Limit of quasilocal mass at spatial infinity

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Abstract

We study the limit of quasilocal mass defined in [4] and [5] for a family of spacelike 2-surfaces in spacetime. In particular, we show the limit coincides with the ADM mass at spatial infinity. The limit for coordinate spheres of a boosted slice of the Schwarzchild solution is computed explicitly and shown to give the expected energy-momentum four-vector.

1 Review of the definition of quasilocal energy

[1] [2] In [4] and [5], we define a notion of quasilocal mass for spacelike 2-surfaces in a spacetime. Given an isometric embedding of a 2-surface into $\mathbb{R}^{3,1}$ and a future timelike unit vector (observer) in $\mathbb{R}^{3,1}$, we associated a quasilocal energy with respect to a canonical gauge. Minimizing among the reference data gives the quasilocal mass and the quasilocal energy-momentum four-vector. We prove that the mass has the important positivity property and it vanishes for surfaces in $\mathbb{R}^{3,1}$. The expression for the mass is nevertheless rather nonlinear and complicated. In this article, we show that for a family of surfaces going out to spatial infinity, the expression indeed gets "linearized" and gives a well-defined energy-momentum four-vector.

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First of all, we recall the definition of quasilocal energy in [4]. Let Σ be a spacelike 2-surface in a time-orientable spacetime N. Consider a reference isometric embedding $\Sigma \hookrightarrow \mathbb{R}^{3,1}$. Fix a future timelike unit vector t_0^ν in $\mathbb{R}^{3,1}$. We decompose t_0^ν along $\Sigma \subset \mathbb{R}^{3,1}$ into $t_0^\nu = N_0 u_0^\nu + N_0^\nu$ in which N_0 is the lapse function, N_0^ν is the shift vector, and u_0^ν is the future timelike unit normal vector field along $\Sigma \subset \mathbb{R}^{3,1}$ determined by this decomposition. We also take the spacelike outward pointing unit normal v_0^ν that is orthogonal to u_0^ν along $\Sigma \subset \mathbb{R}^{3,1}$. (u_0^ν, v_0^ν) is the reference gauge for $\Sigma \subset \mathbb{R}^{3,1}$ with respect to t_0^ν. To compute the quasilocal energy, we also need the canonical gauge $(\bar{u}^\nu, \bar{v}^\nu)$ along $\Sigma \subset N$. $\bar{u}^\nu$ is characterized as the unique future timelike unit normal vector field along $\Sigma \subset N$ such that

$$h_\nu \bar{u}^\nu = (h_0)_\nu u_0^\nu, \tag{1.1}$$

where h^ν is the mean curvature vector of $\Sigma \subset N$ and h_0^ν is the mean curvature vector of $\Sigma \subset \mathbb{R}^{3,1}$. $\bar{v}^\nu$ is the spacelike unit normal vector that is orthogonal to $\bar{u}^\nu$ and satisfies $\bar{v}^\nu h_\nu < 0$. Take a spacelike hypersurface $\Omega_0 \subset \mathbb{R}^{3,1}$ spanned by $\Sigma \subset \mathbb{R}^{3,1}$ and v_0^ν, and a spacelike hypersurface $\bar{\Omega} \subset N$ spanned by $\Sigma \subset N$ and $\bar{v}^\nu$. Let k_0 be the mean curvature of Σ with respect to Ω_0 and $\bar{k}$ be the mean curvature of Σ with respect to $\bar{\Omega}$. Also denote by $(K_0)_{\mu\nu}$ and $\bar{K}_{\mu\nu}$ the extrinsic curvatures of Ω_0 and $\bar{\Omega}$, respectively. These data depend only on the gauges along Σ but not on the hypersurfaces. Quasilocal energy in the canonical gauge (see equation (6) in [4]) is defined to be

$$\frac{1}{8\pi}\int_\Sigma (k_0 - \bar{k})N_0 - (v_0^\mu (K_0)_{\mu\nu} - \bar{v}^\mu \bar{K}_{\mu\nu})N_0^\nu. \tag{1.2}$$

We shall rewrite the quasilocal energy in terms of the mean curvature gauge. In order to do so, we adopt a different set of notations from [5]. Set $T_0 = t_0^\nu$, $H_0 = h_0^\nu$, $H = h^\nu$, $\breve{e}_3 = v_0^\nu$, $\breve{e}_4 = u_0^\nu$, $\bar{e}_3 = \bar{v}^\nu$, $\bar{e}_4 = \bar{u}^\nu$. Denote by $X : \Sigma \to \mathbb{R}^{3,1}$ the position vector of the isometric embedding and by $\tau = \langle X, T_0\rangle$ the restriction of the time function associated with T_0. $T_0 = \sqrt{1+|\nabla\tau|^2}\breve{e}_4 - \nabla\tau$ and thus $N_0 = \sqrt{1+|\nabla\tau|^2}$ and $N_0^\nu = -\nabla\tau$. The quasilocal energy becomes

$$\frac{1}{8\pi}\int_\Sigma (-\langle H_0, \breve{e}_3\rangle + \langle H, \bar{e}_3\rangle)\sqrt{1+|\nabla\tau|^2} - (\langle \nabla^{\mathbb{R}^{3,1}}_{-\nabla\tau}\breve{e}_4, \breve{e}_3\rangle - \langle \nabla^N_{-\nabla\tau}\bar{e}_4, \bar{e}_3\rangle). \tag{1.3}$$

Suppose the mean curvature vector H_0 of Σ in $\mathbb{R}^{3,1}$ is spacelike. Let $e_3^{H_0} = \frac{-H_0}{|H_0|}$ be the unit vector in the direction of $-H_0$ and $e_4^{H_0}$ the future-directed time-like unit normal vector with $\langle e_3^{H_0}, e_4^{H_0}\rangle = 0$. The relation between the two gauges is

$$e_3^{H_0} = \cosh\theta_0 \breve{e}_3 + \sinh\theta_0 \breve{e}_4, \text{ and } e_4^{H_0} = \sinh\theta_0 \breve{e}_3 + \cosh\theta_0 \breve{e}_4$$

for some $\theta_0 \in \mathbb{R}$. Since $\Delta\tau = -\langle H_0, T_0\rangle$, we derive

$$\sinh\theta_0 = \frac{-\Delta\tau}{|H_0|\sqrt{1+|\nabla\tau|^2}}. \tag{1.4}$$

Therefore,

$$\langle \nabla^{\mathbb{R}^{3,1}}_{\nabla\tau} \breve{e}_4, \breve{e}_3\rangle = -\nabla\theta_0 \cdot \nabla\tau + \langle \nabla^{\mathbb{R}^{3,1}}_{\nabla\tau} e_4^{H_0}, e_3^{H_0}\rangle.$$

The canonical gauge condition (1.1)

$$\langle H_0, \breve{e}_4\rangle = \langle H, \bar{e}_4\rangle$$

implies $e^H = \frac{-H}{|H|}$ is given by

$$e_3^H = \cosh\theta \bar{e}_3 + \sinh\theta \bar{e}_4 \text{ with } \sinh\theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}.$$

Expression (1.3) can now be rewritten in terms of the mean curvature gauge.

To summarize, let $\Sigma \subset N$ be a spacelike 2-surface in a spacetime N and let $X : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ be a reference isometric embedding of Σ into the Minkowski space. For any given future timelike constant unit vector $T_0 \in \mathbb{R}^{3,1}$, the time function on Σ is denoted by $\tau = -\langle X, T_0\rangle$. Let H be the mean curvature vector of Σ in N, we assume H is spacelike. Let J be the future timelike unit normal vector field along Σ in N which is dual to H along the light cone in the normal bundle of Σ in N. Denote by H_0 and J_0 the corresponding data on the isometric embedding in $\mathbb{R}^{3,1}$. Again, H_0 is assume to be spacelike in $\mathbb{R}^{3,1}$. The quasilocal energy of Σ with respect to the pair (X, T_0) is given by

$$\begin{aligned} E(\Sigma, X, T_0) = \frac{1}{8\pi}\int_\Sigma & \sqrt{|H_0|^2(1+|\nabla\tau|^2)+(\Delta\tau)^2} - \sqrt{|H|^2(1+|\nabla\tau|^2)+(\Delta\tau)^2} \\ & - \Delta\tau\left[\sinh^{-1}\left(\frac{\Delta\tau}{\sqrt{1+|\nabla\tau|^2}|H_0|}\right) - \sinh^{-1}\left(\frac{\Delta\tau}{\sqrt{1+|\nabla\tau|^2}|H|}\right)\right] \\ & - \langle \nabla^{\mathbb{R}^{3,1}}_{\nabla\tau}\frac{J_0}{|H_0|}, \frac{H_0}{|H_0|}\rangle + \langle \nabla^N_{\nabla\tau}\frac{J}{|H|}, \frac{H}{|H|}\rangle dv_\Sigma, \end{aligned} \tag{1.5}$$

where $\Delta\tau$ is the Laplacian of τ on Σ (with respect to the induced metric), and ∇^N and $\nabla^{\mathbb{R}^{3,1}}$ are the covariant derivatives on N and $\mathbb{R}^{3,1}$, respectively, and $\nabla\tau$ is the gradient of τ on Σ (with respect to the induced metric again), considered as a tangent vector field on Σ. In the expressions for the last two integrands, we push forward $\nabla\tau$ by the embeddings and identify it as vector fields along Σ in $\mathbb{R}^{3,1}$ and N, respectively.

2 General formula for the limit of quasilocal energy

Fix $R > 0$ and suppose Σ_r, $R < r < \infty$, is a family of closed 2-surfaces in N, and X_r is a family of isometric embeddings of Σ_r into $\mathbb{R}^{3,1}$. In the following theorem, we derive an expression for the limit of $E(\Sigma_r, X_r, T_0)$.

Theorem 2.1 *Suppose the mean curvature vectors of Σ_r and of the image of X_r in $\mathbb{R}^{3,1}$ are both spacelike for $r > R_0$ and $\frac{|H|}{|H_0|} \to 1$ as $r \to \infty$. Then the limit of $E(\Sigma_r, X_r, T_0)$ as $r \to \infty$ (if exists) is the same as the limit of*

$$\frac{1}{8\pi}\int_{\Sigma_r} -\langle T_0, \frac{J_0}{|H_0|}\rangle(|H_0| - |H|) - \langle \nabla^{\mathbb{R}^{3,1}}_{\nabla\tau}\frac{J_0}{|H_0|}, \frac{H_0}{|H_0|}\rangle + \langle \nabla^N_{\nabla\tau}\frac{J}{|H|}, \frac{H}{|H|}\rangle dv_r.$$

Proof. We compute

$$\Delta\tau = -H_0 \cdot T_0 = |H_0|\langle e_3^{H_0}, T_0\rangle \tag{2.1}$$

and

$$|\nabla\tau|^2 = -1 + \langle e_4^{H_0}, T_0\rangle^2 - \langle e_3^{H_0}, T_0\rangle^2 \tag{2.2}$$

where $e_3^{H_0} = \frac{-H_0}{|H_0|}$ and $e_4^{H_0} = \frac{J_0}{|H_0|}$ is the future timelike unit normal dual to $e_3^{H_0}$ along the image of X in $\mathbb{R}^{3,1}$.

Rationalize the expression

$$\sqrt{|H_0|^2(1+|\nabla\tau|^2) + (\Delta\tau)^2} - \sqrt{|H|^2(1+|\nabla\tau|^2) + (\Delta\tau)^2}$$

as

$$(|H_0| - |H|)\frac{(|H_0| + |H|)(1+|\nabla\tau|^2)}{\sqrt{|H_0|^2(1+|\nabla\tau|^2) + (\Delta\tau)^2} + \sqrt{|H|^2(1+|\nabla\tau|^2) + (\Delta\tau)^2}}.$$

By assumption $\frac{|H|}{|H_0|} \to 1$ at infinity, the limit as $r \to \infty$ is thus the same as the limit of

$$\frac{1}{8\pi}\int_{\Sigma_r} \frac{\langle e_4^{H_0}, T_0\rangle^2 - \langle e_3^{H_0}, T_0\rangle^2}{-\langle e_4^{H_0}, T_0\rangle}(|H_0| - |H|)dv_{\Sigma_r}. \tag{2.3}$$

Next we study the term

$$-\Delta\tau\left[\sinh^{-1}(\frac{\Delta\tau}{\sqrt{1+|\nabla\tau|^2}|H_0|}) - \sinh^{-1}(\frac{\Delta\tau}{\sqrt{1+|\nabla\tau|^2}|H|})\right]$$

by rewriting it as

$$-\Delta\tau\left[\sinh^{-1}(\frac{\Delta\tau}{\sqrt{1+|\nabla\rho|^2}|H_0|}) - \sinh^{-1}(\frac{\Delta\tau}{\sqrt{1+|\nabla\tau|^2}|H_0|}\frac{|H_0|}{|H|})\right].$$

Note that

$$\frac{\sinh^{-1} A - \sinh^{-1}(A(1+x))}{x} \to \frac{-A}{\sqrt{1+A^2}}$$

as $x \to 0$. With $x = \frac{|H_0|}{|H|} - 1 \to 0$, the limit of the second term is thus the same as the limit of

$$\frac{1}{8\pi}\int_{\Sigma_r} \frac{\langle e_3^{H_0}, T_0\rangle^2}{-\langle e_4^{H_0}, T_0\rangle}(|H_0| - |H|)dv_r. \tag{2.4}$$

The theorem is proved by combining (2.3) and (2.4). $\square$

Suppose the image of the isometric embedding lies X_r in $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$, then $e_4^{H_0} = \frac{J_0}{|H_0|}$ is a constant vector and the $\langle \nabla^{\mathbb{R}^{3,1}}_{\nabla\tau} \frac{J_0}{|H_0|}, \frac{H_0}{|H_0|}\rangle$ term vanishes. In this case, $e_3^{H_0}$ coincide with the outward unit normal of the embedding in $\mathbb{R}^3$.

Corollary 2.1 *Suppose the reference isometric embedding is in $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ and $\frac{|H|}{|H_0|} \to 1$ as $r \to \infty$, then the limit of the quasilocal energy with respect to $T_0 = (\sqrt{1+|a|^2}, a^1, a^2, a^3)$ with $|a|^2 = \sum_{i=1}^3 (a^i)^2$ is*

$$(\sqrt{1+|a|^2})\frac{1}{8\pi}\int_{\Sigma_r} |H_0| - |H| dv_r + \frac{1}{8\pi}\int_{\Sigma_r} \langle \nabla^N_{\nabla\tau}\frac{J}{|H|}, \frac{H}{|H|}\rangle dv_r. \tag{2.5}$$

Suppose the isometric embedding for Σ_r is given by $X_r = (X^1, X^2, X^3) : \Sigma \to \mathbb{R}^3$ and consider $X^i, i = 1, 2, 3$ as functions on Σ_r. Thus $\nabla\tau = -\sum_{i=1}^{3} a^i \nabla X^i$ and we obtain a limiting quasilocal energy-momentum four-vector (e, p_1, p_2, p_3) as the limit of

$$\begin{aligned} e &= \lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} |H_0| - |H| dv_r \\ p_i &= \lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} \langle \nabla^N_{-\nabla X^i} \frac{J}{|H|}, \frac{H}{|H|} \rangle dv_r, i = 1, 2, 3. \end{aligned} \tag{2.6}$$

3 Relating to ADM energy-momentum

Let (M, g_{ij}, p_{ij}) be an asymptotically flat hypersurface in a spacetime N. Thus there exists a compact set $K \subset M$ such that $M \backslash K$ is diffeomorphic to a union of complements of balls in $\mathbb{R}^3$ (ends) such that $g_{ij} = \delta_{ij} + a_{ij}$ with $a_{ij} = O(\frac{1}{r})$, $\partial_k(a_{ij}) = O(\frac{1}{r^2})$, $\partial_l\partial_k(a_{ij}) = O(\frac{1}{r^3})$, and $p_{ij} = O(\frac{1}{r^2})$, $\partial_k(p_{ij}) = O(\frac{1}{r^3})$ on each end of $M \backslash K$.

The ADM energy momentum (Arnowitt-Deser-Misner) of an end of M is the four vector (E, P_1, P_2, P_3) where

$$E = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj}) \nu^i dv_r$$

is the total energy and

$$P_k = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} 2(p_{ik} - \delta_{ik} p_{jj}) \nu^i dv_r$$

is the total momentum. Here S_r is a coordinate sphere of radius r on the end and ν is the outward unit normal of S_r.

The positive mass theorem (Schoen-Yau [3], Witten [6]) asserts that under the dominant energy condition, the four-vector (E, P_1, P_2, P_3) is future timelike, i.e.

$$E \geq 0 \text{ and } -E^2 + P_1^2 + P_2^2 + P_3^2 \leq 0.$$

In the following, we prove that for coordinate spheres of radius r, the limit of the quasilocal energy momentum (2.6) is the same as the ADM energy-momentum.

Theorem 3.1 *Suppose S_r is the coordinate sphere of radius r in an end of an asymptotically flat three-manifold (M, g_{ij}, p_{ij}) and (E, P_1, P_2, P_3) is the ADM energy-momentum four vector of this end, then*

$$\lim_{r\to\infty} E(S_r, X_r, T_0) = \sqrt{1+|a|^2}E + \sum_{i=1}^{3} a^i P_i$$

where X_r is the (unique) isometric embedding of S_r into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ and $T_0 = (\sqrt{1+|a|^2}, a^1, a^2, a^3)$ is an arbitrary constant timelike unit vector.

Proof. Denote by e_0 the future timelike unit normal of the hypersurface M and ν the unit outward normal of the coordinate sphere S_r. Let (y^1, y^2, y^3) be the asymptotically flat coordinates on the end. S_r is given by $(y^1)^2 + (y^2)^2 + (y^3)^2 = r^2$ and we denote the embedding of S_r into M by Y. Since $p_{ij} = O(\frac{1}{r^2})$, we have $\langle H, e_0\rangle = O(\frac{1}{r^2})$. It is known that $\langle H, \nu\rangle = \frac{2}{r} + O(\frac{1}{r^2})$ (see for example [1]). Since $H = \langle H, \nu\rangle\nu - \langle H, e_0\rangle e_0$, we estimate

$$|H| - |\langle H, \nu\rangle| = O(\frac{1}{r_3}).$$

Therefore,

$$\lim_{r\to\infty} \int_{S_r} |H_0| - |H| dv_r = \lim_{r\to\infty} \int_{S_r} |H_0| - |\langle H, \nu\rangle| dv_r,$$

i.e, the Brown-York energy and the Liu-Yau energy have the same limit at spatial infinity. It is known that (see for example [1] and the reference therein) the Brown-York energy approaches the ADM energy E at spatial infinity.

Now it suffices to prove

$$\sum_{i=1}^{3} a^i P_i = \sum_{i=1}^{3} a^i p_i = \frac{1}{8\pi} \int_{\Sigma_r} \langle \nabla^N_{\nabla\tau} \frac{J}{|H|}, \frac{H}{|H|}\rangle dv_r.$$

By definition, the ADM momentum is

$$\sum_{i=1}^{3} a^i P_i = \frac{1}{8\pi} \int_{S_r} p(a^i \frac{\partial}{\partial y^i}, \nu) - (trp)\langle a^i \frac{\partial}{\partial y^i}, \nu\rangle dv_r.$$

We decompose $a^i\frac{\partial}{\partial y^i} = (a^i\frac{\partial}{\partial y^i})^\top + \langle a^i\frac{\partial}{\partial y^i}, \nu\rangle\nu$ and the integrand becomes

$$p((a^i\frac{\partial}{\partial y^i})^\top, \nu) + \langle a^i\frac{\partial}{\partial y^i}, \nu\rangle(p(\nu,\nu) - (trp)).$$

By the definition of the mean curvature vector H, we obtain $p(\nu,\nu) - (trp) = \langle H, e_0\rangle$. Therefore the ADM momentum term is

$$\sum_{i=1}^{3} a^i P_i = \frac{1}{8\pi}\int_{S_r} \langle \nabla^N_{(a^i\frac{\partial}{\partial y^i})^\top} e_0, \nu\rangle + \langle H, e_0\rangle\langle a^i\frac{\partial}{\partial y^i}, \nu\rangle dv_r. \tag{3.1}$$

Now we turn to the limit of the quasilocal energy momentum. We can express the normal vector fields H and J in terms of ν and e_0 as

$$H = \langle H, \nu\rangle\nu - \langle H, e_0\rangle e_0 \text{ and } J = -\langle H, \nu\rangle e_0 + \langle H, e_0\rangle\nu.$$

We compute

$$\langle \nabla^N_{\nabla\tau}\frac{J}{|H|}, \frac{H}{|H|}\rangle = -\nabla\tau\cdot\nabla\sinh^{-1}(\frac{\langle H, e_0\rangle}{|H|}) - \langle\nabla^N_{\nabla\tau}e_0, \nu\rangle.$$

Integrating by parts gives

$$\frac{1}{8\pi}\int_{S_r}\langle \nabla^N_{\nabla\tau}\frac{J}{|H|}, \frac{H}{|H|}\rangle dv_r = \frac{1}{8\pi}\int_{S_r} -\langle\nabla^N_{\nabla\tau}e_0, \nu\rangle + \Delta\tau\sinh^{-1}(\frac{\langle H, e_0\rangle}{|H|})dv_r.$$

Plug in (2.1), the second integrand on the right hand side becomes

$$\langle e_3^{H_0}, T_0\rangle|H_0|\sinh^{-1}(\frac{\langle H, e_0\rangle}{|H|}).$$

Recall the asymptotics

$$\langle H, e_0\rangle = O(\frac{1}{r^2}), |H| = \frac{2}{r} + O(\frac{1}{r^2}), |H_0| = \frac{2}{r} + O(\frac{1}{r^2})$$

and $\sinh^{-1} x \sim x$ if $x << 1$. We see the limit of $\frac{1}{8\pi}\int_{S_r}\langle\nabla^N_{\nabla\tau}\frac{J}{|H|}, \frac{H}{|H|}\rangle dv_r$ is the same as

$$\lim_{r\to\infty}\frac{1}{8\pi}\int_{S_r} -\langle\nabla^N_{\nabla\tau}e_0, \nu\rangle + \langle H, e_0\rangle\langle e_3^{H_0}, T_0\rangle dv_r. \tag{3.2}$$

Now we can compare (3.1) and (3.2). Write out the tangential part of $a^i\frac{\partial}{\partial y^i}$,

$$(a^i\frac{\partial}{\partial y^i})^\top = \langle a^i\frac{\partial}{\partial y^i}, \frac{\partial Y}{\partial u^a}\rangle \sigma^{ab}\frac{\partial Y}{\partial u^b} = a^i g_{ij}\frac{\partial Y^j}{\partial u^a}\sigma^{ab}\frac{\partial Y}{\partial u^b}.$$

On the other hand, as $\tau = -a^i X^i$, and the push-forward of $\nabla\tau$ becomes

$$\nabla\tau = -a^i\frac{\partial X^i}{\partial u^a}\sigma^{ab}\frac{\partial Y}{\partial u^b}.$$

The isometric embeddings satisfy (see for example [1])

$$|X^i - Y^i| = O(1) \text{ and } |e_3^{H_0} - \nu| = O(\frac{1}{r}).$$

From these, we deduce that (3.2) is the same as the limit of the right hand side of (3.1) and the theorem is proved.

□

4 Explicit computation in a boosted slice of Schwarzchild's solution

In this section, we compute the limit of quasilocal energy-momentum for coordinate spheres of a boosted slice of Schwarzchild's solution.

4.1 Asymptotics of the geometry of coordinate spheres

Let (y^0, y^1, y^2, y^3) be the isotropic coordinates of Schwarzchild's solution in which the spacetime metric is of the form:

$$G_{\alpha\beta}dy^\alpha dy^\beta = -\frac{1}{F^2}(dy^0)^2 + \frac{1}{G^2}\sum_{i=1}^3 (dy^i)^2$$

with

$$F^2 = \frac{(1+\frac{M}{2\rho})^2}{(1-\frac{M}{2\rho})^2},\ \ G^2 = \frac{1}{(1+\frac{M}{2\rho})^4}$$

and $\rho^2 = \sum_{i=1}^3 (y^i)^2$. Given $\gamma > 0$ and β which satisfy $\gamma^2 - \beta^2\gamma^2 = 1$, consider the surface Σ_{r_0} defined by

$$\gamma y^0 - \beta\gamma y^3 = 0$$

and

$$(y^1)^2 + (y^2)^2 + (\gamma y^3 - \beta\gamma y^0)^2 = r_0^2$$

with $r_0 \to \infty$. With the coordinate change $(y^0)' = \gamma y^0 - \beta\gamma y^3$ and $(y^3)' = \gamma y^3 - \beta\gamma y^0$, these surfaces are coordinate spheres of radius r_0 in the asymptotically flat slice $\gamma y^0 - \beta\gamma y^3 = 0$

We parametrize the 2-surfaces Σ_{r_0} by

$$\begin{aligned}
y^0 &= \beta\gamma r_0 \cos\theta \\
y^1 &= r_0 \sin\theta \sin\phi \\
y^2 &= r_0 \sin\theta \cos\phi \\
y^3 &= \gamma r_0 \cos\theta.
\end{aligned}$$

Denote the embedding of Σ_{r_0} into Schwarzchild's solution by $Y = (y^0, y^1, y^2, y^3)$. In terms of local coordinates $u^1 = \theta$ and $u^2 = \phi$ on the surface, the induced metric on Σ_{r_0} is

$$\sigma_{ab} = r_0^2[1 + \frac{2M}{\rho}(1 + 2\beta^2\gamma^2 \sin^2\theta)]d\theta^2 + r_0^2(1 + \frac{2M}{\rho})\sin^2\theta d\phi^2 + O(r_0), \quad (4.1)$$

and

$$\sqrt{\det \sigma_{ab}} = r_0^2|\sin\theta|[1 + \frac{2M}{\rho}(1 + \beta^2\gamma^2 \sin^2\theta)] + O(r_0). \quad (4.2)$$

The mean curvature vector $H = H^\gamma \frac{\partial}{\partial y^\gamma}$ of Σ_{r_0} is by definition:

$$H^\gamma = \sigma^{ab}(\frac{\partial^2 y^\gamma}{\partial u^a \partial u^b} + \Gamma^\gamma_{\alpha\beta}\frac{\partial y^\alpha}{\partial u^a}\frac{\partial y^\beta}{\partial u^b})(\delta^\gamma_\beta - \Pi^\gamma_\beta)$$

where $\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols of the metric $G_{\alpha\beta}$ and $\Pi^\gamma_\beta = G_{\beta\alpha}\sigma^{ab}\frac{\partial y^\alpha}{\partial u^a}\frac{\partial y^\gamma}{\partial u^b}$ is the projection operator on the tangent space of Σ_{r_0}. The asymptotic expansion of $\Gamma^\gamma_{\alpha\beta}$ can be computed from the asymptotic expansion of $G_{\alpha\beta}$.

Denote by $\tilde{y}^\alpha = \frac{y^\alpha}{r_0}$ and $\tilde{\rho} = \frac{\rho}{r_0}$, which are both scaling invariant now. We shall use the following frames along Σ_{r_0} to express the mean curvature vector:

$$\begin{aligned}
N &= \tilde{y}^\alpha \frac{\partial}{\partial y^\alpha}, \\
B &= \gamma(\frac{\partial}{\partial y^0} + \beta\frac{\partial}{\partial y^3}), \text{ and} \\
T &= \frac{\partial \tilde{y}^\alpha}{\partial\theta}\frac{\partial}{\partial y^\alpha}.
\end{aligned}$$

We notice that T is a tangent vector field to Σ_{r_0} while N and B are only asymptotically normal in the sense that $\langle N,T\rangle = O(\frac{1}{r_0})$ and $\langle B,T\rangle = O(\frac{1}{r_0})$. We also have $\langle N,N\rangle = 1+O(\frac{1}{r_0})$, $\langle B,B\rangle = -1+O(\frac{1}{r_0})$, and $\langle T,T\rangle = 1+O(\frac{1}{r_0})$.

A straightforward calculation gives

Lemma 4.1

$$H = \frac{-2}{r_0}N + \frac{1}{r_0^2}(\mathfrak{n}N + \mathfrak{t}T + \mathfrak{b}B) + O(\frac{1}{r_0^3})$$

with

$$\begin{aligned}
\mathfrak{n} &= \frac{M}{\tilde{\rho}^3}(6 + 6\beta^2\gamma^2 + 2\beta^4\gamma^4\sin^2\theta\cos^2\theta),\\
\mathfrak{t} &= \frac{M}{\tilde{\rho}^3}(-8\tilde{\rho}^2)(\beta^2\gamma^2\sin\theta\cos\theta), \ \textit{and}\\
\mathfrak{b} &= \frac{M}{\tilde{\rho}^3}(2\beta\gamma^2\cos\theta)(\beta^2\gamma^2\sin^2\theta - 1).
\end{aligned}$$

From here, we compute the norm of $|H|$,

Proposition 4.1

$$|H| = \frac{2}{r_0} + \frac{1}{r_0^2}[\frac{2M}{\tilde{\rho}}(1 + 2\beta^2\gamma^2\cos^2\theta) - \mathfrak{n}] + O(\frac{1}{r_0^3}).$$

Let J be the future-directed timelike normal vector that is dual to H along the light cone in the normal bundle.

Lemma 4.2 *J is given by*

$$\begin{aligned}
&\frac{2}{r_0}B - \frac{1}{r_0^2}[\frac{-2M(1 + 2\beta^2\gamma^2\cos^2\theta + \gamma^2 + \beta^2\gamma^2)}{\tilde{\rho}} + \mathfrak{n}]B\\
&+ \frac{1}{r_0^2}[\frac{8M}{\tilde{\rho}}(\beta\gamma^2\sin\theta)T - (\mathfrak{b} + \frac{8M\beta\gamma^2\cos\theta}{\tilde{\rho}})N].
\end{aligned}$$

Proof. The coefficients of J are determined by the following equations:

$$-\langle J,J\rangle = \langle H,H\rangle,$$

$$\langle J, H\rangle = 0$$

and

$$\langle J, T\rangle = 0.$$

□

With this explicit formula, we compute the coefficients of the connection form of the normal bundle in the mean curvature vector gauge:

Proposition 4.2

$$\langle \nabla_{\frac{\partial Y}{\partial \theta}} J, H\rangle = \frac{1}{r_0^3}(2\mathfrak{b}' + \frac{8M}{\tilde{\rho}}\beta\gamma^2 \sin\theta) + O(\frac{1}{r_0^4})$$

and

$$\langle \nabla_{\frac{\partial Y}{\partial \phi}} J, H\rangle = O(\frac{1}{r_0^4}).$$

It turns out the second term does not contribute to the limit of the quasilocal energy.

4.2 Total mean curvature of isometric embedding

We consider the isometric embedding of a general axially symmetric metric into $\mathbb{R}^3$. The metric is of the form

$$r_0^2 P^2(r_0,\theta) d\theta^2 + r_0^2 Q^2(r_0,\theta)\sin^2\theta d\phi^2$$

with

$$P(r_0,\theta) = 1 + O(\frac{1}{r_0}),\ \text{and}\ Q(r_0,\theta) = 1 + O(\frac{1}{r_0}).$$

Suppose the isometric embedding is given by

$$X = (u(r_0,\theta)\sin\phi, u(r_0,\theta)\cos\phi, v(r_0,\theta)).$$

Thus

$$\frac{\partial X}{\partial \theta} = (\frac{\partial u}{\partial \theta}\sin\phi, \frac{\partial u}{\partial \theta}\cos\phi, \frac{\partial v}{\partial \theta})$$

and

$$\frac{\partial X}{\partial \phi} = (u\cos\phi, -u\sin\phi, 0).$$

It is not hard to see

Lemma 4.3 *u and v are given by*

$$(\frac{\partial u}{\partial \theta})^2 + (\frac{\partial v}{\partial \theta})^2 = r_0^2 P^2$$

and

$$u^2 = r_0^2 Q^2 \sin^2 \theta.$$

Proposition 4.3 *The mean curvature of the isometric embedding of the metric $r_0^2 P^2(r_0,\theta)d\theta^2 + r_0^2 Q^2(r_0,\theta)\sin^2\theta d\phi^2$ into $\mathbb{R}^3$ is given by*

$$H_0 = -(\frac{1}{r_0^3 P^3})(\frac{\partial^2 u}{\partial \theta^2}\frac{\partial v}{\partial \theta} + \frac{\partial^2 v}{\partial \theta^2}\frac{\partial u}{\partial \theta}) + (\frac{1}{r_0^2 PQ\sin\theta})\frac{\partial v}{\partial \theta}.$$

where

$$u(r_0,\theta) = r_0 Q(r_0,\theta)\sin\theta,$$

and

$$(\frac{\partial v}{\partial \theta})^2 = r_0^2 P^2 - (\frac{\partial u}{\partial \theta})^2 = r_0^2[P^2 - (\frac{\partial Q}{\partial \theta}\sin\theta + Q\cos\theta)^2].$$

Now suppose

$$P = 1 + \frac{p}{r_0} + O(\frac{1}{r_0^2}), p = p(\theta)$$

and

$$Q = 1 + \frac{q}{r_0} + O(\frac{1}{r_0^2}), q = q(\theta).$$

The asymptotic expansion of the mean curvature is found to be

$$H_0 = \frac{2}{r_0} - \frac{1}{r^2}(2p + \frac{\cos\theta}{\sin\theta}(2q' - p') + q''). \tag{4.3}$$

Comparing with (4.1), we deduce that in our case

$$p = \frac{M}{\tilde{\rho}}(1 + 2\beta^2\gamma^2\sin^2\theta) \text{ and } q = \frac{M}{\tilde{\rho}}.$$

u and v can be solved explicitly:

$$u = r_0\sin\theta + \frac{M}{\tilde{\rho}}\sin\theta \text{ and } v = r_0\cos\theta + \frac{M}{\tilde{\rho}}\cos\theta + 2M\beta\gamma\sinh^{-1}(\beta\gamma\cos\theta). \tag{4.4}$$

Plug in the expression of p and q into (4.3) and integrate by parts, we obtain

Proposition 4.4

$$\int_{\Sigma_{r_0}} H_0 dv_{r_0} = 8\pi r_0 + 2\pi M \int_0^{2\pi} \frac{1+\beta^2\gamma^2\sin^2\theta}{\tilde{\rho}} |\sin\theta| d\theta + O(\frac{1}{r_0}).$$

This calculation is compatible with Lemma 2.4 in [1].

4.3 Evaluating the quasilocal energy

We are ready to compute the limit of the Liu-Yau mass:

Proposition 4.5

$$\int_{\Sigma_{r_0}} (H_0 - |H|) dv_{r_0} = 8\pi\gamma M + O(\frac{1}{r_0}).$$

Proof. Combine Proposition 4.1 and Proposition 4.4, we obtain

$$\int_{\Sigma_{r_0}} (H_0 - |H|) dv_{r_0} = \pi M \int_0^{2\pi} \frac{2+4\beta^2\gamma^2 - 6\beta^2\gamma^2\cos^2\theta - 4\beta^4\gamma^4\cos^4\theta}{\tilde{\rho}} |\sin\theta| d\theta + O(\frac{1}{r_0}).$$

The integral can be evaluate by the substitution $\beta\gamma\cos\theta = \sinh y$. $\square$

Now we turn to the momentum part. Suppose $T_0 = (\sqrt{1+|a|^2}, a^1, a^2, a^3)$, $|a|^2 = \sum_{i=1}^3 (a^i)^2$ is a future timelike unit vector and the isometric embedding into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ is given by $X = (0, u\sin\phi, u\cos\phi, v)$. We know from (4.4) that $u = r\sin\theta + O(1)$ and $v = r\cos\theta + O(1)$. The gradient of τ is given by $\nabla\tau = \frac{\partial\tau}{\partial u^a}\sigma^{ab}\frac{\partial Y}{\partial u^b}$. We compute

$$\begin{aligned}\int_{\Sigma_{r_0}} \langle \nabla_{\nabla\tau} \frac{J}{|H|}, \frac{H}{|H|} \rangle dv_{r_0} = & -a^1 \int_{\Sigma_{r_0}} \frac{1}{|H|^2} (u\sin\theta)' \sigma^{\theta\theta} \langle \nabla_{\frac{\partial Y}{\partial\theta}} J, H \rangle dv_{r_0} \\ & - a^2 \int_{\Sigma_{r_0}} \frac{1}{|H|^2} (u\cos\theta)' \sigma^{\theta\theta} \langle \nabla_{\frac{\partial Y}{\partial\theta}} J, H \rangle dv_{r_0} \\ & - a^3 \int_{\Sigma_{r_0}} \frac{1}{|H|^2} v' \sigma^{\theta\theta} \langle \nabla_{\frac{\partial Y}{\partial\theta}} J, H \rangle dv_{r_0} + O(\frac{1}{r_0}).\end{aligned} \tag{4.5}$$

These integrals can be evaluated and we obtain

Proposition 4.6

$$\int_{\Sigma_{r_0}} \langle \nabla^N_{\nabla\tau} \frac{J}{|H|}, \frac{H}{|H|} \rangle dv_{r_0} = a^3 8\pi\beta\gamma M + O(\frac{1}{r_0}).$$

Proof. By Proposition 4.2, $\langle \nabla_{\frac{\partial Y}{\partial \theta}} J, H\rangle$ is of the order $\frac{1}{r_0^3}$ while u and v are both of order r_0, we have

$$\begin{aligned}
&\int_{\Sigma_{r_0}} \frac{1}{|H|^2}(u\sin\theta)'\sigma^{\theta\theta}\langle \nabla_{\frac{\partial Y}{\partial \theta}} J, H\rangle dv_{r_0}\\
&= \int_0^{\pi}\int_0^{2\pi} \frac{r_0^2}{4}(r_0\sin^2\theta)'\frac{1}{r_0^2}\langle \nabla_{\frac{\partial Y}{\partial \theta}} J, H\rangle r_0^2|\sin\theta|d\theta d\phi\\
&= \frac{\pi}{4}\int_0^{2\pi}(\sin^2\theta)'[r_0^3\langle \nabla_{\frac{\partial Y}{\partial \theta}} J, H\rangle]|\sin\theta|d\theta.
\end{aligned}$$

Therefore the first integral on the right hand side of (4.5) is

$$-a^1\frac{\pi}{4}\int_0^{2\pi}(\sin^2\theta)'(2\mathfrak{b}' + \frac{8M}{\tilde\rho}\beta\gamma^2\sin\theta)|\sin\theta|d\theta$$

where

$$\mathfrak{b} = \frac{M}{\tilde\rho^3}(2\beta\gamma^2\cos\theta)(\beta^2\gamma^2\sin^2\theta - 1),$$

and the second one is

$$-a^2\frac{\pi}{4}\int_0^{2\pi}(\sin\theta\cos\theta)'(2\mathfrak{b}' + \frac{8M}{\tilde\rho}\beta\gamma^2\sin\theta)|\sin\theta|d\theta.$$

Both integrate to zero as they are of the form $\int_0^{2\pi}(\cos\theta)F(\cos^2\theta)|\sin\theta|d\theta$ or $\int_0^{2\pi}(\sin\theta)F(\cos^2\theta)|\sin\theta|d\theta$ for some smooth function F of $\cos^2\theta$. The last integral becomes

$$a^3\frac{\pi}{4}\int_0^{2\pi}\sin\theta(2\mathfrak{b}' + \frac{8M}{\tilde\rho}\beta\gamma^2\sin\theta)|\sin\theta|d\theta$$

which can be simplified as

$$a^3 4\pi M\beta\gamma^2\int_0^{\pi}\frac{\sin\theta}{\tilde\rho^3}d\theta.$$

Using the same substitution $\beta\gamma\cos\theta = \sinh y$, the integral is

$$a^3 8\pi\beta\gamma M.$$

□

Therefore the limit of the quasilocal energy (1.5) is

$$(\sqrt{1+|a|^2})\gamma M + a^3\beta\gamma M.$$

Recall that $\gamma^2 - \beta^2\gamma^2 = 1$. Minimizing this expression among all $T_0 = (\sqrt{1+|a|^2}, a^1, a^2, a^3)$, we see the minimum is achieved at $(\gamma, 0, 0, -\beta\gamma)$ and the minimum value is M. The limit of the quasilocal energy-momentum is thus $M(\gamma, 0, 0, -\beta\gamma)$.

References

[1] X.-Q. Fan, Y. Shi, and L.-F. Tam, *Large-sphere and small-sphere limits of the Brown-York mass.* Comm. Anal. Geom. **17** (2009), no. 1, 37-72.

[2] C.-C. M. Liu and S.-T. Yau, *Positivity of quasilocal mass,* Phys. Rev. Lett. **90** (2003), no. 23, 231102.

[3] R. Schoen and S.-T. Yau, *Positivity of the total mass of a general space-time,* Phys. Rev. Lett. **43** (1979), no. 20, 1457–1459.

[4] M.-T. Wang and S.-T. Yau, *Quasilocal mass in general relativity.* Phys. Rev. Lett. **102** (2009), no. 2, no. 021101.

[5] M.-T. Wang and S.-T. Yau, *Isometric embeddings into the Minkowski space and new quasi-local mass.* Comm. Math. Phys. **288** (2009), 919-942.

[6] E. Witten, *A new proof of the positive energy theorem.* Comm. Math. Phys. **80** (1981), no. 3, 381–402.