begin{aligned} &c_i c_{i+1} = -c_{i+1}c_i = c_{i+3}, \quad c_{i+1}c_{i+3} = -c_{i+3}c_{i+1} = c_i,\\ &c_{i+3}c_i = -c_i c_{i+3} = c_{i+1}, \quad c_i^2 = -1 \text{ for } i \in \mathbf{Z}_7. \end{aligned}$$

$\mathbf{K}$ is a noncommutative and nonassociative normed division algebra with the conjugation $x \mapsto \bar{x}$ and the canonical inner product $(\ ,\ )$ defined respectively by

$$\overline{x_0 + \sum_{i=1}^{7} x_i c_i} = x_0 - \sum_{i=1}^{7} x_i c_i, \qquad (\sum_{i=0}^{7} x_i c_i, \sum_{i=0}^{7} y_i c_i) = \sum_{i=0}^{7} x_i y_i.$$

We extend the conjugation and the inner product $\mathbf{C}$-linearly to the complexified algebra $\mathbf{K}^{\mathbf{C}}$ of $\mathbf{K}$ and denote them by the same notions $x \mapsto \bar{x}$ and $(\ ,\ )$ respectively.

11.2. **Exceptional Jordan algebra.** The exceptional Jordan algebra $H_3(\mathbf{K})$ is defined as the set

$$H_3(\mathbf{K}) = \{u \in M_3(\mathbf{K}) | \bar{u}^t = u\},$$

with the Jordan product

$$u \circ v = \frac{1}{2}(uv + vu), \qquad \text{for} \quad u, v \in H_3(\mathbf{K}).$$

The real dimension of $H_3(\mathbf{K})$ is 27 and a typical element

$$u = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \xi_2 & x_1\\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \qquad \xi_i \in \mathbf{R}, x_i \in \mathbf{K} \tag{11.1}$$

of $H_3(\mathbf{K})$ will be denoted by

$$u = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + x_1 u_1 + x_2 u_2 + x_3 u_3.$$

In $H_3(\mathbf{K})$, we define the trace $\mathrm{tr}(u)$ and an inner product (u,v) respectively by

$$\mathrm{tr}(u) = \xi_1 + \xi_2 + \xi_3, \qquad (u,v) := \mathrm{tr}(u \circ v).$$

for each $u, v \in H_3(\mathbf{K})$. Moreover, the Freudenthal product $u \times v$ is defined by

$$u \times v := \frac{1}{2}(2u \circ v - \mathrm{tr}(u)v - \mathrm{tr}(v)u + (\mathrm{tr}(u)\mathrm{tr}(v) - (u,v))\mathrm{I}_3),$$

where I_3 is the 3-order identity matrix, and a trilinear form (u,v,w) and the determinant $\det u$ are defined respectively by

$$(u,v,w) = (u, v \times w), \quad \det u = \frac{1}{3}(u,u,u).$$

Put

$$SH_3(\mathbf{K}) = \{u \in M_3(\mathbf{K}) | \bar{u}^t = -u, \mathrm{tr}(u) = 0\}.$$

An element $u \in SH_3(\mathbf{K})$ of the form

$$u = \begin{pmatrix} z_1 & x_3 & -\bar{x}_2 \\ -\bar{x}_3 & z_2 & x_1 \\ x_2 & -\bar{x}_1 & z_3 \end{pmatrix}, \quad z_i, x_i \in \mathbf{K}, \bar{z}_i = -z_i, \Sigma z_i = 0 \tag{11.2}$$

is denoted by

$$u = z_1 e_1 + z_2 e_2 + z_3 e_3 + x_1 \bar{u}_1 + x_2 \bar{u}_2 + x_3 \bar{u}_3.$$

Now we define two injective linear maps $R : H_3(\mathbf{K}) \to \mathfrak{gl}(H_3(\mathbf{K}))$ and $D : SH_3(\mathbf{K}) \to \mathfrak{gl}(H_3(\mathbf{K}))$ respectively by

$$\begin{aligned} R(u)v &= u \circ v = \frac{1}{2}(uv + vu), \quad \text{for } u, v \in H_3(\mathbf{K}), \\ D(u)v &= \frac{1}{2}[u,v] = \frac{1}{2}(uv - vu), \quad \text{for } u \in SH_3(\mathbf{K}), v \in H_3(\mathbf{K}). \end{aligned} \tag{11.3}$$

Denote by $\mathfrak{D}$ and $\mathfrak{R}$ the images of D and R in $\mathfrak{gl}(H_3(\mathbf{K}))$. Introduce some subspaces of $\mathfrak{D}$ and $\mathfrak{R}$ in the following:

$$\begin{aligned} \mathfrak{D}_0 &= \{\delta \in \mathfrak{D} | \delta(e_i) = 0 \, (i = 1,2,3)\}, \\ \mathfrak{D}_i &= \{D(x\bar{u}_i) | x \in \mathbf{K}\} \quad \text{for } i = 1,2,3, \\ \mathfrak{R}_0 &= \{R(\textstyle\sum \xi_i e_i) | \xi_i \in \mathbf{R}, \sum \xi_i = 0\}, \\ \mathfrak{R}_i &= \{R(x u_i) | x \in \mathbf{K}\} \quad \text{for } i = 1,2,3. \end{aligned}$$

Remark that $\dim \mathfrak{D}_0 = 28$, $\dim \mathfrak{D}_1 = \dim \mathfrak{D}_2 = \dim \mathfrak{D}_3 = 8$, $\dim \mathfrak{R}_0 = 2$ and $\dim \mathfrak{R}_1 = \dim \mathfrak{R}_2 = \dim \mathfrak{R}_3 = 8$. Moreover, it is easy to know that $\mathfrak{D}_0$ is a subalgebra of $\mathfrak{gl}(H_3(\mathbf{K}))$ generated by the set $\{D(\Sigma z_i e_i) | z_i \in \mathbf{K}, \bar{z}_i = -z_i, \Sigma z_i = 0\}$. In fact, $\mathfrak{D}_0$ is isomorphic to the Lie algebra $\mathfrak{o}(8)$ and its basis can be chosen as $\{D_{i,r} (1 \le r \le 7), D_{i,pq} (1 \le p < q \le 7)\}$ for $i = 1, 2$ or 3 ([11], [22], [51]). We now explain in details by using Ise's notions ([22], p.82). Put

$$D_{i,r} = D(c_r(-e_j + e_k)), \quad (1 \le i \le 3, 1 \le r \le 7),$$

and

$$D_{i,pq} = [D_{i,p}, D_{i,q}], \quad (1 \le i \le 3, 1 \le p, q \le 7), \tag{11.4}$$

where $\{i,j,k\}$ is a cyclic permutation of $\{1,2,3\}$. Then from

$$D(\sum_{i=1}^{3} z_i e_i)(v) = \frac{1}{2} \sum_{\{i,j,k\}} (z_j x_i - x_i z_k) u_i,$$

we can obtain

$$\begin{cases} D_{i,r}(xu_i) = \begin{cases} -c_r u_i, & \text{if } x = c_0 \\ c_0 u_i, & \text{if } x = c_r \\ 0, & \text{if } x = c_q (q \neq r), \end{cases} \\ \\ D_{i,r}(xu_j) = \frac{1}{2}(c_r x) u_j, \\ \\ D_{i,r}(xu_k) = \frac{1}{2}(x c_r) u_k, \end{cases}$$

and

$$\begin{cases} D_{i,pq}(xu_i) = \begin{cases} c_q u_i, & \text{if } x = c_p \\ -c_p u_i, & \text{if } x = c_q \\ 0, & \text{if } x = c_r (r \leq 0, \neq p, q), \end{cases} \\ \\ D_{i,pq}(xu_j) = \frac{1}{2}\{c_p(c_q x)\} u_j, \\ \\ D_{i,pq}(xu_k) = \frac{1}{2}\{(x c_q) c_p\} u_k. \end{cases}$$

These mean that every $D_{i,r}, D_{i,pq}$ leave $\mathfrak{T}_i = \{xu_i | x \in \mathbf{K}\}$ invariant $(1 \leq i \leq 3, 1 \leq p, q, r \leq 7)$ and identifying $\mathfrak{T}_i$ with $\mathbf{K}$, it represents a skew-symmetric matrix with respect to the basis $\{c_0, c_1, \cdots, c_7\}$; namely $D_{i,r} = E_{0r} - E_{r0}$ and $D_{i,pq} = E_{qp} - E_{pq}$, where E_{pq} denotes the 8×8 matrix with all 0-components except (p,q)-component, 1. Moreover,

$$[D_{i,r}, D_{i,pq}] = D_{i,p}\delta_{qr} - D_{i,q}\delta_{rp}, \tag{11.5}$$

$$[D_{i,pq}, D_{i,rs}] = D_{i,pr}\delta_{sq} + D_{i,qs}\delta_{pr} + D_{i,rq}\delta_{sp} + D_{i,sp}\delta_{rq}, \tag{11.6}$$

where $1 \leq i \leq 3$ and $1 \leq p, q, r, s \leq 7$. Particulary, we have

$$[D_{i,r}, D_{i,pq}] = 0, \quad [D_{i,pq}, D_{i,rs}] = 0,$$

if p, q, r, s are all different each other. Denote the real linear space spanned by all $D_{i,r}, D_{i,pq}$ $(1 \leq p, q, r \leq 7)$ by $\mathfrak{D}_{i,0}$. Then all $\mathfrak{D}_{i,0} (1 \leq i \leq 3)$ are isomorphic to each other, and they are isomorphic to the Lie algebra $\mathfrak{o}(8)$. We shall use $\mathfrak{D}_0 = \mathfrak{D}_{1,0}$ in the next.

Let

$$H_3(\mathbf{K})^{\mathbf{C}} := H_3(\mathbf{K}) + \sqrt{-1} H_3(\mathbf{K})$$

be the complexification of $H_3(\mathbf{K})$. Then there are two complex conjugations on $H_3(\mathbf{K})^{\mathbf{C}}$, namely,

$$\overline{u_1 + \sqrt{-1} u_2} = \bar{u}_1 + \sqrt{-1}\bar{u}_2, \quad \tau(u_1 + \sqrt{-1} u_2) = u_1 - \sqrt{-1} u_2,$$

where $u_1, u_2 \in H_3(\mathbf{K})$. Then $H_3(\mathbf{K})^{\mathbf{C}}$ is canonically identified with

$$H_3(\mathbf{K}^{\mathbf{C}}) = \{u \in M_3(\mathbf{K}^{\mathbf{C}}) | \bar{u}^t = u\}.$$

An element $u \in H_3(\mathbf{K}^{\mathbf{C}})$ of the form (11.1), with $\xi_i \in \mathbf{C}$, $x_i \in \mathbf{K}^{\mathbf{C}}$, is still denoted by $u = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + x_1 u_1 + x_2 u_2 + x_3 u_3$. The standard Hermitian inner product $\langle \ , \ \rangle$ of $H_3(\mathbf{K}^{\mathbf{C}})$ is defined by

$$\langle u, v \rangle := (\tau u, v).$$

Meanwhile, the complexification $SH_3(\mathbf{K})^{\mathbf{C}}$ of $SH_3(\mathbf{K})$ is identified with

$$SH_3(\mathbf{K}^{\mathbf{C}}) = \{u \in M_3(\mathbf{K}^{\mathbf{C}}) | \bar{u}^t = -u, \mathrm{tr}(u) = 0\},$$

whose element u of the form (11.2), with $z_i, x_i \in \mathbf{K}^{\mathbf{C}}$, is still denoted by $u = z_1 e_1 + z_2 e_2 + z_3 e_3 + x_1 \bar{u}_1 + x_2 \bar{u}_2 + x_3 \bar{u}_3$. Then $D(u) \in \mathfrak{gl}(H_3(\mathbf{K}^{\mathbf{C}}))$ for $u \in SH_3(\mathbf{K}^{\mathbf{C}})$ and $R(u) \in \mathfrak{gl}(H_3(\mathbf{K}^{\mathbf{C}}))$ for $u \in H_3(\mathbf{K}^{\mathbf{C}})$ can be defined by the same formula as (11.3).

11.3. **Basic formulas in $\mathfrak{e}_6$.**

Lemma 11.1. *For $v = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + x_1 u_1 + x_2 u_2 + x_3 u_3 \in H_3(\mathbf{K}^{\mathbf{C}})$, we have*

$$\begin{aligned}
R(\sum \eta_l e_l) v &= \eta_1 \xi_1 e_1 + \eta_2 \xi_2 e_2 + \eta_3 \xi_3 e_3 + \frac{1}{2}(\eta_2 + \eta_3) x_1 u_1 \\
&\quad + \frac{1}{2}(\eta_3 + \eta_1) x_2 u_2 + \frac{1}{2}(\eta_1 + \eta_2) x_3 u_3, \\
D(\sum z_l e_l) v &= \frac{1}{2}(z_2 x_1 - x_1 z_3) u_1 + \frac{1}{2}(z_3 x_2 - x_2 z_1) u_2 + \frac{1}{2}(z_1 x_3 - x_3 z_2) u_3, \\
D(x \bar{u}_i) v &= (x, x_i)(e_j - e_k) + \frac{1}{2}(\xi_k - \xi_j) x u_i - \frac{1}{2}(\bar{x} \bar{x}_k) u_j + \frac{1}{2}(\bar{x}_j \bar{x}) u_k, \\
R(x u_i) v &= (x, x_i)(e_j + e_k) + \frac{1}{2}(\xi_j + \xi_k) x u_i + \frac{1}{2} \bar{x} \bar{x}_k u_j + \frac{1}{2} \bar{x}_j \bar{x} u_k,
\end{aligned}$$

where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$.

The relations (11.4), (11.5), (11.6) and the following list give commutation rules for $\mathfrak{e}_6^{\mathbf{C}}$. Here, $x, y, z_i \in \mathbf{K}^{\mathbf{C}}$, $\bar{z}_i = -z_i$ for $i = 1, 2, 3$, $\sum_i z_i = 0$, and $\xi_1, \xi_2, \xi_3 \in \mathbf{C}$ with $\sum_l \xi_l = 0$. In formulae (11.7) (11.15), (i, j, k) is a cyclic permutation of $(1, 2, 3)$. In formulae (11.17) and (11.18), $i = 1, 2, 3$.

$$[R(xu_i), R(yu_j)] = -(1/2) D(\overline{xy}\, \bar{u}_k), \tag{11.7}$$

$$[R(xu_i), D(yu_j)] = [D(x\bar{u}_i), R(yu_j)] = (1/2) R(\overline{xy}\, \bar{u}_k), \tag{11.8}$$

$$[D(x\bar{u}_i), D(y\bar{u}_j)] = -(1/2) D(\overline{xy} \bar{u}_k), \tag{11.9}$$

$$[D(x\bar{u}_i), R(y\bar{u}_i)] = (x, y) R(e_j - e_k), \tag{11.10}$$

$$\Big[R(\sum \xi_l e_l), R(x\bar{u}_i)\Big] = (1/2)(\xi_j - \xi_k) D(x\bar{u}_i), \tag{11.11}$$

$$\Big[R(\sum \xi_l e_l), D(x\bar{u}_i)\Big] = (1/2)(\xi_j - \xi_k) R(x\bar{u}_i), \tag{11.12}$$

$$\Big[D(\sum z_l e_l), D(x\bar{u}_i)\Big] = (1/2) D((z_j x - x z_k)\bar{u}_i), \tag{11.13}$$

$$\Big[D(\sum z_l e_l), R(x\bar{u}_i)\Big] = (1/2) R((z_j x - x z_k) u_i), \tag{11.14}$$

$$\begin{aligned}
[R(xu_i), R(yu_i)] &= -[D(x\bar{u}_i), D(y\bar{u}_i)] \\
&= D((\frac{y+\bar{y}}{2}\frac{x-\bar{x}}{2} - \frac{x+\bar{x}}{2}\frac{y-\bar{y}}{2})(e_j - e_k)) \\
&\quad -[D(\frac{x-\bar{x}}{2}(e_j - e_k)), D(\frac{y-\bar{y}}{2}(e_j - e_k))],
\end{aligned} \tag{11.15}$$

$$\big[\mathfrak{R}_0^{\mathbf{C}}, \mathfrak{R}_0^{\mathbf{C}} + \mathfrak{D}_0^{\mathbf{C}}\big] = \{0\}, \tag{11.16}$$

$$[R(xu_i), [R(xu_i), R(yu_i)]] = R(((x, x)y - (x, y)x)u_i), \tag{11.17}$$

$$(11.18) \qquad [D(x\bar{u}_i), [D(x\bar{u}_i), D(y\bar{u}_i)]] = D(((x,y)x - (x,x)y)\bar{u}_i).$$

Remark that the Killing-Cartan form B of $\mathfrak{e}_6^{\mathbf{C}}$ is given by ([22], p.88 or [51], p.74)

$$(11.19) \qquad B(u,v) = 4\mathrm{tr}(uv),$$

for each $u, v \in \mathfrak{e}_6^{\mathbf{C}} \subset \mathfrak{gl}(H_3(\mathbf{K})^{\mathbf{C}})$.

11.4. **Realization of $E_6/(U(1) \cdot Spin(10))$.** We recall and combine the settings given in [51] and [22]. It is known that F_4 is defined to be the automorphism group of the Jordan algebra $H_3(\mathbf{K})$:

$$\begin{aligned} F_4 &:= \{\alpha \in GL(H_3(\mathbf{K})) \mid \alpha(u \circ v) = \alpha u \circ \alpha v\} \\ &= \{\alpha \in GL(H_3(\mathbf{K})) \mid \det(\alpha u) = \det u, (\alpha u, \alpha v) = (u,v)\}. \end{aligned}$$

Its Lie algebra $\mathfrak{f}_4$ is thus given by

$$\mathfrak{f}_4 := \{\delta \in \mathfrak{gl}(H_3(\mathbf{K})) \mid \delta(u \circ v) = \delta u \circ v + u \circ \delta v\},$$

which is isomorphic to $\mathfrak{D} = \mathfrak{D}_0 + \mathfrak{D}_1 + \mathfrak{D}_2 + \mathfrak{D}_3$. We are interested in the following two subgroups of F_4:

$$\begin{aligned} (F_4)_{e_1} &:= \{\alpha \in F_4 \mid \alpha e_1 = e_1\} \cong Spin(9), \\ (F_4)_{e_1,e_2,e_3} &:= \{\alpha \in F_4 \mid \alpha e_i = e_i \ (i = 1,2,3)\} \cong Spin(8). \end{aligned}$$

Note that the Lie algebra of $(F_4)_{e_1,e_2,e_3}$ is $\mathfrak{D}_0$.

The groups $E_6^{\mathbf{C}}$ and E_6 are defined by

$$\begin{aligned} E_6^{\mathbf{C}} &:= \{\alpha \in GL_{\mathbf{C}}(H_3(\mathbf{K})^{\mathbf{C}}) \mid \det(\alpha u) = \det(u)\}, \\ E_6 &:= \{\alpha \in GL_{\mathbf{C}}(H_3(\mathbf{K})^{\mathbf{C}}) \mid \det(\alpha u) = \det u, \langle \alpha u, \alpha v\rangle = \langle u, v\rangle\}, \end{aligned}$$

respectively. Hence F_4 is a subgroup of E_6. The Lie algebras of $E_6^{\mathbf{C}}$ and E_6 are given respectively by

$$\begin{aligned} \mathfrak{e}_6^{\mathbf{C}} &:= \{\phi \in \mathfrak{gl}_{\mathbf{C}}(H_3(\mathbf{K})^{\mathbf{C}}) \mid (\phi u, u, u) = 0\}, \\ \mathfrak{e}_6 &:= \{\phi \in \mathfrak{gl}_{\mathbf{C}}(H_3(\mathbf{K})^{\mathbf{C}}) \mid (\phi u, u, u) = 0, \langle \phi u, v\rangle + \langle u, \phi v\rangle = 0\}. \end{aligned}$$

It can be shown ([51], p.68) that any element $\phi \in \mathfrak{e}_6^{\mathbf{C}}$ is uniquely expressed as

$$\phi = \delta + \varsigma, \quad \delta \in \mathfrak{D}^{\mathbf{C}}, \quad \varsigma \in \mathfrak{R}^{\mathbf{C}},$$

where $\mathfrak{D}^{\mathbf{C}}$ and $\mathfrak{R}^{\mathbf{C}}$ denote the complexifications of $\mathfrak{D}$ and $\mathfrak{R}$ respectively. So we get the so-called Chevalley-Schafer model ([11]) of $\mathfrak{e}_6^{\mathbf{C}}$: $\mathfrak{e}_6^{\mathbf{C}} = \mathfrak{D}^{\mathbf{C}} + \mathfrak{R}^{\mathbf{C}}$ as a subalgebra of $\mathfrak{gl}(H_3(\mathbf{K})^{\mathbf{C}})$. The inclusion $\phi : \mathfrak{e}_6^{\mathbf{C}} \subset \mathfrak{gl}(H_3(\mathbf{K})^{\mathbf{C}})$ is a 27-dimensional irreducible representation of $\mathfrak{e}_6^{\mathbf{C}}$. Furthermore, any element $\phi \in \mathfrak{e}_6$ is uniquely expressed as

$$\phi = \delta + \sqrt{-1}\varsigma, \quad \delta \in \mathfrak{D}, \quad \varsigma \in \mathfrak{R}.$$

Equivalently, $\mathfrak{e}_6 := \mathfrak{D} + \sqrt{-1}\mathfrak{R}$.

Consider a $\mathbf{C}$-linear transformation σ of $H_3(\mathbf{K})^{\mathbf{C}}$ defined by

$$\sigma \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}.$$

Then $\sigma \in E_6$ and $\sigma^2 = 1$. σ induces an involutive automorphism of E_6 by $\alpha \mapsto \sigma\alpha\sigma$, which we also still denote it by σ. In order to investigate the subgroup $(E_6)^\sigma$ of all fixed points of σ,

$$(E_6)^\sigma = \{\alpha \in E_6 \mid \sigma\alpha = \alpha\}, \tag{11.20}$$

consider two subgroups

$$(E_6)_{e_1} = \{\alpha \in E_6 \mid \alpha e_1 = e_1\} \cong Spin(10)$$

and

$$U(1) = \{\phi(\theta) \in GL_{\mathbf{C}}(H_3(\mathbf{K})^{\mathbf{C}}) \mid \theta = e^{\sqrt{-1}t/2}, t \in \mathbf{R}\}, \tag{11.21}$$

where $\phi(\theta) := \exp(t\sqrt{-1}R(2e_1 - e_2 - e_3)) \in GL_{\mathbf{C}}(H_3(\mathbf{K})^{\mathbf{C}})$ and

$$\phi(\theta)\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \theta^4\xi_1 & \theta x_3 & \theta\bar{x}_2 \\ \theta\bar{x}_3 & \theta^{-2}\xi_2 & \theta^{-2}x_1 \\ \theta x_2 & \theta^{-2}\bar{x}_1 & \theta^{-2}\xi_3 \end{pmatrix}. \tag{11.22}$$

Here the subgroups $U(1)$ and $Spin(10)$ of $(E_6)^\sigma$ are elementwise commutative. Define a mapping

$$p : \tilde{K} = U(1) \times Spin(10) \ni (\theta, \alpha) \longmapsto \phi(\theta)\alpha \in K = (E)^\sigma,$$

which is a surjective Lie group homomorphism. Since

$$U(1) \cap Spin(10) = \{1 = \phi(1), \phi(-1), \phi(\sqrt{-1}), \phi(-\sqrt{-1})\},$$

we have $\mathrm{Ker}(p) = \{(1, \phi(1)), (-1, \phi(-1)), (\sqrt{-1}, \phi(-\sqrt{-1})), (-\sqrt{-1}, \phi(\sqrt{-1}))\}$, which is isomorphic to $\mathbf{Z}_4$. Thus

$$K = (E_6)^\sigma = \tilde{K}/\mathbf{Z}_4 = (U(1) \times Spin(10))/\mathbf{Z}_4,$$

and $U/K = E_6/(U(1) \cdot Spin(10))$. Correspondingly,

$$\begin{aligned} \mathfrak{k} = (\mathfrak{e}_6)_\sigma &= \{\phi \in \mathfrak{e}_6 \mid \sigma_*\phi = \phi\} \\ &= (\mathfrak{e}_6)_{e_1} + \mathbf{R}\sqrt{-1}R(2e_1 - e_2 - e_3). \end{aligned}$$

Since for any $\phi \in \mathfrak{e}_6$ there exist $u \in SH_3(\mathbf{K})$ and $v \in H_3(\mathbf{K})$ such that

$$\phi e_1 = D(u)(e_1) + \sqrt{-1}R(v)(e_1),$$

it holds that $\phi e_1 = 0$ if and only if

$$u = z_1e_1 + z_2e_2 + z_3e_3 + a_1\bar{u}_1 \in SH_3(\mathbf{K}),\ v = \xi_2e_2 + \xi_3e_3 + x_1u_1 \in H_3(\mathbf{K}),$$

where $a_1, x_1 \in \mathbf{K}$, $\xi_2, \xi_3 \in \mathbf{R}$ with $\xi_2 + \xi_3 = 0$. Hence

$$\begin{aligned} (\mathfrak{e}_6)_{e_1} &:= \{\phi \in \mathfrak{e}_6 \mid \phi e_1 = 0\} \\ &= \mathfrak{D}_0 + \mathfrak{D}_1 + \mathbf{R}\sqrt{-1}R(e_2 - e_3) + \sqrt{-1}\mathfrak{R}_1 \\ &\cong \mathfrak{o}(10) \end{aligned}$$

Therefore, we have the Cartan decomposition of a compact simple Lie algebra $\mathfrak{u} = \mathfrak{e}_6$ of type EIII:

$$\begin{aligned} \mathfrak{u} &= \mathfrak{e}_6 = \mathfrak{D} + \sqrt{-1}\mathfrak{R}, \\ \mathfrak{k} &= (\mathfrak{e}_6)_\sigma = \mathfrak{D}_0 + \mathfrak{D}_1 + \sqrt{-1}\mathfrak{R}_0 + \sqrt{-1}\mathfrak{R}_1, \\ \mathfrak{p} &= (\mathfrak{e})_{-\sigma} = \mathfrak{D}_2 + \mathfrak{D}_3 + \sqrt{-1}\mathfrak{R}_2 + \sqrt{-1}\mathfrak{R}_3, \end{aligned}$$

where $\mathfrak{k}$ is isomorphic to $\mathfrak{u}(1)+\mathfrak{o}(10)$,

$$[\mathfrak{k},\mathfrak{k}]=\mathfrak{D}_0+\mathfrak{D}_1+\sqrt{-1}\mathbf{R}R(e_2-e_3)+\sqrt{-1}\mathfrak{R}_1=(\mathfrak{e}_6)_{e_1}$$

is isomorphic to $\mathfrak{o}(10)$ and the center of $\mathfrak{k}$ is spanned by

$$Z=\sqrt{-1}R(2e_1-e_2-e_3).$$

On the other hand, a compact Hermitian symmetric space of type EIII can be defined by ([1, p74-75])

$$\mathrm{EIII}=\{u\in H_3(\mathbf{K})^{\mathbf{C}}\mid u\times u=0,u\neq 0\}/\mathbf{C}^*\subset P(H_3(\mathbf{K})^{\mathbf{C}}),$$

which is considered as a compact complex submanifold embedded in a complex projective space $\mathbf{C}P^{26}$. Since E_6 acts transitively on EIII and the isotropy subgroup of E_6 at $o=[e_1]$ is $(E_6)^\sigma$, we know that $\mathrm{EIII}\cong E_6/(E_6)^\sigma=E_6/(U(1)\cdot Spin(10))$. The tangent vector space $T_o(U/K)$ at o can be identified with a vector subspace

$$\begin{aligned}T_o(\mathrm{EIII})&\cong\{u\in H_3(\mathbf{K})^{\mathbf{C}}\mid u\times e_1=0,\langle u,e_1\rangle=0\}\\&=\{x_2u_2+x_3u_3\mid x_2,x_3\in\mathbf{K}^{\mathbf{C}}\}.\end{aligned}$$

The differential of the natural projection $p:U=E_6\to U/K=\mathrm{EIII}$ induces a linear isomorphism $p_*:\mathfrak{p}\to T_o(\mathrm{EIII})$. Then $p_*(\phi)=\phi(e_1)$ and

$$\begin{aligned}&p_*\Big(2(D(x_2\bar{u}_2)-D(x_3\bar{u}_3))+2\sqrt{-1}(R(x_2'u_2)+R(x_3'u_3))\Big)\\&=(x_2+\sqrt{-1}x_2')u_2+(x_3+\sqrt{-1}x_3')u_3.\end{aligned}\tag{11.23}$$

11.5. **Restricted root systems of EIII.** Define $H_1,H_2\in\mathfrak{p}$ by

$$\begin{aligned}H_1&=D\bar{u}_2+\sqrt{-1}R(c_4u_2),\\H_2&=D\bar{u}_2-\sqrt{-1}R(c_4u_2).\end{aligned}$$

Then by (11.10), $[H_1,H_2]=0$. Hence

$$\mathfrak{a}=\{H(\xi_1,\xi_2)=\xi_1H_1+\xi_2H_2|\xi_1,\xi_2\in\mathbf{R}\}\tag{11.24}$$

is a maximal abelian subalgebra in $\mathfrak{p}$. Remark that this maximal abelian subalgebra $\mathfrak{a}$ is different from the one given by M. Ise and used in [42]. Then by direct computations using (11.4)-(11.18), we get the following restricted root system decomposition of $\mathfrak{k}$ and $\mathfrak{p}$:

$$\begin{aligned}\mathfrak{k}&=\mathfrak{k}_0+\mathfrak{k}_{2\xi_1}+\mathfrak{k}_{2\xi_2}+\mathfrak{k}_{\xi_1+\xi_2}+\mathfrak{k}_{\xi_1-\xi_2}+\mathfrak{k}_{\xi_1}+\mathfrak{k}_{\xi_2},\\\mathfrak{p}&=\mathfrak{a}+\mathfrak{p}_{2\xi_1}+\mathfrak{p}_{2\xi_2}+\mathfrak{p}_{\xi_1+\xi_2}+\mathfrak{p}_{\xi_1-\xi_2}+\mathfrak{p}_{\xi_1}+\mathfrak{p}_{\xi_2},\end{aligned}$$

where

$$\begin{aligned}\mathfrak{k}_0=&\{X\in\mathfrak{k}\mid[X,H]=0\quad\text{for each }H\in\mathfrak{a}\},\\=&\operatorname{span}_{\mathbf{R}}\{\sqrt{-1}R(e_1-2e_2+e_3)\}+\operatorname{span}_{\mathbf{R}}\{-D_{1,4}+D_{1,12},D_{1,12}+D_{1,36},\\&D_{1,36}+D_{1,57},-D_{1,1}+D_{1,24},-D_{1,2}-D_{1,14},-D_{1,3}+D_{1,46},-D_{1,5}-D_{1,47},\\&-D_{1,6}+D_{1,34},-D_{1,7}+D_{1,45},D_{1,13}-D_{1,26},D_{1,15}+D_{1,27},D_{1,16}+D_{1,23},\\&D_{1,17}-D_{1,25},D_{1,35}-D_{1,67},D_{1,37}-D_{1,56}\},\end{aligned}$$

$$
\begin{aligned}
\mathfrak{k}_{2\xi_1} &= \mathrm{span}_{\mathbf{R}}\{\frac{1}{2}(-D_{1,4} - D_{1,12} + D_{1,36} - D_{1,57}) + \sqrt{-1}R(e_3 - e_1)\},\\
\mathfrak{k}_{2\xi_2} &= \mathrm{span}_{\mathbf{R}}\{\frac{1}{2}(D_{1,4} + D_{1,12} - D_{1,36} + D_{1,57}) + \sqrt{-1}R(e_3 - e_1)\},
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{k}_{\xi_1+\xi_2} = \mathrm{span}_{\mathbf{R}}\{&-D_{1,1} - D_{1,24} - D_{1,37} - D_{1,56} = 2D_{2,1},\\
&- D_{1,2} + D_{1,14} - D_{1,35} - D_{1,67} = 2D_{2,2},\\
&- D_{1,3} - D_{1,46} + D_{1,17} + D_{1,25} = 2D_{2,3},\\
&- D_{1,5} + D_{1,47} + D_{1,16} - D_{1,23} = 2D_{2,5},\\
&- D_{1,6} - D_{1,34} - D_{1,15} + D_{1,27} = 2D_{2,6},\\
&- D_{1,7} - D_{1,45} - D_{1,13} - D_{1,26} = 2D_{2,7}\},
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{k}_{\xi_1-\xi_2} = \mathrm{span}_{\mathbf{R}}\{&-D_{1,1} - D_{1,24} + D_{1,37} + D_{1,56} = 2D_{2,24},\\
&- D_{1,2} + D_{1,14} + D_{1,35} + D_{1,67} = 2D_{2,14},\\
&- D_{1,3} - D_{1,46} - D_{1,25} - D_{1,17} = -2D_{2,46},\\
&- D_{1,5} + D_{1,47} - D_{1,16} + D_{1,23} = 2D_{2,47},\\
&- D_{1,6} - D_{1,34} + D_{1,15} - D_{1,27} = -2D_{2,34},\\
&- D_{1,7} - D_{1,45} + D_{1,13} + D_{1,26} = -2D_{2,45}\},
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{k}_{\xi_1} &= \mathrm{span}_{\mathbf{R}}\{D(x_1\bar{u}_1) + \sqrt{-1}R(y_1u_1),\ (x_1, y_1) = (1, c_4), (c_1, -c_2), (c_2, c_1),\\
&\quad (c_3, c_6), (c_4, -1), (c_5, -c_7), (c_6, -c_3), (c_7, c_5)\},\\
\mathfrak{k}_{\xi_2} &= \mathrm{span}_{\mathbf{R}}\{D(x_1\bar{u}_1) + \sqrt{-1}R(y_1u_1),\ (x_1, y_1) = (1, -c_4), (c_1, c_2), (c_2, -c_1),\\
&\quad (c_3, -c_6), (c_4, 1), (c_5, c_7), (c_6, c_3), (c_7, -c_5)\},
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{p}_{2\xi_1} &= \mathrm{span}_{\mathbf{R}}\{D(c_4\bar{u}_2) - \sqrt{-1}Ru_2\},\\
\mathfrak{p}_{2\xi_2} &= \mathrm{span}_{\mathbf{R}}\{D(c_4\bar{u}_2) + \sqrt{-1}Ru_2\},\\
\mathfrak{p}_{\xi_1+\xi_2} &= \mathrm{span}_{\mathbf{R}}\{D(c_i\bar{u}_2),\ i = 1, 2, 3, 5, 6, 7\},\\
\mathfrak{p}_{\xi_1-\xi_2} &= \mathrm{span}_{\mathbf{R}}\{\sqrt{-1}R(c_iu_2),\ i = 1, 2, 3, 5, 6, 7\},\\
\mathfrak{p}_{\xi_1} &= \mathrm{span}_{\mathbf{R}}\{D(x_3\bar{u}_3) + \sqrt{-1}R(y_3u_3),\ (x_3, y_3) = (1, c_4), (c_1, c_2), (c_2, c_1),\\
&\quad (c_3, c_6), (c_4, -1), (c_5, -c_7), (c_6, -c_3), (c_7, c_5)\}\\
\mathfrak{p}_{\xi_2} &= \mathrm{span}_{\mathbf{R}}\{D(x_3\bar{u}_3) + \sqrt{-1}R(y_3u_3),\ (x_3, y_3) = (1, -c_4), (c_1, -c_2), (c_2, c_1),\\
&\quad (c_3, c_6), (c_4, 1), (c_5, -c_7), (c_6, -c_3), (c_7, c_5)\}.
\end{aligned}
$$

Thus we see that

$$
\begin{aligned}
&\mathfrak{k}_0 = \mathfrak{k}_0' + \mathfrak{c}(\mathfrak{k}_0) = \mathfrak{k}_0' + \mathbf{R}\sqrt{-1}R(e_1 - 2e_2 + e_3) \cong \mathfrak{so}(6) + \mathbf{R},\\
&\mathfrak{k}_1 := \mathfrak{k}_0 + \mathfrak{k}_{2\xi_1} + \mathfrak{k}_{2\xi_2}\\
&\quad = \mathfrak{k}_0' + \mathbf{R}\sqrt{-1}R(e_1 - 2e_2 + e_3) + \mathbf{R}(D_{1,4} + D_{1,12} - D_{1,36} + D_{1,57})\\
&\qquad\qquad + \mathbf{R}\sqrt{-1}R(e_3 - e_1)\\
&\quad \cong \mathfrak{so}(6) + \mathbf{R} + \mathbf{R} + \mathbf{R},
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{k}_2 &:= \mathfrak{k}_1 + \mathfrak{k}_{\xi_1+\xi_2} + \mathfrak{k}_{\xi_1-\xi_2} = \mathfrak{D}_0 + \sqrt{-1}\mathfrak{R}_0 \\
&= \mathfrak{D}_0 + \mathbf{R}\sqrt{-1}R(e_1 - 2e_2 + e_3) + \mathbf{R}\sqrt{-1}R(e_3 - e_1) \\
&= \mathfrak{D}_0 + \mathbf{R}\sqrt{-1}R(e_2 - e_3) + \mathbf{R}\sqrt{-1}R(2e_1 - e_2 - e_3) \\
&\cong \mathfrak{so}(8) + \mathbf{R} + \mathbf{R}, \\
\mathfrak{k} &:= \mathfrak{k}_2 + \mathfrak{k}_{\xi_1} + \mathfrak{k}_{\xi_2} = \mathfrak{D}_0 + \sqrt{-1}\mathfrak{R}_0 + \mathfrak{D}_1 + \sqrt{-1}\mathfrak{R}_1 \\
&= (\mathfrak{D}_0 + \mathfrak{D}_1 + \sqrt{-1}\mathfrak{R}_1 + \mathbf{R}\sqrt{-1}R(e_2 - e_3)) + \mathbf{R}\sqrt{-1}R(2e_1 - e_2 - e_3) \\
&= \mathfrak{k}' + \mathfrak{c}(\mathfrak{k}) \cong \mathfrak{so}(10) + \mathbf{R}.
\end{aligned}
$$

Correspondingly, consider the subgroup

$$\tilde{K}_2 = U(1) \times Spin(2) \times Spin(8) \subset \tilde{K} = U(1) \times Spin(10),$$

where $U(1)$ is given by (11.21), $Spin(2) \subset Spin(10) \cong (E_6)_{e_1}$ is generated by

$$
\alpha_{23}(t) := \exp(t\sqrt{-1}R(e_2 - e_3)) :
$$
$$
\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 & e^{\frac{t\sqrt{-1}}{2}}x_3 & e^{-\frac{t\sqrt{-1}}{2}}\bar{x}_2 \\ e^{\frac{t\sqrt{-1}}{2}}\bar{x}_3 & e^{t\sqrt{-1}}\xi_2 & x_1 \\ e^{-\frac{t\sqrt{-1}}{2}}x_2 & \bar{x}_1 & e^{-t\sqrt{-1}}\xi_3 \end{pmatrix},
$$

and $Spin(8) = (E_6)_{e_1,e_2,e_3}$ whose Lie algebra is just $\mathfrak{D}_0$. Therefore,

$$Spin(2) \cap Spin(8) = \{\alpha_{23}(t) \mid e^{t\sqrt{-1}} = 1\} = \{\alpha_{23}(0), \alpha_{23}(2\pi)\}.$$

Then the natural projection

$$
\begin{aligned}
p_2 : \quad & Spin(2) \times Spin(8) \to K_2' \\
& (\alpha_{23}(t), \beta) \mapsto \alpha_{23}(t)\beta
\end{aligned}
$$

has a kernel

$$
\begin{aligned}
\ker p_2 &= \{(\alpha_{23}(t), \alpha_{23}(t)^{-1}) \mid t = 2k\pi, k \in \mathbf{Z}\} \\
&= \{(\alpha_{23}(0), \alpha_{23}(0)), (\alpha_{23}(2\pi), \alpha_{23}(2\pi))\} \cong \mathbf{Z}_2.
\end{aligned}
$$

Hence $K_2' \cong (Spin(2) \times Spin(8))/\mathbf{Z}_2$.

On the other hand, we also have

$$\tilde{K}_2 = S^1 \times Spin(2) \times Spin(8),$$

where S^1 is generated by

$$
\exp(t\sqrt{-1}R(e_1 - 2e_2 + e_3)) :
$$
$$
\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} e^{t\sqrt{-1}}\xi_1 & e^{-\frac{t\sqrt{-1}}{2}}x_3 & e^{t\sqrt{-1}}\bar{x}_2 \\ e^{-\frac{t\sqrt{-1}}{2}}\bar{x}_3 & e^{-2t\sqrt{-1}}\xi_2 & e^{-\frac{t\sqrt{-1}}{2}}x_1 \\ e^{t\sqrt{-1}}x_2 & e^{-\frac{t\sqrt{-1}}{2}}\bar{x}_1 & e^{t\sqrt{-1}}\xi_3 \end{pmatrix},
$$

$Spin(2) \subset E_6$ is generated by

$$\alpha_{31}(t) := \exp(t\sqrt{-1}R(e_3 - e_1)) :$$

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} e^{-t\sqrt{-1}}\xi_1 & e^{-\frac{t\sqrt{-1}}{2}}x_3 & \bar{x}_2 \\ e^{-\frac{t\sqrt{-1}}{2}}\bar{x}_3 & \xi_2 & e^{\frac{t\sqrt{-1}}{2}}x_1 \\ x_2 & e^{\frac{t\sqrt{-1}}{2}}\bar{x}_1 & e^{t\sqrt{-1}}\xi_3 \end{pmatrix}.$$

and $Spin(8) = (E_6)_{e_1,e_2,e_3}$. Here $Spin(2) \times Spin(8) \subset (E_6)_{e_2} \cong Spin(10)$. Similarly, here

$$Spin(2) \cap Spin(8) = \{\alpha_{31}(t) \mid e^{t\sqrt{-1}} = 1\} = \{\alpha_{31}(0), \alpha_{31}(2\pi)\}.$$

Then the natural projection

$$\begin{aligned} p_2' : \quad & Spin(2) \times Spin(8) \to K_2' \\ & (\alpha_{31}(t), \beta) \mapsto \alpha_{31}(t)\beta \end{aligned}$$

has a kernel

$$\begin{aligned} \ker p_2' &= \{(\alpha_{31}(t), \alpha_{31}(t)^{-1}) \mid t = 2k\pi, k \in \mathbf{Z}\} \\ &= \{(\alpha_{31}(0), \alpha_{31}(0)), (\alpha_{31}(2\pi), \alpha_{31}(2\pi))\} \cong \mathbf{Z}_2. \end{aligned}$$

Thus,

$$\begin{aligned} K_2 &= (S^1 \times (Spin(2) \cdot Spin(8)))/\mathbf{Z}_4, \\ Spin(2) \cdot Spin(8) &= (Spin(2) \times Spin(8))/\mathbf{Z}_2. \end{aligned}$$

Furthermore, we have

$$Spin(8) \supset Spin(2) \cdot Spin(6) \cong (Spin(2) \times Spin(6))/\mathbf{Z}_2,$$

where

$$\begin{aligned} Spin(8) = \{(\alpha_1, \alpha_2, \alpha_3) \in SO(\mathbf{K}) \times SO(\mathbf{K}) \times SO(\mathbf{K}) \mid \\ (\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})} \text{ for each } x, y \in \mathbf{K}\} \end{aligned}$$

acts on $H_3(\mathbf{K})$ by

$$(\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} := \begin{pmatrix} \xi_1 & \alpha_3 x_3 & \overline{\alpha_2 x_2} \\ \overline{\alpha_3 x_3} & \xi_2 & \alpha_1 x_1 \\ \alpha_2 x_2 & \overline{\alpha_1 x_1} & \xi_3 \end{pmatrix},$$

$$Spin(2) := \{(\alpha_1, \alpha_2, \alpha_3) \in Spin(8) \mid \alpha_2(c_i) = c_i, \text{ if } i \neq 0, 4\}$$

is generated by $D_{1,4} + D_{1,12} - D_{1,36} + D_{1,57}$ and

$$Spin(6) := \{(\alpha_1, \alpha_2, \alpha_3) \in Spin(8) \mid \alpha_2(1) = 1, \alpha_2(c_4) = c_4\}$$

is generated by $\mathfrak{k}_0'$. Notice

$$Spin(2) \cap Spin(6) = \{(\mathrm{Id}, \mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id})\},$$

we see that $\mathbf{Z}_2 = \{((\mathrm{Id}, \mathrm{Id}, \mathrm{Id}), (\mathrm{Id}, \mathrm{Id}, \mathrm{Id})), ((-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id}), (-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id}))\}$. Thus, the connected compact Lie subgroup K_1 of K generated by $\mathfrak{k}_1$ is

$$K_1 = (S^1 \times (Spin(2) \cdot (Spin(2) \cdot Spin(6))))/\mathbf{Z}_4.$$

Moreover,

$$S^1 \cap Spin(6) = \{(\mathrm{Id}, \mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id})\},$$

hence the connected compact Lie group K_0 of K generated by $\mathfrak{k}_0$ is

$$K_0 = (S^1 \times Spin(6))/\mathbf{Z}_2,$$

where $\mathbf{Z}_2 = \{((\mathrm{Id},\mathrm{Id},\mathrm{Id}),(\mathrm{Id},\mathrm{Id},\mathrm{Id})),((-\mathrm{Id},\mathrm{Id},-\mathrm{Id}),(-\mathrm{Id},\mathrm{Id},-\mathrm{Id}))\}$.

11.6. **Isotropy representation of** $(E_6, U(1)\cdot Spin(10))$**.** Via the linear isomorphism $p_* : \mathfrak{p} \to T_o(\mathrm{EIII})$ given by (11.23), we can describe the isotropy representation of $(E_6, U(1)\cdot Spin(10))$.

Lemma 11.2. (1) *For each $a \in K$ and each $\xi \in \mathfrak{p}$,*

$$p_*(\mathrm{Ad}(a)\xi) = (\mathrm{Ad}(a)\xi)(e_1) = (a \circ \xi \circ a^{-1})(e_1).$$

(2) *For each $T \in \mathfrak{k}$ and each $\xi \in \mathfrak{p}$,*

$$p_*(\mathrm{ad}(T)\xi) = p_*([T,\xi]) = ([T,\xi])(e_1).$$

The restriction $(\rho_K, V = H_3(\mathbf{K}^{\mathbf{C}}))$ of Chevally-Schafer's representation $(\tilde{\rho}, H_3(\mathbf{K}^{\mathbf{C}}))$ of E_6 to K can be decomposed into three irreducible representations

$$(\rho_K, V) = (\rho_1, V_1) \oplus (\rho_2, V_2) \oplus (\rho_3, V_3),$$

where V_1, V_2 and V_3 are given as follows:

$$\begin{aligned}
V_1 &= \{\xi e_1 \mid \xi \in \mathbf{C}\},\\
V_2 &= (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma} = \{x_2 u_2 + x_3 u_3 \mid x_2, x_3 \in \mathbf{K}^{\mathbf{C}}\} \cong T_o(\mathrm{EIII}),\\
V_3 &= H_2(\mathbf{K}^{\mathbf{C}}) = \{\xi_2 e_2 + \xi_3 e_3 + x_1 u_1 \mid x_1 \in \mathbf{K}^{\mathbf{C}}, \xi_2, \xi_3 \in \mathbf{C}\},
\end{aligned}$$

and $V_1 \oplus V_3 = (H_3(\mathbf{K}^{\mathbf{C}}))_{\sigma}$. ρ_1 is a scalar representation, the restriction of ρ_2 to $Spin(10)$ is equivalent to one of the half-spin representations of $Spin(10,\mathbf{C})$, called Δ_{10}^{+}, and the restriction of ρ_3 to $Spin(10)$ is equivalent to the standard representation of $Spin(10,\mathbf{C})$.

Now we discuss the linear isotropy action of an element $\phi(\theta) = \exp(t\sqrt{-1}R(2e_1 - e_2 - e_3)) : H_3(\mathbf{K}^{\mathbf{C}}) \to H_3(\mathbf{K}^{\mathbf{C}})$ generating the center $U(1)$ of K on both $\mathfrak{p}$ and $V_2 = (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$, which are identified with $T_o(\mathrm{EIII})$.

Using the formula (11.22) and Lemma 11.2, we compute

$$\begin{aligned}
p_*(\mathrm{Ad}(\phi(\theta))D(x_2\bar{u}_2)) &= \theta^{-3} p_*(D(x_2\bar{u}_2)),\\
p_*(\mathrm{Ad}(\phi(\theta))R(x_2 u_2)) &= \theta^{-3} p_*(R(x_2 u_2)),\\
p_*(Ad(\phi(\theta))D(x_3\bar{u}_3)) &= \theta^{-3} p_*(D(x_3\bar{u}_3)),\\
p_*(Ad(\phi(\theta))R(x_3 u_3)) &= \theta^{-3} p_*(R(x_3 u_3)).
\end{aligned}$$

On the other hand, the tangent vector space $T_o(\mathrm{EIII})$ at $o = [e_1] \in \mathrm{EIII}) \subset P(H_3(\mathbf{K})^{\mathbf{C}})$ is identified with the vector subspace $V_2 = (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$, which is a horizontal vector subspace at a point e_1 under the Hopf fibration $H_3(\mathbf{K})^{\mathbf{C}} \supset S^{53}(1) \to P(H_3(\mathbf{K})^{\mathbf{C}})$. By the formula (11.22) we see that a vector $x_2u_2 + x_3u_3 \in (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$ at a point e_1 in a vector space $H_3(\mathbf{K})^{\mathbf{C}}$ representing a tangent vector of EIII at $o = [e_1]$ is moved by the linear action of $\phi(\theta)$ to a vector $\theta x_2 u_2 + \theta x_3 u_3 \in (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$ at $\theta^4 e_1$. Thus its corresponding tangent vector of EIII at $o = [e_1]$ must be $\theta^{-4}(\theta x_2 u_2 + \theta x_3 u_3) = \theta^{-3}(x_2 u_2 + x_3 u_3) \in V_2 = (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$ at e_1. Hence the linear isotropy action of $\phi(\theta)$ on $V_2 = (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$ is given by the multiplication by θ^{-3} on $V_2 = (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$.

Therefore the linear isotropy representation of $(E_6, U(1)\cdot Spin(10))$ is $(\mu_3 \otimes_{\mathbf{C}} \triangle_{10}^{+})_{\mathbf{R}}$.

11.7. **The subgroup $K_{[\mathfrak{a}]}$.** The maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is described as follows:

$$\begin{aligned}\mathfrak{a} &= \mathbf{R}H_1 \oplus \mathbf{R}H_2 \\ &= \mathbf{R}(D\bar{u}_2 + \sqrt{-1}R(\mathbf{c}_4 u_2)) \oplus \mathbf{R}(D\bar{u}_2 - \sqrt{-1}R(\mathbf{c}_4 u_2)).\end{aligned}$$

and

$$p_*(\mathfrak{a}) = \mathbf{R}(1 + \sqrt{-1}\mathbf{c}_4)u_2 \oplus \mathbf{R}(1 - \sqrt{-1}\mathbf{c}_4)u_2. \tag{11.25}$$

We will use the map $\varphi : Sp(4) \to E_6$ given by Yokota ([51]) and the known results for $(Sp(4), Sp(2) \times Sp(2))$ case to find a generator of $K_{[\mathfrak{a}]}$ here.

The Cayley algebra $\mathbf{K}$ naturally contains the field $\mathbf{H}$ of quaternions as

$$\mathbf{H} = \{x_0 + x_2c_2 + x_3c_3 + x_5c_5 | x_i \in \mathbf{R}\}.$$

Any element $x \in \mathbf{K}$ can be expressed by

$$\begin{aligned}x &= x_0 + x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 + x_5c_5 + x_6c_6 + x_7c_7 \\ &= (x_0 + x_2c_2 + x_3c_3 + x_5c_5) + (x_4 + x_1c_2 + x_6c_3 - x_7c_5)c_4 \\ &=: m + a\mathbf{e} \in \mathbf{H} \oplus \mathbf{H}\mathbf{e} = \mathbf{K},\end{aligned}$$

where $m := x_0 + x_2c_2 + x_3c_3 + x_5c_5 \in \mathbf{H}$, $a := x_4 + x_1c_2 + x_6c_3 - x_7c_5 \in \mathbf{H}$ and $\mathbf{e} := c_4$. In $\mathbf{H} \oplus \mathbf{H}\mathbf{e}$, we define a multiplication by

$$(m + a\mathbf{e})(n + b\mathbf{e}) = (mn - \bar{b}a) + (a\bar{n} + bm)\mathbf{e}.$$

More explicitly,

$$(a\mathbf{e})n = (a\bar{n})\mathbf{e}, \quad m(b\mathbf{e}) = (bm)\mathbf{e}, \quad (a\mathbf{e})(b\mathbf{e}) = -\bar{b}a.$$

We can also define a conjugation and an $\mathbf{R}$-linear transformation γ on $\mathbf{H} \oplus \mathbf{H}\mathbf{e}$ respectively by

$$\overline{m + a\mathbf{e}} = \bar{m} - a\mathbf{e}, \qquad \gamma(m + a\mathbf{e}) = m - a\mathbf{e}.$$

Thus $\gamma \in G_2 = \{\alpha \in \mathrm{Iso}_{\mathbf{R}}(\mathbf{K}) \mid \alpha(xy) = \alpha(x)\alpha(y)\}$. Consider an $\mathbf{R}$-linear transformation of $H_3(\mathbf{K})$, denoted still by γ, defined by

$$\gamma \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} := \begin{pmatrix} \xi_1 & \gamma x_3 & \overline{\gamma x_2} \\ \overline{\gamma x_3} & \xi_2 & \gamma x_1 \\ \gamma x_2 & \overline{\gamma x_1} & \xi_3 \end{pmatrix},$$

for $x_i \in \mathbf{K}$ $(i = 1, 2, 3)$. Thus $\gamma \in F_4 = \{\alpha \in \mathrm{Iso}_{\mathbf{R}}(H_3(\mathbf{K})) \mid \alpha(X \circ Y) = \alpha(X) \circ \alpha(Y)\}$. Any element

$$X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + \begin{pmatrix} 0 & a_3\mathbf{e} & -a_2\mathbf{e} \\ -a_3\mathbf{e} & 0 & a_1\mathbf{e} \\ a_2\mathbf{e} & -a_1\mathbf{e} & 0 \end{pmatrix},$$

of $H_3(\mathbf{K})$, where $x_i = m_i + a_i\mathbf{e} \in \mathbf{H} \oplus \mathbf{H}\mathbf{e} = \mathbf{K}$ and $\xi_i \in \mathbf{R}$, can be identified with the element

$$\begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + (a_1, a_2, a_3)$$

in $H_3(\mathbf{H}) \oplus \mathbf{H}^3$. Hereafter, there exists an identification $H_3(\mathbf{K}) \cong H_3(\mathbf{H}) \oplus \mathbf{H}^3$.

Let the $\mathbf{C}$-linear mapping $\gamma : H_3(\mathbf{K}^{\mathbf{C}}) \to H_3(\mathbf{K}^{\mathbf{C}})$ be the complexification of $\gamma \in G_2 \subset F_4$. Then $\gamma \in E_6$ and $\gamma^2 = 1$. Recall that τ is the complex conjugation of $H_3(\mathbf{K}^{\mathbf{C}})$ with respect to $H_3(\mathbf{K})$. Consider an involutive complex conjugate linear transformation $\tau\gamma$ of $H_3(\mathbf{K}^{\mathbf{C}})$ and the following subgroup $(E_6)^{\tau\gamma}$ of E_6:

$$(E_6)^{\tau\gamma} = \{\alpha \in E_6 \mid \tau\gamma\alpha = \alpha\tau\gamma\}.$$

Correspondingly, $H_3(\mathbf{K}^{\mathbf{C}})$ can be decomposed into the following two $\mathbf{R}$-vector subspaces:

$$H_3(\mathbf{K}^{\mathbf{C}}) = (H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma} \oplus (H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma},$$

where

$$\begin{aligned}
&(H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma} := \{X \in H_3(\mathbf{K}^{\mathbf{C}}) \mid \tau\gamma X = X\} \\
&= \{ \begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + \sqrt{-1} \begin{pmatrix} 0 & a_3 e & -a_2 e \\ -a_3 e & 0 & a_1 e \\ a_2 e & -a_1 e & 0 \end{pmatrix} \mid \xi_i \in \mathbf{R}, m_i, a_i \in \mathbf{H}\} \\
&= H_3(\mathbf{H}) \oplus \sqrt{-1}\mathbf{H}^3,
\end{aligned}$$

$$\begin{aligned}
&(H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma} := \{X \in H_3(\mathbf{K}^{\mathbf{C}}) \mid \tau\gamma X = -X\} \\
&= \{\sqrt{-1} \begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + \begin{pmatrix} 0 & a_3 e & -a_2 e \\ -a_3 e & 0 & a_1 e \\ a_2 e & -a_1 e & 0 \end{pmatrix} \mid \xi_i \in \mathbf{R}, m_i, a_i \in \mathbf{H}\} \\
&= \sqrt{-1}H_3(\mathbf{H}) \oplus \mathbf{H}^3.
\end{aligned}$$

In particular, $H_3(\mathbf{K}^{\mathbf{C}}) = ((H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma})^{\mathbf{C}}$.

Let $H_4(\mathbf{H})_0 := \{P \in H_4(\mathbf{H}) \mid \mathrm{tr}P = 0\}$. Define a $\mathbf{C}$-linear isomorphism $g : H_3(\mathbf{K}^{\mathbf{C}}) = H_3(\mathbf{H}^{\mathbf{C}}) \oplus (\mathbf{H}^3)^{\mathbf{C}} \to H_4(\mathbf{H})_0^{\mathbf{C}}$ by

$$g(M + \mathbf{a}) := \begin{pmatrix} \frac{1}{2}\mathrm{tr}(M) & \sqrt{-1}\mathbf{a} \\ \sqrt{-1}\mathbf{a}^* & M - \frac{1}{2}\mathrm{tr}(M)\mathrm{I} \end{pmatrix},$$

for $M + \mathbf{a} \in H_3(\mathbf{K}^{\mathbf{C}})$. Then we have

$$\begin{aligned}
g((H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma}) &= H_4(\mathbf{H})_0, \\
g((H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma}) &= \sqrt{-1}H_4(\mathbf{H})_0.
\end{aligned}$$

The mapping $\varphi : Sp(4) \longrightarrow (E_6)^{\tau\gamma} \subset E_6$, defined by $\varphi(A)X := g^{-1}(A(gX)A^*)$ for each $X \in H_3(\mathbf{K}^{\mathbf{C}})$, is a surjective Lie group homomorphism and $\mathrm{Ker}(\varphi) = \{\mathrm{I}, -\mathrm{I}\} \cong \mathbf{Z}_2$. Therefore we obtain

$$Sp(4)/\mathbf{Z}_2 \cong (E_6)^{\tau\gamma}.$$

Consider $\mathbf{R}$-vector subspaces $(H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,\sigma}$, $(H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,-\sigma}$ of $(H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma}$ and $(H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma,\sigma}$, $(H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma,-\sigma}$ of $(H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma}$, which are eigenspaces of σ, respectively given by

$$\begin{aligned}
&(H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,\sigma} = \{X \in H_3(\mathbf{K}^{\mathbf{C}}) | \ \tau\gamma X = X, \sigma X = X\} \\
&= \{ \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & m_1 \\ 0 & \bar{m}_1 & \xi_3 \end{pmatrix} + \sqrt{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_1\mathbf{e} \\ 0 & -a_1\mathbf{e} & 0 \end{pmatrix} \mid \xi_i \in \mathbf{R}, m_1, a_1 \in \mathbf{H}\},
\end{aligned}$$

$$(H_3(\mathbf{K^C}))_{\tau\gamma,-\sigma} = \{X \in H_3(\mathbf{K^C})|\ \tau\gamma X = X, \sigma X = -X\}$$
$$=\{\begin{pmatrix} 0 & m_3 & \bar{m}_2 \\ \bar{m}_3 & 0 & 0 \\ m_2 & 0 & 0 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} 0 & a_3\mathbf{e} & -a_2\mathbf{e} \\ -a_3\mathbf{e} & 0 & 0 \\ a_2\mathbf{e} & 0 & 0 \end{pmatrix} \mid m_2, m_3, a_2, a_3 \in \mathbf{H}\},$$

$$(H_3(\mathbf{K^C}))_{-\tau\gamma,\sigma} = \{X \in H_3(\mathbf{K^C})|\ \tau\gamma X = -X, \sigma X = X\}$$
$$=\{\sqrt{-1}\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & m_1 \\ 0 & \bar{m}_1 & \xi_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_1\mathbf{e} \\ 0 & -a_1\mathbf{e} & 0 \end{pmatrix} \mid \xi_i \in \mathbf{R}, m_1, a_1 \in \mathbf{H}\},$$

$$(H_3(\mathbf{K^C}))_{-\tau\gamma,-\sigma} = \{X \in H_3(\mathbf{K^C})|\ \tau\gamma X = -X, \sigma X = -X\}$$
$$=\{\sqrt{-1}\begin{pmatrix} 0 & m_3 & \bar{m}_2 \\ \bar{m}_3 & 0 & 0 \\ m_2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_3\mathbf{e} & -a_2\mathbf{e} \\ -a_3\mathbf{e} & 0 & 0 \\ a_2\mathbf{e} & 0 & 0 \end{pmatrix} \mid m_2, m_3, a_2, a_3 \in \mathbf{H}\}.$$

Thus we have the following decompositions

$$\begin{aligned} (H_3(\mathbf{K^C}))_{\sigma} &= (H_3(\mathbf{K^C}))_{\tau\gamma,\sigma} \oplus (H_3(\mathbf{K^C}))_{-\tau\gamma,\sigma}, \\ (H_3(\mathbf{K^C}))_{-\sigma} &= (H_3(\mathbf{K^C}))_{\tau\gamma,-\sigma} \oplus (H_3(\mathbf{K^C}))_{-\tau\gamma,-\sigma}. \end{aligned}$$

Note that the images of $(H_3(\mathbf{K^C}))_{\tau\gamma,\sigma}$ and $(H_3(\mathbf{K^C}))_{\tau\gamma,-\sigma}$ of the mapping g defined above can be expressed explicitly as follows:

$$g((H_3(\mathbf{K^C}))_{\tau\gamma,\sigma})$$
$$=\{\begin{pmatrix} \frac{1}{2}(\xi_1+\xi_2+\xi_3) & -a_1 & 0 & 0 \\ -\bar{a}_1 & \frac{1}{2}(\xi_1-\xi_2-\xi_3) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(-\xi_1+\xi_2-\xi_3) & m_1 \\ 0 & 0 & \bar{m}_1 & \frac{1}{2}(-\xi_1-\xi_2+\xi_3) \end{pmatrix}$$
$$\mid \xi_1, \xi_2, \xi_3 \in \mathbf{R}, a_1, m_1 \in \mathbf{H}\},$$

$$g((H_3(\mathbf{K^C}))_{\tau\gamma,-\sigma}) =\{\begin{pmatrix} 0 & 0 & -a_2 & -a_3 \\ 0 & 0 & m_3 & \bar{m}_2 \\ -\bar{a}_2 & \bar{m}_3 & 0 & 0 \\ -\bar{a}_3 & m_2 & 0 & 0 \end{pmatrix} \mid a_2, a_3, m_2, m_3 \in \mathbf{H}\}.$$

For any element $A \in Sp(2) \times Sp(2) \subset Sp(4)$, we can check that $\varphi(A)\sigma = \sigma\varphi(A)$, hence $\varphi(A) \in (E_6)^\sigma$ and we have

$$\varphi : Sp(2) \times Sp(2) \longrightarrow (E_6)^{\tau\gamma,\sigma} \subset (E_6)^\sigma \cong U(1) \cdot Spin(10).$$

Next, the restriction of φ to the subgroup $Sp(1) \times Sp(1) \times Sp(1) \times Sp(1)$ gives

$$\begin{aligned} \varphi : Sp(1) \times Sp(1) \times Sp(1) \times Sp(1) \longrightarrow &\{\alpha \in E_6 \mid \alpha(e_i) = e_i\ (i = 1, 2, 3)\} \\ \cong & Spin(8). \end{aligned}$$

And the group $Sp(1) \times Sp(1)$ can be considered as the diagonal subgroup of $Sp(1) \times Sp(1) \times Sp(1) \times Sp(1)$, namely, each $(a, b) \in Sp(1) \times Sp(1)$ corresponds to $(a, b, a, b) \in Sp(1) \times Sp(1) \times$

$Sp(1) \times Sp(1)$. Thus the restriction of φ to $Sp(1) \times Sp(1)$ is mapped to a subgroup $K_0 = S^1 \cdot Spin(6)$ of $K = E^\sigma = U(1) \cdot Spin(10)$. In fact, for a 2-dimensional $\mathbf{R}$-vector subspace

$$\tilde{\mathfrak{a}} := \{ \begin{pmatrix} 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & m_2 \\ a_2 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix} \mid a_2, m_2 \in \mathbf{R}\} \subset g((H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,-\sigma}),$$

it follows from

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \begin{pmatrix} 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & m_2 \\ a_2 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} a^* & 0 & 0 & 0 \\ 0 & b^* & 0 & 0 \\ 0 & 0 & a^* & 0 \\ 0 & 0 & 0 & b^* \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & m_2 \\ a_2 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix}.$$

that $\tilde{\mathfrak{a}}$ corresponds to the subspace

$$\{ \begin{pmatrix} 0 & 0 & m_2 - \sqrt{-1}a_2\mathbf{e} \\ 0 & 0 & 0 \\ m_2 + \sqrt{-1}a_2\mathbf{e} & 0 & 0 \end{pmatrix} \mid m_2, a_2 \in \mathbf{R}\} \subset \ (H_3(\mathbf{K})^{\mathbf{C}})_{\tau\gamma,-\sigma},$$

which corresponds to the image $\mathfrak{p}_*(\mathfrak{a})$ of the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ under the linear isomorphism $\mathfrak{p}_*$ given by (11.25). It implies that φ maps the subgroup $\check{K}_0 = Sp(1) \times Sp(1)$ for the exceptional symmetric space $(E_6, Sp(4)/\mathbf{Z}_2)$ of type EI to the subgroup $K_0 = S^1 \cdot Spin(6)$ of the exceptional symmetric space $(E_6, U(1) \cdot Spin(10))$ of type EIII.

Recall that

$$\check{k} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \check{K}_{[\tilde{\mathfrak{a}}]} = (Sp(1) \times Sp(1)) \cdot \mathbf{Z}_4$$

is a generator of $\mathbf{Z}_4$. Its adjoint actions on $g((H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,\sigma})$ and $g((H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,-\sigma})$ are given in the following:

$$\check{k} \begin{pmatrix} \frac{1}{2}(\xi_1 + \xi_2 + \xi_3) & -a_1 & 0 & 0 \\ -\bar{a}_1 & \frac{1}{2}(\xi_1 - \xi_2 - \xi_3) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(-\xi_1 + \xi_2 - \xi_3) & m_1 \\ 0 & 0 & \bar{m}_1 & \frac{1}{2}(-\xi_1 - \xi_2 + \xi_3) \end{pmatrix} \check{k}^{-1}$$
$$= \begin{pmatrix} \frac{1}{2}(\xi_1 - \xi_2 - \xi_3) & -\bar{a}_1 & 0 & 0 \\ -a_1 & \frac{1}{2}(\xi_1 + \xi_2 + \xi_3) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(-\xi_1 - \xi_2 + \xi_3) & -\bar{m}_1 \\ 0 & 0 & -m_1 & -\frac{1}{2}(-\xi_1 + \xi_2 - \xi_3) \end{pmatrix},$$

$$\check{k}\begin{pmatrix} 0 & 0 & -a_2 & -a_3 \\ 0 & 0 & m_3 & \bar{m}_2 \\ -\bar{a}_2 & \bar{m}_3 & 0 & 0 \\ -\bar{a}_3 & m_2 & 0 & 0 \end{pmatrix}\check{k}^{-1} = \begin{pmatrix} 0 & 0 & -\bar{m}_2 & m_3 \\ 0 & 0 & a_3 & -a_2 \\ -m_2 & \bar{a}_3 & 0 & 0 \\ \bar{m}_3 & -\bar{a}_2 & 0 & 0 \end{pmatrix}.$$

Taking $(H_3(\mathbf{K^C}))_{\tau\gamma} = (H_3(\mathbf{K^C}))_{\tau\gamma,\sigma} \oplus (H_3(\mathbf{K^C}))_{\tau\gamma,-\sigma}$ and $H_3(\mathbf{K^C}) = ((H_3(\mathbf{K^C}))_{\tau\gamma})^{\mathbf{C}}$ into account, together with the above computation, we know that any element

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & m_3 + \sqrt{-1}a_3\mathbf{e} & \bar{m}_2 - \sqrt{-1}a_2\mathbf{e} \\ \bar{m}_3 - \sqrt{-1}a_3\mathbf{e} & \xi_3 & m_1 + \sqrt{-1}a_1\mathbf{e} \\ m_2 + \sqrt{-1}a_2\mathbf{e} & \bar{m}_1 - \sqrt{-1}a_1\mathbf{e} & \xi_3 \end{pmatrix}$$

in $H_3(\mathbf{K^C})$ is mapped by the adjoint action of $\check{k}$ up to isomorphism to the element

$$\begin{aligned}
&\begin{pmatrix} \xi_1 & a_3 - \sqrt{-1}m_3\mathbf{e} & -a_2 - \sqrt{-1}\bar{m}_2\mathbf{e} \\ \bar{a}_3 + \sqrt{-1}m_3\mathbf{e} & -\xi_2 & -\bar{m}_1 + \sqrt{-1}\bar{a}_1\mathbf{e} \\ -\bar{a}_2 + \sqrt{-1}\bar{m}_2\mathbf{e} & -m_1 - \sqrt{-1}\bar{a}_1\mathbf{e} & -\xi_3 \end{pmatrix} \\
= &\begin{pmatrix} \xi_1 & \sqrt{-1}(-\sqrt{-1}a_3 - m_3\mathbf{e}) & -\sqrt{-1}(-\sqrt{-1}a_2 + \bar{m}_2\mathbf{e}) \\ \sqrt{-1}(-\sqrt{-1}\bar{a}_3 + m_3\mathbf{e}) & -\xi_2 & -(\bar{m}_1 + \sqrt{-1}\overline{\bar{a}_1\mathbf{e}}) \\ -\sqrt{-1}(-\sqrt{-1}\bar{a}_2 - \bar{m}_2\mathbf{e}) & -(m_1 + \sqrt{-1}\bar{a}_1\mathbf{e}) & -\xi_3 \end{pmatrix} \\
= &\ \alpha_{23}(\pi) \circ (\alpha_1, \alpha_2, \alpha_3)(\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}),
\end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3 \in SO(\mathbf{K}) \cong SO(8)$ are defined by

$$\begin{aligned} \alpha_1(m_1 + a_1\mathbf{e}) &:= -(\bar{m}_1 - \bar{a}_1\mathbf{e}), \\ \alpha_2(m_2 + a_2\mathbf{e}) &:= -\bar{a}_2 - \bar{m}_2\mathbf{e}, \\ \alpha_3(m_3 + a_3\mathbf{e}) &:= -a_3 - m_3\mathbf{e}. \end{aligned} \tag{11.26}$$

By simple computation, we know $\alpha_1(m_1 + a_1\mathbf{e})\ \alpha_2(m_2 + a_2\mathbf{e}) = \overline{\alpha_3(\overline{(m_1 + a_1\mathbf{e})(m_2 + a_2\mathbf{e})})}$. Hence, $(\alpha_1, \alpha_2, \alpha_3) \in Spin(8)$. Notice that

$$\begin{aligned} &\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3)(u_2) = \alpha_{23}(\alpha_2(u_2)) = \alpha_{23}(\pi)(-\mathbf{e}u_2) = \sqrt{-1}\mathbf{e}u_2, \\ &\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3)(\sqrt{-1}\mathbf{e}u_2) = \alpha_{23}(\pi)(\alpha_2(-\sqrt{-1}\mathbf{e}u_2)) = \alpha_{23}(\pi)(\sqrt{-1}u_2) = -u_2. \end{aligned}$$

It implies that

$$\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3) \in Spin(2) \cdot Spin(8) \subset (U(1) \times (Spin(2) \cdot Spin(8)))/\mathbf{Z}_4 = K_2$$

induces an isometry of the maximal abelian subspace $\mathfrak{a}$ of order 4 which is a $\pi/2$-rotation of $\mathfrak{a}$, we obtain

$$\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3) \in K_{[\mathfrak{a}]}$$

and it is a generator of $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

11.8. **Description of the Casimir operator.** Define $\langle u, v\rangle_{\mathfrak{u}} := -\mathrm{tr}(uv)$ for each $u, v \in \mathfrak{e}_6 \subset \mathfrak{gl}(H_3(\mathbf{K})^{\mathbf{C}})$. Now the restricted root system is $\Sigma^+(U, K) = \{2\xi_1, 2\xi_2, \xi_1+\xi_2, \xi_1-\xi_2, \xi_1, \xi_2\}$ and

$$H_{\xi_1} = \frac{1}{12}(D(\bar{u}_2) + \sqrt{-1}R(c_4u_2)), \quad H_{\xi_2} = \frac{1}{12}(D(\bar{u}_2) - \sqrt{-1}R(c_4u_2)).$$

With respect to $\langle\ ,\ \rangle_{\mathfrak{u}}$, the lengths of the restricted roots are given by

$$\|\gamma\|^2_{\mathfrak{u}} = \frac{1}{3}, \frac{1}{6} \text{ or } \frac{1}{12}.$$

Then the Casimir operator $\mathcal{C}_L$ with respect to the induced metric $\mathcal{G}^* g^{\mathrm{std}}_{Q_{30}(\mathbf{C})}$ can be expressed as

$$\mathcal{C}_L = 12C_{K/K_0} - 6C_{K_2/K_0} - 3C_{K_1/K_0},$$

where C_{K/K_0}, C_{K_2/K_0} and C_{K_1/K_0} are the Casimir operators of homogeneous spaces K/K_0, K_2/K_0 and K_1/K_0 with respect to the K -invariant metric induced from the metric $\langle\ ,\ \rangle_{\mathfrak{u}}$ of E_6.

11.9. **Descriptions of $D(K)$, $D(K_2)$, $D(K_1)$ and $D(K_0)$.** A maximal torus $\tilde{T}^5$ of $Spin(10)$ can be given by

$$\begin{aligned}\tilde{T}^5 = \{\tilde{t} = &(\cos\theta_1 - e_1e_2\sin\theta_1)\cdot(\cos\theta_2 - e_3e_4\sin\theta_2)\cdot(\cos\theta_3 - e_5e_6\sin\theta_3)\\ &\cdot(\cos\theta_4 - e_7e_8\sin\theta_4)\cdot(\cos\theta_5 - e_9e_{10}\sin\theta_5) \mid \theta_i \in \mathbf{R}\ (i = 1,2,3,4,5)\}.\end{aligned}$$

Under the standard universal $\mathbf{Z}_2$-covering map $p : Spin(10) \to SO(10)$ defined by

$$(p(\alpha))\mathbf{x} := \alpha\cdot\mathbf{x}\cdot{}^t\alpha \in \mathbf{R}^{10} \subset Cl(\mathbf{R}^{10})$$

for each $\alpha \in Spin(10)$ and each $\mathbf{x} \in \mathbf{R}^{10}$, an element of the maximal torus $\tilde{T}^5$ of $Spin(10)$ is mapped to an element in the maximal torus T^5 of $SO(10)$, namely,

$$\begin{aligned}\tilde{T}^5 \ni\ &(\cos\theta_1 - e_1e_2\sin\theta_1)\cdot(\cos\theta_2 - e_3e_4\sin\theta_2)\cdot(\cos\theta_3 - e_5e_6\sin\theta_3)\\ &\cdot(\cos\theta_4 - e_7e_8\sin\theta_4)\cdot(\cos\theta_5 - e_9e_{10}\sin\theta_5)\\ \longmapsto\ &\begin{pmatrix}\begin{pmatrix}\cos 2\theta_1 & -\sin 2\theta_1\\ \sin 2\theta_1 & \cos 2\theta_1\end{pmatrix} & 0 & 0\\ 0 & \cdots & 0\\ 0 & 0 & \begin{pmatrix}\cos 2\theta_5 & -\sin 2\theta_5\\ \sin 2\theta_5 & \cos 2\theta_5\end{pmatrix}\end{pmatrix} \in T^5.\end{aligned}$$

Hence, we have the exponential map as follows:

$$\begin{aligned}\exp : \tilde{\mathfrak{t}} = \mathfrak{t} &= \{(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) \mid \theta_i \in \mathbf{R}\ (i = 1,2,3,4,5)\}\\ &\longrightarrow\\ \tilde{T} = \{\ &(\cos(\theta_1/2) - e_1e_2\sin(\theta_1/2))\cdot(\cos(\theta_2/2) - e_3e_4\sin(\theta_2/2))\\ &\cdot(\cos(\theta_3/2) - e_5e_6\sin(\theta_3/2))\cdot(\cos(\theta_4/2) - e_7e_8\sin(\theta_4/2))\\ &\cdot(\cos(\theta_5/2) - e_9e_{10}\sin(\theta_5/2))\\ &\mid \theta_i \in \mathbf{R}\ (i = 1,2,3,4,5)\} \subset Spin(10).\end{aligned}$$

Thus

$$\begin{aligned}\Gamma(Spin(10)) &= \{\xi = (\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) \in \tilde{\mathfrak{t}} \mid \exp(\xi) = e\}\\ &=\{\xi = 2\pi\ (k_1,k_2,k_3,k_4,k_5) \mid k_i \in \mathbf{Z}\ (i=1,2,3,4,5), \sum_{i=1}^{5} k_i \in 2\mathbf{Z}\} \subset \Gamma(SO(10)).\end{aligned}$$

Denote by y_i $(i = 1, \cdots, 5)$ a linear functional $y_i : \tilde{\mathfrak{t}} \ni \tilde{t} \mapsto \theta_i \in \mathbf{R}$. Then

$$\begin{aligned}D(Spin(10)) = \{\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^*\\ \mid (p_1,\cdots,p_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \text{where } \varepsilon = 0 \text{ or } \frac{1}{2},\\ p_1 \geq p_2 \geq p_3 \geq p_4 \geq |p_5|\} \supset D(SO(10)).\end{aligned}$$

A maximal torus T_K of $K = (U(1) \times Spin(10))/\mathbf{Z}_4$ can be given as follows:

$$\begin{aligned}T_K =&\{(e^{\sqrt{-1}\theta_0}, (\cos\frac{\theta_1}{2} - e_1e_2\sin\frac{\theta_1}{2})(\cos\frac{\theta_2}{2} - e_3e_4\sin\frac{\theta_2}{2})\\ &(\cos\frac{\theta_3}{2} - e_5e_6\sin\frac{\theta_3}{2})(\cos\frac{\theta_4}{2} - e_7e_8\sin\frac{\theta_4}{2})(\cos\frac{\theta_5}{2} - e_9e_{10}\sin\frac{\theta_5}{2}))\\ &\mid \theta_0,\cdots,\theta_5 \in \mathbf{R}\}/\mathbf{Z}_4,\end{aligned}$$

where $t_0 = 2\theta_0$, $t_1 = \theta_1$, $U(1) = \{\exp(t_0\sqrt{-1}R(2e_1 - e_2 - e_3)) \mid t_0 \in \mathbf{R}\}$, $Spin(2) = \{\exp(t_1\sqrt{-1}R(e_2 - e_3)) \mid t_1 \in \mathbf{R}\}$ and

$$\mathbf{Z}_4 := \{(1,1),(-1,-1),(\sqrt{-1}, -e_1e_2\cdots e_{10}),(-\sqrt{-1}, e_1e_2\cdots e_{10})\}.$$

The corresponding maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ is

$$\mathfrak{t} = \{(\theta_0,\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) \mid \theta_i \in \mathbf{R}\ (i = 0,1,2,3,4,5)\}.$$

Then

$$\begin{aligned}\Gamma(K) =&\{\xi = 2\pi(\frac{k_0}{2}, k_1,k_2,k_3,k_4,k_5) + \pi\varepsilon(\frac{1}{2},1,1,1,1,1)\\ &\mid k_0,k_1,k_2,k_3,k_4,k_5 \in \mathbf{Z},\ \varepsilon = 0 \text{ or } 1,\ \sum_{\alpha=0}^{5} k_\alpha \in 2\mathbf{Z}\},\end{aligned}$$

$$\begin{aligned}D(K) =&D((U(1) \times Spin(10))/\mathbf{Z}_4)\\ =&\{\Lambda = p_0y_0 + p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^* \mid\\ &\frac{1}{2}p_0 + p_1 + p_2 + p_3 + p_4 + p_5 \in 2\mathbf{Z},\ p_0 \in \mathbf{Z},\\ &(p_1,p_2,p_3,p_4,p_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2},\\ &p_1 \geq p_2 \geq p_3 \geq p_4 \geq |p_5|\ \}.\end{aligned}$$

Since T_K is also a maximal torus of $K_2 = (U(1)\times(Spin(2)\cdot Spin(8)))/\mathbf{Z}_4 \subset K$, $\Gamma(K_2) = \Gamma(K)$ and

$$\begin{aligned} D(K_2) =& D((U(1) \times Spin(2) \cdot Spin(8))/\mathbf{Z}_4) \\ =& \{\Lambda = p_0y_0 + p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^* \mid \\ & \frac{1}{2}p_0 + p_1 + p_2 + p_3 + p_4 + p_5 \in 2\mathbf{Z},\ p_0 \in \mathbf{Z}, \\ & (p_1, p_2, p_3, p_4, p_5) \in \mathbf{Z}^5 + \varepsilon(1, 1, 1, 1, 1),\ \varepsilon = 0 \text{ or } \frac{1}{2}, \\ & \quad p_2 \geq p_3 \geq p_4 \geq |p_5| \ \}. \end{aligned}$$

On the other hand, $K_2 = (S^1 \times (Spin(2) \cdot Spin(8)))/\mathbf{Z}_4$, where

$$S^1 = \{\exp(\hat{t}_0\sqrt{-1}R(-e_1 + 2e_2 - e_3)) \mid \hat{t}_0 \in \mathbf{R}\},$$
$$Spin(2) = \{\exp(\hat{t}_1\sqrt{-1}R(e_3 - e_1)) \mid \hat{t}_1 \in \mathbf{R}\}$$

and here $Spin(2) \cdot Spin(8) \subset (E_6)_{e_2} \cong Spin(10)$. Since

$$\begin{aligned} & \exp(t_0\sqrt{-1}R(2e_1 - e_2 - e_3)) \cdot \exp(t_1\sqrt{-1}R(e_2 - e_3)) \\ & \quad = \exp(-\frac{t_0 - t_1}{2}\sqrt{-1}R(-e_1 + 2e_2 - e_3)) \cdot \exp(-\frac{3t_0 + t_1}{2}\sqrt{-1}R(e_3 - e_1)), \end{aligned}$$

one can take $\hat{t}_0 = -\frac{t_0-t_1}{2}$, $\hat{t}_1 = -\frac{3t_0+t_1}{2}$ such that the maximal torus $T_{K_2} = T_K$ of K_2 can also be described as

$$\begin{aligned} \hat{T}_{K_2} = T_{K_2} = T_K = \{\hat{t} = (& e^{\sqrt{-1}\hat{\theta}_0}, (\cos\frac{\hat{\theta}_1}{2} - e_1e_2\sin\frac{\hat{\theta}_1}{2})(\cos\frac{\hat{\theta}_2}{2} - e_3e_4\sin\frac{\hat{\theta}_2}{2}) \\ & (\cos\frac{\hat{\theta}_3}{2} - e_5e_6\sin\frac{\hat{\theta}_3}{2})(\cos\frac{\hat{\theta}_4}{2} - e_7e_8\sin\frac{\hat{\theta}_4}{2})(\cos\frac{\hat{\theta}_5}{2} - e_9e_{10}\sin\frac{\hat{\theta}_5}{2})) \\ & \mid \hat{\theta}_0, \cdots, \hat{\theta}_5 \in \mathbf{R}\}/\mathbf{Z}_4, \end{aligned}$$

where $\hat{\theta}_0 = \hat{t}_0/2$, $\hat{\theta}_1 = \hat{t}_1$. Taking account of the triality of $Spin(8) = (E_6)_{e_1,e_2,e_3} \subset (E_6)_{e_1} \cong (E_6)_{e_2} \cong Spin(10)$, we choose a new basis $\hat{y}_i : \hat{t} \mapsto \hat{\theta}_i$ for $\mathfrak{t}^*$ satisfying

$$\begin{aligned} & \hat{y}_0 = -\frac{1}{2}y_0 + \frac{1}{4}y_1, \quad \hat{y}_1 = -3y_0 - \frac{1}{2}y_1, \quad \hat{y}_2 := \frac{1}{2}(y_2 + y_3 + y_4 + y_5), \\ & \hat{y}_3 := \frac{1}{2}(y_2 + y_3 - y_4 - y_5), \quad \hat{y}_4 := \frac{1}{2}(y_2 - y_3 + y_4 - y_5), \\ & \hat{y}_5 := \frac{1}{2}(-y_2 + y_3 + y_4 - y_5). \end{aligned}$$

Thus any weight $\Lambda = p_0y_0 + p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in D(K_2)$ can also be written as $\Lambda = \hat{p}_0\hat{y}_0 + \hat{p}_1\hat{y}_1 + \hat{p}_2\hat{y}_2 + \hat{p}_3\hat{y}_3 + \hat{p}_4\hat{y}_4 + \hat{p}_5\hat{y}_5$, where

$$\begin{aligned} \hat{p}_0 =& -\frac{1}{2}p_0 + 3p_1, \quad \hat{p}_1 = -\frac{1}{4}p_0 - \frac{1}{2}p_1, \quad \hat{p}_2 = \frac{1}{2}(p_2 + p_3 + p_4 + p_5), \\ \hat{p}_3 =& \frac{1}{2}(p_2 + p_3 - p_4 - p_5), \quad \hat{p}_4 = \frac{1}{2}(p_2 - p_3 + p_4 - p_5), \\ \hat{p}_5 =& \frac{1}{2}(-p_2 + p_3 + p_4 - p_5). \end{aligned}$$

Thus $D(K_2)$ has the following another expression:

$$\begin{aligned}
D(K_2) =& D((S^1 \times Spin(2) \cdot Spin(8))/\mathbf{Z}_4) \\
=& \{\Lambda = \hat{p}_0\hat{y}_0 + \hat{p}_1\hat{y}_1 + \hat{p}_2\hat{y}_2 + \hat{p}_3\hat{y}_3 + \hat{p}_4\hat{y}_4 + \hat{p}_5\hat{y}_5 \in \mathfrak{t}^* \mid \\
& \frac{1}{2}\hat{p}_0 + \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4 + \hat{p}_5 \in 2\mathbf{Z},\ \hat{p}_0 \in \mathbf{Z}, \\
& (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2}, \\
& \quad \hat{p}_2 \geq \hat{p}_3 \geq \hat{p}_4 \geq |\hat{p}_5|\ \}.
\end{aligned}$$

Notice that the subgroup $K_1 = (S^1 \times (Spin(2) \cdot (Spin(2) \cdot Spin(6))))/\mathbf{Z}_4$ also has the same maximal torus $T_{K_1} = \hat{T}_{K_2} = T_{K_2} = T_K$ and the corresponding maximal abelian subalgebra $\mathfrak{t}_{\mathfrak{k}_1}$ of $\mathfrak{k}_1$ is

$$\mathfrak{t}_{\mathfrak{k}_1} = \hat{\mathfrak{t}}_{\mathfrak{k}_2} = \{(\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5) \mid \hat{\theta}_i \in \mathbf{R}\ (i = 0,1,2,3,4,5)\} = \mathfrak{t}_{\mathfrak{k}_2} = \mathfrak{t},$$

we get

$$\begin{aligned}
D(K_1) =& \{\Lambda = \hat{p}_0\hat{y}_0 + \hat{p}_1\hat{y}_1 + \hat{p}_2\hat{y}_2 + \hat{p}_3\hat{y}_3 + \hat{p}_4\hat{y}_4 + \hat{p}_5\hat{y}_5 \in \mathfrak{t}^*_{\mathfrak{k}_1} = \mathfrak{t}^* \mid \\
& \frac{1}{2}\hat{p}_0 + \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4 + \hat{p}_5 \in 2\mathbf{Z},\ \hat{p}_0 \in \mathbf{Z}, \\
& (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2}, \\
& \quad \hat{p}_3 \geq \hat{p}_4 \geq |\hat{p}_5|\ \}.
\end{aligned}$$

Finally, the maximal torus of $K_0 = (S^1 \times Spin(6))/\mathbf{Z}_2$ is given as follows:

$$\begin{aligned}
T_{K_0} =& \{(e^{\sqrt{-1}\hat{\theta}_0}, (\cos\frac{\hat{\theta}_3}{2} - e_5e_6\sin\frac{\hat{\theta}_3}{2})(\cos\frac{\hat{\theta}_4}{2} - e_7e_8\sin\frac{\hat{\theta}_4}{2}) \\
& (\cos\frac{\hat{\theta}_5}{2} - e_9e_{10}\sin\frac{\hat{\theta}_5}{2})) \mid \hat{\theta}_i \in \mathbf{R}\ (i = 0,3,4,5)\}/\mathbf{Z}_2\ \subset \hat{T}_{K_2} = T_K
\end{aligned}$$

and the corresponding maximal abelian subalgebra of $\mathfrak{k}_0$ is

$$\mathfrak{t}_{\mathfrak{k}_0} = \{(\hat{\theta}_0, 0, 0, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5) \mid \hat{\theta}_i \in \mathbf{R}\ (i = 0,3,4,5)\} \subset \mathfrak{t}_{\mathfrak{k}_2} = \mathfrak{t}.$$

Then

$$\begin{aligned}
D(K_0) =& \{\Lambda = \hat{q}_0\hat{y}_0 + \hat{q}_3\hat{y}_3 + \hat{q}_4\hat{y}_4 + \hat{q}_5\hat{y}_5 \in \mathfrak{t}^*_{\mathfrak{k}_0} \mid \\
& \frac{1}{2}\hat{q}_0 + \hat{q}_3 + \hat{q}_4 + \hat{q}_5 \in 2\mathbf{Z},\ \hat{q}_0 \in \mathbf{Z}, \\
& (\hat{q}_3, \hat{q}_4, \hat{q}_5) \in \mathbf{Z}^3 + \varepsilon(1,1,1), \varepsilon = 0 \text{ or } \frac{1}{2}, \\
& \quad \hat{q}_3 \geq \hat{q}_4 \geq |\hat{q}_5|\ \}.
\end{aligned}$$

11.10. **Branching Laws.** Based on the branching laws of $(SO(2n+2), SO(2) \times SO(2n))$ obtained by Tsukamoto ([47]), we formulate the following branching laws.

Lemma 11.3 (Branching Law of $(Spin(10), Spin(2) \cdot Spin(8))$). *For each*

$$\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + \delta p_5y_5 \in D(Spin(10)),$$

with $\delta = 1$ *or* -1 *and*

$$(p_1, p_2, p_3, p_4, p_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \ or\ \frac{1}{2},$$
$$p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5 \geq 0,$$

V_Λ *contains an irreducible* $Spin(2) \cdot Spin(8)$*-module with the highest weight*

$$\Lambda' = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 + \delta' q_5 y_5 \in D(Spin(2) \cdot Spin(8))$$

with $\delta' = 1$ *or* -1 *and*

$$(q_1, q_2, q_3, q_4, q_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \ or\ \frac{1}{2},$$
$$q_2 \geq q_3 \geq q_4 \geq q_5 \geq 0,$$

if and only if Λ' *satisfies the following conditions:*

(1)

$$\begin{aligned} &p_1 + 1 > q_2 > p_3 - 1, \\ &p_2 + 1 > q_3 > p_4 - 1, \\ &p_3 + 1 > q_4 > p_5 - 1, \\ &p_4 + 1 > q_5 \geq 0. \end{aligned}$$

(2) *The coefficient of* X^{q_1} *in the following power series expansion in* X *of*

$$X^{\delta\delta'\ell_5}\left(\prod_{i=1}^{4} \frac{X^{\ell_i+1} - X^{-\ell_i-1}}{X - X^{-1}}\right)$$

does not vanish. Here

$$\begin{aligned} &\ell_1 := p_1 - \max\{p_2, q_2\}, \\ &\ell_2 := \min\{p_2, q_2\} - \max\{p_3, q_3\}, \\ &\ell_3 := \min\{p_3, q_3\} - \max\{p_4, q_4\}, \\ &\ell_4 := \min\{p_4, q_4\} - \max\{p_5, q_5\}, \\ &\ell_5 := \min\{p_5, q_5\}. \end{aligned}$$

Moreover its multiplicity is equal to the coefficient of X^{q_1}*.*

Lemma 11.4 (Branching Law of $(Spin(8), Spin(2) \cdot Spin(6))$)**.** *For each*

$$\Lambda = p_2 y_2 + p_3 y_3 + p_4 y_4 + \delta p_5 y_5 \in D(Spin(8)),$$

with $\delta = 1$ *or* -1 *and*

$$(p_2, p_3, p_4, p_5) \in \mathbf{Z}^4 + \varepsilon(1,1,1,1),\ \varepsilon = 0 \ or\ \frac{1}{2},$$
$$p_2 \geq p_3 \geq p_4 \geq p_5 \geq 0,$$

V_Λ *contains an irreducible* $Spin(2) \cdot Spin(6)$*-module with the highest weight*

$$\Lambda' = q_2 y_2 + q_3 y_3 + q_4 y_4 + \delta' p_5 y_5 \in D(Spin(2) \cdot Spin(6))$$

with $\delta' = 1$ *or* -1 *and*

$$(q_2, q_3, q_4, q_5) \in \mathbf{Z}^4 + \varepsilon(1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2},$$
$$q_3 \geq q_4 \geq q_5 \geq 0.$$

if and only if Λ' *satisfies the following conditions:*

(1)

$$\begin{aligned} &p_2 + 1 > q_3 > p_4 - 1, \\ &p_3 + 1 > q_4 > p_5 - 1, \\ &p_4 + 1 > q_5 \geq 0. \end{aligned}$$

(2) *The coefficient of* X^{q_2}

$$X^{\delta\delta'\ell_5}\left(\prod_{i=2}^{4} \frac{X^{\ell_i+1} - X^{-\ell_i-1}}{X - X^{-1}}\right)$$

does not vanish. Here

$$\begin{aligned} \ell_2 &:= p_2 - \max\{p_3, q_3\}, \\ \ell_3 &:= \min\{p_3, q_3\} - \max\{p_4, q_4\}, \\ \ell_4 &:= \min\{p_4, q_4\} - \max\{p_5, q_5\}, \\ \ell_5 &:= \min\{p_5, q_5\}. \end{aligned}$$

Moreover its multiplicity is equal to the coefficient of X^{q_2}*.*

11.11. **Description of** $D(K, K_0)$**.** Let

$$\begin{aligned} \Lambda &= p_0 y_0 + p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + \epsilon p_5 y_5 \in D(K), \\ \Lambda' &= p'_0 y_0 + p'_1 y_1 + p'_2 y_2 + p'_3 y_3 + p'_4 y_4 + \epsilon' p'_5 y_5 \\ &= \hat{p}'_0 \hat{y}_0 + \hat{p}'_1 \hat{y}_1 + \hat{p}'_2 \hat{y}_2 + \hat{p}'_3 \hat{y}_3 + \hat{p}'_4 \hat{y}_4 + \hat{\epsilon}' \hat{p}'_5 \hat{y}_5 \in D(K_2), \\ \Lambda'' &= \hat{p}''_0 \hat{y}_0 + \hat{p}''_1 \hat{y}_1 + \hat{p}''_2 \hat{y}_2 + \hat{p}''_3 \hat{y}_3 + \hat{p}''_4 \hat{y}_4 + \hat{\epsilon}'' \hat{p}''_5 \hat{y}_5 \in D(K_1), \\ \Lambda''' &= \hat{p}'''_0 \hat{y}_0 + \hat{p}'''_3 \hat{y}_3 + \hat{p}'''_4 \hat{y}_4 + \hat{\epsilon}''' \hat{p}'''_5 \hat{y}_5 \in D(K_0). \end{aligned}$$

Assume that the corresponding representation spaces satisfy

$$V_\Lambda \supset W_{\Lambda'} \supset U_{\Lambda''} = U_{\Lambda'''} \neq \{0\}.$$

Suppose that $U_{\Lambda'''} \neq \{0\}$ is a trivial representation of K_0, that is, $\Lambda''' = 0$. Then we have

$$\hat{p}'''_0 = \hat{p}''_0 = 0, \quad \hat{p}'''_3 = \hat{p}''_3 = 0, \quad \hat{p}'''_4 = \hat{p}''_4 = 0, \quad \hat{p}'''_5 = \hat{p}''_5 = 0.$$

Thus $\Lambda'' = \hat{p}''_1 \hat{y}_1 + \hat{p}''_2 \hat{y}_2 \in D(K_1)$ with $\hat{p}''_1, \hat{p}''_2 \in \mathbf{Z}$, $\hat{p}''_1 + \hat{p}''_2 \in 2\mathbf{Z}$.

By the branching law of $(Spin(8), Spin(2) \cdot Spin(6))$, we get

$$\begin{aligned} &\hat{p}'_2 \geq \hat{p}''_3 = 0 \geq \hat{p}'_4, \\ &\hat{p}'_3 \geq \hat{p}''_4 = 0 \geq \hat{p}'_5, \\ &\hat{p}'_4 \geq \hat{p}''_5 = 0 \geq 0. \end{aligned}$$

Thus $(\hat{p}'_4, \hat{p}'_5) = (0,0)$ and $\hat{p}'_2 \geq 0$, $\hat{p}'_3 \geq 0$. It follows that

$$\begin{aligned}
\ell_2 &= \hat{p}'_2 - \max\{\hat{p}'_3, \hat{p}''_3\} = \hat{p}'_2 - \max\{\hat{p}'_3, 0\} = \hat{p}'_2 - \hat{p}'_3, \\
\ell_3 &= \min\{\hat{p}'_3, \hat{p}''_3\} - \max\{\hat{p}'_4, \hat{p}''_4\} = \min\{\hat{p}'_3, 0\} - \max\{0,0\} = 0 - 0 = 0, \\
\ell_4 &= \min\{\hat{p}'_4, \hat{p}''_4\} - \max\{\hat{p}'_5, \hat{p}''_5\} = \min\{0,0\} - \max\{0,0\} = 0 - 0 = 0, \\
\ell_5 &= \min\{\hat{p}'_5, \hat{p}''_5\} = \min\{0,0\} = 0.
\end{aligned}$$

Then the coefficient of $X^{\hat{p}''_2}$ in the (finite) power series expansion in X

$$X^{\hat{\epsilon}'\hat{\epsilon}''\ell_5} \prod_{i=2}^{4} \frac{X^{\ell_i+1} - X^{-\ell_i-1}}{X - X^{-1}} = \frac{X^{\hat{p}'_2 - \hat{p}'_3 + 1} - X^{-(\hat{p}'_2 - \hat{p}'_3) - 1}}{X - X^{-1}}$$

is equal to its multiplicity. Hence we have

$$-(\hat{p}'_2 - \hat{p}'_3) \leq \hat{p}''_2 = \hat{p}'_2 - \hat{p}'_3 - 2i \leq \hat{p}'_2 - \hat{p}'_3$$

for some $i \in \mathbf{Z}$ with $0 \leq i \leq \hat{p}'_2 - \hat{p}'_3$. Moreover, $\hat{p}'_0 = \hat{p}''_0 = 0, \hat{p}'_1 = \hat{p}''_1$. Thus we get

$$\Lambda' = \hat{p}'_1 \hat{y}_1 + \hat{p}'_2 \hat{y}_2 + \hat{p}'_3 \hat{y}_3 \in D(K_2)$$

with

$$\begin{aligned}
&\hat{p}'_1 = \hat{p}''_1, \hat{p}'_2, \hat{p}'_3 \in \mathbf{Z},\ \hat{p}'_1 + \hat{p}'_2 + \hat{p}'_3 \in 2\mathbf{Z}, \\
&-(\hat{p}'_2 - \hat{p}'_3) \leq \hat{p}''_2 = \hat{p}'_2 - \hat{p}'_3 - 2i \leq \hat{p}'_2 - \hat{p}'_3
\end{aligned}$$

for some $i \in \mathbf{Z}$ with $0 \leq i \leq \hat{p}'_2 - \hat{p}'_3$. Therefore,

$$\Lambda' = p'_0 y_0 + p'_1 y_1 + p'_2 y_2 + p'_3 y_3 + p'_4 y_4 + \epsilon' p'_5 y_5 \in D(K_2)$$

with

$$\begin{aligned}
p'_0 &= -\frac{1}{2}\hat{p}'_0 - 3\hat{p}'_1 = -3\hat{p}'_1, \\
p'_1 &= \frac{1}{4}\hat{p}'_0 - \frac{1}{2}\hat{p}'_1 = -\frac{1}{2}\hat{p}'_1, \\
p'_2 &= \frac{1}{2}(\hat{p}'_2 + \hat{p}'_3 + \hat{p}'_4 - \hat{\epsilon}'\hat{p}'_5) = \frac{1}{2}(\hat{p}'_2 + \hat{p}'_3), \\
p'_3 &= \frac{1}{2}(\hat{p}'_2 + \hat{p}'_3 - \hat{p}'_4 + \hat{\epsilon}'\hat{p}'_5) = \frac{1}{2}(\hat{p}'_2 + \hat{p}'_3), \\
p'_4 &= \frac{1}{2}(\hat{p}'_2 - \hat{p}'_3 + \hat{p}'_4 + \hat{\epsilon}'\hat{p}'_5) = \frac{1}{2}(\hat{p}'_2 - \hat{p}'_3), \\
\epsilon' p'_5 &= \frac{1}{2}(\hat{p}'_2 - \hat{p}'_3 - \hat{p}'_4 - \hat{\epsilon}'\hat{p}'_5) = \frac{1}{2}(\hat{p}'_2 - \hat{p}'_3).
\end{aligned}$$

In particular, $\epsilon' = 1$, $p'_2 = p'_3 = \frac{1}{2}(\hat{p}'_2 + \hat{p}'_3)$, $p'_4 = p'_5 = \frac{1}{2}(\hat{p}'_2 - \hat{p}'_3)$.

Then $p_0 = p'_0$ and by the branching laws of $(Spin(10), Spin(2) \cdot Spin(8))$, we get

$$\begin{aligned}
&p_1 \geq p'_2 \geq p_3, \quad p_2 \geq p'_3 = p'_2 \geq p_4, \\
&p_3 \geq p'_4 \geq p_5, \quad p_4 \geq p'_5 = p'_4 \geq 0.
\end{aligned}$$

Thus $p_1 \geq p_2 \geq p_2' = p_3' \geq p_3 \geq p_4 \geq p_4' = p_5' \geq p_5 \geq 0$. It follows that

$$\begin{aligned}
\ell_1 &= p_1 - \max\{p_2, p_2'\} = p_1 - p_2,\\
\ell_2 &= \min\{p_2, p_2'\} - \max\{p_3, p_3'\} = p_2' - p_3' = 0,\\
\ell_3 &= \min\{p_3, p_3'\} - \max\{p_4, p_4'\} = p_3 - p_4,\\
\ell_4 &= \min\{p_4, p_4'\} - \max\{p_5, p_5'\} = p_4' - p_5' = 0,\\
\ell_5 &= \min\{p_5, p_5'\} = p_5.
\end{aligned}$$

Then the coefficient of $X^{p_1'} = X^{-\frac{1}{2}\hat{p}_1'} = X^{-\frac{1}{2}\hat{p}_1''}$ in the (finite) power series expansion in X

$$\begin{aligned}
&X^{\epsilon\epsilon'\ell_5} \prod_{i=1}^{4} \frac{X^{\ell_i+1} - X^{-\ell_i-1}}{X - X^{-1}}\\
=&X^{\epsilon\epsilon' p_5} \frac{X^{p_1-p_2+1} - X^{-(p_1-p_2+1)}}{X - X^{-1}} \frac{X^{p_3-p_4+1} - X^{-(p_3-p_4+1)}}{X - X^{-1}}\\
=&X^{\epsilon\epsilon' p_5} \sum_{i=0}^{p_1-p_2} \sum_{j=0}^{p_3-p_4} X^{(p_1-p_2)+(p_3-p_4)-2(i+j)}
\end{aligned}$$

is equal to its multiplicity.

Then we have $\Lambda = p_0 y_0 + p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + \epsilon p_5 y_5 \in D(K, K_0)$ with $p_0 = p_0' = -3\hat{p}_1' = 6p_1' \in 3\mathbf{Z}$.

11.12. **Eigenvalue computation.** Recall that the standard basis $\mathbf{e}_\alpha$ ($\alpha = 0, 1, \cdots, 5$) of $\mathfrak{t} = \{(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \mid \theta_\alpha \in \mathbf{R}\}$ corresponds to $2\sqrt{-1}R(2e_1 - e_2 - e_3) \in \mathfrak{u}(1)$ and $\sqrt{-1}R(e_2 - e_3), D_{1,4}, D_{1,12}, D_{1,36}, D_{1,57} \in spin(10)$, respectively. With respect to the inner product $\langle u, v\rangle_{\mathfrak{u}} = -\mathrm{tr} uv$ for $u, v \in \mathfrak{k} \subset \mathfrak{e}_6 \subset \mathfrak{gl}(H_3(\mathbf{K})^{\mathbf{C}})$,

$$\langle \mathbf{e}_0, \mathbf{e}_0\rangle = 72, \quad \langle \mathbf{e}_i, \mathbf{e}_i\rangle = 6, \quad \langle \mathbf{e}_\alpha, \mathbf{e}_\beta\rangle = 0,$$

for $1 \leq i \leq 5$ and $0 \leq \alpha \neq \beta \leq 5$. It follows that the inner products of the dual bases $\{y_0, y_1, y_2, y_3, y_4, y_5\}$ of $\mathfrak{t}^*$ corresponding to $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ of $\mathfrak{t}$ are given by

$$\begin{aligned}
&\langle y_\alpha, y_\beta\rangle = 0, \quad (0 \leq \alpha \neq \beta \leq 5),\\
&\langle y_0, y_0\rangle = \frac{1}{72}, \quad \langle y_i, y_j\rangle = \frac{1}{6}, \quad (1 \leq i \neq j \leq 5).
\end{aligned}$$

For

$$\begin{aligned}
\Lambda &= p_0 y_0 + p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + \epsilon p_5 y_5 \in D(K, K_0),\\
\Lambda' &= p_0 y_0 + \frac{p_0}{6} y_1 + p_2' y_2 + p_2' y_3 + p_4' y_4 + p_4' y_5\\
&= -\frac{p_0}{3}\hat{y}_1 + (p_2' + p_4')\hat{y}_2 + (p_2' - p_4')\hat{y}_3 \in D(K_2, K_0),\\
\Lambda'' &= -\frac{p_0}{3}\hat{y}_1 + \hat{p}_2''\hat{y}_2 \in D(K_1, K_0),
\end{aligned}$$

the eigenvalue formulas of the Casimir operators $\mathcal{C}_{K/K_0}$, $\mathcal{C}_{K_2/K_0}$ and $\mathcal{C}_{K_1/K_0}$ with respect to the inner product $\langle\ ,\ \rangle_{\mathfrak{u}}$ are given respectively by

$$\begin{aligned}
-c_\Lambda &= \frac{1}{72}p_0^2 + \frac{1}{6}\{(p_1+8)p_1 + (p_2+6)p_2 + (p_3+4)p_3 + (p_4+2)p_4 + (p_5)^2\},\\
-c_{\Lambda'} &= \frac{1}{72}(p_0')^2 + \frac{1}{6}\{(p_1')^2 + (p_2'+6)p_2' + (p_3'+4)p_3' + (p_4'+2)p_4' + (p_5')^2\}\\
&= \frac{1}{72}(\hat{p}_0')^2 + \frac{1}{6}\{(\hat{p}_1')^2 + (\hat{p}_2'+6)\hat{p}_2' + (\hat{p}_3'+4)\hat{p}_3' + (\hat{p}_4'+2)\hat{p}_4' + (\hat{p}_5')^2\}\\
&= \frac{1}{72}(p_0)^2 + \frac{1}{6}\{(\frac{1}{6}p_0)^2 + (p_2'+6)p_2' + (p_2'+4)p_2' + (p_4'+2)p_4' + (p_4')^2\},\\
-c_{\Lambda''} &= \frac{1}{72}(\hat{p}_0'')^2 + \frac{1}{6}\{(\hat{p}_1'')^2 + (\hat{p}_2'')^2 + (\hat{p}_3''+4)\hat{p}_3'' + (\hat{p}_4''+2)\hat{p}_4'' + (\hat{p}_5'')^2\}\\
&= \frac{1}{6}\{(\frac{1}{3}p_0)^2 + (\hat{p}_2'')^2\}.
\end{aligned}$$

Then for each $\Lambda \in D(K, K_0)$, we have the following eigenvalue formula

$$\begin{aligned}
-c_L &= -12c_\Lambda + 6c_{\Lambda'} + 3c_{\Lambda''}\\
&= 2\{(p_1+8)p_1 + (p_2+6)p_2 + (p_3+4)p_3 + (p_4+2)p_4 + (p_5)^2\}\\
&\quad - \{(p_2'+6)p_2' + (p_2'+4)p_2' + (p_4'+2)p_4' + (p_4')^2\} - \frac{1}{2}(\hat{p}_2'')^2\\
&= 2(p_1+8)p_1 + 2((p_2)^2 - (p_2')^2) + 12p_2 - 10p_2' + 2(p_3)^2 + 8p_3\\
&\quad + 2((p_4)^2 - (p_4')^2) + 4p_4 - 2p_4' + 2(p_5)^2 - \frac{1}{2}(\hat{p}_2'')^2\\
&= 2(p_1+8)p_1 + 2((p_2)^2 - (p_2')^2) + 2p_2 + 10(p_2 - p_2') + 2(p_3)^2 + 8p_3\\
&\quad + 2((p_4)^2 - (p_4')^2) + 2p_4 + 2(p_4 - p_4') + 2(p_5)^2 - \frac{1}{2}(\hat{p}_2'')^2\\
&\geq 2(p_1+8)p_1 + 2p_2 + 2(p_3)^2 - \frac{1}{2}(\hat{p}_2'')^2 + 8p_3 + 2p_4 + 2(p_5)^2\\
&= 2(p_1+8)p_1 + 2p_2 + (2(p_5')^2 - \frac{1}{2}(\hat{p}_2'')^2) + 8p_3 + 2p_4 + 2(p_5)^2\\
&\geq 2(p_1+8)p_1 + 2p_2 + 8p_3 + 2p_4 + 2(p_5)^2,
\end{aligned}$$

where the equalities hold if and only if $p_2 = p_2'$, $p_4 = p_4'$, $2p_3 = 2p_4 = 2p_4' = 2p_5' = |\hat{p}_2''|$ since we have

$$\begin{aligned}
&p_1 \geq p_2 \geq p_2' = p_3' \geq p_3 \geq p_4 \geq p_4' = p_5' \geq p_5 \geq 0,\\
&-2p_4' = -2p_5' = -(\hat{p}_2' - \hat{p}_3') \leq \hat{p}_2'' \leq \hat{p}_2' - \hat{p}_3' = 2p_5' = 2p_4'.
\end{aligned}$$

Notice that if $p_1 = 0$, then $-c_L = 0$ and if $p_1 \geq 2$, then $-c_L \geq 40 > 30$. In case $p_1 = \frac{3}{2}$, the possible $\Lambda = (p_0, p_1, p_2, p_3, p_4, p_5) \in D(K, K_0)$ are

$$
\begin{aligned}
&(p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}), (p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2}), (p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}), (p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{1}{2}),\\
&(p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}), (p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}), (p_0, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (p_0, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}),\\
&(p_0, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (p_0, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}).
\end{aligned}
$$

In these cases, the eigenvalue of the Casimir operator is given by

$$
\begin{aligned}
-c_L \geq & 2(p_1 + 8)p_1 + 2p_2 + 8p_3 + 2p_4 + 2(p_5)^2\\
\geq & 2 \cdot (\frac{3}{2} + 8) \cdot \frac{3}{2} + 2 \cdot \frac{1}{2} + 8 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} + 2 \cdot (\frac{1}{2})^2\\
= & 35 > 30.
\end{aligned}
$$

Hence in order to decide the Hamiltonian stability, i.e., to compare the first eigenvalue $-c_L$ and 30, we can only concern on the cases when $p_1 = \frac{1}{2}$ or 1.

It follows from the description of $D(K, K_0)$ that the element in $D(K, K_0)$ when $p_1 = \frac{1}{2}$ is given by

$$
(p_0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \text{ or } (p_0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})
$$

and the element in $D(K, K_0)$ for $p_1 = 1$ is given by

$$
\begin{aligned}
&(p_0, 1, 0, 0, 0, 0), \quad (p_0, 1, 1, 0, 0, 0), \quad (p_0, 1, 1, 1, 0, 0),\\
&(p_0, 1, 1, 1, 1, 0), \quad (p_0, 1, 1, 1, 1, 1) \text{ or } (p_0, 1, 1, 1, 1, -1).
\end{aligned}
$$

Using the branching laws, the descriptions of $D(K_2, K_0)$, $D(K_1, K_0)$ and the eigenvalue formula given above, by direct computation we get the following small eigenvalues in the above cases.

Λ	Λ'	Λ''	$-c_L$
$3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, -1, 1, 0, 0, 0$	15
$3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, -1, -1, 0, 0, 0$	15
$-3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	$-3, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, 1, 1, 0, 0, 0$	15
$-3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	$-3, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, 1, -1, 0, 0, 0$	15
$6, 1, 0, 0, 0, 0$	$6, 1, 0, 0, 0, 0$	$0, -2, 0, 0, 0, 0$	18
$-6, 1, 0, 0, 0, 0$	$-6, -1, 0, 0, 0, 0$	$0, 2, 0, 0, 0, 0$	18
$0, 1, 1, 0, 0, 0$	$0, 0, 0, 0, 0, 0$	$0, 0, 0, 0, 0, 0$	32
$0, 1, 1, 0, 0, 0$	$0, 0, 1, 1, 0, 0$	$0, 0, 0, 0, 0, 0$	20
$6, 1, 1, 1, 0, 0$	$6, 1, 1, 1, 0, 0$	$0, -2, 0, 0, 0, 0$	30
$-6, 1, 1, 1, 0, 0$	$-6, -1, 1, 1, 0, 0$	$0, 2, 0, 0, 0, 0$	30
$0, 1, 1, 1, 1, 0$	$0, 0, 1, 1, 0, 0$	$0, 0, 0, 0, 0, 0$	36
$0, 1, 1, 1, 1, 0$	$0, 0, 1, 1, 1, 1$	$0, 0, 0, 0, 0, 0$	32
$0, 1, 1, 1, 1, 0$	$0, 0, 1, 1, 1, 1$	$0, 0, 2, 0, 0, 0$	30
$0, 1, 1, 1, 1, 0$	$0, 0, 1, 1, 1, 1$	$0, 0, -2, 0, 0, 0$	30
$6, 1, 1, 1, 1, 1$	$6, 1, 1, 1, 1, 1$	$0, -2, 2, 0, 0, 0$	32
$6, 1, 1, 1, 1, 1$	$6, 1, 1, 1, 1, 1$	$0, -2, -2, 0, 0, 0$	32
$6, 1, 1, 1, 1, 1$	$6, 1, 1, 1, 1, 1$	$0, -2, 0, 0, 0, 0$	34
$-6, 1, 1, 1, 1, -1$	$-6, -1, 1, 1, 1, 1$	$0, 2, 2, 0, 0, 0$	32
$-6, 1, 1, 1, 1, -1$	$-6, -1, 1, 1, 1, 1$	$0, 2, -2, 0, 0, 0$	32
$-6, 1, 1, 1, 1, -1$	$-6, -1, 1, 1, 1, 1$	$0, 2, 0, 0, 0, 0$	34

Here, $\Lambda = (p_0, p_1, p_2, p_3, p_4, p_5) \in D(K, K_0)$, $\Lambda' = (p'_0, p'_1, p'_2, p'_3, p'_4, p'_5) \in D(K_2, K_0)$ and $\Lambda'' = (\hat{p}''_0, \hat{p}''_1, \hat{p}''_2, \hat{p}''_3, \hat{p}''_4, \hat{p}''_5) \in D(K_1, K_0)$.

Since $\Lambda_1 = (3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ corresponds to the complexified isotropy representation of EIII and it is conjugate to $\Lambda_2 = (-3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, we see that $\Lambda_1, \Lambda_2 \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (p_0, p_1, p_2, p_3, p_4, p_5) = (6, 1, 0, 0, 0, 0) \in D(K, K_0)$. Then by the branching laws we get $\Lambda' = 6y_0 + y_1 \in D(K_2, K_0)$, $\Lambda'' = -2\hat{y}_1 \in D(K_1, K_0)$ and $\Lambda''' = 0 \in D(K_0)$. Hence, the eigenvalue of the Casimir operator is $-c_L = 18 < 30$.

On the other hand,

$$
\begin{aligned}
V_\Lambda \cong & \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix} \mid \xi_2, \xi_3 \in \mathbf{C}, x_1 \in \mathbf{K}^{\mathbf{C}} \right\} \cong \mathbf{C}^{10} \\
& \supset W_{\Lambda'} = U_{\Lambda''} = U_{\Lambda'''} = (V_\Lambda)_{K_0}
\end{aligned}
$$

and $\rho_\Lambda = \mu_6 \boxtimes \sigma_{\mathbf{C}^{10}}$, where $\sigma_{\mathbf{C}^{10}}$ denotes the standard representation of $SO(10)$, and for each $\phi(\theta) \in U(1)$,

$$
\mu_6(\phi(\theta)) \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix} = \theta^{-6} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix},
$$

where $\theta = e^{\sqrt{-1}t_0/2}$. Since for any $\exp(\hat{t}_0\sqrt{-1}R(e_1 - 2e_2 + e_3)) \in S^1 \subset K_0$,

$$\begin{aligned}
&\exp(\hat{t}_0\sqrt{-1}R(e_1 - 2e_2 + e_3)) \\
&= \exp(\hat{t}_0\frac{1}{2}\sqrt{-1}R(2e_1 - e_2 - e_3))\exp(-\hat{t}_0\frac{3}{2}\sqrt{-1}R(e_2 - e_3)) \\
&\in U(1)\cdot Spin(2) \subset K,
\end{aligned}$$

we compute

$$\begin{aligned}
&\rho_\Lambda(\exp(\hat{t}_0\sqrt{-1}R(e_1 - 2e_2 + e_3)))\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix} \\
=&(\mu_6 \boxtimes \sigma_{\mathbf{C}^{10}})(\exp(\hat{t}_0\sqrt{-1}R(e_1 - 2e_2 + e_3)))\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix} \\
=&\mu_6(\exp(\hat{t}_0\frac{1}{2}\sqrt{-1}R(2e_1 - e_2 - e_3)))\alpha_{23}(-\hat{t}_0\frac{3}{2})\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix} \\
=&(e^{\sqrt{-1}\frac{1}{2}\hat{t}_0\frac{1}{2}})^{-6}\begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-\sqrt{-1}\hat{t}_0\frac{3}{2}}\xi_2 & x_1 \\ 0 & \bar{x}_1 & e^{\sqrt{-1}\hat{t}_0\frac{3}{2}}\xi_3 \end{pmatrix} \\
=&e^{-\sqrt{-1}\frac{3}{2}\hat{t}_0}\begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-\sqrt{-1}\hat{t}_0\frac{3}{2}}\xi_2 & x_1 \\ 0 & \bar{x}_1 & e^{\sqrt{-1}\hat{t}_0\frac{3}{2}}\xi_3 \end{pmatrix} \\
=&\begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-\sqrt{-1}3\hat{t}_0}\xi_2 & e^{-\sqrt{-1}\frac{3}{2}\hat{t}_0}x_1 \\ 0 & e^{-\sqrt{-1}\frac{3}{2}\hat{t}_0}\bar{x}_1 & \xi_3 \end{pmatrix}.
\end{aligned}$$

In particular,

$$\rho_\Lambda(\exp(\hat{t}_0\sqrt{-1}R(e_1 - 2e_2 + e_3)))\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}$$

for each $\hat{t}_0 \in \mathbf{R}$. Hence,

$$(V_\Lambda)_{K_0} \cong \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix} \mid \xi_3 \in \mathbf{C}\right\}.$$

But as a generator of $\mathbf{Z}_4$ of $K_{[\mathfrak{a}]}$, the action of $\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3) \in K_{[\mathfrak{a}]}$ given by (11.26) is

$$\begin{aligned}
&\rho_\Lambda(\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3))\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix} \\
=&(\alpha_{23}(\pi))\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\xi_3 \end{pmatrix}.
\end{aligned}$$

Therefore $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}$ and $\Lambda = 6y_0 + y_1 \notin D(K, K_{[\mathfrak{a}]})$. Similarly, $\Lambda = -6y_0 + y_1 \notin D(K, K_{[\mathfrak{a}]})$.

Suppose $\Lambda = (p_0, p_1, p_2, p_3, p_4, p_5) = (0, 1, 1, 0, 0, 0) \in D(K, K_0)$. Then by the branching laws we get

$$\begin{aligned} \Lambda' &= (p'_0, p'_1, p'_2, p'_3, p'_4, p'_5) = (0, 0, 1, 1, 0, 0) \in D(K_2, K_0), \\ \Lambda'' &= (\hat{p}''_0, \hat{p}''_1, \hat{p}''_2, \hat{p}''_3, \hat{p}''_4, \hat{p}''_5) = (0, 0, 0, 0, 0, 0) \in D(K_1, K_0). \end{aligned}$$

Here $\rho'_{\Lambda'} = \mathrm{Id} \boxtimes \mathrm{Id} \boxtimes \mathrm{Ad}^{\mathbf{C}}_{Spin(8)} = \mathrm{Id} \boxtimes \mathrm{Id} \boxtimes \mathrm{Ad}^{\mathbf{C}}_{SO(8)} \in \mathcal{D}(K_2)$. Notice that $W_{\Lambda'} = \mathfrak{o}(8)^{\mathbf{C}} = \mathfrak{o}(2)^{\mathbf{C}} \oplus \mathfrak{o}(6)^{\mathbf{C}} \oplus M(2, 6; \mathbf{R})^{\mathbf{C}}$, and the subgroups $U(1)$ and $Spin(2)$ of $K_2 = (U(1) \times (Spin(2) \cdot Spin(8))/\mathbf{Z}_4$ acts trivially on $\mathfrak{o}(8)^{\mathbf{C}}$. The subgroup $Spin(6)$ of $Spin(2) \cdot Spin(6)$ acts trivially on $\mathfrak{o}(2)^{\mathbf{C}}$, hence $(W_{\Lambda'})_{K_0} = \mathfrak{o}(2)^{\mathbf{C}}$. For $\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3) \in K_{[\mathfrak{a}]}$ a generator of $\mathbf{Z}_4$ given in (11.26), $\alpha_{23}(\pi)$ and $(\alpha_1, \alpha_2, \alpha_3)$ commute to each other. $\alpha_{23}(\pi) \in Spin(2)$ acts trivially on $\mathfrak{o}(2)^{\mathbf{C}}$. α_2 of $(\alpha_1, \alpha_2, \alpha_3)$ acts on $\mathbf{R}1 + \mathbf{R}\mathbf{e}$ as $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and preserves the vector subspace orthogonally complementary to $\mathbf{R}1 + \mathbf{R}\mathbf{e}$ in $\mathbf{K} \cong \mathbf{R}^8$. Thus the $Spin(2)$-factor of $(\alpha_1, \alpha_2, \alpha_3)$ in $Spin(2) \cdot Spin(6)$ corresponds to $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in O(2)$. Since its adjoint action of on $\mathfrak{o}(2)^{\mathbf{C}}$ is $-\mathrm{Id}$, the adjoint action of $(\alpha_1, \alpha_2, \alpha_3) \in Spin(8)$ is not trivial on $\mathfrak{o}(2)^{\mathbf{C}}$. Hence $(W_{\Lambda'})_{K_{[\mathfrak{a}]}} = \{0\}$ and in particular we obtain $\Lambda = y_1 + y_2 \notin D(K, K_{[\mathfrak{a}]})$.

Suppose $\Lambda = (p_0, p_1, p_2, p_3, p_4, p_5) = (6, 1, 1, 1, 0, 0) \in D(K, K_0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 120$. By the branching laws we get $\Lambda' = 6y_0 + y_1 + y_2 + y_3 = -2\hat{y}_1 + \hat{y}_2 + \hat{y}_3 \in D(K_2, K_0)$, $\Lambda'' = -2\hat{y}_1 \in D(K_1, K_0)$ and $\Lambda''' = 0 \in D(K_0)$. Hence, the eigenvalue of the Casimir operator is $-c_L = 30$.

On the other hand, $\rho'_{\Lambda'} = \mathrm{Id} \boxtimes \mu_{-2} \boxtimes \mathrm{Ad}^{\mathbf{C}}_{Spin(8)} = \mathrm{Id} \boxtimes \mu_{-2} \boxtimes \mathrm{Ad}^{\mathbf{C}}_{SO(8)} \in \mathcal{D}(K_2)$. Here $W_{\Lambda'} = \mathfrak{o}(8)^{\mathbf{C}} = \mathfrak{o}(2)^{\mathbf{C}} \oplus \mathfrak{o}(6)^{\mathbf{C}} \oplus M(2, 6; \mathbf{R})^{\mathbf{C}}$. Same as the previous case, we get $(W_{\Lambda'})_{K_0} = \mathfrak{o}(2)^{\mathbf{C}}$. Notice that for the generator $\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3)$ of $\mathbf{Z}_4$ in $K_{[\mathfrak{a}]}$ given by (11.26), the action of $\alpha_{23}(\pi) \in Spin(2)$ on $H_3(\mathbf{K}^{\mathbf{C}})$ is given by

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 & \sqrt{-1}x_3 & -\sqrt{-1}\bar{x}_2 \\ \sqrt{-1}\bar{x}_3 & -\xi_2 & x_1 \\ -\sqrt{-1}x_2 & \bar{x}_1 & -\xi_3 \end{pmatrix}.$$

In particular, $\alpha_{23}(\pi)$ transforms u_2 to $-\sqrt{-1}u_2$ and $\mathbf{e}u_2$ to $-\sqrt{-1}\mathbf{e}u_2$, which says that $\alpha_{23}(\pi)$ acts on $\mathfrak{o}(2) \cong \mathbf{R}1 + \mathbf{R}\mathbf{e}$ as the matrix multiplication by $\begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$. Thus $\mu_{-2}(\alpha_{23}(\pi))$ acts on $\mathfrak{o}(2) \cong \mathbf{R}1 + \mathbf{R}\mathbf{e}$ is just the matrix multiplication by $-\mathrm{Id}$. On the other hand, α_2 of $(\alpha_1, \alpha_2, \alpha_3)$ acts on $\mathbf{R}1 + \mathbf{R}e$ as $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Thus the $Spin(2)$-factor of $(\alpha_1, \alpha_2, \alpha_3)$ in $Spin(2) \cdot Spin(6)$ corresponds to $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in O(2)$. Hence its adjoint action on $\mathfrak{o}(2)^{\mathbf{C}}$ is $-\mathrm{Id}$. Therefore, $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \mathfrak{o}(2)^{\mathbf{C}}$, i.e., $\Lambda = 6y_0 + y_1 + y_2 + y_3 \in D(K, K_{[\mathfrak{a}]}) = \mathfrak{o}(2)^{\mathbf{C}}$. Thus $\Lambda = 6y_0 + y_1 + y_2 + y_3 \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 1. Similarly, $\Lambda = -6y_0 + y_1 + y_2 + y_3 \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 1.

Suppose $\Lambda = (p_0, p_1, p_2, p_3, p_4, p_5) = (0, 1, 1, 1, 1, 0) \in D(K, K_0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 210$. By the branching laws we get the following decomposition of V_Λ into irreducible modules of K_2

and K_1:

$$\begin{aligned}&V_{\Lambda(0,1,1,1,1,0)}\\ =&W_{\Lambda_1'(0,0,1,1,1,1)}\oplus W_{\Lambda_2'(0,0,1,1,0,0)}\\ =&(U_{\Lambda_1''(0,0,0,0,0,0)}\oplus U_{\Lambda_1''(0,0,2,0,0,0)}\oplus U_{\Lambda_1''(0,0,-2,0,0,0)})\oplus U_{\Lambda_2''(0,0,0,0,0,0)}.\end{aligned}$$

Then the Casimir operator $-\mathcal{C}_L$ has eigenvalues $-c_L = 32, 30, 30$ or 36 along this decomposition.

On the other hand, $\Lambda_1' = 2\hat{y}_2 \in D(K_2, K_0)$, $W_{\Lambda_1'} \cong S_0^2(\mathbf{C}^8) \cong S_0^2(\mathbf{K}^8)$ and

$$W_{\Lambda_1'} \cap (V_\Lambda)_{K_0} = U_{\Lambda_1''(0,0,0,0,0,0)} \oplus (U_{\Lambda_1''(0,0,2,0,0,0)})_{K_0} \oplus (U_{\Lambda_1''(0,0,-2,0,0,0)})_{K_0}.$$

Recall that $\{1, c_1, \cdots, c_7\}$ denote the standard basis of the Cayley algebra $\mathbf{K}$ and $\mathbf{e} := c_4$. Then

$$3(1\cdot 1+\mathbf{e}\cdot\mathbf{e}) - (c_1\cdot c_1 + c_2\cdot c_2 + c_3\cdot c_3 + c_5\cdot c_5 + c_6\cdot c_6 + c_7\cdot c_7) \in S_0^2(\mathbf{K}^{\mathbf{C}}).$$

For any $A = \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix} \in SO(2)$, $A(1,\mathbf{e}) = (1,\mathbf{e})\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}$. Hence

$$\begin{aligned}A(1\cdot 1) =&(\cos t1+\sin t\mathbf{e})\cdot(\cos t1+\sin t\mathbf{e})\\ =&\cos^2 t(1\cdot 1)+\sin^2 t(\mathbf{e}\cdot\mathbf{e})+2\sin t\cos t(1\cdot\mathbf{e}),\\ A(\mathbf{e}\cdot\mathbf{e}) =&(-\sin t1+\cos t\mathbf{e})\cdot(-\sin t1+\cos t\mathbf{e})\\ =&\sin^2 t(1\cdot 1)+\cos^2 t(\mathbf{e}\cdot\mathbf{e})-2\sin t\cos t(1\cdot\mathbf{e}),\\ A(1\cdot\mathbf{e}) =&(\cos t1+\sin t\mathbf{e})\cdot(-\sin t1+\cos t\mathbf{e})\\ =&-\frac{1}{2}\sin 2t(1\cdot 1-\mathbf{e}\cdot\mathbf{e})+\cos 2t(1\cdot\mathbf{e}).\end{aligned}$$

In particular, $A(1\cdot 1+\mathbf{e}\cdot\mathbf{e}) = 1\cdot 1+\mathbf{e}\cdot\mathbf{e}$ and

$$\begin{aligned}&A(3(1\cdot 1+\mathbf{e}\cdot\mathbf{e}) - (c_1\cdot c_1 + c_2\cdot c_2 + c_3\cdot c_3 + c_5\cdot c_5 + c_6\cdot c_6 + c_7\cdot c_7))\\ =&3(1\cdot 1+\mathbf{e}\cdot\mathbf{e}) - (c_1\cdot c_1 + c_2\cdot c_2 + c_3\cdot c_3 + c_5\cdot c_5 + c_6\cdot c_6 + c_7\cdot c_7).\end{aligned}$$

Thus, $3(1\cdot 1+\mathbf{e}\cdot\mathbf{e}) - (c_1\cdot c_1 + c_2\cdot c_2 + c_3\cdot c_3 + c_5\cdot c_5 + c_6\cdot c_6 + c_7\cdot c_7) \in U_{\Lambda''(0,0,0,0,0,0)}$. On the other hand, $1\cdot 1-\mathbf{e}\cdot\mathbf{e}-2\sqrt{-1}(1\cdot\mathbf{e})$, $1\cdot 1-\mathbf{e}\cdot\mathbf{e}+2\sqrt{-1}(1\cdot\mathbf{e}) \in S_0^2(\mathbf{K}^{\mathbf{C}})$, and we see that

$$\begin{aligned}A(1\cdot 1-\mathbf{e}\cdot\mathbf{e}-2\sqrt{-1}1\cdot\mathbf{e}) = \mathbf{e}^{\sqrt{-1}2t}(1\cdot 1-\mathbf{e}\cdot\mathbf{e}-2\sqrt{-1}1\cdot\mathbf{e}),\\ A(1\cdot 1-\mathbf{e}\cdot\mathbf{e}+2\sqrt{-1}1\cdot\mathbf{e}) = \mathbf{e}^{-\sqrt{-1}2t}(1\cdot 1-\mathbf{e}\cdot\mathbf{e}+2\sqrt{-1}1\cdot\mathbf{e}).\end{aligned}$$

Hence, $1\cdot 1-\mathbf{e}\cdot\mathbf{e}-2\sqrt{-1}1\cdot\mathbf{e} \in U_{\Lambda''(0,0,2,0,0,0)}$, $1\cdot 1-e\cdot\mathbf{e}+2\sqrt{-1}1\cdot\mathbf{e} \in U_{\Lambda''(0,0,-2,0,0,0)}$. Therefore,

$$\begin{aligned}&(V_\Lambda)_{K_0}\cap W_{\Lambda_1'}\\ =\ &\mathbf{C}(3(1\cdot 1+\mathbf{e}\cdot\mathbf{e}) - (c_1\cdot c_1 + c_2\cdot c_2 + c_3\cdot c_3 + c_5\cdot c_5 + c_6\cdot c_6 + c_7\cdot c_7))\\ &\oplus\mathbf{C}(1\cdot 1-\mathbf{e}\cdot\mathbf{e}-2\sqrt{-1}1\cdot\mathbf{e})\\ &\oplus\mathbf{C}(1\cdot 1-\mathbf{e}\cdot\mathbf{e}+2\sqrt{-1}1\cdot\mathbf{e}).\end{aligned}$$

Since the action of the generator $\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3)$ is given by

$$(\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3))(2\sqrt{-1}(1 \cdot \mathbf{e})) = 2(\sqrt{-1}\mathbf{e} \cdot (-1)) = -2\sqrt{-1}(1 \cdot \mathbf{e}),$$
$$(\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3))(1 \cdot 1 - \mathbf{e} \cdot \mathbf{e}) = 1 \cdot 1 - \mathbf{e} \cdot \mathbf{e},$$
$$(\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3))(1 \cdot 1 + \mathbf{e} \cdot \mathbf{e}) = -(1 \cdot 1 + \mathbf{e} \cdot \mathbf{e}),$$

we obtain

$$(V_\Lambda)_{K_{[\mathfrak{a}]}} \cap W_{\Lambda_1'} = \mathbf{C}(1 \cdot 1 - \mathbf{e} \cdot \mathbf{e}),$$

and thus $\Lambda = y_1+y_2+y_3+y_4 \in D(K, K_{[\mathfrak{a}]})$, which has eigenvalue 30 of $-\mathcal{C}_L$ with the multiplicity 1. Therefore,

$$\begin{aligned} n(L^{30}) &= \dim_{\mathbf{C}} V_{(6,1,1,1,0,0)} + \dim_{\mathbf{C}} V_{(-6,1,1,1,0,0)} + \dim_{\mathbf{C}} V_{(0,1,1,1,0,0)} \\ &= 120 + 120 + 210 = 450 \\ &= \dim SO(32) - \dim U(1) \cdot Spin(10) = n_{kl}(L^{30}). \end{aligned}$$

Then we conclude that

Theorem. *The Gauss image*

$$L^{30} = (U(1) \cdot Spin(10))/(S^1 \cdot Spin(6) \cdot \mathbf{Z}_4) \subset Q_{30}(\mathbf{C})$$

is strictly Hamiltonian stable.

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