Lectures on mean curvature flows in higher codimensions

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Abstract

Mean curvature flows of hypersurfaces have been extensively studied and there are various different approaches and many beautiful results. However, relatively little is known about mean curvature flows of submanifolds of higher codimensions. This notes starts with some basic materials on submanifold geometry, and then introduces mean curvature flows in general dimensions and co-dimensions. The related techniques in the so called "blow-up" analysis are also discussed. At the end, we present some global existence and convergence results for mean curvature flows of two-dimensional surfaces in four-dimensional ambient spaces.

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1. Basic materials

1.1 Connections, curvature, and the Laplacian

A Riemannian manifold is a differentiable manifold equipped with a smooth inner product on the tangent bundle. Suppose M is a Riemannian manifold of dimensional N with a Riemannian metric $\langle \cdot, \cdot \rangle$.

There is a unique Levi-Civita connection ∇ on the tangent bundle of M that is compatible with the differentiable structure, i.e.

$$\nabla_X Y - \nabla_Y X = [X, Y] = XY - YX$$

and with the metric, i.e.

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle.$$

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We can extend this connection to the sections of any tensor bundles by requiring the Leibnitz rule and the compatibility with contractions of tensors.

Suppose X, Y, Z, W, U, and V are tangent vector fields on M. The Riemannian curvature tensor is defined to be

$$R(X,Y)Z \equiv -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

$R(X,Y,Z,W) \equiv \langle R(X,Y)Z, W \rangle$ has the following symmetries:

$$R(X,Y,Z,W) = -R(Y,X,Z,W) \text{ and } R(X,Y,Z,W) = R(Z,W,X,Y).$$

The first Bianchi identity:

$$R(X,Y,Z,W) + R(X,Z,W,Y) + R(X,W,Y,Z) = 0.$$

The second Bianchi identity:

$$(\nabla_X R)(U,V,Y,Z) + (\nabla_Y R)(U,V,Z,X) + (\nabla_Z R)(U,V,X,Y) = 0.$$

Given a smooth function f on M, the gradient of f, ∇f, is a tangent vector field that satisfies $\langle \nabla f, X \rangle = df(X)$ for any X. The Hessian of f, denoted by $\nabla^2 f$, is a symmetric (0,2)-tensor defined by

$$\nabla^2_{X,Y} f \equiv XY f - (\nabla_X Y)f.$$

For a (k,l) tensor T, we define a $(k, l+2)$ tensor $\nabla^2 T$ by

$$\nabla^2_{X,Y} T \equiv \nabla_X \nabla_Y T - \nabla_{\nabla_X Y} T.$$

This is no longer symmetric in X and Y and the commutator is the Riemannian curvature tensor.

New tensors can be constructed by contracting old ones. We can pick any orthonormal basis $\{e_A\}_{A=1}^N$ of the tangent bundle for contractions. The contracted tensor will be independent of the choice of such e_A. Notice that the summation convention, i.e. repeated indexes are summed, is followed throughout the article. From the Riemannian curvature tensor, we define the Ricci curvature $Ric(X,Y) \equiv R(X, e_A, Y, e_A)$ and the scalar curvature $R \equiv Ric(e_A, e_A)$.

The Laplacian of f is defined to be the trace of $\nabla^2 f$, i.e.

$$\Delta f \equiv tr(\nabla^2 f) = \nabla^2_{e_A, e_A} f = \langle \nabla_{e_A} \nabla f, e_A \rangle = (\nabla_{e_A} df)(e_A).$$

The rough Laplacian of T is then

$$\Delta T \equiv tr \nabla^2 T = \nabla^2_{e_A, e_A} T.$$

We can always use normal coordinates to simplify the expression of a tensor at a given point p. Given any orthonormal basis of $T_p M$, there exists a coordinate $\{y^A\}$ near p and such that $e_A = \frac{\partial}{\partial y^A}$ and $\nabla_{e_A} e_B = 0$ at the point p.

1.2 Immersed submanifolds and the second fundamental forms

Given an immersion $F : \Sigma \to M$ of an n-dimensional smooth manifold Σ into M. Suppose $\{y^A\}_{A=1}^N$ is a local coordinate system on M with metric tensor $\Lambda_{AB} = \langle \frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \rangle$. Because F is an immersion, the tangent space of Σ at p, $T_p\Sigma$ can be identified with $F_* T_p\Sigma$, the vector subspace of $T_{F(p)}M$ spanned by $\{F_*(\frac{\partial}{\partial x^i})\}_{i=1}^n$. We denote $F^A = y^A \circ F$, then $F_*(\frac{\partial}{\partial x^i}) = \frac{\partial F^A}{\partial x^i} \frac{\partial}{\partial y^A}$. The metric Λ_{AB} induces a Riemannian metric g_{ij} on Σ defined by

$$g_{ij} = \langle F_*(\frac{\partial}{\partial x^i}), F_*(\frac{\partial}{\partial x^j}) \rangle = \frac{\partial F^A}{\partial x^i} \frac{\partial F^B}{\partial x^j} \Lambda_{AB}.$$

The orthogonal complement of $F_* T_p\Sigma$ in $T_{F(p)}M$, denoted by $N_p\Sigma$ is called the normal space of Σ in M at the point p. Given a vector $V \in T_{F(p)}M$, we can define the projections of V onto $F_* T_p\Sigma$ and $N_p\Sigma$, called the tangential component $V^\top$ and the normal component $V^\perp$, respectively. They are

$$V^\top = \langle V, F_*(\frac{\partial}{\partial x^i}) \rangle g^{ij} F_*(\frac{\partial}{\partial x^j}) \in F_* T_p\Sigma,$$

and

$$V^\perp = V - V^\top \in N_p\Sigma.$$

The Levi-Civita connection ∇^M on M induces a connection on Σ by

$$\nabla^\Sigma_X Y \equiv (\nabla^M_X Y)^\top,$$

for X and Y tangent to Σ. It is not hard to check that ∇^Σ is the Levi-Civita connection of the induced metric g_{ij} on Σ.

Given a normal vector field V along Σ. The induced connection on the normal bundle, the normal connection, is defined to be

$$\nabla^\perp_X V \equiv (\nabla^M_X V)^\perp.$$

The curvature of the normal bundle is

$$R^\perp(X,Y)V \equiv -\nabla^\perp_X \nabla^\perp_Y V + \nabla^\perp_Y \nabla^\perp_X V + \nabla^\perp_{[X,Y]} V.$$

The second fundamental form is defined to be

$$\mathrm{II}(X,Y) \equiv (\nabla^M_X Y)^\perp,$$

as a section of the tensor bundle $T^*\Sigma \otimes T^*\Sigma \otimes N\Sigma$. Choose orthonormal bases $\{e_i\}_{i=1}^n$ for $T_p\Sigma$ and $\{e_\alpha\}_{\alpha=n+1}^N$ for $N_p\Sigma$, the components of the second fundamental form are

$$h_{\alpha ij} = \langle \nabla^M_{e_i} e_j, e_\alpha \rangle = \langle \mathrm{II}(e_i, e_j), e_\alpha \rangle.$$

The mean curvature vector is the trace of the second fundamental form,

$$H \equiv \mathrm{II}(e_i, e_i) = g^{ij}(\nabla^M_{F_*(\frac{\partial}{\partial x^i})} F_*(\frac{\partial}{\partial x^j}))^{\perp}.$$

Suppose the ambient space M is $\mathbb{R}^N$ and let

$$F = (F^1(x^1, \cdots, x^n), \cdots, F^N(x^1, \cdots, x^n))$$

be the position vector of the immersion. In this case, we have

$$\Delta^{\Sigma} F = \Delta^{\Sigma}(F^1, \cdots, F^N) = (\Delta^{\Sigma} F^1, \cdots, \Delta^{\Sigma} F^N) = H,$$

where Δ^{Σ} is the Laplace operator with respect to the induced metric g_{ij} on Σ.

Denote the Riemannian curvature tensor of g_{ij} by R^{Σ}. We recall the Gauss equation

$$\langle R^{\Sigma}(X,Y)Z, W \rangle = \langle R^M(X,Y)Z, W \rangle + \langle \mathrm{II}(X,Z), \mathrm{II}(Y,W) \rangle - \langle \mathrm{II}(X,W), \mathrm{II}(Y,Z) \rangle, \tag{1.1}$$

and the Codazzi equation

$$(\nabla_X \mathrm{II})(Y, Z) - (\nabla_Y \mathrm{II})(X, Z) = -(R^M(X,Y)Z)^{\perp}. \tag{1.2}$$

Here ∇ on II is the connection on the tensor bundle $T^*\Sigma \otimes T^*\Sigma \otimes N\Sigma$, thus

$$(\nabla_X \mathrm{II})(Y, Z) = \nabla^{\perp}_X \mathrm{II}(Y, Z) - \mathrm{II}(\nabla^{\Sigma}_X Y, Z) - \mathrm{II}(Y, \nabla^{\Sigma}_X Z). \tag{1.3}$$

1.3 First variation formula

Recall the volume of Σ with respect to the metric g is

$$Vol(\Sigma) = \int_{\Sigma} \sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^n$$

Take a normal vector field V along Σ and consider a family of immersion $F : \Sigma \times [0, \epsilon) \to M$ such that $\frac{\partial F}{\partial s}|_{s=0} = V$. Denote $\Sigma_s = F(\Sigma, s)$.

To consider the change of volume

$$\frac{d}{ds}|_{s=0} Vol(\Sigma_s) = \frac{d}{ds}|_{s=0} \int_{\Sigma} \sqrt{\det g_{ij}(s)} dx^1 \wedge \cdots \wedge dx^n,$$

we compute

$$\frac{d}{ds} \sqrt{\det g_{ij}} = \frac{1}{2}(\det g_{ij})^{-\frac{1}{2}}(\frac{d}{ds} g_{ij})(\det g_{ij}),$$

and

$$\frac{d}{ds} g_{ij} = -2 \langle V, \nabla^M_{F_*(\frac{\partial}{\partial x^i})}, F_*(\frac{\partial}{\partial x^j}) \rangle.$$

Therefore,

$$\frac{d}{ds}|_{s=0} Vol(\Sigma_s) = -\int_{\Sigma} \langle V, H \rangle \sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^n$$

and the mean curvature vector is formally the gradient of the negative of the volume functional on the space of submanifolds.

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2. Mean curvature flow

2.1   The equation

Let $F_0 : \Sigma \to M$ be a smooth immersion. Consider $F : \Sigma \times [0, T) \to M$ that satisfies

$$\begin{cases} \frac{\partial F}{\partial t} = H \\ F(\cdot, 0) = F_0(\cdot) \end{cases}$$

F is called the mean curvature flow of $F_0(\Sigma)$ (in the normal direction). By the first variation formula, we have

$$\frac{d}{dt}\sqrt{\det g_{ij}} = -|H|^2 \sqrt{\det g_{ij}}.$$

Denote $F(\cdot, t)$ by $F_t : \Sigma \to M$. Because $\det g_{ij} > 0$ at $t = 0$, F_t remains an immersion as long as $|H|^2$ is bounded.

It is not hard to established the short time existence of the flow in the case of Σ is compact. Notice that the mean curvature flow is not strictly parabolic, suppose ϕ_t, $t \in [0, T)$ is a family of diffeomorphism of Σ. Consider $\bar{F}_t = F_t \circ \phi_t : \Sigma \to M$, the image of $\bar{F}_t$ is the same as that of F_t. But $\frac{\partial \bar{F}_t}{\partial t}$ has both the tangential and normal components. Therefore the general form of mean curvature flow is

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = H.$$

On the other hand, given a general mean curvature flow $\left(\frac{\partial \bar{F}}{\partial t}\right)^{\perp} = H$, we can always find a reparametrization ϕ_t so that $F_t = \bar{F}_t \circ \phi_t$ satisfies $\frac{\partial F}{\partial t} = H$.

Suppose $M = \mathbb{R}^{n+1}$ and Σ is the the graph of a function $f(x^1, \cdots, x^n)$ on $\{x^{n+1} = 0\}$. We can represent the hypersurface in the parametric form $F(x^1, \cdots, x^n, t) = (x^1, \cdots, x^n, f(x^1, \cdots, x^n, t))$. Denote by ν the upward unit normal $\frac{(-\nabla f, 1)}{\sqrt{1 + |\nabla f|^2}}$ on the graph, by the mean curvature flow equation $\left(\frac{\partial F}{\partial t}\right)^{\perp} = H$, we derive

$$\langle \frac{\partial F}{\partial t}, \nu \rangle \nu = H.$$

The mean curvature flow of the graph of f becomes a quasi-linear parabolic equation for f:

$$\frac{1}{\sqrt{1 + |\nabla f|^2}} \frac{\partial f}{\partial t} = div\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right).$$

This equation was studied extensively in [2] and [3].

In general, suppose $F : \Sigma \times [0, T) \to \mathbb{R}^N$ and $F(x^1, \cdots, x^n, t)$ is a mean curvature flow. As $H = (g^{ij} \frac{\partial^2 F}{\partial x^i \partial x^j})^{\perp}$, the mean curvature flow equation is

$$\frac{\partial F}{\partial t} = \left(g^{ij} \frac{\partial^2 F}{\partial x^i \partial x^j}\right)^{\perp}.$$

Given any vector $V \in \mathbb{R}^N$, we compute

$$V^\perp = V - V^\top = V - \langle V, \frac{\partial F}{\partial x^k} \rangle g^{kl} \frac{\partial F}{\partial x^l} = V^A \frac{\partial}{\partial y^A} - V^B \frac{\partial F^B}{\partial x^k} g^{kl} \frac{\partial F^C}{\partial x^l} \frac{\partial}{\partial y^C}$$

Therefore, the mean curvature flow can be expressed as

$$\frac{\partial F^B}{\partial t} = g^{ij} \frac{\partial^2 F^A}{\partial x^i \partial x^j} (\delta_{AB} - \frac{\partial F^B}{\partial x^k} g^{kl} \frac{\partial F^A}{\partial x^l}).$$

2.2 Finite time singularity

Suppose F_0 is the standard imbedding of the unit sphere S^n in $\mathbb{R}^N$ and $F(x,t) = \phi(t)F_0(x)$ is a mean curvature flow in the normal direction. As

$$\frac{\partial F}{\partial t} = \phi'(t)F_0(x)$$

and the mean curvature vector of $F(x,t)$ is

$$H(x,t) = \frac{n}{\phi(t)}(-F_0(x)),$$

the mean curvature flow is reduced to the ordinary differential equation

$$\phi'(t) = -\frac{n}{\phi(t)}.$$

We solve $\phi(t) = \sqrt{r_0^2 - 2nt}$ where r_0 is the radius of $F(x,0)$. Therefore,

$$F(x,t) = \sqrt{r_0^2 - 2nt} F_0(x).$$

$F(x,t)$ shrinks to a point at $t = \frac{r_0^2}{2n}$. This is the limiting behavior of the mean curvature flow of any compact convex hypersurface in $\mathbb{R}^N$, see [4] and [5].

Indeed, it is very easy to show the mean curvature flow of any compact submanifolds in $\mathbb{R}^N$ must develop finite time singularity. Let $F : \Sigma \times [0,T) \to \mathbb{R}^N$ be such a flow for a compact n-dimensional Σ. By the formula of the mean curvature vector, we have $\frac{d}{dt}F = \Delta_t F$, where Δ_t is the Laplace operator with respect to the induced metric on $F_t(\Sigma)$. We compute

$$\frac{d}{dt}|F|^2 = \Delta_t|F|^2 - 2n. \tag{2.1}$$

Therefore $|F|^2 + 2nt$ satisfies the heat equation:

$$\frac{d}{dt}(|F|^2 + 2nt) = \Delta_t(|F|^2 + 2nt). \tag{2.2}$$

Suppose $F_0(\Sigma)$ is contained in a sphere $S^{N-1}(r_0)$ of radius r_0 centered at the origin and thus $|F|^2 \le r_0^2$ at $t = 0$. It follows from the maximum principle that $|F|^2 + 2nt \le r_0^2$ must be preserved as long as the flow exists smoothly. In particular, the flow must develop singularity before the time $t = \frac{r_0^2}{2n}$.

3. Blow-up analysis

3.1 Backward heat kernel and monotonicity formula

Fix $y_0 \in \mathbb{R}^N$ and $t_0 \in \mathbb{R}$ and consider the backward heat kernel at (y_0, t_0),

$$\rho_{y_0,t_0}(y,t) = \frac{1}{(4\pi(t_0-t))^{n/2}} \exp\left(-\frac{|y-y_0|^2}{4(t_0-t)}\right) \tag{3.1}$$

defined on $\mathbb{R}^N \times (-\infty, t_0)$.

Suppose $F : \Sigma \times [0, t_0) \to \mathbb{R}^N$ is a mean curvature flow, then Huisken's monotonicity formula [5] says

$$\frac{d}{dt} \int_{\Sigma_t} \rho_{y_0,t_0} d\mu_t = -\int_{\Sigma_t} \rho_{y_0,t_0} \left|\frac{F^\perp}{2(t_0-t)} + H\right|^2 d\mu_t \leq 0. \tag{3.2}$$

Hence $\lim_{t \to t_0} \int_{\Sigma_t} \rho_{y_0,t_0} d\mu_t$ exists.

For a general ambient manifold M of dimensional N, we fix an isometric embedding $i : M \to \mathbb{R}^{N+m}$. Given an immersion $F : \Sigma \to M$, we consider $\bar{F} = i \circ F$ as an immersed submanifold in $\mathbb{R}^{N+m}$. Denote by H the mean curvature vector of Σ with respect to M, i.e.

$$H = (\nabla^M_{e_i} e_i)^\perp \in TM$$

for an orthonormal basis $\{e_i\}$ of $T\Sigma$. Denote by $\bar{H}$ the mean curvature vector of Σ with respect to $\mathbb{R}^{N+m}$, i.e.

$$\bar{H} = (\nabla^{\mathbb{R}^{N+m}}_{e_i} e_i)^\perp \in T\mathbb{R}^{N+m}.$$

Therefore,

$$\bar{H} - H = \mathrm{II}^M(e_i, e_i) = -E, \tag{3.3}$$

where II^M is the second fundamental form of M in $\mathbb{R}^{N+m}$. $|E|$ is bounded if M has bounded geometry. The equation of the mean curvature flow becomes

$$\frac{\partial \bar{F}}{\partial t} = \bar{H} + E. \tag{3.4}$$

The following monotonicity formula for a general ambient manifold was derived by White in [18].

Proposition 3.1 *For the mean curvature flow (3.4) on $[0, t_0)$, if we isometrically embed M into $\mathbb{R}^{N+m}$ and take the backward heat kernel ρ_{y_0,t_0} for $y_0 \in \mathbb{R}^{N+m}$, then*

$$\frac{d}{dt} \int_{\Sigma_t} \rho_{y_0,t_0} d\mu_t \leq C - \int_{\Sigma_t} \left|H + \frac{1}{2(t_0-t)}\bar{F}^\perp + \frac{E}{2}\right|^2 \rho_{y_0,t_0} d\mu_t \tag{3.5}$$

for some constant C. Here $\bar{F}^\perp$ is the component of the position vector $\bar{F} \in \mathbb{R}^{N+m}$ in the normal space of Σ_t in $\mathbb{R}^{N+m}$.

Definition 3.1 *The density at (y_0, t_0) is defined to be $\Theta(y_0, t_0) = \lim_{t \to t_0} \int_{\Sigma_t} \rho_{y_0, t_0} d\mu_t$.*

We will use the following pointwise formula which can be found in [14]

$$\frac{d}{dt}\rho_{y_0,t_0} = -\Delta^{\Sigma_t}\rho_{y_0,t_0} - \rho_{y_0,t_0}\Big(\frac{|\bar{F}^\perp|^2}{4(t_0-t)^2} + \frac{\bar{F}^\perp \cdot \bar{H}}{t_0-t} + \frac{\bar{F}^\perp \cdot E}{2(t_0-t)}\Big) \qquad (3.6)$$

The minus sign in front of the Laplacian in equation (3.6) indicates that ρ_{y_0,t_0} satisfies the backward heat equation.

3.2 Synopsis of singularities

**Definition 3.2** *Given a mean curvature flow $F : \Sigma \times [0, t_0) \to M$, the image $\mathcal{F}$ of the map*

$$\begin{array}{rccc} F \times 1: & \Sigma \times [0,t_0) & \to & M \times \mathbb{R} \\ & (x,t) & \mapsto & (F(x,t),t) \end{array}$$

in $M \times \mathbb{R}$ is called the space-time track of the flow. If M is isometrically embedded into $\mathbb{R}^{N+m}$, we can take the image in $\mathbb{R}^{N+m} \times \mathbb{R}$ as the space-time track.

Definition 3.3 *The parabolic dilation of scale $\lambda > 0$ at (y_0, t_0) is given by*

$$\begin{array}{rccc} D_\lambda: & \mathbb{R}^{N+m} \times \mathbb{R} & \to & \mathbb{R}^{N+m} \times \mathbb{R} \\ & (y,t) & \mapsto & (\lambda(y-y_0), \lambda^2(t-t_0)) \\ & t \in [0,t_0) & \mapsto & s = \lambda^2(t-t_0) \in [-\lambda^2 t_0, 0) \end{array}$$

Set $\Sigma_s^\lambda = D_\lambda(\Sigma_t - y_0) = \lambda(\Sigma_{t_0 + \frac{s}{\lambda^2}} - y_0)$ for $t = t_0 + \frac{s}{\lambda^2}$. If $M = \mathbb{R}^N$, the mean curvature flow equation is preserved by D_λ. That is, if $\mathcal{F}$ is the spacetime track of a mean curvature flow, so is $D_\lambda \mathcal{F}$.
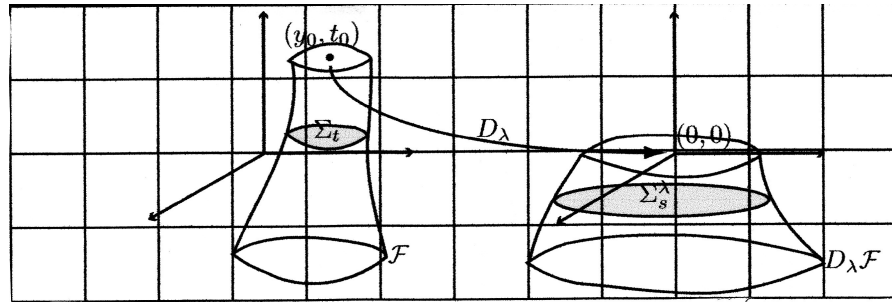
Proposition 3.2 *$\int_{\Sigma_t} \rho_{y_0,t_0} d\mu_t$ is invariant under the parabolic dilation. That is to say,*

$$\int_{\Sigma_t} \rho_{y_0,t_0} d\mu_t = \int_{\Sigma_s^\lambda} \rho_{0,0} d\mu_s^\lambda$$

where μ_s^λ is the volume form of the Σ_s^λ.

The mean curvature flow extends smoothly to $t_0 + \delta$ if the second fundamental form is bounded $\sup_{\Sigma_t} |II| < C$ as $t \to t_0$. Therefore if a singularity is forming at t_0, then $\sup_{\Sigma_t} |II| \to \infty$ as $t \to t_0$.

Definition 3.4 (type I singularity) *A singularity $t = t_0$ is called a type I singularity if there is a $C > 0$ such that*

$$\sup_{\Sigma_t} |II|^2 \leq \frac{C}{t_0 - t}$$

for all $t < t_0$.

The shrinking sphere $F(x,t) = \sqrt{r_0^2 - 2t} \, F_0(x)$ is a type I singularity at $t_0 = \frac{r_0^2}{2n}$. For any $t < t_0$, the radius of the sphere is $r = \sqrt{r_0^2 - 2nt}$ and thus the principal curvature k satisfies

$$k^2 = \frac{1}{r^2} = \frac{1}{r_0^2 - 2nt} = \frac{1}{2n(t_0 - t)}.$$

Consider the parabolic dilation of the spacetime track of a type I singularity, we calculate

$$|II|^2(\Sigma_s^\lambda) = \frac{1}{\lambda^2} |II|^2(\Sigma_t) = \frac{-1}{s}(t_0 - t)|II|^2(\Sigma_t) \leq \frac{-1}{s}C \ , \text{ for } s \in [-\lambda^2 t_0, 0).$$

Hence for any fixed s, the second fundamental form $|II|^2(\Sigma_s^\lambda)$ is bounded. By Arzelà-Ascoli theorem, on any compact subset of space time, there exists smoothly convergent subsequence of $\{D_\lambda \mathcal{F}\}$ as $\lambda \to \infty$. We thus have:

Proposition 3.3 *If there is a type I singularity at $t = t_0$, there exists a subsequence $\{\lambda_i\}$ such that $D_{\lambda_i} \mathcal{F} \to \mathcal{F}_\infty$, the spacetime track of a smooth flow that exists on $(-\infty, 0)$.*

After the parabolic dilation, with $t = t_0 + \frac{s}{\lambda^2}$, the monotonicity formula (3.5) becomes

$$\frac{d}{ds} \int_{\Sigma_s^\lambda} \rho_{0,0} d\mu_s^\lambda \leq \frac{C}{\lambda^2} - \int_{\Sigma_s^\lambda} \left| H_s^\lambda + \frac{-1}{2s}(\bar{F}_s^\lambda)^\perp + \frac{E}{2\lambda} \right|^2 \rho_{0,0} d\mu_s^\lambda.$$

Consider the $s_0 < 0$ slice and integrate both sides from $s_0 - \tau$ to s_0 for $\tau > 0$ and λ large:

$$\int_{s_0 - \tau}^{s_0} \int_{\Sigma_s^\lambda} \left| H_s^\lambda - \frac{1}{2s}(\bar{F}_s^\lambda)^\perp + \frac{E}{2\lambda} \right|^2 \rho_{0,0} \, d\mu_s^\lambda ds$$

$$\leq \frac{C\tau}{\lambda^2} - \int_{\Sigma_{s_0}^\lambda} \rho_{0,0} \, d\mu_{s_0}^\lambda + \int_{\Sigma_{s_0-\tau}^\lambda} \rho_{0,0} \, d\mu_{s_0-\tau}^\lambda$$

Take $\lambda \to \infty$. Because $\int_{\Sigma_{s_0}^\lambda} \rho_{0,0} \, d\mu_{s_0}^\lambda = \int_{\Sigma_t} \rho_{y_0,t_0} \, d\mu_t$ where $t = t_0 + \frac{s_0}{\lambda^2}$,

$$\lim_{\lambda \to \infty} \int_{\Sigma_{s_0}^\lambda} \rho_{0,0} \, d\mu_{s_0}^\lambda = \lim_{t \to t_0} \int_{\Sigma_t} \rho_{y_0,t_0} \, d\mu_t = \lim_{\lambda \to \infty} \int_{\Sigma_{s_0-\tau}^\lambda} \rho_{0,0} \, d\mu_{s_0-\tau}^\lambda.$$

Therefore,

$$\lim_{\lambda \to \infty} \int_{s_0-\tau}^{s_0} \int_{\Sigma_s^\lambda} \left| H_s^\lambda - \frac{1}{2s}(\bar{F}_s^\lambda)^\perp + \frac{E}{2\lambda} \right|^2 \rho_{0,0} \, d\mu_s^\lambda ds = 0$$

Therefore if t_0 is a type I singularity, a subsequence $D_{\lambda_i}\mathcal{F} \to \mathcal{F}_\infty$ smoothly and $H = \frac{1}{2s}F^\perp$ on $\mathcal{F}_\infty$ for $s \in (s_0 - \tau, s_0)$. Take $s_0 \to 0$ and $\tau \to \infty$, we obtain Huisken's theorem [5]:

Theorem 3.1 *If the singularity is of type I, then there exists a subsequence λ_i such that $D_{\lambda_i}\mathcal{F} \to \mathcal{F}_\infty$ smoothly and $H = \frac{1}{2s}F^\perp, -\infty < s < 0$ on $\mathcal{F}_\infty$.*

Let F be a solution of the mean curvature flow that satisfies $H(x, s) = \frac{1}{2s}F^\perp(x, s)$ on $-\infty < s < 0$ then F must be of the form $F(x, s) = \sqrt{-s}F(x)$. Even if the singularity is not of type I (so the scaled $|II|$ is not necessarily bounded), we still get weak solution with $H = \frac{1}{2s}F^\perp$ (see [7]). A theorem of Huisken [5] shows any compact smooth convex hypersurface in $\mathbb{R}^N$ satisfying $H = -F^\perp$ is the unit sphere.

4. Applications to deformations of symplectomorphisms of Riemann surfaces

4.1 Introduction

We apply the mean curvature flow to obtain a result of the deformation retract of symplectomorphism groups of Riemann surfaces.

Theorem 4.1 *Let Σ^1 and Σ^2 be two compact closed Riemann surfaces with metrics of the same constant curvature c. Let ω_1 and ω_2 be the volume forms of Σ^1 and Σ^2, respectively. Consider a map $f : \Sigma^1 \to \Sigma^2$ that satisfies $f^*\omega_2 = \omega_1$, i.e. f is an area-preserving map or a symplectomorphism. Denote by Σ_t the mean curvature flow of the graph of f in $M = \Sigma^1 \times \Sigma^2$. We have*

1. *Σ_t exists smoothly for all $t > 0$ and converges smoothly to Σ_∞ as $t \to \infty$.*

2. *Each Σ_t is the graph of a symplectomorphism $f_t : \Sigma^1 \to \Sigma^2$ and f_t converges smoothly to a symplectomorphism $f_\infty : \Sigma^1 \to \Sigma^2$ as $t \to \infty$.*

Moreover,

$$f_\infty \text{ is } \begin{cases} \text{an isometry} & \text{if } c > 0 \\ \text{a linear map} & \text{if } c = 0 \\ \text{a harmonic diffeormophism} & \text{if } c < 0 \end{cases}$$

The existence of such minimal symplectic maps in the $c < 0$ case was proved by Schoen in [10] (see also [8]). This theorem was proved in [15] and [16]. A different proof in the case $c \leq 0$ was proved by Smoczyk [13] assuming an extra angle condition.

Both ω_1 and ω_2 can be extended to parallel forms on M. Denote $\omega_1 - \omega_2$ by ω' and $\omega_1 + \omega_2$ by ω'', which are parallel forms on M as well. That Σ_t is the graph of a symplectomorphism f_t can be characterized by $*_{\Sigma_t}\omega' = 0$, $*_{\Sigma_t}\omega'' > 0$ where $*_{\Sigma_t}$ is the Hodge star operator on Σ_t. To prove that f_t being a symplectomorphism is preserved, it suffices to show both conditions are preserved along the mean curvature flow. In the following, we compute the evolution equations of these quantities.

4.2   Derivation of evolution equations

In this section, we assume M is a four-dimensional Riemannian manifold and Σ is a closed Riemann surface embedded in M.

Lemma 4.1 *Let ω be a parallel 2-form on M and $\eta = *_{\Sigma}\omega$, then*

$$\Delta^{\Sigma}\eta = -\eta|II|^2 + \omega(\nabla_{e_1}^{\perp} H, e_2) + \omega(e_1, \nabla_{e_2}^{\perp} H)$$
$$- \omega((R^M(e_1, e_k)e_k)^{\perp}, e_2) - \omega(e_1, (R^M(e_2, e_k)e_k)^{\perp}) + 2\omega(II(e_k, e_1), II(e_k, e_2))$$

where $\{e_i\} = \{e_1, e_2\} \in T_p\Sigma$ and $\{e_\alpha\} = \{e_3, e_4\} \in N_p\Sigma$ are oriented orthonormal bases and $II(e_i, e_j) = (\nabla_{e_i}^M e_j)^{\perp}$ is the second fundamental form of Σ.

Proof.   The right hand side is independent of the choice of frames and thus a tensor (it does depend on the orientation though). At any $p \in \Sigma$, we can choose an orthonormal frame $\{e_1, e_2\}$ such that $\nabla_{e_i}^{\Sigma} e_j = 0$ at p. It is not hard to check that $\Delta^{\Sigma}\eta = *\Delta^{\Sigma}\omega$ where the Δ^{Σ} on the right hand side is the rough-Laplacian on two forms. Therefore $\Delta^{\Sigma}\eta$ is equal to

$$(\nabla_{e_k}^{\Sigma}\nabla_{e_k}^{\Sigma}\omega)(e_1, e_2) = e_k\big((\nabla_{e_k}^{\Sigma}\omega)(e_1, e_2)\big) - (\nabla_{e_k}^{\Sigma}\omega)(\nabla_{e_k}^{\Sigma}e_1, e_2) - (\nabla_{e_k}^{\Sigma}\omega)(e_1, \nabla_{e_k}^{\Sigma}e_2)$$
$$= e_k\big(e_k(\omega(e_1, e_2)) - \omega(\nabla_{e_k}^{\Sigma}e_1, e_2) - \omega(e_1, \nabla_{e_k}^{\Sigma}e_2)\big).$$

Since ω is parallel on M, we derive

$$\Delta^{\Sigma}\eta = e_k\big(\omega(\nabla_{e_k}^M e_1, e_2) + \omega(e_1, \nabla_{e_k}^M e_2) - \omega(\nabla_{e_k}^{\Sigma}e_1, e_2) - \omega(e_1, \nabla_{e_k}^{\Sigma}e_2)\big)$$
$$= e_k\big(\omega(II(e_k, e_1), e_2) + \omega(e_1, II(e_k, e_2))\big).$$

Applying $\nabla^M\omega = 0$ again, the last expression can be decomposed into

$$\omega(\nabla_{e_k}^M II(e_k, e_1), e_2) + \omega(II(e_k, e_1), \nabla_{e_k}^M e_2) + \omega(\nabla_{e_k}^M e_1, II(e_k, e_2)) + \omega(e_1, \nabla_{e_k}^M II(e_k, e_2))$$
$$= \omega((\nabla_{e_k}^M II(e_k, e_1))^{\top}, e_2) + \omega(\nabla_{e_k}^{\perp} II(e_k, e_1), e_2)$$
$$+ \omega(e_1, (\nabla_{e_k}^M II(e_k, e_2))^{\top}) + \omega(e_1, \nabla_{e_k}^{\perp} II(e_k, e_2)) + 2\omega(II(e_k, e_1), II(e_k, e_2)).$$

We compute

$$\omega((\nabla_{e_k}^M II(e_k, e_1))^{\top}, e_2) = \langle \nabla_{e_k}^M II(e_k, e_1), e_1 \rangle \eta = -\sum_{k=1}^{2}|II(e_k, e_1)|^2 \eta$$

and likewise for the other term $\omega(e_1, (\nabla_{e_k}^M II(e_k, e_2))^\top)$. On the other hand,

$$\omega(\nabla_{e_k}^\perp II(e_k, e_1), e_2) = \omega((\nabla_{e_k} II)(e_k, e_1), e_2) = \omega((\nabla_{e_k} II)(e_1, e_k), e_2)$$

where the formula for the connection on II (1.3) is used. Applying the Codazzi equation (1.2), we see this is equal to

$$\omega((\nabla_{e_1} II)(e_k, e_k), e_2) - \omega((R^M(e_k, e_1)e_k)^\perp, e_2) = \omega(\nabla_{e_1}^\perp H, e_2) - \omega((R^M(e_k, e_1)e_k)^\perp, e_2).$$

The formula is proved.

$\square$

We compute the evolution equation of $*_{\Sigma_t}\omega$ along a mean curvature flow.

Proposition 4.1 *Let Σ_t be mean curvature flow of a closed embedded Riemann surface in M. Let ω be a parallel 2-form on M and $\eta = *_{\Sigma_t}\omega$, then*

$$(\frac{d}{dt} - \Delta^{\Sigma_t})\eta = |II|^2\eta - 2\omega(II(e_k, e_1), II(e_k, e_2))$$
$$+ \omega((R^M(e_1, e_k)e_k)^\perp, e_2) + \omega(e_1, (R^M(e_2, e_k)e_k)^\perp)$$

Proof. We parametrize Σ_t by $F : \Sigma \times [0, T) \to M$ with $\frac{\partial F}{\partial t} = H$ and fix a local oriented coordinate system x^1, x^2 on Σ. Write $\eta = \frac{1}{\sqrt{\det g_{ij}}}\omega(F_*(\frac{\partial}{\partial x^1}), F_*(\frac{\partial}{\partial x^2}))$ and compute

$$\frac{d}{dt}\eta = \omega((\nabla_{e_1}H)^\perp, e_2) - \omega((\nabla_{e_2}H)^\perp, e_1).$$

Combine this with the formula of $\Delta^{\Sigma_t}\eta$ in Lemma 4.1, we obtain the desired formula. $\square$

Definition 4.1 *Given a Riemannian manifold M, we say a two-form ω is of Kähler type on M if $\omega(X, Y) = \langle JX, Y \rangle$ for a parallel complex structure $J(J^2 = -I)$ that is compatible with the metric, i.e $\langle JX, JY \rangle = \langle X, Y \rangle$.*

Both ω' and ω'' in the previous section are of Kähler type on $M = \Sigma^1 \times \Sigma^2$.

Lemma 4.2 *If ω is of Kähler type on M and J is the corresponding complex structure, then*

$$\omega((R^M(e_1, e_k)e_k)^\perp, e_2) + \omega(e_1, (R^M(e_2, e_k)e_k)^\perp) = (1 - \eta^2)Ric^M(Je_1, e_2).$$

Proof. Recall the following identity

$$\mathrm{Ric}^M(JX, Y) = \frac{1}{2}\sum_{A=1}^{4} R^M(X, Y, e_A, Je_A)$$

for any orthonormal frame $\{e_A\}_{A=1}^{4}$ of T_pM. Denote $\omega(e_A, e_B) = \omega_{AB}$ and $R^M(e_A, e_B, e_C, e_D) = R_{ABCD}$. When $\eta \neq \pm 1$, we can choose an oriented orthonormal frame $\{e_1, e_2, e_3, e_4\}$ with $e_1, e_2 \in T_p\Sigma$ and $e_3, e_4 \in N_p\Sigma$, such that

$$\omega_{12} = \omega_{34} = \eta, \omega_{13} = -\omega_{24} = \sqrt{1 - \eta^2}, \text{and } \omega_{23} = \omega_{14} = 0. \qquad (4.1)$$

We compute

$$\mathrm{Ric}^M(Je_1, e_2) = \frac{1}{2}R^M(e_1, e_2, e_A, Je_A) = \sum_{A<B}\omega_{AB}R_{12AB}.$$

Plug in (4.1), we obtain

$$\mathrm{Ric}^M(Je_1, e_2) = \eta(R_{1212} + R_{1234}) + \sqrt{1-\eta^2}(R_{1213} - R_{1224}).$$

The J invariance of the curvature operator implies $R^M(e_1, e_2, e_1, e_2) = R^M(e_1, e_2, Je_1, Je_2)$. Plug in (4.1) again, we obtain

$$(1-\eta^2)(R_{1212} + R_{1234}) = \eta\sqrt{1-\eta^2}(R_{1213} - R_{1224}).$$

Therefore,

$$\mathrm{Ric}^M(Je_1, e_2) = \frac{1}{\sqrt{1-\eta^2}}(R_{1213} - R_{1224}).$$

On the other hand,

$$\omega((R^M(e_1, e_k)e_k)^\perp, e_2) + \omega(e_1, (R^M(e_2, e_k)e_k)^\perp) = R_{1kk4}\omega_{42} + R_{2kk3}\omega_{13}.$$

The lemma is proved by recalling (4.1) again. $\qquad\qquad\square$

The terms that involve the second fundamental form in Prop 4.1 can also be simplified using (4.1)

$$\begin{aligned}
-2\omega(h_{\alpha 1k}e_\alpha, h_{\beta 2k}e_\beta) &= -2h_{\alpha 1k}h_{\beta 2k}\omega(e_\alpha, e_\beta)\\
&= (-2h_{31k}h_{42k} + 2h_{41k}h_{32k})\omega(e_3, e_4)\\
&= (-2h_{31k}h_{42k} + 2h_{41k}h_{32k})\eta.
\end{aligned}$$

We have thus proved the following theorem.

Theorem 4.2 *Suppose ω is of Kähler type on M and J is the corresponding complex structure, then*

$$(\frac{d}{dt} - \Delta^{\Sigma_t})\eta = ((h_{31k} - h_{42k})^2 + (h_{41k} + h_{32k})^2)\eta + (1-\eta^2)Ric^M(Je_1, e_2)$$

In particular, if the Riemannian metric g on M is Einstein with $Ric^M = cg$,

$$(\frac{d}{dt} - \Delta^{\Sigma_t})\eta = ((h_{31k} - h_{42k})^2 + (h_{41k} + h_{32k})^2)\eta + c\eta(1-\eta^2). \qquad (4.2)$$

4.3 Long time existence

We apply Theorem 4.2 to $M = \Sigma^1 \times \Sigma^2$ on which both ω' and ω'' are of Kähler type. The maximum principle implies $*_{\Sigma_t}\omega' = 0$ is preserved along the mean curvature flow and thus Σ_t remains Lagrangian. We remark that in [13] Smoczyk

proved that being Lagrangian is preserved by the mean curvature flow in any Kähler-Einstein manifold.

Set $\eta = *_{\Sigma_t}\omega''$ and apply the maximum principle, we see that $\eta > 0$ is preserved, or Σ_t remains the graph of a symplectomorphism. Compare η with the solution of the ordinary differential equation

$$\frac{d}{dt}f = cf(1 - f^2),$$

we deduce

$$\eta(x,t) \geq \frac{\alpha e^{ct}}{\sqrt{1 + \alpha^2 e^{2ct}}} \text{ , where } \alpha > 0 \text{ is defined by} \frac{\alpha}{\sqrt{1 + \alpha^2}} = \min_{\Sigma_0} \eta. \quad (4.3)$$

Let J' be the parallel complex structure associated with ω' and $\{e_1, e_2\}$ be an oriented orthonormal basis on Σ_t. Take $e_3 = J'(e_1), e_4 = J'(e_2)$, then $\{e_3, e_4\}$ is an orthonormal basis for the normal space $N\Sigma_t$. The second fundamental has the following symmetries

$$\langle \mathrm{II}(X,Y), J'Z \rangle = \langle \mathrm{II}(X, J'Y), Z \rangle = \langle \mathrm{II}(J'X, Y), Z \rangle.$$

These symmetries imply

$$|H|^2 \leq \frac{4}{3}|\mathrm{II}|^2. \quad (4.4)$$

The evolution equation (4.2) becomes

$$(\frac{d}{dt} - \Delta^{\Sigma_t})\eta = \eta(2|\mathrm{II}|^2 - |H|^2) + c\eta(1 - \eta^2). \quad (4.5)$$

We fix an isometric embedding $i : M \to \mathbb{R}^N$. Let F be the mean curvature flow of Σ in M and denote $i \circ F$ by $\bar{F}$. We use the notation in §3.1.

For $y_0 \in \mathbb{R}^N$, suppose (y_0, t_0) is a singularity of the mean curvature. We consider the backward heat kernel ρ_{y_0, t_0} defined in §3.1.

Denote $Q = 1 - \eta =$ and $P = -\eta(2|\mathrm{II}|^2 - |H|^2) - c\eta(1 - \eta^2)$. Recalling equations (3.6) and (4.5), we derive

$$\frac{d}{dt}\int_{\Sigma_t} Q\rho_{y_0, t_0} d\mu_t = \int_{\Sigma_t}(\Delta^{\Sigma_t}Q + P)\rho_{y_0, t_0} d\mu_t - \int_{\Sigma_t} Q\rho_{y_0, t_0}(\bar{H} \cdot (\bar{H} + E))d\mu_t$$

$$+ \int_{\Sigma_t} Q(-\Delta^{\Sigma_t}\rho_{y_0, t_0} - \rho_{y_0, t_0}(\frac{|\bar{F}^\perp|^2}{4(t_0 - t)^2} + \frac{\bar{F}^\perp \cdot \bar{H}}{t_0 - t} + \frac{\bar{F}^\perp \cdot E}{2(t_0 - t)}))d\mu_t$$

where we also use $\frac{d}{dt}d\mu_t = -|H|^2 d\mu_t = -\bar{H} \cdot (\bar{H} + E)d\mu_t$.

Integrating by parts, collecting terms and completing squares, we obtain

$$\frac{d}{dt}\int_{\Sigma_t} Q\rho_{y_0, t_0} d\mu_t$$

$$= \int_{\Sigma_t}\left(P\rho_{y_0, t_0} - Q\rho_{y_0, t_0}(\frac{|\bar{F}^\perp|^2}{4(t_0 - t)^2} + \frac{\bar{F}^\perp \cdot \bar{H}}{t_0 - t} + \frac{\bar{F}^\perp \cdot E}{2(t_0 - t)} + |\bar{H}|^2 + \bar{H} \cdot E)\right)d\mu_t$$

$$= \int_{\Sigma_t}\left(P\rho_{y_0, t_0} - Q\rho_{y_0, t_0}(|\bar{H} + \frac{\bar{F}^\perp}{2(t_0 - t)} + E|^2 - |E|^2)\right)d\mu_t.$$

Recall (4.4), $0 \le Q \le 1$, and that $\lim_{t \to t_0} \int_{\Sigma_t} \rho_{y_0,t_0} d\mu_t$ exists, we arrive at

$$\frac{d}{dt} \int_{\Sigma_t} (1-\eta)\rho_{y_0,t_0} d\mu_t \le C - \frac{2}{3} \int_{\Sigma_t} \eta |II|^2 \rho_{y_0,t_0} d\mu_t$$

for some constant $C > 0$. Moreover, for $t < \infty$, we know that $\eta > \delta > 0$, thus

$$\frac{d}{dt} \int_{\Sigma_t} (1-\eta)\rho_{y_0,t_0} d\mu_t \le C - C_\delta \int_{\Sigma_t} |II|^2 \rho_{y_0,t_0} d\mu_t \qquad \forall t \in [0,t_0) \qquad (4.6)$$

From here we deduce that $\lim_{t \to t_0} \int_{\Sigma_t} (1-\eta)\rho_{y_0,t_0} d\mu_t$ exists.

Consider the parabolic dilation at (y_0, t_0):

$$D_\lambda: \quad \begin{array}{ccc} \mathbb{R}^N \times [0,t_0) & \to & \mathbb{R}^N \times [-\lambda^2 t_0, 0) \\ (y,t) & \mapsto & (\lambda(y-y_0), \lambda^2(t-t_0)) \end{array}.$$

Notice that η is invariant under D_λ. The inequality (4.6) on Σ_s^λ becomes

$$\frac{d}{ds} \int_{\Sigma_s^\lambda} (1-\eta)\rho_{0,0} d\mu_s^\lambda \le \frac{C}{\lambda^2} - C_\delta \int_{\Sigma_s^\lambda} |II|^2 \rho_{0,0} d\mu_s^\lambda.$$

Fix $s_0 < 0, \tau > 0$, for λ large, we integrate both sides over the interval $[s_0 - \tau, s_0]$ and obtain

$$\int_{s_0-\tau}^{s_0} \int_{\Sigma_s^\lambda} |II|^2 \rho_{0,0} d\mu_s^\lambda ds \le \frac{C'}{\lambda^2} + C'' \int_{\Sigma_{s_0-\tau}^\lambda} (1-\eta)\rho_{0,0} d\mu_{s_0-\tau}^\lambda - C'' \int_{\Sigma_{s_0}^\lambda} (1-\eta)\rho_{0,0} d\mu_{s_0}^\lambda.$$

Since the limit $\lim_{t \to t_0} \int_{\Sigma_t} (1-\eta)\rho_{y_0,t_0} d\mu_t$ exists, argue as before we can pick a sequence $\lambda_i \to \infty$ such that $\Sigma_s^{\lambda_i} \to \Sigma_s^\infty$ for all $s \in (-\infty, 0)$. In fact, we have

$$\int_{s_0-\tau}^{s_0} \int_{\Sigma_s^{\lambda_i}} |II|^2 \rho_{0,0} d\mu_s^{\lambda_i} ds \le C(i)$$

where $C(i) \to 0$ as $i \to \infty$. We first choose $\tau_i \to 0$ such that $\frac{C(i)}{\tau_i} \to 0$, and then choose $s_i \in [s_0 - \tau_i, s_0]$ so that

$$\int_{\Sigma_{s_i}^{\lambda_i}} |II|^2 \rho_{0,0} d\mu_{s_i}^{\lambda_i} \le \frac{C(i)}{\tau_i} \to 0$$

Suppose $\Sigma_{s_i}^{\lambda_i}$ is given by $\bar{F}_{s_i}^{\lambda_i} : \Sigma \to \mathbb{R}^N$, thus

$$\rho_{0,0}(F_{s_i}^{\lambda_i}) = \frac{1}{4\pi(-s_i)} \exp\left(-\frac{|\bar{F}_{s_i}^{\lambda_i}|^2}{4(-s_i)}\right).$$

Then for any $R > 0$,

$$\int_{\Sigma_{s_i}^{\lambda_i}} \rho_{0,0} |II|^2 d\mu_{s_i}^{\lambda_i} \ge \int_{\Sigma_{s_i}^{\lambda_i} \cap B_R(0)} \rho_{0,0} |II|^2 d\mu_{s_i}^{\lambda_i} \ge C \exp\left(-\frac{R^2}{2}\right) \int_{\Sigma_{s_i}^{\lambda_i} \cap B_R(0)} |II|^2 d\mu_{s_i}^{\lambda_i}$$

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Hence, on any compact set $K \subset \mathbb{R}^N$,

$$\lim_{i \to \infty} \int_{\Sigma_{s_i}^{\lambda_i} \cap K} |\mathrm{II}|^2 d\mu_{s_i}^{\lambda_i} = 0$$

We can take a coordinate neighborhood Ω of $\pi_1(y_0) \in \Sigma^1$ where $\pi_1 : M \to \Sigma^1$ is the projection onto the first factor. $\Sigma_{s_i}^{\lambda_i}$ is the graph of $\tilde{u}_i : \lambda_i \Omega \to \lambda_i \Sigma_2$. Since η is bounded and $\int_{\Sigma_{s_i}^{\lambda_i}} |\mathrm{II}|^2 d\mu_{s_i}^{\lambda_i} \to 0$,

$$|D\tilde{u}_i| \leq C \text{ and } \int_{\Omega} |D^2 \tilde{u}_i|^2 \to 0$$

Therefore $\tilde{u}_i \to \tilde{u}_\infty$ in $C^\alpha \bigcap W^{1,2}$, $\tilde{u}_\infty$ is a linear map, and

$$\lim_{i \to \infty} \int_{\Sigma_{s_i}^{\lambda_i}} \rho_{0,0} d\mu_{s_i}^{\lambda_i} = \int \rho_{0,0} d\mu_{s_0}^\infty = 1$$

We thus found a sequence such that $\lim_{t_i \to t_0} \int \rho_{y_0,t_0} d\mu_t$ exists and equals to 1. By White's regularity theorem [19], the second fundamental form $|\mathrm{II}|$ is bounded as $t \to t_0$.

4.4   Smooth convergence as $t \to \infty$

By the general convergence theorem of Simon [11], it suffices to have an uniform bound on $\sup_{\Sigma_t} |\mathrm{II}|^2$ as $t \to \infty$.

We first prove the following differential inequality.

Lemma 4.3
$$\frac{d}{dt} \int_{\Sigma_t} \frac{|H|^2}{\eta} d\mu_t \leq c \int_{\Sigma_t} \frac{|H|^2}{\eta} d\mu_t$$

Proof.   It is not hard to derive:

$$(\frac{d}{dt} - \Delta)|H|^2 = -2|\nabla H|^2 + 2\sum_{i,j}(\sum_\alpha H_\alpha h_{\alpha ij})^2 + c(2 - \eta^2)|H|^2$$

where $H = H_\alpha e_\alpha$ for an orthonormal basis e_α of the normal space of Σ_t. Combining this equation with (4.5) and applying the inequalities $\sum_{i,j}(\sum_\alpha H_\alpha h_{\alpha ij})^2 \leq |\mathrm{II}|^2|H|^2$ and $|\nabla|H|| \leq |\nabla H|$, the inequality follows. $\square$

The following proposition, which follows from a standard point-picking lemma, will be useful in the proof too.

Proposition 4.2 *Suppose $\sup_{\Sigma_t} |\mathrm{II}|^2$ is not bounded as $t \to \infty$, then there exists a blow-up flow $\widetilde{S}_\infty \subset \mathbb{R}^4 \times \mathbb{R}$ defined on the whole $(-\infty, \infty)$ with uniformly bounded second fundamental form and $|\widetilde{\mathrm{II}}|(0,0) = 1$. Indeed, each t slice of $\widetilde{S}_\infty$ is a graph of a symplectomorphism from $\mathbb{C}$ to $\mathbb{C}$.*

4.5 $c > 0$ case

We prove by contradiction and look at the blow-up flow $\widetilde{S}_\infty$. (4.3) implies $\eta \equiv 1$ on $\widetilde{S}_\infty$. The $t = 0$ slice of $\widetilde{S}_\infty$ is a graph of a symplectomorphism from $\mathbb{C}$ to $\mathbb{C}$ given by

$$(x, y) \mapsto (f(x, y), g(x, y))$$

and satisfies

$$f_x g_y - f_y g_x = 1 \text{ and } \frac{2}{\sqrt{2 + f_x^2 + f_y^2 + g_x^2 + g_y^2}} = 1.$$

We derive that $h = f + \sqrt{-1}g$ must be holomorphic with $|\frac{\partial h}{\partial z}| = 1$, $z = x + \sqrt{-1}y$. Thus h is of the form $h = e^{\sqrt{-1}\theta}z + C$ where θ and C are constants. The graph of h has zero second fundamental form and this contradicts with $|\tilde{\mathrm{II}}|(0, 0) = 1$.

4.6 $c = 0$ case

In this case by (4.3), η has a positive lower bound. We have

$$\int_{\Sigma_t} |H|^2 d\mu_t \leq \int_{\Sigma_t} \frac{|H|^2}{\eta} d\mu_t \leq C \int_{\Sigma_t} |H|^2 d\mu_t \tag{4.7}$$

for some constant C.

Since $\int_0^\infty \int_{\Sigma_t} |H|^2 d\mu_t dt < \infty$, there exists a subsequence t_i such that $\int_{\Sigma_{t_i}} |H|^2 d\mu_{t_i} \to 0$ and thus $\int_{\Sigma_{t_i}} \frac{|H|^2}{\eta} d\mu_{t_i} \to 0$ as well. Because $\int_{\Sigma_t} \frac{|H|^2}{\eta} d\mu_t$ is non-increasing, this implies $\int_{\Sigma_t} \frac{|H|^2}{\eta} d\mu_t \to 0$ for the continuous parameter t as it approaches ∞. Together with (4.7) this implies $\int_{\Sigma_t} |H|^2 d\mu_t \to 0$ as $t \to \infty$. By the Gauss-Bonnet theorem, $\int_{\Sigma_t} |\mathrm{II}|^2 d\mu_t = \int_{\Sigma_t} |H|^2 d\mu_t \to 0$. The ϵ regularity theorem in [6] (see also [1]) implies $\sup_{\Sigma_t} |\mathrm{II}|^2$ is uniformly bounded.

4.7 $c < 0$ case

We may assume $c = -1$. In this case we have

$$\frac{d}{dt} \int_{\Sigma_t} \frac{|H|^2}{\eta} d\mu_t \leq -\int_{\Sigma_t} \frac{|H|^2}{\eta} d\mu_t$$

or

$$\int_{\Sigma_t} \frac{|H|^2}{\eta} d\mu_t \leq Ce^{-t}$$

for some constant C.

Since $\eta \leq 1$, we have

$$\int_{\Sigma_t} |H|^2 d\mu_t \leq Ce^{-t}.$$

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Suppose the second fundamental form is unbounded. Since $\int_{\underline{\Sigma}_t} |H|^2 d\mu_t \leq Ce^{-t}$, the integral of $|H|^2$ on each time-slice of the blow-up flow $\widetilde{S}_\infty$ vanishes. Therefore, we obtain a minimal area-preserving map. A result of Ni [9] shows this is a linear diffeomorphism. This contradicts to the fact that $|\tilde{\mathrm{II}}|(0,0) = 1$ again.

5. Acknowledgement

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