A convergence result of the Lagrangian mean curvature flow

Mu-Tao Wang *

Aug. 15, 2005

email: mtwang@math.columbia.edu

Abstract

We prove the mean curvature flow of the graph of a symplectomorphism between Riemann surfaces converges smoothly as time approaches infinity.

1 Introduction

Let Σ_1 and Σ_2 be two homeomorphic compact Riemann surfaces without boundary. We assume Σ_1 and Σ_2 are both equipped with Rimannian metrics of the same constant curvature c, $c = -1, 0$, or 1. Let ω_1 and ω_2 denote the volume or symplectic forms of Σ_1 and Σ_2, respectively. The Riemannian product space $\Sigma_1 \times \Sigma_2$ is denoted by M. We take $\omega' = \omega_1 - \omega_2$ to be the Kähler form of M and M becomes a Kähler-Einstein manifold with the Ricci form $Ric = c\omega'$. Let Σ be the graph of a symplectomorphism $f : \Sigma_1 \to \Sigma_2$, i.e. $f^*\omega_2 = \omega_1$. Σ can be considered as a Lagrangian submanifold with respect to the symplectic form ω'.

The mean curvature flow deforms the initial surface $\Sigma^0 = \Sigma$ in the direction of its mean curvature vector. Denote by Σ^t the time slice of the flow at t. That Σ^t remains a Lagrangian submanifold follows from a result of Smoczyk [9]. The long-time existence and convergence problems of this flow were studied in [10] and [12].

*The author is partially supported by National Science Foundation Grant DMS0104163 and DMS0306049 and an Alfred P. Sloan Research Fellowship.

In [12], the author proved the long time existence of the flow and showed that Σ^t for $t > 0$ remains the graph of a symplectomorphism f_t. When $c = 1$, the author proved the C^∞ convergence as $t \to \infty$. However, only C^0 convergence was achieved in the case when $c = -1$ or 0.

Independently, in [10], Smoczyk studied the case when $c = -1$ or 0 assuming an extra angle condition. He discovered a curvature estimate and showed that the second fundamental form is uniformly bounded under this condition, and thus established the long time existence and C^∞ convergence at infinity.

In view of the above results, it is interesting to see whether the C^∞ convergence of the flow does require the angle condition. In this paper, we show this assumption is unnecessary.

Theorem 1.1 *Let (Σ_1, ω_1) and (Σ_2, ω_2) be two homeomorphic compact Riemann surface of the same constant curvature $c = -1, 0,$ or 1. Suppose Σ is the graph of a symplectomorphism $f : \Sigma_1 \to \Sigma_2$ as a Lagrangian submanifold of $M = (\Sigma_1 \times \Sigma_2, \omega_1 - \omega_2)$ and Σ^t is the mean curvature flow with initial surface $\Sigma^0 = \Sigma$. Then Σ^t remains the graph of a symplectomorphism f_t along the mean curvature flow. The flow exists smoothly for all time and Σ^t converges smoothly to a minimal Lagrangian submanifold as $t \to \infty$.*

The long time existence part was already proved in [12]. The smooth convergence was established through a new integral estimate (Lemma 3.1) related to the second variation formula. This estimate is most useful when $c = -1$ or 0. We remark the existence of such minimal Lagrangian submanifold was proved using variational method by Schoen [7] (see also Lee [5]).

The author would like to thank Tom Ilmanen and Andre Neves for useful discussions.

2 Background material

First we recall some formulas from [12]. The restriction of the Kähler form ω' to Σ^t gives a time-dependent function $\eta = *\omega'$. Since Σ^t is Lagrangian, $*\omega' = 2 * \omega_1$. $*\omega_1$ is indeed the Jacobian of the projection π_1 from M to Σ_1 when restricted to Σ and $\eta > 0$ if and only if Σ is locally a graph over Σ_1. η satisfies the following evolution equation:

$$\frac{d}{dt}\eta = \Delta\eta + \eta[2|A|^2 - |H|^2] + c\eta(1 - \eta^2) \tag{2.1}$$

along the mean curvature flow.

Notice that $0 < \eta \le 1$. By the equation of η and the comparison theorem for parabolic equations, we get

$$\eta(x,t) \ge \frac{\alpha e^{ct}}{\sqrt{1+\alpha^2 e^{2ct}}} \tag{2.2}$$

where $\alpha > 0$ is given by $\frac{\alpha}{\sqrt{1+\alpha^2}} = \min_{\Sigma_0} \eta$. Therefore $\eta(x,t)$ converges uniformly to 1 when $c = 1$ and is nondecreasing when $c = 0$. In any case, η has a positive lower bound at any finite time and thus Σ_t remains the graph of a symplectomorphism.

Using the fact that the second fundamental form for Lagrangian submanifold is a fully symmetric three tensor, one derives

$$|H|^2 \le \frac{4}{3}|A|^2.$$

Plug this into (2.1) and we obtain

$$\frac{d}{dt}\eta \ge \Delta\eta + \frac{2}{3}|A|^2\eta + c\eta(1-\eta^2) \tag{2.3}$$

In [12], we apply blow-up analysis to this equation to show there exists a weak blow-up limit with vanishing $\int |A|^2$. This together with the lower bound of η shows the limit is a flat space and White's regularity theorem [13] implies the blow-up center is a regular point. This proves the long-time existence of the flow.

3 A monotonicity lemma

In this section, we derive a new monotonicity formula. First $|H|^2$ satisfies the following evolution equation:

$$(\frac{d}{dt} - \Delta)|H|^2 = -2|\nabla H|^2 + 2\sum_{ij}(\sum_k H_k h_{kij})^2 + c(2-\eta^2)|H|^2 \tag{3.1}$$

where the symmetric three-tensor h_{ijk} is the second fundamental form and $H_k = h_{iik}$, the trace of the second fundamental form, is the component of the mean curvature vector after identifying the tangent bundle and the normal bundle through J.

We remark that both equations (2.1) and (3.1) are derived in Lemma 5.3 of [10] where p in [10] and η are related by $\eta^2 = \frac{4}{p}$ and $S = 2c$.

We claim the following differential inequality is true:

Lemma 3.1

$$\frac{d}{dt}\int_{\Sigma_t} \frac{|H|^2}{\eta} \leq c\int_{\Sigma_t} \frac{|H|^2}{\eta}$$

Proof.

The proof is a direct computation by combining equations (2.1) and (3.1). We compute

$$\begin{aligned}\frac{d}{dt}\frac{|H|^2}{\eta} &= \frac{\eta\Delta|H|^2 - |H|^2\Delta\eta}{\eta^2} - 2\frac{|\nabla H|^2}{\eta} \\ &\quad + \frac{2\sum_{ij}(\sum_k H_k h_{kij})^2 - 2|H|^2|A|^2 + |H|^4}{\eta} + c\frac{|H|^2}{\eta}.\end{aligned}$$

Now

$$\Delta\frac{|H|^2}{\eta} = \frac{\eta\Delta|H|^2 - |H|^2\Delta\eta}{\eta^2} - \frac{2\eta\nabla\eta(\eta\nabla|H|^2 - |H|^2\nabla\eta)}{\eta^4}.$$

We plug this into the previous equation and obtain

$$\begin{aligned}\frac{d}{dt}\frac{|H|^2}{\eta} &= \Delta\frac{|H|^2}{\eta} + \frac{2\eta\nabla\eta(\eta\nabla|H|^2 - |H|^2\nabla\eta)}{\eta^4} - 2\frac{|\nabla H|^2}{\eta} \\ &\quad + \frac{2\sum_{ij}(\sum_k H_k h_{kij})^2 - 2|H|^2|A|^2 + |H|^4}{\eta} + c\frac{|H|^2}{\eta}.\end{aligned}$$

Rearranging terms, we arrive at

$$\begin{aligned}\frac{d}{dt}\frac{|H|^2}{\eta} &= \Delta\frac{|H|^2}{\eta} + \frac{4\eta|H|\nabla\eta\cdot\nabla|H| - 2|\nabla\eta|^2|H|^2 - 2\eta^2|\nabla H|^2}{\eta^3} \\ &\quad + \frac{2\sum_{ij}(\sum_k H_k h_{kij})^2 - 2|H|^2|A|^2 + |H|^4}{\eta} + c\frac{|H|^2}{\eta}.\end{aligned}$$

Integrate this identity and we have

$$\frac{d}{dt}\int_{\Sigma_t}\frac{|H|^2}{\eta}=\int_{\Sigma_t}\frac{4\eta|H|\nabla\eta\cdot\nabla|H|-2|\nabla\eta|^2|H|^2-2\eta^2|\nabla H|^2}{\eta^3}$$
$$+\int_{\Sigma_t}\frac{2\sum_{ij}(\sum_k H_k h_{kij})^2-2|H|^2|A|^2}{\eta}+c\int_{\Sigma_t}\frac{|H|^2}{\eta}.$$

We use $|\nabla|H||\leq|\nabla H|$ in the first summand on the right hand side and complete the square

$$\frac{4\eta|H|\nabla\eta\cdot\nabla|H|-2|\nabla\eta|^2|H|^2-2\eta^2|\nabla|H||^2}{\eta^3}=-2\frac{|\nabla\eta|H|-\eta\nabla|H||^2}{\eta^3}.$$

At last, we apply Cauchy-Schwarz inequality to the second summand and the differential inequality is proved.

□

4 Proof of the theorem

The smooth convergence in the case when $c=1$, i.e. when Σ_1 and Σ_2 are both standard S^2, was proved in [12].

We prove the C^∞ convergence in the case $c=0$ and $c=-1$ in the following. By the general convergence theorem of Simon [8], it suffices to show $|A|^2$ is bounded independent of time.

In the case when $c=0$, by (2.2), η has a positive lower bound. We have

$$\int_{\Sigma_t}|H|^2\leq\int_{\Sigma_t}\frac{|H|^2}{\eta}\leq K_1\int_{\Sigma_t}|H|^2 \tag{4.1}$$

for some constant K_1.

Since $\int_0^\infty\int_{\Sigma_t}|H|^2<\infty$, there exists a subsequence t_i such that $\int_{\Sigma_{t_i}}|H|^2\to 0$ and thus $\int_{\Sigma_{t_i}}\frac{|H|^2}{\eta}\to 0$ as well. Because $\int_{\Sigma_t}\frac{|H|^2}{\eta}$ is non-increasing, this implies $\int_{\Sigma_t}\frac{|H|^2}{\eta}\to 0$ for the continuous parameter t as it approaches ∞. Together with (4.1) this implies $\int_{\Sigma_t}|H|^2\to 0$ as $t\to\infty$. By the Gauss formula, $\int_{\Sigma_t}|A|^2=\int_{\Sigma_t}|H|^2\to 0$. The ϵ regularity theorem in [4] (see also [2]) implies $\sup_{\Sigma_t}|A|^2$ is uniformly bounded.

In the case when $c=-1$, we have

$$\frac{d}{dt}\int_{\Sigma_t}\frac{|H|^2}{\eta}\leq-\int_{\Sigma_t}\frac{|H|^2}{\eta}$$

or

$$\int_{\Sigma_t}\frac{|H|^2}{\eta}\leq K_2e^{-t}$$

for some constant K_2,

Since $\eta\leq 1$, we have

$$\int_{\Sigma_t}|H|^2\leq K_2e^{-t}.$$

This implies $\Sigma_t\to\Sigma_\infty$ in Radon measure. Indeed, for any function ϕ on M with compact support, it is easy to see

$$\frac{d}{dt}\int_{\Sigma_t}\phi=\int_{\Sigma_t}\nabla^M\phi\cdot H+\int_{\Sigma_t}\phi|H|^2,$$

and thus

$$\int_{\Sigma_t}\phi\to\int_{\Sigma_\infty}\phi.$$

exponentially. Also the limit measure Σ_∞ is unique.

The argument in [12] shows the limit Σ_∞ is smooth. It seems one can adapt the proof of the local regularity theorem of Ecker (Theorem 5.3) [2] or the original local regularity theorem of Brakke [1] to get the uniform bound on second fundamental form. This does require versions of these theorem in a general ambient Riemannian manifold.

We circumvent this step by quoting a theorem in minimal surfaces. Suppose the second fundamental form is unbounded. The blow-up procedure in Proposition 3.1 of [12] produces a limiting flow that exists on $(-\infty,\infty)$. The flow has uniformly bounded second fundamental form $A(x,t)$ and $|A|(0,0)=1$. It is not hard to see each slice is the graph of an area-preserving map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Since $\int_{\Sigma_t}|H|^2\leq K_2e^{-t}$, the limiting flow will satisfies

$$\int|H|^2\equiv 0$$

Therefore, we obtain a minimal area-preserving map. A result of Ni [6] generalizing Schoen's theorem [7] shows this is a linear diffeomorphism. This contradicts to the fact that $|A|(0,0)=1$.

References

[1] K. A. Brakke, *The motion of a surface by its mean curvature.* Mathematical Notes, 20. Princeton University Press, Princeton, N.J., 1978.

[2] K. Ecker, *Regularity theory for mean curvature flow.* Progress in Nonlinear Differential Equations and their Applications, 57. Birkhuser Boston, Inc., Boston, MA, 2004.

[3] C. J. Earle and J. Eells, *The diffeomorphism group of a compact Riemann surface.* Bull. Amer. Math. Soc. 73 1967 557–559.

[4] T. Ilmanen, *Singularities of mean curvature flow of surfaces.* preprint, 1997.

[5] Y.-I. Lee, *Lagrangian minimal surfaces in Khler-Einstein surfaces of negative scalar curvature.* Comm. Anal. Geom. 2 (1994), no. 4, 579–592.

[6] L. Ni, *A Bernstein type theorem for minimal volume preserving maps.* Proc. Amer. Math. Soc. 130 (2002), no. 4, 1207–1210.

[7] R. Schoen, *The role of harmonic mappings in rigidity and deformation problems.* Complex geometry (Osaka, 1990), 179–200, Lecture Notes in Pure and Appl. Math., 143, Dekker, New York, 1993.

[8] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems.* Ann. of Math. (2) 118 (1983), no. 3, 525–571.

[9] K. Smoczyk, *Der Lagrangesche mittlere Krümmungsfluß (The Lagrangian mean curvature flow).* Habilitation thesis (English with German preface), University of Leipzig, Germany (1999), 102 pages.

[10] K. Smoczyk, *Angle theorems for the Lagrangian mean curvature flow.* Math. Z. 240 (2002), no. 4, 849–883.

[11] M.-T. Wang, *Mean curvature flow of surfaces in Einstein four-Manifolds.* J. Differential Geom. **57** (2001), no. 2, 301-338.

[12] M.-T. Wang, *Deforming area preserving diffeomorphism of surfaces by mean curvature flow.* Math. Res. Lett. 8 (2001), no.5-6, 651-662.

[13] B. White, *A local regularity theorem for classical mean curvature flow.* preprint, 1999, revised 2002.