CHARACTERIZATIONS OF UMBILIC HYPERSURFACES IN WARPED PRODUCT MANIFOLDS

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ABSTRACT. We consider closed orientable hypersurfaces in a wide class of warped product manifolds which include space forms, deSitter-Schwarzschild and Reissner-Nordström manifolds. By using a new integral formula or Brendle's Heintze-Karcher type inequality, we present some new characterizations of umbilic hypersurfaces. These results can be viewed as generalizations of the classical Jellet-Liebmann theorem and the Alexandrov theorem in Euclidean space.

1. INTRODUCTION

The characterization of hypersurfaces with constant mean curvature in warped product manifolds has attracted much attention recently. There are at least three types of results. Classical Jellet-Liebmann theorem, also referred to as the Liebmann-Süss theorem, asserts that any closed star-shaped (or convex) immersed hypersurface in Euclidean space with constant mean curvature is a round sphere. This has been generalized to a class of warped products by Montiel [9]. Similar results are also obtained for hypersurfaces with constant higher order mean curvature or Weingarten hypersurfaces in warped products (see [1, 5, 12]).

The classical Alexandrov theorem states that any closed embedded hypersurface of constant mean curvature in Euclidean space is a round sphere. This was generalized to a class of warped product manifolds by Brendle [4]. The key step in his proof is the Minkowski type formula and a Heintze-Karcher type inequality, which also works for Weingarten hypersurfaces (c.f. [5, 12]). In [6], Kwong-Lee-Pyo proved Alexandrov type results for closed embedded hypersurfaces with radially symmetric higher order mean curvature in a class of warped products.

2010 *Mathematics Subject Classification.* 53C44, 53C40, 52A20.

Key words and phrases. umbilic, k-th mean curvature, warped products.

This work was supported by National Natural Science Foundation of China (Grant No. 11671223 and Grant No. 11831005).

From a variational point of view, hypersurfaces of constant mean curvature in a Riemannian manifold are critical points of the area functional under variations preserving a certain enclosed volume (see [3, 2]). Under the assumption of stability, constant mean curvature hypersurfaces in warped products are studied in [8, 11] etc.

In this paper, we prove Jellet-Liebmann type theorems and an Alexandrov type theorem for certain closed hypersurfaces including constant mean curvature hypersurfaces in some class of warped product manifolds. Throughout this paper, we assume that $\bar{M}^{n+1} = [0, \bar{r}) \times_\lambda P^n$ $(0 < \bar{r} \leq \infty)$ is a warped product manifold endowed with a metric

$$\bar{g} = dr^2 + \lambda^2(r) g^P,$$

where (P, g^P) is an n-dimensional closed Riemannian manifold $(n \geq 2)$ and $\lambda : [0, \bar{r}) \to [0, +\infty)$ is a smooth positive function, called the warping function. We first consider a hypersurface $x : M^n \to \bar{M}^{n+1}$ immersed in a warped product $\bar{M}^{n+1}$ whose mean curvature H satisfies

$$H = \phi(r), \tag{1}$$

where $\phi(r) = x^*(\Phi(r))$ and $\Phi(r)$ is a radially symmetric positive function on $\bar{M}$; or

$$H^{-\alpha} = \langle \lambda \partial_r, \nu \rangle, \tag{2}$$

where ν is a normal vector of M and $\alpha > 0$ is a constant.

Notice that hypersurfaces satisfying (1) are critical points of the area functional under variations preserving a weighted volume (see Appendix A). And hypersurfaces satisfying (2) are self-similar solutions to the curvature flow expanding by $H^{-\alpha}$. Both of them can be regard as generalizations of constant mean curvature hypersurfaces.

Our first main results are Jellet-Liebmann type theorems of these hypersurfaces.

Theorem 1.1. *Suppose that $(\bar{M}^{n+1}, \bar{g})$ is a warped product manifold satisfying*

$$\mathrm{Ric}^P \geq (n-1)(\lambda'^2 - \lambda\lambda'') g^P,$$

and $x : M \to \bar{M}$ is an immersion of a closed orientable hypersurface M^n in $\bar{M}$. If $x(M)$ is star-shaped and satisfies

$$\langle \nabla H, \partial_r \rangle \leq 0, \tag{3}$$

then $x(M)$ must be totally umbilic.

Remark 1.2. If M has constant mean curvature, Theorem 1.1 reduces to the Jellett-Liebmann type theorem proved by Montiel [9].

Applying Theorem 1.1 to hypersurfaces satisfying (1), we obtain the following

Corollary 1.3. *Under the same assumption of Theorem 1.1, if $x(M)$ is star-shaped and satisfies*

$$H = \phi(r),$$

where $\phi(r) = x^*(\Phi(r))$ and $\Phi(r)$ is a positive non-increasing function of r, then $x(M)$ must be totally umbilic.

Remark 1.4. An Alexandrov type theorem for the above hypersurfaces under the embeddedness assumption was obtained by Kwong-Lee-Pyo [6].

The following example shows that the non-increasing assumption on $\Phi(r)$ is necessary.

Example 1.5. Let Σ be an ellipsoid given by

$$\Sigma = \{y \in \mathbb{R}^{n+1} | y_1^2 + ... + y_n^2 + \frac{y_{n+1}^2}{a^2} = 1\}.$$

The mean curvature of Σ is

$$H = \frac{a}{n\sqrt{a^2+1-r^2}}(n - 1 + \frac{1}{a^2+1-r^2}).$$

It is easy to check

$$\Phi = \frac{a}{n\sqrt{a^2+1-r^2}}(n - 1 + \frac{1}{a^2+1-r^2})$$

is increasing for r.

Another application of Theorem 1.1 is about hypersurfaces satisfying (2).

Corollary 1.6. *Under the same assumption of Theorem 1.1, if $x(M)$ is strictly convex and satisfies*

$$H^{-\alpha} = \langle \lambda \partial_r, \nu \rangle,$$

where $\alpha > 0$ is a constant, then $x(M)$ is a slice $\{r_0\} \times P$ for some $r_0 \in (0, \bar{r})$.

Similar results hold for higher order mean curvature under stronger assumptions. Let $\sigma_k(\kappa)$ denote the k-th elementary symmetric polynomial of principal curvatures $\kappa = (\kappa_1, ..., \kappa_n)$ of $x(M)$, i.e.,

$$\sigma_k(\kappa) = \sum_{1 \le i_1 < ... < i_k \le n} \kappa_{i_1} \cdots \kappa_{i_k}.$$

Thus, the k-th mean curvature is given by $H_k = \frac{1}{\binom{n}{k}}\sigma_k(\kappa)$. A hypersurface $x(M)$ is k-convex, if, at any point of M, principal curvatures

$$\kappa \in \Gamma_k := \{\mu \in \mathbb{R}^n | \sigma_i(\mu) > 0, \text{ for } 1 \leq i \leq k\}.$$

Theorem 1.7. *Suppose that $\bar{M}^{n+1} = [0, \bar{r}) \times_\lambda P^n$ is a warped product manifold, where (P, g^P) is a closed Riemannian manifold with constant sectional curvature ϵ and*

$$\frac{\lambda(r)''}{\lambda(r)} + \frac{\epsilon - \lambda(r)'^2}{\lambda(r)^2} \geq 0. \tag{4}$$

Let $x : M \to \bar{M}$ be an immersion of a closed orientable hypersurface M^n in $\bar{M}$. For any fixed k with $2 \leq k \leq n-1$, if $x(M)$ is k-convex, star-shaped and satisfies

$$\langle \nabla H_k, \partial_r \rangle \leq 0,$$

then $x(M)$ must be totally umbilic.

If we require that the inequality in (4) is strict as in [4, 5] and $H_k =$ constant, we obtain

Corollary 1.8. *Suppose that $\bar{M}^{n+1} = [0, \bar{r}) \times_\lambda P^n$ is a warped product manifold, where (P, g^P) is a closed Riemannian manifold with constant sectional curvature ϵ and*

$$\frac{\lambda(r)''}{\lambda(r)} + \frac{\epsilon - \lambda(r)'^2}{\lambda(r)^2} > 0. \tag{5}$$

Let $x : M \to \bar{M}$ be an immersion of a closed orientable hypersurface M^n in $\bar{M}$. For any fixed k with $2 \leq k \leq n-1$, if $x(M)$ is k-convex, star-shaped and $H_k =$ constant, then $x(M)$ is a slice $\{r_0\} \times P$ for some $r_0 \in (0, \bar{r})$.

Remark 1.9. The above corollary implies that the embeddedness condition in Theorem 2 of [5] is not necessary.

Corollary 1.10. *Under the same assumption of Theorem 1.7, if for any fixed k with $2 \leq k \leq n-1$, $x(M)$ is k-convex, star-shaped and satisfies*

$$H_k = \phi(r),$$

where $\phi(r) = x^*(\Phi(r))$ and $\Phi(r)$ is a positive non-increasing function of r, then $x(M)$ must be totally umbilic.

Corollary 1.11. *Under the same assumption of Theorem 1.7, if for any fixed k with $2 \leq k \leq n-1$, $x(M)$ is strictly convex and satisfies*

$$H_k^{-\alpha} = \langle \lambda \partial_r, \nu \rangle,$$

where $\alpha > 0$ is a constant, then $x(M)$ is a slice $\{r_0\} \times P$ for some $r_0 \in (0, \bar{r})$.

Now we turn to the warped product manifold $\bar{M}^{n+1} = [0, \bar{r}) \times_\lambda P^n$, where (P, g^P) is with a closed Riemannian manifold with constant sectional curvature ϵ. As in [4, 12], we list four conditions of the warping function $\lambda : [0, \bar{r}) \to [0, +\infty)$:

(C1) $\lambda'(0) = 0$ and $\lambda''(0) > 0$.
(C2) $\lambda'(r) > 0$ for all $r \in (0, \bar{r})$.
(C3) The function

$$2\frac{\lambda''(r)}{\lambda(r)} - (n-1)\frac{\epsilon - \lambda'(r)^2}{\lambda(r)^2}$$

is non-decreasing for $r \in (0, \bar{r})$.
(C4) We have

$$\frac{\lambda''(r)}{\lambda(r)} + \frac{\epsilon - \lambda'(r)^2}{\lambda(r)^2} > 0$$

for all $r \in (0, \bar{r})$.

Instead of star-sharpness or convexity, under the embeddedness assumption, we study hypersurfaces satisfying

$$H_k^{-\alpha}\lambda' = \langle \lambda \partial_r, \nu \rangle, \tag{6}$$

and prove the following Alexandrov type theorem.

Theorem 1.12. *Suppose that $(\bar{M}, \bar{g})$ is a warped product manifold satisfying conditions (C1)-(C4). Let $x : M \to \bar{M}$ be an immersion of a connected closed embedded orientable hypersurface M^n in $\bar{M}$. If $H_k > 0$ and $x(M)$ satisfies*

$$H_k^{-\alpha}\lambda' = \langle \lambda \partial_r, \nu \rangle, \tag{7}$$

for any fixed k with $1 \le k \le n$ and $\alpha \ge \frac{1}{k}$, then $x(M)$ is a slice $\{r_0\} \times P$ for some $r_0 \in (0, \bar{r})$.

Remark 1.13. It is interesting to compare Theorem 1.12 and Corollary 1.11 in the special case when $P = \mathbb{S}^n$ and $\lambda(r) = r$, i.e. $\bar{M}$ is Euclidean space $\mathbb{R}^{n+1}$. With embeddedness and less convexity requirement (only $H_k > 0$), Theorem 1.12 leads to the conclusion including the case when $k = n$.

Remark 1.14. Throughout the paper, the assumptions for the ambient spaces $\bar{M}$ are satisfied by space forms, the deSitter-Schwarzschild, the Reissner-Nordström manifolds and many other manifolds (c.f. [4]).

The paper is organized as follows. In Section 2, we list some useful properties of warped products. In Section 3, we derive an integral formula which is the key to the proof of our main theorems. In Section 4, we present the proofs of Theorem 1.1, Theorem 1.7 and the corollaries. In Section 5, we prove Theorem 1.12. In Appendix A, we show that a hypersurface with a given positive mean curvature function is the critical point of the area functional under variations preserving weighted volume. Throughout the paper, the summation convention is used unless otherwise stated.

2. Preliminaries of warped products

In this section, we list some basic properties of warped products $(\bar{M} = [0, \bar{r}) \times_\lambda P^n, \bar{g})$ given above (see [10]).

Proposition 2.1. *Suppose $U, V \in \Gamma(TP)$. The Levi-Civita connection $\bar{\nabla}$ of a warped product $(\bar{M} = [0, \bar{r}) \times_\lambda P, \bar{g})$ satisfies*

i) $\bar{\nabla}_{\partial_r} \partial_r = 0$,

ii) $\bar{\nabla}_{\partial_r} V = \bar{\nabla}_V \partial_r = \frac{\lambda'}{\lambda} V$,

iii) $\bar{\nabla}_V U = \nabla^P_V U - \frac{\lambda'}{\lambda} \bar{g}(V, U) \partial_r$,

where ∇^P is the Levi-Civita connection of (P, g^P).

Remark 2.2. From the preceding proposition, we know that any slice $\{r\} \times P$ in a warped product $\bar{M} = [0, \bar{r}) \times_\lambda P$ is totally umbilic.

Proposition 2.3. *Suppose $Y_1, Y_2, Y_3, Y_4 \in \Gamma(TP)$. The $(0, 4)$-Riemannian curvature tensor $\overline{\mathrm{Rm}}$ of a warped product $(\bar{M} = [0, \bar{r}) \times_\lambda P, \bar{g})$ satisfies*

i) $\overline{\mathrm{Rm}}(\partial_r, Y_1, \partial_r, Y_2) = -\frac{\lambda''}{\lambda} \bar{g}(Y_1, Y_2)$,

ii) $\overline{\mathrm{Rm}}(\partial_r, Y_1, Y_2, Y_3) = 0$,

iii) $\overline{\mathrm{Rm}}(Y_1, Y_2, Y_3, Y_4) = \lambda^2 \mathrm{Rm}^P(Y_1, Y_2, Y_3, Y_4) - \frac{\lambda'^2}{\lambda^2} (\bar{g}(Y_1, Y_3)\bar{g}(Y_2, Y_4) - \bar{g}(Y_2, Y_3)\bar{g}(Y_1, Y_4))$,

where Rm^P is the $(0, 4)$-Riemannian curvature tensor of (P, g^P).

Proposition 2.4. *Suppose $U, V \in \Gamma(TP)$. The Ricci curvature tensor $\overline{\mathrm{Ric}}$ of a warped product $(\bar{M} = [0, \bar{r}) \times_\lambda P, \bar{g})$ satisfies*

i) $\overline{\mathrm{Ric}}(\partial_r, \partial_r) = -n\frac{\lambda''}{\lambda}$,

ii) $\overline{\mathrm{Ric}}(\partial_r, V) = 0$,

iii) $\overline{\mathrm{Ric}}(V, U) = \mathrm{Ric}^P(V, U) - \left(\frac{\lambda''}{\lambda} + (n-1)\frac{\lambda'^2}{\lambda^2}\right) \bar{g}(V, U)$,

where Ric^P is the Ricci curvature tensor of (P, g^P).

For the convenience, we introduce the Kulkarni-Nomizu product $\owedge$. For any two $(0, 2)$-type symmetric tensors h and w, $h \owedge w$ is the 4-tensor given by

$$(h \owedge w)(X_1, X_2, X_3, X_4) = h(X_1, X_3)w(X_2, X_4) + h(X_2, X_4)w(X_1, X_3) \\ - h(X_1, X_4)w(X_2, X_3) - h(X_2, X_3)w(X_1, X_4).$$

The following result is a corollary of Proposition 2.3 through a straightforward calculation and the proof is given for the completeness.

Proposition 2.5. *Suppose (P, g^P) is a Riemannian manifold with constant sectional curvature ϵ. The Riemannian curvature tensor $\overline{\mathrm{Rm}}$ of a warped product $\bar{M} = [0, \bar{r}) \times_\lambda P$ can be expressed as follows:*

$$\overline{\mathrm{Rm}} = \frac{\epsilon - \lambda'^2}{2\lambda^2}\bar{g} \owedge \bar{g} - \left(\frac{\lambda''}{\lambda} + \frac{\epsilon - \lambda'^2}{\lambda^2}\right) \bar{g} \owedge dr^2. \tag{8}$$

Proof. Let $e_A, e_B, e_C, e_D \in \Gamma(T\bar{M})$, $r_A = \bar{g}(e_A, \partial_r)$ and $e_A^* = e_A - r_A\partial_r$. Using Proposition 2.3, we have

$$\begin{aligned}
&\overline{\mathrm{Rm}}(e_A, e_B, e_C, e_D) \\
&= \overline{\mathrm{Rm}}(e_A^*, e_B^*, e_C^*, e_D^*) + \overline{\mathrm{Rm}}(e_A^*, r_B\partial_r, e_C^*, r_D\partial_r) + \overline{\mathrm{Rm}}(e_A^*, r_B\partial_r, r_C\partial_r, e_D^*) \\
&\quad + \overline{\mathrm{Rm}}(r_A\partial_r, e_B^*, e_C^*, r_D\partial_r) + \overline{\mathrm{Rm}}(r_A\partial_r, e_B^*, r_C\partial_r, e_D^*) \\
&= \lambda^2\mathrm{Rm}^P(e_A^*, e_B^*, e_C^*, e_D^*) - \frac{\lambda'^2}{\lambda^2}\Big(\bar{g}(e_A^*, e_C^*)\bar{g}(e_B^*, e_D^*) - \bar{g}(e_B^*, e_C^*)\bar{g}(e_A^*, e_D^*)\Big) \\
&\quad - \frac{\lambda''}{\lambda}\Big(r_Br_D\bar{g}(e_A^*, e_C^*) - r_Br_C\bar{g}(e_A^*, e_D^*) - r_Ar_D\bar{g}(e_B^*, e_C^*) + r_Ar_C\bar{g}(e_B^*, e_D^*)\Big).
\end{aligned}$$

Since

$$\bar{g}(e_A^*, e_C^*) = \bar{g}(e_A, e_C) - r_Ar_C = (\bar{g} - dr^2)(e_A, e_C),$$

we know

$$\begin{aligned}
&\bar{g}(e_A^*, e_C^*)\bar{g}(e_B^*, e_D^*) - \bar{g}(e_B^*, e_C^*)\bar{g}(e_A^*, e_D^*) \\
&\quad = \frac{1}{2}(\bar{g} - dr^2) \owedge (\bar{g} - dr^2)(e_A, e_B, e_C, e_D)
\end{aligned}$$

and

$$\begin{aligned}
&r_Br_D\bar{g}(e_A^*, e_C^*) - r_Br_C\bar{g}(e_A^*, e_D^*) - r_Ar_D\bar{g}(e_B^*, e_C^*) + r_Ar_C\bar{g}(e_B^*, e_D^*) \\
&\quad = (\bar{g} - dr^2) \owedge dr^2(e_A, e_B, e_C, e_D).
\end{aligned}$$

Using the sectional curvatures of (P, g^P) is a constant ϵ, i.e., $\mathrm{Rm}^P = \frac{\epsilon}{2}g^P \owedge g^P$, we have

(9)

$$\lambda^2\mathrm{Rm}^P(e_A^*, e_B^*, e_C^*, e_D^*) = \frac{\epsilon}{2\lambda^2}(\bar{g} - dr^2) \owedge (\bar{g} - dr^2)(e_A, e_B, e_C, e_D).$$

Combining these together, we obtain

$$\overline{\mathrm{Rm}} = \frac{\epsilon - \lambda'^2}{2\lambda^2} \bar{g} \owedge \bar{g} - \left(\frac{\epsilon - \lambda'^2}{\lambda^2} + \frac{\lambda''}{\lambda} \right) \bar{g} \owedge dr^2.$$

$\square$

3. An integral formula

In this section, we obtain an integral formula by the divergence theorem, which is the key to the proof of our main results.

Let $x : M \to \bar{M}$ be an immersion of a closed orientable hypersurface M^n into a warped product $\bar{M}^{n+1} = [0, \bar{r}) \times_\lambda P^n$ endowed with a metric $\bar{g} = dr^2 + \lambda^2(r) g^P$. Let ν be a normal vector field of M and $h = (h_{ij})$ denote the second fundamental form with respect to an orthogonal frame $\{e_1, ..., e_n\}$ on M defined by

$$h_{ij} = \langle \nabla_i x, \nabla_j \nu \rangle.$$

The principal curvatures $\kappa = (\kappa_1, ..., \kappa_n)$ are the eigenvalues of h. Thus the k-th elementary symmetric polynomials of principal curvatures can be expressed as follows

$$\sigma_k(\kappa(h)) = \frac{1}{k!} \delta^{i_1 \dots i_k}_{j_1 \dots j_k} h_{i_1 j_1} \cdots h_{i_k j_k},$$

where $\delta^{i_1 \dots i_n}_{j_1 \dots j_k}$ is the generalized Kronecker symbol. Let $\sigma_{k;i}(\kappa)$ denote $\sigma_k(\kappa)$ with $\kappa_i = 0$ and $\sigma_{k;ij}(\kappa)$, with $i \neq j$, denote the symmetric function $\sigma_k(\kappa)$ with $\kappa_i = \kappa_j = 0$.

The following proposition is from a standard calculation (see also [5, 6]) and the proof is given for the completeness.

Proposition 3.1. *Under an orthonormal frame such that $h_{ij} = \kappa_i \delta_{ij}$, we have the following equality*

$$\sum_i \nabla_i \left(\frac{\partial \sigma_k(h)}{\partial h_{ij}} \right) = - \sum_{p \neq j} \bar{R}_{\nu p j p} \sigma_{k-2;jp}(\kappa)$$

for any fixed j and $2 \leq k \leq n$.

Proof. Let $\tilde{h} = I + th$. Then,

$$\sigma_n(\tilde{h}) = \sigma_n(I + th) = \sum_{k=0}^{n} t^k \sigma_k(h). \tag{10}$$

Using

$$\frac{\partial \sigma_n(\tilde{h})}{\partial h_{ij}} = t(\tilde{h}^{-1})_{ij} \sigma_n(\tilde{h}),$$

$$\sum_{i=1}^{n} \nabla_i (\tilde{h}^{-1})_{ij} = -t(\tilde{h}^{-1})_{ip}(\tilde{h}^{-1})_{qj} \nabla_i h_{pq},$$

for arbitrary t and the Codazzi equation

$$\nabla_i h_{pq} = \nabla_q h_{pi} + \bar{R}_{\nu pqi},$$

we have

$$\begin{aligned} t^k \nabla_i (\frac{\partial \sigma_k(h)}{\partial h_{ij}}) &= \nabla_i (\frac{\partial \sigma_n(\tilde{h})}{\partial h_{ij}}) \\ &= t^2 \sigma_n(\tilde{h}) \left(-(\tilde{h}^{-1})_{ip}(\tilde{h}^{-1})_{qj} \nabla_i h_{pq} + (\tilde{h}^{-1})_{ij}(h^{-1})_{pq} \nabla_i h_{pq} \right) \\ &= t^2 (\tilde{h}^{-1})_{pq}(\tilde{h}^{-1})_{ij} \sigma_n(\tilde{h}) \left(-\nabla_q h_{pi} + \nabla_i h_{pq} \right) \\ &= t^2 (\tilde{h}^{-1})_{pq}(\tilde{h}^{-1})_{ij} \sigma_n(\tilde{h}) \bar{R}_{\nu pqi}. \end{aligned} \tag{11}$$

Now we choose a local orthonormal frame $\{e_1, \cdots, e_n\}$ such that $h_{ij} = \kappa_i \delta_{ij}$. Then

$$(\tilde{h}^{-1})_{ij} = \frac{\delta_{ij}}{1 + t\kappa_i}.$$

Thus,

$$\begin{aligned} (\tilde{h}^{-1})_{pq}(\tilde{h}^{-1})_{ij} \sigma_n(\tilde{h}) \bar{R}_{\nu pqi} &= -\frac{\bar{R}_{\nu pjp}}{(1 + t\kappa_j)(1 + t\kappa_p)} \prod_{l=1}^{n} (1 + t\kappa_l) \\ &= -\bar{R}_{\nu pjp} \prod_{l \in \{1, \ldots, n\} \setminus \{j,p\}} (1 + t\kappa_l) \\ &= -\bar{R}_{\nu pjp} \sum_{k=2}^{n} t^{k-2} \sigma_{k-2;jp}(\kappa). \end{aligned}$$

Combining with (11), we have

$$\sum_{k=1}^{n} t^k \nabla_i (\frac{\partial \sigma_k(h)}{\partial h_{ij}}) = -\bar{R}_{\nu pjp} \sum_{k=2}^{n} t^k \sigma_{k-2;jp}(\kappa).$$

Comparing the coefficients of t^k, we have

$$\sum_i \nabla_i (\frac{\partial \sigma_k(h)}{\partial h_{ij}}) = -\sum_{p \neq j} \bar{R}_{\nu pjp} \sigma_{k-2;jp}(\kappa) \tag{12}$$

for each $k \in \{2, ..., n\}$. □

Denote $\eta = x^*(\int_0^r \lambda(s) ds)$ and $u = \langle \lambda \partial_r, \nu \rangle$. We have the following integral formula.

Lemma 3.2. *Suppose $x(M)$ is a closed hypersurface of $\bar{M}$. The following equality holds*

$$\int_M \{ -(n-k)\langle \nabla\sigma_k, \lambda\partial_r\rangle + ((n-k)\sigma_1\sigma_k - n(k+1)\sigma_{k+1})u - n\bar{R}_{\nu pjp}\langle\lambda\partial_r, e_j\rangle\sigma_{k-1;jp}\}d\mu = 0.$$

Proof. From a straightforward calculation, we have

$$\begin{aligned}
\nabla_i(k\sigma_k\nabla_i\eta - n\frac{\partial\sigma_k}{\partial h_{ij}}\nabla_j u) &= k\langle\nabla\sigma_k, \nabla\eta\rangle + k\sigma_k\Delta\eta - n\nabla_i(\frac{\partial\sigma_k}{\partial h_{ij}})\nabla_j u - n\frac{\partial\sigma_k}{\partial h_{ij}}\nabla_i\nabla_j u \\
&= k\langle\nabla\sigma_k, \nabla\eta\rangle + k\sigma_k(n\lambda' - \sigma_1 u) - n\nabla_i(\frac{\partial\sigma_k}{\partial h_{ij}})\langle\lambda\partial_r, h_{jl}e_l\rangle \\
&\quad + n\frac{\partial\sigma_k}{\partial h_{ij}}(-\lambda' h_{ij} - \langle\lambda\partial_r, h_{jli}e_l\rangle + h_{jl}h_{li}u) \\
&= k\langle\nabla\sigma_k, \lambda\partial_r\rangle + (n-k)\sigma_k\sigma_1 u - n(k+1)\sigma_{k+1}u - n\nabla_i(\frac{\partial\sigma_k}{\partial h_{ij}})\langle\lambda\partial_r, h_{jl}e_l\rangle \\
&\quad - n\langle\lambda\partial_r, \nabla\sigma_k\rangle - n\frac{\partial\sigma_k}{\partial h_{ij}}\langle\lambda\partial_r, e_l\rangle\bar{R}_{\nu jli}.
\end{aligned}$$

Using Proposition 3.1, we know that

$$\begin{aligned}
&-n\nabla_i(\frac{\partial\sigma_k}{\partial h_{ij}})\langle\lambda\partial_r, h_{jl}e_l\rangle - n\frac{\partial\sigma_k}{\partial h_{ij}}\langle\lambda\partial_r, e_l\rangle\bar{R}_{\nu jli} \\
&= n\bar{R}_{\nu pjp}\sigma_{k-2;jp}\langle\lambda\partial_r, \kappa_j e_j\rangle - n\sigma_{k-1;i}\langle\lambda\partial_r, e_l\rangle\bar{R}_{\nu ili} \\
&= n\bar{R}_{\nu pjp}\langle\lambda\partial_r, e_j\rangle(\sigma_{k-2;jp}\kappa_j - \sigma_{k-1;p}) \\
&= -n\bar{R}_{\nu pjp}\langle\lambda\partial_r, e_j\rangle\sigma_{k-1;jp}.
\end{aligned}$$

Combining these equalities and using divergence theorem, we finish the proof. ∎

4. Proof of the main theorems

Proof of Theorem 1.1. Using Lemma 3.2 for $k = 1$, we know that

$$\int_M \{-n(n-1)\langle\nabla H, \lambda\partial_r\rangle + ((n-1)\sigma_1^2 - 2n\sigma_2)u - n\overline{\mathrm{Ric}}(\nu, \lambda\partial_r^\top)\}d\mu = 0, \tag{13}$$

where $\partial_r^\top$ deontes the tangent part of ∂_r.

From Newton inequality and $x(M)$ is star-shaped ($\langle\partial_r, \nu\rangle > 0$), we have

$$((n-1)\sigma_1^2 - 2n\sigma_2)u \geq 0. \tag{14}$$

And the equality of (14) occurs if and only $\kappa_1 = ... = \kappa_n$.

Let $\nu^P = \nu - \langle \nu, \partial_r \rangle \partial_r$. Since

$$\begin{aligned}
&\overline{\mathrm{Ric}}(\nu, \lambda\partial_r^\top) = \overline{\mathrm{Ric}}(\nu, \lambda\partial_r) - \overline{\mathrm{Ric}}(\nu, \nu)u \\
&\quad = u\left(-n\frac{\lambda''}{\lambda} + n\frac{\lambda''}{\lambda}\langle \partial_r, \nu\rangle^2 - \mathrm{Ric}^P(\nu^P, \nu^P) + \left(\frac{\lambda''}{\lambda} + (n-1)\frac{\lambda'^2}{\lambda^2}\right)|\nu^P|^2\right) \\
&\quad = -u\left(\mathrm{Ric}^P(\nu^P, \nu^P) + (n-1)\left(\frac{\lambda''}{\lambda} - \frac{\lambda'^2}{\lambda^2}\right)|\nu^P|^2\right) \\
&\quad = -u\left(\mathrm{Ric}^P(\nu^P, \nu^P) + (n-1)\left(\lambda\lambda'' - \lambda'^2\right)g^P(\nu^P, \nu^P)\right),
\end{aligned}$$

we know $\overline{\mathrm{Ric}}(\nu, \lambda\partial_r^\top) \le 0$ by assumption.

Combining these estimates with $\langle \nabla H, \partial_r \rangle \le 0$, we obtain the left hand side (13) is nonnegative. These implies the inequalities are actually equalities at any point of M. Thus, $x(M)$ is totally umbilic. $\square$

Remark 4.1. In the previous proof, if $\mathrm{Ric}^P > (n-1)(\lambda'^2 - \lambda\lambda'')g^P$, we also obtain $\partial_r^\top = 0$. This means ∂_r is the normal vector of $x(M)$ which implies $x(M)$ is a slice.

Proof of Theorem 1.7. Under the assumption, it follows from (8) that

$$\bar{R}_{\nu pjp} = -\left(\frac{\lambda''}{\lambda} + \frac{\epsilon - \lambda'^2}{\lambda^2}\right)\langle \partial_r, \nu\rangle \langle \partial_r, e_j\rangle,$$

for any fixed p and $j \neq p$. Using

$$\sum_{p\neq j} \sigma_{k-1;jp} = (n-k)\sigma_{k-1;j},$$

we know

$$\begin{aligned}
&-\bar{R}_{\nu pjp}\langle \lambda\partial_r, e_j\rangle \sigma_{k-1;jp} = u\left(\frac{\lambda''}{\lambda} + \frac{\epsilon - \lambda'^2}{\lambda^2}\right)\langle \partial_r, e_j\rangle^2 \sigma_{k-1;jp} \\
&= (n-k)u\left(\frac{\lambda''}{\lambda} + \frac{\epsilon - \lambda'^2}{\lambda^2}\right)\langle \partial_r, e_j\rangle^2 \sigma_{k-1;j} \ge 0,
\end{aligned}$$

where the inequality follows from that $\frac{\lambda''}{\lambda} + \frac{\epsilon - \lambda'^2}{\lambda^2} \ge 0$, $x(M)$ is star-shaped and k-convex.

Similar to the previous proof, we know

$$-(n-k)\langle \nabla \sigma_k, \lambda\partial_r\rangle \ge 0$$

and

$$((n-k)\sigma_1\sigma_k - n(k+1)\sigma_{k+1})u \ge 0.$$

But Lemma 3.2 shows that the integral of these terms are zero. Thus, we know all these inequalities are actually equalities which implies $x(M)$ is totally umbilic. The rest of the proof is similar to the previous one. $\square$

Proof of Corollary 1.8. Since $H_k =$ constant, we know $\nabla\sigma_k = 0$. From the proof of Theorem 1.7,

$$-\bar{R}_{\nu pjp}\langle\lambda\partial_r, e_j\rangle\sigma_{k-1;jp} = 0.$$

Combining with the condition (5), we obtain $|\partial_r^\top| = 0$, which implies $x(M)$ is a slice. □

Proof of Corollary 1.3 and Corollary 1.10. From $H_k = \phi(r)$, by direct calculation, we have

$$\langle\nabla H_k, \partial_r\rangle = \phi'|\partial_r^\top|^2.$$

Since $\Phi(r)$ is non-increasing, we know $\phi'(r) \leq 0$. Thus,

$$\langle\nabla H_k, \partial_r\rangle \leq 0.$$

By Theorem 1.1 or Theorem 1.7, we finish the proof. □

Proof of Corollary 1.6 and Corollary 1.11. From $H_k^{-\alpha} = u$, we have

$$\langle\nabla H_k, \partial_r\rangle = -\frac{1}{\alpha}u^{-\frac{1}{\alpha}-1}\langle\nabla u, \partial_r\rangle = -\frac{1}{\alpha}u^{-\frac{1}{\alpha}-1}\lambda\kappa_i\langle e_i, \partial_r\rangle^2.$$

Since $x(M)$ is strictly convex, from $u = H_k^{-\alpha}$, we know $u > 0$. By $\alpha > 0$,

$$\langle\nabla H_k, \partial_r\rangle \leq 0.$$

From the proof of Theorem 1.1 or Theorem 1.7, we know

$$\langle\nabla H_k, \partial_r\rangle = 0.$$

This implies ∂_r is the normal vector of $x(M)$, which means $x(M)$ is a slice.

□

5. Proof of Theorem 1.12

In this section we give the proof of Theorem 1.12. By Lemma 2.3 in [7], we know $x(M)$ is k-convex from $H_k > 0$. Thus, Maclaurin's inequality

$$H_k^{\frac{1}{k}} \leq H_{k-1}^{\frac{1}{k-1}} \tag{15}$$

holds. From Brendle's Heintze-Karcher type inequality established in [4]

$$\int_M u d\mu \leq \int_M \frac{\lambda'}{H_1} d\mu$$

and the Minkowski type formula (see [5])

$$\int_M H_k u d\mu \geq \int_M H_{k-1}\lambda' d\mu,$$

combining with Maclaurin's inequality and $H_k^{-\alpha}\lambda' = u$, we obtain

$$\int_M H_k^{-\alpha}\lambda' d\mu \leq \int_M \frac{\lambda'}{H_1} d\mu \leq \int_M H_k^{-\frac{1}{k}}\lambda' d\mu \tag{16}$$

and

$$\int_M H_k^{1-\alpha}\lambda' d\mu \geq \int_M H_{k-1}\lambda' d\mu. \tag{17}$$

By Hölder's inequality, Maclaurin's inequality (15) and (17), we have

$$\begin{aligned}\int_M H_k^{-\frac{1}{k}}\lambda' d\mu &\leq \left(\int_M H_{k-1}^{1-p} H_k^{-\frac{p}{k}}\lambda' d\mu\right)^{\frac{1}{p}} \left(\int_M H_{k-1}\lambda' d\mu\right)^{\frac{p-1}{p}} \\ &\leq \left(\int_M H_k^{\frac{-kp-1+k}{k}}\lambda' d\mu\right)^{\frac{1}{p}} \left(\int_M H_k^{1-\alpha}\lambda' d\mu\right)^{\frac{p-1}{p}}.\end{aligned}$$

Choose p such that $p - 1 + \frac{1}{k} = \alpha$, then $p = \frac{k\alpha+k-1}{k}$. Notice that $p \geq 1$ implies $\alpha \geq \frac{1}{k}$. The above inequality becomes

$$\int_M H_k^{-\frac{1}{k}}\lambda' d\mu \leq \left(\int_M H_k^{-\alpha}\lambda' d\mu\right)^{\frac{k}{k\alpha+k-1}} \left(\int_M H_k^{1-\alpha}\lambda' d\mu\right)^{\frac{k\alpha-1}{k\alpha+k-1}}. \tag{18}$$

Using Hölder's inequality, (15) and (17) again as before, we obtain

$$\int_M H_k^{1-\alpha}\lambda' d\mu \leq \left(\int_M H_k^{\frac{-1+k+p-pk\alpha}{k}}\lambda' d\mu\right)^{\frac{1}{p}} \left(\int_M H_k^{1-\alpha}\lambda' d\mu\right)^{\frac{p-1}{p}}.$$

Equivalently,

$$\int_M H_k^{1-\alpha}\lambda' d\mu \leq \int_M H_k^{\frac{-1+k+p-pk\alpha}{k}}\lambda' d\mu.$$

Now we choose p such that $\frac{-1+k+p-pk\alpha}{k} = -\alpha$. Then $p = \frac{k\alpha-1+k}{k\alpha-1}$. Thus, the above inequality is

$$\int_M H_k^{1-\alpha}\lambda' d\mu \leq \int_M H_k^{-\alpha}\lambda' d\mu.$$

Substituting the above inequality into (18), we obtain

$$\int_M H_k^{-\frac{1}{k}}\lambda' d\mu \leq \int_M H_k^{-\alpha}\lambda' d\mu.$$

Combining the above inequality with (16), we know that the equality of the Heintze-Karcher type inequality occurs. As [4], we finish the proof.

Appendix A.

Let $\bar{M}^{n+1}$ be an oriented Riemannian manifold and let $x : M^n \to \bar{M}^{n+1}$ be an immersion of a closed smooth n-dimensional manifold M into $\bar{M}^{n+1}$. Suppose that a smooth map $X : (-\epsilon, \epsilon) \times M \to \bar{M}$ is a normal variation satisfying

$$\frac{\partial}{\partial t} X = -f\nu,$$

where f is a smooth function on M and ν is the unit normal of $X(t, M)$.

We introduce the weighted volume $V : (-\epsilon, \epsilon) \to \mathbb{R}$ by

$$V(t) = \int_{[0,t]\times M} X^*(e^{\Psi} d\bar{\mu}), \tag{19}$$

where $d\bar{\mu}$ is a standard volume element of $\bar{M}$ and Ψ is a smooth function on $\bar{M}$. Thus $V(t)$ represents the (oriented) weighted volume sweeping by M on the time interval $[0, t)$. By the same calculations as in [2], we have

$$V'(t) = \int_M f X^*(e^{\Psi}) d\mu, \tag{20}$$

where $d\mu$ is the volume element of M with respect to the induced metric.

Definition A.1. A variation of $x : M \to \bar{M}$ is called weighted volume-preserving variation if $V(t) \equiv 0$.

Denote

$$J(t) = A(t) + nH_0 V(t), \tag{21}$$

where $A(t) = \int_M d\mu$ and $H_0 = \frac{\int_M H d\mu}{\int_M X^*(e^{\Psi}) d\mu}$.

Then

$$J'(0) = \int_M nf(-H + H_0 e^{\psi}) d\mu,$$

where ψ denotes $x^*(\Psi)$.

Proposition A.2. *The following three statements are equivalent:*

(i) The mean curvature of $x(M)$ satisfies $H = Ce^{\psi}$ for a constant C.
(ii) For all weighted volume-preserving variations, $A'(0) = 0$.
(iii) For arbitrary variations, $J'(0) = 0$.

Proof. It is easy to check $(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$. Now, we show $(ii) \Rightarrow (i)$. We can choose $f = -He^{-\psi} + H_0$ since $\int_M fe^{\psi} d\mu = 0$. From

$$0 = J'(0) = \int_M n(-He^{-\psi} + H_0)^2 e^{\psi} d\mu,$$

we know $-He^{-\psi} + H_0 = 0$. Thus, we finish the proof. □

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