CURVATURE DECAY ESTIMATES OF GRAPHICAL MEAN CURVATURE FLOW IN HIGHER CODIMENSIONS

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ABSTRACT. We derive pointwise curvature estimates for graphical mean curvature flows in higher codimensions. To the best of our knowledge, this is the first such estimates without assuming smallness of first derivatives of the defining map. An immediate application is a convergence theorem of the mean curvature flow of the graph of an area decreasing map between flat Riemann surfaces.

1. INTRODUCTION

Let Σ_1 and Σ_2 be two compact Riemannian manifolds and $M = \Sigma_1 \times \Sigma_2$ be the product manifold. We consider a smooth map $f : \Sigma_1 \to \Sigma_2$ and denote the graph of f by Σ; Σ is a submanifold of M by the embedding $id \times f$. We study the deformation of f by the mean curvature flow. The idea is to deform Σ along its mean curvature vector field in M with the hope that Σ will remain a graph. This is the negative gradient flow of the volume functional and a stationary point is a "minimal map" introduced by Schoen in [RS].

To describe previous results, we recall the differential of f, df, at each point of Σ_1 is a linear map between the tangent spaces. The Riemannian structures enable us to define the adjoint of df. Let $\{\lambda_i\}$ denote the eigenvalues of $\sqrt{(df)^T df}$, or the singular values of df, where $(df)^T$ is the adjoint of df. Note that λ_i is always nonnegative. We say f is

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an *area decreasing map* if $\lambda_i\lambda_j < 1$ for any $i \neq j$ at each point. In particular, f is area-decreasing if df has rank one everywhere.

In [TW] it was proved that the area decreasing condition is preserved along the mean curvature flow and that the following global existence and convergence theorem holds.

Theorem ([TW], 2004). *Let Σ_1 and Σ_2 be compact Riemannian manifolds of constant sectional curvatures k_1 and k_2 respectively. Suppose $k_1 \geq |k_2|$, $k_1 + k_2 \geq 0$ and $dim(\Sigma_1) \geq 2$. If f is a smooth area decreasing map from Σ_1 to Σ_2, the mean curvature flow of the graph of f remains the graph of an area decreasing map and exists for all time. Moreover, if $k_1 + k_2 > 0$ then it converges smoothly to the graph of a constant map.*

This result has been generalized to allow more general curvature conditions [LL, SHS]. For example, the convergence part can be established when $k_1 + k_2 = 0$ and $k_1 \geq |k_2| > 0$ in [LL]. An important ingredient of these proofs is to use the positivity of k_1 to show that the gradient of f approaches zero as $t \to \infty$. In [SHS] the convergence follows, if the sectional curvatures $\sec_{\Sigma_1}, \sec_{\Sigma_2}$ of Σ_1 and Σ_2 are not necessarily constant and satisfy

$$\sec_{\Sigma_1} > -\sigma, \quad \mathrm{Ric}_{\Sigma_1} \geq (n-1)\sigma \geq (n-1)\sec_{\Sigma_2}$$

for some positive constant σ, where $n = \dim(\Sigma_1)$. In this case the positivity of Ric_{Σ_1} is important to get the convergence. However, in all cases mentioned above the convergence part in the case $k_1 = k_2 = 0$ remains an open standing problem.

In general, the global existence and convergence of a mean curvature flow relies on the boundedness of the second fundamental form. In the above theorem, the boundedness of the second fundamental form is obtained by an indirect blow-up argument, see [W, W3, TW]. While the idea of the proof of convergence is to use the positivity of $k_1 + k_2$ (or k_1 resp. Ric_{Σ_1}) to show that the gradient of f is approaching zero, which in turn gives the boundedness of the second fundamental form when the flow exists for sufficiently long time. In [SHS2] mean curvature estimates are shown in case of length decreasing maps ($\lambda_i < 1$). Other curvature estimates for higher co-dimensional graphical mean curvature flows have been obtained under various conditions [CCH, CCY]. However, to the best of our knowledge, there is no direct pointwise curvature estimate for higher codimensional mean curvature flow without assuming smallness conditions on first derivatives. In this paper, we prove pointwise estimates without making any smallness assumption on the gradient of f. As a result, the convergence of the flow can be established in dimension two when $k_1 = k_2 = 0$.

Theorem 1. *Let (Σ_1, g_1) and (Σ_2, g_2) be complete flat Riemann surfaces, Σ_1 being compact. Suppose $\Sigma \subset (\Sigma_1 \times \Sigma_2, g_1 \times g_2)$ is the graph of an area decreasing map $f : \Sigma_1 \to \Sigma_2$ and let Σ_t denote its mean curvature flow with initial surface $\Sigma_0 = \Sigma$. Then Σ_t remains the graph of an area decreasing map f_t along the mean curvature flow. The flow exists smoothly for all time and Σ_t converges smoothly to a totally geodesic submanifold as $t \to \infty$. Moreover, we have the following mean curvature decay estimate*

$$t|H|^2 \leq \frac{2}{\alpha}$$

where $\alpha = inf_{\Sigma_0} \frac{2(1-\lambda_1^2\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)} > 0$ and λ_1 and λ_2 are the singular values of df.

Remark 1.1. *Let $f : \Sigma_1 \to \Sigma_2$ be an arbitrary smooth map between flat Riemann surfaces (Σ_1, g_1), (Σ_2, g_2) and suppose Σ_1 is compact. Then there exists a constant $c > 0$ such that all singular values of f satisfy $\lambda_i\lambda_j < c^2$. The map $f : (\Sigma_1, g_1) \to (\Sigma_2, c^{-2}g_2)$ becomes area decreasing and we can apply Theorem 1 to this case since the new metric $\tilde{g}_2 = c^{-2}g_2$ is still flat.*

As in [SW], consider the symplectic structure $dx^1 \wedge dy^1 + dx^2 \wedge dy^2$ on $(T^2, \{x^i\}_{i=1,2}) \times (T^2, \{y^j\}_{j=1,2})$ and suppose Σ is Lagrangian with respect to this symplectic structure. A stronger decay estimate on the second fundamental can be obtained in this case:

Theorem 2. *Let $f : T^2 \to T^2$ be an area decreasing map such that its graph Σ is a Lagrangian submanifold in $T^2 \times T^2$ with respect to the above symplectic structure, then the same conclusion as in Theorem 1 holds and*

$$t|A|^2 \leq C_\alpha,$$

where C_α is a positive constant that only depends on α.

We first revisit the curvature estimates in codimension one by Ecker and Huisken [EH]. A direct generalization of their estimate only works in the higher codimensional case when the gradient of the defining function is small enough. However, we were able to reformulate their estimates in a different way that can be adapted to the higher codimensional case. It turns out in higher codimensions a more sophisticated approach has to be developed to accommodate the complexity of the normal bundle.

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2. Ecker and Huisken's estimates in codimension one

In this section, we slightly rewrite the estimate in [EH] so it can be adapted to the higher codimensional situation in later sections. Consider the mean curvature flow of the graph of a function $f:\mathbb{R}^n\to\mathbb{R}$ and let $v=\sqrt{1+|Df|^2}$. Recall that the evolution equations of v and $|A|^2$ are

$$\begin{aligned}(\frac{d}{dt}-\Delta)v =& -|A|^2 v - 2\frac{|\nabla v|^2}{v} \text{ and}\\ (\frac{d}{dt}-\Delta)|A|^2 =& -2|\nabla A|^2+2|A|^4.\end{aligned}$$

We obtain the evolution equation of $\ln v^2$ as

$$(\frac{d}{dt}-\Delta)\ln(v^2) = -2|A|^2-\frac{1}{2}|\nabla\ln(v^2)|^2. \tag{2.1}$$

Using $|\nabla A|^2 \geq |\nabla|A||^2 = \frac{|A|^2}{4}|\nabla\ln|A|^2|^2$, we have $(\frac{d}{dt}-\Delta)|A|^2 \leq -\frac{|A|^2}{2}|\nabla\ln|A|^2|^2+2|A|^4$. Taking ln of $|A|^2$, we obtain

$$(\frac{d}{dt}-\Delta)\ln(|A|^2)\leq 2|A|^2+\frac{1}{2}|\nabla\ln|A|^2|^2. \tag{2.2}$$

As in [EH], equations (2.1) and (2.2) together imply a sup norm bound for $|A|^2v^2$.

The following differential inequality for $\ln(\delta t|A|^2+\epsilon)$, which is similar to equation (2.2), gives a decay estimate of $|A|^2$.

Lemma 2.1. *Given any $\epsilon>0$ and $\delta<2\epsilon$. Then*

$$(\frac{d}{dt}-\Delta)\ln(\delta t|A|^2+\epsilon)\leq 2|A|^2+\frac{1}{2}|\nabla\ln(\delta t|A|^2+\epsilon)|^2.$$

Proof. Using $\triangle \ln(\delta t|A|^2+\epsilon) = \frac{\delta t \triangle |A|^2}{\delta t|A|^2+\epsilon} - \frac{\delta^2 t^2 |\nabla |A|^2|^2}{(\delta t|A|^2+\epsilon)^2}$, we compute the evolution equation of $\ln(\delta t|A|^2+\epsilon)$:

$$
\begin{aligned}
& (\frac{d}{dt} - \Delta) \ln(\delta t|A|^2+\epsilon) \\
= \;& \frac{1}{\delta t|A|^2+\epsilon}(\frac{d}{dt} - \Delta)(\delta t|A|^2+\epsilon) + |\nabla \ln(\delta t|A|^2+\epsilon)|^2 \\
\leq \;& \frac{1}{\delta t|A|^2+\epsilon}\Big(\delta|A|^2 + \delta t(2|A|^4 - \frac{|\nabla |A|^2|^2}{2|A|^2})\Big) + |\nabla \ln(\delta t|A|^2+\epsilon)|^2 \\
\leq \;& \frac{\delta|A|^2 + 2[\delta t|A|^2+\epsilon]|A|^2 - 2\epsilon|A|^2}{\delta t|A|^2+\epsilon} - \frac{1}{2}\frac{\delta t|\nabla |A|^2|^2}{(\delta t|A|^2+\epsilon)|A|^2} \\
& + |\nabla \ln(\delta t|A|^2+\epsilon)|^2 \\
\leq \;& 2|A|^2 + \frac{\delta|A|^2 - 2\epsilon|A|^2}{\delta t|A|^2+\epsilon} - \frac{1}{2}\frac{\delta t|\nabla |A|^2|^2}{(\delta t|A|^2+\epsilon)|A|^2} + |\nabla \ln(\delta t|A|^2+\epsilon)|^2 \\
\leq \;& 2|A|^2 + \frac{1}{2}|\nabla \ln(\delta t|A|^2+\epsilon)|^2.
\end{aligned}
$$

Here we use $\delta - 2\epsilon < 0$ and

$$-\frac{1}{2}\frac{\delta t|\nabla |A|^2|^2}{(\delta t|A|^2+\epsilon)|A|^2} + \frac{1}{2}|\nabla \ln(\delta t|A|^2+\epsilon)|^2 \leq 0.$$

□

Theorem 3. *$sup_{\Sigma_t}(t|A|^2) \leq v_0^2$ where $v_0 = sup_{\Sigma_0} v > 0$.*

Proof. From the evolution equation of v, we have $sup_{\Sigma_t} v^2 \leq v_0^2$. Choosing $\varepsilon = 1$ and $\delta = 1$ in the previous Lemma and combining with equation (2.1), we derive

$$
\begin{aligned}
(\frac{d}{dt} - \Delta) \ln((t|A|^2+1)v^2) &\leq \frac{1}{2}|\nabla \ln(t|A|^2+1)|^2 - \frac{1}{2}|\nabla \ln(v^2)|^2 \\
&\leq \frac{1}{2}\nabla \ln(\frac{t|A|^2+1}{v^2}) \cdot \nabla \ln((t|A|^2+1)v^2).
\end{aligned}
$$

The maximum principle implies

$$sup_{\Sigma_t}\left((t|A|^2+1)v^2\right) \leq sup_{\Sigma_0} v^2.$$

Therefore $t|A|^2 \leq (t|A|^2+1)v^2 \leq v_0^2$. □

3. Estimates in higher codimensions

Our basic set-up here is a mean curvature flow $F : \Sigma \times [0, T) \to M$ of an n dimensional submanifold Σ inside an $n+m$ dimensional flat

Riemannian manifold M. Given any tensor on M, we may consider the pull-back tensor by F_t and consider the evolution equation with respect to the time-dependent induced metric on $F_t(\Sigma) = \Sigma_t$. For the purpose of applying the maximum principle, it suffices to derive the equation at a space-time point. We write all geometric quantities in terms of orthonormal frames, keeping in mind all quantities are defined independent of choices of frames. At any point $p \in \Sigma_t$, we choose any orthonormal frames $\{e_i\}_{i=1,\cdots,n}$ for $T_p\Sigma_t$ and $\{e_\alpha\}_{\alpha=n+1,\cdots,n+m}$ for the normal space $N_p\Sigma_t$. The second fundamental form $h_{\alpha ij}$ is denoted by $h_{\alpha ij} = \langle \nabla^M_{e_i} e_j, e_\alpha\rangle$ and the mean curvature vector is denoted by $H_\alpha = \sum_i h_{\alpha ii}$. For any j, k, we pretend

$$h_{n+i,jk} = 0$$

if $i > m$. Also we denote $|A|^2 = \sum_{\alpha,i,j} h^2_{\alpha ij}$ and $|H|^2 = \sum_\alpha H^2_\alpha$.

First, we recall the evolution equations for $|H|^2$ and $|A|^2$. The following proposition is taken from Corollary 3.8 and Corollary 3.9 from the survey paper [S2]. Here we assume the ambient manifold M is flat.

Proposition 3.1. *For a mean curvature flow $F : \Sigma \times [0, T) \to M$ of any dimension, the quantities $|A|^2$ and $|H|^2$ satisfy the following equations along the mean curvature flow:*

$$\begin{aligned}&\frac{d}{dt}|A|^2 = \Delta|A|^2 - 2|\nabla^\perp A|^2\\ &+2\sum_{\alpha,\gamma,i,m}(\sum_k h_{\alpha ik}h_{\gamma mk} - h_{\alpha mk}h_{\gamma ik})^2 + 2\sum_{i,j,m,k}(\sum_\alpha h_{\alpha ij}h_{\alpha mk})^2\end{aligned} \tag{3.1}$$

and

$$\frac{d}{dt}|H|^2 = \Delta|H|^2 - 2|\nabla^\perp H|^2 + 2\sum_{i,k}(\sum_\alpha H_\alpha h_{\alpha ik})^2. \tag{3.2}$$

Using Theorem 1 from [LL2], we have

$$2\sum_{\alpha,\gamma,i,m}(\sum_k h_{\alpha ik}h_{\gamma mk} - h_{\alpha mk}h_{\gamma ik})^2 + 2\sum_{i,j,m,k}(\sum_\alpha h_{\alpha ij}h_{\alpha mk})^2 \le 3|A|^4.$$

(This improves the prior bound of $4|A|^4$ used in [W]). Using

$$2\sum_{i,k}(\sum_\alpha H_\alpha h_{\alpha ik})^2 \le 2|A|^2|H|^2,$$

we obtain the next lemma.

Lemma 3.1. *We have the following differential inequalities for* $|A|^2$ *and* $|H|^2$ *if the ambient manifold* M *is flat.*

$$
\begin{aligned}
(\frac{d}{dt}-\Delta)|A|^2 &\leq -2|\nabla^{\perp}A|^2+3|A|^4,\\
(\frac{d}{dt}-\Delta)|H|^2 &\leq -2|\nabla^{\perp}H|^2+2|A|^2|H|^2.
\end{aligned}
\tag{3.3}
$$

We note that the term $3|A|^4$ in the first equation is different from the term $2|A|^4$ in the codimension one case. It cannot be improved unless the normal bundle is flat. This creates a major difficulty in attempting to generalize the codimension one estimate to the higher codimension case.

In the following, we derive differential inequalities for various geometric quantities which will be used for curvature decay estimates in §4.

Lemma 3.2. *Given any* $\epsilon > 0$ *and* $0 < \delta \leq \frac{2\epsilon}{n}$*, we have the following differential inequalities*

$$
\begin{aligned}
(\frac{d}{dt}-\Delta)\ln(\delta t|H|^2+\epsilon) &\leq 2|A|^2+\frac{|\nabla \ln(\delta t|H|^2+\epsilon)|^2}{2},\\
(\frac{d}{dt}-\Delta)\ln(\delta t|A|^2+\epsilon) &\leq 3|A|^2+\frac{|\nabla \ln(\delta t|A|^2+\epsilon)|^2}{2}.
\end{aligned}
\tag{3.4}
$$

Proof. From Lemma 3.1, we have

$$
\begin{aligned}
(\frac{d}{dt}-\Delta)\ln(|H|^2) =&\frac{1}{|H|^2}(\frac{d}{dt}-\Delta)|H|^2+|\nabla \ln|H|^2|^2\\
\leq&\frac{1}{|H|^2}\Big(2|A|^2|H|^2-2|\nabla|H||^2\Big)+|\nabla \ln|H|^2|^2\\
\leq&2|A|^2+\frac{1}{2}|\nabla \ln|H|^2|^2.
\end{aligned}
$$

In the last step, we have used $-\frac{2|\nabla|H||^2}{|H|^2}=-\frac{|\nabla \ln|H|^2|^2}{2}$.

Using $(\frac{d}{dt}-\Delta)(\delta t|H|^2+\epsilon)=\delta|H|^2+\delta t(\frac{d}{dt}-\Delta)|H|^2$ and Lemma 3.1, we compute the evolution equation of $\ln(\delta t|H|^2+\epsilon)$ to obtain

$$
\begin{aligned}
&(\frac{d}{dt}-\Delta)\ln(\delta t|H|^2+\epsilon)\\
=&\frac{1}{\delta t|H|^2+\epsilon}(\frac{d}{dt}-\Delta)(\delta t|H|^2+\epsilon)+|\nabla\ln(\delta t|H|^2+\epsilon)|^2\\
\le&\frac{1}{\delta t|H|^2+\epsilon}\Big(\delta|H|^2+\delta t(2|A|^2|H|^2-2|\nabla|H||^2)\Big)+|\nabla\ln(\delta t|H|^2+\epsilon)|^2\\
=&\frac{\delta|H|^2+2(\delta t|H|^2+\epsilon)|A|^2-2\epsilon|A|^2}{\delta t|H|^2+\epsilon}-\frac{2\delta t|\nabla|H||^2}{\delta t|H|^2+\epsilon}+|\nabla\ln(\delta t|H|^2+\epsilon)|^2\\
=&2|A|^2+\frac{\delta|H|^2-2\epsilon|A|^2}{\delta t|H|^2+\epsilon}-\frac{\delta t|\nabla|H|^2|^2}{2(\delta t|H|^2+\epsilon)|H|^2}+|\nabla\ln(\delta t|H|^2+\epsilon)|^2.
\end{aligned}
$$

In the last step, we have used $|\nabla|H||^2=\frac{|\nabla|H|^2|^2}{4|H|^2}$. Since $|H|^2\le n|A|^2$ and $-\frac{\delta t|\nabla|H||^2}{2(\delta t|H|^2+\epsilon)|H|^2}+\frac{1}{2}|\nabla\ln(\delta t|H|^2+\epsilon)|^2\le 0$, we can choose $\delta\le\frac{2\epsilon}{n}$ and get

$$(\frac{d}{dt}-\Delta)\ln(\delta t|H|^2+\epsilon)\le 2|A|^2+\frac{1}{2}|\nabla\ln(\delta t|H|^2+\epsilon)|^2.$$

The rest of the Lemma can be proved in a similar fashion to Lemma 2.1. □

To derive an a priori curvature estimate, we need to find the right geometric quantity to counteract the quadratic growth of the second fundamental forms in Lemma 3.2. When $n=2$, we are able to find the right quantity to establish the a prioi mean curvature estimate.

In [TW], a parallel symmetric two tensor S is introduced to study the area decreasing map. We first recall some basic notations and definitions from Section 3 and Section 4 in [TW].

When $M=\Sigma_1\times\Sigma_2$ is the product of Σ_1 and Σ_2, we denote the projections by $\pi_1:M\to\Sigma_1$ and $\pi_2:M\to\Sigma_2$. By abusing notations, we also denote the differentials by $\pi_1:T_pM\to T_{\pi_1(p)}\Sigma_1$ and $\pi_1:T_pM\to T_{\pi_2(p)}\Sigma_2$ at any point $p\in M$.

When Σ is the graph of $f:\Sigma_1\to\Sigma_2$, the equation at each point can be written in terms of the singular values of df and special bases adapted to df. Denote the singular values of df, or eigenvalues of $\sqrt{(df)^T df}$, by $\{\lambda_i\}_{i=1,\cdots,n}$. Let r denote the rank of df. We can rearrange them so that $\lambda_i=0$ when i is greater than r. By singular value decomposition, there exist orthonormal bases $\{a_i\}_{i=1,\cdots,n}$ for $T_{\pi_1(p)}\Sigma_1$ and $\{a_\alpha\}_{\alpha=n+1,\cdots,n+m}$ for $T_{\pi_2(p)}\Sigma_2$ such that

$$df(a_i) = \lambda_i a_{n+i}$$

for i less than or equal to r and $df(a_i) = 0$ for i greater than r. Moreover,

$$e_i = \begin{cases} \frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i a_{n+i}) & \text{if } 1 \le i \le r \\ a_i & \text{if } r+1 \le i \le n \end{cases} \tag{3.5}$$

becomes an orthonormal basis for $T_p\Sigma$ and

$$e_{n+p} = \begin{cases} \frac{1}{\sqrt{1+\lambda_p^2}}(a_{n+p} - \lambda_p a_p) & \text{if } 1 \le p \le r \\ a_{n+p} & \text{if } r+1 \le p \le m \end{cases} \tag{3.6}$$

becomes an orthonormal basis for $N_p\Sigma$.

The tangent space of $M = \Sigma_1 \times \Sigma_2$ is identified with $T\Sigma_1 \oplus T\Sigma_2$. Let π_1 and π_2 denote the projection onto the first and second summand in the splitting. We define the parallel symmetric two-tensor S by

$$S(X,Y) = \langle \pi_1(X), \pi_1(Y)\rangle \; - \langle \pi_2(X), \pi_2(Y)\rangle \tag{3.7}$$

for any $X, Y \in TM$.

Let Σ be the graph of $f : \Sigma_1 \to \Sigma_1 \times \Sigma_2$. S restricts to a symmetric two-tensor on Σ and we can represent S in terms of the orthonormal basis (3.5).

Let r denote the rank of df. By (3.5), it is not hard to check

$$\pi_1(e_i) = \frac{a_i}{\sqrt{1+{\lambda_i}^2}}\,, \pi_2(e_i) = \frac{\lambda_i a_{n+i}}{\sqrt{1+{\lambda_i}^2}} \text{ for } 1 \le i \le r\,, \quad \text{and } \pi_1(e_i) = a_i\,, \pi_2(e_i) = 0 \text{ for } r+1 \le i \le n. \tag{3.8}$$

Similarly, by (3.6) we have

$$\pi_1(e_{n+p}) = \frac{-\lambda_p a_p}{\sqrt{1+{\lambda_p}^2}}\,, \pi_2(e_{n+p}) = \frac{a_{n+p}}{\sqrt{1+{\lambda_p}^2}} \text{ for } 1 \le p \le r\,, \quad \text{and } \pi_1(e_{n+p}) = 0\,, \pi_2(e_{n+p}) = a_{n+p} \text{ for } r+1 \le p \le m\,. \tag{3.9}$$

From the definition of S, we have

$$S(e_i, e_j) = \frac{1-\lambda_i^2}{1+{\lambda_i}^2}\delta_{ij}\,. \tag{3.10}$$

In particular, the eigenvalues of S are

$$\frac{1-{\lambda_i}^2}{1+{\lambda_i}^2},\ i=1,\cdots,n. \tag{3.11}$$

Now, at each point we express S in terms of the orthonormal basis $\{e_i\}_{i=1,\cdots,n}$ and $\{e_\alpha\}_{\alpha=n+1,\cdots,n+m}$. Let $I_{k\times k}$ denote a k by k identity matrix. Then S can be written in the block form

$$S=\Big(S(e_k,e_l)\Big)_{1\le k,l\le n+m}=\begin{pmatrix} B & 0 & D & 0\\ 0 & I_{n-r\times n-r} & 0 & 0\\ D & 0 & -B & 0\\ 0 & 0 & 0 & -I_{m-r\times m-r}\end{pmatrix} \tag{3.12}$$

where B and D are r by r matrices with $B_{ij}=S(e_i,e_j)=\frac{1-\lambda_i^2}{1+\lambda_i^2}\delta_{ij}$ and $D_{ij}=S(e_i,e_{n+j})=\frac{-2\lambda_i}{1+\lambda_i^2}\delta_{ij}$ for $1\le i,j\le r$.

Next we recall the evolution equation of parallel two-tensors from [SW]. Given a parallel two-tensor S on M, we consider the evolution of S restricted to Σ_t. This is a family of time-dependent symmetric two tensors on Σ_t.

Proposition 3.2. *Let S be a parallel two-tensor on M. Then the pull-back of S to Σ_t satisfies the following equation.*

$$\begin{aligned}(\frac{d}{dt}-\Delta)S_{ij}&=-h_{\alpha il}H_\alpha S_{lj}-h_{\alpha jl}H_\alpha S_{li}\\&+R_{kik\alpha}S_{\alpha j}+R_{kjk\alpha}S_{\alpha i}\\&+h_{\alpha kl}h_{\alpha ki}S_{lj}+h_{\alpha kl}h_{\alpha kj}S_{li}-2h_{\alpha ki}h_{\beta kj}S_{\alpha\beta}\end{aligned} \tag{3.13}$$

where Δ is the rough Laplacian on two-tensors over Σ_t and $S_{\alpha i}=S(e_\alpha,e_i)$, $S_{\alpha\beta}=S(e_\alpha,e_\beta)$, and $R_{kik\alpha}=R(e_k,e_i,e_k,e_\alpha)$ is the curvature of M.

The evolution equations (3.13) of S can be written in terms of evolving orthonormal frames as in Hamilton [H]. If the orthonormal frames

$$F=\{F_1,\cdots,F_a,\cdots,F_n\} \tag{3.14}$$

are given in local coordinates by

$$F_a=F_a^i\frac{\partial}{\partial x_i}\,.$$

To keep them orthonormal, i.e. $g_{ij}F_a^iF_b^j=\delta_{ab}$, we evolve F by the formula

$$\frac{\partial}{\partial t}F_a^i=g^{ij}g^{\alpha\beta}h_{\alpha jl}H_\beta F_a^l\,.$$

Let $S_{ab} = S_{ij}F^i_aF^j_b$ be the components of S in F. Then S_{ab} satisfies the following equation

$$(\frac{d}{dt} - \Delta)S_{ab} = R_{caca}S_{\alpha b} + R_{cbc\alpha}S_{\alpha a} + h_{\alpha cd}h_{\alpha ca}S_{db} + h_{\alpha cd}h_{\alpha cb}S_{da} - 2h_{\alpha ca}h_{\beta cb}S_{\alpha\beta}\,. \tag{3.15}$$

In the following, we will compute the evolution of $Tr(S) = \sum_{i,j} g^{ij}S_{ij}$ in the case Σ_1, Σ_2 are flat Riemann surfaces, i.e. where the curvature tensor $R = 0$. By equation (3.10), $Tr(S) = \frac{1-\lambda_1^2}{1+\lambda_1^2} + \frac{1-\lambda_2^2}{1+\lambda_2^2} = \frac{2(1-\lambda_1^2\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}$. The next proposition gives a new proof that the area decreasing condition is preserved along the mean curvature. In addition, the equation satisfied by $\ln Tr(S)$ plays a critical role in next section's curvature estimates.

Proposition 3.3. *The quantity $Tr(S)$ satisfies the following equation*

$$\begin{aligned}&(\frac{d}{dt} - \Delta)\ln Tr(S)\\ =&2|A|^2 + \frac{|\nabla \ln(TrS)|^2}{2} + \frac{2\sum_{c=1}^2 (T_{11}h_{4c2} + T_{22}h_{3c1})^2}{Tr(S)^2},\end{aligned} \tag{3.16}$$

where $T_{11} = \frac{2\lambda_1}{(1+\lambda_1^2)}$ and $T_{22} = \frac{2\lambda_2}{(1+\lambda_2^2)}$.

Proof. Using the evolution equation of S (with respect to an orthonormal frame) in equation (3.15) and $S(e_{2+i}, e_{2+j}) = -S_{ij}$, we derive

$$\begin{aligned}&(\frac{d}{dt} - \Delta)Tr(S)\\ =&\sum_{a,b}\delta^{ab}(\sum_{\alpha,c,d} h_{\alpha cd}h_{\alpha ca}S_{db} + h_{\alpha cd}h_{\alpha cb}S_{da} \; - 2h_{\alpha ca}h_{\beta cb}S_{\alpha\beta})\\ =&\sum_a(\sum_{\alpha,c} 2h^2_{\alpha ca}S_{aa} - 2h^2_{\alpha ca}S_{\alpha\alpha})\\ =&\sum_a(\sum_{p,c} 2h^2_{2+p\ ca}S_{aa} + 2h^2_{2+p\ ca}S_{pp})\\ =&\sum_{p,c}\left[2h^2_{2+p\ c1}S_{11} + 2h^2_{2+p\ c1}S_{pp} + 2h^2_{2+p\ c2}S_{22} + 2h^2_{2+p\ c2}S_{pp}\right]\\ =&\sum_c\left[(2h^2_{4c1} + 2h^2_{3c2})(S_{11} + S_{22}) + 4h^2_{3c1}S_{11} + 4h^2_{4c2}S_{22}\right]\end{aligned}$$

Using $|A|^2 = \sum_{c=1}^{2}(h_{4c1}^2 + h_{3c2}^2 + h_{3c1}^2 + h_{4c2}^2)$, we obtain

$$(\frac{d}{dt} - \Delta)Tr(S) = 2|A|^2 Tr(S) + 2(S_{11} - S_{22})\sum_{c=1}^{2}(h_{3c1}^2 - h_{4c2}^2). \quad (3.17)$$

We claim the following relation holds:

$$\begin{aligned} &4Tr(S)(S_{11} - S_{22})\sum_{c=1}^{2}(h_{3c1}^2 - h_{4c2}^2) + |\nabla Tr(S)|^2 \\ &= 4\sum_{c=1}^{2}(T_{11}h_{4c2} + T_{22}h_{3c1})^2. \end{aligned} \quad (3.18)$$

Equation (3.16) follows from equations (3.17) and (3.18).

In the rest of the proof, we verify equation (3.18). The covariant derivative of the restriction of S on Σ can be computed by

$$\begin{aligned} & (\nabla_{e_k} S)(e_i, e_j) \\ = \ & e_k(S(e_i, e_j)) - S(\nabla_{e_k} e_i, e_j) - S(e_i, \nabla_{e_k} e_j) \\ = \ & S(\nabla^M_{e_k} e_i - \nabla_{e_k} e_i, e_j) - S(e_i, \nabla^M_{e_k} e_j - \nabla_{e_k} e_j) \\ = \ & h_{\alpha ki} S_{\alpha j} + h_{\beta kj} S_{\beta i}. \end{aligned}$$

Since $S_{2+p\ l} = -\frac{2\lambda_p \delta_{pl}}{(1+\lambda_p^2)}$, we derive

$$\begin{aligned} S_{ij,k} = & -\frac{2h_{2+p\ ki}\lambda_p\delta_{pj}}{(1+\lambda_p^2)} - \frac{2h_{2+p\ kj}\lambda_p\delta_{pi}}{(1+\lambda_p^2)} \\ = & -h_{2+p\ ki}T_{pp}\delta_{pj} - h_{2+p\ kj}T_{pp}\delta_{pi} \\ = & -h_{2+j\ ki}T_{jj} - h_{2+i\ kj}T_{ii}. \end{aligned} \quad (3.19)$$

In particular, $\nabla_k(S_{ii}) = -2T_{ii}h_{2+i\ ki}$ and

$$|\nabla Tr(S)|^2 = 4\sum_{k=1}^{2}(T_{11}^2 h_{3k1}^2 + 2T_{11}T_{22}h_{3k1}h_{4k2} + T_{22}^2 h_{4k2}^2).$$

We compute

$$
\begin{aligned}
&4(S_{11}^2 - S_{22}^2)\sum_{k=1}^{2}(h_{3k1}^2 - h_{4k2}^2) + |\nabla Tr(S)|^2 \\
&= 4(S_{11}^2 - S_{22}^2)\sum_{k=1}^{2}(h_{3k1}^2 - h_{4k2}^2) \\
&\quad + 4\sum_{k=1}^{2}(T_{11}^2 h_{3k1}^2 + 2T_{11}T_{22}h_{3k1}h_{4k2} + T_{22}^2 h_{4k2}^2) \\
&= 4\sum_{k=1}^{2}(T_{11}h_{4k2} + T_{22}h_{3k1})^2,
\end{aligned}
$$

where we use the fact that $S_{11}^2 + T_{11}^2 = S_{22}^2 + T_{22}^2 = 1$ to complete the square in the last equality. This verifies (3.18). ☐

4. Proof of Theorems

4.1. **Proof of Theorem 1.** Since $Tr(S) = \frac{2(1-\lambda_1^2\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}$, by Proposition 3.3 and the maximum principle, we deduce that the area decreasing condition is preserved by the mean curvature flow and $inf_{\Sigma_t} Tr(S) \geq \alpha$, where $\alpha = inf_{\Sigma_0} Tr(S)$. On any Σ_t, $t > 0$, $Tr(S) \geq \alpha > 0$ and $(1+\lambda_1^2)(1+\lambda_2^2) < \frac{2}{\alpha}$. Thus Σ_t remains as the graph of an area decreasing map.

Next we derive the mean curvature decay estimate. Combining the first equation in Lemma 3.2 (with $\delta = \epsilon = 1$) and Proposition 3.3, we obtain

$$
(\frac{d}{dt} - \Delta)\ln\frac{(t|H|^2+1)}{Tr(S)} \leq \frac{1}{2}\nabla \ln\frac{(t|H|^2+1)}{Tr(S)} \cdot \nabla \ln\left[(t|H|^2+1)Tr(S)\right].
$$

By the maximum principle, we have

$$
sup_{\Sigma_t}\ln\frac{(t|H|^2+1)}{Tr(S)} \leq sup_{\Sigma_0}\ln\frac{1}{Tr(S)}.
$$

Note that f_t remains area decreasing and

$$
Tr(S) = \frac{2(1-\lambda_1^2\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)} < 2.
$$

This implies that $t|H|^2 \leq Tr(S) sup_{\Sigma_0}\frac{1}{Tr(S)} \leq sup_{\Sigma_0}\frac{2}{Tr(S)} = \frac{2}{\alpha}$. If Σ_2 is compact we already have longtime existence of the flow by the methods in [W, W3, TW]. In case Σ_2 is complete and non-compact we can use the mean curvature estimate to obtain a C^0-estimate on finite time intervals and then we may proceed as in [SHS2] to get longtime existence.

Now we can prove the C^∞ convergence using the mean curvature decay estimate. By the Gauss formula, $\int_{\Sigma_t} |A|^2 d\mu_t = \int_{\Sigma_t} |H|^2 d\mu_t \to 0$. Therefore $\int_{\Sigma_t} |A|^2 d\mu_t$ is sufficiently small when t is large enough, the ϵ regularity theorem in [TI] (see also [E]) implies $\sup_{\Sigma_t} |A|^2$ is uniformly bounded. The general convergence theorem of Simon [LS] implies C^∞ convergence of Σ_t to a minimal submanifold Σ_∞, which is totally geodesic by the Gauss formula again.

4.2. **Proof of Theorem 2.** Now we prove a decay estimate for the second fundamental form in the case when Σ is a Lagrangian submanifold. It is well known that Σ_t remains as a Lagrangian submanifold in $T^2 \times T^2$ from [S]. Using the first equation in Lemma 3.1 and Proposition 3.3, we derive

$$
\begin{aligned}
&(\frac{d}{dt} - \Delta)\ln(\frac{|A|^2}{Tr(S)^2})\\
\leq&\frac{|\nabla \ln(|A|^2)|^2}{2} - |\nabla \ln Tr(S)|^2 - |A|^2 - \frac{4}{Tr(S)^2}\sum_{c=1}^2 (T_{11}h_{4c2} + T_{22}h_{3c1})^2.\\
=&\frac{1}{2}\nabla \ln(\frac{|A|^2}{Tr(S)^2}) \cdot \nabla \ln(|A|^2 Tr(S)^2) - |A|^2\\
+&|\nabla \ln Tr(S)|^2 - \frac{4}{Tr(S)^2}\sum_{c=1}^2 (T_{11}h_{4c2} + T_{22}h_{3c1})^2.
\end{aligned}
$$

We estimate the last two terms on the right hand side. From equation (3.18), we obtain

$$
\begin{aligned}
&|\nabla \ln Tr(S)|^2 - \frac{4}{Tr(S)^2}\sum_{c=1}^2 (T_{11}h_{4c2} + T_{22}h_{3c1})^2\\
&= -\frac{4(S_{11} - S_{22})}{Tr(S)}\sum_{c=1}^2 (h_{3c1}^2 - h_{4c2}^2).
\end{aligned}
$$

Let J be the standard almost complex structure on $(T^2, \{x^i\}_{i=1,2}) \times (T^2, \{y^j\}_{j=1,2})$ that maps from the tangent space of the first component to the tangent space of the second one, and vice versa. Suppose f is the defining map of a Lagrangian surface Σ in $(T^2, \{x^i\}_{i=1,2}) \times$

$(T^2, \{y^j\}_{j=1,2})$ with respect to $dx^1 \wedge dy^1 + dx^2 \wedge dy^2$. By the Lagrangian condition, Jdf is a self-adjoint map on the tangent space of $(T^2, \{x^i\}_{i=1,2})$. Therefore, there exists an orthonormal basis $\{a_i\}_{i=1,2}$ such that

$$df(a_i) = \lambda_i J(a_i), i = 1, 2.$$

We can then choose $a_{2+i} = J(a_i)$ in equations (3.5) and (3.6) such that

$$e_i = \frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i J(a_i))$$

and

$$e_{2+i} = \frac{1}{\sqrt{1+\lambda_i^2}}(J(a_i) - \lambda_i a_i), i = 1, 2.$$

From these expressions, it is easy to check that $e_3 = J(e_1)$ and $e_4 = J(e_2)$. Since Σ_t remains Lagrangian, such orthonormal frames can be picked at any point on Σ_t. Because J is parallel,

$$\langle \nabla^M_{e_c} e_2, e_3 \rangle = \langle \nabla^M_{e_c} e_1, e_4 \rangle$$

and we have $h^2_{3c2} = h^2_{4c1}$ for $c = 1, 2$. Therefore,

$$\Big| \sum_{c=1}^{2} (h^2_{3c1} - h^2_{4c2}) \Big| \le |h_{311} - h_{322}||H_3| + |h_{411} - h_{422}||H_4| \le 2\sqrt{2}|A||H|$$

and

$$||\nabla \ln Tr(S)|^2 - \frac{4}{Tr(S)^2} \sum_{c=1}^{2} (T_{11} h_{4c2} + T_{22} h_{3c1})^2| \le C|A||H|,$$

where C depends only on $\alpha = inf_{\Sigma_0} \frac{2(1-\lambda_1^2\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)} > 0$ on the initial surface. For example, $C = \frac{16\sqrt{2}}{\alpha}$ suffices. To this end,

$$\begin{aligned}(\frac{d}{dt} - \Delta) \ln(\frac{|A|^2}{Tr(S)^2}) \quad &\le \quad \frac{1}{2} \nabla \ln(\frac{|A|^2}{Tr(S)^2}) \cdot \nabla \ln(|A|^2 Tr(S)^2) \\ &\quad -|A|^2 + C|A||H|.\end{aligned}$$

Note that $-|A|^2 \le -\alpha^2 \frac{|A|^2}{Tr(S)^2}$ and $|H|^2 \le \frac{2}{t\alpha}$ from Theorem 1, we derive

$$-|A|^2 + C|A||H| \le -\frac{1}{2}|A|^2 + \frac{C^2}{2}|H|^2 \le -\frac{\alpha^2 |A|^2}{2Tr(S)^2} + \frac{C^2}{\alpha t}.$$

Therefore, the quantity $\frac{|A|^2}{Tr(S)^2}$ satisfies the following differential inequality:

$$
\begin{aligned}
&(\frac{d}{dt}-\Delta)(\frac{|A|^2}{Tr(S)^2})\\
\leq&\frac{1}{2}\nabla(\frac{|A|^2}{Tr(S)^2})\cdot\nabla\ln(|A|^2Tr(S)^2)+(-\frac{\alpha^2|A|^2}{2Tr(S)^2}+\frac{C^2}{\alpha t})\frac{|A|^2}{Tr(S)^2}.
\end{aligned}
$$

Consider the ODE $u'=(-\frac{\alpha^2}{2}u+\frac{C^2}{\alpha t})u$. Note that $w=\frac{2(1+\frac{C^2}{\alpha})}{\alpha^2 t}$ is a solution to $u'=(-\frac{\alpha^2}{2}u+\frac{C^2}{\alpha t})u$ and $\lim_{t\to 0^+}w(t)=\infty$. Thus we have $sup_{\Sigma_t}(\frac{|A|^2}{Tr(S)^2})\leq\frac{2(1+\frac{C^2}{\alpha})}{\alpha^2 t}$ and $|A|^2\leq\frac{8(1+\frac{C^2}{\alpha})}{\alpha^2 t}$.

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