AN ENERGY FUNCTIONAL FOR LAGRANGIAN TORI IN $\mathbb{C}P^2$

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Abstract. A two-dimensional periodic Schrödingier operator is associated with every Lagrangian torus in the complex projective plane $\mathbb{C}P^2$. Using this operator we introduce an energy functional on the set of Lagrangian tori. It turns out this energy functional coincides with the Willmore functional W^- introduced by Montiel and Urbano. We study the energy functional on a family of Hamiltonian-minimal Lagrangian tori and support the Montiel–Urbano conjecture that the minimum of the functional is achieved by the Clifford torus. We also study deformations of minimal Lagrangian tori and show that if a deformation preserves the conformal type of the torus, then it also preserves the area, i.e. preserves the value of the energy functional. In particular, the deformations generated by Novikov–Veselov equations preserve the area of minimal Lagrangian tori.

1. Introduction and main results

In this paper we study Lagrangian tori in the complex projective plane $\mathbb{C}P^2$. The paper consists of two parts. In the first part we introduce an energy functional E on the set of Lagrangian tori. The value of E on the torus is an integral of the potential of the associated Schrödinger operator. It turns out that it coincides with the Willmore functional W^- introduced by Montiel and Urbano [14] from the twistor decomposition of the classical Willmore functional $E(\Sigma) = W^-(\Sigma) = 2\int_\Sigma d\sigma + \frac{1}{4}\int_\Sigma |\mathbf{H}|^2 d\sigma$ (see Lemma 1.4 bellow). We study this energy functional on a family of the Hamiltonian-minimal (H-minimal) Lagrangian tori constructed in [12] and support the Montiel–Urbano conjecture that the minimum of the functional is achieved by the Clifford torus. This conjecture can be considered as a Lagrangian analogue of the famous Willmore conjecture for tori in $\mathbb{R}^3$, which was proved by Marques and Neves [10].

The existence of the Schrödinger operator associated with Lagrangian tori allows one to use methods of spectral theory and integrable systems to study Lagrangian tori. In the second part of the paper we study minimal Lagrangian tori as an important subclass. More precisely, we study the deformations of minimal Lagrangian tori preserving the value of the energy functional, i.e. preserving the area. Such deformations give rise to eigenfunctions of the Laplace–Beltrami operator on minimal Lagrangian tori with eigenvalue 6. Using the Novikov–Veselov (NV) hierarchy [19] we propose a method of finding such eigenfunctions. We also prove that if a

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deformation of minimal Lagrangian tori preserves the conformal type of the torus then it preserves the area of the torus. As an example of such deformations one can consider deformations defined by higher equations from the NV hierarchy.

Let Σ be a closed Lagrangian surface immersed in $\mathbb{C}P^2$ with the holomorphic sectional curvature 4. Let x, y denote local conformal coordinates such that the induced metric of Σ is given by

$$ds^2 = 2e^{v(x,y)}(dx^2 + dy^2). \tag{1.1}$$

Let $r : U \to S^5$ be a local horizontal lift of the immersion defined on an open subset U of Σ, where $S^5 \subset \mathbb{C}^3$ is the unit sphere. Since Σ is Lagrangian and x, y are conformal coordinates, we have

$$\langle r, r \rangle = 1, \quad \langle r_x, r \rangle = \langle r_y, r \rangle = \langle r_x, r_y \rangle = 0, \tag{1.2}$$

$$\langle r_x, r_x \rangle = \langle r_y, r_y \rangle = 2e^v, \tag{1.3}$$

where $\langle \, , \, \rangle$ denotes the Hermitian inner product on $\mathbb{C}^3$. Hence

$$\tilde{R} = \begin{pmatrix} r \\ \frac{r_x}{|r_x|} \\ \frac{r_y}{|r_y|} \end{pmatrix} \in \mathrm{U}(3).$$

Then one can define a local function $\beta : U \to \mathbb{R}$, called the *Lagrangian angle* of Σ, by $e^{i\beta(x,y)} = \det \tilde{R}$. Consequently,

$$R = \begin{pmatrix} r \\ \frac{1}{\sqrt{2}} e^{-\frac{v}{2} - i\frac{\beta}{2}} r_x \\ \frac{1}{\sqrt{2}} e^{-\frac{v}{2} - i\frac{\beta}{2}} r_y \end{pmatrix} \in \mathrm{SU}(3). \tag{1.4}$$

By direct calculations, from

$$R_x = AR, \qquad R_y = BR, \tag{1.5}$$

where $A, B \in \mathrm{su}(3)$ (see Section 3), one can obtain the following lemma.

Lemma 1.1 ([11]). *Any local horizontal lift r of a Lagrangian surface in $\mathbb{C}P^2$ satisfies the Schrödinger equation $Lr = 0$, where*

$$L = (\partial_x - \frac{i\beta_x}{2})^2 + (\partial_y - \frac{i\beta_y}{2})^2 + V(x, y),$$

with the potential

$$V = 4e^v + \frac{1}{4}(\beta_x^2 + \beta_y^2) + \frac{i}{2}\Delta\beta,$$

$\Delta = \partial_{xx} + \partial_{yy}$.

Now assume that Σ is a Lagrangian torus given by the mapping

$$r : \mathbb{R}^2 \to S^5,$$

where r satisfies (1.2) and (1.3). Then the potential V is a doubly periodic function with respect to a lattice of periods $\Lambda \subset \mathbb{R}^2$ and r is a Bloch eigenfunction of the Schrödinger operator L, i.e.,

$$r((x, y) + \mathbf{e}_s) = e^{ip_s} r(x, y), \;\; p_s \in \mathbb{R}, \;\; s = 1, 2,$$

where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the basis of Λ. We thus can introduce the energy of the Lagrangian torus as follows.

Definition 1.2. The energy $E(\Sigma)$ of the Lagrangian torus $\Sigma \subset \mathbb{C}P^2$ is defined by

$$E(\Sigma) = \int_\Sigma V dx \wedge dy.$$

Remark 1.3. Similarly one can define the energy functional of arbitrary closed Lagrangian surfaces in $\mathbb{C}P^2$.

It turns out that the energy functional has the following geometric meaning.

Lemma 1.4. *The energy of a Lagrangian torus is*

$$E(\Sigma) = W^-(\Sigma) = 2\int_\Sigma d\sigma + \frac{1}{4}\int_\Sigma |\mathbf{H}|^2 d\sigma, \tag{1.6}$$

where $d\sigma = 2e^v dx \wedge dy$ is the area element of Σ and $\mathbf{H} = \mathrm{tr} h$ is the mean curvature vector field in terms of the second fundamental form h.

Remark 1.5. In [14] $W^-(\Sigma) = 2\int_\Sigma d\sigma + \int_\Sigma |\mathbf{H}|^2 d\sigma$, but there another definition of the mean curvature vector $\mathbf{H} = \frac{1}{2}\mathrm{tr} h$ is used.

Remark 1.6. From now we will use both energy functional and Willmore functional for the same notion (1.6).

Let Σ_{r_1,r_2,r_3} be a homogeneous torus in $\mathbb{C}P^2$, where r_1, r_2, r_3 are positive numbers such that $r_1^2 + r_2^2 + r_3^2 = 1$. Any homogeneous torus can be obtained as the image of the Hopf projection $\mathcal{H} : S^5 \to \mathbb{C}P^2$ of the 3-torus

$$\{(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}, r_3 e^{i\varphi_3}), \varphi_j \in \mathbb{R}\} \subset S^5.$$

Every homogeneous torus is H-minimal Lagrangian in $\mathbb{C}P^2$, i.e. it is a critical point of the area functional under Hamiltonian deformations (see [15]). Among homogeneous tori there is a minimal torus $\Sigma_{\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}}$, called the Clifford torus in $\mathbb{C}P^2$ and denoted by Σ_{Cl}. One can easily calculate the energy of a homogeneous torus.

Lemma 1.7. *The identity holds*

$$E(\Sigma_{r_1,r_2,r_3}) = \frac{\pi^2(1-r_1^2)(1-r_2^2)(1-r_3^2)}{r_1 r_2 r_3}.$$

Montiel and Urbano observed [14] that among Lagrangian homogeneous tori in $\mathbb{C}P^2$, the Clifford torus attains the minimum of the energy functional

$$E(\Sigma_{Cl}) = 2Area(\Sigma_{Cl}) = \frac{8\pi^2}{3\sqrt{3}}.$$

From this observation they proposed the following conjecture.

Conjecture 1.8. The Clifford torus attains the minimum of the Willmore functional among all Lagrangian tori in $\mathbb{C}P^2$.

Let us consider another family of H-minimal Lagrangian tori in $\mathbb{C}P^2$ (see [12]). Let $\Sigma_{m,n,k} \subset \mathbb{C}P^2$ be given as the image of the surface

$$\{(u_1 e^{2\pi imy}, u_2 e^{2\pi iny}, u_3 e^{2\pi iky)}\} \subset S^5$$

under the Hopf projection, where $(u_1, u_2, u_3) \in \mathbb{R}^3$ such that

$$u_1^2 + u_2^2 + u_3^2 = 1,$$

$$mu_1^2 + nu_2^2 + ku_3^2 = 0,$$

with integers $m \geq n > 0$ and $k < 0$. Such surface $\Sigma_{m,n,k}$ is an (immersed or embedded) H-minimal Lagrangian torus or a Klein bottle. The topology of $\Sigma_{m,n,k}$ depends on whether the involution

$$(u_1, u_2, u_3) \to (u_1 \cos(m\pi), u_2 \cos(n\pi), u_3 \cos(k\pi))$$

preserves the orientation of the surface $mu_1^2 + nu_2^2 + ku_3^2 = 0$ in $\mathbb{R}^3$. We obtain

Theorem 1.9. *The energy of $\Sigma_{m,n,k}$ is greater than the energy of the Clifford torus*

$$E(\Sigma_{m,n,k}) > E(\Sigma_{Cl}).$$

Theorem 1.9 supports the conjecture 1.8.

Remark 1.10. M. Haskins estimated the area of a minimal Lagrangian torus in $\mathbb{C}P^2$ in [5]. It follows from his estimate that Conjecture 1.8 is valid for tori with spectral curves of large genus. For arbitrary minimal Lagrangian tori the conjecture remains open.

In the second part of the paper we study deformations of minimal Lagrangian tori in $\mathbb{C}P^2$.

Theorem 1.11. *Let Σ_0 be a minimal Lagrangian torus in $\mathbb{C}P^2$. Suppose that Σ_t is a smooth deformation of Σ_0 preserving the conformal type of the initial surface such that Σ_t is still minimal Lagrangian. Then the area of Σ_t is preserved, i.e.,*

$$\mathrm{Area}(\Sigma_t) = \mathrm{Area}(\Sigma_0).$$

Remarkable examples of such deformations are those generated by the NV hierarchy. Let $\Sigma \subset \mathbb{C}P^2$ be a minimal Lagrangian torus. On Σ there are coordinates x, y such that the induced metric is of the form $ds^2 = 2e^{v(x,y)}(dx^2 + dy^2)$, where v satisfies the Tzizéica equation

$$\Delta v = 4(e^{-2v} - e^v). \tag{1.7}$$

Let $r : \mathbb{R}^2 \to S^5$ be a horizontal lift map for Σ satisfying (1.2) and (1.3).

Theorem 1.12 ([11])**.** *There is a mapping $\tilde{r}(t) : \mathbb{R}^2 \to S^5$, $t = (t_1, t_2, \dots)$, $\tilde{r}(0) = r$, defining a minimal Lagrangian torus $\Sigma_t \subset \mathbb{C}P^2$ such that $\Sigma_0 = \Sigma$. The map $\tilde{r}$ satisfies the equations*

$$L\tilde{r} = \Delta\tilde{r} + 4e^{\tilde{v}}\tilde{r} = 0,$$
$$\partial_{t_n}\tilde{r} = A_{2n+1}\tilde{r},$$

where A_{2n+1} is a differential operator of order $(2n+1)$ in x, y and $ds^2 = 2e^{\tilde{v}(x,y,t)}(dx^2 + dy^2)$ is the induced metric on Σ_t. The deformation $\tilde{r}(t)$ preserves the conformal type of the torus and the spectral curve of the Schrödinger operator L. The function $\tilde{V} = 4e^{\tilde{v}}$ with $\tilde{v}(0) = v$ satisfies the NV hierarchy

$$\frac{\partial L}{\partial t_n} = [A_{2n+1}, L] + B_{2n-2}L,$$

where B_{2n-2} is a differential operator of order $2n - 2$.

Thus Theorems 1.11 and 1.12 lead to the following corollary.

Corollary 1.13. *Deformations of minimal Lagrangian tori given by the Novikov–Veselov hierarchy (see Theorem 1.12) preserve the area of tori.*

Taimanov [17], [18] proved that the modified NV equation defines a deformation of tori in $\mathbb{R}^3$, which preserves the value of the Willmore functional. The tori of revolutions are preserved by this deformation. It would be interesting to prove that the first NV equation defines a deformation of arbitrary Lagrangian tori in $\mathbb{C}P^2$. In this case the spectral curve of the corresponding Schrödinger operator is preserved and the energy is also preserved. Moreover, since the first NV deformation preserves the minimal Lagrangian tori, it gives a support of Conjecture 1.8.

Deformations of minimal Lagrangian tori are related to eigenfunctions of the Laplace–Beltrami operator with eigenvalue 6. The preceding theorem provides a method to find such eigenfunctions. In the next theorem two eigenfunctions related to the second NV equation are found. Other eigenfunctions can be found with the help of explicit calculations related to the higher NV equations.

Theorem 1.14. *Let $ds^2 = 2e^{v(x,y)}(dx^2 + dy^2)$ be a metric on a surface Σ and v satisfy the Tzizéica equation. Then functions*

$$\begin{aligned} \alpha_1 &= v_x^2 - v_y^2 + v_{xx} - v_{yy}, \\ \alpha_2 &= v_x v_y + v_{xy} \end{aligned}$$

are eigenfunctions of the Laplace–Beltrami operator with eigenvalue 6.

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2. An energy functional for Lagrangian tori in $\mathbb{C}P^2$

In this section we prove Lemmas 1.4, 1.7 and Theorem 1.9.

2.1. **Proof of Lemma 1.4.**

For a given Lagrangian surface $\Sigma \subset \mathbb{C}P^2$, its mean curvature vector field $\mathbf{H} = \mathrm{tr}_g h$ satisifies

$$\mathbf{H} = J\mathrm{grad}\beta,$$

where h is the second fundamental form, J is the standard complex structure on $\mathbb{C}P^2$ and $\mathrm{grad}\beta$ is the gradient of the Lagrangian angle β with respect to the induced metric $ds^2 = 2e^v(dx^2 + dy^2)$. Thus

$$|\mathbf{H}|^2 = |J\mathrm{grad}\beta|^2 = |\mathrm{grad}\beta|^2 = \frac{e^{-v}}{2}(\beta_x^2 + \beta_y^2),$$

and

$$\begin{aligned} E(\Sigma) &= \int_\Sigma V dx \wedge dy = \int_\Sigma 4e^v dx \wedge dy + \frac{1}{4}\int_\Sigma 2e^v|\mathbf{H}|^2 dx \wedge dy + \frac{i}{2}\int_\Sigma \triangle\beta dx \wedge dy \\ &= 2\int_\Sigma d\sigma + \frac{1}{4}\int_\Sigma |\mathbf{H}|^2 d\sigma. \end{aligned}$$

2.2. **Proof of Lemma 1.7.**

Let $\Sigma_{r_1,r_2,r_3} \subset \mathbb{C}P^2$ be a homogeneous torus. Then by choosing of appropriate coordinates and taking automorphisms of $\mathbb{C}P^2$, the horizontal lift $r : \mathbb{R}^2 \to S^5$ of Σ_{r_1,r_2,r_3} can be given by

$$r(x,y) = (r_1 e^{2\pi i x}, r_2 e^{2\pi i(a_1 x + b_1 y)}, r_3 e^{2\pi i(a_2 x + b_2 y)}),$$

where $r_1^2 + r_2^2 + r_3^2 = 1$. It follows from (1.2) and (1.3) that

$$a_1 = a_2 = -\frac{r_1^2}{r_2^2 + r_3^2}, \quad b_1 = \frac{r_1 r_3}{r_2(r_2^2 + r_3^2)}, \quad b_2 = -\frac{r_1 r_2}{r_3(r_2^2 + r_3^2)}.$$

By direct calculations, we obtain that the lattice of periods Λ for $\mathcal{H}\circ r$ is $\Lambda = \{\mathbb{Z}\mathbf{e}_1 + \mathbb{Z}\mathbf{e}_2\} \subset \mathbb{R}^2$, where

$$\mathbf{e}_1 = (r_2^2 + r_3^2, 0), \quad \mathbf{e}_2 = (r_3^2, \frac{r_2 r_3}{r_1}).$$

By using $\langle r_x, r_x\rangle = \langle r_y, r_y\rangle = \frac{4\pi^2 r_1^2}{r_2^2+r_3^2}$, we have

$$\begin{aligned} ds^2 &= \frac{4\pi^2 r_1^2}{r_2^2+r_3^2}(dx^2+dy^2), \\ \int_{\Sigma_{r_1,r_2,r_3}} d\sigma &= 4\pi^2 r_1 r_2 r_3, \\ \beta &= 2\pi\frac{1-3r_1^2}{r_2^2+r_3^2}x - 2\pi\frac{r_1(r_2^2-r_3^2)}{r_2 r_3(r_2^2+r_3^2)}y, \\ V &= \frac{\pi^2(1-r_2^2)(r_1^2+r_2^2)}{r_2^2 r_3^2}. \end{aligned}$$

Hence the energy of Σ_{r_1,r_2,r_3} is

$$\begin{aligned} E(\Sigma_{r_1,r_2,r_3}) &= 8\pi^2 r_1 r_2 r_3 + \pi^2\frac{(r_1^2 r_2^4 + r_2^2 r_3^2 - 8r_1^2 r_2^2 r_3^2 + 9r_1^4 r_2^2 r_3^2 + r_1^2 r_3^4)}{r_1 r_2 r_3 (r_2^2+r_3^2)} \\ &= \frac{\pi^2(1-r_1^2)(1-r_2^2)(1-r_3^2)}{r_1 r_2 r_3}. \end{aligned}$$

Thus it is easy to find that the minimum of $E(\Sigma_{r_1,r_2,r_3})$ is attained on the Clifford torus $\Sigma_{\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}}$.

2.3. **Proof of Theorem 1.9.** By taking an appropriate parametrization of the curve given by

$$u_1^2 + u_2^2 + u_3^2 = 1,$$
$$mu_1^2 + nu_2^2 + ku_3^2 = 0,$$

where $m \geq n > 0$ and $k < 0$ are constant integers, we obtain

$$\psi(u, y) = (u_1 e^{2\pi i m y}, u_2 e^{2\pi i n y}, u_3 e^{2\pi i k y}) \subset S^5,$$

with

$$u_1 = \sin x\sqrt{\frac{k}{k-m}}, \quad u_2 = \cos x\sqrt{\frac{k}{k-n}}, \quad u_3 = \sqrt{\frac{n\cos^2 x}{n-k} + \frac{m\sin^2 x}{m-k}}.$$

Thus we have a horizontal lift of the surface in $\mathbb{C}P^2$ given by

$$r(x, y) = (u_1(x)e^{2\pi i m y}, u_2(x)e^{2\pi i n y}, u_3(x)e^{2\pi i k y}).$$

Now let us consider the torus as given above, denoted by $\Sigma_{m,n,k}$.

By straightforward calculations, we obtain that the induced metric on $\Sigma_{m,n,k}$ is

$$ds^2 = 2e^{v_1(x)}dx^2 + 2e^{v_2(x)}dy^2,$$

where

$$\begin{aligned} 2e^{v_1(x)} &= -\frac{k(m+n-(m-n)\cos(2x))}{2mn - k(m+n) + k(m-n)\cos(2x)}, \\ 2e^{v_2(x)} &= -2k\pi^2(m+n-(m-n)\cos(2x)), \end{aligned}$$

and the Lagrangian angle is $\beta = 2\pi(k+m+n)y + \frac{\pi}{2}$.

From above we know that $\mathbf{e}_1 = (2\pi, 0)$ and $\mathbf{e}_2 = (0, \frac{1}{p})$ give the basis of the lattice of the periods, where $p = (m-k, n-k)$ is the largest common factor of $m-k$ and $n-k$. Moreover,

$$\begin{aligned} A(\Sigma_{m,n,k}) &= \int_{\Sigma_{m,n,k}} d\sigma = \int_{[0,2\pi]\times[0,\frac{1}{p}]} 2e^{\frac{v_1+v_2}{2}} dx \wedge dy \\ &= \frac{1}{p}\int_0^{2\pi} \frac{-\sqrt{2}k\pi(m+n-(m-n)\cos 2x)}{\sqrt{2mn-k(m+n)+k(m-n)\cos 2x}} dx, \\ \int_{\Sigma_{m,n,k}} |\mathbf{H}|^2 d\sigma &= \int_{[0,2\pi]\times[0,\frac{1}{p}]} |\mathbf{H}|^2 2e^{\frac{v_1+v_2}{2}} dx \wedge dy \\ &= \frac{1}{p}\int_0^{2\pi} \frac{2\sqrt{2}(k+m+n)^2\pi}{\sqrt{2mn-k(m+n)+k(m-n)\cos 2x}} dx, \\ E(\Sigma_{m,n,k}) &= 2A(\Sigma_{m,n,k}) + \frac{1}{4}\int_{\Sigma_{m,n,k}} |\mathbf{H}|^2 d\sigma \\ &= \frac{1}{p}\frac{\pi}{\sqrt{2}}\int_0^{2\pi} \frac{4k(m-n)\cos 2x + (m+n-k)^2}{\sqrt{2mn-k(m+n)+k(m-n)\cos 2x}} dx. \end{aligned}$$

It follows from $k < 0$ and $n - k = pn_0$ with some positive integer n_0 that

$$\begin{aligned} E(\Sigma_{m,n,k}) &\geq \frac{1}{p}\frac{\pi}{\sqrt{2}}\int_0^{2\pi} \frac{4k(m-n)\cos 2x + (m+n-k)^2}{\sqrt{2mn-k(m+n)-k(m-n)}} \\ &= \frac{\pi^2}{p}\frac{(m+n-k)^2}{\sqrt{m(n-k)}} = \frac{\pi^2}{p}\frac{(m+n-k)(m+pn_0)}{\sqrt{mpn_0}} \\ &\geq \frac{2\pi^2}{p}(n+m-k) = 2(\frac{m}{p}+n_0)\pi^2 > 2\pi^2 > \frac{8}{3\sqrt{3}}\pi^2. \end{aligned}$$

Thus the proof of Theorem 1.9 is completed.

3. DEFORMATIONS OF MINIMAL LAGRANGIAN TORI

Now consider a Lagrangian torus in $\mathbb{C}P^2$ defined by the composition of the maps $r : \mathbb{R}^2 \to S^5$ and the Hopf projection $\mathcal{H} : S^5 \to \mathbb{C}P^2$, where r satisfies (1.2) and (1.3). Then the associated frame R given by (1.4) satisfies (1.5), where

$$A = \begin{pmatrix} 0 & \sqrt{2}e^{\frac{v+i\beta}{2}} & 0 \\ -\sqrt{2}e^{\frac{v-i\beta}{2}} & iFe^{-v} & i(\frac{\beta_y}{2}+e^{-v}G) - \frac{v_y}{2} \\ 0 & i(\frac{\beta_y}{2}+e^{-v}G) + \frac{v_y}{2} & -iFe^{-v} \end{pmatrix} \in \mathrm{su}(3),$$

$$B = \begin{pmatrix} 0 & 0 & \sqrt{2}e^{\frac{v+i\beta}{2}} \\ 0 & iGe^{-v} & i(\frac{\beta_x}{2}-e^{-v}F) + \frac{v_x}{2} \\ -\sqrt{2}e^{\frac{v-i\beta}{2}} & i(\frac{\beta_x}{2}-e^{-v}F) - \frac{v_x}{2} & -iGe^{-v} \end{pmatrix} \in \mathrm{su}(3),$$

with real functions F and G given by

$$F = -\frac{1}{2i}(\langle r_{xy}, r_y\rangle - e^v(v_x + i\beta_x))$$

and

$$G = \frac{1}{2i}(\langle r_{xy}, r_x\rangle - e^v(v_y + i\beta_y)).$$

The compatibility condition

$$A_y - B_x + [A,B] = 0$$

leads to the following equations (see [11] and also [8], [9])

$$\begin{aligned} 2F_x + 2G_y &= (\beta_{xx} - \beta_{yy})e^v, \\ 2F_y - 2G_x &= (\beta_x v_y + \beta_y v_x)e^v, \\ \triangle v &= 4(F^2+G^2)e^{-2v} - 4e^v - 2(F\beta_x - G\beta_y)e^{-v}. \end{aligned}$$

3.1. **Minimal Lagrangian tori.** Now we assume that Σ is minimal with $\beta = 0$. From the above equations it follows that F and G are constants. With the appropriate change of coordinates (a homothety and a rotation) one can assume that $F = 1$ and $G = 0$. Hence

$$A = \begin{pmatrix} 0 & \sqrt{2}e^{\frac{v}{2}} & 0 \\ -\sqrt{2}e^{\frac{v}{2}} & ie^{-v} & -\frac{v_y}{2} \\ 0 & \frac{v_y}{2} & -ie^{-v} \end{pmatrix} \in \mathrm{su}(3), \tag{3.1}$$

$$B = \begin{pmatrix} 0 & 0 & \sqrt{2}e^{\frac{v}{2}} \\ 0 & 0 & -ie^{-v} + \frac{v_x}{2} \\ -\sqrt{2}e^{\frac{v}{2}} & -ie^{-v} - \frac{v_x}{2} & 0 \end{pmatrix} \in \mathrm{su}(3), \tag{3.2}$$

and v satisfies the Tzizéica equation (1.7). Smooth periodic solutions of the Tzizéica equation are finite-gap solutions [16]. Minimal Lagrangian tori were studied in [3]–[6]. Assume that we have a deformation Σ_t of Σ, $\Sigma_0 = \Sigma$, given by the mapping

$$r(t) : \mathbb{R}^2 \to S^5$$

with the induced metric $ds^2 = 2e^{v(x,y,t)}(dx^2 + dy^2)$, where $v(x,y,t)$ satisfies (1.7). We have $R_t = TR$, where

$$T = \begin{pmatrix} is & a_1 + ib_1 & a_2 + ib_2 \\ -a_1 + ib_1 & is_1 & a_3 + ib_3 \\ -a_2 + ib_2 & -a_3 + ib_3 & -i(s+s_1) \end{pmatrix} \in \mathrm{su}(3), \tag{3.3}$$

with functions $s, s_1, a_1, a_2, a_3, b_1, b_2$ and b_3 depending on x, y and t. From the compatibility conditions

$$A_t - T_x + [A,T] = 0, \qquad B_t - T_y + [B,T] = 0, \tag{3.4}$$

we obtain the identities

$$b_1 = \frac{e^{-\frac{v}{2}}s_x}{2\sqrt{2}}, \qquad b_2 = \frac{e^{-\frac{v}{2}}s_y}{2\sqrt{2}},$$

$$a_3 = \frac{e^{-2v}(\sqrt{2}s_y - 2e^{\frac{3}{2}v}(a_1 v_y - 2a_{2x}))}{4\sqrt{2}},$$

$$b_3 = \frac{e^{-\frac{3}{2}v}(-8a_2 - \sqrt{2}e^{\frac{1}{2}v}(s_x v_y + s_y v_x - 2s_{xy}))}{8\sqrt{2}},$$

$$s_1 = \frac{e^{-\frac{3}{2}v}(8a_1 + \sqrt{2}e^{\frac{1}{2}v}(8e^v s + s_y v_y - s_x v_x + 2s_{xx}))}{8\sqrt{2}}$$

and the following overdetermined system of equations for s, a_1, a_2 which determine the deformation:

$$v_t + \frac{1}{2}e^{-2v}(-\sqrt{2}e^{\frac{3}{2}v}a_2 v_y - 2\sqrt{2}e^{\frac{3}{2}v}a_{1x} + s_x) = 0, \tag{3.5}$$

$$\triangle s+12e^{v}s=0, \tag{3.6}$$

$$2a_{1x}-2a_{2y}-a_1v_x+a_2v_y-\sqrt{2}e^{-\frac{3}{2}v}s_x=0, \tag{3.7}$$

$$2a_{1y}+2a_{2x}-a_1v_y-a_2v_x+\sqrt{2}e^{-\frac{3}{2}v}s_y=0, \tag{3.8}$$

$$\sqrt{2}e^{\frac{3v}{2}}a_2v_y+2\sqrt{2}e^{\frac{3v}{2}}a_{1x}-s_x+2e^{3v}(b_3v_y+s_x+s_{1x})=0, \tag{3.9}$$

$$2e^{-v}a_3-s_{1y}+b_3v_x=0. \tag{3.10}$$

Now we can give a geometric interpretation of s.

Remark 3.1.
(1) The deformation we obtained above is a Hamiltonian deformation with the Hamiltonian $\frac{s}{2}$. In fact,

$$r_t^{\perp}=\frac{b_1}{\sqrt{2}e^{\frac{v}{2}}}ir_x+\frac{b_2}{\sqrt{2}e^{\frac{v}{2}}}ir_y=\frac{e^{-v}}{4}(s_xir_x+s_yir_y)=\mathrm{grad}\frac{s}{2},$$

where $r_t^{\perp}$ denotes the normal component of the velocity vector r_t.
(2) Observe that (3.6) can be rewritten as

$$\triangle_{LB}s=6s,$$

where $\triangle_{LB}$ is the Laplace–Beltrami operator with respect to the induced metric on Σ_t. In other words, the function s is an eigenfunction of the Laplace–Beltrami operator with the eigenvalue 6.

Proof of Theorem 1.11. Using (3.5) and (3.7), we get

$$v_t \quad = \quad \frac{e^{-\frac{v}{2}}(2a_{2y}+a_2v_y+2a_{1x}+a_1v_x)}{2\sqrt{2}}.$$

Considering the area form $d\sigma=2e^{v}dx\wedge dy$, we set

$$\Omega=\partial_t(2e^{v})dx\wedge dy=\frac{e^{\frac{v}{2}}(2a_{2y}+a_2v_y+2a_{1x}+a_1v_x)}{\sqrt{2}}dx\wedge dy.$$

It turns out that $\Omega=d\omega$, where

$$\omega=\sqrt{2}e^{\frac{v}{2}}(a_1dy-a_2dx).$$

If Σ_t is a smooth deformation of Σ_0 preserving the conformal type of Σ_0, then

$$\frac{d}{dt}\int_{\Lambda}2e^{v}dx\wedge dy=\int_{\Lambda}\Omega=\int_{\Lambda}d\omega=0,$$

where Λ is a lattice of periods. The proof is completed.

□

3.2. **The Novikov–Veselov hierarchy and deformations of minimal Lagrangian tori.** In this subsection we consider an example of a deformation of minimal Lagrangian tori defined by the second NV equation. In particular we give an explicit solution of the system (3.5)–(3.10) in terms of the function v defining the induced metric (1.1) of the torus.

Let us recall the NV hierarchy. Let L be a Schrödinger operator

$$L = \partial_z \partial_{\bar{z}} + V(z, \bar{z}),$$

and

$$A_{2n+1} = \partial_z^{2n+1} + u_{2n-1}\partial_z^{2n-1} + \cdots + u_1\partial_z + \partial_{\bar{z}}^{2n+1} + w_{2n-1}\partial_{\bar{z}}^{2n-1} + \cdots + w_1\partial_{\bar{z}},$$

where $u_j = u_j(z, \bar{z}), w_j = w_j(z, \bar{z})$. The operator $\partial_{t_n} - A_{2n+1}$ defines an evolution equation for the eigenfunction r of the Schrödinger operator $Lr = 0$ by

$$\partial_{t_n} r = A_{2n+1} r.$$

The n-th NV equation is

$$\frac{\partial L}{\partial t_n} = [A_{2n+1}, L] + B_{2n-2}L,$$

where B_{2n-2} is a differential operator of order $2n-2$. Note that V and the coefficients of A_{2n+1} are the unknowns. In the case of the periodic Schrödinger operator L the NV equation preserves its spectral curve.

When $n = 1$, as obtained in [13], we could take $V = e^{v(z,\bar{z},t_1)}$, where v is a real function satisfying the Tzizéica equation, and

$$\begin{aligned} A_3 &= \partial_z^3 + \partial_{\bar{z}}^3 - (v_z^2 + v_{zz})\partial_z - (v_{\bar{z}}^2 + v_{\bar{z}\bar{z}})\partial_{\bar{z}}, \\ B_0 &= -\partial_z(v_z^2 + v_{zz}) - \partial_{\bar{z}}(v_{\bar{z}}^2 + v_{\bar{z}\bar{z}}). \end{aligned}$$

It turns out the first NV equation reduces to $v_{t_1} = 0$. Hence it cannot provide a deformation of minimal Lagrangian tori in $\mathbb{C}P^2$.

For the case where $n = 2$, as discussed in [13], assume

$$A_5 = \partial_z^5 + u_3\partial_z^3 + u_{3z}\partial_z^2 + u_1\partial_z + \partial_{\bar{z}}^5 + w_3\partial_{\bar{z}}^3 + w_{3\bar{z}}\partial_{\bar{z}}^2 + w_1\partial_{\bar{z}},$$

$$B_2 = u_{3z}\partial_z^2 + u_{3zz}\partial_z + w_{3\bar{z}}\partial_{\bar{z}}^2 + w_{3\bar{z}\bar{z}}\partial_{\bar{z}} + u_{1z} + w_{1\bar{z}},$$

the second NV equation reduces to

$$u_{3\bar{z}} = 5V_z, \qquad w_{3z} = 5V_{\bar{z}}, \tag{3.11}$$

$$u_{1\bar{z}} = 10V_{zzz} + 3u_3V_z + u_{3z}V - u_{3zz\bar{z}}, \tag{3.12}$$

$$w_{1z} = 10V_{\bar{z}\bar{z}\bar{z}} + 3w_3V_{\bar{z}} + w_{3\bar{z}}V - w_{3z\bar{z}\bar{z}}, \tag{3.13}$$

$$\begin{aligned} V_{t_2} = {} & \partial_z^5 V + u_3V_{zzz} + 2u_{3z}V_{zz} + (u_1 + u_{3zz})V_z + u_{1z}V \\ & + \partial_{\bar{z}}^5 V + w_3V_{\bar{z}\bar{z}\bar{z}} + 2w_{3\bar{z}}V_{\bar{z}\bar{z}} + (w_1 + w_{3\bar{z}\bar{z}})V_{\bar{z}} + w_{1\bar{z}}V. \end{aligned} \tag{3.14}$$

Moreover, by the argument of [13], the equations (3.11)–(3.13) can be solved explicitly in the following way.

Theorem 3.2 ([13]). *Let $v(z,\bar z,t_2)$ be a real function satisfying the Tzizéica equation and $V = e^{v(z,\bar z,t_2)}$. Then the functions*

$$u_3 = -\frac{5}{3}(v_z^2 + v_{zz}), \quad w_3 = \bar u_3,$$
$$u_1 = \frac{5}{9}v_z^4 + \frac{10}{9}v_z^2 v_{zz} - \frac{5}{3}v_z^2 - \frac{20}{9}v_z v_{zzz} - \frac{10}{9}v_{zzzz}, \qquad w_1 = \bar u_1,$$

satisfy equations (3.11)–(3.13). The second Novikov–Veselov equation (3.14) attains the following form

$$v_{t_2} = h + \bar h, \tag{3.15}$$
$$h = \frac{1}{9}(5v_1v_2^2 + 5v_1^2v_3 - 5v_2v_3 - v_1^5 - v_5),$$

where $v_j = \partial_z^j v$.

Let us consider a minimal Lagrangian torus defined by $r : \mathbb{R}^2 \to S^5$ such that $R_x = AR, R_y = BR$, matrices A, B have the form (3.1), (3.2). If we rewrite the NV deformation

$$\partial_{t_2} r = A_5 r$$

in terms of the frame R, we get the corresponding matrix $T \in \mathrm{su}(3)$ such that $R_t = TR$. The matrix T is given by (3.3), where

$$\begin{aligned}
a_1 \;&=\; \frac{1}{72\sqrt{2}}e^{\frac{v}{2}}(144e^{-v} - v_y^4 - 3v_y^2 - v_{yyyy} - v_x^4 + 12v_{xy}^2 - 3v_xv_{xyy}\\
&\quad -2v_{yy}(2v_x^2 - 3v_{xx}) - 2v_y^2(2v_{yy} + 3v_x^2 - 2v_{xx}) - 4v_x^2v_{xx} + 3v_{xx}^2\\
&\quad -v_y(v_{yyy} - 16v_xv_{xy} - 3v_{xxy}) - 6v_{xxyy} - v_xv_{xxx} - v_{xxxx}),\\
a_2 \;&=\; \frac{1}{72\sqrt{2}}e^{\frac{v}{2}}(v_xv_{yyy} - 4v_y^3v_x + 8v_y^2v_{xy} - 4v_{xyyy} - 4v_{xy}(3v_{yy} + 2v_x^2 - 3v_{xx})\\
&\quad -3v_xv_{xxy} + v_y(3v_{xyy} + 4v_x(2v_{yy} + v_x^2 - 2v_{xx}) - v_{xxx}) + 4v_{xxxy}),\\
s \;&=\; \frac{1}{3}(v_x^2 - v_y^2 + v_{xx} - v_{yy}).
\end{aligned}$$

By direct calculation one can check that a_1, a_2 and s provide the solution to the equations (3.5)–(3.10). In other words, equations (3.4) follow from the Tzizéica equation (1.7) and the second NV equation (3.15).

Proof of Theorem 1.14. Given v satisfying the Tzizéica equation, it is easy to see that the function $\eta = v_z^2 + v_{zz}$ satisfies (3.6), which means that η is an eigenfunction of the Laplace-Beltrami operator with eigenvalue 6. Thus $s = \alpha_1 = 4\mathrm{Re}\eta, \alpha_2 = -2\mathrm{Im}\eta$ give two real eigenfunctions of the Laplace-Beltrami operator with eigenvalue 6 on the minimal Lagrangian torus. □

References

[1] Carberry, E., McIntosh. I.: Minimal Lagrangian 2-tori in $\mathbb{C}P^2$ come in real families of every dimension. J. London Math. Soc. (2) **69**:2, 531–544 (2004)

[2] Castro, I., Urbano, F.: New examples of minimal Lagrangian tori in the complex projective plane. Manuscripta Math. **85**, 265–281 (1994)

[3] Dorfmeister, J.F., Ma, H.: Explicit expressions for the Iwasawa factors, the metric and the monodromy matrices for minimal Lagrangian surfaces in $\mathbb{C}P^2$. Dynamical systems, number theory and applications, 19–47, World Sci. Publ., Hackensack, NJ (2016)

[4] Dorfmeister, J.F., Ma, H.: A new look at equivariant minimal Lagrangian surfaces in $\mathbb{C}P^2$. Geometry and topology of manifolds, 97–125, Springer Proc. Math. Stat., 154, Springer, (2016)
[5] Haskins, H.: The geometric complexity of special Lagrangian T^2-cones. Invent. Math., **157**:1, 11–70 (2004)
[6] Joyce, D.: Special Lagrangian Submanifolds with Isolated Conical Singularities. v. Survey and Applications. J. Differential Geom. **63**:2, 279–347 (2003)
[7] Ma, H., Ma, Y.: Totally real minimal tori in $\mathbb{C}P^2$. Math. Z. **249**:2, 241–267, (2005)
[8] Ma, H.: Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}P^2$. Ann. Globl. Anal. Geom. **27**, 1–16 (2005)
[9] Ma, H., Schmies, M.: Examples of Hamiltonian stationary Lagrangian tori in $\mathbb{C}P^2$. Geom. Dedicata. **118**, 173–183 (2006)
[10] Marques, F.C., Neves, A.: Min-max theory and the Willmore conjecture. Ann. Math. (2) **179**, 683–782 (2014)
[11] Mironov, A.E.: The Novikov–Veselov hierarchy of equations and integrable deformations of minimal Lagrangian tori in $\mathbb{C}P^2$. (Russian) Sib. Elektron. Mat. Izv. **1**, 38–46 (also arXiv:math/0607700v1) (2004)
[12] Mironov, A.E.: New examples of Hamilton-minimal and minimal Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{C}P^n$. Sbornik: Mathematics. **195**:1, 85–96 (2004)
[13] Mironov, A.E.: Relationship between symmetries of the Tzitzéica equation and the Novikov-Veselov hierarchy. Mathematical Notes. **82**:4, 569–572 (2007)
[14] Montiel, S., Urbano, F.: A Willmore functional for compact surfaces in the complex projective plane. J. Reine Angew. Math., **546**, 139–154 (2002)
[15] Oh. Y.: Volume minimization of Lagrangian submanifolds under Hamiltonian deformations. Math. Z. **212**, 175–192 (1993)
[16] Sharipov, R.A.: Minimal tori in the five-dimensional sphere in $\mathbb{C}^3$. Theoretical and Mathematical Physics. **87**:1, 363–369 (1991)
[17] Taimanov. I.A.: Two-dimensional Dirac operator and the theory of surfaces. Russian Math. Surveys. **61**:1 79–159 (2006)
[18] Taimanov. I.A.: Modified Novikov–Veselov equation and differential geometry of surfaces. Amer. Math. Soc. Transl. Ser. 2. **179**, 133–151 (1997)
[19] Veselov, A.P., Novikov, S.P.: Finite-gap two-dimensional potential Schrödinger operators. Explicit formulas and evolution equations. (Russian) Dokl. Akad. Nauk SSSR. **279**:1, 20–24 (1984)

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