Mean Curvature Flows and Isotopy of Maps Between Spheres

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Abstract

Let f be a smooth map between unit spheres of possibly different dimensions. We prove the global existence and convergence of the mean curvature flow of the graph of f under various conditions. A corollary is that any area-decreasing map between unit spheres (of possibly different dimensions) is isotopic to a constant map.

1 Introduction

Let Σ_1 and Σ_2 be two compact Riemannian manifolds and $M = \Sigma_1 \times \Sigma_2$ be the product manifold. We consider a smooth map $f : \Sigma_1 \to \Sigma_2$ and denote the graph of f by Σ; Σ is a submanifold of M by the embedding $id \times f$. In [17], [18], and [19], the second author studies the deformation of f by the mean curvature flow (see also the work of Chen-Li-Tian [2]). The idea is to deform Σ along the direction of its mean curvature vector in M with the hope that Σ will remain a graph. This is the negative gradient flow of the volume functional and a stationary point is a "minimal map" introduced by Schoen in [12]. In [19], the second author proves various long-time existence and convergence results of graphical mean curvature flows in arbitrary codimensions under assumptions on the Jacobian of the projection from Σ to Σ_1. This quantity is denoted by $*\Omega$ in [19] and $*\Omega > 0$ if and only if Σ is a graph over Σ_1 by the implicit function theorem. A crucial observation in [19] is that $*\Omega$ is a monotone quantity under the mean curvature flow when $*\Omega > \frac{1}{\sqrt{2}}$.

In this paper, we discover new positive geometric quantities preserved by the graphical mean curvature flow. To describe these results, we recall the differential of f, df, at each point of Σ_1 is a linear map between the tangent spaces. The Riemannian structures enables us to define the adjoint of df. Let $\{\lambda_i\}$ denote the eigenvalues of $\sqrt{(df)^T df}$, or the singular values of df, where $(df)^T$ is the adjoint of df. Note that λ_i is always nonnegative. We say f is an *area decreasing map* if $\lambda_i \lambda_j < 1$ for any $i \neq j$ at each point. In particular, f is area-decreasing if the df has rank one everywhere. Under this condition, the second author proves the Bernstein type theorem [21] and interior gradient estimates [22] for solutions of the minimal surface system. It is also proved in [23] that the set of graphs of area-decreasing linear transformations forms a convex subset of the Grassmannian. We prove that this condition is preserved along the mean curvature flow and the following global existence and convergence theorem.

Theorem A. *Let Σ_1 and Σ_2 be compact Riemannian manifolds of constant curvature k_1 and k_2 respectively. Suppose $k_1 \geq |k_2|$, $k_1 + k_2 > 0$ and $dim(\Sigma_1) \geq 2$. If f is a smooth area decreasing map from Σ_1 to Σ_2, the mean curvature flow of the graph of f remains the graph of an area decreasing map, exists for all time, and converges smoothly to the graph of a constant map.*

We remark that the condition $k_1 \geq |k_2|$ is enough to prove the long time existence of the flow. The following is an application to determine when a map between spheres is homotopically trivial.

Corollary A *Any area-decreasing map from S^n to S^m with $n \geq 2$ is homotopically trivial.*

When $m = 1$, the area-decreasing condition always holds and the above statement follows from the fact that $\pi_n(S^1)$ is trivial for $n \geq 2$. We remark that the result when $m = 2$ is proved by the second author in [20] using a somewhat different method. The higher homotopy groups $\pi_n(S^m)$ has been computed in many cases and it is known that homotopically nontrivial maps do exist when $n \geq m$. Since an area-decreasing map may still be surjective when $n > m$, we do not know any topological method that would imply such a conclusion.

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2 Preliminaries

In this section, we recall notations and formulae for mean curvature flows. Let $f : \Sigma_1 \to \Sigma_2$ be a smooth map between Riemannian manifolds. The graph of f is an embedded submanifold Σ in $M = \Sigma_1 \times \Sigma_2$. At any point of Σ, the tangent space of M, TM splits into the direct sum of the tangent space of Σ, $T\Sigma$ and the normal space $N\Sigma$, the orthogonal complement of the tangent space $T\Sigma$ in TM. There are isomorphisms $T\Sigma_1 \to T\Sigma$ by $X \mapsto X + df(X)$ and $T\Sigma_2 \to N\Sigma$ by $Y \mapsto Y - (df)^T(Y)$ where $(df)^T : T\Sigma_2 \to T\Sigma_1$ is the adjoint of df.

We assume the mean curvature flow of Σ can be written as a graph of f_t for $t \in [0, \epsilon)$ and derive the equation satisfied by f_t. The mean curvature flow is given by a smooth family of immersions F_t of Σ into M which satisfies

$$(\frac{\partial F}{\partial t})^\perp = H$$

where H is the mean curvature vector in M and $(\cdot)^\perp$ denotes the projection onto the normal space $N\Sigma$. Notice that we do not require $\frac{\partial F}{\partial t}$ is in the normal direction since the difference is only a tangential diffeomorphism (see for example White [24] for the issue of parametrization). By the definition of the mean curvature vector, this equation is equivalent to

$$(\frac{\partial F}{\partial t})^\perp = (\Lambda^{ij} \nabla^M_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^j})^\perp$$

where Λ^{ij} is the inverse to the induced metric $\Lambda_{ij} = \langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \rangle$ on Σ.

In terms of coordinates $\{y^A\}_{A=1\cdots n+m}$ on M, we have

$$\Lambda^{ij} \nabla^M_{\frac{\partial F}{\partial x^j}} \frac{\partial F}{\partial x^i} = \Lambda^{ij} (\frac{\partial^2 F^A}{\partial x^i \partial x^j} + \frac{\partial F^B}{\partial x^i} \frac{\partial F^C}{\partial x^j} \Gamma^A_{BC}) \frac{\partial}{\partial y^A}$$

where Γ^A_{BC} is the Christoffel symbol of M and thus

$$(\Lambda^{ij} \nabla^M_{\frac{\partial F}{\partial x^j}} \frac{\partial F}{\partial x^i})^\perp = \Lambda^{ij} (\frac{\partial^2 F^A}{\partial x^i \partial x^j} + \frac{\partial F^B}{\partial x^i} \frac{\partial F^C}{\partial x^j} \Gamma^A_{BC} - \tilde{\Gamma}^k_{ij} \frac{\partial F^A}{\partial x^k}) \frac{\partial}{\partial y^A}$$

where $\tilde{\Gamma}^k_{ij}$ is the Christoffel symbol of the induced metric on Σ.

By assumption, the embedding is given by the graph of f_t. We fix a coordinate system $\{x^i\}$ on Σ_1 and consider $F : \Sigma_1 \times [0, T) \to M$ given by

$$F(x^1, \cdots, x^n, t) = (x^1, \cdots, x^n, f^{n+1}, \cdots, f^{n+m}).$$

We shall use $i, j, k, l \cdots = 1 \cdots n$ and $\alpha, \beta, \gamma = n+1 \cdots n+m$ for the indices. Of course $f^\alpha = f^\alpha(x^1, \cdots, x^n, t)$ is time-dependent.

Therefore $\frac{\partial F}{\partial t} = \frac{\partial f^\alpha}{\partial t} \frac{\partial}{\partial y^\alpha}$ and

$$\Lambda^{ij}(\frac{\partial^2 F^A}{\partial x^i \partial x^j} + \frac{\partial F^B}{\partial x^i} \frac{\partial F^C}{\partial x^j} \Gamma^A_{BC}) \frac{\partial}{\partial y^A} = \Lambda^{ij}(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} \frac{\partial}{\partial y^\alpha} + \Gamma^l_{ij} \frac{\partial}{\partial y^l} + \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j} \Gamma^\alpha_{\beta\gamma} \frac{\partial}{\partial y^\alpha}).$$

Thus the mean curvature flow equation is equivalent to the normal part of

$$[\frac{\partial f^\alpha}{\partial t} - \Lambda^{ij}(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j} \Gamma^\alpha_{\beta\gamma})] \frac{\partial}{\partial y^\alpha} - \Lambda^{ij} \Gamma^l_{ij} \frac{\partial}{\partial y^l}$$

is zero.

Now given any vector $a^i \frac{\partial}{\partial y^i} + b^\alpha \frac{\partial}{\partial y^\alpha}$, the equation that the normal part being zero is equivalent to

$$b^\alpha - a^i \frac{\partial f^\alpha}{\partial x^i} = 0 \tag{2.1}$$

for each α. Therefore we obtain the evolution equation for f

$$\frac{\partial f^\alpha}{\partial t} - \Lambda^{ij}(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j} \Gamma^\alpha_{\beta\gamma} + \Gamma^k_{ij} \frac{\partial f^\alpha}{\partial x^k}) = 0. \tag{2.2}$$

where Λ^{ij} is the inverse to $g_{ij} + h_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j}$ and $g_{ij} = \langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \rangle$ and $h_{\alpha\beta} = \langle \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta} \rangle$ are the Riemannian metrics on Σ_1 and Σ_2, respectively. Γ^k_{ij} and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of g_{ij} and $h_{\alpha\beta}$ respectively.

(2.2) is a nonlinear parabolic system and the usual derivative estimates do not apply to this equations. However, the second author in [19] identifies a geometric quantity in terms of the derivatives of f^α that satisfies the maximum principle; this quantity and its evolution equation are recalled in the next section.

3 Two evolution equations

In this section, we recall two evolution equations along the mean curvature flow. The basic set-up is a mean curvature flow $F : \Sigma \times [0, T) \to M$ of an n dimensional submanifold Σ inside an $n+m$ dimensional Riemannian manifold M. Given any parallel tensor on M, we may consider the pull-back tensor by F_t and consider the evolution equation with respect to the time-dependent induced metric on $F_t(\Sigma) = \Sigma_t$. For the purpose of applying maximum principle, it suffices to derive the equation at a space-time point. We write all geometric quantities in terms of orthonormal frames keeping in mind all quantities are defined independent of choices of frames. At any point $p \in \Sigma_t$, we choose any orthonormal frames $\{e_i\}_{i=1\cdots n}$ for $T_p\Sigma_t$ and $\{e_\alpha\}_{\alpha=n+1\cdots n+m}$ for $N_p\Sigma_t$. The second fundamental form $h_{\alpha ij}$ is denoted by $h_{\alpha ij} = \langle \nabla^M_{e_i} e_j, e_\alpha \rangle$ and the mean curvature vector is denoted by $H_\alpha = \sum_i h_{\alpha ii}$. For any j, k, we pretend

$$h_{n+i,jk} = 0$$

if $i > m$.

When $M = \Sigma_1 \times \Sigma_2$ is the product of Σ_1 and Σ_2, we denote the projections by $\pi_1 : M \to \Sigma_1$ and $\pi_2 : M \to \Sigma_2$. By abusing notations, we also denote the differentials by $\pi_1 : T_pM \to T_{\pi_1(p)}\Sigma_1$ and $\pi_1 : T_pM \to T_{\pi_2(p)}\Sigma_2$ at any point $p \in M$. The volume form Ω of Σ_1 can be extended to a parallel n-form on M. For an oriented orthonormal basis $e_1 \cdots e_n$ of $T_p\Sigma$, $\Omega(e_1, \cdots, e_n) = \Omega(\pi_1(e_1), \cdots, \pi_1(e_n))$ is the Jacobian of the projection from $T_p\Sigma$ to $T_{\pi_1(p)}\Sigma_1$. This can also be considered as the pairing between the n-form Ω and the n-vector $e_1 \wedge \cdots \wedge e_n$ representing $T_p\Sigma$. We use $*\Omega$ to denote this function as p varies along Σ. By the implicit function theorem, $*\Omega > 0$ at p if and only if Σ is locally a graph over Σ_1 at p. The evolution equation of $*\Omega$ is calculated in Proposition 3.2 of [19].

When Σ is the graph of $f : \Sigma_1 \to \Sigma_2$, the equation at each point can be written in terms of singular values of df and special bases adapted to df. Denote the singular values of df, or eigenvalues of $(df)^T df$, by $\{\lambda_i\}_{i=1\cdots n}$. Let r denote the rank of df. We can rearrange them so that $\lambda_i = 0$ when i is greater than r. By singular value decomposition, there exist orthonormal bases $\{a_i\}_{i=1\cdots n}$ for $T_{\pi_1(p)}\Sigma_1$ and $\{a_\alpha\}_{\alpha=n+1\cdots n+m}$ for $T_{\pi_2(p)}\Sigma_2$ such that

$$df(a_i) = \lambda_i a_{n+i}$$

for i less than or equal to r and $df(a_i) = 0$ for i greater than r. Moreover,

$$e_i = \begin{cases} \frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i a_{n+i}) & \text{if } 1 \le i \le r \\ a_i & \text{if } r+1 \le i \le n \end{cases} \tag{3.1}$$

becomes an orthonormal basis for $T_p\Sigma$ and

$$e_{n+p} = \begin{cases} \frac{1}{\sqrt{1+\lambda_p^2}}(a_{n+p} - \lambda_p a_p) & \text{if } 1 \le p \le r \\ a_{n+p} & \text{if } r+1 \le p \le m \end{cases} \tag{3.2}$$

becomes an orthonormal basis for $N_p\Sigma$.

In terms of the singular values λ_i,

$$*\,\Omega = \frac{1}{\sqrt{\prod_{i=1}^n (1+\lambda_i^2)}} \tag{3.3}$$

With all the notations understood, the following result is essentially derived in Proposition 3.2 of [19] by noting that $(\ln *\Omega)_k = -(\sum_i \lambda_i h_{n+i,ik})$.

Proposition 3.1 *Suppose $M = \Sigma_1 \times \Sigma_2$ and Σ_1 and Σ_2 are compact Riemannian manifolds of constant curvature k_1 and k_2 respectively. With respect to the particular bases given by the singular value decomposition of df, $\ln *\Omega$ satisfies the following equation.*

$$\begin{aligned}(\frac{d}{dt} - \Delta)\ln *\Omega = &\sum_{\alpha,i,k} h_{\alpha ik}^2 + \sum_{k,i} \lambda_i^2 h_{n+i,ik}^2 + 2\sum_{k,i<j} \lambda_i\lambda_j h_{n+j,ik} h_{n+i,jk} \\ &+ \sum_i \frac{\lambda_i^2}{1+\lambda_i^2}\Big[(k_1+k_2)(\sum_{j\ne i}\frac{1}{1+\lambda_j^2}) + k_2(1-n)\Big]\end{aligned} \tag{3.4}$$

Next we recall the evolution equation of parallel two tensors from [15]. The calculation indeed already appears in [17]. The equation will be used later to obtain more refined information. Given a parallel two-tensor S on M, we consider the evolution of S restricted to Σ_t. This is a family of time-dependent symmetric two tensors on Σ_t.

Proposition 3.2 *Let S be a parallel two-tensor on M. Then the pull-back of S to Σ_t satisfies the following equation.*

$$
\begin{aligned}
(\frac{d}{dt}-\Delta)S_{ij} = & -h_{\alpha il}H_\alpha S_{lj} - h_{\alpha jl}H_\alpha S_{li} \\
& + R_{kik\alpha}S_{\alpha j} + R_{kjk\alpha}S_{\alpha i} \\
& + h_{\alpha kl}h_{\alpha ki}S_{lj} + h_{\alpha kl}h_{\alpha kj}S_{li} - 2h_{\alpha ki}h_{\beta kj}S_{\alpha\beta}
\end{aligned}
\tag{3.5}
$$

where Δ is the rough Laplacian on two-tensors over Σ_t and $S_{\alpha i} = S(e_\alpha, e_i)$, $S_{\alpha\beta} = S(e_\alpha, e_\beta)$, and $R_{kik\alpha} = R(e_k, e_i, e_k, e_\alpha)$ is the curvature of M.

The evolution equations (3.5) of S can be written in terms of evolving orthonormal frames as in Hamilton [8]. If the orthonormal frames

$$
F = \{F_1, \cdots, F_a, \cdots, F_n\}
\tag{3.6}
$$

are given in local coordinates by

$$
F_a = F_a^i \frac{\partial}{\partial x_i} .
$$

To keep them orthonormal, i.e. $g_{ij}F_a^iF_b^j = \delta_{ab}$, we evolve F by the formula

$$
\frac{\partial}{\partial t}F_a^i = g^{ij}g^{\alpha\beta}h_{\alpha jl}H_\beta F_a^l .
$$

Let $S_{ab} = S_{ij}F_a^iF_b^j$ be the components of S in F. Then S_{ab} satisfies the following equation

$$
\begin{aligned}
(\frac{d}{dt}-\Delta)S_{ab} = & \; R_{cac\alpha}S_{\alpha b} + R_{cbc\alpha}S_{\alpha a} \\
& + h_{\alpha cd}h_{\alpha ca}S_{db} + h_{\alpha cd}h_{\alpha cb}S_{da} \\
& - 2h_{\alpha ca}h_{\beta cb}S_{\alpha\beta} .
\end{aligned}
\tag{3.7}
$$

4 Preserving the distance-decreasing condition

In this section, we show the condition $|df| < 1$, or each singular value $\lambda_i < 1$, is preserved by the mean curvature flow. This result will not be used in proof of the Theorem A. But the proof of Theorem A depends on the computation in this section. The tangent space of $M = \Sigma_1 \times \Sigma_2$ is identified with $T\Sigma_1 \oplus T\Sigma_2$. Let π_1 and π_2 denote the projection onto the first and second summand in the splitting. We define the parallel symmetric two-tensor S by

$$S(X,Y) = \langle \pi_1(X), \pi_1(Y) \rangle \ - \langle \pi_2(X), \pi_2(Y) \rangle \tag{4.1}$$

for any $X, Y \in TM$.

Let Σ be the graph of $f : \Sigma_1 \to \Sigma_1 \times \Sigma_2$. S restricts to a symmetric two-tensor on Σ and we can represent S in terms of the orthonormal basis (3.1).

Let r denote the rank of df. By (3.1), it is not hard to check

$$\begin{aligned} &\pi_1(e_i) = \frac{a_i}{\sqrt{1+{\lambda_i}^2}} \ , \pi_2(e_i) = \frac{\lambda_i a_{n+i}}{\sqrt{1+{\lambda_i}^2}} \ \text{ for } 1 \le i \le r \ , \\ &\text{and } \pi_1(e_i) = \ a_i \ , \pi_2(e_i) = \ 0 \text{ for } r+1 \le i \le n. \end{aligned} \tag{4.2}$$

Similarly, by (3.2) we have

$$\begin{aligned} &\pi_1(e_{n+p}) = \frac{-\lambda_p a_p}{\sqrt{1+{\lambda_p}^2}} \ , \pi_2(e_{n+p}) = \frac{a_{n+p}}{\sqrt{1+{\lambda_p}^2}} \text{ for } 1 \le p \le r \ , \\ &\text{and } \pi_1(e_{n+p}) = 0 \ , \pi_2(e_{n+p}) = a_{n+p} \text{ for } r+1 \le p \le m \ . \end{aligned} \tag{4.3}$$

From the definition of S, we have

$$S(e_i, e_j) = \frac{1-\lambda_i^2}{1+{\lambda_i}^2} \delta_{ij} \ . \tag{4.4}$$

In particular, the eigenvalues of S are

$$\frac{1-{\lambda_i}^2}{1+{\lambda_i}^2}, \ i = 1 \cdots n \ . \tag{4.5}$$

Notice that S is positive-definite if and only if

$$|\lambda_i| < 1$$

for any singular value λ_i of df.

Now, at each point we express S in terms of the orthonormal basis $\{e_i\}_{i=1\cdots n}$ and $\{e_\alpha\}_{\alpha=n+1\cdots n+m}$. Let $I_{k\times k}$ denote a k by k identity matrix. Then S can be written in the block form

$$S = \Big(S(e_k, e_l)\Big)_{1\le k,l\le n+m} = \begin{pmatrix} B & 0 & D & 0 \\ 0 & I_{n-r\times n-r} & 0 & 0 \\ D & 0 & -B & 0 \\ 0 & 0 & 0 & -I_{m-r\times m-r} \end{pmatrix} \tag{4.6}$$

where B and D are r by r matrices with $B_{ij} = S(e_i, e_j) = \frac{1-\lambda_i^2}{1+\lambda_i^2}\delta_{ij}$ and $D_{ij} = S(e_i, e_{n+j}) = \frac{-2\lambda_i}{1+\lambda_i^2}\delta_{ij}$ for $1 \le i, j \le r$. We show that the positivity of S is preserved by the mean curvature flow. We remark that a similar positive definite tensor has been considered for the Lagrangian mean curvature flow in Smoczyk [14] and Smoczyk-Wang [15]. The following lemma shows that the distance decreasing condition is preserved by the mean curvature flow if $k_1 \ge |k_2|$.

Lemma 4.1 *The condition*

$$T_{ij} = S_{ij} - \epsilon g_{ij} > 0 \quad \textit{for some } \epsilon \ge 0 \tag{4.7}$$

is preserved by the mean curvature flow if $k_1 \ge |k_2|$.

Proof. We compute the evolution equation for T_{ij}. From Proposition (3.2) and

$$\frac{\partial}{\partial t} g_{ij} = -2h_{\alpha ij}H_\alpha\,,$$

we have

$$\begin{aligned}(\frac{d}{dt} - \Delta)T_{ij} = &-h_{\alpha il}H_\alpha T_{lj} - h_{\alpha jl}H_\alpha T_{li} + R_{kik\alpha}S_{\alpha j} + R_{kjk\alpha}S_{\alpha i} \\ &+ h_{\alpha kl}h_{\alpha ki}T_{lj} + h_{\alpha kl}h_{\alpha kj}T_{li} + 2\epsilon h_{\alpha ki}h_{\alpha kj} - 2h_{\alpha ki}h_{\beta kj}S_{\alpha\beta}.\end{aligned} \tag{4.8}$$

To apply Hamilton's maximum principle, it suffices to prove that $N_{ij}V^iV^j \ge 0$ for any null eigenvector V of T_{ij}, where N_{ij} is the right hand side of (4.8). Since V is a null eigenvector of T_{ij}, it satisfies $\sum_j T_{ij}V^j = 0$ for any i, and thus $N_{ij}V^iV^j$ is equal to

$$2\epsilon h_{\alpha ki}h_{\alpha kj}V^iV^j + 2R_{kik\alpha}S_{\alpha j}V^iV^j - 2h_{\alpha ki}h_{\beta kj}S_{\alpha\beta}V^iV^j\,. \tag{4.9}$$

Obviously, the first term of (4.9) is nonnegative. Applying the relation in (4.6) to the last term of (4.9) we obtain

$$-2h_{\alpha ki}h_{\beta kj}S_{\alpha\beta}V^iV^j = \sum_{1\leq p,q\leq r} 2h_{n+pki}h_{n+qkj}S_{pq}V^iV^j + \sum_{r+1\leq p,q\leq m} 2h_{n+pki}h_{n+qkj}V^iV^j\,.$$

Since $T_{pq} \geq 0$ implies that $S_{pq} \geq \epsilon g_{pq}$, we obtain $-2h_{\alpha ki}h_{\beta kj}S_{\alpha\beta}V^iV^j \geq 0$. In the next lemma we show that $R_{kik\alpha}S_{\alpha j}$ is nonnegative definite whenever S_{ij} is under the curvature assumption $k_1 \geq |k_2|$.

□

Lemma 4.2

$$R_{kik\alpha}S_{\alpha j} = \frac{\lambda_i^2}{(1+\lambda_i^2)^2}\left[(k_1-k_2)(n-1)+(k_1+k_2)\sum_{k\neq i}\frac{1-\lambda_k^2}{1+\lambda_k^2}\right]\delta_{ij}. \quad (4.10)$$

Proof. We follow the calculation of the curvature terms in [19].

$$\begin{aligned}
&\sum_k R(e_\alpha, e_k, e_k, e_i)\\
=&\sum_k R_1(\pi_1(e_\alpha),\pi_1(e_k),\pi_1(e_k),\pi_1(e_i)) + R_2(\pi_2(e_\alpha),\pi_2(e_k),\pi_2(e_k),\pi_2(e_i))\\
=&\sum_k k_1\Big[\langle\pi_1(e_\alpha),\pi_1(e_k)\rangle\langle\pi_1(e_k),\pi_1(e_i)\rangle - \langle\pi_1(e_\alpha),\pi_1(e_i)\rangle\langle\pi_1(e_k),\pi_1(e_k)\rangle\Big]\\
&+k_2\Big[\langle\pi_2(e_\alpha),\pi_2(e_k)\rangle\langle\pi_2(e_k),\pi_2(e_i)\rangle - \langle\pi_2(e_\alpha),\pi_2(e_i)\rangle\langle\pi_2(e_k),\pi_2(e_k)\rangle\Big]\,.
\end{aligned}$$

Notice that $\langle\pi_2(X),\pi_2(Y)\rangle = \langle X,Y\rangle - \langle\pi_1(X),\pi_1(Y)\rangle$ since $T\Sigma_1 \perp T\Sigma_2$. Therefore

$$\begin{aligned}
&\sum_k R(e_\alpha, e_k, e_k, e_i)\\
=&\sum_k (k_1+k_2)\Big[\langle\pi_1(e_\alpha),\pi_1(e_k)\rangle\langle\pi_1(e_k),\pi_1(e_i)\rangle - \langle\pi_1(e_\alpha),\pi_1(e_i)\rangle|\pi_1(e_k)|^2\Big]\\
&+k_2(n-1)\langle\pi_1(e_\alpha),\pi_1(e_i)\rangle
\end{aligned}$$

Now use $\pi_1(e_\alpha) = -\lambda_p \pi_1(e_p)\delta_{\alpha,n+p}$ and $S(e_j, e_{n+p}) = -\frac{2\lambda_j \delta_{jp}}{1+\lambda_j{}^2}$ in (4.6), we have

$$\sum_{\alpha,k} R_{kik\alpha} S_{\alpha j} = -\sum_{p,k} R_{n+p,kki} S_{n+p,j}$$
$$= \sum_{p,k} \Big\{ \lambda_p (k_1 + k_2) \left[\langle \pi_1(e_p), \pi_1(e_k)\rangle \langle \pi_1(e_k), \pi_1(e_i)\rangle - \langle \pi_1(e_p), \pi_1(e_i)\rangle |\pi_1(e_k)|^2 \right]$$
$$+ \lambda_p k_2 (n-1) \langle \pi_1(e_p), \pi_1(e_i)\rangle \Big\} S_{n+p,j}$$
$$= -\frac{2\lambda_i^2}{1+\lambda_i{}^2} \Big\{ (k_1+k_2) \Big[\frac{\delta_{ij}}{(1+\lambda_i{}^2)^2} - \frac{\delta_{ij}}{1+\lambda_i^2} \sum_k |\pi_1(e_k)|^2 \Big]$$
$$+ k_2(n-1) \frac{\delta_{ij}}{1+\lambda_i{}^2} \Big\}.$$

Recall that $|\pi_1(e_k)|^2 = \frac{1}{1+\lambda_k^2}$ and we obtain

$$R_{kik\alpha} S_{\alpha j} = \frac{2\lambda_i^2 \delta_{ij}}{(1+\lambda_i^2)^2} \left[(k_1+k_2) (\sum_{k \neq i} \frac{1}{1+\lambda_k^2}) + k_2(1-n) \right].$$

This can be further simplified by noting

$$(k_1+k_2)(\sum_{k\neq i} \frac{1}{1+\lambda_k^2}) + k_2(1-n) = \frac{(k_1-k_2)(n-1)}{2} + (k_1+k_2) \sum_{k \neq i} \frac{1-\lambda_k^2}{2(1+\lambda_k^2)} \tag{4.11}$$

where we use the following identity for each i

$$(\sum_{k\neq i} \frac{1}{1+\lambda_k^2}) - \frac{n-1}{2} = \sum_{k \neq i} (\frac{1}{1+\lambda_k^2} - \frac{1}{2}) = \sum_{k\neq i} \frac{1-\lambda_k^2}{2(1+\lambda_k^2)}.$$

□

5 Preserving the area-decreasing condition

In this section, we show that the area decreasing condition is preserved along the mean curvature flow. In the following, we require that $n = dim(\Sigma_1) \geq 2$. By (4.5), the sum of any two eigenvalues of S is

$$\frac{1-\lambda_i^2}{1+\lambda_i^2} + \frac{1-\lambda_j^2}{1+\lambda_j^2} = \frac{2(1-\lambda_i^2\lambda_j^2)}{(1+\lambda_i^2)(1+\lambda_j^2)} \ . \tag{5.1}$$

Therefore the area decreasing condition $|\lambda_i \lambda_j| < 1$ for $i \neq j$ is equivalent to the two-positivity of S, i.e. the sum of any two eigenvalues is positive. We remark that curvature operator being two-positive is preserved by the Ricci flow, see Chen [1] or Hamilton [8] for detail.

The two-positivity of a symmetric two tensor P can be related to the convexity of another tensor $P^{[2]}$ associated with P. The following notation is adopted from Caffarelli-Nirenberg-Spruck [3]. Let P be a self-adjoint operator on an n-dimensional inner product space. From P we can construct a new self-adjoint operator

$$P^{[k]} = \sum_{i=1}^{k} 1 \otimes \cdots \otimes \underset{i}{P} \otimes \cdots \otimes 1$$

acting on the exterior powers Λ^k by

$$P^{[k]}(\omega_1 \wedge \cdots \wedge \omega_k) = \sum_{i=1}^{k} \omega_1 \wedge \cdots \wedge P(\omega_i) \wedge \cdots \wedge \omega_k \ .$$

With the definition of $P^{[k]}$, we have the following lemma.

Lemma 5.1 *Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be the eigenvalues of P with corresponding eigenvectors $v_1 \cdots v_n$. Then $P^{[k]}$ has eigenvalues $\mu_{i_1} + \cdots + \mu_{i_k}$ and eigenvectors $v_{i_1} \wedge \cdots \wedge v_{i_k}$, $i_1 < i_2 \cdots < i_k$.*

Recall that the Riemannian metric g and S are both in $T\Sigma \odot T\Sigma$, the space of symmetric two tensor on Σ. We can identify S with a self-adjoint operator on the tangent bundle through the metric g. Therefore $S^{[2]}$ and $g^{[2]}$ are both sections of $(\Lambda^2(T\Sigma))^* \odot \Lambda^2(T\Sigma)$ associated to S and g respectively. We shall use orthonormal frames in the following calculation; this has the advantage that g is the identity matrix and we will not distinguish lower index and upper index. With the above interpretation and (5.1), we have the following lemma.

Lemma 5.2 *The area decreasing condition is equivalent to the convexity of $S^{[2]}$.*

To show that the area decreasing condition is preserved, it suffices to prove that the convexity of $S^{[2]}$ is preserved. In fact, we prove the stronger result that the convexity of $S^{[2]} - \epsilon g^{[2]}$ for $\epsilon > 0$ is preserved.

We compute the evolution equation of $S^{[2]}-\epsilon g^{[2]}$ in terms of the evolving orthonormal frames $\{F_a\}_{a=1\cdots n}$ introduced earlier in (3.6). We will use indices $a,b,\cdots$ to denote components in the evolving frames. Denote $S_{ab}=S(F_a,F_b)$ and $g_{ab}=g(F_a,F_b)=\delta_{ab}$. Since $\{F_a\wedge F_b\}_{a<b}$ form a basis for $\Lambda^2T\Sigma$, we have

$$\begin{aligned} S^{[2]}(F_a\wedge F_b) &= S(F_a)\wedge F_b+F_a\wedge S(F_b)=S_{ac}F_c\wedge F_b+F_a\wedge S_{ac}F_c\\ &= \sum_{c<d}(S_{ac}\delta_{bd}+S_{bd}\delta_{ac}-S_{ad}\delta_{bc}-S_{bc}\delta_{ad})F_c\wedge F_d \quad \text{and}\\ g^{[2]}(F_a\wedge F_b) &= \sum_{c<d}(2\delta_{ac}\delta_{bd}-2\delta_{ad}\delta_{bc})F_c\wedge F_d\,. \end{aligned} \tag{5.2}$$

We denote $S^{[2]}_{(ab)(cd)}=(S_{ac}\delta_{bd}+S_{bd}\delta_{ac}-S_{ad}\delta_{bc}-S_{bc}\delta_{ad})$ and $g^{[2]}_{(ab)(cd)}=2\delta_{ac}\delta_{bd}-2\delta_{ad}\delta_{bc}$. Thus the evolution equation of $S^{[2]}-\epsilon g^{[2]}$ in terms of the evolving orthonormal frames is

$$\begin{aligned} &(\frac{d}{dt}-\Delta)(S_{ac}\delta_{bd}+S_{bd}\delta_{ac}-S_{ad}\delta_{bc}-S_{bc}\delta_{ad}-2\epsilon\delta_{ac}\delta_{bd}+2\epsilon\delta_{ad}\delta_{bc})\\ &=\ R_{eae\alpha}S_{\alpha c}\delta_{bd}+R_{ece\alpha}S_{\alpha a}\delta_{bd}+R_{ebe\alpha}S_{\alpha d}\delta_{ac}+R_{ede\alpha}S_{\alpha b}\delta_{ac}\\ &-R_{eae\alpha}S_{\alpha d}\delta_{bc}-R_{ede\alpha}S_{\alpha a}\delta_{bc}-R_{ebe\alpha}S_{\alpha c}\delta_{ad}-R_{ece\alpha}S_{\alpha b}\delta_{ad}\\ &+\ h_{\alpha ef}h_{\alpha ea}S_{fc}\delta_{bd}+h_{\alpha ef}h_{\alpha ec}S_{fa}\delta_{bd}+h_{\alpha ef}h_{\alpha eb}S_{fd}\delta_{ac}+h_{\alpha ef}h_{\alpha ed}S_{fb}\delta_{ac}\\ &-\ h_{\alpha ef}h_{\alpha ea}S_{fd}\delta_{bc}-h_{\alpha ef}h_{\alpha ed}S_{fa}\delta_{bc}-h_{\alpha ef}h_{\alpha eb}S_{fc}\delta_{ad}-h_{\alpha ef}h_{\alpha ec}S_{fb}\delta_{ad}\\ &-\ 2h_{\alpha ea}h_{\beta ec}S_{\alpha\beta}\delta_{bd}-2h_{\alpha eb}h_{\beta ed}S_{\alpha\beta}\delta_{ac}+2h_{\alpha ea}h_{\beta ed}S_{\alpha\beta}\delta_{bc}+2h_{\alpha eb}h_{\beta ec}S_{\alpha\beta}\delta_{ad}\,. \end{aligned} \tag{5.3}$$

Now, we are ready to prove that the area decreasing condition is preserved along the mean curvature flow.

Lemma 5.3 *Under the assumption of Theorem A, with S defined in (4.1)and $S^{[2]}$ defined in (5.2), suppose there exists an $\epsilon>0$ such that*

$$S^{[2]}-\epsilon g^{[2]}\geq 0 \tag{5.4}$$

holds on the initial graph. Then this is preserved along the mean curvature flow.

Proof. Set

$$M_\eta=S^{[2]}-\epsilon g^{[2]}+\eta t g^{[2]}\,.$$

Suppose the mean curvature flow exists on $[0,T)$. Consider any $T_1<T$, it suffices to prove that $M_\eta>0$ on $[0,T_1]$ for all $\eta<\frac{\epsilon}{2T_1}$. If not, there will be a first time $0 < t_0 \leq T_1$ where $M_\eta = S^{[2]} - \epsilon g^{[2]} + \eta t g^{[2]}$ is nonnegative definite and has a null eigenvector $V = V^{ab} F_a \wedge F_b$ at some point $x_0 \in \Sigma_{t_0}$. We extend V^{ab} to a parallel tensor in a neighborhood of x_0 along geodesic emanating out of x_0, and defined V^{ab} on $[0, T)$ independent of t. Define $f = \sum_{a<b,c<d} V^{ab} M_{\eta_{(ab)(cd)}} V^{cd}$, then by (5.2), f equals

$$\sum_{a<b,c<d} \left(S_{ac}g_{bd} + S_{bd}g_{ac} - S_{ad}g_{bc} - S_{bc}g_{ad} + 2(\eta t - \epsilon)(g_{ac}g_{bd} - g_{ad}g_{bc})\right) V^{ab}V^{cd} \; .$$

At (x_0, t_0), we have $f = 0$, $\nabla f = 0$ and $(\frac{d}{dt} - \Delta)f \leq 0$ where ∇ denotes the covariant derivative and Δ denotes the Laplacian on Σ_{t_0}.

We may assume that at (x_0, t_0) the orthonormal frames $\{F_a\}$ is given by $\{e_i\}$ in (3.1). In the following, we use the orthonormal basis $\{e_i\}$ to write down the condition $f = 0$ and $\nabla f = 0$ at (x_0, t_0). The basis $\{e_i\}$ diagonalizes S with eigenvalues $\{\lambda_i\}$ and we order $\{\lambda_i\}$ such that

$$\lambda_1^2 \geq \lambda_2^2 \geq \cdots \geq \lambda_n^2$$

and

$$S_{nn} = \frac{1 - \lambda_n^2}{1 + \lambda_n^2} \geq \cdots \geq S_{22} = \frac{1 - \lambda_2^2}{1 + \lambda_2^2} \geq S_{11} = \frac{1 - \lambda_1^2}{1 + \lambda_1^2} \; . \tag{5.5}$$

It follows from Lemma (5.1) that $\{e_i \wedge e_j\}_{i<j}$ are the eigenvectors of M_η. Thus we may assume that

$$V = e_1 \wedge e_2. \tag{5.6}$$

At (x_0, t_0), the condition $f = 0$ is the same as

$$S_{11} + S_{22} = 2\epsilon - 2\eta t_0 > 0 \; . \tag{5.7}$$

This is equivalent to

$$\frac{2(1 - \lambda_1^2 \lambda_2^2)}{(1 + \lambda_1^2)(1 + \lambda_2^2)} = 2(\epsilon - \eta t_0) > 0.$$

Thus

$$\lambda_1 \lambda_2 < 1 \text{ and } \lambda_i < 1 \text{ for } i \geq 3 \; . \tag{5.8}$$

Next, we compute the covariant derivative of the restriction of S on Σ.

$$\begin{aligned}
& (\nabla_{e_i} S)(e_j, e_k) \\
= \; & e_i(S(e_j, e_k)) - S(\nabla_{e_i} e_j, e_k) - S(e_j, \nabla_{e_i} e_k) \\
= \; & S(\nabla^M_{e_i} e_j - \nabla_{e_i} e_j, e_k) + S(e_j, \nabla^M_{e_i} e_k - \nabla_{e_i} e_k) \\
= \; & h_{\alpha ij} S_{\alpha k} + h_{\beta ik} S_{\beta j} \; .
\end{aligned}$$

So

$$S_{jk,i} = h_{\alpha ij} S_{\alpha k} + h_{\beta ik} S_{\beta j} \; .$$

Recall that V_{ab} is parallel at (x_0, t_0) , $V^{12} = 1$ and all other components of V^{ab} is zero. At (x_0, t_0), $\nabla f = 0$ is equivalent to

$$\begin{aligned}
0 = & \sum_{i<j,k<l} \nabla_{e_p}((S_{ik}\delta_{jl} + S_{jl}\delta_{ik} - S_{il}\delta_{jk} - S_{jk}\delta_{il} + 2(\eta t - \epsilon)(\delta_{ik}\delta_{jl} - \epsilon\delta_{il}\delta_{jk}))V^{ij}V^{kl}) \\
& = \nabla_{e_p} S_{11} + \nabla_{e_p} S_{22} \\
& = 2h_{\alpha p1} S_{\alpha 1} + 2h_{\beta p2} S_{\beta 2} \; .
\end{aligned}$$

Since $S_{n+q,l} = -\frac{2\lambda_q \delta_{ql}}{1+\lambda_q^2}$, we have

$$\frac{\lambda_1}{1+\lambda_1^2} h_{n+1,p1} + \frac{\lambda_2}{1+\lambda_2^2} h_{n+2,p2} = 0 \tag{5.9}$$

for any p.

By (5.3), at (x_0, t_0), we have

$$\begin{aligned}
(\frac{d}{dt} - \Delta) f = & \; 2\eta + \; 2R_{k1k\alpha} S_{\alpha 1} + 2R_{k2k\alpha} S_{\alpha 2} + 2h_{\alpha kj} h_{\alpha k1} S_{j1} + 2h_{\alpha kj} h_{\alpha k2} S_{j2} \\
& - \; 2h_{\alpha k1} h_{\beta k1} S_{\alpha\beta} - 2h_{\alpha k2} h_{\beta k2} S_{\alpha\beta}.
\end{aligned} \tag{5.10}$$

The ambient curvature term can be calculated using Lemma 4.2 and we derive

$$\begin{aligned}
& \sum_{k,\alpha} R_{k1k\alpha} S_{\alpha 1} + R_{k2k\alpha} S_{\alpha 2}. \\
= & \; (k_1 - k_2)(n-1) \sum_{i=1}^{2} \frac{\lambda_i^2}{(1+\lambda_i^2)^2} + (k_1 + k_2) \sum_{i=1}^{2} \frac{\lambda_i^2}{(1+\lambda_i^2)^2} \left[\sum_{j \neq i} \frac{1 - \lambda_j^2}{(1+\lambda_j^2)} \right] \; .
\end{aligned} \tag{5.11}$$

This can be simplified as

$$
\begin{aligned}
&(k_1-k_2)(n-1)\sum_{i=1}^{2}\frac{\lambda_i^2}{(1+\lambda_i^2)^2}+(k_1+k_2)\sum_{i=1}^{2}\frac{\lambda_i^2}{(1+\lambda_i^2)^2}\left[\sum_{j>3}\frac{1-\lambda_j^2}{(1+\lambda_j^2)}\right]\\
&+(k_1+k_2)\left[\frac{\lambda_1^2}{(1+\lambda_1^2)^2}\frac{1-\lambda_2^2}{(1+\lambda_2^2)}+\frac{\lambda_2^2}{(1+\lambda_2^2)^2}\frac{1-\lambda_1^2}{(1+\lambda_1^2)}\right]\\
&=(k_1-k_2)(n-1)\sum_{i=1}^{2}\frac{\lambda_i^2}{(1+\lambda_i^2)^2}+(k_1+k_2)\sum_{i=1}^{2}\frac{\lambda_i^2}{(1+\lambda_i^2)^2}\left[\sum_{j>3}\frac{1-\lambda_j^2}{(1+\lambda_j^2)}\right]\\
&+(k_1+k_2)\left[\frac{(\lambda_1^2+\lambda_2^2)(1-\lambda_1^2\lambda_2^2)}{(1+\lambda_1^2)^2(1+\lambda_2^2)^2}\right]\ .
\end{aligned}
\tag{5.12}
$$

This is nonnegative by equation (5.8).

Using the relations in (4.6) again, the last four terms on the right hand side of (5.10) can be rewritten as

$$
\begin{aligned}
&\sum_{p,k}2h_{n+p,k1}^2S_{11}+2h_{n+p,k2}^2S_{22}+2h_{n+p,k1}^2S_{pp}+2h_{n+p,k2}^2S_{pp}\\
=&\ \sum_{k}(2h_{n+1,k1}^2S_{11}+2h_{n+2,k1}^2S_{11}+2h_{n+1,k2}^2S_{22}+2h_{n+2,k2}^2S_{22}\\
+&\ 2h_{n+1,k1}^2S_{11}+2h_{n+2,k1}^2S_{22}+2h_{n+1,k2}^2S_{11}+2h_{n+2,k2}^2S_{22})\\
+&\ \sum_{q\geq 3,k}2h_{n+q,k1}^2S_{11}+2h_{n+q,k2}^2S_{22}+2h_{n+q,k1}^2S_{qq}+2h_{n+q,k2}^2S_{qq}\ .
\end{aligned}
\tag{5.13}
$$

Since $S_{ii}\geq S_{11}$ for $i\geq 2$, it is clear that (5.13) is nonnegative if $S_{11}\geq 0$. Otherwise, from (5.7), we may assume that

$$S_{11}<0,\ \ S_{22}>0 \text{ and } S_{11}+S_{22}>0\ . \tag{5.14}$$

In particular, we have $\lambda_2^2<\lambda_1^2$ and $\lambda_1^2\lambda_2^2<1$. From (5.9), we have

$$h_{n+1,p1}^2=\frac{\lambda_2^2(1+\lambda_1^2)^2}{\lambda_1^2(1+\lambda_2^2)^2}h_{n+2,p2}^2.$$

Since $\lambda_2^2<\lambda_1^2$ and $\lambda_1^2\lambda_2^2<1$, we have $\frac{\lambda_2^2(1+\lambda_1^2)^2}{\lambda_1^2(1+\lambda_2^2)^2}<1$. Thus

$$h_{n+1,p1}^2\leq h_{n+2,p2}^2 \text{ for all } p\ \geq 1. \tag{5.15}$$

Recall that $S_{qq} \geq S_{22}$ for $q \geq 3$. The right hand side of (5.13) can be regrouped as

$$\sum_k \left[(4h^2_{n+1,k1}S_{11} + 4h^2_{n+2,k2}S_{22}) + 2h^2_{n+2,k1}(S_{11} + S_{22}) + 2h^2_{n+1,k2}(S_{11} + S_{22}) \right]$$
$$+ \sum_{q \geq 3,k} \left[2h^2_{n+q,k1}(S_{11} + S_{qq}) + 2h^2_{n+q,k2}(S_{22} + S_{qq}) \right] .$$

This is nonnegative by (5.5) ,(5.14), and (5.15). Thus, we have $(\frac{d}{dt} - \Delta)f \geq 2\eta > 0$ at (x_0, t_0) and this is a contradiction. $\square$

Remark: The condition $S^{[2]} - \epsilon g^{[2]} \geq 0$ is equivalent to $\frac{(1-\lambda_i^2\lambda_j^2)}{(1+\lambda_i^2)(1+\lambda_j^2)} \geq \epsilon$ for all $i \neq j$. In particular, we have $\lambda_i^2 \leq \frac{1-\epsilon}{\epsilon}$. This implies that the Lipschitz norm of f is preserved along the mean curvature flow.

6 Long time existence and convergence

In this section, we prove Theorem A using the evolution equation (3.4) of $\ln *\Omega$.

Proof of Theorem A. Since $|\lambda_i\lambda_j| < 1$ for $i \neq j$ and Σ_1 is compact, we can find an $\epsilon > 0$ such that $\frac{(1-\lambda_i^2\lambda_j^2)}{(1+\lambda_i^2)(1+\lambda_j^2)} \geq \epsilon$ for all $i \neq j$. By Lemma (5.3), the condition $\frac{(1-\lambda_i^2\lambda_j^2)}{(1+\lambda_i^2)(1+\lambda_j^2)} \geq \epsilon$ for all $i \neq j$ is preserved along the mean curvature flow. In particular, we have $|\lambda_i\lambda_j| \leq \sqrt{1-\epsilon}$ and $\lambda_i^2 \leq \frac{1-\epsilon}{\epsilon}$. This implies Σ_t remains the graph of a map $f_t : \Sigma_1 \to \Sigma_2$ whenever the flow exists. Each f_t has uniformly bounded $|df_t|$.

We look at the evolution equation (3.4) of $\ln *\Omega$. The quadratic terms of the second fundamental form in equation (3.4) is

$$\sum_{\alpha,i,k} h^2_{\alpha ik} + \sum_{k,i} \lambda_i^2 h^2_{n+i,ik} + 2\sum_{k,i<j} \lambda_i\lambda_j h_{n+j,ik}h_{n+i,jk}$$
$$= \delta|A|^2 + \sum_{k,i} \lambda_i^2 h^2_{n+i,ik} + (1-\delta)|A|^2 + 2\sum_{k,i<j} \lambda_i\lambda_j h_{n+j,ik}h_{n+i,jk} \; .$$

Let $1 - \delta = \sqrt{1-\epsilon}$. Using $|\lambda_i\lambda_j| \leq 1 - \delta$, we conclude that this term is bounded below by $\delta|A|^2$.

By equation (4.11), the curvature term in (3.4) equals

$$\frac{(k_1-k_2)(n-1)}{2}\sum_{i=1}^{n}\frac{\lambda_i^2}{1+\lambda_i^2}+(k_1+k_2)\sum_{i=1}^{n}\frac{\lambda_i^2}{1+\lambda_i^2}\left[\sum_{j\neq i}\frac{1-\lambda_j^2}{2(1+\lambda_j^2)}\right]. \tag{6.1}$$

The second term on the right hand side of (6.1) can be simplified as

$$\begin{aligned}\sum_{i=1}^{n}\frac{\lambda_i^2}{1+\lambda_i^2}\left[\sum_{j\neq i}\frac{1-\lambda_j^2}{2(1+\lambda_j^2)}\right] &= \sum_{i=1}^{n}\sum_{i\neq j}\frac{\lambda_i^2-\lambda_i^2\lambda_j^2}{2(1+\lambda_i^2)(1+\lambda_j^2)}\\ =\sum_{i<j}\frac{\lambda_i^2+\lambda_j^2-2\lambda_i^2\lambda_j^2}{2(1+\lambda_i^2)(1+\lambda_j^2)}.\end{aligned} \tag{6.2}$$

This is non-negative because $|\lambda_i\lambda_j| \leq 1-\delta$. Thus $\ln *\Omega$ satisfies the following differential inequality with $k_1 \geq |k_2|$:

$$\frac{d}{dt}\ln *\Omega \geq \Delta \ln *\Omega + \delta|A|^2 . \tag{6.3}$$

According to the maximum principle for parabolic equations, $\min_{\Sigma_t}\ln *\Omega$ is nondecreasing in time. In particular, $*\Omega \geq min_{\Sigma_0} * \Omega = \Omega_0$ is preserved and $*\Omega$ has a positive lower bound. Let $u = \frac{\ln *\Omega - \ln\Omega_0 + c}{-\ln\Omega_0 + c}$ where c is a positive number such that $-\ln\Omega_0 + c > 0$. Recall that $0 < *\Omega \leq 1$. This implies that $0 < u \leq 1$ and u satisfies the following differential inequality

$$\frac{d}{dt}u \geq \Delta u + \frac{\delta}{-\ln\Omega_0 + c}|A|^2.$$

Because u is also invariant under parabolic dilation, it follows from the blow-up analysis in the proof of Theorem A [19] that the mean curvature flow of the graph of f remains a graph and exists for all time under the assumption that $k_1 \geq |k_2|$.

Using $\lambda_i^2 \leq \frac{1-\epsilon}{\epsilon}$ and $\lambda_i\lambda_j \leq \sqrt{1-\epsilon}$, it is not hard to show

$$(k_1+k_2)\sum_{i<j}\frac{\lambda_i^2+\lambda_j^2-2\lambda_i^2\lambda_j^2}{2(1+\lambda_i^2)(1+\lambda_j^2)} \geq c_1\sum_{i=1}^{n}\lambda_i^2 \geq c_1 \ln\prod_{i=1}^{n}(1+\lambda_i^2) \tag{6.4}$$

where c_1 is a constant that depends on ϵ, k_1 and k_2.

Recall equation (3.3) and we obtain

$$\frac{d}{dt} \ln *\Omega \geq \Delta \ln *\Omega - c_3 \ln *\Omega \ .$$

By the comparison theorem for parabolic equations, $\min_{\Sigma_t} \ln *\Omega$ is non-decreasing in t and $\min_{\Sigma_t} \ln *\Omega \to 0$ as $t \to \infty$. This implies that $\min_{\Sigma_t} *\Omega \to 1$ and $\max |\lambda_i| \to 0$ as $t \to \infty$. We can then apply Theorem B in [19] to conclude smooth convergence to a constant map at infinity.

$\square$

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