On Graphic Bernstein Type Results in Higher Codimension

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Abstract

Let Σ be a minimal submanifold of $\mathbb{R}^{n+m}$ that can be represented as the graph of a smooth map $f : \mathbb{R}^n \mapsto \mathbb{R}^m$. We apply a formula we derived in the study of mean curvature flow to obtain conditions under which Σ must be an affine subspace. Our result covers all known ones in the general case. The conditions are stated in terms of the singular values of df.

1 Introduction

The well-known Bernstein theorem asserts any complete minimal surface that can be written as the graph of a function on $\mathbb{R}^2$ must be a plane. This result has been generalized to $\mathbb{R}^n$, for $n \leq 7$ and general dimension under various growth condition, see [2] and the reference therein for the codimension one case. In this note, we study the higher codimension case, i.e. minimal submanifolds in $\mathbb{R}^{n+m}$ that can be written as graphs of vector-valued functions $f : \mathbb{R}^n \mapsto \mathbb{R}^m$.

For higher codimension Bernstein type problems, there are general results of [3], [5] and [7]. The idea in these papers is to find a subharmonic function whose vanishing implies Σ is totally geodesic. Under the assumption that f is of bounded gradient, one can perform blow down analysis to reduce to minimal cone case. Maximum principle together with Allard regularity theorem then complete the argument. A fundamental fact is the Gauss map of a minimal submanifold is a harmonic map, so any convex function on the Grassmannian renders a subharmonic function. The work in [3], [5], and [7] consists of delicate analysis of the geometry of Grassmannian in order to locate the maximal region where a convex function exists . The condition for Bernstein type reuslts is in terms of the function

$$*\Omega_1 = \frac{1}{\sqrt{\det(I + (df)^t df)}}$$

This can be considered as the inner product of the tangent space of Σ and the domain $\mathbb{R}^n$. In particular, in [3] and [5], it was shown,

$$*\Omega_1 \geq K' > \cos^p(\frac{\pi}{2\sqrt{2}p})$$

where $p = \min(n, m)$ implies Σ is an affine subspace. This bound is later improved in [7] to

$$*\Omega_1 \geq K'' > \frac{1}{2}$$

We adopt a direct approach to calculate explicitly the Laplacian of $\ln *\Omega$. The calculation first appeared to us in [13] in the context of mean curvature flow. The formula determines precisely when $\ln *\Omega$ is a superharmonic function.

Theorem A *Let Σ be a minimal submanifold of $\mathbb{R}^{n+m}$. Suppose Σ can be written as the graph of a smooth map $f : \mathbb{R}^n \mapsto \mathbb{R}^m$. Denote the singular values of df by $\lambda_i, i = 1, \cdots, n$. If there exist $\delta, K > 0$ such that $|\lambda_i\lambda_j| \leq 1-\delta$ for any $i \neq j$ and $*\Omega_1 = \frac{1}{\sqrt{\det(I+(df)^t df)}} \geq K$, then Σ is an affine subspace.*

Notice that K is not necessarily bounded from below. At the end of §3, we discuss how Theorem A implies the results in [3], [5] or [7].

We remark Theorem A also holds for submanifolds with parallel mean curvature vector. This condition for minimal submanifolds is not the optimal one. We state it first because it is simple to apply and seems natural (see the remark at the end of the introduction on the geometric meaning).

The optimal condition, which is in terms of λ_i again, is somewhat more complicate. It will be described explicitly in §2.

Theorem B *Under the assumption of Theorem A. There exists an optimal condition in terms of the singular values of df such that Σ is an affine subspace whenever f satisfies this optimal condition*

Theorem B also implies the result of [1] and [3] that any three-dimensional minimal cone is flat and thus is sharper than Theorem A.

The higher codimension Bernstein type result is not expected to be true in the most generality due to the counterexample of Lawson-Ossermann [9].

Notice that in codimension one case, any graphic minimal hypersurface is stable. It is thus natural to impose stable assumption in higher codimension. A special category of minimal submanifolds are calibrated submanifolds studied by Harvey and Lawson [6]. They are volume minimizing and thus stable. These include holomorphic submanifolds, special Lagrangian submanifolds, assoicative, coassociative, and Cayley submanifold. The special Lagrangian case is studied in [4],[10] [8] [14], and [11]. The Lagrangian condition implies the existence of a potential function F such that $f = \nabla F$. In terms of PDE, the Bernstein type problem asks when an entire solution of

$$Im(\det(I + \sqrt{-1} D^2 F)) = constant \tag{1.1}$$

is a quadratic polynomial. In this case, the condition of Theorem A or B can be stated in terms of the eigenvalues of D^2F.

We remark that conditions on the singular values of df or eigenvalues of D^2F depend on the choice of a subspace (usually a calibrated one) over which Σ is written as a graph. Such conditions correspond to regions in the relevant Grassmannian. A more invariant description is to require the image of the Gauss map lies in the orbit of one of these regions under the respective holonomy group actions.

To demonstrate, let us compare the condition $|\lambda_1\lambda_2| < 1$ and $(1+\lambda_1^2)(1+\lambda_2^2) < 4$ (or $*\Omega_1 = \frac{1}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}} > \frac{1}{2}$) when $n = m = 2$. It is clear the former condition is weaker than the latter one. In this case, $G(2,4) = S^2(\frac{1}{\sqrt{2}}) \times S^2(\frac{1}{\sqrt{2}})$. In $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$, denote the coordinates on the domain $\mathbb{R}^2$ by x^1, x^2 and the coordinates on the target $\mathbb{R}^2$ by y^1, y^2. Then the forms ω_1, ω_2

$$\omega_1 = \frac{1}{\sqrt{2}}(dx^1 \wedge dx^2 - dy^1 \wedge dy^2)$$

$$\omega_1 = \frac{1}{\sqrt{2}}(dx^1 \wedge dx^2 + dy^1 \wedge dy^2)$$

serve as coordinate functions on $G(2,4)$, see for example section 3 in [12]. We may consider them as the height function on each of the two $S^2(\frac{1}{\sqrt{2}})$. The condition $|\lambda_1\lambda_2| < 1$ corresponds to $*\omega_1 > 0$ and $*\omega_2 > 0$, which means the image of the Gauss map lies in the product of the two hemispheres. These are natural geometric condition. It is shown in [12] that under this assumption, the maximum principle not only works in the elliptic case but also in the parabolic one.

Contrary to the codimension one case, the equation of the second fundamental form has not been used in higher codimension to our knowledge . We expect a complete classification theorem of graphic calibrated submanifolds will rely on such estimate.

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2 General results

We first recall a formula whose parabolic version was derived in [13]. To apply the formula in [13] to the current situation, we note that a minimal submanifold corresponds to stationary phase of mean curvature flow.

Let Σ be an n-dimensional submanifold of $\mathbb{R}^{n+m}$ and Ω a parallel n form on $\mathbb{R}^{n+m}$. Around any point $p \in \Sigma$, we choose any orthonormal frames $\{e_i\}_{i=1\cdots n}$ for $T_p\Sigma$ and $\{e_\alpha\}_{\alpha=n+1,\cdots,n+m}$ for $N_p\Sigma$, the normal bundle of Σ . The convention that $i, j, k, \cdots$ denote tangent indexes and $\alpha, \beta, \gamma \cdots$ denote normal indexes is followed.

The second fundamental form of Σ is denoted by $h_{\alpha ij} =< \nabla_{e_i} e_j, e_\alpha >$.

If we assume Σ has parallel mean curvature vector, then $*\Omega = \Omega(e_1, \cdots, e_n)$ satisfies (see Proposition 3.1 in [13])

$$\Delta * \Omega + *\Omega(\sum_{\alpha,l,k} h_{\alpha lk}^2)$$
$$- 2\sum_{\alpha,\beta,k}[\Omega_{\alpha\beta 3\cdots n} h_{\alpha 1k} h_{\beta 2k} + \Omega_{\alpha 2\beta\cdots,n} h_{\alpha 1k} h_{\beta 3k} + \cdots + \Omega_{1\cdots(n-2)\alpha\beta} h_{\alpha(n-1)k} h_{\beta nk}] = 0 \tag{2.1}$$

$$(*\Omega)_k = \sum_\alpha \Omega_{\alpha 2\cdots n} h_{\alpha 1k} + \cdots + \Omega_{1\cdots n-1\alpha} h_{\alpha nk} \tag{2.2}$$

where Δ is the Laplace operator of the induced metric on Σ and $\Omega_{\alpha\beta 3\cdots n} = \Omega(e_\alpha, e_\beta, e_3, \cdots, e_n)$.

Now take $\Omega = dx^1 \wedge \cdots \wedge dx^n$, i.e the volume form of $\mathbb{R}^n$. In this case $*\Omega = \Omega(e_1, \cdots, e_n)$ is exactly the Jacobian of the projection from Σ to $\mathbb{R}^n$. A similar formula in this case appeared in Fischer-Colbrie's paper [3] (Lemma 1.1).

When $m = 1$, i.e. the codimension one case, $\partial_i = \frac{\partial}{\partial x^i} + \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^{n+1}}$ forms a basis for $T_p\Sigma$. Then

$$*\Omega = \frac{\Omega(\partial_1, \cdots, \partial_n)}{\sqrt{\det(\partial_i, \partial_j)}}$$

It is not hard to compute that $*\Omega = \frac{1}{\sqrt{1+|df|^2}}$, i.e. the angle made by the normal vector of Σ and the x_{n+1} axis. Thus formula (2.1) reduces to the classical formula for nonparametric minimal surfaces.

$$\Delta\frac{1}{\sqrt{1+|df|^2}} + \frac{1}{\sqrt{1+|df|^2}}|A|^2 = 0$$

A standard argument will imply the Bernstein type result for bounded $|df|$. In fact, this equation coupled with Simon's identity gives the optimal results in codimension one case, see [2].

In the general dimensional case, we choose a particular basis to represent formula (2.1) for $\Omega = dx^1 \wedge \cdots \wedge dx^n$. Given any p then the differential of f is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. We can use singular value decomposition to find orthonormal bases $\{a_i\}_{i=1\cdots n}$ for $\mathbb{R}^n$ and $\{a_\alpha\}_{\alpha=n+1,\cdots,n+m}$ for $\mathbb{R}^m$ such that

$$df(a_i) = \lambda_i a_{n+i}$$

for $i = 1\cdots n$. Notice that $\lambda_i = 0$ if $i > \min\{n, m\}$.

Now $\{e_i = \frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i a_{n+i})\}_{i=1,\cdots,n}$ forms an orthonormal basis for $T_p\Sigma$ and $\{e_\alpha = \frac{1}{\sqrt{1+\lambda_{\alpha-n}^2}}(a_\alpha - \lambda_{\alpha-n}a_{\alpha-n})\}_{\alpha=n+1,\cdots,n+m}$ an orthonormal basis for $N_p\Sigma$.

It is not hard to see $*\Omega = \frac{1}{\sqrt{\prod_{i=1}^n(1+\lambda_i^2)}}$. Apply these bases to equation (2.1).

Proposition 2.1 *Let Σ be the graph of a smooth map $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ and $\{\lambda_i\}_{i=1\cdots n}$ be the singular values of df. If the mean curvature vector of Σ is parallel, then $*\Omega$ satisfies the following equation.*

$$\Delta * \Omega = - * \Omega\{\sum_{\alpha,l,k} h_{\alpha lk}^2 - 2\sum_{k,i<j}\lambda_i\lambda_j h_{n+i,ik}h_{n+j,jk} + 2\sum_{k,i<j}\lambda_i\lambda_j h_{n+j,ik}h_{n+i,jk}\} \tag{2.3}$$

where Δ is the Laplace operator of the induced metric on Σ. The indexes i, j range between 1 and $\min\{n, m\}$.

Proof of Theorem A.

We shall calculate

$$\Delta(\ln *\Omega) = \frac{*\Omega\Delta(*\Omega) - |\nabla * \Omega|^2}{| * \Omega|^2} \tag{2.4}$$

By formula (2.2), the covariant derivative of $*\Omega$ is .

$$(*\Omega)_k = - * \Omega(\sum_i \lambda_i h_{n+i,ik})$$

Plug this and equation (2.1) into equation (2.4) and we obtain

$$\Delta(\ln *\Omega) = -\{\sum_{\alpha,l,k} h_{\alpha lk}^2 + \sum_{k,i} \lambda_i^2 h_{n+i,ik}^2 + 2\sum_{k,i<j} \lambda_i\lambda_j h_{n+i,jk}h_{n+j,ik}\} \qquad (2.5)$$

From the assumption of the theorem, it is obvious that $\Delta(\ln *\Omega) \leq -\delta_1|A|^2$ by completing square. The condition $\frac{1}{\sqrt{\det(I+(df)^t df)}} \geq K$ means Σ is the graph of a vector-valued function with bounded gradient. Therefore we can perform blow down to get a minimal cone. Notice that these conditions are invariant under scaling. We can apply maximum principle to conclude the minimal cone is flat and then Allard regularity theorem forces Σ to be an affine space.

$\square$

Proof of Theorem B. The optimal condition can be found by considering

$$F(h_{\alpha lk}) = \sum_{\alpha,l,k} h_{\alpha lk}^2 + \sum_{k,i} \lambda_i^2 h_{n+i,ik}^2 + 2\sum_{k,i<j} \lambda_i\lambda_j h_{n+i,jk}h_{n+j,ik}$$

as a quadratic form define on the space of all possible $h_{\alpha lk}$ with $h_{\alpha lk} = h_{\alpha kl}$ and $\sum_k h_{\alpha kk} = 0$ for each α. Now the problem is reduced to determine conditions on $\lambda_i\lambda_j$ to guarantee F is a positive quadratic form with $F(h_{\alpha lk}) \geq \epsilon \sum h_{\alpha lk}^2$ for some $\epsilon > 0$. It is computable, yet it seems hard to find a simple description. At the end of next section, we show how this gives the optimal result when $n = 2$.

$\square$

The condition in Theorem A or B depends on the choice a subspace over which Σ is written a graph. Since being minimal is invariant under the isometry action of $O(n+m)$, a slightly more general condition is the following.

Corollary 2.1 *If Σ is the graph of $f : \mathbb{R}^n \mapsto \mathbb{R}^m$. Denote $A = df$ as an $n \times m$ matrix . If there exists an element $g \in O(n+m)$ of the block form $g = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ such that $P + AR$ is invertible and the singular values of $(P+AR)^{-1}(Q+AS)$ satisfy the optimal condition in Theorem B , then Σ is an affine subspace.*

Proof The tangent space of Σ is represented by $\begin{bmatrix} I & A \end{bmatrix}$. Under the action of g on $\mathbb{R}^{n+m}$, the tangent space of $g(\Sigma)$ becomes $\begin{bmatrix} P+AR & Q+AS \end{bmatrix}$. If $P+AR$ is invertible, then $g(\Sigma)$ is still the graph of some $\bar{f}$ over $\mathbb{R}^n$. Now $d\bar{f} = (P+AR)^{-1}(Q+AS)$. □

When Σ is a Lagrangian submanifold defined as the graph of $\nabla F : \mathbb{R}^n \mapsto \mathbb{R}^n$. Let $A = D^2F$. Note being minimal Lagrangian is invariant under $U(n)$. An element g in $U(n)$ can be expressed as a $2n \times 2n$ block matrix of the form

$$\begin{bmatrix} P & -Q \\ Q & P \end{bmatrix}$$

with

$$PP^t + QQ^t = I, -PQ^t + QP^t = 0$$

where the first n components correspond to the real x^i and the last n components the imaginary y^i.

Rotate $\mathbb{C}^n$ by this element g, the tangent space becomes $\begin{bmatrix} P+AQ & -Q+AP \end{bmatrix}$ If we require $P+AQ$ is invertible then the condition is determined by the eigenvalues of the symmetric matrix $(P+AQ)^{-1}(-Q+AP)$. Please see [11] for the optimal condition in this case.

3 Applications

In the last section, we show how our theorems imply all known general graphic Bernstein type results in higher codimension. In Theorem 1 of [7], the authors proved a Bernstein type theorem in terms of a bound on

$$\Delta_f = \{\det(\delta_{\alpha\beta} + \sum_i f^i_{x_\alpha}(x) f^i_{x_\beta}(x))\}^{\frac{1}{2}} < 2$$

This result improves those in [3] and [5]. This quantities of course is our $\frac{1}{*\Omega} = \sqrt{\prod_{i=1}^n (1+\lambda_i^2)}$.

It is not hard to check that $\prod_{i=1}^n (1+\lambda_i^2) < 4$ implies $|\lambda_i \lambda_j| < 1$ for $i \neq j$. Therefore, Theorem A implies the result in [7].

As was mentioned above, Theorem A only assumes Σ is of parallel mean curvature vector. The condition is weakened if we further assume Σ is minimal, i.e $\sum_k h_{\alpha kk} = 0$. We demonstrate how it works in two-dimension. Write out the right hand side of equation (2.5) when $n = 2$.

$$|A|^2 + \lambda_1^2(h_{n+1,1,1}^2 + h_{n+1,1,2}^2) + \lambda_2^2(h_{n+2,2,1}^2 + h_{n+2,2,2}^2) + 2\lambda_1\lambda_2(h_{n+1,2,1}h_{n+2,1,1} + h_{n+1,2,2}h_{n+2,1,2}) \tag{3.1}$$

Since Σ is minimal, $h_{n+1,1,1} = -h_{n+1,2,2}$ and $h_{n+2,2,2} = -h_{n+2,1,1}$. This can be completed square and becomes

$$|A|^2 + (\lambda_1 h_{n+1,2,2} + \lambda_2 h_{n+2,1,2})^2 + (\lambda_1 h_{n+1,1,2} + \lambda_2 h_{n+2,1,1})^2$$

This in fact applies to three dimensional minimal cone because the second fundamental form vanishes in one direction and the right hand side of equation (2.5) reduces to the two-dimensional case. This recovers the result of Barbosa [1] and Fischer-Colbrie [3].

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