THE CURVATURE OF GRADIENT RICCI SOLITONS

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Abstract. We study integral and pointwise bounds on the curvature of gradient shrinking Ricci solitons. As applications, we discuss gap and compactness results for gradient shrinkers.

1. Introduction

Our goal in this paper is to obtain further information about the curvature of gradient shrinking Ricci solitons. This is important for a better understanding and ultimately for the classification of these manifolds. The classification of gradient shrinkers is known in dimensions 2 and 3, and assuming locally conformally flatness, in all dimensions $n \geq 4$ (see [14, 13, 6, 15, 20, 12, 2]). Many of the techniques used in these works required some control of the Ricci curvature. For example, in [15] gradient shrinking Ricci solitons which are locally conformally flat were classified assuming an integral condition on the Ricci tensor. This condition and other integral estimates of the curvature were later proved in [12].

Without making the strong assumption of being conformally flat, it is natural to ask whether similar estimates are true for the Riemann curvature tensor. In this paper we are able to prove pointwise estimates on the Riemann curvature, assuming in addition that the Ricci curvature is bounded. We will show that any gradient shrinking Ricci soliton with bounded Ricci curvature has Riemann curvature tensor growing at most polynomially in the distance function. We note that by Shi's local derivative estimates we can then obtain growth estimates on all derivatives of the curvature. This, in particular, proves weighted L^2 estimates for the Riemann curvature tensor and its covariant derivatives.

We point out that for self shrinkers of the mean curvature flow Colding and Minicozzi [8] were able to prove weighted L^2 estimates for the second fundamental form, assuming the mean curvature is positive. These estimates were instrumental in the classification of stable shrinkers. Our estimates can be viewed as parallel to theirs, however the classification of gradient Ricci solitons is still a major open question in the field.

A gradient shrinking Ricci soliton is a Riemannian manifold (M, g) for which there exists a potential function f such that

$$R_{ij} + f_{ij} = \frac{1}{2} g_{ij}. \tag{1}$$

It can be shown directly from the equation that the quantity $|\nabla f|^2 + R - f$ is constant on M hence we can normalize f such that

$$|\nabla f|^2 + R = f.$$

Let us denote with $|Rc|$ and $|Rm|$ the norms of the Ricci and Riemann tensors, respectively. We now state the main result of this paper.

Theorem 1. *Let (M, g, f) be a gradient shrinking Ricci soliton with bounded Ricci curvature. Then the Riemann curvature tensor grows at most polynomially in the distance function i.e.*

$$|Rm|(x) \leq C\left(r(x)+1\right)^{a},$$

for some constant $a > 0$.

We apply this theorem to prove a gap result for gradient shrinkers. We show that if the Ricci curvature is small enough everywhere on M then the soliton is isometric to the Gaussian soliton $\left(\mathbb{R}^n, dx^2, \frac{1}{4}|x|^2\right)$.

Corollary 1. *Let (M, g, f) be a gradient shrinking Ricci soliton. If $|Rc| \leq \frac{1}{100n^2}$ on M then M is isometric to the Gaussian soliton.*

Yokota [18] has obtained a gap theorem for gradient shrinking Ricci solitons, namely he showed that if $(4\pi)^{-\frac{n}{2}} \int_M e^{-f} > 1-\varepsilon_n$ for some ε_n depending on n then M is isometric to the Gaussian soliton.

Another application of our main theorem is in relation to compactness results for Ricci solitons. This topic has been studied recently in both compact ([4, 17, 19]) and noncompact ([10]) settings. We recall a recent result for complete noncompact shrinkers, due to Haslhofer and Müller [10]. Let $\left(M_i, g_i, \bar{f}_i\right)$ a sequence of gradient shrinkers with the potentials $\bar{f}_i$ normalized such that $(4\pi)^{-\frac{n}{2}} \int_{M_i} e^{-\bar{f}_i} = 1$. Consider z_i a point where $\bar{f}_i$ attains its minimum. Assume that Perelman's entropy μ_i has a uniform bound from below i.e. there exists $\bar{\mu}$ such that

$$\mu_i := (4\pi)^{-\frac{n}{2}} \int_{M_i} \left(\left|\nabla \bar{f}_i\right|^2 + R_{g_i} + \bar{f}_i - n\right) e^{-\bar{f}_i} \geq \bar{\mu}.$$

If, moreover, for any i and $r > 0$

$$\int_{B_{z_i}(r)} \left|Rm_{g_i}\right|^{n/2} \leq E(r),$$

then a subsequence of $\left(M_i, g_i, \bar{f}_i, z_i\right)$ converges to an orbifold gradient shrinker in the pointed Cheeger-Gromov sense.

As a consequence of our main theorem, we can show the following result. Here it is not important how we normalize the function f, but we need to take x_0 a point where f assumes its minimum value on M. Moreover, here we also need to take $n \geq 6$. Notice however that in dimensions 2 and 3 shrinking solitons are completely classified and in dimension 4, Haslhofer and Müller have proved that $\int_{B_{z_i}(r)} \left|Rm_{g_i}\right|^2 \leq E(r)$ using the Gauss-Bonnet theorem. However, it seems that their argument is more special and works only in lower dimensions.

Corollary 2. *Let (M, g, f) be a gradient shrinker with $dim M = n \geq 6$ and $|Rc| \leq K$ on M. There exists a r_0 depending on n and K such that if $\int_{B_{x_0}(r_0)} |Rm|^{n/2} \leq L$ for a minimum point x_0 of f, then for any $r > 0$ we have*

$$\int_{B_{x_0}(r)} |Rm|^{n/2} \leq E(r),$$

where E depends on n, K and L.

In particular, this Corollary and the main theorem in [10] implies compactness of shrinkers assuming Ricci curvature bounds and only local curvature bounds.

Corollary 3. *Let $\left(M_i, g_i, \bar{f}_i\right)$ be a sequence of gradient shrinking Ricci solitons normalized by $(4\pi)^{-\frac{n}{2}} \int_{M_i} e^{-\bar{f}_i} = 1$. Assume $\mu_i \geq \bar{\mu}$ and that we have a uniform bound on the Ricci curvature $|Rc_{g_i}| \leq K$. Then there exists r_0 depending on n and K such that if*

$$\sup_i \int_{B_{z_i}(r_0)} |Rm_{g_i}|^{n/2}$$

is finite for z_i a minimum point of $\bar{f}_i$, then a subsequence of $\left(M_i, g_i, \bar{f}_i, z_i\right)$ converges to an orbifold gradient shrinker in the pointed Cheeger-Gromov sense.

2. Proof of the curvature estimate

The idea of the proof of Theorem 1 is the following.

From the Ricci soliton equation (1), we can estimate $\Delta_f |Rm|^2 \geq -c|Rm|^3$, where $\Delta_f = \Delta - \nabla f \cdot \nabla$. It is natural therefore to attempt to use Moser iteration for this problem. Since Ricci is bounded below, the Sobolev constant is uniformly bounded on arbitrary balls of fixed radius $= 1$. Therefore it is known that the Moser iteration will work if we can control the L^p norm (for $p > n/2$) of $|Rm|$ on any ball of radius one. This is quite technical and it is done in the second Lemma below.

To get L^p estimates, we use the Ricci soliton equation and that the Ricci curvature is bounded. At the core of our estimates is a formula that relates the divergence of the Riemann curvature tensor of a Ricci soliton to the gradient of its Ricci curvature.

We first prove some Lemmas. Everywhere in this section (M, g, f) is a gradient shrinking Ricci soliton with bounded Ricci curvature. We use the notation:

$$\begin{aligned} |Rc|^2 &= \sum |R_{ij}|^2, \\ |\nabla Rc|^2 &= \sum |\nabla_k R_{ij}|^2, \\ |Rm|^2 &= \sum |R_{ijkl}|^2, \\ |\nabla Rm|^2 &= \sum |\nabla_h R_{ijkl}|^2. \end{aligned}$$

We recall some basic identities for shrinking Ricci solitons, which are essential in the proof. First, taking trace of the soliton equation we get $R + \Delta f = \frac{n}{2}$. As we have mentioned above, using the Bianchi identities and normalizing f we get $|\nabla f|^2 + R = f$. This normalization of f will be assumed throughout the paper. Other formulas that follow from Bianchi and Ricci identities and the soliton equation are

$$\begin{aligned} \nabla_i R &= 2R_{ij} f_j, \\ \nabla_k R_{jk} &= R_{jk} f_k, \\ \nabla_l R_{ijkl} &= R_{ijkl} f_l, \\ \nabla_j R_{ki} - \nabla_i R_{kj} &= R_{ijkl} f_l. \end{aligned}$$

We will not prove these here since they are quite standard, see e.g. [9].

Lemma 1. *We have:*

$$|\nabla Rc|^2 \leq \frac{1}{2}\Delta |Rc|^2 - \frac{1}{2}\nabla f \cdot \nabla |Rc|^2 + c|Rm| \tag{2}$$

$$|\nabla Rm|^2 \leq \frac{1}{2}\Delta |Rm|^2 - \frac{1}{2}\nabla f \cdot \nabla |Rm|^2 + c|Rm|^3 \tag{3}$$

Proof. It is known (see [15]) that

$$\begin{aligned}\Delta R_{ij} &= \nabla f\cdot\nabla R_{ij}+R_{ij}-2R_{ikjh}R_{kh}\\ \Delta Rm &= \nabla f\cdot\nabla Rm+Rm-2\left(Rm^{2}+Rm^{\#}\right).\end{aligned}$$

The Lemma follows immediately from here. ■

We now prove the following estimate, which is of independent interest.

Lemma 2. *For any $p\geq 2$ there exist positive constants C and a such that*

$$\int_{M}|Rm|^{p}\,(f+1)^{-a}\leq C.$$

In particular, there exist positive constants C and a such that for any $x\in M$ we have:

$$\int_{B_{x}(1)}|Rm|^{p}\leq C\,(r\,(x)+1)^{2a}\,.$$

Proof. Let us denote by $\rho:=2\sqrt{f}$ and for $r>>1$ let

$$D\,(r):=\{x\in M:\ \rho\,(x)\leq r\}\,.$$

Notice that $D\,(r)$ is always compact, in fact approximates well the geodesic ball of radius r when r is large. Here we recall that f has the following asymptotics, see [5]:

$$\left(\frac{1}{2}r\,(x)-c\right)^{2}\leq f\,(x)\leq\left(\frac{1}{2}r\,(x)+c\right)^{2},\ \text{ for }\ r\,(x)\geq r_{0}.$$

Here and below we denote by $r\,(x)$ the distance from x to a fixed point $x_{0}\in M$. Let us point out moreover that if x_{0} is chosen to be a minimum point of f then c and r_{0} will depend only on n, see [10].

We take the following cut-off

$$\phi=\begin{cases}\frac{1}{r^{2}}\left(\frac{1}{4}r^{2}-f\,(x)\right) & \text{if}\quad x\in D\,(r)\\ 0 & \text{if}\quad x\in M\backslash D\,(r)\end{cases}$$

Since the Ricci curvature is bounded on M, let us set

$$K=\sup_{M}|Rc|\,.$$

Let a be a fixed number to be determined later, depending on n,K and p. Consider also q a large enough integer, $q\geq 2p+1$. We will discuss first the case when $p\geq 3$, the case $2\leq p<3$ will follow immediately by Hölder's inequality.

We have, integrating by parts, that

$$\begin{aligned}a\int_{M}|Rm|^{p}\,|\nabla f|^{2}\,(f+1)^{-a-1}\,\phi^{q}&=-\int_{M}|Rm|^{p}\,\nabla f\cdot\nabla\,(f+1)^{-a}\,\phi^{q}\\ &=\int_{M}|Rm|^{p}\,(\Delta f)\,(f+1)^{-a}\,\phi^{q}+\int_{M}|Rm|^{p}\,(f+1)^{-a}\,\nabla f\cdot\nabla\phi^{q}\\ &\quad+\int_{M}\left(\nabla\,|Rm|^{p}\cdot\nabla f\right)(f+1)^{-a}\,\phi^{q}.\end{aligned}\tag{4}$$

Let us explain why we take these functions in (4). We take $|\nabla f|^{2}$ because we want to use integration by parts and the symmetries of the Riemann curvature tensor (which are implied by the soliton equation). For shrinking solitons the factor e^{-f} seems much more convenient than $(f+1)^{-a}$, however the former factor only gives exponential growth control. So to prove weighted L^{2} estimates for the

Riemann curvature it is easier to start from $\int_M |Rm|^p |\nabla f|^2 e^{-f}\phi^q$ and carry all the estimates below, with some useful simplifications, but in order to prove Lemma 2 we need to use the less natural weight $(f+1)^{-a}$, at the expense of more complicated computations.

Everywhere in this section c will denote a constant that depends on n, p, q and K but not on a. We will use C_1, C_2, ... etc. to denote finite constants that have a more complicated dependence, such as on $\sup_\Omega |Rm|$ over some compact set Ω. However, we stress that all the constants c or C_1, C_2... are independent of r.

We now check that:

$$a|\nabla f|^2 (f+1)^{-a-1} - \Delta f (f+1)^{-a} = \left(a\frac{f-R}{f+1} - \left(\frac{n}{2} - R\right)\right)(f+1)^{-a}.$$

Furthermore, since $R \geq 0$ on any gradient Ricci soliton ([7, 3]) we see that there exists a constant r_1 depending on n, K and a (e.g., $r_1 = \sqrt{8a(K+1)}$) such that on $M \backslash D(r_1)$ we have

$$a|\nabla f|^2 (f+1)^{-a-1} - \Delta f (f+1)^{-a} \geq (a-n)(f+1)^{-a}.$$

Notice also that by the choice of cut-off we have:

$$\nabla f \cdot \nabla \phi^q = -\frac{q}{r^2}\phi^{q-1}|\nabla f|^2 \leq 0.$$

Using these simple estimates in (4) it follows that

$$(5) \qquad (a-n)\int_M |Rm|^p (f+1)^{-a}\phi^q \leq \int_M \left(\nabla |Rm|^p \cdot \nabla f\right)(f+1)^{-a}\phi^q + C_1.$$

Here we have set

$$C_1 := \int_{D(r_1)} \left(-a|\nabla f|^2 (f+1)^{-1} + \Delta f + a - n\right)|Rm|^p (f+1)^{-a}\phi^q.$$

Let us compute, using the Bianchi identities:

$$\begin{aligned} \nabla |Rm|^2 \cdot \nabla f &= 2f_h \left(\nabla_h R_{ijkl}\right) R_{ijkl} \\ &= 4f_h \left(\nabla_l R_{ijkh}\right) R_{ijkl} \end{aligned}$$

Therefore, using this in the right hand side of (5) we get

$$
\begin{aligned}
&\int_M \left(\nabla |Rm|^p \cdot \nabla f\right) (f+1)^{-a} \phi^q \\
&= 2p \int_M f_h \left(\nabla_l R_{ijkh}\right) R_{ijkl} |Rm|^{p-2} (f+1)^{-a} \phi^q \\
&= -2p \int_M R_{ijkh} \nabla_l \left(f_h R_{ijkl} |Rm|^{p-2} (f+1)^{-a} \phi^q \right) \\
&= -2p \int_M R_{ijkh} f_{hl} R_{ijkl} |Rm|^{p-2} (f+1)^{-a} \phi^q \\
&\quad -2p \int_M R_{ijkh} f_h \left(\nabla_l R_{ijkl}\right) |Rm|^{p-2} (f+1)^{-a} \phi^q \\
&\quad -2p \int_M R_{ijkh} f_h R_{ijkl} \left(\nabla_l |Rm|^{p-2}\right) (f+1)^{-a} \phi^q \\
&\quad +2ap \int_M R_{ijkh} f_h R_{ijkl} f_l |Rm|^{p-2} (f+1)^{-a-1} \phi^q \\
&\quad -2pq \int_M R_{ijkh} f_h R_{ijkl} \phi_l |Rm|^{p-2} (f+1)^{-a} \phi^{q-1} \\
&= I + II + III + IV + V.
\end{aligned}
$$

It is easy to see that since the Ricci curvature is bounded,

$$
I = -2p \int_M R_{ijkh} f_{hl} R_{ijkl} |Rm|^{p-2} (f+1)^{-a} \phi^q \leq c \int_M |Rm|^p (f+1)^{-a} \phi^q.
$$

Furthermore, using that for a gradient shrinker we have (see [9, 12])

$$
\nabla_l R_{ijkl} = R_{ijkl} f_l
$$

we see that

$$
II + IV = -2p \int_M R_{ijkh} f_h R_{ijkl} f_l |Rm|^{p-2} \left(1 - \frac{a}{f+1}\right) (f+1)^{-a} \phi^q \leq C_2,
$$

where we have set

$$
C_2 := 2p \int_{D\left(2\sqrt{a-1}\right)} |R_{ijkh} f_h|^2 |Rm|^{p-2} \left(\frac{a}{f+1} - 1\right) (f+1)^{-a} \phi^q.
$$

The estimate above follows because if $f(x)+1 > a$ then the integral is negative. Let us point out however that in fact each of $|II|$ and $|IV|$ can be estimated by a similar argument as in the proof of inequality (9) below. Clearly, since a is independent of r, so are the constants C_1 and C_2 obtained so far.

We use the above estimates in (5), and get that

$$
\begin{aligned}
&(a-c) \int_M |Rm|^p (f+1)^{-a} \phi^q \\
&\leq -2p \int_M R_{ijkh} f_h R_{ijkl} \left(\nabla_l |Rm|^{p-2}\right) (f+1)^{-a} \phi^q \\
&\quad + \frac{2pq}{r^2} \int_M |R_{ijkh} f_h|^2 |Rm|^{p-2} (f+1)^{-a} \phi^{q-1} + C_1 + C_2.
\end{aligned} \tag{6}
$$

Recall that for Ricci solitons we have (see e.g. [9, 12])

$$
R_{ijkh} f_h = \nabla_j R_{ik} - \nabla_i R_{kj},
$$

so that we can estimate the first term in (6) by

$$\begin{aligned} (7) \qquad & -2p\int_M R_{ijkh}f_h R_{ijkl}\left(\nabla_l |Rm|^{p-2}\right)(f+1)^{-a}\phi^q \\ & \leq c\int_M |\nabla Rc|\,|\nabla Rm|\,|Rm|^{p-2}(f+1)^{-a}\phi^q \\ & \leq c\int_M |\nabla Rc|^2 |Rm|^{p-1}(f+1)^{-a}\phi^q + c\int_M |\nabla Rm|^2 |Rm|^{p-3}(f+1)^{-a}\phi^q. \end{aligned}$$

We now work on the second term in (6), and we will use below some interpolations which are important throughout the rest of the proof.

$$\begin{aligned} & \frac{1}{r^2}\int_M R_{ijkh}f_h R_{ijkl}f_l |Rm|^{p-2}(f+1)^{-a}\phi^{q-1} \\ &= \frac{2}{r^2}\int_M (\nabla_j R_{ik}) R_{ijkl}f_l |Rm|^{p-2}(f+1)^{-a}\phi^{q-1} \\ &= -\frac{2}{r^2}\int_M R_{ik}\nabla_j\left(R_{ijkl}f_l |Rm|^{p-2}(f+1)^{-a}\phi^{q-1}\right) \\ &= -\frac{2}{r^2}\int_M R_{ik}f_{lj}R_{ijkl}|Rm|^{p-2}(f+1)^{-a}\phi^{q-1} \\ & \quad -\frac{2}{r^2}\int_M R_{ik}f_l(\nabla_j R_{ijkl})|Rm|^{p-2}(f+1)^{-a}\phi^{q-1} \\ & \quad -\frac{2}{r^2}\int_M R_{ik}R_{ijkl}f_l\left(\nabla_j |Rm|^{p-2}\right)(f+1)^{-a}\phi^{q-1} \\ & \quad +\frac{2a}{r^2}\int_M R_{ik}R_{ijkl}f_l f_j |Rm|^{p-2}(f+1)^{-a-1}\phi^{q-1} \\ & \quad -\frac{2(q-1)}{r^2}\int_M R_{ik}R_{ijkl}f_l\phi_j |Rm|^{p-2}(f+1)^{-a}\phi^{q-2} \\ (8) \qquad &= I + II + III + IV + V. \end{aligned}$$

Since the Ricci curvature is bounded, we get

$$\begin{aligned} I &= -\frac{2}{r^2}\int_M R_{ik}f_{lj}R_{ijkl}|Rm|^{p-2}(f+1)^{-a}\phi^{q-1} \\ &\leq \frac{c}{r^2}\int_M |Rm|^{p-1}(f+1)^{-a}\phi^{q-1}. \end{aligned}$$

Using again that $\nabla_l R_{ijkl} = R_{ijkl}f_l$ and that $|\nabla f|^2 \leq f \leq \frac{1}{4}r^2$ on $D(r)$, it follows

$$II \leq c\int_M |Rm|^{p-1}(f+1)^{-a}\phi^{q-1}.$$

Furthermore,

$$\begin{aligned} III &= -\frac{2}{r^2}\int_M R_{ik}R_{ijkl}f_l\left(\nabla_j |Rm|^{p-2}\right)(f+1)^{-a}\phi^{q-1} \\ &\leq c\int_M |\nabla Rm|\,|Rm|^{p-2}(f+1)^{-a}\phi^{q-1} \\ &\leq c\int_M |\nabla Rm|^2 |Rm|^{p-3}(f+1)^{-a}\phi^q + c\int_M |Rm|^{p-1}(f+1)^{-a}\phi^{q-2}. \end{aligned}$$

Similarly,

$$\begin{aligned} IV &= \frac{2a}{r^2}\int_M R_{ik}R_{ijkl}f_lf_j\left|Rm\right|^{p-2}(f+1)^{-a-1}\phi^{q-1} \\ &\leq \frac{ca}{r^2}\int_M \left|Rm\right|^{p-1}(f+1)^{-a}\phi^{q-1} \\ &\leq c\int_M \left|Rm\right|^{p-1}(f+1)^{-a}\phi^{q-1}, \end{aligned}$$

by taking $r \geq \sqrt{a}$. Finally, we also have

$$V \leq \frac{c}{r^2}\int_M \left|Rm\right|^{p-1}(f+1)^{-a}\phi^{q-2}.$$

Using these estimates in (8) we get

$$\begin{aligned} \frac{2pq}{r^2}\int_M \left|R_{ijkh}f_h\right|^2\left|Rm\right|^{p-2}(f+1)^{-a}\phi^{q-1} \\ \leq c\int_M \left|\nabla Rm\right|^2\left|Rm\right|^{p-3}(f+1)^{-a}\phi^{q} \\ +c\int_M \left|Rm\right|^{p-1}(f+1)^{-a}\phi^{q-2}. \end{aligned} \tag{9}$$

Notice moreover that we can interpolate, using Young's inequality:

$$\begin{aligned} \int_M \left|Rm\right|^{p-1}(f+1)^{-a}\phi^{q-2} &\leq \int_M \left|Rm\right|^{p-1}\phi^{q\frac{p-1}{p}}\phi^{\frac{q}{p}-2}(f+1)^{-a} \\ &\leq \varepsilon\int_M \left|Rm\right|^{p}(f+1)^{-a}\phi^{q} + c(\varepsilon)\int_M (f+1)^{-a}\phi^{q-2p}. \end{aligned} \tag{10}$$

We want to use this in (9), hence here we can take $\varepsilon = 1$. Moreover, to guarantee that ϕ^{q-2p} is well defined, we take $q \geq 2p+1$.

Plug (10) in (9) and then use (9) and (7) in (6); it results that

$$\begin{aligned} (a-c)\int_M \left|Rm\right|^{p}(f+1)^{-a}\phi^{q} \leq c\int_M \left|\nabla Rc\right|^2\left|Rm\right|^{p-1}(f+1)^{-a}\phi^{q} \\ +c\int_M \left|\nabla Rm\right|^2\left|Rm\right|^{p-3}(f+1)^{-a}\phi^{q} + C_1 + C_2 + C_0. \end{aligned} \tag{11}$$

We have denoted with

$$C_0 := c\int_M (f+1)^{-a},$$

and observe that taking $q = 2p+1$ and taking a such that $a > \frac{n}{2}+1$ then C_0 is a finite constant, independent of r. Indeed it is known that the volume growth of M is polynomial, see [5].

We finish the proof by estimating each of the two terms in the right hand side of (11). Start with the first, which by (2) we have:

$$\begin{aligned} 2\int_M \left|\nabla Rc\right|^2\left|Rm\right|^{p-1}(f+1)^{-a}\phi^{q} \leq \int_M \left(\Delta\left|Rc\right|^2\right)\left|Rm\right|^{p-1}(f+1)^{-a}\phi^{q} \\ -\int_M \left(\nabla f\cdot\nabla\left|Rc\right|^2\right)\left|Rm\right|^{p-1}(f+1)^{-a}\phi^{q} + c\int_M \left|Rm\right|^{p}(f+1)^{-a}\phi^{q} \end{aligned} \tag{12}$$

Let us observe that

$$\begin{aligned}
&\int_M \left(\Delta |Rc|^2\right) |Rm|^{p-1} (f+1)^{-a} \phi^q \\
&= -\int_M \nabla |Rc|^2 \cdot \nabla \left(|Rm|^{p-1} (f+1)^{-a} \phi^q\right) \\
&= -\int_M \nabla |Rc|^2 \cdot \nabla |Rm|^{p-1} (f+1)^{-a} \phi^q \\
&+a \int_M \left(\nabla |Rc|^2 \cdot \nabla f\right) |Rm|^{p-1} (f+1)^{-a-1} \phi^q \\
&-q \int_M \left(\nabla |Rc|^2 \cdot \nabla \phi\right) |Rm|^{p-1} (f+1)^{-a} \phi^{q-1} \\
&\leq c \int_M |\nabla Rc| \, |\nabla Rm| \, |Rm|^{p-2} (f+1)^{-a} \phi^q \\
&+ca \int_M |\nabla Rc| \, |Rm|^{p-1} (f+1)^{-a} \phi^q \\
&+\frac{c}{r} \int_M |\nabla Rc| \, |Rm|^{p-1} (f+1)^{-a} \phi^{q-1} \\
&\leq c \int_M |\nabla Rc| \, |\nabla Rm| \, |Rm|^{p-2} (f+1)^{-a} \phi^q \\
&+ca \int_M |\nabla Rc| \, |Rm|^{p-1} (f+1)^{-a} \phi^{q-1}.
\end{aligned}$$

Furthermore, let us use that

$$\begin{aligned}
& c \int_M |\nabla Rc| \, |\nabla Rm| \, |Rm|^{p-2} (f+1)^{-a} \phi^q \\
\leq \; & \frac{1}{4} \int_M |\nabla Rc|^2 |Rm|^{p-1} (f+1)^{-a} \phi^q \\
& +c \int_M |\nabla Rm|^2 |Rm|^{p-3} (f+1)^{-a} \phi^q,
\end{aligned}$$

and, similarly,

$$\begin{aligned}
& ca \int_M |\nabla Rc| \, |Rm|^{p-1} (f+1)^{-a} \phi^{q-1} \\
\leq \; & \frac{1}{4} \int_M |\nabla Rc|^2 |Rm|^{p-1} (f+1)^{-a} \phi^q \\
& +ca^2 \int_M |Rm|^{p-1} (f+1)^{-a} \phi^{q-2}.
\end{aligned}$$

We conclude from above that

$$\begin{aligned}
& \int_M \left(\Delta |Rc|^2\right) |Rm|^{p-1} (f+1)^{-a} \phi^q \\
\leq \; & \frac{1}{2} \int_M |\nabla Rc|^2 |Rm|^{p-1} (f+1)^{-a} \phi^q + c \int_M |\nabla Rm|^2 |Rm|^{p-3} (f+1)^{-a} \phi^q \\
& +ca^2 \int_M |Rm|^{p-1} (f+1)^{-a} \phi^{q-2}.
\end{aligned}$$

Moreover,

$$\begin{aligned} & -\int_M \left(\nabla f\cdot\nabla\left|Rc\right|^2\right)\left|Rm\right|^{p-1}(f+1)^{-a}\phi^q \\ & \leq c\int_M \left|\nabla Rc\right|\left|Rm\right|^{p-1}\left|\nabla f\right|(f+1)^{-a}\phi^q \\ \leq \frac{1}{2}\int_M & \left|\nabla Rc\right|^2\left|Rm\right|^{p-1}(f+1)^{-a}\phi^q + c\int_M \left|Rm\right|^{p-1}\left|\nabla f\right|^2(f+1)^{-a}\phi^q. \end{aligned}$$

We can use the same idea as in (10) to bound:

$$\begin{aligned} & \int_M \left|Rm\right|^{p-1}\left|\nabla f\right|^2(f+1)^{-a}\phi^q \\ \leq\quad & c\int_M \left|Rm\right|^p(f+1)^{-a}\phi^q + c\int_M (f+1)^{-a}\left|\nabla f\right|^{2p}\phi^q. \end{aligned}$$

Once again, we can take a large enough e.g. $a>\frac{n}{2}+p+1$ so that

$$\int_M (f+1)^{-a}\left|\nabla f\right|^{2p} < \int_M (f+1)^{-a+p} < \infty.$$

We also use (10) for $\varepsilon=\frac{1}{a^2}$ to get

$$\begin{aligned} & ca^2\int_M \left|Rm\right|^{p-1}(f+1)^{-a}\phi^{q-2} \\ \leq\quad & c\int_M \left|Rm\right|^p(f+1)^{-a}\phi^q + ca^{2p}\int_M (f+1)^{-a}\phi^{q-2p}. \end{aligned}$$

Therefore, plugging all in (12) gives

$$\begin{aligned} \int_M \left|\nabla Rc\right|^2\left|Rm\right|^{p-1}(f+1)^{-a}\phi^q &\leq c\int_M \left|\nabla Rm\right|^2\left|Rm\right|^{p-3}(f+1)^{-a}\phi^q \\ &+c\int_M \left|Rm\right|^p(f+1)^{-a}\phi^q + ca^{2p}\int_M (f+1)^{-a+p}, \end{aligned}$$

which by (11) yields

$$(a-c)\int_M \left|Rm\right|^p(f+1)^{-a}\phi^q \leq c\int_M \left|\nabla Rm\right|^2\left|Rm\right|^{p-3}(f+1)^{-a}\phi^q + C, \tag{13}$$

where

$$\begin{aligned} C &:= C_1+C_2+C_3, \\ C_3 &:= ca^{2p}\int_M (f+1)^{-a+p}. \end{aligned}$$

Finally, let us use (3) to check that

$$\begin{aligned} 2\int_M \left|\nabla Rm\right|^2\left|Rm\right|^{p-3}(f+1)^{-a}\phi^q &\leq \int_M \left(\Delta\left|Rm\right|^2\right)\left|Rm\right|^{p-3}(f+1)^{-a}\phi^q \\ -\int_M \left(\nabla f\cdot\nabla\left|Rm\right|^2\right)\left|Rm\right|^{p-3}(f+1)^{-a}\phi^q &+ c\int_M \left|Rm\right|^p(f+1)^{-a}\phi^q. \end{aligned} \tag{14}$$

The first term in the formula above is

$$\begin{aligned}
&\int_M \left(\Delta |Rm|^2\right) |Rm|^{p-3} (f+1)^{-a} \phi^q \\
&= - \int_M \left(\nabla |Rm|^2 \cdot \nabla |Rm|^{p-3}\right) (f+1)^{-a} \phi^q \\
&+a \int_M \left(\nabla |Rm|^2 \cdot \nabla f\right) |Rm|^{p-3} (f+1)^{-a-1} \phi^q \\
&-q \int_M \left(\nabla |Rm|^2 \cdot \nabla \phi\right) |Rm|^{p-3} (f+1)^{-a} \phi^{q-1}.
\end{aligned}$$

Choosing $p \geq 3$ guarantees that

$$- \int_M \left(\nabla |Rm|^2 \cdot \nabla |Rm|^{p-3}\right) (f+1)^{-a} \phi^q \leq 0.$$

On the other hand,

$$\begin{aligned}
&a \int_M \left(\nabla |Rm|^2 \cdot \nabla f\right) |Rm|^{p-3} (f+1)^{-a-1} \phi^q \\
&\leq ac \int_M |\nabla Rm| \, |Rm|^{p-2} (f+1)^{-a} \phi^q \\
&\leq \frac{1}{4} \int_M |\nabla Rm|^2 |Rm|^{p-3} (f+1)^{-a} \phi^q + ca^2 \int_M |Rm|^{p-1} (f+1)^{-a} \phi^q.
\end{aligned}$$

Similarly we find:

$$\begin{aligned}
&-q \int_M \left(\nabla |Rm|^2 \cdot \nabla \phi\right) |Rm|^{p-3} (f+1)^{-a} \phi^{q-1} \\
&\leq \frac{1}{4} \int_M |\nabla Rm|^2 |Rm|^{p-3} (f+1)^{-a} \phi^q \\
&+c \int_M |Rm|^{p-1} (f+1)^{-a} \phi^{q-2}.
\end{aligned}$$

Notice moreover that

$$\begin{aligned}
&- \int_M \left(\nabla f \cdot \nabla |Rm|^2\right) |Rm|^{p-3} (f+1)^{-a} \phi^q \\
&\leq 2 \int_M |\nabla Rm| \, |\nabla f| \, |Rm|^{p-2} (f+1)^{-a} \phi^q \\
&\leq \frac{1}{2} \int_M |\nabla Rm|^2 |Rm|^{p-3} (f+1)^{-a} \phi^q + c \int_M |Rm|^{p-1} |\nabla f|^2 (f+1)^{-a} \phi^q.
\end{aligned}$$

Using these estimates in (14) we get:

$$\begin{aligned}
\int_M |\nabla Rm|^2 |Rm|^{p-3} (f+1)^{-a} \phi^q &\leq c \int_M |Rm|^p (f+1)^{-a} \phi^q \\
&+ca^2 \int_M |Rm|^{p-1} (f+1)^{-a+1} \phi^{q-2}.
\end{aligned}$$

From interpolation as in formula (10), for $\varepsilon = \frac{1}{a^2}$, it follows:

$$\begin{aligned} & ca^2 \int_M |Rm|^{p-1} (f+1)^{-a+1} \phi^{q-2} \\ \leq \;& c \int_M |Rm|^p (f+1)^{-a} \phi^q + ca^{2p} \int_M (f+1)^{-a+p} \phi^{q-2p}. \end{aligned}$$

Therefore, we have proved:

$$\int_M |\nabla Rm|^2 |Rm|^{p-3} (f+1)^{-a} \phi^q \leq c \int_M |Rm|^p (f+1)^{-a} \phi^q + ca^{2p} \int_M (f+1)^{-a+p}.$$

This, by (13) implies that

$$(a-c) \int_M |Rm|^p (f+1)^{-a} \phi^q \leq C.$$

Recall that c is a constant depending only on n, p and K while a is a sufficiently large arbitrary number. This shows that there exists a depending on n, p and K such that

$$\int_M |Rm|^p (f+1)^{-a} \phi^q \leq C.$$

To conclude the proof of the Lemma notice that if

$$\rho(x) \leq \frac{1}{2} r,$$

then

$$\phi(x) \geq \frac{3}{16},$$

hence this shows that

$$\int_{D\left(\frac{1}{2}r\right)} |Rm|^p (f+1)^{-a} \leq C.$$

Since C is independent of r, making $r \to \infty$ we get

$$\int_M |Rm|^p (f+1)^{-a} \leq C.$$

Moreover, from $C = C_1 + C_2 + C_3$ and the expressions for these constants we see that in fact we have the estimate

$$\int_M |Rm|^p (f+1)^{-a} \leq c \int_{D(r_0)} |Rm|^p (f+1)^{-a} + c \int_M (f+1)^{-a+p},$$

where r_0 is a fixed number, depending on n, p and K.

This proves the Lemma in the case $p \geq 3$. We can interpolate as in (10) to get the claim for any $p \geq 2$. The claim that $\int_{B_x(1)} |Rm|^p \leq C (r(x)+1)^{2a}$ follows immediately from the asymptotics of f, see above. ■

We are now ready to finish the proof of the Theorem. We will be brief here, since this part of the proof is standard.

From (3) we infer that

$$\begin{aligned} \Delta |Rm|^2 &\geq 2 |\nabla Rm|^2 + \nabla f \cdot \nabla |Rm|^2 - c |Rm|^3 \\ &\geq -c \left(|Rm| + |\nabla f|^2 \right) |Rm|^2. \end{aligned}$$

Therefore, if we denote

$$u := c \left(|Rm| + |\nabla f|^2 \right),$$

then

$$\Delta |Rm|^2 \geq -u |Rm|^2 .$$

Since we assumed the Ricci curvature is bounded below, there exists a uniform bound on the Sobolev constant of the ball $B_x(1)$. More exactly, for any φ with support in $B_x(1)$ we have:

$$\left(\int_{B_x(1)} \varphi^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq C_S \int_{B_x(1)} \left(|\nabla \varphi|^2 + \varphi^2 \right).$$

Indeed, we can use the Sobolev inequality in [16] and notice that we can control the volume of $B_x(1)$ from below uniformly in x since M satisfies a log Sobolev inequality and the Ricci curvature is bounded, see [1]. The constant C_S depends only on n, the Ricci curvature lower bound on M and on $\inf_{x \in M} vol(B_x(1))$. As proved by Carillo and Ni in [1], $vol(B_x(1)) \geq k > 0$, where k depends on $K = \sup_M |Rc|$ and on Perelman's μ invariant. We remark that with our normalization of f, the μ invariant can be computed as $\mu = \log \left((4\pi)^{\frac{n}{2}} \int_M e^{-f} \right) < \infty$.

Then the standard Moser iteration, see [11], implies that

$$|Rm|^2(x) \leq A \int_{B_x(1)} |Rm|^2,$$

where

$$A := \bar{C} \left(\int_{B_x(1)} u^n + 1 \right),$$

for a constant $\bar{C}$ depending only on n and C_S. Since we showed in Lemma 2 that $\int_{B_x(1)} u^p$ grows at most polynomially in $r(x)$, the theorem follows from here.■

3. Gap and compactness theorems

In this section we prove the gap theorem and the compactness theorem of shrinking Ricci solitons, based on the estimates proved above. Since everywhere in this section the Ricci curvature is bounded, we can apply Theorem 1 to see that $|Rm|(x) \leq C(1 + r(x))^{2a}$. We follow a similar argument as in Theorem 1, this time using the weight e^{-f} and paying more attention to the dependence on the Ricci curvature bound. The computations will be simpler, and we will use many times the identities

$$\begin{aligned} \nabla_l \left(R_{ijkl} e^{-f} \right) &= 0, \\ \nabla_j \left(R_{ij} e^{-f} \right) &= 0. \end{aligned}$$

Clearly, we do not need to use a cut-off here, since all the curvature terms will be integrable with respect to e^{-f}, by Theorem 1. As in the previous section, we take

$$K = \sup_M |Rc|.$$

We also assume that $K > 0$, since otherwise there is nothing to prove. We use the notation $\Delta_f = \Delta - \nabla f \cdot \nabla$ and note that Δ_f is self adjoint with respect to the weighted volume $e^{-f} dv$.

Since

$$\Delta_f(f) = \Delta f - |\nabla f|^2 = \frac{n}{2} - f,$$

it follows that

$$\int_M |Rm|^p \left(f - \frac{n}{2}\right) e^{-f} = -\int_M |Rm|^p \Delta_f(f) e^{-f} = \int_M \nabla f \cdot \nabla |Rm|^p e^{-f} \tag{15}$$
$$= p\int_M \nabla_h R_{ijkl} R_{ijkl} f_h |Rm|^{p-2} e^{-f} = 2p \int_M \nabla_l R_{ijkh} R_{ijkl} f_h |Rm|^{p-2} e^{-f}$$
$$= -2p \int_M R_{ijkh} f_{hl} R_{ijkl} |Rm|^{p-2} e^{-f} - 2p\int_M R_{ijkh} f_h R_{ijkl} \nabla_l \left(|Rm|^{p-2}\right) e^{-f}.$$

As in Theorem 1, we take $p \geq 3$. Using the soliton equation and the Ricci curvature bound $|Rc| \leq K$, we find that

$$\begin{aligned} R_{ijkh} f_{hl} R_{ijkl} &= \frac{1}{2}|Rm|^2 - R_{ijkh} R_{ijkl} R_{hl} \\ &\geq \left(\frac{1}{2} - K\right)|Rm|^2. \end{aligned}$$

We can also estimate

$$-2p\int_M R_{ijkh} f_h R_{ijkl} \nabla_l \left(|Rm|^{p-2}\right) e^{-f} = -4p\int_M \left(\nabla_j R_{ik}\right) R_{ijkl} \nabla_l \left(|Rm|^{p-2}\right) e^{-f}$$
$$\leq 4p(p-2)\int_M |\nabla Rc| \, |\nabla Rm| \, |Rm|^{p-2} e^{-f}.$$

Then (15) shows that

$$\int_M \left(f - \frac{n}{2} + p(1-2K)\right)|Rm|^p e^{-f} \leq 4p^2 \int_M |\nabla Rc|\,|\nabla Rm|\,|Rm|^{p-2} e^{-f}. \tag{16}$$

We estimate the right hand side of (16) as follows:

$$2\int_M |\nabla Rc|\,|\nabla Rm|\,|Rm|^{p-2} e^{-f} \leq \frac{1}{pK}\int_M |\nabla Rc|^2 |Rm|^{p-1} e^{-f} \tag{17}$$
$$+pK\int_M |\nabla Rm|^2 |Rm|^{p-3} e^{-f}.$$

Furthermore, we have

$$\begin{aligned} \Delta_f |Rc|^2 &= 2|\nabla Rc|^2 + 2|Rc|^2 - 4R_{ikjh} R_{ij} R_{kh} \\ &\geq 2|\nabla Rc|^2 - 4K^2 |Rm|. \end{aligned}$$

Consequently,

$$\int_M |\nabla Rc|^2 |Rm|^{p-1} e^{-f} \leq \frac{1}{2}\int_M \left(\Delta_f |Rc|^2\right)|Rm|^{p-1} e^{-f} + 2K^2 \int_M |Rm|^p e^{-f}$$
$$= -\frac{1}{2}\int_M \nabla |Rc|^2 \cdot \nabla |Rm|^{p-1} e^{-f} + 2K^2\int_M |Rm|^p e^{-f}$$
$$\leq (p-1)K\int_M |\nabla Rc|\,|\nabla Rm|\,|Rm|^{p-2} e^{-f} + 2K^2 \int_M |Rm|^p e^{-f}.$$

Using this in (17) we get

$$\int_M |\nabla Rc|\,|\nabla Rm|\,|Rm|^{p-2} e^{-f} \leq \frac{2K}{p}\int_M |Rm|^p e^{-f} + pK\int_M |\nabla Rm|^2 |Rm|^{p-3} e^{-f},$$

which, after plugging into (16), yields

$$\int_M \left(f - \frac{n}{2} + p(1-10K)\right)|Rm|^p e^{-f} \leq 4p^3 K\int_M |\nabla Rm|^2 |Rm|^{p-3} e^{-f}. \tag{18}$$

Finally, we have:

$$2\int_M |\nabla Rm|^2 |Rm|^{p-3} e^{-f} \le \int_M \left(\Delta_f |Rm|^2\right) |Rm|^{p-3} e^{-f} + 20\int_M |Rm|^p e^{-f}.$$

Indeed, to see this one only has to check the details of the proof of (3). Integrating by parts and using that $p \ge 3$ we find

$$\int_M \left(\Delta_f |Rm|^2\right) |Rm|^{p-3} e^{-f} = -\int_M \left(\nabla |Rm|^2 \cdot \nabla |Rm|^{p-3}\right) e^{-f} \le 0.$$

Therefore, from (18) we conclude that for $p \ge 3$ we have:

$$\int_M \left(f - \frac{n}{2} + p\left(1 - 50p^2K\right)\right) |Rm|^p e^{-f} \le 0. \tag{19}$$

We are now ready to prove the Corollaries.

Proof of Corollary 1.

Let us take $p = n$. We check from (19) that if $K \le \frac{1}{100n^2}$ then

$$\int_M f |Rm|^n e^{-f} \le 0.$$

Recall that f is normalized such that $|\nabla f|^2 + R = f$, and in particular, since any gradient shrinker has $R \ge 0$ (see [7, 3]) it follows that $f \ge 0$. Thus the above inequality implies that M is flat i.e. (M, g, f) is the Gaussian soliton $\left(\mathbb{R}^n, dx^2, \frac{1}{4}|x|^2\right)$.■

Proof of Corollary 2.

We take $p = \frac{n}{2}$ in (19) to see that

$$\int_M \left(f - 7n^3K\right) |Rm|^{\frac{n}{2}} e^{-f} \le 0. \tag{20}$$

We fix x_0 a point where f achieves its minimum on M. Then we have (see [5, 10])

$$\frac{1}{4}\left[\left(d(x_0, x) - 5n\right)_+\right]^2 \le f(x) \le \frac{1}{4}\left(d(x_0, x) + \sqrt{2n}\right)^2, \tag{21}$$

where $a_+ := \max\{0, a\}$.

Let us set $r_0 := 6n + \sqrt{28n^3K}$. Using this in (20) we see that:

$$\begin{aligned} \int_{M\setminus B_{x_0}(r_0)} |Rm|^{\frac{n}{2}} e^{-f} &\le \int_{f \ge 7n^3K+1} |Rm|^{\frac{n}{2}} e^{-f} \\ \le \int_{f\ge 7n^3K+1} \left(f - 7n^3K\right) |Rm|^{\frac{n}{2}} e^{-f} &\le 7n^3K \int_{f \le 7n^3K+1} |Rm|^{\frac{n}{2}} e^{-f} \\ &\le 7n^3K \int_{B_{x_0}(r_0)} |Rm|^{\frac{n}{2}} \le 7n^3KL. \end{aligned}$$

Using again (21) shows that for any $r > 0$

$$\begin{aligned} \int_{B_{x_0}(r)} |Rm|^{\frac{n}{2}} &\le E(r), \text{ for} \\ E(r) &:= 7n^3KLe^{\frac{1}{4}\left(r+\sqrt{2n}\right)^2}. \end{aligned}$$

This proves Corollary 2. ■

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