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# Quasilocal mass from a mathematical perspective

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ABSTRACT. Quasilocal mass in general relativity is a notion defined for a closed spacelike 2-surface in spacetime. In this note, we explain the definition in [**23**] and [**24**] from a mathematical viewpoint, emphasizing the connection to differential geometry and nonlinear partial differential equations. We also discuss a minimax interpretation of the definition and compare with other notions of quasilocal mass.

## 1. Surface Hamiltonian

Our subject of study is a two-dimensional closed embedded spacelike surface $\Sigma$ in spacetime $N$, and thus the induced metric is Riemannian. We also assume $\Sigma$ bounds a spacelike region $\Omega$ in spacetime. These are surfaces on which the notion of quasilocal energy/mass is defined in relativity. [1]

We denote the Lorentz metric on $N$ by $\langle \cdot, \cdot \rangle_N$ and the connection by $\nabla^N$. Let $e_4$ be the future unit timelike normal vector field of $\Omega$ and $P(\cdot,\cdot)$ be the second fundamental form of $\Omega$ with respect to $e_4$. Let $e_3$ denote the outward unit spacelike normal of $\Sigma$ with respect to $\Omega$. We also choose orthonormal frames $\{e_1, e_2\}$ tangent to $\Sigma$. The mean curvature vector $H$ is the unique normal vector field that is the normal part of $\sum_{a=1}^{2} \nabla^N_{e_a} e_a$. Denote by $k = \sum_{a=1}^{2} \langle \nabla^N_{e_a} e_3, e_a \rangle$ the mean curvature of $\Sigma$ with respect $e_3$ and, $p = tr_\Sigma P = \sum_{a=1}^{2} \langle \nabla^N_{e_a} e_4, e_a \rangle$, then

$$H = -k e_3 + p\, e_4.$$

We can reflect $H$ along the light cone of the normal bundle to get

$$J = k e_4 - p\, e_3.$$

$H$ and $J$ are well-defined independent of the choice of frames. For a standard round 2-sphere in the Minkowski space $\mathbb{R}^{3,1}$ bounding a standard 3-ball, the mean curvature vector $H$ is inward pointing ($k > 0$) and $J$ is future pointing.

Let $T$ be a future timelike unit vector field which generates a unit timelike translation in spacetime, the surface Hamiltonian derived by Brown and York [**2, 3**]

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[1]Even in the Minkowski space, there are closed spacelike two-surfaces that do not bound any spacelike hypersurface (the author thanks Marcus Khuri for pointing out reference [**10**]).

(see also [**8**]) is given by

$$\frac{1}{8\pi}\int_\Sigma \langle J, T^\perp\rangle_N + \langle \nabla^N_{e_3} e_4, T^\top\rangle_N \tag{1.1}$$

where $T^\top$ is the tangential part of $T$ (shift vector) and $T^\perp$ is the normal part of $T$ along $\Sigma$. The last term $\langle \nabla^N_{e_3} e_4, T^\top\rangle_N$ can be expressed by the second fundamental form as $P(e_3, T^\top)$. This expression apparently depends on the vector field $T$ and the normal gauge $\{e_3, e_4\}$ along $\Sigma$.

When the spacetime is the Minkowski space $\mathbb{R}^{3,1}$, we can take $T_0$ to be a *constant* future timelike unit vector. Let $L$ be the orthogonal complement to $T_0$ in $\mathbb{R}^{3,1}$ which is isometric to $\mathbb{R}^3$. It is not hard to see that the projection of $\Sigma$ onto $L$ is an embedded closed surface $\hat{\Sigma}$ with a well-defined mean curvature $\hat{H}$ with respect to the outward normal of $\hat{\Sigma}$. $T_0$ picks up a distinguished normal gauge $\{\breve{e}_3, \breve{e}_4\}$ along $\Sigma$.

Proposition 1.1. For a closed spacelike 2-surface $\Sigma$ in the Minkowski space which bounds a spacelike hypersurface, there exists a unique orthogonal normal gauge $\{\breve{e}_3, \breve{e}_4\}$ along $\Sigma$ such that $\breve{e}_3$ is a outward spacelike unit normal and $\breve{e}_4$ is a future timelike unit normal and they satisfy

$$\frac{1}{8\pi}\int_\Sigma \langle J, T_0^\perp\rangle_{\mathbb{R}^{3,1}} + \langle \nabla^{\mathbb{R}^{3,1}}_{\breve{e}_3} \breve{e}_4, T_0^\top\rangle_{\mathbb{R}^{3,1}} = -\frac{1}{8\pi}\int_{\hat{\Sigma}} \hat{H}. \tag{1.2}$$

In fact, denote by $\tau$ the restriction of time function defined by $T_0$ to $\Sigma$ and by $\nabla\tau$ the gradient vector field of $\tau$ on $\Sigma$ (with respect to the induced metric), we have

$$T_0 = \sqrt{1+|\nabla\tau|^2}\,\breve{e}_4 - \nabla\tau.$$

The lapse and shift are given by $\sqrt{1+|\nabla\tau|^2}$ and $T_0^\top = -\nabla\tau$, respectively.

Proof. Proposition 3.1 of [**24**] (see also [**7**]). □

Obviously, no such relation as (1.2) holds when the ambient space is a general spacetime.

For a closed embedded 2-surface $\Sigma$ in a general spacetime, to define the quasilocal energy by the Hamilton-Jacobi method, one needs to find a reference embedding of $\Sigma$ in a reference spacetime. The quasilocal energy is obtained by subtracting the reference surface Hamiltonian from the physical one.

The idea in [**23, 24**] is to use an isometric embedding of the surface into $\mathbb{R}^{3,1}$ as a reference and emigrate a constant future timelike vector $T_0$ in $\mathbb{R}^{3,1}$ back to the physical spacetime $N$. When the mean curvature vector $H$ of $\Sigma$ in $N$ is spacelike, $T_0$ picks up a "canonical gauge" $\{\bar{e}_3, \bar{e}_4\}$ in the physical spacetime, and determines a future timelike unit vector field $T$ along $\Sigma$ in $N$ with the same lapse and shift as $T_0$. Isometric embedding anchors the intrinsic geometry, and allows one to compare the difference of the extrinsic geometry caused by the spacetime curvature.

## 2. Isometric embedding into the Minkowski space $\mathbb{R}^{3,1}$

The well-known Weyl's embedding problem is about isometric embeddings into $\mathbb{R}^3$ in which one asks for an embedding $\hat{X} : S^2 \to \mathbb{R}^3$ whose induced metric is a given Riemannian metric $\sigma$ on $S^2$. The equation can be written compactly as

$$\langle d\hat{X}, d\hat{X}\rangle_{\mathbb{R}^3} = \sigma.$$

In terms of local coordinates $u, v$ on $S^2$, there are three equations corresponding to three components $E, F$, and $G$ in $\sigma = Edu^2 + 2Fdudv + Gdv^2$ and three unknown functions $\hat{X} = (\hat{X}_1, \hat{X}_2, \hat{X}_3)$. The problem has a very satisfactory answer by Nirenberg [**16**] and Pogorelov [**17**] when the Gauss curvature of $\sigma$ is positive:

THEOREM 2.1. *If the Gauss curvature of $\sigma$ is positive, there exists a unique isometric embedding $\hat{X}$ of $\sigma$ up to rigid motion of $\mathbb{R}^3$.*

Here we are interested in isometric embeddings into $\mathbb{R}^{3,1}$. In this case, there are four unknowns functions $X = (X_0, X_1, X_2, X_3)$ and one more equation is needed to give a well-determined system. As $X_0$ plays the role of the time function and can be distinguished from other coordinate functions, it is natural to prescribe the time function. Before we state the solution, here are two observations:

1. Suppose $\Sigma$ is a spacelike 2-surface in $\mathbb{R}^{3,1}$ which bounds a spacelike hypersurface. Consider the projection from $\mathbb{R}^{3,1}$ to $\mathbb{R}^3$ given by $(X_0, X_1, X_2, X_3)$ to $(X_1, X_2, X_3)$. The image of the projection of $\Sigma$ is a well-defined embedded surface $\hat{\Sigma}$ in $\mathbb{R}^3$. In fact, if the induced metric on $\Sigma$ is $\sigma$, the induced metric on $\hat{\Sigma}$ is $\sigma + (dX_0)^2$. As opposed to a projection in Euclidean space which is a contraction, a projection in the Minkowski space is an expansion.

2. Take any function $\tau$ on $(S^2, \sigma)$, then $\hat{\sigma} = \sigma + (d\tau)^2$ is another Riemannian metric, the Gauss curvature $\hat{K}$ of $\hat{\sigma}$ is related to the Gauss curvature $K$ of $\sigma$ by

$$\widehat{K} = (1 + |\nabla\tau|^2)^{-1}[K + (1 + |\nabla\tau|^2)^{-1} \det(\nabla^2\tau)], \tag{2.1}$$

where $\nabla$ is the gradient operator and $\nabla^2$ is the Hessian operator, with respect to $\sigma$.

With these two observations, the following proposition can be proved easily (see[**24**]):

PROPOSITION 2.2. Suppose $\tau$ is a function on $(S^2, \sigma)$ with $\hat{K} > 0$, then there exists a unique isometric embedding $X$ into $\mathbb{R}^{3,1}$ of $\sigma$ such that the time function $X_0 = \tau$.

PROOF. We first isometrically embed $\hat{\sigma}$ into $\mathbb{R}^3$ to get a closed convex surface $\hat{\Sigma}$, and then take the graph of $\tau$ over $\hat{\Sigma}$ in $\mathbb{R}^{3,1}$. □

We say such an isometric embedding has convex shadow as $\hat{\Sigma}$ is a convex surface in $\mathbb{R}^3$.

## 3. Minimax definition of quasilocal mass

Let $\Sigma$ a closed embedded spacelike 2-surface which bounds a spacelike region in spacetime $N$. Let $X$ be an isometric embedding of the induced metric on $\Sigma$ into $\mathbb{R}^{3,1}$, and $T_0$ be a constant future timelike unit vector in $\mathbb{R}^{3,1}$. We used the normal gauge $\{\breve{e}_3, \breve{e}_4\}$ from Proposition 1.1 for $(X, T_0)$ to compute the reference surface Hamiltonian. The question now is how to choose $T, e_3, e_4$ along the surface in physical spacetime for the physical surface Hamiltonian(1.1).

We require that $T$ and $T_0$ should have the same lapse and shift, and thus

$$T^\perp = \sqrt{1 + |\nabla\tau|^2} e_4 \text{ and } T^\top = -\nabla\tau \tag{3.1}$$

for an orthogonal normal gauge $\{e_3, e_4\}$ along $\Sigma$ to be determined. Suppose we fix a spacelike domain $\Omega$ bounded by $\Sigma$. Let $e_4^\Omega$ be the future timelike unit normal of $\Omega$ and $e_3^\Omega$ be the outward spacelike normal of $\Sigma$ with respect to $\Omega$. All other orthogonal normal gauges $\{e_3, e_4\}$ (with $e_3$ outward spacelike unit and $e_4$ future timelike unit) can be written in terms of $\{e_3^\Omega, e_4^\Omega\}$ by $e_4 = \cosh\phi\, e_4^\Omega - \sinh\phi\, e_3^\Omega$ and $e_3 = \cosh\phi\, e_3^\Omega - \sinh\phi\, e_4^\Omega$ for a function $\phi$. With $J = k\, e_4^\Omega - p\, e_3^\Omega$, the physical surface Hamiltonian is computed as

$$\mathfrak{H}(T, e_3, e_4) = -\frac{1}{8\pi}\int_\Sigma \sqrt{1+|\nabla\tau|^2}(k\cosh\phi - p\sinh\phi) + (\Delta\tau)\phi + \langle\nabla^N_{e_3^\Omega} e_4^\Omega, \nabla\tau\rangle_N \tag{3.2}$$

where $\Delta$ is the Laplace operator on $\Sigma$. It is not hard to check that when $k^2 > p^2$ (i.e. the mean curvature vector $H$ is spacelike), the minimum of (3.2) as a function of $\phi$ is achieved when $\phi = \bar\phi$ satisfies

$$\sqrt{1+|\nabla\tau|^2}(k\sinh\bar\phi - p\cosh\bar\phi) + \Delta\tau = 0.$$

The corresponding $\bar e_3, \bar e_4$, and $\bar T$ are

$$\begin{aligned}\bar e_4 &= \cosh\bar\phi\, e_4^\Omega - \sinh\bar\phi\, e_3^\Omega,\\ \bar e_3 &= \cosh\bar\phi\, e_3^\Omega - \sinh\bar\phi\, e_4^\Omega\end{aligned}$$

and

$$\bar T = \sqrt{1+|\nabla\tau|^2}\bar e_4 - \nabla\tau.$$

Since $H = -k\, e_3^\Omega + p\, e_4^\Omega$, we have

$$\langle H, \bar e_4\rangle_N = k\,\sinh\bar\phi - p\,\cosh\bar\phi = \frac{-\Delta\tau}{\sqrt{1+|\nabla\tau|^2}}$$

and

$$\langle H, \bar T\rangle_N = -\Delta\tau. \tag{3.3}$$

$\{\bar e_3, \bar e_4\}$ is the canonical gauge referred in [**23**]. On the reference side, the time function $\tau$ can be expressed in terms of the embedding $X$ and $T_0$ with $\tau = -\langle X, T_0\rangle$. The mean curvature vector $H_0$ of a surface in Minkowski space is $H_0 = \Delta X$, therefore the condition of canonical gauge in fact means

$$\langle H_0, T_0\rangle_{\mathbb{R}^{3,1}} = \langle H, \bar T\rangle_N, \tag{3.4}$$

i.e. the area change when the surface moves in $\mathbb{R}^{3,1}$ along the direction of $T_0$ is the same as the area change when the surface moves in spacetime $N$ along the direction of $\bar T$. Or, in physical terms, the "expansions" with respect to observers $T_0$ and $\bar T_0$ are the same.

Definition 3.1. Let $\Sigma$ a closed embedded spacelike 2-surface which bounds a spacelike region in spacetime $N$. Let $X$ be an isometric embedding of the induced metric on $\Sigma$ into $\mathbb{R}^{3,1}$, and $T_0$ be a constant future timelike unit vector in $\mathbb{R}^{3,1}$. The quasilocal energy $E(\Sigma, X, T_0)$ of $\Sigma$ with respect to $(X, T_0)$ is defined to be the supremum of

$$\frac{1}{8\pi}\int_{\hat\Sigma}\hat H - \mathfrak{H}(T, e_3, e_4)$$

among all $T$ satisfying (3.1) and all orthogonal normal gauges $\{e_3, e_4\}$ so that $e_3$ is an outward pointing unit spacelike normal and $e_4$ is a future pointing unit timelike normal.

We summarize in the following proposition proved in [**24**]:

Proposition 3.2. When the mean curvature vector of $\Sigma$ in $N$ is spacelike. The quasilocal energy is achieved at the (unique) canonical gauge.

The pair $(X, T_0)$ should be considered as a quasilocal observer. In general relativity, the rest mass can be recovered by minimizing energy seen by various observers. Thus we defined the quasilocal mass by minimization.

DEFINITION 3.3. Same notation as in Definition 3.1, the quasilocal mass of $\Sigma$ is defined to be the infimum of $E(\Sigma, X, T_0)$ among all $(X, T_0)$.

Strictly speaking, we should restrict $(X, T_0)$ to those admissible pairs in [**23, 24**]. We specialize to the case when mean curvature vector is spacelike and thus the canonical gauge is uniquely defined in [**23, 24**]. Note the minimax definition for quasilocal mass works even if the mean curvature vector is not spacelike and it picks up the right expression at a marginally trapped surface on which the mean curvature vector is null.

In the case when the mean curvature vector is spacelike and inward pointing, we can use $\{-\frac{H}{|H|}, \frac{J}{|H|}\}$ as a reference normal gauge. This is the case for a small sphere near a point or a large sphere near asymptotically flat infinity. Denote the connection one-form $\alpha_H$ on $\Sigma$ by

$$\alpha_H(Y) = \langle \nabla^N_Y \frac{J}{|H|}, \frac{H}{|H|} \rangle_N$$

for any tangent vector field $Y$.

The quasilocal energy of $\Sigma$ with respect to $(X, T_0)$ can be expressed as:

$$E(\Sigma, X, T_0) = \frac{1}{8\pi} \int_{\hat{\Sigma}} \hat{H} - \frac{1}{8\pi} \int_{\Sigma} |H| \sqrt{1 + |\nabla \tau|^2} \cosh \theta + (\Delta \tau)\, \theta - \alpha_H(\nabla \tau),$$

where $\sinh \theta = \frac{-\Delta \tau}{|H| \sqrt{1+|\nabla \tau|^2}}$.

This is zero if $\Sigma$ is in the Minkowski space and. We proved in [**24**] :

THEOREM 3.4. *$E(\Sigma, X, T_0) \geq 0$ if $\Sigma$ bounds a regular spacelike region in a spacetime which satisfies the dominant energy condition and $(X, T_0)$ is an admissible pair in the sense of* [**23, 24**].

## 4. Searching for optimal isometric embedding

We can describe the evaluation of quasilocal mass as a variational problem for an optimal isometric embedding into $\mathbb{R}^{3,1}$. Given an spacelike 2-surface $\Sigma$ in a general spacetime, we assume the mean curvature vector is spacelike and inward pointing. Therefore the induced metric, the Lorentz norm of the mean curvature vector, and the connection one form $(\sigma, |H|, \alpha_H)$ are given and intrinsically defined on the surface $\Sigma$. The question can be formulated as given a Riemannian metric, a positive function, and a one-form, is there a surface in $\mathbb{R}^{3,1}$ that best matches these data?

We can view the quasilocal energy as a functional on the space of isometric embeddings of $\sigma$ into $\mathbb{R}^{3,1}$ and study the minimization problem. To be more specific, we consider an admissible pair $(X, T_0)$ in which $X$ is an isometric embedding of $\sigma$ into $\mathbb{R}^{3,1}$ and $T_0$ is a future timelike unit vector. We ask that $X$ has spacelike mean curvature vector and has convex shadow $\hat{\Sigma}$ in the direction of $T_0$.

Given $(\sigma, |H|, \alpha_H)$, the Euler Lagrange equation for quasilocal energy $E(\Sigma, X, T_0)$ at $(X, T_0)$ with $\tau = -\langle X, T_0\rangle_{\mathbb{R}^{3,1}}$ is

(4.1)
$$-(\widehat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd})\frac{\nabla_b\nabla_a\tau}{\sqrt{1+|\nabla\tau|^2}} + div_\sigma(\frac{\nabla\tau}{\sqrt{1+|\nabla\tau|^2}}\cosh\theta|H| - \nabla\theta - \alpha_H) = 0$$

where $\sinh\theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}$, $\hat{\sigma}$ is the induced metric on $\hat{\Sigma}$, and $\hat{h}_{ab}$ is the second fundamental form of $\hat{\Sigma}$.

Coupling with the isometric embedding equation $\langle dX, dX\rangle_{\mathbb{R}^{3,1}} = \sigma$ or $\langle d\hat{X}, d\hat{X}\rangle_{\mathbb{R}^3} = \sigma + (d\tau)^2$, they form what we called the optimal isometric embedding system. There are exactly four unknown functions (coordinates) and four equations.

The equation should be read in the following way. First take a function $\tau$ on $\Sigma$, consider $\hat{\sigma} = \sigma + (d\tau)^2$ on $\Sigma$. Isometrically embed $\hat{\sigma}$ into $\mathbb{R}^3$ and pick up $\hat{h}_{ab}$ and $\hat{H}$ from this isometric embedding. We look for $\tau$ that satisfies (4.1), a fourth-order elliptic equation.

## 5. Solving the equation for isolated systems

In general relativity, an isolated system is an asymptotically flat spacetime on which gravitation is weak near infinity. There are two notions of asymptotic flatness which correspond to null infinity and spatial infinity. In both cases, the infinity is foliated by a one-parameter family of spacelike 2-surface $\Sigma_r$, $r \in [r_0, \infty)$ such that

$$\begin{aligned}\sigma &= r^2\tilde{\sigma} + O(r),\\ |H| &= \frac{2}{r} + O(r^{-2}),\\ div_\sigma\alpha_H &= O(r^{-3}),\end{aligned} \tag{5.1}$$

where $\tilde{\sigma}$ is the standard round metric on a unit 2-sphere. When $r$ is large enough, $\Sigma_r$ has positive Gauss curvature, and there exists a unique isometric embedding $X_r : \Sigma_r \to \mathbb{R}^3 \subset \mathbb{R}^{3,1}$. $X_r$ solves the optimal embedding equation up to the top order and this is enough to evaluate the limit of the quasilocal energy. Our analysis of limit of quasilocal mass at infinity only relies on the existence of a family of 2-surfaces with such data, and thus we may as well take this as a definition of asymptotically flat spacetime which is coordinate independent.

In [**25**], we show the limit of $E(\Sigma_r, \hat{X}_r, T_0)$ is the same as the limit of

$$\sqrt{1+|a|^2}\frac{1}{8\pi}\int_{\Sigma_r}(H_0 - |H|) + \sum_{i=1}^{3} a^i \frac{1}{8\pi}\int_{\Sigma_r}(\hat{X}^i div_\sigma \alpha_H), \tag{5.2}$$

where $(\hat{X}^1, \hat{X}^2, \hat{X}^3)$ are the components of the isometric embedding $\hat{X}_r$, $H_0$ is the mean curvature of the image of $\hat{X}_r$ in $\mathbb{R}^3$, and $T_0 = (\sqrt{1+|a|^2}, a^1, a^2, a^3)$. It is not hard to see that $\hat{X}_r$ approaches a round sphere as $r \to \infty$ and thus $\hat{X}^i = r\tilde{X}^i + O(1)$ for $i = 1, 2, 3$ where $\tilde{X}^1$, $\tilde{X}^2$, $\tilde{X}^3$ are the three standard first eigenfunctions on the unit 2-sphere $S^2$. In particular, this is a coordinate independent expression that recovers the ADM and Bondi-Sachs energy-momentum at spatial and null infinity, respectively [**25, 5**].

Remarkably, expression (5.2) is a *linear* in $T_0$. Thus the quasilocal energy $E(\Sigma_r, \hat{X}_r, T_0)$, a priori nonlinear in $T_0$, gets linearized at infinity, acquires

Lorentzian symmetry, and defines an energy-momentum 4-vector. Expand

$$div_\sigma \alpha_H = \frac{v}{r^3} + O(r^{-4})$$

for a function $v$ on $S^2$, the components of the momentum vector are given by

$$\frac{1}{8\pi}\int_{S^2} \tilde{X}_i v = P_i, i = 1, 2, 3. \tag{5.3}$$

The calculation of quasilocal energy, which only picks up the top order term, is stable with respect to variation of the reference isometric embedding. We will take an $O(1)$ perturbation of the isometric embedding $\hat{X}_r$ into $\mathbb{R}^3$ in the time direction and solve the next order of the optimal isometric embedding equation within this family. Thus

$$X^0 = \tau = \tau^{(1)} + O(r^{-1})$$

for a function $\tau^{(1)}$ on $S^2$. The $O(r^{-3})$ term of the optimal embedding equation is

$$\frac{1}{2}\Delta_{S^2}(\Delta_{S^2} + 2)\tau^{(1)} = v.$$

The equation is solvable as long as $v$ is perpendicular to the kernel of $\Delta_{S^2} + 2$, or the linear space spanned by $\tilde{X}^1, \tilde{X}^2$ and $\tilde{X}^3$. In view of (5.3), nonzero "momentum" vector is the obstruction to solvability.

Recall that $(X_r, T_0)$ plays the role of an observer. In classical relativity, energy is relative and mass is absolute. Given a system with energy-momentum 4-vector $(E, P_1, P_2, P_3)$, energy depends on the observer and mass $m = \sqrt{E^2 - \sum_i P_i^2}$. If the observer is at rest with the system, $m = E$ and the momentum is eliminated. We observe that in our case the observer $T_0$ needs to be aligned with the energy-momentum vector in order to solve the optimal embedding equation. This can be achieved by boosting the reference embedding $\hat{X}_r$. The discussion covers the spatial infinity case discussed in [**25**] as well. We boost $X_r$ in $\mathbb{R}^{3,1}$ by a family of elements of $SO(3, 1)$ so the images are close to a totally geodesic slice that is orthogonal to the total energy-momentum $(E, P_1, P_2, P_3)$.

The second variation of the quasilocal energy at each order is given by

$$\int_{S^2} (\Delta_{S^2}\delta\tau + \frac{1}{2}\Delta_{S^2}\Delta_{S^2}\delta\tau)\delta\tau,$$

where $\delta\tau$ is the variation of $\tau$.

Since the operator $\Delta_{S^2} + \frac{1}{2}\Delta_{S^2}\Delta_{S^2}$ is positive, this shows the solution we obtained is locally energy-minimizing. It turns out the Euler-Lagrange equation can be solved term-by-term (assuming analyticity in $r$) and all terms are locally energy-minimizing.

THEOREM 5.1. [**5**] *The optimal isometric embedding problem for any isolated system can be solved, and the solution is a local minimizer (assuming analyticity in $r$).*

## 6. Relations with other masses

In this section, we compare the Brown-York mass and the Liu-Yau mass with the new quasilocal mass, and discuss their relations.

One of the most promising approaches to the definition of quasilocal mass had been the one proposed by Brown-York [**2, 3**] in which the definition was motivated by using the Hamiltonian formulation of general relativity (see also

Hawking-Horowitz [8]). The Brown-York mass depends on a spacelike hypersurface $\Omega$ bounded by $\Sigma$ and is defined to be

$$\frac{1}{8\pi}\left(\int_\Sigma H_0 - \int_\Sigma H_\Omega\right)$$

where $H_\Omega$ is the mean curvature of $\Sigma$ with respect to $\Omega$, and $H_0$ is the mean curvature of an isometric embedding of $\Sigma$ into $\mathbb{R}^3$. Under the assumptions that $H_\Omega > 0$, $\Sigma$ has positive Gauss curvature $K > 0$, and $\Omega$ has non-negative scalar curvature, Shi and Tam [21] prove the Brown-York mass is non-negative. The Brown-York mass appears to be most useful in the time-symmetric case, i.e. when the second fundamental form of $\Omega$ in $N$ vanishes.

Motivated by geometric consideration in [27], Liu and Yau [11] (see also Kijowski [9], Booth-Mann[1], and Epp [6]) introduce a mass that is gauge independent. The Liu-Yau mass is

$$\frac{1}{8\pi}\left(\int_\Sigma H_0 - \int_\Sigma |H|\right)$$

where $H_0$ is the same as the one in the definition of the Brown-York mass, and $H$ is the mean curvature vector of $\Sigma$ in spacetime. The positivity is proved under the assumptions that $H$ is spacelike, $K > 0$, and the spacetime satisfies the dominant energy condition. However, it was pointed out by ÓMurchadha *et al.* [14] that the Liu-Yau mass, as well as the Brown-York mass, can be strictly positive for a surface in the Minkowski space. In contrast, the new quasilocal mass is zero for surfaces in the Minkowski space. In addition,

1) In the definitions of the Brown-York or the Liu-Yau mass, the lapse-shift is chosen to be $(1, 0)$. Our lapse-shift $(\sqrt{1 + |\nabla\tau|^2}, -\nabla\tau)$ comes from the reference isometric embedding, and the surface Hamiltonian corresponds to unit time translation in $T$. The future timelike unit vector field $T$ should be interpreted as a fleet of observers which need not be orthogonal to the surface $\Sigma$.

2) The canonical gauge which corresponds to the natural condition that the expansions of the surface in the reference and physical spacetime are the same is adopted in our definition. This gauge arises naturally from the minimax definition of quasilocal mass.

3) We obtain an optimal isometric embedding of $\Sigma$ in $\mathbb{R}^{3,1}$ through the data $(\sigma, |H|, \alpha_H)$. The optimal isometric embedding thus provides a diffeomorphism from $\Sigma$ onto a closed spacelike 2-surface in the Minkowski space. On the asymptotically flat region (in the sense that there exists a family of 2-surfaces that satisfies (5.1)), the optimal isometric embedding gives a diffeomorphism from the region onto a spacelike hypersurface in the Minskowki space. In particular, the total energy-momentum expression we have is independent of the coordinate system at infinity.

This is equivalent to saying that our approach takes an asymptotically flat hypersurface in the Minkowski space as a reference, while in the usual formulation of ADM mass in terms of asymptotically flat coordinates takes the totally geodesic $\mathbb{R}^3$ in the Minkowski space as a reference.

Suppose we look the hypersurface in $\mathbb{R}^{3,1}$ defined by $t = f(r)$ with $f(r) = r^k + o(r^k)$. When $k \geq \frac{1}{2}$ , the ADM mass and the limit of the Brown-York mass both diverge. The new quasilocal mass is the same as the Liu-Yau mass and both are equal to zero. In fact, the reference isometric embedding recovers the hypersurface itself.

A natural question is whether the new quasilocal mass is the same as the Brown-York mass in the time symmetric case. Since $\alpha_H = 0$ for as a surface $\Sigma$ in a time-symmetric spacelike region $\Omega$, $\tau = 0$ satisfies (4.1) and thus the isometric embedding $\hat{X}$ into $\mathbb{R}^3$ with $T_0 = (1,0,0,0)$ is a critical point of the quasilocal energy. Suppose $H_\Omega > 0$, the second variation of quasilocal energy at $(\hat{X}, (1,0,0,0))$ (with $\tau = 0$) was computed in [**13**]:

$$\int_\Sigma \frac{(\Delta\eta)^2}{H_\Omega} + (H_0 - H_\Omega)|\nabla\eta|^2 - \mathbb{II}_0(\nabla\eta, \nabla\eta) \tag{6.1}$$

for any function $\eta = \delta\tau$ on $\Sigma$. Here $\mathbb{II}_0$ is the second fundamental form of the image surface of $\hat{X}$ in $\mathbb{R}^3$.

Miao-Tam-Nie [**13**] proved that (6.1) is non-negative assuming the pointwise inequality $H_0 \geq H_\Omega$ holds. This shows that the Brown-York mass is a local minimum of our quasilocal energy in this case. In fact, it is not hard to see that the argument applies to the case when $div_\sigma \alpha_H = 0$, and thus the Liu-Yau mass is a local minimum of our quasilocal energy under the pointwise assumption $H_0 \geq |H|$.

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